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The Penner matrix model and c=1 strings

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The steepest descent solution of the Penner matrix model has a one-cut eigenvalue support. Criticality results when the two branch points of this support coalesce. The support is then a closed contour in the complex eigenvalue plane. Simple generalizations of the Penner model have multi-cut solutions. For these models, the eigenvalue support at criticality is also a closed contour, but consisting of several cuts. We solve the simplest such model, which we call the KT model, in the double-scaling limit. Its free energy is a Legendre transform of the free energy of the c=1 string compactified to the critical radius of the Kosterlitz–Thouless phase transition.

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1. Introduction

The Penner matrix model was introduced as a means of generating the Euler characteristic of the moduli space of Riemann surfaces[1]. Recently Distler and Vafa have shown[2,3] that the double scaling critical limit of this model is related to the c=1 string. In particular, the free energy is a Legendre transform of the free energy of the c=1 string compactified to the self-dual radius, as computed by Gross and Klebanov in the singlet sector[4].

This result motivated us to analyze the Penner matrix model more closely, using the techniques successfully applied to matrix models with polynomial potentials[5–11]. We find that criticality in the Penner model has a number of new and interesting features. We also construct two distinct generalizations of the Penner model; one of these appears to provide a realization of the c=1 string compactified to any integer multiple of the self-dual radius.

2. Eigenvalue analysis of the Penner matrix model

In the naive large N limit (i.e. the spherical limit) a hermitian matrix model with potential $V(\Phi)$ is dominated by the steepest descent configuration for the eigenvalues $\lambda(x)=\lambda(i/N)=\sqrt{N}\lambda_i$, which is described (in general) by a normalized density measure $d\rho(\lambda)$ which has support on some contour $\mathcal{C}$ in the complex λ-plane. Following [12] and [9], the spherical solution is conveniently analyzed in terms of the generating function

$$F(\lambda) = \int d\rho(\mu)\frac{1}{\lambda-\mu} \tag{2.1}$$

and the primitive

$$G(\lambda) = \int^{\lambda} d\mu\,(V'(\mu) - 2F(\mu)) \tag{2.2}$$

We will first consider one-cut solutions, for which the support $\mathcal{C}$ consists of a single connected component. Thus $\mathcal{C}$ has two endpoints, at $b=\lambda(0)$ and $a=\lambda(1)$. The function $G(\lambda)$ has branch points at a and b, and is pure imaginary along $\mathcal{C}$: $G[\lambda(x)] = \pm 2i\pi x$. Thus, along $\mathcal{C}$, $G'(\lambda)$ is proportional to the eigenvalue density $u(\lambda)=dx/d\lambda$:

$$G'(\lambda) = \pm 2i\pi u(\lambda) \tag{2.3}$$

Furthermore $G(\lambda)$ can be interpreted as the action of a single eigenvalue. Thus the steepest descent solution is stable only if the real part of $G(\lambda)$ is positive along the *entire* integration contour for λ in the original path integral [9].

In Penner's matrix model, the potential is nonpolynomial:

$$V(\lambda) = \lambda + \log(1 - \lambda) \tag{2.4}$$

We can solve for the generating function $F(\lambda)$ using the following ansatz:

$$F(\lambda) = \frac{1}{2}\left[V'(\lambda) - \frac{f}{\lambda - 1}\sqrt{(\lambda - a)(\lambda - b)}\right] \tag{2.5}$$

This ansatz has two branch points which will give rise to a single branch cut. It has three undetermined parameters: f, a, and b. These parameters can be determined from a consideration of the analytic structure and asymptotic form of F.

From the expression (2.1) for F in terms of the spectral density, we can see that for $|\lambda| \to \infty$ we must have $F \to 1/\lambda$. This requires us to choose a branch of the square root in (2.5) which goes as λ at large λ, without branch cuts at infinity. It also puts two conditions on the parameters from matching the coefficients of λ^0 and λ^{-1}. This leaves one of the parameters undetermined.

In addition to these requirements, (2.1) also shows that F should not be singular at $\lambda = 1$, if we assume that the spectral density is not singular. This provides one additional constraint on the free parameters. For the potential we are considering, this is sufficient to completely determine F. We can solve these constraints explicitly:

$$\begin{aligned} 0 &= 1 - f \\ 1 &= \tfrac{1}{2}(a + b)f \\ 0 &= 1 - f\sqrt{(1 - a)(1 - b)} \end{aligned} \tag{2.6}$$

It is easy to see that these equations imply a=b=2.

