Fermions on Colliding Branes

Yu-ichi Takamizu [1], Gary Gibbons [2] and Kei-ichi Maeda [3]

[1,3] *Department of Physics, Waseda University, Okubo 3-4-1, Shinjuku, Tokyo 169-8555, Japan*
[2] *DAMTP, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 OWA, UK*

Abstract

We study the behaviour of five-dimensional fermions localized on branes, which we describe by domain walls, when two parallel branes collide in a five-dimensional Minkowski background spacetime. We find that most fermions are localized on both branes as a whole even after collision. However, how much fermions are localized on which brane depends sensitively on the incident velocity and the coupling constants unless the fermions exist on both branes.

1 Introduction

In the 80's, one may regard our universe as a domain wall, or more generally a brane in a higher dimensional universe. The idea is that the fermionic chiral matter making up the standard model is composed of trapped zero modes. A similar mechanism is used in models, such as the Horawa-Witten model, in which two domain walls are present. The existence of models with more than one brane suggests that branes may collide, and it is natural to suppose that the Big Bang is associated with the collision [1]. This raises the fascinating questions of what happens to the localized fermions during such collisions? In this study we shall embark on what we believe is the first study of this question by solving numerically the Dirac equation for a fermions coupled via Yukawa interaction to a system of two colliding domain walls. Kink-anti-kink collisions, have recently been studied numerically [2, 3]. One may extend the treatment to include gravity [4, 5] but in this paper we shall, for the sake of our preliminary study, work throughout with gravity switched off.

2 Fermions on moving branes

We start with a discussion of five-dimensional (5D) four-component fermions in a time-dependent domain wall in 5D Minkowski spacetime. As a domain wall, we adopt a 5D real scalar field Φ with an appropriate potential $V(\Phi)$. The 5D Dirac equation with a Yukawa coupling term $g\Phi\bar{\Psi}\Psi$ is given by

$$(\Gamma^{\hat{A}}\partial_{\hat{A}} + g\Phi)\Psi = 0, \quad (\hat{A} = 0, 1, 2, 3, 5)\,, \tag{1}$$

where Ψ is a 5D four-component fermion. $\Gamma^{\hat{A}}$ are the Dirac matrices in 5D Minkowski spacetime satisfying the anticommutation relations. We introduce two chiral fermion states

$$\Psi_\pm = \frac{1}{2}\left(1 \pm \Gamma^{\hat{5}}\right)\Psi\,, \quad \Psi_+ = \begin{pmatrix} \psi_+ \\ \psi_+ \end{pmatrix}, \quad \Psi_- = \begin{pmatrix} \psi_- \\ -\psi_- \end{pmatrix}, \tag{2}$$

where ψ_+ and ψ_- are two-component spinors. The Dirac equation (1) is now reduced to

$$(\pm\partial_{\hat{5}} + g\Phi)\,\psi_\pm + \Gamma^{\hat{\mu}}\partial_{\hat{\mu}}\psi_\mp = 0\,. \tag{3}$$

As for a domain wall, now we assume the potential form is given by $V(\Phi) = \frac{\lambda}{4}\left(\Phi^2 - \eta^2\right)^2$. Then a domain wall solution is given by $\Phi = \epsilon \tanh(z/D)$ where $\epsilon = \pm$ correspond to a kink and an anti-kink

[1] E-mail:takamizu@gravity.phys.waseda.ac.jp
[2] E-mail:G.W.Gibbons@damtp.cam.ac.uk
[3] E-mail:maeda@waseda.jp

solutions and $D=\sqrt{2/\lambda}$ is the width of a domain wall. As for a fermion, in the case of a static domain wall, separating variables as $\psi_\pm = \overset{(4)}{\psi}_\pm(x^\mu) f_\pm(z)$ and assuming massless chiral fermions on a brane, i.e. $\Gamma^{\hat\mu}\partial_\mu \overset{(4)}{\psi}_\pm(x^\mu)=0$, we find the solutions are

$$f_\pm \propto \left[\cosh\left(\frac{z}{D}\right)\right]^{\mp\epsilon g D} . \tag{4}$$

