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Giulio Caciotta and Francesco Nicolò

Non Linear Perturbations of Kerr Spacetime in External Regions and the Peeling Decay

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Abstract We prove, outside the influence region of a ball of radius R_0 centred in the origin of the initial data hypersurface, Σ_0, the existence of global solutions near to Kerr spacetime, provided that the initial data are sufficiently near to those of Kerr. This external region is the "far" part of the outer region of the perturbed Kerr spacetime. Moreover, if we assume that the corrections to the Kerr metric decay sufficiently fast, $o(r^{-3})$, we prove that the various null components of the Riemann tensor decay in agreement with the "Peeling theorem".

1 Introduction

1.1 The Problem and the Results

The problem of the global stability for the Kerr spacetime is a very difficult and open problem. The more difficult issue is that of proving the existence of solutions of the vacuum Einstein equations with initial data "near to Kerr" in the whole outer region up to the event horizon.[1]

What is known up to now relative to the whole outer region are some relevant uniform boundedness results for solutions to the wave equation in the Kerr spacetime used as a background spacetime (see Dafermos and Rodnianski (7), Klainerman (9)).[2]

Giulio Caciotta and Francesco Nicolò
Dipartimento di Matematica
Università degli Studi di Roma "Tor Vergata"
Via della Ricerca Scientifica
00133 Rome, Italy
giulioc42@gmail.com;nicolo@axp.mat.uniroma2.it

[1] Which is also an unknown of the problem.

[2] See also for the $J = 0$ case, (1) and references therein.

If we consider the existence problem in an external region sufficiently far from the Kerr event horizon for a slow rotating Kerr spacetime, the result is included in the version of Minkowski stability result proved by S. Klainerman and one of the present author, F. Nicolò, see (10) and also (6). In this case F. Nicolò, recently proved (15), that the asymptotic behaviour of the Riemann components is in agreement with the "Peeling theorem" if the corrections to the Kerr initial data decay sufficiently fast.

In this paper, we prove the non-linear stability of the Kerr spacetime for any $J \le M^2$ and an appropriate class of initial data near to Kerr, in an external region[3] where[4] $r \ge R_0$ and $M/R_0 \le \lambda$ with λ sufficiently small. Moreover, if we restrict the class of initial data, once subtracted the Kerr part, to those which decay toward the spacelike infinity faster than r^{-3}, we prove that the null asymptotic decay of the Riemann tensor is in agreement with the "Peeling theorem". Our main result is, in a somewhat preliminary version, the detailed version in Sect. 4,

Theorem 11 *Assume that initial data are given on Σ_0 such that, outside of a ball centred in the origin of radius R_0, they are different from the "Kerr initial data of a Kerr spacetime with mass M satisfying*

$$\frac{M}{R_0} \ll 1, \quad J \le M^2$$

for some metric corrections decaying faster than r^{-3} towards spacelike infinity together with its derivatives up to an order $q \ge 4$, namely[5]

$$g_{ij} = g_{ij}^{(\mathrm{Kerr})} + o_{q+1}\big(r^{-(3+\frac{\gamma}{2})}\big), \quad k_{ij} = k_{ij}^{(\mathrm{Kerr})} + o_q\big(r^{-(4+\frac{\gamma}{2})}\big) \tag{1.1}$$

where $\gamma > 0$. Let us assume that the metric correction δg_{ij} and the second fundamental form correction δk_{ij} are sufficiently small, namely that we can define a function made by L^2 norms on Σ_0 of these quantities and require it to be small:

$$\mathcal{J}(\Sigma_0, R_0; \delta^{(3)}\mathbf{g}, \delta\mathbf{k}) \le \varepsilon, \tag{1.2}$$

then this initial data set has a unique development, $\widetilde{\mathcal{M}}$, defined outside the domain of influence of B_{R_0}. Moreover, $\widetilde{\mathcal{M}}$ can be foliated by a canonical double-null foliation $\{C(u), \underline{C}(\underline{u})\}$ whose outgoing leaves $C(u)$ are complete[6] *and the various null components of the Riemann tensor relative to a null frame associated with this foliation decay as expected from the Peeling theorem.*

The proof of this result depends on many previous results: the proof of the stability of the Minkowski spacetime in the external region by Klainerman and Nicolò, in (10), a result in turn based on the seminal work by Christodoulou and Klainerman (6), and concerning the peeling decay, important ideas of the proof come from the previous work by Klainerman and Nicolò (11) and the recent (15).

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[3] See the remark after the statement of Theorem 41 in Sect. 4

[4] r is a radial coordinate which will be defined later on.

[5] The components of the metric tensor written in dimensional coordinates. $f = o_q(r^{-a})$ means that f asymptotically behaves as $o(r^{-a})$ and its partial derivatives $\partial^k f$, up to order q behave as $o(r^{-a-k})$.

[6] By this we mean that the null geodesics generating $C(u)$ can be indefinitely extended toward the future.

We observe that the two results proved in this work, the global stability in the external region and the asymptotic decay in agreement with the peeling, are basically independent; relaxing the decay conditions on the initial data we can prove the stability with a worst null asymptotic decay. In this paper the two results are proved together, but it is easy to enlarge the class of initial data to prove only the first one, proceeding as in (11) where it has been shown in detail, for the perturbed Minkowski spacetime, how the spacelike decays of the initial data are connected to the null decay of the Riemann components.

Therefore, as many steps to prove Theorem 11 have been discussed in previous works, we prove here the new part of this result, namely Theorem 31 which is the core result to obtain, via a bootstrap mechanism, the global existence and the decay satisfying the "Peeling theorem".

In the remaining part of the introduction we examine the difficulties one encounters to prove this result and how they have been solved.

1.2 The Global Existence in an External Region Around the Kerr Spacetime

To understand the problem of perturbing around the Kerr spacetime it is appropriate to remember how the problem of perturbing around the Minkowski spacetime has been solved. The general strategy is a "bootstrap mechanism": one proves (given a local existence result) that there is a finite region V, whose metric satisfies the Einstein equations, endowed with some specific properties, mainly that some norms associated with the metric components and its derivatives are bounded by a (small) constant, then assumes that the largest possible region, V_*, where these "bootstrap bounds", hold is finite and proves that, if the initial data are sufficiently small, the previous bounds can be improved. Therefore this region can be extended, and to avoid a contradiction, has to coincide with the whole spacetime.

More precisely, we assume that V_* is endowed with a foliation made by outgoing and incoming null cones, $\{C(\lambda)\}$ and $\{\underline{C}(\nu)\}$ and that the norms we assume bounded are those relative to the connection coefficients and to the components of the Riemann tensor. The central part of the proof is, therefore, to show that these norms can have better bounds. To do it we use in the manifold V_* the structure equations and the Bianchi equations. The structure equations are transport equations for the connection coefficients along the incoming and outgoing cones and elliptic Hodge systems on the two-dimensional surface intersections of the incoming and an outgoing cones, $S = C \cap \underline{C}$. These equations are inhomogeneous equations whose inhomogeneous part depends on the Riemann components. The Bianchi equations, at their turn, can be written as transport equations for the Riemann components along the cones whose inhomogeneous part is made by products of the Riemann components and the connection coefficients.

To use these equations a sort of "linearization" is done, namely we consider the Riemann components as external sources satisfying the "bootstrap bounds" and show, using the equations for the connection coefficients, that the bounds of the connection coefficients can be improved. Then we control the Riemann components; to do it, we use in a crucial way the fact that it is possible to define for the Riemann components a set of norms, globally denoted by $\mathcal{Q}$, weighted L^2 integrals along the outgoing and incoming cones, of the Bel-Robinson tensor, which play the role of the energy norms associated with the Bianchi equations.

Assuming the "bootstrap bounds" for the connection coefficients, we prove that these energy norms are bounded by the corresponding initial data norms and using the Bianchi equations obtain finally, better norms for the Riemann components.

If we ask ourselves how to transport this strategy to perturb non-linearly around the Kerr spacetime solution instead than around the Minkowski one, we realize that the main difference is, in broad terms, that now we are perturbing around a solution different from zero, while the Minkowski spacetime can be considered a "zero solution". That is, in the Minkowski spacetime all the connection coefficients are identically zero with the exception of the second null fundamental forms χ and $\underline{\chi}$ (which, due to the spherical symmetry, reduce to the two scalar functions $\mathrm{tr}\chi = 2r^{-1}$ and $\mathrm{tr}\underline{\chi} = -2r^{-1}$). Moreover, all the Riemann components are identically zero. Therefore, to prove that the norms are bounded and small we do not have to subtract their Minkowski part (with the exception of $\mathrm{tr}\chi$ and $\mathrm{tr}\underline{\chi}$).

The Kerr spacetime is not a "zero solution" and some kind of subtraction has to be done. This "subtraction" mechanism is delicate as we are not looking for a linearly perturbed solution,[7] and it is realized through four different steps:

(i) We state the "bootstrap assumptions" in V_* for the corrections of the connection coefficients and the Riemann tensor, that is for the various components to which we have subtracted their Kerr parts.[8] Symbolically,

$$\delta O = O - O^{(\mathrm{Kerr})}, \quad \delta R = R - R^{(\mathrm{Kerr})} \tag{1.3}$$

and in some detail, see (10, Chapter 3), for all the definitions,

$$\delta\chi = \chi - \chi^{(\mathrm{Kerr})}, \quad \delta\underline{\chi} = \underline{\chi} - \underline{\chi}^{(\mathrm{Kerr})}, \quad \delta\zeta = \zeta - \zeta^{(\mathrm{Kerr})} \ldots$$
$$\delta\alpha = \alpha - \alpha^{(\mathrm{Kerr})}, \quad \delta\beta = \beta - \beta^{(\mathrm{Kerr})}, \ldots$$

(ii) Due to (i) we write the structure equation (in the V_* region) instead that for the connection coefficients, $\chi, \underline{\chi}, \zeta, \omega, \underline{\omega} \ldots$ for their corrections.[9] Recalling the proof for the Minkowski stability, these equations have inhomogeneous terms which depend on the Riemann tensor, and in this case, on the correction to the Riemann tensor

$$\delta R = R - R^{(\mathrm{Kerr})}.$$

We use these modified structure equations to obtain better estimates for the norms of these connection coefficient corrections, δO, provided we have a control for the norms of the Riemann components corrections, δR.

(iii) The third step consists in obtaining estimates for the Riemann components corrections. This requires to subtract the Kerr part of the Riemann tensor. This cannot be done in a direct way as the basic step to control the Riemann norms is to prove the boundedness of the energy type norms, $\mathcal{Q}$.

We would need analogous norms for the δR corrections again with a positive integrand,[10] to get from them estimates for the (correction of the) null

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[7] Observe that also the linear perturbation around Kerr has some problems due again to the fact that the Riemann tensor in Kerr spacetime is different from zero, see (1) and reference therein.

[8] More precisely the Kerr part "projected" on the V_* foliation, see the details in Sect. 2.2.2.

[9] The technical details for this "Kerr decoupling" for the connection coefficients are discussed later on.

[10] See the explicit expression of the $\mathcal{Q}$ norms in Minkowski case in (10, Chapter 3).

Riemann components, but to define them is a difficult and unsolved task. We proceed in a different way based on the fact that the Kerr spacetime is stationary and $\frac{\partial}{\partial t}$ is a Killing vector field. Therefore, if, instead of considering the Riemann components, we consider their time derivatives, they do not depend anymore on the Kerr part of the Riemann tensor, their initial data can have a better decay, and if we control their $\mathcal{Q}$ norms we can obtain a good control of the δR norms in V_* and also a good asymptotic decay along the null directions. This argument is not rigorous as in the perturbed Kerr spacetime $\frac{\partial}{\partial t}$ is not anymore a Killing vector field, but it turns out that the basic idea can be implemented in the following sense:

Instead of the time derivative of the Riemann tensor we define its (modified[11]) Lie derivative $\hat{\mathcal{L}}_{T_0}R$, where T_0, whose precise definition will be given later on and is equal to $\frac{\partial}{\partial t}$ in the Kerr spacetime, is not anymore a Killing vector field, but only "nearly Killing",[12] then we define some $\tilde{\mathcal{Q}}$ norms relative to $\hat{\mathcal{L}}_{T_0}R$ with appropriate weights in the integrand and prove that they are bounded in terms of the corresponding quantities written in terms of the initial data; from it we can prove, after quite a few steps, that the δR norms are smaller than that assumed in the bootstrap assumptions and satisfy appropriate decays.

In V_* we can build a null frame, $\{e_3, e_4, e_1, e_2\}$, adapted to the foliation, where e_1, e_2 are vector fields orthonormal and tangent to the two-dimensional surfaces $S = C(\lambda) \cap \underline{C}(\nu)$ while e_3, e_4 are proportional to the null geodesics generating the cones and in the coordinates $\{u, \underline{u}, \omega^1, \omega^2\}$[13] have the following expressions:

$$e_4 = \frac{1}{\Omega}\left(\frac{\partial}{\partial \underline{u}} + X\right); \quad \varepsilon_3 = \frac{1}{\Omega}\left(\frac{\partial}{\partial u} + X_{(\mathrm{Kerr})}\right), \tag{1.4}$$

while if we were considering the Kerr metric (see Sect. 2 for greater details),

$$\begin{aligned} e_4^{(\mathrm{Kerr})} &= \frac{1}{\Omega_{(\mathrm{Kerr})}}\left(\frac{\partial}{\partial \underline{u}} + X_{(\mathrm{Kerr})}\right); \\ \varepsilon_3^{(\mathrm{Kerr})} &= \frac{1}{\Omega_{(\mathrm{Kerr})}}\left(\frac{\partial}{\partial u} + X_{(\mathrm{Kerr})}\right). \end{aligned} \tag{1.5}$$

with

$$X_{(\mathrm{Kerr})} = \omega_B \frac{\partial}{\partial \omega^2} = \omega_B \frac{\partial}{\partial \phi},$$

where ω_B and $\Omega_{(\mathrm{Kerr})}$ are defined in Eq. 2.33, From this we define

$$T_0 = \frac{\Omega}{2}(e_3 + e_4) - \frac{X + X_{(\mathrm{Kerr})}}{2} = \frac{\partial}{\partial u} + \frac{\partial}{\partial \underline{u}}. \tag{1.6}$$

It is important to point out that the integrands of the $\tilde{\mathcal{Q}}$ norms have to be a sum of non-negative terms (see for instance (10), Chapter 3 equations (3.5.1),...,(3.5.3)),

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[11] See later for its precise definition.

[12] With "nearly Killing" we mean that its deformation tensor is small with respect to some Sobolev norms.

[13] u and $\underline{u}$ are the affine parameters of the null geodesics generating the null cones, ω^1, ω^2 are the angular variables.

which requires that the Bel-Robinson tensor has to be saturated by appropriate vector fields, linear combinations of e_3, e_4 with positive weights. Therefore, as in (10) we saturate the Bel-Robinson tensor with the following vector fields:

$$T=\frac{1}{2}(e_4+e_3),\quad S=\frac{1}{2}(\tau_+e_4+\tau_-e_3),\quad K=\frac{1}{2}(\tau_+^2e_4+\tau_-^2e_3) \tag{1.7}$$

where

$$\tau_+=\sqrt{1+\underline{u}^2},\quad \tau_-=\sqrt{1+u^2}.$$

These vector fields, when we perturb Minkowski spacetime, are nearly Killing; here, they are not Killing vectors even in the Kerr spacetime.[14] The relevance of the T,S,K vector fields in the present case is connected to the fact that they are non spacelike fields in the region outside the ergosphere and made by e_3 and e_4, the null vectors of the frame adapted to the foliation which have the property that the fields $N=\Omega e_4$, $\underline{N}=\Omega e_3$ are equivariant vector fields.

1.3 The Decay of the Riemann Components, the Peeling

Beside the proof of the Kerr stability in a region with $r\geq R_0\gg M$, we prove that the null Riemann components have a null asymptotic decay consistent with the "Peeling theorem". This result has already been obtained in (15), if we restrict ourselves to the perturbation of a very slow rotating Kerr spacetime or to a "very external region" which was defined through the condition

$$M\leq\tilde{\lambda}R_0^{\frac{1}{2}} \tag{1.8}$$

where $\tilde{\lambda}$ is a small number depending on the smallness of the initial data. As discussed in (15), the advantage of restricting to this "much farther" region is that we do not have to prove again a global existence result as the Kerr part of the metric can itself be considered a perturbation of the Minkowski one satisfying the conditions of (10). In this paper, once we have proved a global existence result, the way to prove again the "peeling decay", now in a much larger region, is basically the same as the one discussed in (15). We sketch now the main ideas involved and we refer to (15) and to next sections for a more detailed discussion.

In (6; 10), the null asymptotic behaviour of some of the null components of the Riemann tensor, specifically the α and the β components, see later for their definitions, is different from the one expected from the "Peeling Theorem", (16), as the proved decay is slower. More precisely, the components α and β[15] do not follow the "Peeling theorem" decaying both as $r^{-\frac{7}{2}}$ while we expect r^{-5} and r^{-4}, respectively.[16] In a subsequent paper, (11), S. Klainerman and F. Nicolò proved that the decay suggested from the "Peeling theorem" could be obtained assuming stronger spacelike decays for the initial data. Unfortunately, that result required an

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[14] Perturbing Minkowski spacetime we can choose $T=T_0$ as in Minkowski spacetime $T=\frac{\partial}{\partial t}$.

[15] Components relative to a null frame adapted to the null outgoing and incoming cones which foliate the "external region".

[16] In principle some log powers can be present, see Kroon (13).

initial data decay too strong for proving the “peeling decay” in spacetimes near to Kerr. To show how this result can be improved let us first recall it in some detail. In (11) the following result was proved:

Theorem 1 *Let assume that on Σ_0/B the metric and the second fundamental form have the following asymptotic behaviour*[17]

$$\begin{aligned} g_{ij} &= g_{S_{ij}} + O_{q+1}(r^{-(3+\varepsilon)}) \\ k_{ij} &= O_q(r^{-(4+\varepsilon)}) \end{aligned} \tag{1.9}$$

where g_S denotes the restriction of the Schwarzschild metric on the initial hypersurface. Let us assume that a smallness condition for the initial data is satisfied.[18] *Then along the outgoing null hypersurfaces $C(u)$ (of the external region) the following limits hold, with $\varepsilon' < \varepsilon$ and $\underline{u}$ and u the generalization of the Finkelstein variables $u = t - r_*$, $\underline{u} = t + r_*$ in the Schwarzschild spacetime*[19]*:*

$$\begin{aligned} &\lim_{C(u);\underline{u}\to\infty} r(1+|u|)^{(4+\varepsilon)}|\underline{\alpha}| = C_0, \quad \lim_{C(u);\underline{u}\to\infty} r^2(1+|u|)^{(3+\varepsilon)}|\underline{\beta}| = C_0 \\ &\lim_{C(u);\underline{u}\to\infty} r^3|\rho| = C_0, \quad \lim_{C(u);\underline{u}\to\infty} r^3|\sigma| = C_0 \\ &\lim_{C(u);\underline{u}\to\infty} r^4(1+|u|)^{(1+\varepsilon)}|\beta| = C_0, \quad \sup_{(u,\underline{u})\in\mathcal{K}} r^5(1+|u|)^{\varepsilon'}|\alpha| \le C_0. \end{aligned} \tag{1.10}$$

This result was obtained, basically, in two steps. First, we proved that a family of energy-type norms, $\tilde{\mathcal{Q}}$ of the same type as those used to prove the global existence near Minkowski in (10), but with a different weight in the integrand, were bounded in terms of the initial data. The new weights are the previous weights multiplied by a function $|u|^\gamma$ with appropriate $\gamma > 0$, and the central point is that the extra terms appearing in the “Error” of the $\tilde{\mathcal{Q}}$ norms have a definite sign and can be discarded. This allowed to prove that the various null components of the Riemann tensor, beside the decay in r, have a decay factor in the $|u|$ variable. In the second crucial step it was proved that, integrating along the incoming cones, the extra decay in the $|u|$ variable can be transformed in an extra decay in the r variable proving the final result.

This result cannot be immediately translated to the present case as the required decay for the initial data does not admit initial data near Kerr. The way out, see (15), is the one used to prove global stability: instead of looking directly to the decay of the Riemann components, we look at the decay of their (modified) Lie derivative with respect to the nearly Killing vector field T_0. This basically subtracts the Kerr part, allows to prove the boundedness of the modified $\tilde{\mathcal{Q}}$ norms, proves that, after a “time” integration, δR satisfies bounds which allow to extend the region V_* proving global existence and finally shows that the null components of the Riemann tensor satisfy the peeling decay.

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[17] Here, $f = O_q(r^{-a})$ means that f asymptotically behaves as $O(r^{-a})$ and its partial derivatives $\partial^k f$, up to order q behave as $O(r^{-a-k})$. Here, with g_{ij} we mean the components written in Cartesian coordinates.

[18] The details of the smallness condition are in (11).

[19] $\underline{\alpha}, \underline{\beta}, \dots$ are the null components of the Riemann tensor defined with respect to a null frame adapted to the double null foliation, see (10, Chapter 3), and later on. The norm $|\cdot|$ is, in this case, the sup norm relative to the S^2-sections of $C(u)$.

Summarizing the previous discussion, we can say that together with the proof of the "peeling decay" the relevant result of this paper is that we are able to extend the external region where the global solution of a non linear Kerr perturbation does exist, with respect to (10; 15). Nevertheless it has to be pointed out that, even technically improving the kind of estimates we are using here, this strategy does not allow us to cover the whole outer region, (that is from the event horizon on). This should be evident already looking at (1) for the Schwarzschild case where it is clear that the control of the "error", crucial to control the "energy-norms", is very problematic, even in the linear case, around the so-called "photosphere region". Therefore, this is the more fundamental reason why the assumption $\frac{M}{R_0} \ll 1$ is required in Theorem 11, and it is obvious that a different approach is required to cope with this region in the non-linear case.

2 The Bootstrap Assumptions

The main difference with (10) and also (11) is that, in this case, the initial data we are considering are a perturbation of Kerr initial data (in Σ_0/B_{R_0}) and therefore, we do not assume the ADM mass small.

We denote with $\mathscr{O}$ the connection coefficient norms defined as in (10) and we make specific assumptions on them. We denote by $\mathscr{R}$ the norms associated with the various null Riemann components, where $\rho-\overline{\rho}$ and $\sigma-\overline{\sigma}$ are in place of ρ and σ. The choice of norms as $|\cdot|_\infty$, $|\cdot|_{p,S}$ or $|\cdot|_{L^2}$ norms follows exactly the pattern of (10); here we are interested in their weight factors and their smallness; therefore we denote all the norms with $|\cdot|$.

2.1 The Null Canonical Foliation and the Metric in V_*

The Kerr connection coefficients satisfy transport equations similar to those in (10) with respect to a double null cone foliation of the Kerr spacetime. On the other side, the connection coefficients of the perturbed Kerr spacetime satisfy the transport equations with respect to a double null cone foliation of the perturbed Kerr spacetime. This implies that to subtract the Kerr part (to these transport equations) we have to control the difference between the double null cone foliation of the Kerr spacetime and the one associated with the perturbed Kerr and prove this difference "small". This also has to be part of the bootstrap mechanism and requires that the bootstrap assumptions on the connection coefficient terms imply analogous estimates at the level of the corrections to the metric components, we denote hereafter globally $\delta\mathscr{O}^{(0)}$.

The double null cone foliation: We assume that a, possibly finite, region V_* exists whose boundary is made by the union of $V_*\cap\Sigma_0$, ∂V_{*1} and ∂V_{*2}, the first part being a spacelike hypersurface the second and the third two null hypersurfaces, the first incoming and the second outgoing; we also assume that in this region we have a metric whose components satisfy the Einstein vacuum equations. We can solve the eikonal equation

$$g^{\mu\nu}\partial_\mu w\partial_\nu w=0 \tag{2.11}$$

choosing as initial data a function[20] $\underline{u}_0$ on $V_* \cap \Sigma_0$ or a function u_0 on ∂V_{*1}. Let us call $\underline{u}(p)$ and $u(p)$ these solutions, respectively. The level hypersurfaces $\underline{u}(p) = \underline{u}, u(p) = u$ define two family of null hypersurfaces we call null cones in analogy with the Minkowski case and denote $\underline{C}(u), C(\underline{u})$. These two families form the "double null cone foliation". The null tangent vector fields of the geodesics generating the C and $\underline{C}$ null "cones" are

$$L^\mu = -g^{\mu\nu}\partial_\nu u \frac{\partial}{\partial x^\nu}; \quad \underline{L}^\mu = -g^{\mu\nu}\partial_\nu \underline{u} \frac{\partial}{\partial x^\nu} \tag{2.12}$$

and the "lapse function" Ω is defined through the relation

$$\mathbf{g}(L, \underline{L}) = -(2\Omega^2)^{-1}.$$

Associated with the double null cone foliation we define two null fields $\{e_3, e_4\}$:

$$e_4 = 2\Omega L, \quad e_3 = 2\Omega \underline{L} \tag{2.13}$$

such that

$$\mathbf{g}(e_3, e_4) = -2.$$

Given the double null foliation we define $S(u, \underline{u}) = C(u) \cap \underline{C}(\underline{u})$. On each $S(u, \underline{u})$ we define two vector fields $\{e_a\}, a \in \{1, 2\}$ orthonormal to e_3, e_4, obtaining at each point $p \in V_*$ a null orthonormal frame. The foliation made by the two-dimensional surfaces $\{S(u, \underline{u})\}$ is null outgoing and null incoming integrable. This means that the distributions $\triangle$ and $\underline{\triangle}$ made by $\{e_4, e_1, e_2\}$ and by $\{e_3, e_1, e_2\}$, respectively, are integrable. Moreover the integrability property of the S-foliation implies that the connection coefficients ξ and $\underline{\xi}$, see for their definitions equations 2.44, are identically zero and that the second null fundamental forms are symmetric. The frame $\{e_4, e_3, e_1, e_2\}$ is called the "adapted (to the double null foliation) frame". One can have different double null foliations choosing different "initial data", and among the initial data we choose some specific ones we will discuss later on, see Sects. 3.5, 3.6 and also (10, Chapter 3), and the associated foliation will be called "double null canonical foliation". In conclusion, the first of the "bootstrap assumptions" we are stating is the following one:

(Assumption I): V_* **is endowed with a double null canonical foliation.**

2.1.1 The "Adapted" Coordinates and the Adapted Null Frame

Theorem 21 *let us assume* V_* *be endowed with a double null cone foliation*; *then, in the "adapted" coordinates the metric tensor has the following form*:

$$\begin{aligned}\mathbf{g}(\cdot,\cdot) = & -4\Omega^2 \mathrm{d}u\mathrm{d}\underline{u} + \gamma_{ab}\left(\mathrm{d}\omega^a - ({X_{(\mathrm{Kerr})}}^a \mathrm{d}u + X^a \mathrm{d}\underline{u})\right) \\ & \times \left(\mathrm{d}\omega^b - ({X_{(\mathrm{Kerr})}}^b \mathrm{d}u + X^b \mathrm{d}\underline{u})\right),\end{aligned} \tag{2.14}$$

where

$$\Omega = \sqrt{-\frac{\mathbf{g}(L, \underline{L})}{2}}, \quad X = X^a \frac{\partial}{\partial \omega^a}, \quad X_{(\mathrm{Kerr})} = \omega_B \frac{\partial}{\partial \omega^2} \tag{2.15}$$

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[20] $\underline{u}_0$ defines an appropriate radial foliation of Σ_0 / B_{R_0}, see Sect. 3.6.