We thus discover that criticality in the Penner model is the result of the coalescing of the two branch points of $G'(\lambda)$. This is qualitatively different from the critical behavior of polynomial hermitian matrix models, which results[9] from the coalescing of a branch point of $G'(\lambda)$ with a *zero* of $G'(\lambda)$.

The approach to criticality can be studied by introducing an overall coupling $1/\gamma$ in front of the potential $V(\lambda)$ (γ is related to the cosmological constant). For γ=1 we recover the critical Penner model with a=b=2. For γ slightly less than one, i.e.: $\gamma = 1-\delta^2\mu$,

$\delta \ll 1$, the branch points are split slightly above and below the real axis: $a,b \simeq 2 \pm 2i\delta\sqrt{\mu}$. The branch cut (which coincides with the eigenvalue support C) lies along some curve connecting a and b. If this curve corresponded to, say, a straight line connecting a to b, we would obtain the pathological result that the eigenvalue support shrinks to a point at criticality. However, such behavior is not consistent with our solution for $F(\lambda)$. Recall that the asymptotic behavior of $F(\lambda)$ requires that $arg[\sqrt{(\lambda - a)(\lambda - b)}]$ vanishes as $\lambda \to +\infty$ on the real axis. On the other hand, our solution of (2.6) requires that $arg[\sqrt{(\lambda - a)(\lambda - b)}]$ also vanishes as λ approaches 1 on the real axis. So it cannot be the case that the branch cut crosses the real axis anywhere on the interval $[1, \infty)$.

Thus we have discovered that consistency requires that the branch cut will *not* disappear when $a \to b$ but will instead form a closed loop encircling the point λ=1 and passing through λ=2. Therefore the eigenvalue support will not shrink to a point, as has been previously suggested[13]. Since the eigenvalue support C corresponds to a contour of $Re(G(\lambda))$=0, we can plot it after computing $G(\lambda)$. First we obtain $G'(\lambda)$ at criticality:

$$G'(\lambda) = V'(\lambda) - 2F(\lambda) = \frac{\sqrt{(\lambda - 2)^2}}{\lambda - 1} \tag{2.7}$$

Outside of the branch cut loop, the square root is just equal to $\lambda-2$. So in this case we can easily integrate to give

$$G(\lambda) = (\lambda - 2) - \log(\lambda - 1) + 2\pi i \tag{2.8}$$

where the integration constant has been chosen such that $G(\lambda)$ runs from 0 to $2\pi i$ as we move just outside the square root branch cut from b to a. The branch cut of the logarithm in (2.8) is chosen to extend along the positive real axis. From (2.8) we obtain the eigenvalue support C, which is plotted in fig. 1. Note that, although a=b at criticality, these endpoints are on opposite sides of the log branch cut.

Away from exact criticality, the correct scaling is given by

$$\begin{aligned} \gamma &= 1 - \delta^2\mu \\ \lambda &= 2 - \delta z \\ a, b &\simeq 2 \pm 2i\delta\sqrt{\mu} - 2\delta^2\mu \end{aligned} \tag{2.9}$$

Which gives

$$G'(z) = \delta\sqrt{z^2 + 4\mu} \tag{2.10}$$

As mentioned above, for $\mu>0$ the endpoints a and b are split above and below the real axis. In addition, the $Re(G)=0$ contour C' of fig. 1 now consists of three segments, and is still connected to C at the points a and b. $Re(G)$ is positive in the region to the right of C'. Thus it is easy to see that there exists a smooth curve for which $ReG\geq 0$, which includes C, and which approaches C' asymptotically (from the right). Such a curve can be used to define the original path integral of the matrix model, and guarantees stability of the large N steepest descent solution. Note also that, along such a curve, $arg(\lambda) = \pm i$ asymptotically. Thus this definition of the model is equivalent, asymptotically, to replacing $V(\lambda)$ by $V(i\lambda)$ in (2.4), which is, in turn, a common definition of the Penner model.

We conclude this section with a discussion of multi-cut solutions[14–18] and other critical behaviors. Clearly for the Penner potential (2.4) a multi-cut ansatz for $F(\lambda)$ cannot be consistent. However we may wish to consider a generalization of the Penner model in which the linear term in (2.4) is replaced by an arbitrary polynomial of degree n:

$$\begin{aligned} V(\lambda) &= U(\lambda) + \log(1-\lambda) \\ U(\lambda) &= \sum_{k=1}^{n} \frac{g_k}{k}\lambda^k \end{aligned} \tag{2.11}$$