Hence the positive-chiral (the negative-chiral) fermion is localized for a kink (an anti-kink) but is not localized for an anti-kink (a kink). To discuss fermions at collision of branes, we first discuss fermions on a domain wall moving with a constant velocity. Since 3-space is flat, we expand the wave functions by Fourier series. In what follows, we shall consider only low energy fermions, that is, we assume that $\vec{k}\approx 0$. The equations we have to solve are now

$$i\partial_0\psi_\pm = \left(\mp\partial_5 + g\Phi\right)\psi_\mp . \tag{5}$$

Since up- and down-components of $\psi_\pm$ are decoupled, we discuss only up-components here. With this ansatz, we can construct a localized fermion wave function on a moving domain wall with a constant velocity v. We find for a kink with velocity v,

$$\psi_+^{(\mathrm{K})}(z,t;v)=\sqrt{\frac{\gamma+1}{2}}\tilde{\psi}^{(\mathrm{K})}\left(\gamma(z-vt)\right) , \quad \psi_-^{(\mathrm{K})}(z,t;v)=i\frac{\gamma v}{\gamma+1}\sqrt{\frac{\gamma+1}{2}}\tilde{\psi}^{(\mathrm{K})}\left(\gamma(z-vt)\right) \tag{6}$$

and we also find solution for an anti-kink, where $\tilde{\psi}^{(\mathrm{K})}(\tilde{z})=f_+(\tilde{z})$ are static wave functions of chiral fermions localized on static kink. If a domain wall is given by a kink [an anti-kink], we have only the positive-chiral fermions in a comoving frame [the negative-chiral fermions]. However, from Eqs (6), we find that the negative-chiral modes [positive-chiral modes] also appear in this boosted Lorentz frame. The above wave functions on a moving domain wall can be used for setting the initial data for colliding domain walls.

3 Fermions on colliding domain wall

We construct our initial data as follows. Provide a kink solution at $z=-z_0$ and an anti-kink solution at $z=z_0$, which are separated by a large distance and approaching each other with the same speed v. We can set up as an initial profile for the scalar field Φ and fermions Ψ;

$$\Phi(z,t)=\Phi^{(\mathrm{K})}(z+z_0,t;v)+\Phi^{(\mathrm{A})}(z-z_0,t;-v)-1 , \tag{7}$$

$$\hat{\Psi}=\Psi_{\mathrm{in}}^{(\mathrm{K})}(x,z+z_0;v)a_{\mathrm{K}}+\Psi_{\mathrm{in}}^{(\mathrm{A})}(x,z-z_0;-v)a_{\mathrm{A}}+\Psi_{\mathrm{in}}^{(\mathrm{B})}(x,z)a_{\mathrm{B}} , \tag{8}$$

where $\Phi^{(\mathrm{K,A})}(z,t;v)=\pm\tanh(\gamma(z-vt)/D)$, and $\Psi_{\mathrm{in}}^{(\mathrm{K})}(x,z;v)$ and $\Psi_{\mathrm{in}}^{(\mathrm{A})}(x,z;-v)$ are the wave function of right-moving localized fermion on a kink and those of left-moving one on an anti-kink. We also denote the bulk fermions symbolically by $\Psi_{\mathrm{in}}^{(\mathrm{B})}(x,z)$. To quantize the fermion fields, we define annihilation operators of localized fermions on a kink and on an anti-kink by

$$a_{\mathrm{K}}=\langle\Psi^{(\mathrm{K})},\Psi\rangle \quad \text{and} \quad a_{\mathrm{A}}=\langle\Psi^{(\mathrm{A})},\Psi\rangle \tag{9}$$

Now we can set up an initial state for fermion by creation-annihilation operators. We shall call a domain wall associated with fermions a fermion wall, and a domain wall in vacuum a vacuum wall. We shall discuss two cases: one is collision of two fermion walls, and the other is collision of fermion and vacuum walls. For initial state of fermions, we consider two states;

$$|\mathrm{KA}\rangle\equiv a_{\mathrm{A}}^\dagger a_{\mathrm{K}}^\dagger|0\rangle \quad \text{and} \quad |\mathrm{K0}\rangle\equiv a_{\mathrm{K}}^\dagger|0\rangle \tag{10}$$

where $|0\rangle$ is a fermion vacuum state. We discuss behaviour of fermions at collision. After collision of two domain walls, each wall will recede to infinity with almost the same velocity as the initial one v. We define final fermion states as

$$\hat{\Psi}=\Psi_{\mathrm{out}}^{(\mathrm{K})}(x,z;-v)b_{\mathrm{K}}+\Psi_{\mathrm{out}}^{(\mathrm{A})}(x,z;v)b_{\mathrm{A}}+\Psi_{\mathrm{out}}^{(\mathrm{B})}(x,z)b_{\mathrm{B}} , \tag{11}$$

where $b_{\rm K}$, $b_{\rm A}$ and $b_{\rm B}$ are annihilation operators of those fermion states. We find the relations between ingoing and outgoing states by solving the Dirac equation (5);

$$b_{\rm K} = \alpha_{\rm K} a_{\rm K} + \beta_{\rm A} a_{\rm A}\,, \quad b_{\rm A} = \alpha_{\rm A} a_{\rm A} + \beta_{\rm K} a_{\rm K}\,, \tag{12}$$