Proof. Let us choose $u(p)$ and $\underline{u}(p)$ as coordinates. As they satisfy the eikonal equation it follows that

$$g^{uu} = g^{\underline{u}\underline{u}} = 0. \tag{2.16}$$

Therefore, we have from (2.12), where the coordinates $\{x^a\}$ are still generic ones,

$$L = -g^{u\underline{u}}\left(\frac{\partial}{\partial\underline{u}} + \frac{g^{au}}{g^{u\underline{u}}}\frac{\partial}{\partial x^a}\right); \quad \underline{L} = -g^{u\underline{u}}\left(\frac{\partial}{\partial u} + \frac{g^{a\underline{u}}}{g^{u\underline{u}}}\frac{\partial}{\partial x^a}\right). \tag{2.17}$$

The vector fields, see (10, Chapter 3),

$$N = \left(\frac{\partial}{\partial\underline{u}} + \frac{g^{au}}{g^{u\underline{u}}}\frac{\partial}{\partial x^a}\right); \quad \underline{N} = \left(\frac{\partial}{\partial u} + \frac{g^{a\underline{u}}}{g^{u\underline{u}}}\frac{\partial}{\partial x^a}\right) \tag{2.18}$$

are equivariant. This means that the diffeomorphism generated by them sends a surface $S(u,\underline{u}) = C(u)\cap\underline{C}(\underline{u})$ to another surface S on the same outgoing or incoming cone, respectively. To specify the "angular" coordinates $\{x^1,x^2\}$, we proceed in two steps. We consider the diffeomorphism generated by $\underline{N}$, we denote it $\underline{\Phi}_\lambda$, which sends $S(u,\underline{u})$ to $S(u+\lambda,\underline{u})$. Let $p\in S(u,\underline{u})$ there exist a point $p_0\in\underline{C}\cap\Sigma_0$ such that $p = \underline{\Phi}(u;p_0)$. Let us denote the "angular" coordinates on Σ_0, ω_0^1, ω_0^2 and make the following choice for the angular coordinates of p:

$$x^1(p) = \omega_0^1(p_0), \quad x^2(p) = \omega_0^2(p_0).$$

Therefore, the integral curve of the vector field $\underline{N}$, $\underline{\Phi}(\lambda;p_0)$, in these coordinates is

$$\begin{aligned} \underline{\Phi}^u(\lambda;\{\underline{u},\omega_0^b\}) &= \lambda \\ \underline{\Phi}^{\underline{u}}(\lambda;\{\underline{u},\omega_0^b\}) &= \underline{u} \\ \underline{\Phi}^a(\lambda;\{\underline{u},\omega_0^b\}) &= \omega_0^a \end{aligned} \tag{2.19}$$

and

$$\underline{N} = \frac{\partial}{\partial u}.$$

To have an expression for N and $\underline{N}$ as similar as possible to the one in Kerr spacetime, in the Pretorius–Israel coordinates, see (8), we perform a change of coordinates $\{x^\mu\}\equiv\{u,\underline{u},x^1,x^2\} \;\rightarrow\; \{u,\underline{u},\omega^1,\omega^2\}\equiv\{y^\mu\}$ where

$$\omega^a = x^a + f^a(\lambda,\underline{u},\{\omega^b\}). \tag{2.20}$$

and from it

$$\underline{N} = \underline{N}^\mu\frac{\partial}{\partial x^\mu} = \left(\underline{N}^\mu\frac{\partial y^\nu}{\partial x^\mu}\right)\frac{\partial}{\partial y^\nu} = \frac{\partial}{\partial u} + \frac{\partial f^a}{\partial\lambda}\frac{\partial}{\partial\omega^a}. \tag{2.21}$$

We choose $f^a(\lambda,\underline{u},\{\omega^b\})$ such that

$$\frac{\partial f^a}{\partial\lambda} = \delta_2^a\,\omega_B(\lambda,\underline{u},\omega^1) \tag{2.22}$$

where the explicit expression of the function $\omega_B(\lambda, \underline{u}, \omega^1)$ will be given later on. Therefore, with this change of coordinates,

$$\underline{N} = \frac{\partial}{\partial u} + \omega_B \frac{\partial}{\partial \omega^2} \equiv \frac{\partial}{\partial u} + X_{(\mathrm{Kerr})}. \tag{2.23}$$

Once $\underline{N}$ has been defined there is no more freedom in the expression of the equivariant vector field N; an explicit calculation, see (10, Chapter 3), gives

$$[N, \underline{N}] = -4\Omega^2 \zeta(e_a) e_a, \tag{2.24}$$

with ζ, the "torsion" connection coefficient, which in the adapted frame is

$$\zeta(e_a) = \frac{1}{2} \mathbf{g}(\mathbf{D}_{e_a} e_4, e_3). \tag{2.25}$$

$$\text{Therefore } N = \frac{\partial}{\partial \underline{u}} + X \quad \text{where } X = \omega_B \frac{\partial}{\partial \phi} + \delta X = X_{(\mathrm{Kerr})} + \delta X \tag{2.26}$$

and δX satisfies the equation, which will be needed later on,

$$\partial_N X_{(\mathrm{Kerr})} - \partial_{\underline{N}}(X_{(\mathrm{Kerr})} + \delta X) = -4\Omega^2 \zeta(e_a) e_a. \tag{2.27}$$

Associated with the double null cone foliation we define an adapted null orthonormal frame:

$$Le_4 = 2\Omega L = \frac{1}{\Omega} N, \quad e_3 = 2\Omega \underline{L} = \frac{1}{\Omega} \underline{N}, \; e_1 = e_1^1 \frac{\partial}{\partial \omega^1}, \quad e_2 = e_2^2 \frac{\partial}{\partial \omega^2}, \tag{2.28}$$

where e_1, e_2 are S tangent vector fields orthonormal and orthogonal to e_3, e_4. At the generic point of V_*, of the coordinates $\{u, \underline{u}, \theta, \phi\}$, the inverse metric is

$$\begin{aligned} g^{\mu\nu} &= -2\left(e_4^\mu e_3^\nu + e_3^\mu e_4^\nu\right) + \sum_{a=1}^{2} e_a^\mu e_a^\nu \\ &= -\frac{2}{\Omega^2}\left(N^\mu \underline{N}^\nu + \underline{N}^\mu N^\nu\right) + \sum_{a=1}^{2} e_a^\mu e_a^\nu \\ &= -\frac{1}{2\Omega^2}\left[\left(\delta_{\underline{u}}^\mu \delta_u^\nu + \delta_u^\mu \delta_{\underline{u}}^\nu\right) + X^c\left(\delta_c^\mu \delta_u^\nu + \delta_c^\nu \delta_u^\mu\right) + {X_{(\mathrm{Kerr})}}^d\left(\delta_d^\mu \delta_{\underline{u}}^\nu + \delta_d^\nu \delta_{\underline{u}}^\mu\right)\right] \\ &\quad + \gamma^{ab} \delta_a^\mu \delta_b^\nu. \end{aligned} \tag{2.29}$$

and the metric tensor is

$$\begin{aligned} \mathbf{g}(\cdot,\cdot) = &-4\Omega^2 \mathrm{d}u \mathrm{d}\underline{u} + \gamma_{ab}\left(\mathrm{d}\omega^a - ({X_{(\mathrm{Kerr})}}^a \mathrm{d}u + X^a \mathrm{d}\underline{u})\right) \\ &\times \left(\mathrm{d}\omega^b - ({X_{(\mathrm{Kerr})}}^b \mathrm{d}u + X^b \mathrm{d}\underline{u})\right). \end{aligned} \tag{2.30}$$

The previous result has still a certain arbitrariness as e_3, e_4 depend on the foliation which in turn depends on the choice of the initial conditions on $V_* \cap \Sigma_0$ and on

∂V_{*1}. The natural choice is that of a foliation “near” to the Pretorius–Israel one used in the Kerr spacetime. Let us recall some aspects of this foliation. In (8)

$$\underline{u}=\frac{t+r_*}{2};\quad u=\frac{t-r_*}{2}, \tag{2.31}$$

where r_* is the solution of the eikonal equation described there.[21] With these definitions the Kerr null frame “adapted” to the P-I double null foliation is

$$\begin{aligned}
e_3^{(\mathrm{Kerr})} &= 2\Omega^{(\mathrm{Kerr})}\underline{L}=\frac{R}{\sqrt{\Delta}}(\partial_t-\partial_{r_*}+\omega_B\partial_\phi)=\frac{1}{\Omega^{(\mathrm{Kerr})}}\left(\frac{\partial}{\partial u}+\omega_B\frac{\partial}{\partial\phi}\right)\\
e_4^{(\mathrm{Kerr})} &= 2\Omega^{(\mathrm{Kerr})}L=\frac{R}{\sqrt{\Delta}}(\partial_t+\partial_{r_*}+\omega_B\partial_\phi)=\frac{1}{\Omega^{(\mathrm{Kerr})}}\left(\frac{\partial}{\partial \underline{u}}+\omega_B\frac{\partial}{\partial\phi}\right)\\
e_1^{(\mathrm{Kerr})} &= \frac{R}{\mathscr{L}\sin 2\theta_*}\frac{\partial}{\partial\theta_*}=\frac{R}{\mathscr{L}}\frac{\partial}{\partial\lambda},\quad e_2^{(\mathrm{Kerr})}=\frac{1}{R\sin\theta}\frac{\partial}{\partial\phi},
\end{aligned} \tag{2.32}$$

where

$$\begin{aligned}
\omega_B &= \frac{2mar_b}{\Sigma R^2},\quad \Omega=\sqrt{\frac{\Delta}{R^2}},\ \lambda=\sin^2\theta_*.\\
\Delta &= r_b^2+a^2-2Mr,\quad \Sigma=r_b^2+a^2\cos^2\theta\\
\Sigma R^2 &= (r_b^2+a^2)^2-\Delta a^2\sin^2\theta
\end{aligned} \tag{2.33}$$

and r_b is the Boyer–Lindquist radial coordinate. $\mathscr{L}$ and λ are defined in (8) (see also equations 3.127). Denoting $\{\theta_*,\phi\}$[22] again $\{\omega^1,\omega^2\}$, a point of the Kerr spacetime is specified assigning $\{u,\underline{u},\omega^1,\omega^2\}$. The null hypersurfaces $u=const$, $\underline{u}=const$ define the double null foliation. As done before, starting from the null orthonormal frame, the Kerr metric in the $\{u,\underline{u},\omega^1,\omega^2\}$ coordinates is

$$\begin{aligned}
\mathbf{g}_{(\mathrm{Kerr})}(\cdot,\cdot) &= -4\Omega^2_{(\mathrm{Kerr})}\mathrm{d}u\mathrm{d}\underline{u}\\
&\quad+\gamma_{ab}^{(\mathrm{Kerr})}\Big(\mathrm{d}\omega^a-X^a_{(\mathrm{Kerr})}(\mathrm{d}u+\mathrm{d}\underline{u})\Big)\Big(\mathrm{d}\omega^b-X^b_{(\mathrm{Kerr})}(\mathrm{d}u+\mathrm{d}\underline{u})\Big).
\end{aligned} \tag{2.34}$$

where

$$\begin{aligned}
X_{(\mathrm{Kerr})} &= \omega_B\frac{\partial}{\partial\phi},\quad \Omega_{(\mathrm{Kerr})}=\sqrt{\frac{\Delta}{R^2}}\\
\gamma_{11} &= \frac{\mathscr{L}^2(\sin 2\theta_*)^2}{R^2},\quad \gamma_{12}=0,\quad \gamma_{22}=R^2\sin^2\theta.
\end{aligned} \tag{2.35}$$

The equivariant vector fields in the Kerr spacetime

$$N^{(\mathrm{Kerr})}=\left(\frac{\partial}{\partial\underline{u}}+\omega_B\frac{\partial}{\partial\phi}\right),\quad \underline{N}^{(\mathrm{Kerr})}=\left(\frac{\partial}{\partial u}+\omega_B\frac{\partial}{\partial\phi}\right) \tag{2.36}$$

---

[21] To specify it uniquely we still have to give the initial data, we discuss it later on.

[22] Observe that in the Pretorius–Israel frame the variable which stays near to the spherical coordinate θ of Minkowski spacetime is θ_* and not θ.

satisfy

$$[N^{(\mathrm{Kerr})},\underline{N}^{(\mathrm{Kerr})}]=-4\Omega^2\zeta^{(\mathrm{Kerr})}(e_a)e_a$$

with

$$\zeta^{(\mathrm{Kerr})}(e_a)e_a=\left(-\frac{QR\sin\theta}{2\Sigma}\partial_{r_b}\omega_B\right)e_\phi. \tag{2.37}$$

It is now possible to compare the foliation associated with the Kerr spacetime and to the perturbed Kerr spacetime. If we endow V_* with the metric (2.34), $(V_*,\mathbf{g}_{(\mathrm{Kerr})})$ is a submanifold of the Kerr spacetime; if the metric associated with V_* is (2.30), then $(V_*,\mathbf{g})$ is a submanifold of the global perturbed Kerr spacetime whose existence we are proving. Comparing the two metrics (2.34) and (2.30) [23] we see that, at the metric level, the components which are modified due to the initial data perturbation of the "Kerr initial data" are

$$\delta\Omega=\Omega-\Omega_{(\mathrm{Kerr})},\quad \delta X^a=X^a-X^a_{(\mathrm{Kerr})},\quad \delta\gamma_{ab}=\gamma_{ab}-\gamma^{(\mathrm{Kerr})}_{ab}. \tag{2.38}$$

We will have to prove that these corrections are "small" once the connection coefficients satisfy the "Bootstrap assumptions". Therefore, we define the following norms, we denote globally $\delta\mathcal{O}^{(0)}$, which will be proved small later on:

$$\begin{aligned}&|r^2|u|^{1+\delta}\delta\Omega|_\infty\equiv\sup_{V_*}|r|u|^{2+\delta}\delta\Omega|\\&|r^2|u|^{2+\delta}\delta X|_\infty\equiv\sup_{V_*}\sum_a|r^2|u|^{2+\delta}\delta X^a|,\ ||u|^{1+\delta}\delta\gamma|_\infty\equiv\sup_{V_*}\sum_{ab}||u|^{1+\delta}\delta\gamma_{ab}|.\end{aligned} \tag{2.39}$$

2.2 The Connection Coefficients and Riemann Tensor Bootstrap Assumptions

Once V_* is endowed with a double null canonical foliation we have to specify the remaining bootstrap assumptions in V_*. They are relative to the connection coefficients and to the Riemann components: more precisely to their difference from the analogous quantities in the Kerr spacetime.

2.2.1 Some Dimensional Remarks

All along this paper we use systematically, as a good help, "dimensional" arguments to obtain insight into the various decay factors of the Kerr connection coefficients and the Kerr Riemann tensor; moreover, the dimensionality of all the constants which appear in the theorems and in the various estimates is carefully specified. All the constants c which will appear are adimensional. The only constants with a natural dimension are M, R_0 which both have a length dimension,

---

[23] In the perturbed Kerr spacetime we defined ω_B as in the Kerr spacetime; this requires some care. In fact we define it as a function in the $\{u,\underline{u},\omega^1,\omega^2\}$ coordinates while in Kerr spacetime $\omega_B=\omega_B(r_b,\theta,\phi)$ and r_b,θ,ϕ are Boyer Lindquist coordinates. Therefore, using the Pretorius–Israel coordinates we can rewrite it as a function $\omega_B=\omega_B(r_*,\theta_*,\phi)$, and finally following (8) we express r_* as a function of u and $\underline{u}$. This is the expression we use in $(V_*,\mathbf{g})$.

L^1, and ε and ε_0 which define the smallness of the various energy type norms which have dimension: L^3. Attention has, therefore, to be paid in considering the weight factors which appear in the various norms. They are of the following type: some integer powers of r times $|u|$ factors with exponents $1+\delta, 2+\delta, \dots$ or $5+\gamma$ (in the $\mathcal{Q}$ norms), or $1+\frac{\varepsilon}{2}, 2+\frac{\varepsilon}{2}, \dots$ (see later on). To keep consistent the dimension of the various terms in the various inequalities we have to interpret each term $|u|^{k+\delta}$ with k integer as $|u|^k\frac{|u|^\delta}{R_0^\delta}$, in the same way $|u|^{5+\gamma}$ as $|u|^5\frac{|u|^\gamma}{R_0^\gamma}$ and also $|u|^{k+\frac{\varepsilon}{2}}$ as $|u|^k\frac{|u|^{\frac{\varepsilon}{2}}}{R_0^{\frac{\varepsilon}{2}}}$. We omit hereafter the factors $R_0^{-\delta}, \dots, R_0^{-\gamma}, \dots R_0^{-\frac{\varepsilon}{2}}$ not to burden the notations. Therefore, in all the expressions the terms $|u|^\delta, |u|^\gamma, |u|^{\frac{\varepsilon}{2}}$ have to be considered dimensionless.

2.2.2 The $\mathcal{O}$ Connection Coefficient Norms

We write any connection coefficient as the sum of the Kerr connection coefficient part plus a correction and make assumptions for the estimates of the correction parts. The O connection coefficients are tensor fields tangent to the S two-dimensional surfaces associated with the foliation of the perturbed Kerr spacetime, while the $O^{(\mathrm{Kerr})}$ connection coefficients are tangent to the $S_{(\mathrm{Kerr})}$ two-dimensional surfaces. Therefore, their difference would not be an S-tangent tensor. To avoid this problem we observe that $O^{(\mathrm{Kerr})}$, assuming it, for instance, being a $(0,2)$ covariant tensor, can be written as

$$O^{(\mathrm{Kerr})}_{\mu\nu} = \Pi^{(\mathrm{Kerr})\rho}{}_{\mu}\Pi^{(\mathrm{Kerr})\sigma}{}_{\nu}H^{(\mathrm{Kerr})}_{\rho\sigma}, \tag{2.40}$$

where $H^{(\mathrm{Kerr})}$ is not S-tangent and $\Pi^{(\mathrm{Kerr})}$ projects it on the $TS_{(\mathrm{Kerr})}$ tangent space. Therefore, we substitute $O^{(\mathrm{Kerr})}$ with $\hat{O}$

$$\hat{O}_{\mu\nu} = \Pi^\rho_\mu \Pi^\sigma_\nu H^{(\mathrm{Kerr})}_{\rho\sigma}. \tag{2.41}$$

where Π projects on the TS tangent space. Then we will have to prove that the differences $(\hat{O} - O^{(\mathrm{Kerr})})$ are small (see Sect. 3.4). Therefore, we write each connection coefficient as

$$O = \hat{O} + \delta O$$

and we make the bootstrap assumptions on the norms of the δO parts denoted globally as $\delta\mathcal{O}$, while we control the *sup* norms of the Kerr part. These bootstrap norm estimates control the smallness and the decay along the outgoing or incoming cones. The decay is expressed in terms of a "radial variable" denoted by r defined, at a generic point $p \in V_*$, as

$$r = r(u, \underline{u}) = \frac{1}{\sqrt{4\pi}}|S(u,\underline{u})|^{\frac{1}{2}}. \tag{2.42}$$

where $|S(u,\underline{u})|$ is the area of the surface to which the point p belongs. Observe that this variable, although not far from, is different from the radial variable $r_* = \underline{u} - u$ used by Israel and Pretorius, (8), in the Kerr spacetime and also with respect to the quantity $\underline{u}-u$ of the perturbed Kerr spacetime. All the decay are expressed in this variable and in the $|u|$ variable, and we show, in Sect. 3.4.3, that all the various radial coordinates appearing in the Kerr spacetime and in the perturbed Kerr spacetime stay near to $r(u,\underline{u})$, (of course in this external region). Therefore, hereafter with r, we always mean the quantity defined in (2.42), a well-defined function of u and $\underline{u}$.

Finally, the norms we are using for the connection coefficients are, as in (10), or sup norms, $|\cdot|_\infty$, or (p,S)-norms, $|\cdot|_{p,S}$, with $p\in[2,4]$, where

$$|f|_{p,S}=\left(\int_S|f|^p d\mu_S\right)^{\frac{1}{p}}. \tag{2.43}$$

If f denotes a generic connection coefficient the pointwise norm of the integrand is made with the restriction of the metric on S, γ_{ab}. Even if we are considering the norm of the Kerr part we use the γ metric associated with the perturbed metric, Eq. (2.30), which does not change, apart from a constant, the estimates for the Kerr part. In fact we will show in detail, from the bootstrap assumptions it follows, that the metric γ_{ab} stays near to $\gamma_{ab}^{(\mathrm{Kerr})}$, step IV of Sect. 3.2.

2.2.3 Decay of Kerr Connection Coefficients

We start recalling the general definitions of the connection coefficients, see (10, Chapter 3), for more details:

$$\begin{aligned}
\xi_a &= \frac{1}{2}\mathbf{g}(\mathbf{D}_{e_4}e_4,e_a), \quad \underline{\xi}_a=\frac{1}{2}\mathbf{g}(\mathbf{D}_{e_3}e_3,e_a)\\
\eta_a &= -\frac{1}{2}\mathbf{g}(\mathbf{D}_{e_3}e_a,e_4), \quad \underline{\eta}_a=-\frac{1}{2}\mathbf{g}(\mathbf{D}_{e_4}e_a,e_3)\\
\omega &= -\frac{1}{4}\mathbf{g}(\mathbf{D}_{e_4}e_3,e_4), \quad \underline{\omega}=-\frac{1}{4}\mathbf{g}(\mathbf{D}_{e_3}e_4,e_3)\\
\zeta_a &= \frac{1}{2}g(\mathbf{D}_{e_a}e_4,e_3).
\end{aligned} \tag{2.44}$$

The decay of the Kerr connection coefficients,[24] is the following one[25]:

$$|r\mathrm{tr}\chi^{(\mathrm{Kerr})}|\le\kappa;\quad |r\mathrm{tr}\underline{\chi}^{(\mathrm{Kerr})}|\le\kappa \tag{2.45}$$

$$|r^2\omega^{(\mathrm{Kerr})}|\le\kappa M;\quad |r^2|\underline{\omega}^{(\mathrm{Kerr})}|\le\kappa M \tag{2.46}$$

$$\begin{aligned}
|r^3\hat{\chi}^{(\mathrm{Kerr})}|&\le\kappa aM\le\kappa M^2;\quad |r^3\hat{\underline{\chi}}^{(\mathrm{Kerr})}|\le\kappa aM\le\kappa M^2\\
|r^4\nabla\!\!\!/\,\mathrm{tr}\chi^{(\mathrm{Kerr})}|&\le\kappa aM\le\kappa M^2;\quad |r^4\nabla\!\!\!/\,\mathrm{tr}\underline{\chi}^{(\mathrm{Kerr})}|\le\kappa aM\le\kappa M^2
\end{aligned} \tag{2.47}$$

---

[24] Those for the tangential derivatives follows in the standard way, getting an extra r for each tangential derivative.

[25] As $\mathrm{tr}\chi$ and $\mathrm{tr}\underline{\chi}$ are different from zero even in the Minkowski spacetime their norm estimates does not depend at the lowest order on M, in fact

$$\mathrm{tr}\chi^{(\mathrm{Kerr})}=\frac{O(1)}{r}+\frac{O(M)}{r^2}+\frac{O(M^2)}{r^3}+\cdots,\quad \mathrm{tr}\underline{\chi}^{(\mathrm{Kerr})}=\frac{O(1)}{r}+\frac{O(M)}{r^2}+\frac{O(M^2)}{r^3}+\cdots.$$

$$\begin{aligned}&|r^3\zeta^{(\mathrm{Kerr})}|\leq\kappa aM\leq\kappa M^2\\&|r^4\nabla\!\!\!/\,\omega^{(\mathrm{Kerr})}|\leq\kappa aM\leq\kappa M^2,\quad|r^4\nabla\!\!\!/\,\underline{\omega}^{(\mathrm{Kerr})}|\leq\kappa aM\leq\kappa M^2\end{aligned}$$

where $\kappa>1$ is a definite adimensional constant.

The norm estimates in (2.45) refer to those connection coefficients which are different from zero in Minkowski, those in (2.46) refer to the coefficients different from zero in Schwarzschild, but zero in Minkowski and the norm estimates in (2.47) refer to the connection coefficients or derivatives of connection coefficients different from zero in Kerr, but zero in Schwarzschild and in Minkowski.

These "Kerr decays" can be easily obtained by dimensional arguments recalling that the Kerr metric, written in the Boyer–Lindquist coordinates, in the limit $M\to 0$, a kept fixed and different from zero, reduces to the Minkowski metric written in the "oblate coordinates" and as we know that in Minkowski spacetime all connection coefficients, with the exception of $\mathrm{tr}\chi$ and $\mathrm{tr}\underline{\chi}$, are zero, it follows that performing an $\frac{1}{r}$ expansion of the connection coefficients, all the coefficients of the terms in $\frac{1}{r^k}$ which depend only on a and not on M must be identically zero. The dimensional argument goes as following: we assume the metric written in "cartesian" coordinates, namely coordinates with the dimension of a length: L^1. Then as the metric tensor has dimension L^2 its components $g_{\mu\nu}$ has dimension L^0. Proceeding in the same way it follows that the connection coefficient components have dimension L^{-1} and the same happens for their norms. Therefore, as for instance, $[\chi]=L^{-1}$, it follows that in Kerr it must be

$$\hat{\chi}^{(\mathrm{Kerr})}=\frac{O(aM)}{r^3}.$$

Moreover, as in Kerr spacetime $a=J/M$ and we consider those spacetimes where $J\leq M^2$, it follows that $a\leq M$ and the Kerr coefficient satisfy the condition

$$|r^3\hat{\chi}^{(\mathrm{Kerr})}|\leq\kappa M^2.$$

The bounds for $\omega^{(\mathrm{Kerr})}$ and $\underline{\omega}^{(\mathrm{Kerr})}$ and for $\mathrm{tr}\chi^{(\mathrm{Kerr})}$ and $\mathrm{tr}\underline{\chi}^{(\mathrm{Kerr})}$ are different as they reflect the fact that $\omega^{(\mathrm{Kerr})}$, $\underline{\omega}^{(\mathrm{Kerr})}$ are different from zero also in Schwarzschild and $\mathrm{tr}\chi^{(\mathrm{Kerr})},\mathrm{tr}\underline{\chi}^{(\mathrm{Kerr})}$ even in Minkowski.