Then we take as an ansatz for the generating function:

$$F(\lambda) = \frac{1}{2}\left[\frac{1}{\lambda-1} + U'(\lambda) - \frac{f}{\lambda-1}\sqrt{(\lambda-a_1)(\lambda-a_2)\cdots(\lambda-a_{2n})}\right] \tag{2.12}$$

This is a perfectly general ansatz for F with up to n branch cuts in the square root. Fewer branch cuts may be obtained by allowing some of the parameters a_i to coincide and then defining the square root appropriately. This ansatz has $2n+1$ independent parameters, f and a_i, which must be determined. Conditions on these parameters may be obtained by an analysis similar to that above. Again for large λ, F must go as $1/\lambda$. This gives $n+1$ conditions. There is as before a finiteness condition at $\lambda = 1$. This leaves $n-1$ free parameters. To look for new types of critical behavior, we look for a solution where $a_{2i}=a_{2l-1}$ for $i=1,\ldots,n$. This makes the problem over-determined and it is a non-trivial fact that such a solution exists. In principle such a solution might exhibit multiple loops or other unusual features.

However, we in fact obtain the following results at criticality. Outside of any branch cut loops, we find the simple expression:

$$G(\lambda) = U(\lambda) - \log(\lambda-1) + \text{const} \tag{2.13}$$

This function has only one nontrivial closed contour of $Re(G)$=0. Further, since $G(\lambda) \to -G(\lambda)$ inside this contour, we can verify that there are no nested contours. Thus the eigenvalue support at criticality always consists of a single closed contour.

If this contour is smooth, apart from the right angle where the branch points coalesce, then we have a one-cut solution. This will either be in the universality class of the Penner model, or can exhibit new critical behavior if we arrange for zeroes of $G'(\lambda)$ to coalesce with its branch points. The simplest example of such behavior occurs for the model

$$V(\lambda) = -\frac{1}{2}\lambda^2 + 3\lambda + \log(1-\lambda) \tag{2.14}$$

This is a critical model with eigenvalue density

$$u(\lambda) = \frac{1}{4\pi i}\frac{\sqrt{(\lambda-a)^3(\lambda-b)}}{\lambda-1} \tag{2.15}$$

where a=b=2. We will not analyze such models in this paper, since they have no simple connection to the c=1 string. However, they are interesting in their own right.

If the contour is not smooth, but instead consists of two or more smooth segments, then we obtain a multi-cut solution. The simplest example of a multi-cut solution results from the model

$$V(\lambda) = \frac{1}{2}(\lambda-1)^2 + \log(1-\lambda) \tag{2.16}$$

At criticality we obtain for this model

$$G'(\lambda) = \frac{\sqrt{\lambda^2(\lambda-2)^2}}{\lambda-1} \tag{2.17}$$

and, outside of the branch cut loop,

$$G(\lambda) = \frac{1}{2}\lambda(\lambda-2) - \log(\lambda-1) + 2\pi i \tag{2.18}$$

As in the previous case we take the log branch cut to extend along the positive real axis. The eigenvalue support C here consists of a closed contour surrounding the log branch point, and intersecting the real axis at λ=0 and 2. This is plotted in fig. 2. The contour C has right angles at both λ=0 and 2, and thus consists of two smooth segments. Indeed from (2.17) it is clear that we have a two-cut solution, in the critical limit where both pairs of branch points coalesce simultaneously. One can explicitly verify that off criticality C breaks up into two disconnected pieces, and that it is possible to satisfy the stability criteria of [9]. As we will show, the double scaling limit of this model corresponds to the c=1 string compactified to the critical radius of the Kosterlitz–Thouless phase transition[4]; thus we have dubbed this the "KT model".

The construction just given can be generalized to provide models with n-cut solutions and analogous critical behavior. The simplest set of such models have potentials

$$V(\lambda) = \frac{(-1)^n}{n}(1-\lambda)^n + \log(1-\lambda) \tag{2.19}$$

Because of the symmetry of the potential, $G(\lambda)$ and the eigenvalue support C at criticality will be invariant under $(1-\lambda)\rightarrow exp(2\pi i/n)(1-\lambda)$, modulo appropriate shifts of the log branch cut. Thus C will consist of a single closed contour with n equally spaced smooth segments. Note that, for $n>2$, some of the endpoints of the n branch cuts will be complex. Nevertheless all of the parameters entering the orthogonal polynomial analysis are real, and we expect real free energy and real correlators.

