THE NON-UNIQUENESS OF SUPERTRACE.

by

N.B. BACKHOUSE

Department of Applied Mathematics and Theoretical Physics,
University of Liverpool,
P.O. Box 147 ,
Liverpool, L693BX
U.K.

and

A. G. FELLOURIS

Department of Mathematics,
National Technical University,
Zografou Campus,
Athens 15773
GREECE.

1. INTRODUCTION

The mathematical forms supporting supersymmetry are obtained from the familiar ones of mathematical physics by a Grassmann-algebraic version of complexification, the so-called "Grassmannification". It means the replacement of real or complex numbers with elements from a Grassmann algebra. Some of the basic objects obtained in this way are, for example, supermatrices, supergrours, supermanifolds[1].

In this talk we investigate the linear invariants, generalizing the trace function, for algebras of supermatrices. First we recall [2] some basic results on Grassmann algebras.

Denote by $B\rho(F)$, ρ finite, the Grassmann algebra over the field F generated by the identity and the ρ mutually anticommuting generators θ_i , $i=1,2,\ldots,\rho$. For convenience we suppress mention of the field F, which is either the real or complex numbers. The subspace $B_{\rho,e}$ (resp. $B_{\rho,u}$) is the even (resp. uneven or odd) subspace, consisting of linear combinations of products of an even (resp. odd) number of generators. B_ρ is the direct sum of the $2^{\rho-1}$-dimensional subspaces $B_{\rho,e}$ and $B_{\rho,u}$. The elements of $B_{\rho,e}$ commute with all elements of B_ρ, whereas the elements of $B_{\rho,u}$ mutually anticommute.

There is an alternative decomposition $B_\rho = B_{num} \oplus B_{nil}$, where $B_{num}=F1$ is called the numeric component of B_ρ and B_{nil} consists of all linear combinations of a non-zero number of generators.

Considering B_∞ to be formal linear combinations of finite products of elements of a countably infinite number of independent anticommuting generators, all these definitions can be extended to ρ infinite[3].

A supermatrix over B_ρ is a block-form matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad , \tag{1}$$

where the entries in A , D (resp. B , C) belong to $B_{\rho,e}$ (resp. $B_{\rho,u}$). We denote by $gl(m,n)$ the full matrix algebra over B_ρ, consisting of supermatrices M, where A, D have sizes $m\times m$ and $n\times n$, respectively.

The algebra of fully Grassmannified matrices of size $n\times n$ is isomorphic to a subalgebra $q(\rho;n)$ or simply $q(n)$ of $gl(\rho;n,n)$ consisting of the so-called Q-type supermatrices M with $A = D$ and $B = C$[4].

The supertrace function is a map $str: gl(m,n) \to B_{\rho,e}$ defined by

$$strM = trA - trD \, , \tag{2}$$

where tr is the usual trace function. The basic property, implying invariance of supertrace under equivalence by an invertible supermatrix, is

$$str(MN) = str(NM), \tag{3}$$

for all M , N in $gl(m,n)$.

The ω-supertrace is a map $str_\omega : q(n) \to B_{\rho,e}$ defined[4] by

$$\mathrm{str}_\omega M = \omega \mathrm{tr} B, \qquad (4)$$

where ω is an arbitrary element in $B_{\rho,u}$ and tr is the usual trace function.

The problem we consider in this talk is the following: are there any other functions which satisfy the invariance condition (3)? Obviously, without any further condition we cannot expect an interesting answer. Bearing in mind that $gl(m,n)$ is a $B_{\rho,e}$-module and that the supertrace function (2) is $B_{\rho,e}$-linear, it is reasonable to require that this condition holds. Then we find a generalized supertrace function which is a linear combination of the supertrace and a modestly distinct function, for ρ finite. For $\rho=\infty$ the supertrace function is unique.

Considering the subalgebra $q(n)$ of $gl(n,n)$ we find that the ω-supertrace is the unique invariant form for $\rho=\infty$. However, the ω-supertrace is not unique for ρ finite.

Next we review our results[5)] on the various types of trace functions, starting with a well-known result[6)] for the full matrix algebra $gl(n;F)$, where f is the field of real or complex numbers.