2.2.4 The $\delta\mathcal{O}$ Connection Coefficient Norms

The assumptions

$$\delta\mathcal{O}\leq\varepsilon_0 \tag{2.48}$$

summarize the conditions on the connection coefficients and their tangential derivatives, $\not\nabla^l$, with $0 \le l \le 4$,[26]

$$
\begin{aligned}
&|r^{2+l}|u|^{2+\delta}\not\nabla^l \delta \mathrm{tr}\chi| \le \varepsilon_0;\;\; |r^{1+l}|u|^{3+\delta}\not\nabla^l \delta \mathrm{tr}\underline{\chi}| \le \varepsilon_0\\
&|r^{2+l}|u|^{2+\delta}\not\nabla^l \delta \hat{\chi}| \le \varepsilon_0;\;\; |r^{1+l}|u|^{3+\delta}\not\nabla^l \delta \underline{\hat{\chi}}| \le \varepsilon_0\\
&|r^{2+l}|u|^{2+\delta}\not\nabla^l \delta \zeta| \le \varepsilon_0 \qquad\qquad (2.49)\\
&|r^{2+l}|u|^{2+\delta}\not\nabla^l \delta \omega| \le \varepsilon_0,\;\; |r^{1+l}|u|^{3+\delta}\not\nabla^l \delta \underline{\omega}| \le \varepsilon_0.
\end{aligned}
$$

Remark **1** The choice of the δ coefficient is connected to the decay of the Riemann tensor. $\delta > 0$ is required to prove the decay in agreement with the peeling. $\delta \le 0$ could also be chosen, as discussed in the introduction, but, in this case, a weaker decay for (some components of) the Riemann tensor follows.[27]
In conclusion the second of the "bootstrap assumptions" we are requiring is the following one:

(Assumption II): In V_* the connection coefficient norms satisfy the following bounds: $\delta\mathcal{O} \le \varepsilon_0$.

2.2.5 The $\delta\mathcal{R}$ Null Riemann Component Norms

We start recalling the general definitions, (10, Chapter 3), of the null components of the Riemann tensor with respect to the adapted null frame, where W is a generic Weyl tensor and X,Y are S-tangent vector fields,

$$
\begin{aligned}
\alpha(W)(X,Y) &= W(X,e_4,Y,e_4), &\quad \beta(W)(X) &= \frac{1}{2}W(X,e_4,e_3,e_4)\\
\rho(W) &= \frac{1}{4}W(e_3,e_4,e_3,e_4), &\quad \sigma(W) &= \frac{1}{4}{}^{\star}W(e_3,e_4,e_3,e_4) \qquad (2.50)\\
\underline{\beta}(W)(X) &= \frac{1}{2}W(X,e_3,e_3,e_4), &\quad \underline{\alpha}(W)(X,Y) &= W(X,e_3,Y,e_3).
\end{aligned}
$$

As done for the connection coefficients, to prove our result we "subtract" the Kerr part of the Riemann tensor. Therefore, writing all the various components of the Riemann tensor as

$$R = R^{(\mathrm{Kerr})} + \delta R$$

we state bootstrap assumptions on the norms relative to the "correction part", we denote globally $\delta\mathcal{R}$. The norms bounded in the "bootstrap assumptions" are the

---

[26] The bounds foe η and $\underline{\eta}$ follow from these ones in the adapted frame, see for details (10, Chapter 3).

[27] The factor $|u|^{\delta}$ could also describe, symbolically, a log factor, for instance $|r^2|u|^{2+\delta}\delta\mathrm{tr}\chi| \le \varepsilon_0$ could mean

$$|\delta \mathrm{tr}\chi| \le c\varepsilon_0 \frac{1}{(\log|u|/R_0)^{\delta} r^2 |u|^2}.$$

sup norms of the various null components of δR and their tangential derivatives.[28] In a compact way

$$\delta\mathscr{R} \leq \varepsilon_0, \tag{2.51}$$

and in more detail, with $l \leq q-1$:

$$\begin{aligned}
&\sup_{V_*} r^{5+l}|u|^{\frac{\varepsilon}{2}}|\nabla\!\!\!\!/^{\,l}\alpha(\delta R)| \leq \varepsilon_0, \quad \sup_{V_*} r^{4+l}|u|^{1+\frac{\varepsilon}{2}}|\nabla\!\!\!\!/^{\,l}\beta(\delta R)| \leq \varepsilon_0\\
&\sup_{V_*} r^{3+l}|u|^{2+\frac{\varepsilon}{2}}|\nabla\!\!\!\!/^{\,l}(\rho(\delta R)-\overline{\rho(\delta R)})| \leq \varepsilon_0,\\
&\sup r^{3+l}|u|^{2+\frac{\varepsilon}{2}}|\nabla\!\!\!\!/^{\,l}(\sigma(\delta R)-\overline{\sigma(\delta R)})| \leq \varepsilon_0\\
&\sup_{V_*} r^{2+l}|u|^{3+\frac{\varepsilon}{2}}|\nabla\!\!\!\!/^{\,l}\underline{\beta}(\delta R)| \leq \varepsilon_0, \quad \sup_{V_*} r^{1+l}|u|^{4+\frac{\varepsilon}{2}}|\nabla\!\!\!\!/^{\,l}\underline{\alpha}(\delta R)| \leq \varepsilon_0.
\end{aligned} \tag{2.52}$$

Finally, we add a bootstrap assumption for $\overline{\rho(\delta R)}$,

$$\sup_{V_*}|r^3|u|^{2+\frac{\varepsilon}{2}}\overline{\rho(\delta R)}| \leq \varepsilon_0, \tag{2.53}$$

where $\overline{\rho}$ denotes the average over the S surface.

Remarks **1** (i) The exponent $\frac{\varepsilon}{2}$ appearing here in the decay factors has to be assumed, at the end, equal to δ to complete the proof of the main theorem, Theorem 31. This ε has not to be confused with the ε denoting the smallness of the initial data in (1.2).

(ii) The estimate for $\alpha(\delta R)$ is not yet the estimate for $\alpha(R)$ as we have to add $\alpha(R^{(\mathrm{Kerr})})$, the same, of course, for all the remaining components. In fact, all the Riemann components contain also a Kerr part, even the components different from ρ and σ as we are not using the principal null direction frame, see (2) and later on for its precise definition. The precise estimates of the Kerr parts are given later on in Sect. 3.3.5.

In conclusion the third "bootstrap assumption" is the following one:

(Assumption III): In V_* the norms of the components of the correction to the Kerr Riemann tensor, δR, satisfy the following bounds: $\delta\mathscr{R} \leq \varepsilon_0$.

3 The Results

3.1 The Main Theorem

Theorem 31 *If the bootstrap assumptions,* (2.48), (2.51), (2.53) *hold with* $\delta = \frac{\varepsilon}{2}$, *the following smallness assumptions are satisfied,*

$$\frac{\varepsilon_0^2}{R_0^3} < \varepsilon < \varepsilon_0, \frac{M}{R_0} \ll 1 \tag{3.54}$$

---

[28] Once we control the tangential derivatives, via the Bianchi equations we control also the derivatives along e_3 and e_4, see (10) for the more delicate control of $\mathbf{D}_{e_4}\alpha$ and $\mathbf{D}_{e_3}\underline{\alpha}$.

and the smallness of the initial data is controlled by a function $\mathscr{J}$ *explicitly defined in Sect.* 3.6*, Eq.* (3.232)*,*

$$\mathscr{J}(\Sigma_0, R_0; \delta^{(3)}\mathbf{g}, \delta^{(3)}\mathbf{k}) \leq \varepsilon, \tag{3.55}$$

then in the V_* *region the following bounds are satisfied:*

$$\delta\mathscr{O} \leq \frac{\varepsilon_0}{2}, \quad \delta\mathscr{R} \leq \frac{\varepsilon_0}{2}, \quad \sup_{V_*} |r^3 |u|^{2+\frac{\varepsilon}{2}} \overline{\rho(\delta R)}| \leq \frac{\varepsilon_0}{2}. \tag{3.56}$$

This theorem is the main step to prove the global existence and the peeling properties in the external region defined by the condition $|u| \geq R_0$ (see also Sect. 3.7).

3.2 The Structure of the Proof of Theorem 31

The proof of Theorem 31 is divided into many steps we list in the following:

I step: The definition of the $\tilde{\mathscr{Q}}$ norms

We denote $\tilde{\mathscr{Q}}$ the $\mathscr{Q}$ norms associated with $\tilde{R} = \hat{\mathscr{L}}_{T_0} R$; their explicit expression will be given in the sequel.

II step: the estimate of of the $\tilde{\mathcal{Q}}$ norms and of $\overline{\rho(\tilde{R})}$
We prove that under the bootstrap assumptions (2.48), (2.51) and initial data such that on Σ_0:

$$\tilde{\mathcal{Q}}_{\Sigma_0} \le \varepsilon^2; \qquad \sup_{\Sigma_0/B_{R_0}} |r^{6+\frac{\varepsilon}{2}}\overline{\rho(\tilde{R})}| \le c\varepsilon$$

we have

$$\tilde{\mathcal{Q}} \le c\left(\varepsilon^2 + \frac{M}{R_0}\varepsilon_0^2\right); \quad \sup_{V_*}|r^3|u|^{3+\frac{\varepsilon}{2}}\overline{\rho(\tilde{R})}| \le \frac{\varepsilon_0}{2}. \tag{3.57}$$

III step: Proceeding basically as in (15), we prove that from the results in the first two steps all the norms $\delta\mathcal{R}$ are bounded by $\frac{\varepsilon_0}{2}$. The proof is divided into many consecutive lemmas; it requires also the control, based on the $\delta\mathcal{O}$ bootstrap assumptions, of the deformation tensor ${}^{(T_0)}\pi$.

IV step: We complete the proof of Theorem 31 showing that, using the results of the previous steps, we have

$$\delta\mathcal{O} \le \frac{\varepsilon_0}{2}. \tag{3.58}$$

This last step is divided in three parts to do in a precise order, namely:

(a) Using the bootstrap assumptions for the $\delta\mathcal{O}$ norms, $\delta\mathcal{O} \le \varepsilon_0$, we prove that, under the initial data assumptions, the metric component corrections norms, defined in (2.39), satisfy,

$$\delta\mathcal{O}^{(0)} \le c\varepsilon_0. \tag{3.59}$$

(b) Using this result plus the initial data assumptions and the bootstrap assumptions, we obtain that the connection coefficient norms satisfy

$$\delta\mathcal{O} \le \frac{\varepsilon_0}{2}. \tag{3.60}$$

(c) Finally, we prove that under this result also the metric components corrections have better estimates, namely

$$\delta\mathcal{O}^{(0)} \le \frac{\varepsilon_0}{2}. \tag{3.61}$$

This last result will be needed with all the other ones to perform step VI.

V step: In this step we collect all the required initial data conditions and show how they can be satisfied imposing a global decay condition for the initial data and the smallness condition (3.55).

VI step: In this step we recall how, by a partial local existence theorem, in view of the previous results we can extend the region V_* showing that it has to coincide with the whole spacetime.

3.3 Proof of the Various Steps

We call a constant “independent” if it does not depend on $a, M, \varepsilon_0, \varepsilon$.

3.3.1 I Step: The Definition of the $\tilde{\mathscr{Q}}$ Norms

These norms, we denote $\tilde{Q}$, are analogous to the Q norms defined in (10, Chapter 3):

$$\begin{aligned}
\widetilde{\mathscr{Q}}(u,\underline{u}) &= \widetilde{\mathscr{Q}}_1(u,\underline{u}) + \widetilde{\mathscr{Q}}_2(u,\underline{u}) + \sum_{i=2}^{q-1} \widetilde{\mathscr{Q}}_{(q)}(u,\underline{u}) \\
\underline{\widetilde{\mathscr{Q}}}(u,\underline{u}) &= \underline{\widetilde{\mathscr{Q}}}_1(u,\underline{u}) + \underline{\widetilde{\mathscr{Q}}}_2(u,\underline{u}) + \sum_{i=2}^{q-1} \underline{\widetilde{\mathscr{Q}}}_{(q)}(u,\underline{u}),
\end{aligned} \tag{3.62}$$

where, with $S(u,\underline{u}) \subset V_*$, we denote

$$\begin{aligned}
V(u,\underline{u}) &= J^{(-)}(S(u,\underline{u})) \cap V_*, \\
\widetilde{\mathscr{Q}}_1(u,\underline{u}) &\equiv \int_{C(u)\cap V(u,\underline{u})} |u|^{5+\gamma} Q(\hat{\mathscr{L}}_T \tilde{R})(\bar{K},\bar{K},\bar{K},e_4) \\
&\quad + \int_{C(u)\cap V(u,\underline{u})} |u|^{5+\gamma} Q(\hat{\mathscr{L}}_O \tilde{R})(\bar{K},\bar{K},T,e_4) \\
\widetilde{\mathscr{Q}}_2(u,\underline{u}) &\equiv \int_{C(u)\cap V(u,\underline{u})} |u|^{5+\gamma} Q(\hat{\mathscr{L}}_O \hat{\mathscr{L}}_T \tilde{R})(\bar{K},\bar{K},\bar{K},e_4) \\
&\quad + \int_{C(u)\cap V(u,\underline{u})} |u|^{5+\gamma} Q(\hat{\mathscr{L}}_O^2 \tilde{R})(\bar{K},\bar{K},T,e_4) \\
&\quad + \int_{C(u)\cap V(u,\underline{u})} |u|^{5+\gamma} Q(\hat{\mathscr{L}}_S \hat{\mathscr{L}}_T \tilde{R})(\bar{K},\bar{K},\bar{K},e_4)
\end{aligned} \tag{3.63}$$

$$\begin{aligned}
\widetilde{\mathscr{Q}}_{(q)}(u,\underline{u}) &\equiv \sum_{i=2}^{q-1} \left\{ \int_{C(u)\cap V(u,\underline{u})} |u|^{5+\gamma} Q(\hat{\mathscr{L}}_T \hat{\mathscr{L}}_O^i \tilde{R})(\bar{K},\bar{K},\bar{K},e_4) \right. \\
&\quad + \int_{C(u)\cap V(u,\underline{u})} |u|^{5+\gamma} Q(\hat{\mathscr{L}}_O^{i+1} \tilde{R})(\bar{K},\bar{K},T,e_4) \\
&\quad \left. + \int_{C(u)\cap V(u,\underline{u})} |u|^{5+\gamma} Q(\hat{\mathscr{L}}_S \hat{\mathscr{L}}_T \hat{\mathscr{L}}_O^{i-1} \tilde{R})(\bar{K},\bar{K},\bar{K},e_4) \right\}
\end{aligned} \tag{3.64}$$

$$\begin{aligned}
\underline{\widetilde{\mathscr{Q}}}_1(u,\underline{u}) &\equiv \int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} |u|^{5+\gamma} Q(\hat{\mathscr{L}}_T \tilde{R})(\bar{K},\bar{K},\bar{K},e_3) \\
&\quad + \int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} |u|^{5+\gamma} Q(\hat{\mathscr{L}}_O \tilde{R})(\bar{K},\bar{K},T,e_3),
\end{aligned}$$

$$\begin{aligned}\underline{\widetilde{\mathscr{Q}}}_2(u,\underline{u}) \equiv & \int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} |u|^{5+\gamma} Q(\hat{\mathscr{L}}_O\hat{\mathscr{L}}_T\tilde{R})(\bar{K},\bar{K},\bar{K},e_3)\\ &+\int_{\underline{C}(u)\cap V(u,\underline{u})} |u|^{5+\gamma} Q(\hat{\mathscr{L}}_O^2\tilde{R})(\bar{K},\bar{K},T,e_3)\\ &+\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} |u|^{5+\gamma} Q(\hat{\mathscr{L}}_S\hat{\mathscr{L}}_T\tilde{R})(\bar{K},\bar{K},\bar{K},e_3)\end{aligned} \tag{3.65}$$

$$\begin{aligned}\underline{\widetilde{\mathscr{Q}}}_{(q)}(u,\underline{u}) \equiv \sum_{i=2}^{q-1}\Bigg\{ & \int_{C(\lambda)\cap V(\lambda,\nu)} |u|^{5+\gamma} Q(\hat{\mathscr{L}}_T\hat{\mathscr{L}}_O^i\tilde{R})(\bar{K},\bar{K},\bar{K},e_3)\\ &+\int_{C(u)\cap V(u,\underline{u})} |u|^{5+\gamma} Q(\hat{\mathscr{L}}_O^{i+1}\tilde{R})(\bar{K},\bar{K},T_0,e_3)\\ &+\int_{C(u)\cap V(u,\underline{u})} |u|^{5+\gamma} Q(\hat{\mathscr{L}}_S\hat{\mathscr{L}}_{T_0}\hat{\mathscr{L}}_O^{i-1}\tilde{R})(\bar{K},\bar{K},\bar{K},e_3)\Bigg\}.\end{aligned} \tag{3.66}$$

and

$$\begin{aligned}\widetilde{\mathscr{Q}}_1(\Sigma_0) \equiv & \int_{\Sigma_0\cap V_*} |u|^{5+\gamma} Q(\hat{\mathscr{L}}_T\tilde{R})(\bar{K},\bar{K},\bar{K},T)\\ &+\int_{\Sigma_0\cap V_*} |u|^{5+\gamma} Q(\hat{\mathscr{L}}_O\tilde{R})(\bar{K},\bar{K},T,T)\end{aligned} \tag{3.67}$$

$$\begin{aligned}\widetilde{\mathscr{Q}}_2(\Sigma_0) \equiv & \int_{\Sigma_0\cap V_*} |u|^{5+\gamma} Q(\hat{\mathscr{L}}_O\hat{\mathscr{L}}_T\tilde{R})(\bar{K},\bar{K},\bar{K},T)\\ &+\int_{\Sigma_0\cap V_*} |u|^{5+\gamma} Q(\hat{\mathscr{L}}_O^2\tilde{R})(\bar{K},\bar{K},T,T)\\ &+\int_{\Sigma_0\cap V_*} |u|^{5+\gamma} Q(\hat{\mathscr{L}}_S\hat{\mathscr{L}}_T\tilde{R})(\bar{K},\bar{K},\bar{K},T).\end{aligned} \tag{3.68}$$

$$\begin{aligned}\sum_{i=2}^{q-1}\widetilde{\mathscr{Q}}_{(q)}(\Sigma_0) \equiv \sum_{i=2}^{q-1}\Bigg\{ & \int_{\Sigma_0\cap V_*} |u|^{5+\gamma} Q(\hat{\mathscr{L}}_T\hat{\mathscr{L}}_O^i\tilde{R})(\bar{K},\bar{K},\bar{K},T)\\ &+\int_{\Sigma_0\cap V_*} |\lambda|^{5+\gamma} Q(\hat{\mathscr{L}}_O^{i+1}\tilde{R})(\bar{K},\bar{K},T,T)\\ &+\int_{\Sigma_0\cap V_*} |\lambda|^{5+\gamma} Q(\hat{\mathscr{L}}_S\hat{\mathscr{L}}_T\hat{\mathscr{L}}_O^{i-1}\tilde{R})(\bar{K},\bar{K},\bar{K},T)\Bigg\}\end{aligned} \tag{3.69}$$

We also introduce the following quantity:

$$\widetilde{\mathscr{Q}}_{V_*} \equiv \sup_{\{u,\underline{u}|S(u,\underline{u})\subseteq V_*\}} \{\widetilde{\mathscr{Q}}(u,\underline{u}) + \underline{\widetilde{\mathscr{Q}}}(u,\underline{u})\}. \tag{3.70}$$

The initial conditions have to be chosen in such a way that the $\widetilde{\mathscr{Q}}, \underline{\widetilde{\mathscr{Q}}}$ norms are bounded on Σ_0:

$$\widetilde{\mathscr{Q}}_1(\Sigma_0) + \widetilde{\mathscr{Q}}_2(\Sigma_0) + \sum_{i=2}^{q-1}\widetilde{\mathscr{Q}}_{(q)}(\Sigma_0) \leq \varepsilon^2. \tag{3.71}$$

Let us look at the differences with the similar norms in (10, Chapter 3):

(i) In these $\tilde{\mathcal{Q}}$ norms a term associated with $\overline{\rho(\tilde{R})}$ is not present, differently from the definitions in (10).

(ii) We are considering these norms associated with the Weyl tensor $\tilde{R} = \hat{\mathcal{L}}_{T_0}R$ which can be interpreted as the "time derivative" of R; this is required, as discussed in the introduction, to "subtract" the Kerr part of the Riemann tensor.

(iii) The weights of the $\tilde{\mathcal{Q}}$ norms have an extra decay factor, $|u|^{5+\gamma}$, with $\gamma > 0$. It has the effect, in a complicated way shown in the following lemmas, of giving a better decay for the various null components of $\hat{\mathcal{L}}_{T_0}R$. This improved decay is required, first, to guarantee that the decay of Riemann tensor be in agreement with the "peeling"; second, to recover the estimates for the Riemann tensor by "time integration".[29]

Once we prove that these norms are bounded in V_*, the null components of $\hat{\mathcal{L}}_{T_0}R$, $\alpha, \beta, \ldots$ decay with an exponent $5+\frac{\varepsilon}{2}, 4+\frac{\varepsilon}{2}, 3+\frac{\varepsilon}{2}, \ldots$. depending on the null components we are considering, with $\varepsilon = \gamma$. Observe that, assigned γ in the $\tilde{\mathcal{Q}}$ norms, the $\tilde{\mathcal{Q}}|_{\Sigma_0}$ norms are bounded if the Riemann components on Σ_0 decay with an $\hat{\varepsilon} > \gamma$, implying that the correction of the metric components have to decay on Σ_0 toward spacelike infinity as $O(r^{-(3+\frac{\hat{\varepsilon}}{2})})$.[30]

Finally, the connection coefficients satisfy the structure equations which depend on the Riemann tensor so that the exponent factor δ must be equal to $\gamma/2$, see later on, and δ must coincide with the exponent $\varepsilon/2$, see Lemmas 32, 33, 35. In conclusion, in the bootstrap assumptions for the $\delta\mathcal{O}$ norms we have to choose

$$\gamma = 2\delta = \varepsilon \in (0, \hat{\varepsilon}).$$

---

[29] To perform the "time integration" the $|u|$ exponent could be less the $5+\gamma$. To have the same decay as in (10), $2+\gamma$ would be enough.

[30] Therefore, as starting from the $\tilde{\mathcal{Q}}$ norm with a weight $|u|^{5+\gamma}$ it follows that the sup norm estimates for the Riemann components have the same factor γ; this implies that there is a loss of decay going from the initial data decay to the decay of the Riemann components in the whole V_* and therefore, on the whole spacetime.

(iv) In the definition of $\tilde{R}=\hat{\mathscr{L}}_{T_0}R$ the vector field T_0 is

$$T_0=\frac{\Omega}{2}(e_3+e_4)-\frac{X_{(\text{Kerr})}+X}{2}. \tag{3.72}$$

T_0 is a "nearly" Killing vector field if the corrections to the Kerr metric are small and equal to $\frac{\partial}{\partial t}$ in Kerr spacetime. Vice versa in the $\tilde{\mathscr{Q}}$ norms definition

$$T=\frac{\Omega}{2}(e_3+e_4) \tag{3.73}$$

is one of the vector fields saturating the Bel-Robinson tensor and also appearing in the various Lie derivatives of $\tilde{R}$. The reason for the use of both T and T_0 is that to go back from $\tilde{R}$ to the "time derivatives" of the components of R controlling both the norm smallness and the "peeling decay" we need to use T_0 which is nearly a Killing vector field. Vice versa to obtain from the control of the $\tilde{\mathscr{Q}}$ norms the control of the null components norms of $\tilde{R}$ it is crucial to have their integrands positive which requires to use the vector field $T=\frac{\Omega}{2}(e_3+e_4)$.

3.3.2 II Step: The Estimate of $\overline{\rho(\tilde{R})}$ and of $\tilde{\mathscr{Q}}$ in V_*

Theorem 32 *Let the bootstrap assumptions* 2.48 *hold in V_*; assume that on Σ_0 the initial data satisfy*

$$\sup_{\Sigma_0/B_{R_0}}|r^{6+\frac{\varepsilon}{2}}\overline{\rho(\tilde{R})}|\leq\varepsilon;\quad \tilde{\mathscr{Q}}_{\Sigma_0}\leq\varepsilon^2 \tag{3.74}$$

with $\varepsilon=\gamma$; then in V_* the following estimates hold:

$$\sup_{V_*}|r^3|u|^{3+\frac{\varepsilon}{2}}\overline{\rho(\tilde{R})}|\leq c_1^{\frac{1}{2}}\left(\varepsilon+\left(\frac{M}{R_0}\right)^{\frac{1}{2}}\varepsilon_0\right);\quad \tilde{\mathscr{Q}}\leq c_1\left(\varepsilon^2+\frac{M}{R_0}\varepsilon_0^2\right) \tag{3.75}$$

where c is an adimensional constant independent from M.

Proof. The proof similar to the analogous proof in (15) uses a bootstrap inside the region V_*. The rationale for it is the following: to prove the estimate for $\overline{\rho(\tilde{R})}$ we need to control the other null components of $\tilde{R}$. These are controlled in terms of the $\tilde{Q}$ norms which at their turn require, to be controlled, an estimate for $\overline{\rho(\tilde{R})}$.[31]

Let $V_*=J^{(-)}(S(u(R_0),\underline{u}_*))$, we consider $\tilde{V}=J^{(-)}(S(u(R_0),\underline{\tilde{u}}))\subset V_*$ where $\underline{\tilde{u}}\leq\underline{u}_*$ is the largest value of the variable $\underline{u}$ such that in $\tilde{V}$ the following estimates hold[32]:

$$\sup_{\tilde{V}}|r^3|u|^{3+\frac{\varepsilon}{2}}\overline{\rho(\tilde{R})}|\leq c_1^{\frac{1}{2}}\left(\varepsilon+\left(\frac{M}{R_0}\right)^{\frac{1}{2}}\varepsilon_0\right);\quad \tilde{\mathscr{Q}}\leq c_1\left(\varepsilon^2+\frac{M}{R_0}\varepsilon_0^2\right) \tag{3.76}$$

---

[31] The bootstrap we use here is different from the bootstrap we need to extend the region V_*; in fact the main bootstrap assumptions refer to the null components of $\delta R=R-R^{(\text{Kerr})}$ and not to those of $\tilde{R}=\hat{\mathscr{L}}_{T_0}R$.

[32] $u(R_0)$ is the value of $u|_{\Sigma_0}=r_*$ associated to $r=R_0$.

with $c_1 > 1$.