3. Logarithmic scaling violations

Now we would like to relate the unusual features of the eigenvalue analysis described above to another characteristic feature of the Penner model on the sphere: namely, the presence of logarithmic scaling violations in the free energy. To do this we compute the spherical contribution to the free energy for arbitrary $\gamma<1$, using the expression[12]:

$$F_0 = -\int_{C_1} d\lambda u(\lambda)V(\lambda) + \frac{1}{2}\int_{C_1} d\lambda \int_{C_1} d\mu u(\lambda)u(\mu)\log(\lambda-\mu)^2 \tag{3.1}$$

where C_1 is the contour shown in fig. 3 and has clockwise orientation; C_1 coincides with the eigenvalue support taken just inside the square root branch cut. The eigenvalue density $u(\lambda)$ along C_1 is given by:

$$u(\lambda) = -\frac{1}{2\pi i\gamma}\frac{1}{\lambda-1}\sqrt{(\lambda-a)(\lambda-b)} \tag{3.2}$$

with the endpoints given by $a,b = 2\gamma\pm 2\sqrt{\gamma(\gamma-1)}$, respectively.

Since the contour C_1 is given by the solution of a transcendental equation, we proceed by deforming the integration contour while keeping track of the log and pole singularities of the integrand at $\lambda=1$. Using the same integration by parts tricks employed in [12], one sees that (3.1) can be rewritten as follows:

$$\begin{aligned} F_0 = &-\frac{1}{2}\int_{C_2} d\lambda u(\lambda)\left[\frac{\lambda}{\gamma} - \log(\lambda-a)^2\right] \\ &-\frac{1}{2\gamma}\int_{C_8} d\lambda u(\lambda)\log(1-\lambda) + \frac{1}{2\gamma}\left[\frac{1}{\gamma} - \log(1-a)^2\right] \end{aligned} \tag{3.3}$$

where C_2 and C_3 are the contours shown in fig. 3, and the last term is the contribution from deforming C_1 past the simple pole at $\lambda=1$.

Now, in the critical limit, the contour C_2 shrinks to a point. Thus, in the first integral in (3.3), we need only consider the $\log(\lambda-a)^2$ term, which has an endpoint singularity. Similarly, the only source of log scaling violations in the second integral in (3.3) arises from the difference of the two horizontal segments of C_3. These contributions can be computed from elementary integrals. We thus obtain

$$\begin{aligned} F_0 =& \frac{1}{2\gamma^2}\log\left[2\gamma(1-\gamma)+\sqrt{\gamma(1-\gamma)}\right] - \frac{(1-2\gamma)}{4\gamma^2}\log\left[\frac{\gamma}{(1-\gamma)}\right] \\ &+ Re\int_{2\gamma}^{a} d\lambda u(\lambda)\log(\lambda-a)^2 \\ &+ \text{terms regular as}\, \gamma \to 1 \end{aligned} \tag{3.4}$$

Let us write $\gamma = 1-\mu$ and evaluate the above expression for small μ. The behavior of the integral is easily shown to be

$$Re\int_{2\gamma}^{a} d\lambda u(\lambda)\log(\lambda-a)^2 \to \frac{1}{2}(\mu-\mu^2+\ldots)\log\mu + \text{regular} \tag{3.5}$$

and thus

$$F_0 \to \frac{1}{2}\mu^2\log\mu + \ldots \tag{3.6}$$

which agrees precisely with the known logarithmic scaling violation of the Penner model on the sphere.

4. Double scaling limit and string equations

As noted in [19,2], the free energy of the Penner model can be derived directly from the matrix integral by the method of orthogonal polynomials [20]. We wish to evaluate

$$Z = \int dM e^{-\frac{N}{\gamma}Tr(M+\log(1-M))} \tag{4.1}$$

It is convenient to change variables to $\Phi = 1-M$. As described in [2], we will restrict the integration to positive definite Φ and assume that $\alpha = -N/\gamma$ is positive. At the end, we analytically continue our solution to negative α and recover the critical Penner model. Diagonalizing Φ and integrating over the angular variables yields the familiar form

$$Z = e^{\frac{-N^2}{\gamma}} \int_0^\infty \prod_{i=1}^{N} d\lambda_i\ \lambda_i^\alpha\ e^{-\alpha\lambda_i}\ det(P_i^{(\alpha)}(\lambda_j)) \tag{4.2}$$

The $P_n^{(\alpha)}(\lambda)$ are easily recognized as the associated Laguerre polynomials, normalized so that $P_n^{(\alpha)} = \lambda^n + \cdots$, and satisfying the orthogonality relation

$$\int_0^\infty d\lambda e^{-\alpha\lambda}\lambda^\alpha P_n^{(\alpha)} P_m^{(\alpha)} = \delta_{n,m}\left[\alpha^{-2n-\alpha-1}\,\Gamma(n+\alpha+1)\Gamma(n+1)\right] \tag{4.3}$$

They satisfy the recursion relation

$$\lambda P_n^{(\alpha)}(\lambda) = P_{n+1}^{(\alpha)}(\lambda) + \frac{2n+\alpha+1}{\alpha} P_n^{(\alpha)}(\lambda) + \frac{n(n+\alpha)}{\alpha^2} P_n^{(\alpha)}(\lambda) \tag{4.4}$$

We will denote the recursion coefficients above by S_n and R_n, respectively. Using the R_n's, one can immediately obtain an (unregulated) expression for the free energy of the Penner model[2].