Using the Bogoliubov coefficients $\alpha_{\rm K}, \beta_{\rm K}$ and $\alpha_{\rm A}, \beta_{\rm A}$, we obtain the expectation values of fermion number on a kink and an anti-kink after collision as

$$\langle N_{\rm K}\rangle \equiv \langle {\rm KA}|b^\dagger_{\rm K} b_{\rm K}|{\rm KA}\rangle = |\alpha_{\rm K}|^2 + |\beta_{\rm A}|^2\,, \tag{13}$$

$$\langle N_{\rm A}\rangle \equiv \langle {\rm KA}|b^\dagger_{\rm A} b_{\rm A}|{\rm KA}\rangle = |\alpha_{\rm A}|^2 + |\beta_{\rm K}|^2\,, \tag{14}$$

for the case of $|{\rm KA}\rangle$. If the initial state is $|{\rm K0}\rangle$, we find

$$\langle N_{\rm K}\rangle \equiv \langle {\rm K0}|b^\dagger_{\rm K} b_{\rm K}|{\rm K0}\rangle = |\alpha_{\rm K}|^2\,, \tag{15}$$

$$\langle N_{\rm A}\rangle \equiv \langle {\rm K0}|b^\dagger_{\rm A} b_{\rm A}|{\rm K0}\rangle = |\beta_{\rm K}|^2\,. \tag{16}$$

In order to obtain the Bogoliubov coefficients, we have to solve the equations for domain wall Φ [3] and fermion Ψ numerically. From the solution (4), we find the fermions are localized within the domain wall width D if $g \gtrsim 2/D$. When $g < 2/D$, fermions leak out from the domain wall. Hence, in this paper, we analyze for the case of $g \geq 2$ with setting $D = 1$, but leave υ free. To obtain the Bogoliubov coefficients, we solve the Dirac equation for the collision of fermion-vacuum walls. We shall give numerical results only for the case that positive chiral fermions are initially localized on a kink. Because of z-reflection symmetry, we find the same Bogoliubov coefficients for the case that negative chiral fermions are initially localized on an anti-kink, i.e. $|\alpha_{\rm K}|^2 = |\alpha_{\rm A}|^2$. The Bogoliubov coefficients depend on the initial wall velocity. In Table 1, we summarize our results for different values of velocity and Yukawa coupling constant.

υ	$g=2$			$g=2.5$		
	$\|\alpha_{\rm K}\|^2$	$\|\beta_{\rm K}\|^2$	$\|\gamma_{\rm K}\|^2$	$\|\alpha_{\rm K}\|^2$	$\|\beta_{\rm K}\|^2$	$\|\gamma_{\rm K}\|^2$
0.3	0.94	0.056	0.004	0.47	0.53	0.00
0.4	0.87	0.12	0.01	0.57	0.40	0.03
0.6	0.69	0.30	0.01	0.78	0.17	0.05
0.8	0.42	0.55	0.03	0.88	0.02	0.10

Table 1: The Bogoliubov coefficients of fermion wave functions localized on each domain wall after collision ($|\alpha_{\rm K}|^2$ and $|\beta_{\rm K}|^2$) with respect to the initial velocity υ. We also show the amount of fermions escaped into bulk space ($|\gamma_{\rm K}|^2 = 1 - (|\alpha_{\rm K}|^2 + |\beta_{\rm K}|^2)$).