The transport equation for $r^3\overline{\rho}(\tilde{R})$ is,[33]

$$\frac{d}{\mathrm{d}\lambda}(r^3\overline{\rho}) = \frac{1}{|S(\lambda,\nu)|}\int_{S(\lambda,\nu)} G$$

where

$$G \equiv \left[\frac{(\overline{\Omega tr\underline{\chi}} - \Omega\mathrm{tr}\underline{\chi})}{2} r^3(\rho - \overline{\rho}) + \Omega r^3 \left(\mathrm{d}\!\!/\mathrm{iv}\Omega \cdot \underline{\beta} - 2\eta \cdot \underline{\beta} - \frac{1}{2}\hat{\chi} \cdot \underline{\alpha} + \zeta \cdot \underline{\beta}\right)\right]. \tag{3.77}$$

Integrating along the incoming cones we obtain

$$|r^3\overline{\rho}|(u,\underline{u}) \leq |r^3\overline{\rho}|(u_0,\underline{u}) + \int_{u_0}^{u} |r^3 G|(u',\underline{u}) \tag{3.78}$$

Using the connection coefficients sup norm estimates 2.45, 2.46, 2.47, the bootstrap assumptions 2.48, recalling that in the external region $|u| \leq r$, we have

$$\begin{aligned} |r^3 G| &\leq c\left(\frac{M^2}{r^2|u|}|r^3(\rho - \overline{\rho})| + \frac{M^2}{r^2|u|}|r^2|u|\underline{\beta}| + \frac{M^2}{r^2|u|}|r|u|^2\underline{\alpha}|_{p,S}\right) \\ &\leq c\left(\frac{M^2}{r^2|u|^{4+\frac{\varepsilon}{2}}}|r^3|u|^{3+\frac{\varepsilon}{2}}(\rho - \overline{\rho})| + \frac{M^2}{r^2|u|^{4+\frac{\varepsilon}{2}}}|r^2|u|^{4+\frac{\varepsilon}{2}}\underline{\beta}|\right. \\ &\qquad \left. + \frac{M^2}{r^2|u|^{4+\frac{\varepsilon}{2}}}|r|u|^{5+\frac{\varepsilon}{2}}\underline{\alpha}|\right) \end{aligned} \tag{3.79}$$

The norms of the $\tilde{R}$ null components in (3.79) can be bounded by the $\tilde{\mathcal{Q}}^{\frac{1}{2}}$ quantity exactly as it was done in (10, Chapter 5), with the only difference that R is substituted by $\tilde{R}$ and $\mathcal{Q}$ by $\tilde{\mathcal{Q}}$. Therefore, using the second inequality in (3.76),

$$\begin{aligned} &|r^3|u|^{3+\frac{\varepsilon}{2}}\overline{\rho}|(u,\underline{u}) \\ &\quad \leq \left(|r^{6+\frac{\varepsilon}{2}}\overline{\rho}|(u_0,\underline{u}) + c\frac{M^2}{R_0^2}\tilde{\mathcal{Q}}^{\frac{1}{2}}\right) \\ &\quad \leq \left(\varepsilon + c\frac{M^2}{R_0^2}c_1^{\frac{1}{2}}\left(\varepsilon + \left(\frac{M}{R_0}\right)^{\frac{1}{2}}\varepsilon_0\right)\right) \\ &\quad \leq c\left(1 + \frac{M^2}{R_0^2}c_1^{\frac{1}{2}}\right)\varepsilon + cc_1^{\frac{1}{2}}\left(\frac{M}{R_0}\right)^{\frac{5}{2}}\varepsilon_0 \\ &\quad < c_1^{\frac{1}{2}}\left(\varepsilon + \left(\frac{M}{R_0}\right)^{\frac{1}{2}}\varepsilon_0\right) \end{aligned} \tag{3.80}$$

---

[33] The derivation of this equation is in Ref. (10, Chapter 5).

provided ε and M/R_0 have been chosen sufficiently small and $c_1 > c^2$, so that

$$\left(1+c\frac{M^2}{R_0^2}c_1^{\frac{1}{2}}\right) < c_1^{\frac{1}{2}}, \quad c\left(\frac{M}{R_0}\right)^2 < 1. \tag{3.81}$$

The control of the $\tilde{\mathscr{Q}}$ norms in V_*. We are left to prove that in $\tilde{V}$ the second condition of (3.76)

$$\tilde{\mathscr{Q}} \leq c_1\left(\varepsilon^2 + \frac{M}{R_0}\varepsilon_0^2\right),$$

can be improved. This requires the estimate of the error $\tilde{\mathscr{E}} = \tilde{\mathscr{Q}} - \tilde{\mathscr{Q}}_{\Sigma_0}$. The estimate of the error proceeds basically as in (10), the main differences being that in the definition of the $\tilde{\mathscr{Q}}$ norms there are extra weight factors. Let us study some error terms analogous to those discussed in (10, Chapter 6).

Let $V \subset \tilde{V}$ and consider one of the terms in $\int_{V_{(u,\underline{u})}} |Div\tilde{Q}(\hat{\mathscr{L}}_O\tilde{R})_{\beta\gamma\delta}$ $(K^\beta K^\gamma T^\delta)|$, namely the first term of

$$\begin{aligned}\int_{V_{(u,\underline{u})}} \tau_+^4|u|^{5+\gamma}D(O,\tilde{R})_{444} = \frac{1}{2}\int_{V_{(u,\underline{u})}} &\tau_+^4|u|^{5+\gamma}\alpha(\hat{\mathscr{L}}_O\tilde{R})\cdot\Theta(O,\tilde{R}) \\ -\int_{V_{(u,\underline{u})}} &\tau_+^4|u|^{5+\gamma}\beta(\hat{\mathscr{L}}_O\tilde{R})\cdot\Xi(O,\tilde{R})\end{aligned} \tag{3.82}$$

We have

$$\begin{aligned}\left|\int_{V_{(u,\underline{u})}} \tau_+^4|u|^{5+\gamma}\alpha(\hat{\mathscr{L}}_O\tilde{R})\cdot\Theta(O,\tilde{R})\right| &\leq \left(\sup_V \int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^4|u|^{5+\gamma}|\alpha(\hat{\mathscr{L}}_O\tilde{R})|^2\right)^{\frac{1}{2}} \\ \times \int_{u_0}^{u} \mathrm{d}u' &\left(\int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^4|u|^{5+\gamma}|\Theta(O,\tilde{R})|^2\right)^{\frac{1}{2}} \\ \leq c\tilde{\mathscr{Q}}_V^{\frac{1}{2}}\int_{u_0}^{u}\mathrm{d}u'\sum_{i=0}^{3} &\left(\int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^4|u|^{5+\gamma}|\Theta^{(i)}(O,\tilde{R})|^2\right)^{\frac{1}{2}}.\end{aligned} \tag{3.83}$$

The term with $i=0$ is the only one not present in the (10) error estimates, the reason being that in that case the Riemann tensor $R_{\mu\nu\rho\sigma}$ satisfied the Bianchi equation $D^\mu R_{\mu\nu\rho\sigma} = 0$, while here

$$D^\mu \mathscr{L}_{T_0} R_{\mu\nu\rho\sigma} \neq 0.$$

We show in detail how to control it; for all the remaining error terms, we refer to (5) where a detailed discussion on the other terms is done and to (10, Chapter 6), where for the $\mathscr{Q}$ norms, the control of the error terms is done in a very accurate way.

The term in the error with $i = 0$, absent in (10),[34] is proportional to

$$\tilde{\mathscr{Q}}_V^{\frac{1}{2}} \int_{u_0}^{u} \mathrm{d}u' \left(\int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^4 |u|^{5+\gamma} |\Theta^{(0)}(O,\tilde{R})|^2 \right)^{\frac{1}{2}} . \tag{3.84}$$

This term arises from the first term of

$$D^\alpha(\hat{\mathscr{L}}_O \tilde{R})_{\beta\gamma\delta} = D^\alpha(\hat{\mathscr{L}}_O \hat{\mathscr{L}}_{T_0} R)_{\beta\gamma\delta} = \hat{\mathscr{L}}_O D^\alpha(\hat{\mathscr{L}}_{T_0} R)_{\beta\gamma\delta} + \sum_{i=1}^{3} J^i(O;\tilde{R})_{\beta\gamma\delta} . \tag{3.85}$$

As, see (10), Chapter 6, Eqs. (6.1.6)

$$\hat{\mathscr{L}}_O D^\alpha(\hat{\mathscr{L}}_{T_0} R) = \hat{\mathscr{L}}_O J(T_0,R) = \hat{\mathscr{L}}_O (J^1(T_0,R) + J^2(T_0,R) + J^3(T_0,R)) \tag{3.86}$$

the term we have to estimate is

$$\tilde{\mathscr{Q}}_V^{\frac{1}{2}} \int_{u_0}^{u} \mathrm{d}u' \left(\int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^4 |u|^{5+\gamma} |\hat{\mathscr{L}}_O J(T_0,R)|^2 \right)^{\frac{1}{2}} . \tag{3.87}$$

This term has to be treated in a different way from the remaining ones as it cannot be bounded in terms of the $\tilde{\mathscr{Q}}$; in fact, it depends on R instead of on $\tilde{R}$. It is easy to recognize that all the terms in which we can decompose $J(T_0,R)$ can be estimated in the same way; therefore, we consider only $J^1(T_0,R)$ and again all the various terms which compose it can be treated in the same way, see (10, pp. 245–247). They all have the structure

$${}^{(T_0)}\pi DR + {}^{(T_0)}\pi \frac{R}{r} \tag{3.88}$$

and recalling that T_0 is nearly Killing, it follows that the generic term of $\hat{\mathscr{L}}_O J^1(T_0,R)$ considering, for simplicity, all these norms being sup norms, satisfies the following bound, with $\sigma > 0$,

$$|\hat{\mathscr{L}}_O J^1(T_0,R)| \leq |{}^{(T_0)}\pi||DR| \leq \frac{\varepsilon_0}{r^2|u|^{2+\delta}} \frac{M}{r^4} \leq c \frac{\varepsilon_0}{r^5|u|^{2+\delta+\sigma}} \frac{M}{R_0^{1-\sigma}} , \tag{3.89}$$

remembering that $\hat{\mathscr{L}}_0$ does not improve the decay. Substituting this estimate in 3.87 we obtain,

$$\begin{aligned} &\tilde{\mathscr{Q}}_V^{\frac{1}{2}} \int_{u_0}^{u} \mathrm{d}u' \left(\int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^4 |u|^{5+\gamma} |\hat{\mathscr{L}}_O J(T_0,R)|^2 \right)^{\frac{1}{2}} \\ &\quad \leq c \tilde{\mathscr{Q}}_V^{\frac{1}{2}} \varepsilon_0 \frac{M}{R_0^{1-\sigma}} \int_{u_0}^{u} \mathrm{d}u' \left(\int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^4 |u|^{5+\gamma} \frac{1}{r^{10+2\sigma}|u|^{4+2\delta+2\sigma}} \right)^{\frac{1}{2}} \end{aligned}$$

---

[34] There is an analogous term which is not present in (10) coming from the error term $\int_{V_{(u,\underline{u})}} |Div\tilde{Q}(\hat{\mathscr{L}}_T \tilde{R})_{\beta\gamma\delta}(K^\beta K^\gamma T^\delta)|$, its estimate goes exactly in the same way and we do not report it here.

$$
\begin{aligned}
&\leq c\widetilde{\mathscr{Q}}_V^{\frac{1}{2}}\varepsilon_0\frac{M}{R_0^{1-\sigma}}\int_{u_0}^{u}\mathrm{d}u'\left(\int_{|u|'}^{\infty}\mathrm{d}r r^6|u'|^{5+\gamma}\frac{1}{r^{10+2\sigma}|u'|^{4+2\delta+2\sigma}}\right)^{\frac{1}{2}}\\
&\leq c\widetilde{\mathscr{Q}}_V^{\frac{1}{2}}\varepsilon_0\frac{M}{R_0^{1-\sigma}}\int_{u_0}^{u}\mathrm{d}u'|u'|^{\frac{1}{2}+\frac{\gamma}{2}-\delta}\left(\int_{|u|'}^{\infty}\mathrm{d}r\frac{1}{r^{4+2\sigma}}\right)^{\frac{1}{2}} \qquad (3.90)
\end{aligned}
$$

Let us choose[35]

$$
\delta \geq \frac{\gamma}{2}\,; \qquad (3.91)
$$

then the last integral satisfies the following inequality:

$$
\begin{aligned}
&\widetilde{\mathscr{Q}}_V^{\frac{1}{2}}\int_{u_0}^{u}\mathrm{d}u'\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^4|u|^{5+\gamma}|\hat{\mathscr{L}}_O J(T_0,R)|^2\right)^{\frac{1}{2}}\\
&\quad\leq c\widetilde{\mathscr{Q}}_V^{\frac{1}{2}}\varepsilon_0\frac{M}{R_0^{1-\sigma}}\int_{u_0}^{u}\mathrm{d}u'|u'|^{\frac{1}{2}}\frac{1}{|u'|^{\frac{3}{2}+\sigma}} \leq c\widetilde{\mathscr{Q}}_V^{\frac{1}{2}}\varepsilon_0\frac{M}{R_0}\\
&\quad\leq c\left(\widetilde{\mathscr{Q}}_V\frac{M}{R_0}+\varepsilon_0^2\frac{M}{R_0}\right). \qquad (3.92)
\end{aligned}
$$

If this were the only error term we could write

$$
|\tilde{\mathscr{E}}| \leq c\left(\widetilde{\mathscr{Q}}_V\frac{M}{R_0}+\varepsilon_0^2\frac{M}{R_0}\right) \qquad (3.93)
$$

and

$$
\widetilde{\mathscr{Q}}_V \leq \widetilde{\mathscr{Q}}_{\Sigma_0} + c\frac{M}{R_0}\widetilde{\mathscr{Q}}_V + c\frac{M}{R_0}\varepsilon_0^2 \qquad (3.94)
$$

implying

$$
\widetilde{\mathscr{Q}}_V \leq \frac{1}{1-c\frac{M}{R_0}}\widetilde{\mathscr{Q}}_{\Sigma_0} + \frac{c}{\left(1-c\frac{M}{R_0}\right)}\frac{M}{R_0}\varepsilon_0^2 < \frac{c_1}{2}\left(\varepsilon^2+\frac{M}{R_0}\varepsilon_0^2\right) \qquad (3.95)
$$

choosing c_1 appropriately. Clearly the estimates discussed here and in (5) for some of the error terms to prove the boundedness of the $\tilde{\mathscr{Q}}$ norms, are far from giving a complete proof of the result; in fact, the error term is made by a great number of integrals, more than one hundred, and their complete estimates would take a huge number of pages. On the other side, these estimates have been done in a complete way for the $\mathscr{Q}$ norms in (10, Chapter 6), and what we want to point out here is that, apart some differences explicitly examined, the way of controlling the error terms follows exactly the same pattern and therefore the present proof is just a consequence of the proof given there.

---

[35] To complete the bootstrap procedure we need the equal sign, see Sect. 3.4.2.

3.3.3 III Step: The Control of the $\delta\mathscr{R}$ Norms

In the previous step we have proved estimates (3.75). In this step using these estimates we prove estimates (3.56) for all the null components of the correction to the Riemann tensor $\delta R = R - R^{(\mathrm{Kerr})}$.

Theorem 33 *Assume that in* V_* *the estimates* (2.48) *and* (2.51) *hold; then, in* V_*, *we have*

$$\delta\mathscr{R} \leq \frac{\varepsilon_0}{2}. \tag{3.96}$$

Proof. The proof goes basically as in (15). We repeat here the main steps stating the various lemmas which prove the theorem.

Remark **2** In the following Lemmas 31,. . . ,36 we prove the estimates for the sup norms of the null components of δR. To complete the bootstrap we need analogous estimates for their derivatives up to $l = q-1$. Again, apart from notational complications, the more delicate part is the control of the non-differentiated components. The remaining estimates are just a repetition and follow the pattern discussed in (10).

Lemma 31 *Assuming the estimates* (2.48) *and the condition*

$$\frac{M}{R_0} \ll 1 \tag{3.97}$$

it follows that the various null components of $\tilde{R} = \hat{\mathscr{L}}_{T_0} R$ *satisfy the following inequalities with an "independent" constant* $c_2 > c_1$ *and* $\varepsilon = \gamma$:

$$\begin{aligned}
\sup_{\mathscr{K}} r^{\frac{7}{2}} |u|^{\frac{5}{2}+\frac{\varepsilon}{2}} |\alpha(\hat{\mathscr{L}}_{T_0} R)| &\leq c_2 \left(\varepsilon + \frac{\varepsilon_0}{N_0}\right)\\
\sup_{\mathscr{K}} r^{\frac{7}{2}} |u|^{\frac{5}{2}+\frac{\varepsilon}{2}} |\beta(\hat{\mathscr{L}}_{T_0} R)| &\leq c_2 \left(\varepsilon + \frac{\varepsilon_0}{N_0}\right)\\
\sup_{\mathscr{K}} r^{3} |u|^{3+\frac{\varepsilon}{2}} |\rho(\hat{\mathscr{L}}_{T_0} R) - \overline{\rho(\hat{\mathscr{L}}_{T_0} R)}| &\leq c_2 \left(\varepsilon + \frac{\varepsilon_0}{N_0}\right)\\
\sup_{\mathscr{K}} r^{3} |u|^{3+\frac{\varepsilon}{2}} |\sigma(\hat{\mathscr{L}}_{T_0} R) - \overline{\sigma(\hat{\mathscr{L}}_{T_0} R)}| &\leq c_2 \left(\varepsilon + \frac{\varepsilon_0}{N_0}\right)\\
\sup_{\mathscr{K}} r^{2} |u|^{4+\frac{\varepsilon}{2}} |\underline{\beta}(\hat{\mathscr{L}}_{T_0} R)| &\leq c_2 \left(\varepsilon + \frac{\varepsilon_0}{N_0}\right)\\
\sup_{\mathscr{K}} r |u|^{5+\frac{\varepsilon}{2}} |\underline{\alpha}(\hat{\mathscr{L}}_{T_0} R)| &\leq c_2 \left(\varepsilon + \frac{\varepsilon_0}{N_0}\right).
\end{aligned} \tag{3.98}$$

Proof. Under the estimates 2.48 and the condition

$$\frac{M}{R_0} < \frac{1}{N_0^2} \ll 1$$

with N_0 positive integer, we proved in Theorem 32 the inequalities

$$
\begin{aligned}
\tilde{\mathcal{Q}} \leq c_1\left(\varepsilon^2+\frac{M}{R_0}\varepsilon_0^2\right) &\leq c_1\left(\varepsilon^2+\frac{\varepsilon_0^2}{N_0^2}\right) \leq c_1\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right)^2 \\
\sup_{V_*}|r^3|u|^{3+\frac{\varepsilon}{2}}\overline{\rho(\tilde{R})}| &\leq c_1^{\frac{1}{2}}\left(\varepsilon+\left(\frac{M}{R_0}\right)^{\frac{1}{2}}\varepsilon_0\right) \leq c_1^{\frac{1}{2}}\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right)
\end{aligned}
\tag{3.99}
$$

From it proceeding as in Chapter 5 of (10) the thesis follows.

Next lemma shows that integrating along the incoming cones in V_*, we can transform the decay in $|u|$ proved in the previous lemma in a better decay in r.
Lemma 32 *From the results of Lemma* 31, *using assumptions* (2.48) *and also the condition*

$$
\kappa\frac{M}{R_0} \leq 1,
$$

the following estimates hold:

$$
\begin{aligned}
\sup_{V_*} r^5|u|^{1+\frac{\varepsilon}{2}}|\alpha(\hat{\mathcal{L}}_{T_0}R)| &\leq \tilde{c}_4\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right) \\
\sup_{V_*} r^4|u|^{2+\frac{\varepsilon}{2}}|\beta(\hat{\mathcal{L}}_{T_0}R)| &\leq \tilde{c}_3\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right) \\
\sup_{V_*} r^3|u|^{3+\frac{\varepsilon}{2}}|\rho(\hat{\mathcal{L}}_{T_0}R)-\overline{\rho(\hat{\mathcal{L}}_{T_0}R)}| &\leq c_3\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right) \\
\sup_{V_*} r^3|u|^{3+\frac{\varepsilon}{2}}|\sigma(\hat{\mathcal{L}}_{T_0}R)-\overline{\sigma(\hat{\mathcal{L}}_{T_0}R)}| &\leq c_3\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right) \\
\sup_{V_*} r^2|u|^{4+\frac{\varepsilon}{2}}\underline{\beta}(\hat{\mathcal{L}}_{T_0}R) &\leq c_3\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right) \\
\sup_{V_*} r|u|^{5+\frac{\varepsilon}{2}}|\underline{\alpha}(\hat{\mathcal{L}}_{T_0}R)| &\leq c_3\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right).
\end{aligned}
\tag{3.100}
$$

with an "independent" constant $c_3 > c_2 > c_1 > c_0$, *and* $\tilde{c}_3, \tilde{c}_4$ *satisfying*

$$
\tilde{c}_3 \geq c(1+c''+c'''+c'''') \tag{3.101}
$$

where c *is a constant* > 1, *which can be different in different inequalities, where* c''', c'''' *satisfy*

$$
c''' \geq c\kappa\frac{M^2}{R_0^2}\left(c_2+c_0\frac{M}{R_0}\right), \quad c'''' \geq cc_2\kappa\frac{M^2}{R_0^2} \tag{3.102}
$$

$$
\tilde{c}_4 \geq c(1+c_4) \tag{3.103}
$$

and

$$
c_4 \geq \tilde{c}_3\left(1+\kappa\frac{M^2}{R_0^2}\right)+\kappa c_1^{\frac{1}{2}}\frac{M^2}{R_0^2}. \tag{3.104}
$$

Proof. This Lemma is basically Theorem 2.3 of (15) (see also (11) where the way of transforming the $|u|$ decay in r decay was first introduced). As it is one of the central step of the whole result we repeat its proof for the β component.

Proof of the second line of (3.100). From the Bianchi equations, see (3.2.8) of (10), it follows that $\beta_a = \beta(\hat{\mathscr{L}}_{T_0}R)(e_a)$ satisfies, along the incoming null hypersurface $\underline{C}(\nu)$, the evolution equation[36]

$$\frac{\partial \beta_a}{\partial \lambda} + \Omega \mathrm{tr}\underline{\chi}\beta_a = 2\Omega\underline{\omega}\beta_a + \Omega\left[\not\nabla_a\rho + {}^\star\not\nabla_a\sigma + 2(\hat{\chi}\cdot\underline{\beta})_a + 3(\eta\rho + {}^\star\eta\sigma)_a\right] \tag{3.105}$$

From this equation, see Chapter 4 of (10), we obtain the following inequality:

$$\begin{aligned}\frac{d}{\mathrm{d}\lambda}|r^{(2-\frac{2}{p})}\beta|_{p,S} \leq{}& ||2\Omega\underline{\omega} - (1-1/p)(\Omega\mathrm{tr}\underline{\chi} - \overline{\Omega\mathrm{tr}\underline{\chi}})||_\infty |r^{(2-\frac{2}{p})}\beta|_{p,S}\\ &+\|\Omega\|_\infty\left(|r^{(2-\frac{2}{p})}\not\nabla\rho|_{p,S} + 3|r^{(2-\frac{2}{p})}\eta\rho|_{p,S} + |r^{(2-\frac{2}{p})}\tilde{F}|_{p,S}\right),\end{aligned} \tag{3.106}$$

where $\tilde{F}(\cdot) = 2\hat{\chi}\cdot\underline{\beta} + ({}^\star\not\nabla\sigma + 3{}^\star\eta\sigma)$.[37] Integrating along $\underline{C}(\nu)$, with $\lambda_1 = u|_{\underline{C}(\nu)\cap\Sigma_0}$, we obtain

$$\begin{aligned}|r^{(2-\frac{2}{p})}\beta|_{p,S}(\lambda,\nu) \leq{}& |r^{(2-\frac{2}{p})}\beta|_{p,S}(\lambda_1)\\ &+\int_{\lambda_1}^{\lambda} ||2\Omega\underline{\omega} - (1-1/p)(\Omega\mathrm{tr}\underline{\chi} - \overline{\Omega\mathrm{tr}\underline{\chi}})||_\infty |r^{(2-\frac{2}{p})}\beta|_{p,S}(\lambda',\nu)\\ &+\|\Omega\|_\infty\left(\int_{\lambda_1}^{\lambda}|r^{(2-\frac{2}{p})}\not\nabla\rho|_{p,S} + 3\int_{\lambda_1}^{\lambda}|r^{(2-\frac{2}{p})}\eta\rho|_{p,S} + \frac{1}{2}\int_{\lambda_1}^{\lambda}|r^{(2-\frac{2}{p})}\tilde{F}|_{p,S}\right).\end{aligned}$$

In V_* the previous assumptions imply the following estimates:

$$\|r|\lambda|\Omega\underline{\omega}\|_\infty \leq \kappa M \quad \text{and} \quad \|r|\lambda|\Omega(\mathrm{tr}\underline{\chi} - \overline{\Omega\mathrm{tr}\underline{\chi}})\|_\infty \leq \kappa M.$$

Therefore, we can apply the Gronwall's Lemma obtaining:[38]

$$\begin{aligned}|r^{2-\frac{2}{p}}\beta|_{p,S}(\lambda,\nu) \leq c\Bigg[&|r^{2-\frac{2}{p}}\beta|_{p,S}(\lambda_1) + \|\Omega\|_\infty\Bigg(\int_{\lambda_1}^{\lambda}|r^{2-\frac{2}{p}}\not\nabla\rho|_{p,S}\\ &+ 3\int_{\lambda_1}^{\lambda}|r^{2-\frac{2}{p}}\eta\rho|_{p,S} + \frac{1}{2}\int_{\lambda_1}^{\lambda}|r^{2-\frac{2}{p}}\tilde{F}|_{p,S}\Bigg)\Bigg]\end{aligned} \tag{3.107}$$

---

[36] All the notations used in this paper without an explicit definition are those already introduced in (10). The moving frame compatible with equation 3.105 is the Fermi transported one (see the detailed discussion in (10, Chapter 3)).

[37] The term ${}^\star\not\nabla\sigma + 3{}^\star\eta\sigma$ behaves as the term $\not\nabla_a\rho + 3\eta_a\rho$ and, therefore, we will not consider it explicitly.

[38] The constant c can be chosen as an "independent" constant for the following reason: in this application of the Gronwall Lemma the constant c has to bound the following exponent

$$exp\left\{\int_{\lambda_1}^{\infty}\frac{\kappa M}{\lambda^2}\right\} \leq \exp\frac{kM}{R_0},$$

therefore, under the assumption of the lemma we can choose $c \geq e$.

From inequality (3.59) and the explicit expression of $\Omega^{(\mathrm{Kerr})}$, $\|\Omega\|_\infty \le c$, multiplying both sides by $r^2|\lambda|^{2+\frac{\varepsilon''}{2}}$, with $\varepsilon > \varepsilon'' > 0$, remembering that $r(\lambda,\nu) < r(\lambda_1,\nu)$ and $|\lambda| < |\lambda_1|$, we obtain

$$\begin{aligned}|r^{4-\frac{2}{p}}|\lambda|^{2+\frac{\varepsilon''}{2}}\beta|_{p,S}(\lambda,\nu) \le c\Bigg(&|r^{4-\frac{2}{p}}|\lambda|^{2+\frac{\varepsilon''}{2}}\beta|_{p,S}(\lambda_1)\\ &+\int_{\lambda_1}^{\lambda}|r^{4-\frac{2}{p}}|\lambda'|^{2+\frac{\varepsilon''}{2}}\not\nabla\rho|_{p,S}+3\int_{\lambda_1}^{\lambda}|r^{4-\frac{2}{p}}|\lambda'|^{2+\frac{\varepsilon''}{2}}\eta\rho|_{p,S}\\ &+\frac{1}{2}\int_{\lambda_1}^{\lambda}|r^{4-\frac{2}{p}}|\lambda'|^{2+\frac{\varepsilon''}{2}}\tilde{F}|_{p,S}\Bigg).\end{aligned} \tag{3.108}$$

We examine the integrals in (3.108).