We will now show how the recursion coefficients could have been obtained directly from the string equations. This is a useful exercise in instructing us how to handle generalizations of the Penner model or other hermitian matrix models with nonpolynomial potentials. First consider the string equations on the sphere. In the notation of [21], one of these can be written

$$\frac{2n+1}{\alpha} = \left\langle n|\hat{\Phi}V'(\hat{\Phi})|n\right\rangle = S_n - 1 \tag{4.5}$$

which immediately yields S_n. The second of the string equations is

$$0 = \left\langle n|V'(\hat{\Phi})|n\right\rangle = 1 - \left\langle n|\hat{\Phi}^{-1}|n\right\rangle \tag{4.6}$$

This can be evaluated as a formal power series; alternatively, (4.6) can be rewritten as follows:

$$0 = 1 - \frac{1}{2\pi i}\oint_{|w|=1} dw \frac{1}{(w-w_+)(w-w_-)} \tag{4.7}$$

where $w_\pm = \frac{1}{2}[-S_n \pm \sqrt{S_n^2 - 4R_n}]$. For α positive, only the pole at $w{=}w_+$ contributes, and we obtain

$$1 = \frac{1}{\sqrt{S_n^2 - 4R_n}}\ , \quad S_n = \frac{2n}{\alpha} + 1\ , \quad \text{and } R_n = \frac{n}{\alpha}(\frac{n}{\alpha}+1) \tag{4.8}$$

Comparing (4.8) with (4.4), we see that the string equations on the sphere give the exact result. This means that the full string equations in the double scaling limit are trivial. Writing $x{=}n/N$, $\epsilon{=}1/N{=}\nu\delta^2$, this limit is defined by:

$$\begin{aligned} \gamma = 1 - \delta^2\nu\mu \quad &, \quad x = 1 - \delta^2\nu z \\ R = -\delta^2\rho \quad &, \quad S = -1 - \delta^2\sigma \end{aligned} \tag{4.9}$$

and the string equations are

$$\rho \;=\; \nu(z+\mu)\;,\quad \sigma \;=\; -2\rho \tag{4.10}$$

It is easy to generalize these results to the case where we add arbitrary polynomial terms to the matrix potential. Consider the matrix potential

$$V(\Phi) \;=\; \sum_{k=1}^{l}\frac{g_k}{k}\Phi^k - \log\Phi \tag{4.11}$$

The string equations in the spherical limit can be written

$$\frac{n}{\alpha} = \sqrt{R_n}\left\langle n-1|V'(\hat\Phi)|n\right\rangle$$

$$= -\frac{1}{2}(1+t) + \sum_{k=1}^{l-1}\frac{2}{k+1}g_{k+1}R_nS_n^{k-1}t^{1-k}C_{k-1}^{(3/2)}(t) \tag{4.12a}$$

$$0 = \langle n|V'(\Phi)|n\rangle = \frac{-1}{\sqrt{S_n^2-4R_n}} + \sum_{k=1}^{l} g_kS_n^{k-1}t^{1-k}P_{k-1}(t) \tag{4.12b}$$

where $t = -S_n/\sqrt{S_n^2-4R_n}$, while the $C_k^{(3/2)}$ and the P_k are Gegenbauer and Legendre polynomials, respectively. These models will exhibit critical behavior when the branch points of the eigenvalue support coalesce. This requires[22] that $R_c = 0$, which implies $t=1$. From (4.12) we then obtain only a weak constraint on the potential, namely

$$U'(S_c) \;=\; 1\;,\qquad \text{where } U(\Phi) \;=\; V(\Phi)+\log(\Phi) \tag{4.13}$$

If we consider the double scaling limit given by (4.9), we find models in the same universality class as the Penner model. The full string equations are again trivial. To see this, we write the exact (all orders) expression for the matrix element of (4.12a) :

$$\gamma x = \frac{1}{2}(1+t) - \frac{1}{2\pi i}\oint_{|w|=1} dwU'\left[w+\frac{1}{w}R(x+\epsilon w\frac{d}{dw}) + S(x+\epsilon w\frac{d}{dw})\right] \tag{4.14}$$

where $R(x+\epsilon w(d/dw)) = R_c - \delta^2\rho(x) - \delta^2\rho'(x)\epsilon w(d/dw) + \ldots$, and similarly for S. Because R_c=0, only one term in (4.14) survives the integration at order δ^2:

$$\nu(z+\mu) = [1+U''(S_c)]\,\rho \tag{4.15}$$

Thus (4.11) and (4.13) define a large class of critical models which are essentially equivalent to the Penner model.