For the coupling constant $g = 2$, $|\alpha_{\rm K}|^2$ and $|\beta_{\rm K}|^2$ are almost equal (0.44 and 0.55), but for $g = 2.5$, most fermions remain on the kink ($|\alpha_{\rm K}|^2 = 0.88$ and $|\beta_{\rm K}|^2 = 0.02$). We find that the Bogoliubov coefficients depend sensitively on the coupling constant g as well as the velocity υ. In Fig. 1, we shows the g-dependence. Since the wave function is changed at collision, when the background scalar field evolves in a complicated way, one might think that the behaviour of wave function would be difficult to describe analytically. However, we may understand the qualitative behaviour in terms of the naive estimation. As a result, we obtain the formula;

$$|\alpha_{\rm K}|^2, |\beta_{\rm K}|^2 \approx \frac{1}{2}\left[1 \pm \sin\left(2\varepsilon g \Phi_c \Delta t + C_0\right)\right]\,, \tag{17}$$

where $\varepsilon = \pm 1$ and C_0 is an integration constant. Comparing the numerical data and the formula (17) with $\Phi_c \approx -1.5$, we find the fitting curves in Fig. 1 ($\varepsilon = -1$, $\Delta t \approx 1.4$ and $C_0 = -1.2$). This fitting formula explains our numerical results very well.

Finally, we can evaluate the expectation values of fermion numbers after collision as follows. For the initial state of fermions, we consider two cases: case (a) collision of two fermion walls $|{\rm KA}\rangle$ and case (b) collision of fermion and vacuum walls $|{\rm K0}\rangle$. In the case (a), we find

$$\langle N_{\rm K}\rangle = |\alpha_{\rm K}|^2 + |\beta_{\rm A}|^2 = |\alpha_{\rm K}|^2 + |\beta_{\rm K}|^2 \approx 1\,, \quad \langle N_{\rm A}\rangle = |\alpha_{\rm A}|^2 + |\beta_{\rm K}|^2 = |\alpha_{\rm A}|^2 + |\beta_{\rm A}|^2 \approx 1\,. \tag{18}$$

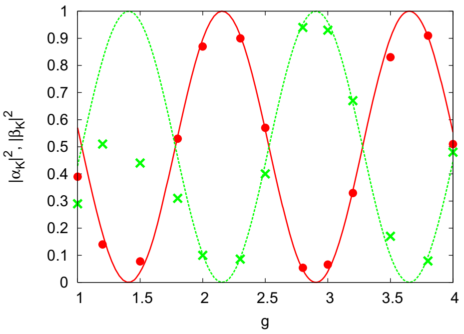

Figure 1: The Bogoliubov coefficients ($|\alpha_{\rm K}|^2, |\beta_{\rm K}|^2$) with $v = 0.4$ in terms of a coupling constant g. The circle and the cross denote $|\alpha_{\rm K}|^2$ and $|\beta_{\rm K}|^2$ respectively. Two sine curves ($|\alpha_{\rm K}|^2, |\beta_{\rm K}|^2 \approx [1 \pm \sin(4.2g - 1.2))]/2$ show the formula (17) with the best-fit parameters.

We find that most fermions on domain walls remain on both walls even after the collision. A small amount of fermions escapes into the bulk spacetime at collision. In the case (b), however, we obtain

$$\langle N_{\rm K} \rangle = |\alpha_{\rm K}|^2 \;\; , \;\; \langle N_{\rm A} \rangle = |\beta_{\rm K}|^2 \, . \tag{19}$$

Since the Bogoliubov coefficients depend sensitively on both the velocity v and the coupling constant g, the amount of fermions on each wall is determined by the fundamental model as well as the details of the collision of the domain walls.

4 Conclusion

We have studied the behaviour of five-dimensional fermions localized on domain walls, when two parallel walls collide in five-dimensional Minkowski background spacetime. We analyzed the dynamical behaviour of fermions during the collision of fermion-fermion branes (case (a)) and that of fermion-vacuum walls (case (b)). In case (a), we find that most fermions are localized on both branes even after collision. In case (b), however, some fermions jump up to the vacuum brane at collision. The amount of fermions localized on which brane depends sensitively on the incident velocity v and the coupling constants $g/\sqrt{\lambda}$. The detailed discussion is shown in [6].

References

[1] J. Khoury, B. A. Ovrut, P. J. Steinhardt and N. Turok, Phys. Rev. D **64**, 123522 (2001) [hep-th/0103239]

[2] P. Anninos, S. Oliveira and R. A. Matzner, Phys. Rev. D **44**, 1147 (1991).

[3] Y. Takamizu and K. Maeda, Phys. Rev. D **70** (2004) 123514 [hep-th/0406235].

[4] Y. Takamizu and K. Maeda, Phys. Rev. D **73** (2006) 103508 [hep-th/0603076].

[5] G.W. Gibbons, H. Lu, C.N. Pope, Phys. Rev. Lett. **94**, 131602 (2005) [hep-th/0501117].

[6] G. Gibbons, K. Maeda and Y. Takamizu, arXiv:hep-th/0610286.