This first integral has the following estimate we prove in the Appendix:

$$\sup_{\mathscr{K}}\left|r^{4-\frac{2}{p}}|\lambda|^{3+\frac{\varepsilon}{2}}\not\nabla\rho(\tilde{R})\right|_{p,S} \le c'\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right). \tag{3.109}$$

Therefore,

$$\begin{aligned}\int_{\lambda_1}^{\lambda}|r^{4-\frac{2}{p}}|\lambda'|^{2+\frac{\varepsilon''}{2}}\not\nabla\rho(\tilde{R})|_{p,S} &\le c'\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right)\int_{\lambda_1}^{\lambda}\frac{1}{|\lambda'|^{1+\frac{\varepsilon-\varepsilon''}{2}}}\\ &\le c''\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right)\frac{1}{|\lambda|^{\frac{\varepsilon-\varepsilon''}{2}}}.\end{aligned} \tag{3.110}$$

To estimate the second integral, observe, recalling assumption (2.48), that η satisfies

$$|r^{2-2/p}|\lambda|\eta|_{p,S}(\lambda,\nu) \le \kappa M^2, \quad p\in[2,\infty]. \tag{3.111}$$

Using the previous estimate for $\overline{\rho(\tilde{R})}$ in V_* and the results of the previous lemma we can conclude that

$$\sup_{V_*}\left|r^3|\lambda|^{3+\frac{\varepsilon}{2}}\rho(\tilde{R})\right| \le \left(c_2+c_0\frac{M}{R_0}\right)\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right);$$

it follows immediately

$$\begin{aligned}\int_{\lambda_1}^{\lambda}|r^{4-\frac{2}{p}}|\lambda'|^{2+\frac{\varepsilon''}{2}}\eta\rho|_{p,S} &\le \kappa M^2\left(c_2+c_0\frac{M}{R_0}\right)\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right)\cdot\int_{\lambda_1}^{\lambda}\frac{1}{r|\lambda'|^{2+\frac{\varepsilon-\varepsilon''}{2}}}\\ &\le c\kappa\frac{M^2}{R_0^2}\left(c_2+c_0\frac{M}{R_0}\right)\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right)\frac{1}{|\lambda|^{\frac{\varepsilon-\varepsilon''}{2}}}\\ &\le c'''\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right)\frac{1}{|\lambda|^{\frac{\varepsilon-\varepsilon''}{2}}}\end{aligned}$$

with

$$c''' \ge c\kappa\frac{M^2}{R_0^2}\left(c_2+c_0\frac{M}{R_0}\right). \tag{3.112}$$

To estimate the third integral, from the expression $\tilde{F}(\cdot) = {}^\star\nabla\!\!\!\!/\,\sigma + 3{}^\star\eta\sigma + 2\hat{\chi}\cdot\underline{\beta}$ and the previous remark concerning ${}^\star\nabla\!\!\!\!/\,\sigma + 3{}^\star\eta\sigma$, we are left to prove that

$$\int_{\lambda_1}^{\lambda} |r^{4-\frac{2}{p}}|\lambda'|^{2+\frac{\varepsilon''}{2}}\hat{\chi}\underline{\beta}|_{p,S} \leq c \tag{3.113}$$

This is easy, as, from the estimates (3.98) and (2.48), we have

$$\begin{aligned}
&\sup_{V_*}|r^2|\lambda|^{4+\frac{\varepsilon}{2}}\underline{\beta}| \leq c_2\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right),\\
&\sup_{V_*}||\lambda|r^{2-2/p}\hat{\chi}|_{p,S} \leq \kappa M^2 \quad\; p\in[2,\infty].
\end{aligned} \tag{3.114}$$

Therefore,

$$\begin{aligned}
\int_{\lambda_1}^{\lambda} |r^{4-\frac{2}{p}}|\lambda'|^{2+\frac{\varepsilon''}{2}}\tilde{F}|_{p,S} &\leq c_2\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right)\kappa M^2\int_{\lambda_1}^{\lambda}\frac{1}{|\lambda'|^{3+\frac{\varepsilon-\varepsilon''}{2}}}\\
&\leq cc_2\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right)\kappa\frac{M^2}{R_0^2}\frac{1}{|\lambda|^{\frac{\varepsilon-\varepsilon''}{2}}}\\
&\leq c''''\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right)\frac{1}{|\lambda|^{\frac{\varepsilon-\varepsilon''}{2}}}
\end{aligned} \tag{3.115}$$

with

$$c'''' \geq cc_2\kappa\frac{M^2}{R_0^2} \geq cc_2\kappa\frac{M^2}{R_0^2}. \tag{3.116}$$

Collecting all these estimates for the integrals in (3.108), we infer that

$$\begin{aligned}
&|r^{4-\frac{2}{p}}|\lambda|^{2+\frac{\varepsilon''}{2}}\beta|_{p,S}(\lambda,\nu)\\
&\quad\leq c\left(|r^{4-\frac{2}{p}}|\lambda|^{2+\frac{\varepsilon''}{2}}\beta|_{p,S}(\lambda_1)+(c''+c'''+c'''')\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right)\frac{1}{|\lambda|^{\frac{\varepsilon-\varepsilon''}{2}}}\right)\\
&\quad\leq c(1+c''+c'''+c'''')\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right)\frac{1}{|\lambda|^{\frac{\varepsilon-\varepsilon''}{2}}}\\
&\quad\leq \tilde{c}_3\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right)\frac{1}{|\lambda|^{\frac{\varepsilon-\varepsilon''}{2}}}
\end{aligned} \tag{3.117}$$

and finally,

$$|r^{4-\frac{2}{p}}|\lambda|^{2+\frac{\varepsilon}{2}}\beta|_{p,S}(\lambda,\nu) \leq \tilde{c}_3\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right) \tag{3.118}$$

where we used the initial data assumptions $\mathcal{Q}_{\Sigma_0} \leq \varepsilon$ which implies, assuming $\varepsilon \leq \gamma$,

$$|r^{4-\frac{2}{p}}|\lambda|^{2+\frac{\varepsilon}{2}}\beta|_{p,S}(\lambda_1,\underline{u}=|\lambda_1|) \leq c\varepsilon.$$

To prove the sup estimate in (3.100) we have to repeat for $\not\nabla\beta$ the previous estimate for β. This requires the transport equation for $\not\nabla\beta$ along the $\underline{C}$ "cones" which in turn requires the control of an extra derivative for ρ and σ. This is the reason why we need a greater regularity in the initial data which translate in the definition of $\mathcal{Q}$ norms with more Lie derivatives than in (10). We do not write the proof as it goes, with the obvious changes, exactly as for the β estimate. Therefore, we have proved the following inequality:

$$\sup_{\mathcal{K}} r^4|u|^{2+\frac{\varepsilon}{2}}|\beta(\hat{\mathcal{L}}_T R)| \leq \tilde{c}_3\left(\varepsilon + \frac{\varepsilon_0}{N_0}\right) \tag{3.119}$$

with

$$\tilde{c}_3 \geq c(1+c''+c'''+c'''') \tag{3.120}$$

where c''', c'''' satisfy

$$c''' \geq c\kappa\frac{M^2}{R_0^2}\left(c_2 + c_0\frac{M}{R_0}\right), \quad c'''' \geq cc_2\kappa\frac{M^2}{R_0^2}. \tag{3.121}$$

The analogous estimate for α follows the same lines; we do not report it here and we refer to (5) where it is written in all the details.

Next Lemma allows us to go from the estimates of $\alpha(\hat{\mathcal{L}}_{T_0}\tilde{R}),\ldots$ to the estimates for $\alpha(\mathcal{L}_{T_0}\tilde{R}),\ldots$; it is (a simplified version of[39]) Theorem 2.4 of (15). To prove it we have to use some norm estimates for the components of the ${}^{(T_0)}\pi$ deformation tensor based on the $\delta\mathcal{O}$ bootstrap assumptions. Moreover we also use the estimates of the norms of the Riemann tensor $R = R^{(\mathrm{Kerr})} + \delta R$, which requires the control of the various components of the Kerr Riemann tensor.

3.3.4 The ${}^{(T_0)}\pi$ Deformation Tensor Estimates

The explicit expressions of the null components of the ${}^{(T_0)}\pi$ deformation tensor are

$$\begin{aligned}
&{}^{(T_0)}\pi(e_3,e_3) = {}^{(T_0)}\pi(e_4,e_4) = 0\\
&{}^{(T_0)}\pi(e_3,e_4) = 4(\omega+\underline{\omega}) - (\mathbf{g}(\mathbf{D}_3X,e_4)+\mathbf{g}(\mathbf{D}_4X,e_3))\\
&{}^{(T_0)}\pi(e_3,e_a) = 2\Omega\zeta(e_a) - \mathbf{g}(\mathbf{D}_3X,e_a) - \mathbf{g}(\mathbf{D}_aX,e_3)\\
&{}^{(T_0)}\pi(e_4,e_a) = -2\Omega\zeta(e_a) - \mathbf{g}(\mathbf{D}_4X,e_a) - \mathbf{g}(\mathbf{D}_aX,e_4)\\
&{}^{(T_0)}\pi(e_a,e_b) = \Omega(\chi+\underline{\chi})(e_a,e_b) - (\mathbf{g}(\mathbf{D}_aX,e_b)+\mathbf{g}(\mathbf{D}_bX,e_a))
\end{aligned} \tag{3.122}$$

---

[39] The reason is that in V_* the bootstrap assumptions are stronger than those which can be done "ab initio" on the whole spacetime in (15), see the initial discussion in the introduction there.

They are identically zero in Kerr spacetime. Under the bootstrap assumptions 2.48 we prove the following estimates:

$$\begin{aligned} |r^2|u|^{2+\delta}\,{}^{(T_0)}\pi(e_3,e_4)| &\le c\varepsilon_0\\ |r^2|u|^{2+\delta}\,{}^{(T_0)}\pi(e_3,e_a)| &\le c\varepsilon_0\\ |r^2|u|^{2+\delta}\,{}^{(T_0)}\pi(e_4,e_a)| &\le c\varepsilon_0\\ |r^2|u|^{2+\delta}\,{}^{(T_0)}\pi(e_a,e_b)| &\le c\varepsilon_0 \end{aligned} \tag{3.123}$$

The proof follows immediately from the bootstrap assumptions (2.48) and from the estimates for the metric component corrections, (3.59), with the only exception of the estimate of the $\mathbf{D}_4\delta X$ part which requires a separate estimate proved in Lemma 6.1 of (5). Observe that in the Kerr spacetime

$$\begin{aligned} (\mathbf{g}(\mathbf{D}_3X,e_4)+\mathbf{g}(\mathbf{D}_4X,e_3))^{(\mathrm{Kerr})} &= 0\\ (\mathbf{g}(\mathbf{D}_\lambda X,e_\lambda)+\mathbf{g}(\mathbf{D}_\lambda X,e_\lambda))^{(\mathrm{Kerr})} &= 0\\ \big(\mathbf{g}(\mathbf{D}_\phi X,e_\phi)+\mathbf{g}(\mathbf{D}_\phi X,e_\phi)\big)^{(\mathrm{Kerr})} &= 0. \end{aligned} \tag{3.124}$$

This implies that the following combination of the connection coefficients are identically zero in Kerr and, therefore, in the perturbed Kerr they satisfy the following inequalities:

$$\begin{aligned} |r^2|u|^{2+\delta}(\omega+\underline{\omega})| \le c\varepsilon_0, &\quad |r^2|u|^{2+\delta}(\mathrm{tr}\chi+\mathrm{tr}\underline{\chi})| \le c\varepsilon_0\\ R_0^{-\delta}|r^2|u|^{2+\delta}(\chi+\underline{\chi})_{\lambda\lambda}| \le c\varepsilon_0, &\quad R_0^{-\delta}|r^2|u|^{2+\delta}(\chi+\underline{\chi})_{\phi\phi}| \le c\varepsilon_0. \end{aligned} \tag{3.125}$$

Remark **3** The estimates of the ${}^{(T_0)}\pi$ components are required to move from the estimates of the $\hat{\mathscr{L}}_{T_0}R$ components to those of the $\mathscr{L}_{T_0}R$, later to the estimates of the $\partial_{T_0}R$ components and finally to the δR ones. To prove that they have the right smallness and the appropriate decay we need to control the corrections δO. It is at this point that, to close the bootstrap mechanism, we need the transport equations for the δO parts which require the delicate subtraction of the Kerr part from these equations, we discuss in Sect. 3.4.

3.3.5 The Riemann Null Components in the Kerr Spacetime

The “principal null directions” frame $\{l,n,\tilde{e}_\theta,\tilde{e}_\phi\}$, see (2), is made by the following vector fields, where r is the Boyer–Lindquist radial coordinate r_b:

$$\begin{aligned} l &= \frac{r^2+a^2}{\Delta}\frac{\partial}{\partial t}+\frac{a}{\Delta}\frac{\partial}{\partial\phi}+\frac{\partial}{\partial r},\\ n &= \frac{\Delta}{2\Sigma}\left(\frac{r^2+a^2}{\Delta}\frac{\partial}{\partial t}+\frac{a}{\Delta}\frac{\partial}{\partial\phi}-\frac{\partial}{\partial r}\right)\\ \tilde{e}_\theta &= \frac{1}{\sqrt{\Sigma}}\frac{\partial}{\partial\theta},\\ \tilde{e}_\phi &= \frac{1}{\sqrt{\Sigma}}\left(a\sin\theta\frac{\partial}{\partial t}+\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right) \end{aligned}$$

where l, n are the principal null directions. As, in the Petrov classification, the Kerr spacetime is of type D, in this frame the only null Riemann component different from zero are ρ and σ or, in the Newman-Penrose notations, Ψ_2,

$$\Psi_2 = \rho(R) + i\sigma(R) = \frac{1}{(r - i\cos\theta)^3} = \frac{1}{r^3} + i\frac{3a\cos\theta}{r^4} + O\left(\frac{1}{r^5}\right) \tag{3.126}$$

Beside the fact that our initial data are not exactly those of the Kerr spacetime due to corrections $\delta^{(3)}\mathbf{g}$ and $\delta^{(3)}\mathbf{k}$, the null orthonormal frame we use is not the one associated with the principal null directions. Therefore, in the frame introduced in (8), all the Riemann components of the Kerr spacetime are different from zero. The relation between the frame adapted to the (Kerr) double null foliation of V_*, see (2.32), and the "principal null directions" one is

$$\begin{aligned}
&e_4^{(\mathrm{Kerr})}\\
&\quad = \frac{\sqrt{\Delta}}{2\Sigma R}\left[\left(r^2 + a^2 + \frac{R\Sigma r}{Q}\right) l + \left(r^2 + a^2 - \frac{R\Sigma r}{Q}\right)\frac{2\Sigma}{\Delta} n - 2\sqrt{\Sigma} a \sin\theta \tilde{e}_\phi\right]\\
&e_3^{(\mathrm{Kerr})}\\
&\quad = \frac{\sqrt{\Delta}}{2\Sigma R}\left[\left(r^2 + a^2 - \frac{R\Sigma r}{Q}\right) l + \left(r^2 + a^2 + \frac{R\Sigma r}{Q}\right)\frac{2\Sigma}{\Delta} n - 2\sqrt{\Sigma} a \sin\theta \tilde{e}_\phi\right]\\
&\qquad\qquad e_\lambda^{(\mathrm{Kerr})} = \frac{Q}{\sqrt{\Sigma}R}\tilde{e}_\theta - \frac{\Delta P}{2\Sigma R}\left(l - \frac{2\Sigma}{\Delta} n\right),\\
&\qquad\qquad e_\phi^{(\mathrm{Kerr})} = \frac{r^2 + a^2}{\sqrt{\Sigma}R}\tilde{e}_\phi - \frac{\Delta}{2R\Sigma} a \sin\theta\left(l + \frac{2\Sigma}{\Delta} n\right).
\end{aligned}$$

where[40]

$$Q^2 = (r^2 + a^2)^2 - a^2\lambda\triangle,\ P^2 = a^2(\lambda - \sin\theta^2),\ \mathscr{L} = \mu P Q,\ \lambda = \sin^2\theta_*. \tag{3.127}$$

Assuming $M/r \leq M/R_0$ small the previous relations become approximately

$$\begin{aligned}
e_4 &= \left[l + O\left(\frac{M^2}{r^2}\right) n + O\left(\frac{M}{r}\right)\tilde{e}_\phi\right],\\
e_3 &= \left[n + O\left(\frac{M^2}{r^2}\right) l + O\left(\frac{M}{r}\right)\tilde{e}_\phi\right]\\
e_\lambda &= \tilde{e}_\theta + O\left(\frac{M}{r}\right)(l - 2n), \quad e_\phi = \tilde{e}_\phi + O\left(\frac{M}{r}\right)(l + 2n).
\end{aligned} \tag{3.128}$$

---

[40] μ is an integrating factor defined in (8) equation (25) and θ_* is in the $M \to 0$ limit the spherical θ coordinate of the Minkowski spacetime.

Denoting with the upperscript $^{(PN)}$ the Riemann components in the "Principal null directions frame", we have

$$\begin{aligned}\alpha^{(\mathrm{Kerr})}(e_a,e_b) &= R(e_a,e_4,e_b,e_4) = O\left(\frac{M^2}{r^2}\right)\rho^{(PN)} + O\left(\frac{M^2}{r^2}\right)\varepsilon_{ab}\sigma^{(PN)}\\ \beta^{(\mathrm{Kerr})}(e_a) &= 2^{-1}R(e_a,e_4,e_3,e_4) = O\left(\frac{M}{r}\right)\rho^{(PN)} + O\left(\frac{M}{r}\right)\varepsilon_{ab}\sigma^{(PN)}\end{aligned} \tag{3.129}$$

which implies the following estimates:

$$\|r^5\alpha(R^{(\mathrm{Kerr})})\|_\infty \leq cM^3, \quad \|r^4\beta(R^{(\mathrm{Kerr})})\|_\infty \leq cM^2. \tag{3.130}$$

Estimates (3.130) and the "Bootstrap assumptions" for δR imply[41]

$$\|r^5\alpha(R)\|_\infty \leq cM^3 + \tilde{c}\varepsilon_0, \quad \|r^4\beta(R)\|_\infty \leq cM^2 + \tilde{c}\frac{\varepsilon_0}{R_0}. \tag{3.131}$$

We are now in the position to obtain from the $\hat{\mathscr{L}}_{T_0}R$ estimates the $\mathscr{L}_{T_0}R$ ones.

Lemma 33 *Under the same assumptions as in Lemma* 32*, using the results proved there we have in the region V_* the following inequalities*:

$$\begin{aligned}\sup_{\mathscr{K}} r^5|u|^{1+\frac{\varepsilon}{2}}|\alpha(\mathscr{L}_{T_0}R)| &\leq \tilde{c}_6\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right)+c\frac{M^2}{R_0^2}\varepsilon_0\\ \sup_{\mathscr{K}} r^4|u|^{2+\frac{\varepsilon}{2}}|\beta(\mathscr{L}_{T_0}R)| &\leq \tilde{c}_5\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right)+c\frac{M}{R_0}\varepsilon_0\\ \sup_{\mathscr{K}} r^3|u|^{3+\frac{\varepsilon}{2}}|\rho(\mathscr{L}_{T_0}R)-\overline{\rho(\mathscr{L}_{T_0}R)}| &\leq c_5\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right)\\ \sup_{\mathscr{K}} r^3|u|^{3+\frac{\varepsilon}{2}}|\sigma(\mathscr{L}_{T_0}R)-\overline{\sigma(\mathscr{L}_{T_0}R)}| &\leq c_5\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right)\\ \sup_{\mathscr{K}} r^2|u|^{4+\frac{\varepsilon}{2}}|\underline{\beta}(\mathscr{L}_{T_0}R)| &\leq c_5\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right)\\ \sup_{\mathscr{K}} r|u|^{5+\frac{\varepsilon}{2}}|\underline{\alpha}(\mathscr{L}_{T_0}R)| &\leq c_5\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right).\end{aligned}$$

where, we assume $\delta \geq \frac{\varepsilon}{2}$,

$$\tilde{c}_5 \geq \tilde{c}_3+\tilde{c}; \quad \tilde{c}_6 \geq (\tilde{c}_4+c\tilde{c}). \tag{3.132}$$

---

[41] It has to be pointed out that the estimates (3.130) refer to the various components of the (Kerr) Riemann tensor in the orthonormal frame associated with the Kerr spacetime, what in fact we have to consider here are the null Riemann components relative to the null orthonormal frame associated with the perturbed Kerr spacetime (see Eqs. (2.28)). It is easy to see, using the bootstrap assumptions for the metric components, that estimates (3.130) still hold possibly with a different $\tilde{c}$ constant.

Proof. We start recalling the following expressions:

$$\mathscr{L}_{T_0}R = \hat{\mathscr{L}}_{T_0}R + \frac{1}{2}{}^{(T_0)}[R] - \frac{3}{8}(\mathrm{tr}^{(T_0)}\pi)R, \tag{3.133}$$

where

$${}^{(T_0)}[R]_{\alpha\beta\gamma\delta} = {}^{(T_0)}\pi_\alpha^\mu R_{\mu\beta\gamma\delta} + {}^{(T_0)}\pi_\beta^\mu R_{\alpha\mu\gamma\delta} + {}^{(T_0)}\pi_\gamma^\mu R_{\alpha\beta\mu\delta} + {}^{(T_0)}\pi_\delta^\mu R_{\alpha\beta\gamma\mu}. \tag{3.134}$$

From these equations it follows

$$\begin{aligned}
\alpha(\mathscr{L}_{T_0}R)_{ab} &= \alpha(\hat{\mathscr{L}}_{T_0}R)_{ab} + \frac{1}{2}{}^{(T_0)}[R]_{a4b4} - \frac{3}{8}(\mathrm{tr}^{(T_0)}\pi)\alpha(R)_{ab} \\
\beta(\mathscr{L}_{T_0}R)_a &= \beta(\hat{\mathscr{L}}_{T_0}R)_a + \frac{1}{2}{}^{(T_0)}[R]_{a434} - \frac{3}{8}(\mathrm{tr}^{(T_0)}\pi)\beta(R)_a
\end{aligned} \tag{3.135}$$

and, observing that ${}^{(T_0)}\pi_{44} = 0$ we easily obtain

$$\begin{aligned}
&{}^{(T_0)}[R]_{a4b4} \\
&\quad = -\frac{1}{2}{}^{(T_0)}\hat{\pi}_{a4}R_{34b4} + \left({}^{(T_0)}\hat{\pi}_{ac}R_{c4b4} + {}^{(T_0)}\hat{\pi}_{bc}R_{a4c4} + \frac{3}{4}(\mathrm{tr}^{(T_0)}\pi)R_{a4b4}\right) \\
&\quad\quad -\frac{1}{2}{}^{(T_0)}\hat{\pi}_{43}R_{a4b4} + {}^{(T_0)}\hat{\pi}_{4c}(R_{acb4} + R_{a4bc}) - \frac{1}{2}{}^{(T_0)}\hat{\pi}_{b4}R_{a434}.
\end{aligned}$$

Therefore estimating ${}^{(T_0)}[R]_{a4b4}$ we obtain

$$\begin{aligned}
\|{}^{(T_0)}[R]_{a4b4}\|_\infty &\le c\Big((\||{}^{(T_0)}\mathbf{i}|\|_\infty + \|{}^{(T_0)}\mathbf{j}\|_\infty)\|\alpha(R)\|_\infty + \|{}^{(T_0)}\mathbf{m}\|_\infty\|\beta(R)\|_\infty\Big) \\
&\le c\bigg(\frac{1}{r^{2+5}|\lambda|^{2+\delta}}(\|r^2|\lambda|^{2+\delta}{}^{(T_0)}\mathbf{i}\|_\infty + \|r^2|\lambda|^{2+\delta}{}^{(T_0)}\mathbf{j}\|_\infty)\|r^5\alpha(R)\|_\infty \\
&\quad + \frac{1}{r^{2+4}|\lambda|^{2+\delta}}\|r^2|\lambda|^{2+\delta}{}^{(T_0)}\mathbf{m}\|_\infty\|r^4\beta(R)\|_\infty\bigg) \\
&\le c\varepsilon_0\left(\frac{1}{r^7|\lambda|^{2+\delta}}\|r^5\alpha(R)\|_\infty + \frac{1}{r^6|\lambda|^{2+\delta}}\|r^4\beta(R)\|_\infty\right) \\
&\le \frac{c\varepsilon_0}{r^5|\lambda|^{1+\frac{\varepsilon}{2}}}\left(\frac{(cM^3 + \tilde{c}\varepsilon_0)}{r^2|\lambda|^{1+\delta-\frac{\varepsilon}{2}}} + \frac{(cM^2 + \tilde{c}\frac{\varepsilon_0}{R_0})}{r|\lambda|^{1+\delta-\frac{\varepsilon}{2}}}\right)
\end{aligned} \tag{3.136}$$

where in the last two lines we used estimates (3.131) so that, finally,

$$\|r^5|\lambda|^{1+\frac{\varepsilon}{2}}{}^{(T_0)}[R]_{a4b4}\|_\infty \le c\varepsilon_0\left(\frac{(cM^3 + \tilde{c}\varepsilon_0)}{r^2|\lambda|^{1+\delta-\frac{\varepsilon}{2}}} + \frac{(cM^2 + \tilde{c}\frac{\varepsilon_0}{R_0})}{r|\lambda|^{1+\delta-\frac{\varepsilon}{2}}}\right). \tag{3.137}$$

Moreover,

$$\begin{aligned}
\|(\mathrm{tr}^{(T_0)}\pi)\alpha(R)\|_\infty &\le \frac{c}{r^7|\lambda|^{2+\delta}}|r^2|\lambda|^{2+\delta}\mathrm{tr}^{(T_0)}\pi|_\infty\|r^5\alpha(R)\|_\infty \\
&\le \frac{c\varepsilon_0}{r^5|\lambda|^{1+\frac{\varepsilon}{2}}}\frac{(cM^3 + \tilde{c}\varepsilon_0)}{r^2|\lambda|^{1+\delta-\frac{\varepsilon}{2}}}
\end{aligned}$$

and

$$\|r^5|\lambda|^{1+\frac{\varepsilon}{2}}(\mathrm{tr}^{(T_0)}\pi)\alpha(R)\|_\infty \le c\varepsilon_0\frac{(cM^3+\tilde{c}\varepsilon_0)}{r^2|\lambda|^{1+\delta-\frac{\varepsilon}{2}}}. \tag{3.138}$$

Using estimates (3.137) and (3.138) it follows, using condition (3.54), ($\delta \ge \frac{\varepsilon}{2}$),

$$\begin{aligned}
&\sup_{\mathscr{K}}|r^5|u|^{1+\frac{\varepsilon}{2}}\alpha(\mathscr{L}_T R)|\\
&\le \left(\tilde{c}_4\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right)+c\varepsilon_0\left(\frac{(cM^3+\tilde{c}\varepsilon_0)}{r^2|\lambda|^{1+\delta-\frac{\varepsilon}{2}}}+\frac{(cM^2+\tilde{c}\frac{\varepsilon_0}{|\lambda|})}{r|\lambda|^{1+\delta-\frac{\varepsilon}{2}}}\right)\right)\\
&\le \tilde{c}_6\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right)+c\frac{M^2}{R_0^{2+\delta-\frac{\varepsilon}{2}}}\varepsilon_0 \le \tilde{c}_6\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right)+c\frac{M^2}{R_0^2}\varepsilon_0.
\end{aligned} \tag{3.139}$$

The estimates for the β term go in the same way and we do not report them here. For the other terms there is no need to repeat this computation. In fact the R null components already satisfy the peeling and going from $\hat{\mathscr{L}}_{T_0}$ to $\mathscr{L}_{T_0}$ the decay factors do not worsen.[42]

Next step is to obtain from the estimates for $\alpha(\mathscr{L}_{T_0}R), \beta(\mathscr{L}_{T_0}R)$ the estimates for $\partial_{T_0}\alpha(R), \partial_{T_0}\beta(R)$. This requires the control of $[T_0,e_a]$, $[T_0,e_4]$ and $[T_0,e_3]$.