Note that, if the potential is chosen such that the coefficient $1+U''(S_c)$ in (4.15) vanishes, then the double scaling limit of (4.9) is inconsistent. Potentials which satisfy this additional constraint correspond to the multi-cut critical models described previously. The double scaling limits of such models are described in the next section.

Although it is difficult to match scaling operators of the Penner model to those of the c=1 string, it is straightforward to compute the correlators on the sphere. The main ingredient is the general matrix element

$$\left\langle n-m|\hat{\Phi}^k|n\right\rangle = \left(\frac{\sqrt{R}}{S}\right)^m \frac{2m!k!}{m!(k+m)!} S^k t^{m-k} C_{k-m}^{(m+\frac{1}{2})}(t) \tag{4.16}$$

As an example, consider the 1-point function of the operator:

$$\begin{aligned}\langle tr(1-\Phi+\log\Phi)\rangle &= -\sum_{n=1}^{N}\sum_{k=2}^{\infty}\left\langle n|\frac{1}{k}(1-\hat{\Phi})^k|n\right\rangle \\ &= \gamma+\log 2-\frac{(1-\gamma)}{\gamma}\log(1-\gamma)\end{aligned} \tag{4.17}$$

This exhibits the correct log scaling violations. With a little more effort, one can obtain the connected 2-point function of this operator:

$$\langle tr(1-\Phi+\log\Phi)\; tr(1-\Phi+\log\Phi)\rangle = (2-\gamma)(1-\gamma)+\log(1-\gamma) \tag{4.18}$$

5. Double scaling limit of the KT model

Let us write the partition function of the KT model (2.16) in the transformed variables:

$$Z = \int d\Phi e^{\alpha Tr(\frac{1}{2}\Phi^2+\log\Phi)} \tag{5.1}$$

Note that the integral already converges for $\alpha<0$, so that it is not necessary to define the orthogonal polynomials by analytic continuation from positive α. This has no net effect on the string equations.

The KT model is a critical model with $R_c=0$ and $S_c=-1$. It satisfies the constraint (4.13) and the additional constraint $1+U''(S_c)=0$. To obtain a consistent double scaling limit, we modify (4.9) as follows:

$$R=-\delta\rho \quad , \quad S=-1-\delta\sigma \tag{5.2}$$

To obtain one string equation we use

$$\frac{2n+1}{(-\alpha)} = \left\langle n|\hat{\Phi}V'(\hat{\Phi})|n\right\rangle = S_n^2 + R_n + R_{n+1} + 1 \tag{5.3}$$

Solving to order δ^2 we find

$$\sigma = \frac{1}{2}[\rho(z) + \rho(z-1)] \tag{5.4}$$

For our second string equation we write

$$\frac{n}{(-\alpha)} = \sqrt{R_n}\left\langle n-1|V'(\hat{\Phi})|n\right\rangle \tag{5.5}$$

$$= 1 + R_n\left(1 - \frac{1}{S_{n-1}^2}\right) + \frac{R_n}{S_nS_{n-1}}\left(\frac{R_n}{S_nS_{n-1}} + \frac{R_{n-1}}{S_{n-1}S_{n-2}} + \frac{R_{n+1}}{S_{n+1}S_n}\right) + \mathcal{O}(R_n^3)$$

Solving to order δ^2 and employing (5.4) this gives

$$\rho(z)\rho(z-1) = \nu(z+\mu) \tag{5.6}$$

Thus (5.4) and (5.6) are the nonperturbative string equations of the KT model. In contrast to the string equations of polynomial matrix models, which are nonlinear differential equations, here we have nonlinear difference equations. These are solved by the ansatz:

$$\rho(z) = \sqrt{2\nu}\,\frac{\Gamma\left(\frac{z+\mu+2}{2}\right)}{\Gamma\left(\frac{z+\mu+1}{2}\right)} \tag{5.7}$$

This solution corresponds to the following expression for the recursion coefficients:

$$R_n = -\sqrt{\frac{2}{N}}\,\frac{\Gamma\left(\frac{N-n+\mu+2}{2}\right)}{\Gamma\left(\frac{N-n+\mu+1}{2}\right)} \tag{5.8}$$

As we expect for a two-cut model, the solution (5.8) can be regarded as defining two distinct sets of recursion coefficients, depending as n is even or odd.