<u>Theorem 1:</u> Let $f: gl(n;F) \rightarrow F$ be an F-linear function satisfying, for all M,N in $gl(n;F)$,

$$f(MN) = f(NM) \qquad (5).$$

Then, $f(M) = \lambda \mathrm{tr} M$ for all M in $gl(n;F)$, where λ belongs to F and is independent of M. In more detail,

$$\lambda = f(E_{ii}), \qquad (6)$$

for every $i=1,2,...n$, where E_{ii} is the elementary matrix having 1 in the (i,i) position and zero elsewhere.

2. GENERALIZED SUPERTRACE

Bearing in mind that ordinary trace is characterized, up to scalar multiples, by the linearity condition and the invariance relation (5), we try to generalize supertrace function using axioms completely similar to that of Theorem 1. Thus we have the following:

Theorem 2: Let $f: gl(\rho;m,n) \rightarrow B_\rho$ be a function satisfying

(i) $B_{\rho,e}$ - linearity

(ii) $f(MN) = f(NM)$, for all M,N in $gl(\rho;m,n)$.

Then

$$f(M) = a\,\mathrm{str}M + \mu\,\mathrm{tr}D, \tag{7}$$

where a in B_ρ is arbitrary and μ in B_ρ annihilates all even nilpotent elements of B_ρ.

Moreover, μ has $\rho+1$ degrees of freedom if $\rho<\infty$, but $\mu=0$, if $\rho=\infty$.

3. GENERALIZED ω-SUPERTRACE

Clearly, the ordinary supertrace (2) vanishes identically on the subalgebra $q(n)$ of the superalgebra $gl(n,n)$. Moreover, the restriction of the generalized supertrace of §2 to $q(n)$, gives a $B_{\rho,e}$-linear invariant form and it is known that thereis a completely different invariant on $q(n)$, the ω-supertrace (4).

We seek a generalized supertace on $q(n)$ which exhibits both of these supertraces as special cases. We have the following:

Theorem 3: Let $f: q(\rho;n) \rightarrow B_\rho$ be a $B_{\rho,e}$-linear function satisfying the invariance condition

$$f(MN) = f(NM) \tag{8}$$

for all M,N in $q(\rho;n)$. Then

$$f(M) = a\,\mathrm{tr}A + g(B), \tag{9}$$

where a in B_ρ annihilates all even nilpotent elements of B_ρ and

$$g(B) = \sum_{i=1}^{n} g(B_{ii}\,E_{ii}), \tag{10}$$

where

$$B = \sum_{i,j=1}^{n} B_{ij}\,E_{ij} \tag{11}$$

with B_{ij} in $B_{\rho,u}$ and E_{ij} being the elementary matrix having 1 in the (i,j) position and zero elsewhere.

Furthermore, the function g is $B_{\rho,e}$-linear and

$$g(B) = \omega trB + E(B) \ , \tag{12}$$

where ω in B_ρ is a sum of elements of maximum degree $\rho-3$, $\rho>2$ and $E(B)$ consists of higher order terms. More precisely, the function E is $B_{\rho,e}$-linear and its values are linear combinations of terms of degree $\rho-1$ and ρ. This extends to $\rho=2$, if say a term of order (-1) is identically zero. For $\rho=\infty$ we have a=0 and E(B)=0 and the ω-supertrace is completely recovered.

References

1. DeWitt B., "Supermanifolds", Cambridge Univ. Press, Cambridge, 1984.

2. Backhouse N.B. and Fellouris A.G., "On the superdeterminant function for supermatrices",J.Phys.A, <u>17</u>, 1389-95 (1984),ibid,"Grassmann analogs of classical matrix groups" J.Math. Phys., <u>26</u>, 1146-51 (1985).

3. Rogers A., "A global theory of supermanifolds", J.Math. Phys. <u>21</u>, 1352 (1980).

4. Rittenberg V., "A guide to Lie superalgebras", Lecture Notes in Physics, <u>79</u>, 3-21 (1978).

5. Backhouse N.B.and Fellouris A.G., "Generalized supertrace functions", submitted for publication to J. Linear Algebra and Applications.

6. GREUB W., "Linear Algebra", Springer-Verlag, 1981.