Lemma 34 *Under the bootstrap assumptions in the region V_* the following estimates hold:*

$$\begin{aligned}
|\mathbf{g}([T_0,e_a],e_d)| &\le c\left(1+\frac{M^2}{R_0^2}\right)\frac{\varepsilon_0}{r^2|u|^{2+\delta}}\\
|\mathbf{g}([T_0,e_4],e_3)| &\le c\left(1+\frac{M^2}{R_0^2}\right)\frac{\varepsilon_0}{r^2|u|^{2+\delta}}\\
|\mathbf{g}([T_0,e_4],e_d)| &\le c\left[1+\frac{M^2}{R_0^2}\left(1+\frac{\varepsilon_0}{R_0^3}\right)\right]\frac{\varepsilon_0}{r^2|u|^{2+\delta}}\\
|\mathbf{g}([T_0,e_3],e_d)| &\le c\left(1+\frac{M^2}{R_0^2}\right)\frac{\varepsilon_0}{r^2|u|^{2+\delta}}\\
|\mathbf{g}([T_0,e_3],e_4)| &\le c\left(1+\frac{M^2}{R_0^2}\right)\frac{\varepsilon_0}{r^2|u|^{2+\delta}}
\end{aligned}$$

Proof. The proof consists in a long, but elementary set of estimates which together with the explicit expressions of the various commutators are in (5).

In the next Lemma 35 we prove the estimates we are looking for relative to $\partial_{T_0}(\alpha(R)(e_a,e_b))$ and $\partial_{T_0}(\beta(R)(e_a))$.

---

[42] The corrections due to the change of null frames are inglobed in the choice of the c_5 constant.

Lemma 35 *Under the same assumptions as in Lemma* 32*, using the results proved there and in Lemmas* 33, 34*, we have in the region* V_* *the following inequalities:*

$$\begin{aligned}\sup_{\mathscr{K}} r^5|u|^{1+\frac{\varepsilon}{2}}|\partial_{T_0}(\alpha(R)(e_a,e_b))| &\leq \tilde{c}_7\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right)+c\frac{M^2}{R_0^2}\varepsilon_0\\ \sup_{\mathscr{K}} r^4|u|^{2+\frac{\varepsilon}{2}}|\partial_{T_0}(\beta(R)(e_a))| &\leq \tilde{c}_8\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right)+c\frac{M}{R_0}\varepsilon_0,\end{aligned} \tag{3.140}$$

with

$$\tilde{c}_7 \geq (\tilde{c}_6+c)\,; \quad \tilde{c}_8 \geq (\tilde{c}_5+c)\,. \tag{3.141}$$

Proof. It is a long, but elementary estimate obtained starting from the relations

$$\begin{aligned}&\alpha(\mathscr{L}_{T_0}R)(e_a,e_b)=(\mathscr{L}_{T_0}R)(e_a,e_4,e_b,e_4)\\ &\quad=\partial_{T_0}(\alpha(R)(e_a,e_b))+R([T_0,e_a],e_4,e_b,e_4)+R(e_a,[T_0,e_4],e_b,e_4)\\ &\beta(\mathscr{L}_{T_0}R)(e_a)=(\mathscr{L}_{T_0}R)(e_a,e_4,e_3,e_4)=\partial_{T_0}(\beta(R)(e_a))+R([T_0,e_a],e_4,e_3,e_4)\\ &\qquad\qquad+R(e_a,[T_0,e_4],e_3,e_4)+R(e_a,e_4,[T_0,e_3],e_4).\end{aligned} \tag{3.142}$$

We do not report it here, but for details see (5).

3.3.6 The Estimate of δR

The final step consists in integrating along the integral curves of T_0. The $|u|$ weight factors will allow to bound uniformly these integrals. Denoting by $\gamma(s)$ the integral curve of T_0 starting in Σ_0 at a distance r_0^* from the origin and $\alpha(t)=\alpha(t)(e_a,e_b)$,

$$\alpha(\bar{t},\bar{r}_*)=\alpha^{(\mathrm{Kerr})}(0,\bar{r}_{*0})+\delta\alpha(0,\bar{r}_{*0})+\int_0^{\bar{s}}(\partial_{T_0}\alpha)(\gamma(s))\mathrm{d}s \tag{3.143}$$

where

$$\begin{aligned}\bar{r}_* &= r_*(u,\underline{u})=\underline{u}-u=\gamma^{r_*}(\bar{s}), \quad \bar{t}=t(u,\underline{u})=\underline{u}+u=\gamma^0(\bar{s})\\ u &= u(\bar{t},\bar{r}_*)=u(0,r_{*1})=-\frac{r_{*1}}{2}, \quad \underline{u}=\underline{u}(\bar{t},\overline{r_*})=\underline{u}(0,r_{*2})=\frac{r_{*2}}{2}\\ \bar{t} &= t(u,\underline{u})=\frac{r_{*2}-r_{*1}}{2}=\frac{\bar{t}+\overline{r_*}-r_{*1}}{2}.\end{aligned} \tag{3.144}$$

As

$$T_0=\frac{\partial}{\partial\underline{u}}+\frac{\partial}{\partial u} \tag{3.145}$$

it follows that

$$\frac{\mathrm{d}\gamma^\mu}{\mathrm{d}s}=T_0^\mu=\delta_u^\mu+\delta_{\underline{u}}^\mu \tag{3.146}$$

therefore, with these definitions, in the coordinates $\{x^\mu\} = \{u, \underline{u}, \omega^1, \omega^2\}$,

$$\begin{aligned} \frac{d}{\mathrm{d}s} u(\gamma(s)) &= \frac{\mathrm{d}\gamma^\mu}{\mathrm{d}s}\frac{\partial u}{\partial x^\mu} = \frac{\mathrm{d}\gamma^u}{\mathrm{d}s} = 1 \\ \frac{d}{\mathrm{d}s} r_*(\gamma(s)) &= \frac{\mathrm{d}\gamma^\mu}{\mathrm{d}s}\left(\frac{\partial u}{\partial x^\mu} - \frac{\partial \underline{u}}{\partial x^\mu}\right) = \frac{\mathrm{d}\gamma^u}{\mathrm{d}s} - \frac{\mathrm{d}\gamma^{\underline{u}}}{\mathrm{d}s} = 0 \end{aligned} \tag{3.147}$$

and, as ω^1, ω^2 do not change along $\gamma(s)$

$$\frac{d}{\mathrm{d}s} r(\gamma(s)) = \frac{\mathrm{d}r}{\mathrm{d}r_*}\frac{d}{\mathrm{d}s} r_*(\gamma(s)) = 0 \tag{3.148}$$

and

$$\begin{aligned} u(\gamma(s;\bar{r}_*)) &= u(\gamma(0;\bar{r}_*)) + s, \quad \underline{u}(\gamma(s;\bar{r}_*)) = \underline{u}(\gamma(0;\bar{r}_*)) + s \\ t &= u + \underline{u} = (u(\gamma(0;\bar{r}_*)) + \underline{u}(\gamma(0;\bar{r}_*))) + 2s = 2s. \end{aligned} \tag{3.149}$$

It follows

$$\delta\alpha(u,\underline{u}) = \alpha(u,\underline{u}) - \alpha^{(\mathrm{Kerr})}(u,\underline{u}) = \delta\alpha(0,\bar{r}_*) + \int_0^{\bar{s}} (\partial_{T_0}\alpha)(\gamma(s))\mathrm{d}s$$

and

$$\begin{aligned} |\delta\alpha(u,\underline{u})| \leq\ & |\delta\alpha(0,\bar{r}_*)| + \int_0^{\bar{s}} |(\partial_{T_0}\alpha)(\gamma(s))|\mathrm{d}s \leq \hat{c}\frac{\varepsilon}{{r_*}^5} \\ & + \left(\tilde{c}_7\varepsilon + c\frac{M^2}{R_0^2}\varepsilon_0\right)\int_0^{\bar{s}} \frac{1}{r(\gamma(s))^5|u(\gamma(s))|^{1+\frac{\varepsilon}{2}}}\mathrm{d}s. \end{aligned} \tag{3.150}$$

Therefore,

$$\begin{aligned} |r^5\delta\alpha(u,\underline{u})| &\leq \hat{c}\frac{r^5(u,\underline{u})}{r_*^5(u,\underline{u})}\varepsilon + \left(\tilde{c}_7\varepsilon + c\frac{M^2}{R_0^2}\varepsilon_0\right)\int_0^{\bar{s}} \frac{1}{|u(\gamma(s))|^{1+\frac{\varepsilon}{2}}}\mathrm{d}s \\ &\leq \hat{c}\frac{r^5(u,\underline{u})}{r_*^5(u,\underline{u})}\varepsilon + \left(\tilde{c}_7\varepsilon + c\frac{M^2}{R_0^2}\varepsilon_0\right)\int_0^{\bar{s}} \frac{1}{|u(\gamma(0;\bar{r}_*)) + s|^{1+\frac{\varepsilon}{2}}}\mathrm{d}s \\ &\leq \hat{c}_1\varepsilon + c\left(\tilde{c}_7\varepsilon + c\frac{M^2}{R_0^2}\varepsilon_0\right) \leq \tilde{c}_9\left(\varepsilon + \frac{\varepsilon_0}{N_0}\right), \end{aligned} \tag{3.151}$$

where we have chosen $\hat{c}_1$ such that

$$\hat{c}\frac{r^5(u,\underline{u})}{r_*^5(u,\underline{u})} \leq \hat{c}_1, \tag{3.152}$$

which is possible as we have proved that r_* and $r(u,\underline{u})$ stay near, and chosen $\tilde{c}_9$ such that

$$\tilde{c}_9\varepsilon \geq (\hat{c}_1 + \tilde{c}_7)\,\varepsilon. \tag{3.153}$$

The proof for $\delta\beta$ goes exactly in the same way and we do not repeat it. Therefore, we have proved the following lemma:

Lemma 36 *Under the same assumptions as in Lemma* 32*, using the results proved there and in Lemmas* 33, 34, 35 *we have in the region* V_* *the following inequalities:*

$$\begin{aligned}|r^5\delta\alpha(R)| &\leq \tilde{c}_9\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right)\\ |r^4\delta\beta(R)| &\leq \tilde{c}_{10}\left(\varepsilon+\frac{\varepsilon_0}{N_0}\right)\end{aligned} \tag{3.154}$$

where

$$\tilde{c}_9 \geq \hat{c}_1+\tilde{c}_7; \quad \tilde{c}_{10} \geq \hat{c}_2+\tilde{c}_8. \tag{3.155}$$

3.4 IV Step: The Control of the $\delta\mathscr{O}$ Norms

To prove better estimate for these norms in V_*, we have to use the transport equations along the incoming and outgoing cones. The use of the outgoing cones is made, as in (10), obtaining estimates starting from "scri", here the upper boundary of the region V_*, a portion of an incoming cone.

Therefore, first we have to control the not underlined connection coefficients[43] on this last slice and to avoid a loss of derivatives we have to prove the existence, on it, of an appropriate foliation called the "last slice canonical foliation", see (14; 10).[44] The decay factors for the bootstrap assumptions for the various (not underlined) $\delta\mathscr{O}$ norms have to be consistent with the weight factors of the norms we control on the "last slice". The estimates for the underlined connection coefficients are made, vice versa, starting from the initial data hypersurface Σ_0. Also in this case an appropriate (canonical) foliation has to be introduced on Σ_0.[45] Finally, as anticipated in Sect. 3.2 to control the $\delta\mathscr{O}$ norms we need estimates for the corrections to the Kerr metric. These estimates follow from the bootstrap assumptions on $\delta\mathscr{O}$ and are the content of the following lemma:

Lemma 37 *Assume that in* V_* *the norms* $\delta\mathscr{O}$ *satisfy the bootstrap assumptions*

$$\delta\mathscr{O} \leq \varepsilon_0$$

then, assuming for the $\delta\mathscr{O}^{(0)}$ *norms appropriate initial data conditions, see Sect.* 3.6*, the following estimates hold in* V_*:

$$|r|u|^{2+\delta}\delta\Omega|_\infty \leq c\varepsilon_0; \quad |r^2|u|^{2+\delta}\delta X|_\infty \leq c\varepsilon_0; \quad ||u|^{1+\delta}\delta\gamma|_\infty \leq c\varepsilon_0. \tag{3.156}$$

The proof of this lemma is given later on,[46] (see Sect. 3.4.4). It is important to remark the order of the various proofs:

---

[43] Basically we denote as not underlined connection coefficients those coefficients whose transport equations we use are those along the outgoing cones, the opposite for the underlined ones. Remember, as discussed in detail in (10) that the choice of the transport equations to use is not arbitrary and is uniquely fixed by the request of avoiding any loss of derivatives which will make the bootstrap mechanism to fail.

[44] See also (6) where the original idea was first stated.

[45] This could be in principle avoided requiring more regularity for the initial data.

[46] In fact we prove the equivalent Lemma 311.

(a) The bootstrap assumptions $\delta\mathscr{O} \leq \varepsilon_0$ imply, Lemma 37, the metric correction estimates $\delta\mathscr{O}^{(0)} \leq c\varepsilon_0$.
(b) The metric correction estimates $\delta\mathscr{O}^{(0)} \leq c\varepsilon_0$, the initial data assumptions and the Riemann bootstrap assumptions imply better estimates for the $\delta\mathscr{O}$ connection coefficients $\delta\mathscr{O} \leq \frac{\varepsilon_0}{2}$.
(c) The improved estimates for the connection coefficients imply better estimates for the metric corrections (see Lemma 311, $\delta\mathscr{O}^{(0)} \leq \frac{\varepsilon_0}{2}$).

3.4.1 The Control of the $\delta\mathscr{O}$ Norms

The $\delta\mathscr{O}$ norms involve tangential derivatives up to fifth order to prove the bootstrap and the peeling, see (10; 11). The proof we sketch here is restricted to the zero and first derivatives as the control of the higher derivatives is simpler and is basically a repetition of what has been done in (10).

To control the $\delta\mathscr{O}$ norms we subtract from the connection coefficients their Kerr parts to obtain transport equations and Hodge equations for the $\delta\mathscr{O}$ corrections. This operation, we call "Kerr decoupling" is a central step of the whole procedure and we discuss it in some generality.

Let us consider a two-covariant tensor connection coefficient

$$O = O_{\mu\nu}\mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu}. \tag{3.157}$$

O is a tensor field tangent to the two-dimensional surfaces $S(u,\underline{u}) = C(u) \cap \underline{C}(\underline{u})$ intersections of the outgoing and incoming cones of the canonical null cone foliation assumed in V_*; $O_{\mu\nu}$ can be written

$$O_{\mu\nu} = \Pi_{\mu}^{\rho}\Pi_{\nu}^{\sigma}H_{\rho\sigma} \tag{3.158}$$

where H is a $(0,2)$ tensor in $(V_*,\mathbf{g})$, a priori not S-tangent, and Π_{ν}^{μ} projects from TV_* to TS.[47] As an example, look at the second null fundamental form χ,

$$\mathbf{g}(\mathbf{D}_{e_a}e_4,e_b) = \chi(e_a,e_b) = \chi_{\mu\nu}e_a^{\mu}e_b^{\nu}. \tag{3.159}$$

Denoting $\{\theta^b(\cdot)\}$ the one forms dual to the TS orthonormal frame $\{e_a\}$,

$$\begin{aligned}\chi_{\mu\nu} &= \sum_{a,b}\chi(e_a,e_b)\theta_{\mu}^{a}\theta_{\nu}^{b} = \sum_{a,b}\mathbf{g}(\mathbf{D}_{e_a}e_4,e_b)\theta_{\mu}^{a}\theta_{\nu}^{b}\\ &= \sum_{a,b}g_{\rho\sigma}e_a^{\tau}(\mathbf{D}_{\tau}e_4)^{\rho}e_b^{\sigma}\theta_{\mu}^{a}\theta_{\nu}^{b} = \Pi_{\mu}^{\tau}\Pi_{\nu}^{\sigma}(\mathbf{D}_{\tau}e_4)^{\rho}g_{\rho\sigma}\\ &= \Pi_{\mu}^{\tau}(\mathbf{D}_{\tau}e_4)_{\sigma}\Pi_{\nu}^{\sigma}\end{aligned}$$

---

[47] It is important to remark that it is not true that $O = O_{ab}\mathrm{d}\omega^a \otimes \mathrm{d}\omega^b$. This can be easily recognized looking at (3.160). In fact

$$\theta_{\mu}^{a} = \mathbf{g}(e_a,\cdot) = g_{\mu c}e_a^c = \Big(\delta_{\mu}^{a}\gamma_{ac} + \delta_{\mu}^{u}(\gamma_{bc}X_{(\mathrm{Kerr})}^{b}) + \delta_{\mu}^{\underline{u}}(\gamma_{bc}X^{b})\Big)e_a^c.$$

and finally, with $\{x^\mu\} = \{u, \underline{u}, \theta, \phi\}$,

$$\chi = \chi_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \Pi_\mu^\rho \Pi_\nu^\sigma (\mathbf{D}_\rho e_4)_\sigma \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu. \tag{3.160}$$

The generic structure equations for the connection coefficients O are of two types: equations which are "transport equations" along the outgoing or incoming cones and "elliptic Hodge type" equations on the S surfaces. The transport equation has the following general structure, indicating with O a connection coefficient or a tangential derivative:

$$\not{\mathbf{D}}_4 O + k \mathrm{tr}\chi O = F, \tag{3.161}$$

where the integer k depends on the connection coefficient O we are considering, $\not{\mathbf{D}}_4$ is the projection on TS of the differential operator $\mathbf{D}_4 = \mathbf{D}_{e_4}$,

$$(\not{\mathbf{D}}_4 O)_{\mu\nu} = \Pi_\mu^\rho \Pi_\nu^\sigma (\mathbf{D}_4 O)_{\rho\sigma}, \tag{3.162}$$

F is a covariant "S-tangent" tensor whose components are quadratic or cubic functions of the (components of the) connection coefficients and possibly, if O is a tangential derivative of a connection coefficient, of the Riemann tensor,

$$F = F_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu, \quad F_{\mu\nu} = F_{\mu\nu}(\{O\}, \{R\}).$$

The Hodge equations for the connection coefficients or their tangential derivatives have the form[48]

$$\not{\mathrm{d}}\mathrm{iv} O = G + R \tag{3.163}$$

where $\not{\mathrm{d}}\mathrm{iv}$ is the divergence associated with $\not\nabla$, the covariant derivative relative to the induced metric, $\gamma^{(S)}$, on S,

$$\not{\mathrm{d}}\mathrm{iv} O = (\not{\mathrm{d}}\mathrm{iv} O)_\sigma \mathrm{d}x^\sigma, \quad (\not{\mathrm{d}}\mathrm{iv} O)_\sigma = {\gamma^{(S)}}^{\mu\nu} (\not\nabla_\mu O)_{\nu\sigma}, \tag{3.164}$$

G is a covariant tensor and R denotes a null component of the Riemann tensor tangent to S of the same degree. To have a specific example of structure equations with this structure, we consider in (5) the transport equation for $\not\nabla \mathrm{tr}\chi$ and the Hodge equation for $\hat{\chi}$ (see also equations (4.3.6) and (4.3.13) of (10)). To obtain the structure equations for δO, we define $O^{(\mathrm{Kerr})}$, the connection coefficient analogous to O but associated with the Kerr spacetime,

$$O^{(\mathrm{Kerr})} = O^{(\mathrm{Kerr})}_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu.$$

We would like to subtract the Kerr part and define

$$\delta O \equiv O - O^{(\mathrm{Kerr})}. \tag{3.165}$$

We need, nevertheless, a slight modification; in fact in the transport equations for the corrections, the δO terms have to be S-tangent to the S intersections of the outgoing and incoming cones of the foliation in the perturbed Kerr spacetime $(V_*, \mathbf{g})$. Nevertheless, the $O^{(\mathrm{Kerr})}$ tensor field is tangent to the S surfaces, intersections of

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[48] Applied to a two S-tangent covariant tensor field.

the outgoing and incoming cones of the foliation in V_* thought as a region of the Kerr spacetime. This means that, as in 3.158, we have

$$O^{(\mathrm{Kerr})}_{\mu\nu} = \Pi^{(\mathrm{Kerr})}{}^{\rho}{}_{\mu}\Pi^{(\mathrm{Kerr})}{}^{\sigma}{}_{\nu}H^{(\mathrm{Kerr})}_{\rho\sigma} \tag{3.166}$$

which is not S-tangent in $(V_*, \mathbf{g})$. Therefore, instead of (3.165) we define

$$\delta O \equiv O - \hat{O}. \tag{3.167}$$

where
$$\hat{O}_{\mu\nu} = \Pi^{\rho}_{\mu}\Pi^{\sigma}_{\nu}H^{(\mathrm{Kerr})}_{\rho\sigma}. \tag{3.168}$$

To obtain a transport equation for δO we apply to it $\not{\mathbf{D}}_4$ obtaining, recalling that

$$\begin{aligned}
&-\not{\mathbf{D}}^{(\mathrm{Kerr})}_4 O^{(\mathrm{Kerr})} - k\mathrm{tr}\chi^{(\mathrm{Kerr})}O^{(\mathrm{Kerr})} + F^{(\mathrm{Kerr})} = 0,\\
&\not{\mathbf{D}}_4\delta O + k\mathrm{tr}\chi\delta O = H(\delta O^{(0)}, \hat{O}, O^{(\mathrm{Kerr})}) + \delta F(\delta O, \hat{O}, O^{(\mathrm{Kerr})}, \delta R)
\end{aligned} \tag{3.169}$$

where

$$\begin{aligned}
&H(\delta O^{(0)}, \hat{O}, O^{(\mathrm{Kerr})})\Big[-(\not{\mathbf{D}}_4 - \not{\mathbf{D}}^{(\mathrm{Kerr})}_4)\hat{O} - \not{\mathbf{D}}^{(\mathrm{Kerr})}_4(\hat{O} - O^{(\mathrm{Kerr})})\\
&\quad -k\delta\mathrm{tr}\chi\hat{O} - k\mathrm{tr}\chi^{(\mathrm{Kerr})}(\hat{O} - O^{(\mathrm{Kerr})})\Big]\\
&\delta F(\delta O, O^{(\mathrm{Kerr})}, \delta R) = F - F^{(\mathrm{Kerr})}.
\end{aligned} \tag{3.170}$$

Now we proceed as in (10, Chapter 4), and we just sketch the proof. Let $|\delta O|$ be a $|\cdot|_{p,S}$ norm, applying Gronwall inequality and Lemma 4.1.5 of (10) we obtain the following estimate, with $\sigma > 0$:

$$\begin{aligned}
&||u|^{2+\delta}r^{(2-\sigma)-\frac{2}{p}}\delta O|_{p,S}(u,\underline{u}) \leq c_0\Big(||u|^{2+\delta}r^{(2-\sigma)-\frac{2}{p}}\delta O|_{p,S}(u,\underline{u}_*)\\
&\quad + \int_{\underline{u}}^{\underline{u}_*}\Big[||u|^{2+\delta}r^{(2-\sigma)-\frac{2}{p}}H|_{p,S} + ||u|^{2+\delta}r^{(2-\sigma)-\frac{2}{p}}\delta F|_{p,S}\Big](u,\underline{u}')\Big).
\end{aligned} \tag{3.171}$$

To control the right-hand side of (3.171) we have to estimate the norm

$$||u|^{2+\delta}r^{3-\frac{2}{p}}\delta O|_{p,S}(u,\underline{u}_*)$$

on the last slice and the norms of H and δF.[49] The norm on the last slice is discussed later on when we prove the existence of the "last slice canonical foliation". The bounds for the norms $||u|^{2+\delta}r^{3-\frac{2}{p}}H|_{p,S}$ and $||u|^{2+\delta}r^{3-\frac{2}{p}}\delta F|_{p,S}$ are proved in the following lemma:

Lemma 38 *Under the bootstrap assumptions, the following estimates hold in* V_*

$$||u|^{2+\delta}r^{3-\frac{2}{p}}H|_{p,S} \leq c\varepsilon_0\left(\frac{M}{R_0}\right)^2, \quad ||u|^{2+\delta}r^{3-\frac{2}{p}}\delta F|_{p,S} \leq c\varepsilon_0\left(\frac{M}{R_0}\right)^2.$$

---

[49] It i to obtain these estimates that we consider the transport equations starting from the last slice which, therefore, has to be assigned as "initial data", see Sect. 2.1.

Once Lemma 38 is proved, assuming an analogous estimate on the last slice,

$$||u|^{2+\delta}r^{2-\frac{2}{p}}\delta O|_{p,S}(u,\underline{u}_*) \leq c\varepsilon_0\left(\frac{M}{R_0}\right)^2 , \tag{3.172}$$

we obtain, integrating,

$$\begin{aligned} &|||u|^{2+\delta}r^{(2-\sigma)-\frac{2}{p}}\delta O|_{p,S}(u,\underline{u}) \\ &\leq c_0\left(||u|^{2+\delta}r^{(2-\sigma)-\frac{2}{p}}\delta O|_{p,S}(u,\underline{u}_*) + c\varepsilon_0\left(\frac{M}{R_0}\right)^2\int_{\underline{u}}^{\underline{u}_*}\frac{1}{r^{1+\sigma}}\right) \end{aligned} \tag{3.173}$$

and, as $|\underline{u}| \geq r$, choosing R_0 such that $c\left(\frac{M}{R_0}\right)^2 < \frac{1}{2}$, the result

$$||u|^{2+\delta}r^{2-\frac{2}{p}}\delta O|_{p,S}(u,\underline{u}) \leq c\varepsilon_0\left(\frac{M}{R_0}\right)^2 < \frac{\varepsilon_0}{2}, \tag{3.174}$$

proving that the norms $\delta\mathscr{O}$ satisfy in V_* better estimates than those in the "Bootstrap assumptions".