Now we can use the expression:

$$F \sim \log\left(\prod_{1}^{N-1} R_{N-n}^{n}\right) \tag{5.9}$$

and the solution (5.8) to get an expression for the free energy of the KT model:

$$F \sim N\log\Gamma\left(\tfrac{N+\mu+1}{2}\right) - \sum_{n=1}^{N}\log\Gamma\left(\tfrac{n+\mu+1}{2}\right) \tag{5.10}$$

Rewrite this by re-expressing the Γ-functions:

$$\Gamma\Big(\frac{n+\mu+1}{2}\Big) = \begin{cases} \Gamma\Big(\frac{N+\mu+1}{2}\Big)\Big[(\frac{n+1+\mu}{2})(\frac{n+3+\mu}{2})\cdots(\frac{N-1+\mu}{2})\Big]^{-1} & n \text{ even} \\ \Gamma\Big(\frac{N+\mu}{2}\Big)\Big[(\frac{n+1+\mu}{2})(\frac{n+3+\mu}{2})\cdots(\frac{N-2+\mu}{2})\Big]^{-1} & n \text{ odd} \end{cases} \tag{5.11}$$

Then expand the logarithms and recollect the terms to get

$$F \sim \frac{N}{2}\log\frac{\Gamma\Big(\frac{N+\mu+1}{2}\Big)}{\Gamma\Big(\frac{N+\mu}{2}\Big)} + \sum_{k=1}^{N/2-1} k\log(2k+\mu+1)(2k+\mu) \tag{5.12}$$

Let us now make an overall shift $\mu \to \mu - \frac{1}{2}$. Then (5.12), up to irrelevant divergent constants, is equivalent to a Legendre transform[2] of the free energy of the $c=1$ string compactified to twice the self-dual radius[4].

6. Conclusion

Distler and Vafa speculated in [2] that there might be generalizations of the Penner model which correspond to the $c=1$ string compactified to any integer multiple of the self-dual radius. We have shown that the models defined by (2.19) appear to provide such a generalization, and verified this explicitly for the KT model. It would be of great interest to understand this correspondence at a deeper level.

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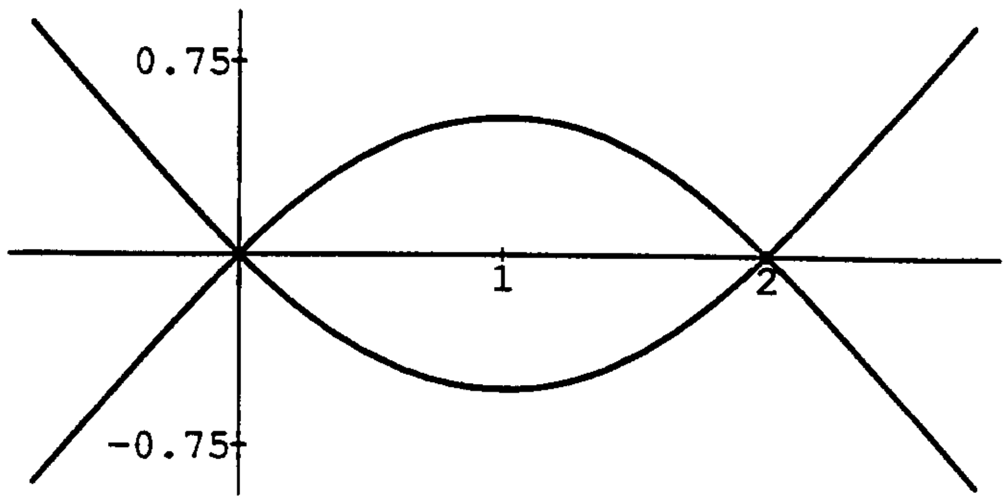
Figure Captions

Fig. 1. Contours of $Re(G(\lambda))$=0 for the Penner model, plotted in the complex λ-plane. The closed contour is the eigenvalue support $\mathcal{C}$. The remaining pieces form the contour $\mathcal{C}'$, discussed in the text.

Fig. 2. Contours of $Re(G(\lambda))$=0 for the KT model, plotted in the complex λ-plane.