Proof of Lemma 38. We do not report it, see for details (5). Although complicated by the need of subtracting the Kerr part, the strategy we have discussed follows the one described in (10). There to complete the bootstrap mechanism, beside the transport equations, also the structure equations which are elliptic Hodge systems on S are used. Also for these equations the Kerr part has to be subtracted. Performing this subtraction there are no new ideas involved, different from those already described, therefore we do not discuss here their Kerr decoupling and the reader can find a detailed discussion in (5). Here we estimate in detail the norm of the tangential derivative of a specific connection coefficient (its correction), $\delta(\nabla\!\!\!\!/\,\mathrm{tr}\chi)$.

3.4.2 Detailed Estimate of the Norm $|r^{3-\frac{2}{p}}|u|^{2+\delta}\delta(\nabla\!\!\!\!/\,tr\chi)|$

We look at the correction $\delta\mathscr{O}$ associated to the S-tangent vector field

$$\Psi\!\!\!\!/ = \Omega^{-1}\left(\nabla\!\!\!\!/\,\mathrm{tr}\chi + \mathrm{tr}\chi\zeta\right)$$

and show in detail, for its transport equation, the structure we sketched in general. The transport equation $\Psi\!\!\!\!/$ satisfies is the following one:

$$\Omega \mathbf{D}\!\!\!\!/_4\Psi\!\!\!\!/ + \frac{3}{2}\Omega\mathrm{tr}\chi\Psi\!\!\!\!/ = F\!\!\!/. \tag{3.175}$$

Proceeding as discussed in general, see Sect. 3.4.1, we write $\Psi\!\!\!\!/$ in the following way:

$$\Psi\!\!\!\!/ = \hat{\Psi\!\!\!\!/} + \delta\Psi\!\!\!\!/ \tag{3.176}$$

where[50]

$$\hat{\slashed{\Psi}} = \frac{1}{\Omega^{(\mathrm{Kerr})}} \left(\slashed{\nabla} \mathrm{tr}\chi^{(\mathrm{Kerr})} + \mathrm{tr}\chi^{(\mathrm{Kerr})} \hat{\zeta} \right) \tag{3.177}$$

and

$$\delta\slashed{\Psi} = -\frac{\delta\Omega}{\Omega\,\Omega^{(\mathrm{Kerr})}} (\slashed{\nabla}\mathrm{tr}\chi + \mathrm{tr}\chi\zeta) + \frac{1}{\Omega^{(\mathrm{Kerr})}} \left(\slashed{\nabla}\delta\mathrm{tr}\chi + \delta\mathrm{tr}\chi\zeta + \mathrm{tr}\chi^{(\mathrm{Kerr})}\delta\zeta \right) \tag{3.178}$$

where

$$\delta\zeta = \zeta - \hat{\zeta}. \tag{3.179}$$

Analogously we write

$$\slashed{F} = \hat{\slashed{F}} + \delta\slashed{F}. \tag{3.180}$$

Therefore, Eq. (3.175) can be written as

$$\begin{aligned}
\Omega\slashed{\mathbf{D}}_4\delta\slashed{\Psi} + \frac{3}{2}\Omega\mathrm{tr}\chi\delta\slashed{\Psi} &= \delta\slashed{F} + (\hat{\slashed{F}} - \slashed{F}^{(\mathrm{Kerr})}) \\
&+ L - \left[\Omega\slashed{\mathbf{D}}_4(\hat{\slashed{\Psi}} - \slashed{\Psi}^{(\mathrm{Kerr})}) + \Omega(\delta\slashed{\mathbf{D}}_4)\slashed{\Psi}^{(\mathrm{Kerr})} + \delta\Omega\slashed{\mathbf{D}}_4^{(\mathrm{Kerr})}\slashed{\Psi}^{(\mathrm{Kerr})} \right] \\
&- \left[\frac{3}{2}\Omega\mathrm{tr}\chi(\hat{\slashed{\Psi}} - \slashed{\Psi}^{(\mathrm{Kerr})}) \right. \\
&\left. + \frac{3}{2}\Omega\delta\mathrm{tr}\chi\slashed{\Psi}^{(\mathrm{Kerr})} + \frac{3}{2}\delta\Omega\mathrm{tr}\chi^{(\mathrm{Kerr})}\slashed{\Psi}^{(\mathrm{Kerr})} + (\hat{\slashed{F}} - \slashed{F}^{(\mathrm{Kerr})}) \right] \\
&+ \left[-\Omega^{(\mathrm{Kerr})}\slashed{\mathbf{D}}_4^{(\mathrm{Kerr})}\slashed{\Psi}^{(\mathrm{Kerr})} - \frac{3}{2}\Omega^{(\mathrm{Kerr})}\mathrm{tr}\chi^{(\mathrm{Kerr})}\slashed{\Psi}^{(\mathrm{Kerr})} + \slashed{F}^{(\mathrm{Kerr})} \right]
\end{aligned} \tag{3.181}$$

where the last line is identically zero, being the structure equation satisfied in Kerr spacetime; therefore, we can rewrite the equation as

$$\Omega\slashed{\mathbf{D}}_4\delta\slashed{\Psi} + \frac{3}{2}\Omega\mathrm{tr}\chi\delta\slashed{\Psi} = \delta\slashed{F} + [(\delta\slashed{\Psi}), \hat{\slashed{\Psi}}, \slashed{\Psi}^{(\mathrm{Kerr})}] \tag{3.182}$$

where

$$\begin{aligned}
&[(\delta\slashed{\Psi}), \hat{\slashed{\Psi}}, \slashed{\Psi}^{(\mathrm{Kerr})}] \\
&\quad = -\left[\Omega(\delta\slashed{\mathbf{D}}_4)\hat{\slashed{\Psi}} + \Omega\slashed{\mathbf{D}}_4^{(\mathrm{Kerr})}(\hat{\slashed{\Psi}} - \slashed{\Psi}^{(\mathrm{Kerr})}) + +\delta\Omega\slashed{\mathbf{D}}_4^{(\mathrm{Kerr})}\slashed{\Psi}^{(\mathrm{Kerr})} \right] \\
&\qquad - \left[\frac{3}{2}\Omega\mathrm{tr}\chi(\hat{\slashed{\Psi}} - \slashed{\Psi}^{(\mathrm{Kerr})}) + \frac{3}{2}\Omega\delta\mathrm{tr}\chi\slashed{\Psi}^{(\mathrm{Kerr})} + \frac{3}{2}\delta\Omega\mathrm{tr}\chi^{(\mathrm{Kerr})}\slashed{\Psi}^{(\mathrm{Kerr})} \right]
\end{aligned}$$

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[50] In fact $\hat{\slashed{\Psi}}_\mu = \Pi_\mu^\nu \left(\Omega^{(\mathrm{Kerr})^{-1}} D_\nu \mathrm{tr}\chi^{(\mathrm{Kerr})} + \mathrm{tr}\chi^{(\mathrm{Kerr})} H_\nu^{(\mathrm{Kerr})} \right)$ and $H_\nu^{(\mathrm{Kerr})} = \left(g_{\mu\rho}^{(\mathrm{Kerr})} (D_\nu^{(\mathrm{Kerr})} e_4^{(\mathrm{Kerr})})^\mu e^{(\mathrm{Kerr})}{}_3^\rho \right)$.

Lemma 39 *Under all the previous assumptions the following inequality holds in* V_*

$$\begin{aligned}
\left| r^{4-\frac{2}{p}}|u|^{2+\delta}[(\delta\not\Psi),\hat{\not\Psi},\not\Psi^{(\mathrm{Kerr})}]\right|_{p,S} &\leq c\Big(|r^{4-\frac{2}{p}}|u|^{2+\delta}(\delta\not\mathbf{D}_4)\hat{\not\Psi}|_{p,S}\\
&\leq +|r^{4-\frac{2}{p}}|u|^{2+\delta}\Omega\not\mathbf{D}_4^{(\mathrm{Kerr})}(\hat{\not\Psi}-\not\Psi^{(\mathrm{Kerr})})|_{p,S}\\
&\quad+|r|u|^{2+\delta}\delta\Omega|_\infty|r^{4-\frac{2}{p}}\not\mathbf{D}_4^{(\mathrm{Kerr})}\not\Psi^{(\mathrm{Kerr})}|_{p,S}\\
&\quad+\frac{3}{2}|r\Omega tr\chi|_\infty|r^{3-\frac{2}{p}}|u|^{2+\delta}(\hat{\not\Psi}-\not\Psi^{(\mathrm{Kerr})})|_{p,S}\\
&\quad+|r^2|u|^{2+\delta}\delta tr\chi|_\infty|r^{3-\frac{2}{p}}\not\Psi^{(\mathrm{Kerr})}|_{p,S}\\
&\quad+|r|u|^{2+\delta}\delta\Omega|_\infty|rtr\chi^{(\mathrm{Kerr})}|_\infty|r^{3-\frac{2}{p}}\not\Psi^{(\mathrm{Kerr})}|_{p,S}\\
&\quad+|r^{4-\frac{2}{p}}|u|^{2+\delta}((\hat{\not F}-\not F^{(\mathrm{Kerr})}))|_{p,S}\Big)\\
&\leq c\varepsilon_0\frac{M^2}{R_0^2}. \qquad (3.183)
\end{aligned}$$

Proof. We do not report here the easy part of the estimates which simply follows from the bootstrap assumptions and the explicit expression of $\not\Psi^{(\mathrm{Kerr})}$. The bound of the norm $|r^{4-\frac{2}{p}}|u|^{2+\delta}(\delta\not\mathbf{D}_4)\hat{\not\Psi}|_{p,S}$ is done exactly as in the proof of Lemma 38 obtaining

$$|r^{4-\frac{2}{p}}|u|^{2+\delta}(\delta\not\mathbf{D}_4)\not\Psi^{(\mathrm{Kerr})}|_{p,S} \leq c\varepsilon_0\frac{M^2}{r^2}. \qquad (3.184)$$

The term $|r^{4-\frac{2}{p}}|u|^{2+\delta}\Omega\not\mathbf{D}_4^{(\mathrm{Kerr})}(\hat{\not\Psi}-\not\Psi^{(\mathrm{Kerr})})|_{p,S}$ requires the control of $|r^{3-\frac{2}{p}}|u|^{2+\delta}(\hat{\not\Psi}-\not\Psi^{(\mathrm{Kerr})})|_{p,S}$, the effect of $\not\mathbf{D}_4^{(\mathrm{Kerr})}$ is only that of adding a power of r in the decay. On the other side again as in the proof of Lemma 38, we have immediately

$$|r^{3-\frac{2}{p}}|u|^{2+\delta}(\hat{\not\Psi}-\not\Psi^{(\mathrm{Kerr})})|_{p,S} \leq c\varepsilon_0\frac{M^2}{R_0^2} \qquad (3.185)$$

so that finally we have

$$\left| r^{4-\frac{2}{p}}|u|^{2+\delta}[(\delta\not\Psi),\hat{\not\Psi},\not\Psi^{(\mathrm{Kerr})}]\right|_{p,S} \leq c\varepsilon_0\frac{M^2}{R_0^2}. \qquad (3.186)$$

The explicit expression of $\delta\not F+(\hat{\not F}-\not F^{(\mathrm{Kerr})})=\not F-\not F^{(\mathrm{Kerr})}$ is

$$\begin{aligned}
\delta\not F = &-\Omega\hat{\chi}\cdot\delta\not\Psi-\Omega\delta\hat{\chi}\not\Psi^{(\mathrm{Kerr})}-\delta\Omega\hat{\chi}^{(\mathrm{Kerr})}\not\Psi^{(\mathrm{Kerr})}\\
&-2\hat{\chi}\not\nabla\delta\hat{\chi}-2\hat{\chi}(\delta\not\nabla)\hat{\chi}^{(\mathrm{Kerr})}-2\delta\hat{\chi}\not\nabla^{(\mathrm{Kerr})}\hat{\chi}^{(\mathrm{Kerr})}\\
&-\eta\hat{\chi}\cdot\delta\hat{\chi}-\eta\delta\hat{\chi}\cdot\hat{\chi}^{(\mathrm{Kerr})}-\delta\eta|\hat{\chi}^{(\mathrm{Kerr})}|^2\\
&+\mathrm{tr}\chi\ \hat{\chi}\cdot\delta\underline{\eta}+\mathrm{tr}\chi\ \delta\hat{\chi}\cdot\underline{\eta}^{(\mathrm{Kerr})}+\delta\mathrm{tr}\chi\ \hat{\chi}^{(\mathrm{Kerr})}\cdot\underline{\eta}^{(\mathrm{Kerr})}\\
&-\mathrm{tr}\chi\delta\beta-\delta\mathrm{tr}\chi\beta^{(\mathrm{Kerr})}. \qquad (3.187)
\end{aligned}$$

The first term of $\delta \not{F}$ has the following estimate:

$$|r^{4-\frac{2}{p}}|u|^{2+\delta}\Omega\hat{\chi}\cdot\delta\not{\psi}|_{p,S} \le c\frac{M^2}{r^3}|r^{4-\frac{2}{p}}|u|^{2+\delta}\delta\not{\psi}| \tag{3.188}$$

The norms of all the other terms satisfy the following estimates, as proved in the Appendix:

$$|r^{4-\frac{2}{p}}|u|^{2+\delta}(\delta\not{F} - \Omega\hat{\chi}\cdot\delta\not{\psi})|_{p,S} \le c\varepsilon_0\left(\frac{M^2}{R_0^2}+\frac{M^4}{R_0^4}\right). \tag{3.189}$$

Again proceeding as we did in general for δO, applying Gronwall inequality and Lemma 4.1.5 of (10) we obtain the following estimate, with $\sigma > 0$:

$$\begin{aligned}||u|^{2+\delta}r^{(3-\sigma)-\frac{2}{p}}\delta\not{\psi}|_{p,S}(u,\underline{u}) &\le c_0\Big(||u|^{2+\delta}r^{(3-\sigma)-\frac{2}{p}}\delta\not{\psi}|_{p,S}(u,\underline{u}_*)\\ &+\int_{\underline{u}}^{\underline{u}_*}\Big[||u|^{2+\delta}r^{(3-\sigma)-\frac{2}{p}}[(\delta U)U^{(\mathrm{Kerr})}]|_{p,S}+||u|^{2+\delta}r^{(3-\sigma)-\frac{2}{p}}\delta\not{F}|_{p,S}\Big](u,\underline{u}')\Big).\end{aligned} \tag{3.190}$$

Integrating, provided $M/R_0 \ll 1$ and provided we have the appropriate last slice canonical estimates, we obtain

$$|\delta\not{\psi}| \le \frac{c\varepsilon}{r^3|u|^{2+\delta}} + c\frac{M}{R_0}\frac{c\varepsilon}{r^3|u|^{2+\delta}} \le \frac{\varepsilon_0}{2r^3|u|^{2+\delta}}. \tag{3.191}$$

if

$$c(\varepsilon + \frac{M}{R_0}\varepsilon_0) \le \frac{\varepsilon_0}{2}. \tag{3.192}$$

Remark **4** Observe that when starting from the transport equation for $\delta\not{\psi}$, Eq. (3.182), we write a transport equation for the $|\cdot|_{p,S}$ norms we use a Fermi transported frame. This Fermi transported frame is used only at this stage, while the "Kerr decoupling" is performed at the level of the tensorial equations.
Next step is to obtain from the knowledge of the bounds for the norms of $\not{\psi}$ the bounds for the norms of $\not{\nabla}\mathrm{tr}\chi$ and for $\hat{\chi}$.[51] $\hat{\chi}$ satisfies the following equation, see (10) Eq. (4.3.13),

$$\not{\mathrm{d}}\mathrm{iv}\hat{\chi} + \zeta\cdot\hat{\chi} - \frac{\Omega}{2}\not{\psi} + \beta = 0. \tag{3.193}$$

Therefore[52]:

$$\not{\mathrm{d}}\mathrm{iv}\delta\hat{\chi} = -(\delta\not{\mathrm{d}}\mathrm{iv})\widehat{(\hat{\chi})} - \zeta\cdot\delta\hat{\chi} - \delta\zeta\cdot\widehat{(\hat{\chi})} + \frac{\Omega}{2}\delta\not{\psi} + \frac{\delta\Omega}{2}\hat{\not{\psi}} - \delta\beta \tag{3.194}$$

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[51] The way we prove this result here is slightly different from what has been done in (10) (see in particular remark 1 at page 132).

[52] The small hat denotes the traceless part of χ the large hat is the one introduced in (3.168).

where $\delta\beta = \beta - \hat{\beta}$ and $\hat{\beta}$ is defined as the corresponding connection coefficients. As we have

$$(\delta \not{\mathrm{d}}\mathrm{iv})\widehat{(\hat{\chi})} = \delta\gamma^{\sigma\mu}\not{\nabla}_{\sigma}\widehat{(\hat{\chi})}_{\mu\nu} + \gamma^{(\mathrm{Kerr})\,\sigma\mu}\delta\not{\nabla}_{\sigma}\widehat{(\hat{\chi})}_{\mu\nu} \tag{3.195}$$

it follows that to use (3.194) to estimate $\not{\nabla}\delta\hat{\chi}$ we need to be able to estimate the right-hand side and for that we need to have only a norm estimate for the first derivatives of $\delta\gamma$ which are present in $\delta\not{\nabla}_{\sigma}$. From (3.194) we obtain immediately the following estimate:

$$\begin{aligned}
&||u|^{2+\delta}r^{3-\frac{2}{p}}\not{\nabla}\delta\hat{\chi}|_{p,S}\\
&\quad\leq c\Big(||u|^{2+\delta}r^{3-\frac{2}{p}}(\delta\not{\mathrm{d}}\mathrm{iv})\widehat{(\hat{\chi})}|_{p,S} + ||u|^{2+\delta}r^{3-\frac{2}{p}}\zeta\cdot\delta\hat{\chi}|_{p,S}\\
&\qquad+||u|^{2+\delta}r^{3-\frac{2}{p}}\delta\zeta\cdot\widehat{(\hat{\chi})}|_{p,S} + ||u|^{2+\delta}r^{3-\frac{2}{p}}\frac{\Omega}{2}\delta\not{\psi}|_{p,S} + ||u|^{2+\delta}r^{3-\frac{2}{p}}\frac{\delta\Omega}{2}\hat{\not{\psi}}|_{p,S}\\
&\qquad+||u|^{2+\delta}r^{3-\frac{2}{p}}\delta\beta|_{p,S}\Big)\\
&\quad\leq c\Big(||u|^{2+\delta}r^{3-\frac{2}{p}}(\delta\not{\mathrm{d}}\mathrm{iv})\widehat{(\hat{\chi})}|_{p,S} + ||u|^{2+\delta}r^{2-\frac{2}{p}}\delta\hat{\chi}|_{p,S}|r\zeta|_{\infty}\\
&\qquad+||u|^{2+\delta}r^{2-\frac{2}{p}}\delta\zeta|_{p,S}|r\widehat{(\hat{\chi})}|_{\infty} + ||u|^{2+\delta}r^{3-\frac{2}{p}}\delta\not{\psi}|_{p,S}|\Omega|_{\infty}\\
&\qquad+||u|^{2+\delta}r^{1-\frac{2}{p}}\delta\Omega|_{p,S}|r^{2}\hat{\not{\psi}}|_{\infty}\Big)\\
&\quad\leq c\left(\varepsilon_0\frac{M^2}{R_0^2} + \tilde{c}_8\varepsilon + c\varepsilon_0\frac{M}{R_0}\right) \leq \frac{\varepsilon_0}{2}
\end{aligned} \tag{3.196}$$

provided

$$c\varepsilon_0\left(\frac{M}{R_0} + \frac{M^2}{R_0^2}\right) + \tilde{c}_8\varepsilon \leq \frac{\varepsilon_0}{2}. \tag{3.197}$$

Remarks **2** (a) The estimates of the quantities with the hat are the same as the one for the Kerr terms. In fact their difference is a small correction as it has been proved in (5).

(b) It is here, to close the bootstrap that we need to require $\delta = \frac{\varepsilon}{2}$,[53] In fact looking at the inequality (3.196) it follows that the term depending on $\delta\beta$ is bounded only if $\delta \leq \frac{\varepsilon}{2}$. As, on the other side, to control the boundedness of the $\tilde{\mathcal{Q}}$ norms, we required that $\delta \geq \frac{\varepsilon}{2}$, (see the footnote after (3.90)); the conclusion is that, in the bootstrap assumptions, we must choose

$$\delta = \frac{\varepsilon}{2}. \tag{3.198}$$

---

[53] If, in the other instances where we required $\delta = \frac{\varepsilon}{2}$, we had avoided it the only price to be paid would have been the presence of dimensional constants like $R_0^{\delta-\frac{\varepsilon}{2}}$.

3.4.3 The Various Radial Coordinates

In this paper various quantities play the role of "radial coordinates"; let us compare them. In the Kerr metric written in the Pretorius Israel coordinates,

$$\begin{aligned}\mathbf{g}_{(\mathrm{Kerr})} &= -4\Omega^2_{(\mathrm{Kerr})}\mathrm{d}u\mathrm{d}\underline{u} + \gamma^{(\mathrm{Kerr})}_{ab}\left(\mathrm{d}\omega^a - X^a_{(\mathrm{Kerr})}(\mathrm{d}u+\mathrm{d}\underline{u})\right)\\ &\quad\times\left(\mathrm{d}\omega^b - X^b_{(\mathrm{Kerr})}(\mathrm{d}u+\mathrm{d}\underline{u})\right)\end{aligned}$$

and in the perturbed Kerr metric,

$$\begin{aligned}\mathbf{g}_{(pert.Kerr)} &= -4\Omega^2\mathrm{d}u\mathrm{d}\underline{u} + \gamma_{ab}\left(\mathrm{d}\omega^a - (X^a_{(\mathrm{Kerr})}\mathrm{d}u + X^a\mathrm{d}\underline{u})\right)\\ &\quad\times\left(\mathrm{d}\omega^b - (X^b_{(\mathrm{Kerr})}\mathrm{d}u + X^b\mathrm{d}\underline{u})\right),\end{aligned}$$

we define, respectively,[54]

$$r_*^{(\mathrm{Kerr})} = \underline{u} - u \quad \text{and} \quad r_* = \underline{u} - u. \tag{3.199}$$

Finally, a radial function $r(u,\underline{u})$ was defined in (2.42) proportional to the square root of the corresponding surface $S(u,\underline{u})$, both for the Kerr and for the perturbed Kerr case,

$$r = r(u,\underline{u}) = (\sqrt{4\pi})^{-\frac{1}{2}}|S(u,\underline{u})|^{\frac{1}{2}}.$$

We use all these radial functions in a interchangeable way; this is possible as we can control their norm differences. In fact it can be easily proved, see (5) for details, that the following inequalities hold:

$$|r_*^{(\mathrm{Kerr})} - r_*(u,\underline{u})| \le c\frac{\varepsilon_0}{r|u|^{1+\delta}} \tag{3.200}$$

$$r_*^{(\mathrm{Kerr})} \le r_{(\mathrm{Kerr})}\left(1 + c\frac{M}{R_0}\right), \tag{3.201}$$

$$|r(u,\underline{u}) - r_{(\mathrm{Kerr})}(u,\underline{u})| \le c\frac{\varepsilon_0}{r|u|^{1+\delta}}. \tag{3.202}$$

Therefore, as expected, if $\frac{M}{R_0} \ll 1$, we can identify $r(u,\underline{u}), r_{(\mathrm{Kerr})}(u,\underline{u})$, $r_*, r_*^{(\mathrm{Kerr})}, r_b$.

---

[54] The $\{r_*^{(\mathrm{Kerr})}, \theta_*\}$ Pretorius Israel coordinates (for Kerr spacetime) can be expressed in terms of the Boyer–Lindquist coordinates, $r_*^{(\mathrm{Kerr})} = r_*^{(\mathrm{Kerr})}(\theta, r_b)$, $\theta_* = \theta^*(\theta, r_b)$ where r_b is the Boyer–Lindquist radial coordinate, see (8).