Fig. 3. Integration contours used in evaluating the free energy on the sphere.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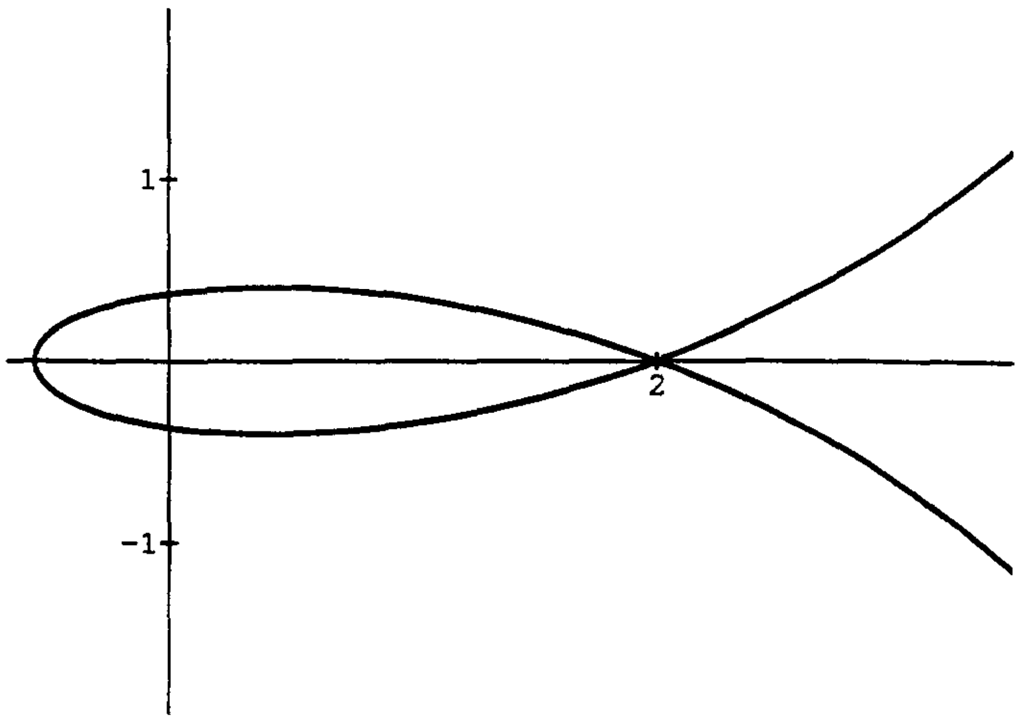
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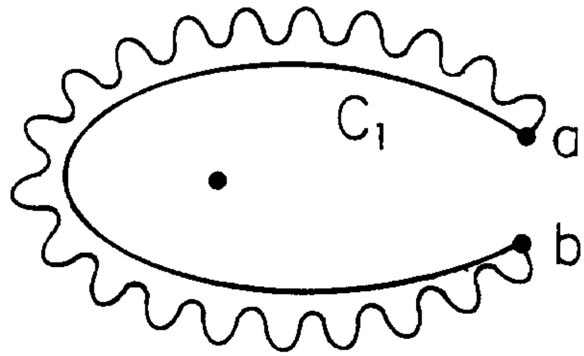
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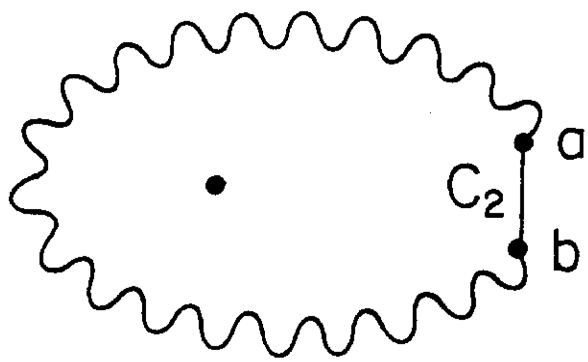
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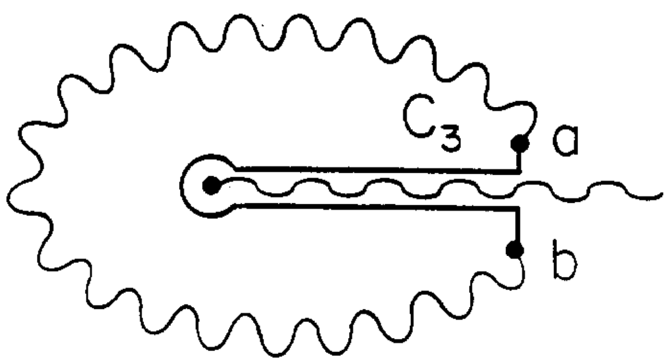
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