3.4.4 The Estimates of the $\delta\mathcal{O}^{(0)}$ Norms

To obtain this result, we have to first find the transport equations for $\delta\Omega$, δX^a and $\delta\gamma_{ab}$. This is the content of the following lemma:

Lemma 310 *The corrections to the Kerr components of the metric $\delta\Omega$, δX^a and $\delta\gamma_{ab}$, satisfy the following equations:*

$$\partial_{\underline{N}}(\delta\Omega) = -2(\Omega^{(\mathrm{Kerr})}+\delta\Omega)^2\delta\underline{\omega}+\left(\frac{\delta\Omega(\Omega+\Omega^{(\mathrm{Kerr})})}{\Omega^{(\mathrm{Kerr})^2}}\right)\partial_{\underline{N}}\Omega^{(\mathrm{Kerr})} \tag{3.203}$$

$$\partial_{\underline{N}}\delta X^a-\Omega(\partial_c X^a_{(\mathrm{Kerr})})\delta X^c = -\Omega\left(\frac{2}{\Omega^{(\mathrm{Kerr})}}\frac{Q\Delta}{\Sigma R^2}\frac{\partial}{\partial r_b}X^a_{(\mathrm{Kerr})}\right)\delta\Omega+4\Omega^2\delta\zeta^a \tag{3.204}$$

$$\begin{aligned}\partial_{\underline{N}}(\delta\gamma_{ab})-\Omega tr\underline{\chi}(\delta\gamma_{ab}) &= -\left[\frac{\partial X^c_{(\mathrm{Kerr})}}{\partial\omega^a}(\delta\gamma_{cb})+\frac{\partial X^c_{(\mathrm{Kerr})}}{\partial\omega^b}(\delta\gamma_{ac})\right]\\ &\quad+\left[\Omega\gamma^{(\mathrm{Kerr})}_{ab}\delta tr\underline{\chi}+\delta\Omega\gamma^{(\mathrm{Kerr})}_{ab}tr\underline{\chi}^{(\mathrm{Kerr})}+2\Omega^{(\mathrm{Kerr})}\delta\hat{\underline{\chi}}_{ab}+2\delta\Omega\hat{\underline{\chi}}^{(\mathrm{Kerr})}_{ab}\right].\end{aligned} \tag{3.205}$$

Proof. **The equation for δX^a:** With the definition of the metric, (2.30), the commutation relation (2.24) has the following aspect:

$$\Omega\left(\partial_{e_4}X^a_{(\mathrm{Kerr})}-\partial_{e_3}X^a\right) = -4\Omega^2\zeta(e_c)e^a_c \tag{3.206}$$

which we can write as

$$\begin{aligned}\Omega\partial_{e_3}\delta X^a &= \Omega\left(\partial_{e_4}X^a_{(\mathrm{Kerr})}-\partial_{e_3}X^a_{(\mathrm{Kerr})}\right)+4\Omega^2\zeta(e_c)e^a_c\\ &= \Omega(\delta e^\mu_4-\delta e^\mu_3)\partial_\mu X^a_{(\mathrm{Kerr})}+4\Omega^2\delta\zeta^a\\ &\quad+\left[\Omega\left(\partial_{\hat{e}_4}X^a_{(\mathrm{Kerr})}-\partial_{\hat{e}_3}X^a_{(\mathrm{Kerr})}\right)+4\Omega^2\zeta^{(\mathrm{Kerr})^a}\right]\\ &= \Omega(\delta e^\mu_4-\delta e^\mu_3)\partial_\mu X^a_{(\mathrm{Kerr})}+4\Omega^2\delta\zeta^a\end{aligned} \tag{3.207}$$

where

$$\delta e_4 = -\frac{\delta\Omega}{\Omega^{(\mathrm{Kerr})}}e_4+\frac{\delta X}{\Omega};\quad \delta e_3 = -\frac{\delta\Omega}{\Omega^{(\mathrm{Kerr})}}e_3. \tag{3.208}$$

Therefore, Eq. (3.207) becomes

$$\begin{aligned}\partial_{e_3}\delta X^a &= (\partial_c X^a_{(\mathrm{Kerr})})\delta X^c\\ &\quad+\frac{\Omega}{\Omega^{(\mathrm{Kerr})}}\left(\partial_{e_3}X^a_{(\mathrm{Kerr})}-\partial_{e_4}X^a_{(\mathrm{Kerr})}\right)\delta\Omega+4\Omega^2\delta\zeta^a.\end{aligned} \tag{3.209}$$

As

$$\begin{aligned}\Omega\left(\partial_{e_3}X^a_{(\mathrm{Kerr})}-\partial_{e_4}X^a_{(\mathrm{Kerr})}\right) &= \left(\partial_u X^a_{(\mathrm{Kerr})}-\partial_{\underline{u}}X^a_{(\mathrm{Kerr})}\right) = -2\frac{\partial}{\partial r_*}X^a_{(\mathrm{Kerr})}\\ &= -2\frac{\partial r_b}{\partial r_*}\frac{\partial}{\partial r_b}X^a_{(\mathrm{Kerr})} = -2\frac{Q\Delta}{\Sigma R^2}\frac{\partial}{\partial r_b}X^a_{(\mathrm{Kerr})}\end{aligned} \tag{3.210}$$

the final expression is,[55]

$$\partial_{e_3}\delta X^a-(\partial_c X^a_{(\mathrm{Kerr})})\delta X^c=-\left(\frac{2}{\Omega^{(\mathrm{Kerr})}}\frac{Q\Delta}{\Sigma R^2}\frac{\partial}{\partial r_b}X^a_{(\mathrm{Kerr})}\right)\delta\Omega+4\Omega^2\delta\zeta^a. \tag{3.211}$$

The equation for $\delta\Omega$: We start from the definition of $\underline{\omega}$ in Eqs. (2.44) $\underline{\omega}=-\frac{1}{2}\mathbf{D}_3\log\Omega$. From it

$$\begin{aligned}-\frac{1}{2}\mathbf{D}_3\log\Omega &= -\frac{1}{2\Omega^2}\partial_{\underline{N}}(\Omega^{(\mathrm{Kerr})}+\delta\Omega)\\ &= -\frac{1}{2\Omega^{(\mathrm{Kerr})^2}}\partial_{\underline{N}}\Omega^{(\mathrm{Kerr})}\\ &\quad -\left(\frac{1}{2\Omega^2}-\frac{1}{2\Omega^{(\mathrm{Kerr})^2}}\right)\partial_{\underline{N}}\Omega^{(\mathrm{Kerr})}-\frac{1}{2\Omega^2}\partial_{\underline{N}}\delta\Omega\end{aligned}$$

and the final equation we have for $\delta\Omega$ is

$$\partial_{\underline{N}}\delta\Omega=-2(\Omega^{(\mathrm{Kerr})}+\delta\Omega)^2\delta\underline{\omega}+\left(\frac{\delta\Omega(\Omega+\Omega^{(\mathrm{Kerr})})}{\Omega^{(\mathrm{Kerr})^2}}\right)\partial_{\underline{N}}\Omega^{(\mathrm{Kerr})}. \tag{3.212}$$

The equation for $\delta\gamma_{ab}$: The definition of the induced metric on the generic $S(u,\underline{u})$, whose components we denote $\{\gamma^{(S)}_{\rho\sigma}\}$, is[56]

$$\Pi^\rho_\mu\Pi^\sigma_\nu g_{\rho\sigma}=\gamma^{(S)}_{\mu\nu} \tag{3.213}$$

where

$$\Pi^\rho_\mu=\delta^\rho_\mu-(\theta^3_\mu e^\rho_3+\theta^4_\mu e^\rho_4). \tag{3.214}$$

Observe that, as the spacetime is not static,[57] $\gamma^{(S)}$, the induced metric on the generic S, is not a 2×2 matrix (with the only γ_{ab} components different from zero);

---

[55] Observe that these equations refer to the metric components in the $\{u,\underline{u},\omega^1,\omega^2\}$ coordinates. We do not have here the previous problem of considering tensor fields tangent to S; therefore, we do not have to introduce the "auxiliary" $\hat{O}$ quantities.

[56] We use here small latin letters $a,b,\ldots$ to indicate the θ,ϕ coordinates; the frame vector fields tangential to S will be indicated with $e_A,e_B\ldots$.

[57] In fact this is not possible even in the Kerr spacetime.

Nevertheless the following holds[58]

$$\gamma^{(S)}_{ab} = \gamma_{ab} = \mathbf{g}\left(\frac{\partial}{\partial\omega^a}, \frac{\partial}{\partial\omega^b}\right), \quad \gamma^{(S)}(e_A, e_B) = \gamma^{(S)}_{ab} e^a_A e^b_B = \gamma_{ab} e^a_A e^b_B. \tag{3.215}$$

Therefore,

$$\begin{aligned}
\partial_{\underline{N}}\gamma_{ab} &= \mathbf{D}_{\underline{N}}\left(\mathbf{g}\left(\frac{\partial}{\partial\omega^a}, \frac{\partial}{\partial\omega^b}\right)\right) = \mathbf{g}\left(\mathbf{D}_{\underline{N}}\frac{\partial}{\partial\omega^a}, \frac{\partial}{\partial\omega^b}\right) + \mathbf{g}\left(\frac{\partial}{\partial\omega^a}, \mathbf{D}_{\underline{N}}\frac{\partial}{\partial\omega^b}\right)\\
&\quad \times \mathbf{g}\left([\underline{N}, \frac{\partial}{\partial\omega^a}], \frac{\partial}{\partial\omega^b}\right) + \mathbf{g}\left(\frac{\partial}{\partial\omega^a}, [\underline{N}, \frac{\partial}{\partial\omega^b}]\right) + \mathbf{g}\left(\mathbf{D}_{\frac{\partial}{\partial\omega^a}}\underline{N}, \frac{\partial}{\partial\omega^b}\right)\\
&\quad + \mathbf{g}\left(\frac{\partial}{\partial\omega^a}, \mathbf{D}_{\frac{\partial}{\partial\omega^b}}\underline{N}\right)\\
&= 2\Omega\chi_{ab} + \mathbf{g}\left([\underline{N}, \frac{\partial}{\partial\omega^a}], \frac{\partial}{\partial\omega^b}\right) + \mathbf{g}\left(\frac{\partial}{\partial\omega^a}, [\underline{N}, \frac{\partial}{\partial\omega^b}]\right).
\end{aligned} \tag{3.216}$$

We have

$$\left[\underline{N}, \frac{\partial}{\partial\omega^a}\right] = -\frac{\partial X^c_{(\mathrm{Kerr})}}{\partial\omega^a}\frac{\partial}{\partial\omega^c}\ ; \tag{3.217}$$

therefore,

$$\partial_{\underline{N}}\gamma_{ab} = 2\Omega\underline{\chi}_{ab} - \frac{\partial X^c_{(\mathrm{Kerr})}}{\partial\omega^a}\gamma_{cb} - \frac{\partial X^c_{(\mathrm{Kerr})}}{\partial\omega^b}\gamma_{ca}. \tag{3.218}$$

Analogously,

$$\partial_{\underline{N}}\gamma^{(\mathrm{Kerr})}_{ab} = 2\Omega^{(\mathrm{Kerr})}\underline{\chi}^{(\mathrm{Kerr})}_{ab} - \frac{\partial X^c_{(\mathrm{Kerr})}}{\partial\omega^a}\gamma^{(\mathrm{Kerr})}_{cb} - \frac{\partial X^c_{(\mathrm{Kerr})}}{\partial\omega^b}\gamma^{(\mathrm{Kerr})}_{ca} \tag{3.219}$$

and subtracting the result follows.

---

[58] To prove it one has to compute explicitly θ^3 and θ^4 and from them the projection components Π^ρ_μ, $\theta^3(\cdot) = -\frac{1}{2}\mathbf{g}(e_4, \cdot)$ therefore $\theta^3_\mu = -\frac{1}{2}g_{\mu\nu}e^\nu_4$

$$\begin{aligned}
\theta^3_u &= -\frac{1}{2}g_{u\underline{u}}e^{\underline{u}}_4 - \frac{1}{2}g_{uc}e^c_4 = -\frac{1}{2\Omega}(-2\Omega^2 + X\cdot X_{(\mathrm{Kerr})}) - \frac{1}{2\Omega}(-\gamma_{cd}X^d_{(\mathrm{Kerr})}X^c) = \Omega\\
\theta^3_{\underline{u}} &= -\frac{1}{2}g_{\underline{u}\underline{u}}e^{\underline{u}}_4 - \frac{1}{2}g_{\underline{u}c}e^c_4 = -\frac{1}{2\Omega}X\cdot X_{(\mathrm{Kerr})} - \frac{1}{2}(-\gamma_{cd}\hat{X}^d\frac{X^c}{\Omega}) = 0\\
\theta^3_a &= -\frac{1}{2}g_{a\underline{u}}e^{\underline{u}}_4 - \frac{1}{2}g_{ac}e^c_4 = -\frac{1}{2\Omega}(-\gamma_{ac}X^c) - \frac{1}{2\Omega}\gamma_{ac}X^c = 0.
\end{aligned}$$

Analogously $\theta^4_{\underline{u}} = \Omega,\ \theta^4_u = \theta^4_c = 0.$

Remark **5** The previous equations seem to imply a potentially dangerous loss of derivatives. In fact on the right-hand side of all the three equations there are terms of the order of first derivatives of the metric components, namely connection coefficients like $\omega, \zeta, \mathrm{tr}\chi$. Nevertheless, this loss of derivatives is not harmful as it does not propagate when we estimate the connection coefficients and their tangential derivatives up to the order we need. A detailed discussion of this delicate aspect is in (5).
Next lemma is the stronger version of Lemma 37 and its proof includes the proof of this lemma (see also Step IV in Sect. 3.2).
Lemma 311 *Assume that in V_* we have already proved that the norms $\delta\mathscr{O}$ satisfy better estimates than those of the bootstrap assumptions, namely*

$$\delta\mathscr{O} \leq \frac{\varepsilon_0}{N_0},$$

with N_0 a large integer number; then, assuming for the $\mathscr{O}^{(0)}$ norms appropriate initial data conditions, we prove that in V_* better estimates hold:

$$|r|u|^{2+\delta}\delta\Omega| \leq \frac{\varepsilon_0}{2};\ \ |r^2|u|^{2+\delta}\delta X| \leq \frac{\varepsilon_0}{2};\ \ ||u|^{1+\delta}\delta\gamma| \leq \frac{\varepsilon_0}{2}. \tag{3.220}$$

Proof. See the Appendix.

3.5 The Last Slice Canonical Foliation

To control $\slashed{\nabla}\delta\mathrm{tr}\chi$, we use transport equation 3.182, and this requires an estimate of $\slashed{\nabla}\delta\mathrm{tr}\chi$ on the last slice of V_* we denote $\underline{C}_*$; this as discussed in (10) and in (14) requires a delicate choice of its foliation. Let us recall the main problem we have to cure: the equation for $\mathrm{tr}\chi$ along an incoming cone is

$$\begin{aligned}&\mathbf{D}_3\mathrm{tr}\chi + \frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi + (\mathbf{D}_3\log\Omega)\mathrm{tr}\chi + \underline{\hat{\chi}}\cdot\hat{\chi} - 2|\zeta|^2 - 4\zeta\cdot\slashed{\nabla}\log\Omega - 2|\slashed{\nabla}\log\Omega|^2\\ &\quad = 2(\slashed{\Delta}\log\Omega + \slashed{\mathrm{div}}\zeta + \rho).\end{aligned} \tag{3.221}$$

Looking at this expression it is clear that there is a loss of derivatives due to the term in the right-hand side. To cure this problem in (10) we require that $\log\Omega$ satisfies the equation

$$\slashed{\Delta}\log\Omega = -\slashed{\mathrm{div}}\zeta - \rho + \overline{\rho}. \tag{3.222}$$

To satisfy it we introduce a background foliation whose leaves are the level surfaces of the affine parameter v, associated with the null geodesic generators of $\underline{C}_*$. Then we look for a new foliation $u_* = u_*(v)$, expressed relatively to the background one, such that, relatively to it, Ω satisfies the equations

$$\begin{aligned}\slashed{\Delta}\log\Omega &= -\slashed{\mathrm{div}}\zeta - \rho + \overline{\rho};\quad \overline{\log 2\Omega} = 0\\ \frac{\mathrm{d}u_*}{\mathrm{d}v} &= (2\Omega^2)^{-1};\quad u_*|_{S_*(0)} = \lambda_1\end{aligned} \tag{3.223}$$

where $S_*(0) = \underline{C}_* \cap \Sigma_0$. Once these conditions are satisfied, the evolution equation for $\mathrm{tr}\chi$ becomes

$$\begin{aligned}&\mathbf{D}_3\mathrm{tr}\chi + \frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi\\ &\quad+(\mathbf{D}_3\log\Omega)\mathrm{tr}\chi + \underline{\hat{\chi}}\cdot\hat{\chi} - 2|\zeta|^2 - 4\zeta\cdot\delta\slashed{\nabla}\log\Omega - 2|\slashed{\nabla}\log\Omega|^2 = 2\overline{\rho}\end{aligned}$$

and the loss of derivatives disappears when we apply $\slashed{\nabla}$. The proof of the existence of this foliation is in (14). It is then clear that the last slice transport equation for $\slashed{\nabla}\mathrm{tr}\chi$ has the following expression:

$$\begin{aligned}&\mathbf{D}_3\slashed{\nabla}\mathrm{tr}\chi + \frac{1}{2}\mathrm{tr}\underline{\chi}\slashed{\nabla}\mathrm{tr}\chi + (\mathbf{D}_3\log\Omega)\slashed{\nabla}\mathrm{tr}\chi + [\slashed{\nabla},\mathbf{D}_3]\mathrm{tr}\chi + \frac{1}{2}(\slashed{\nabla}\mathrm{tr}\underline{\chi})\mathrm{tr}\chi\\ &\quad+(\mathbf{D}_3\slashed{\nabla}\log\Omega)\mathrm{tr}\chi + [\slashed{\nabla},\mathbf{D}_3]\log\Omega\,\mathrm{tr}\chi + \slashed{\nabla}\underline{\hat{\chi}}\cdot\hat{\chi} + \underline{\hat{\chi}}\cdot\slashed{\nabla}\hat{\chi} - 4\zeta\cdot\slashed{\nabla}\zeta\\ &\quad-4\slashed{\nabla}\zeta\cdot\slashed{\nabla}\log\Omega - 4\eta\cdot(-\slashed{\mathrm{div}}\zeta - \rho + \overline{\rho}) = 0.\end{aligned}$$

In the present case the problem has to be worked in a slight different way; let us consider again the transport equation (3.221),

$$\begin{aligned}&\mathbf{D}_3\mathrm{tr}\chi + \frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi + (\mathbf{D}_3\log\Omega)\mathrm{tr}\chi + \underline{\hat{\chi}}\cdot\hat{\chi} - 2|\zeta|^2 - 4\zeta\cdot\slashed{\nabla}\log\Omega - 2|\slashed{\nabla}\log\Omega|^2\\ &\quad= 2(\slashed{\Delta}\log\Omega + \slashed{\mathrm{div}}\zeta + \rho).\end{aligned}$$

and subtract to it the "Kerr part" obtaining

$$\begin{aligned}&\mathbf{D}_3\delta\mathrm{tr}\chi + \frac{1}{2}\mathrm{tr}\underline{\chi}\delta\mathrm{tr}\chi + \frac{1}{2}\delta\mathrm{tr}\underline{\chi}\mathrm{tr}\chi^{(\mathrm{Kerr})} - 2\underline{\omega}\delta\mathrm{tr}\chi - 2\delta\underline{\omega}\mathrm{tr}\chi^{(\mathrm{Kerr})}\\ &\quad+\underline{\hat{\chi}}\cdot\delta\hat{\chi} + \delta\underline{\hat{\chi}}\cdot\hat{\chi}^{(\mathrm{Kerr})} - 2\zeta\cdot\delta\zeta - 2\delta\zeta\cdot\zeta^{(\mathrm{Kerr})} - 4\zeta\cdot\delta\slashed{\nabla}\log\Omega\\ &\quad-4\delta\zeta\cdot(\slashed{\nabla}\log\Omega)^{(\mathrm{Kerr})} - 4\slashed{\nabla}\log\Omega\cdot\delta\slashed{\nabla}\log\Omega - 4\delta\slashed{\nabla}\log\Omega\cdot(\delta\slashed{\nabla}\log\Omega)^{(\mathrm{Kerr})}\\ &\quad= 2(\slashed{\Delta}\delta\log\Omega + (\slashed{\Delta} - \slashed{\Delta}^{(\mathrm{Kerr})})(\log\Omega)^{(\mathrm{Kerr})}\\ &\quad+\slashed{\mathrm{div}}\delta\zeta + (\slashed{\mathrm{div}} - \slashed{\mathrm{div}}^{(\mathrm{Kerr})})\zeta^{(\mathrm{Kerr})} + \delta\rho).\end{aligned}$$

To avoid the loss of derivatives $\delta\log\Omega$ has to satisfy the following equation:

$$\slashed{\Delta}\delta\log\Omega + \slashed{\mathrm{div}}\delta\zeta + \delta\rho - \overline{\delta\rho} = 0, \tag{3.224}$$

and the important point is that the term

$$(\slashed{\Delta} - \slashed{\Delta}^{(\mathrm{Kerr})})(\log\Omega)^{(\mathrm{Kerr})} + (\slashed{\mathrm{div}} - \slashed{\mathrm{div}}^{(\mathrm{Kerr})})\zeta^{(\mathrm{Kerr})}$$

does not produce loss of derivatives as it contains second derivatives only of $(\log\Omega)^{(\mathrm{Kerr})}$ which is a given function.

Using the transport equations for the not underlined connection coefficients on the last slice, $\underline{C}_*$, allows to control their norms on the last slice in terms of the corresponding norms in the intersection $\underline{C}_* \cap \Sigma_0$ and prove they are bounded again by[59]

$$c\left(\varepsilon + \varepsilon_0\frac{M^2}{R_0^2}\right).$$

---

[59] The details of the "last slice problem", first discussed and solved for a different foliation in (6), are discussed in detail in (10), Chapter 7 and in (14).

3.6 V Step: The Initial Data

The global existence and the peeling is proved here assuming a strong regularity for the initial data. We collect in the next subsections all the initial data conditions we have used in the various proofs. Moreover, all these conditions can be expressed in terms of quantities relative only to the initial data hypersurface, namely the three-dimensional metric ${}^{(3)}\mathbf{g}$ and the second fundamental form ${}^{(3)}\mathbf{k}$, together with their covariant derivatives.[60] This will allow to express the initial data smallness conditions requiring that a L^2 integral on Σ_0/B_{R_0}, whose integrand depends only on $\delta{}^{(3)}\mathbf{k} = {}^{(3)}\mathbf{k} - {}^{(3)}\mathbf{k}^{(\mathrm{Kerr})}$ and on $\delta{}^{(3)}\mathrm{Ricci} = {}^{(3)}\mathrm{Ricci} - {}^{(3)}\mathrm{Ricci}^{(\mathrm{Kerr})}$ be sufficiently small.

3.6.1 The Asymptotic Conditions on the Initial Data Metric

From equation 2.30 it follows

$$
\begin{aligned}
&\mathbf{g}|_{\Sigma_0}(\cdot,\cdot)\\
&= \Omega^2 \mathrm{d}r_*^2 + \gamma_{ab}\left(\mathrm{d}\omega^a + \frac{1}{2}({X_{(\mathrm{Kerr})}}^a - X^a)\mathrm{d}r_*\right)\left(\mathrm{d}\omega^b + \frac{1}{2}({X_{(\mathrm{Kerr})}}^b - X^b)\mathrm{d}r_*\right)\\
&= \left(\Omega^2 + \frac{1}{4}\delta X^2\right)\mathrm{d}r_*^2 - \frac{1}{2}\gamma_{ab}\delta X^a \mathrm{d}r_* \mathrm{d}\omega^b + \gamma_{ab}\mathrm{d}\omega^a\mathrm{d}\omega^b\\
&= \left[\Omega^2_{(\mathrm{Kerr})}\mathrm{d}r_*^2 + \gamma^{(\mathrm{Kerr})}_{ab}\mathrm{d}\omega^a\mathrm{d}\omega^b\right]\\
&\quad + \left((2\Omega_{(\mathrm{Kerr})}\delta\Omega + \delta\Omega^2 + \frac{1}{4}\delta X^2)\mathrm{d}r_*^2 - \frac{1}{2}\gamma_{ab}\delta X^a \mathrm{d}r_*\mathrm{d}\omega^b + \delta\gamma_{ab}\mathrm{d}\omega^a\mathrm{d}\omega^b\right)
\end{aligned}
\tag{3.225}
$$

Therefore,

$$
\begin{aligned}
&(\mathbf{g}|_{\Sigma_0} - \mathbf{g}_{(\mathrm{Kerr})}|_{\Sigma_0})(\cdot,\cdot)\\
&= \left(2\Omega_{(\mathrm{Kerr})}\delta\Omega + \delta\Omega^2 + \frac{1}{4}\delta X^2\right)\mathrm{d}r_*^2 - \frac{1}{2}\gamma_{ab}\delta X^a\mathrm{d}r_*\mathrm{d}\omega^b + \delta\gamma_{ab}\mathrm{d}\omega^a\mathrm{d}\omega^b
\end{aligned}
$$

and the asymptotic conditions on the metric components are

$$
\delta\Omega = o_5\left(\frac{1}{r^{3+\gamma}}\right); \quad \delta X^a = o_5\left(\frac{1}{r^{4+\gamma}}\right); \quad \delta\gamma_{ab} = o_5\left(\frac{1}{r^{1+\gamma}}\right), \tag{3.226}
$$

where r is defined as in equation 2.42, and the S's are the two-dimensional surfaces associated with the canonical foliation of Σ_0 we are going to define. We call these conditions “Kerr asymptotic flatness”.

3.6.2 The Smallness Conditions on the Initial Data Metric

The smallness conditions for the metric components are

$$
\sup_{\Sigma_0/B_{R_0}} |r^{3+\delta}\delta\Omega| \le \varepsilon; \quad \sup_{\Sigma_0/B_{R_0}} |r^{4+\delta}\delta X| \le \varepsilon; \quad \sup_{\Sigma_0/B_{R_0}} |r^{1+\delta}\delta\gamma| \le \varepsilon. \tag{3.227}
$$

---

[60] Covariant with respect to the three-dimensional metric ${}^{(3)}\mathbf{g}$.

3.6.3 The Smallness Conditions on the Initial Data Connection Coefficients

As the estimates of the connection coefficients in V_* are obtained in terms of the initial data ones,[61] the initial data have to be such that the following estimates hold, with $l \leq 4$:

$$\begin{aligned}
&|r^{4+l+\delta}\not\nabla^l\delta\mathrm{tr}\chi| \leq \varepsilon;\ \ |r^{4+l+\delta}\not\nabla^l\delta\mathrm{tr}\underline{\chi}| \leq \varepsilon\\
&|r^{4+l+\delta}\not\nabla^l\delta\hat{\chi}| \leq \varepsilon;\ \ |r^{4+l+\delta}\not\nabla^l\delta\underline{\hat{\chi}}| \leq \varepsilon\\
&|r^{4+l+\delta}\not\nabla^l\delta\zeta| \leq \varepsilon\\
&|r^{4+l+\delta}\not\nabla^l\delta\omega| \leq \varepsilon,\ \ |r^{4+l+\delta}\not\nabla^l\delta\underline{\omega}| \leq \varepsilon.
\end{aligned} \tag{3.228}$$

3.6.4 The Smallness Conditions on the Initial Data Riemann Components

Our initial data have to guarantee that the norms of the connection coefficients and their tangential derivatives are small and bounded on Σ_0, but, more than that, the $\tilde{\mathcal{Q}}$ norms defined in terms of initial data have to be finite; this implies that they have to be such that

$$\tilde{\mathcal{Q}}_{\Sigma_0/B_{R_0}} \leq \varepsilon^2 \tag{3.229}$$

and also, on Σ_0/B_{R_0} the following condition must hold:

$$\sup_{\Sigma_0/B_{R_0}} |r^{6+\frac{\gamma}{2}}\overline{\rho(\tilde{R})}|^2 \leq \varepsilon^2. \tag{3.230}$$

3.6.5 The Canonical Foliation on Σ_0

A foliation of Σ_0/B_{R_0} in terms of two-dimensional surfaces $\{S_0\}$ is specified through a function $\underline{u}_0(p)$, the generic S_0 is defined as

$$S_0(\underline{u}_0 = \nu) = \{p \in \Sigma_0/B_{R_0} | \underline{u}_0(p) = \nu\}. \tag{3.231}$$

The solution of the eikonal equation $\underline{u}(p)$ with initial data $\underline{u}_0(p)$ on Σ_0/B_{R_0} is such that the level hypersurfaces $\underline{u}(p) = \nu$ defines the incoming cones of the double null foliation. As the norms of the "underlined" connection coefficients, see (10, Chapter 3) for all the detailed definitions, are estimated, using the transport equations along the incoming cones in terms of the same norms on the initial hypersurface, while the opposite happens for the not underlined coefficients estimated in terms of the norms on "Scri", an analogous, although mild, problem appears in this case. To avoid a loss of derivatives we have to choose an appropriate (canonical) foliation of Σ_0/B_{R_0}. We do not go here through the details as the way to obtain it is analogous to what was done in (10, Chapter 7). Moreover, differently from the last slice case, here in principle we could even avoid the choice of the canonical foliation admitting a loss of regularity going from the initial data to the solution, while this could not be allowed when we prove that V_* can be extended.

---

[61] Remark: see the discussion about the last canonical slice, that the way in which the connection coefficients norms depend on the corresponding ones on the initial data, is different from the underlined and the not underlined ones.

The previous discussion is not completely precise. In fact the intersections of the outgoing cones which are defined starting from “Scri” with the hypersurface Σ_0 are not the S_0 surfaces. This minor, although delicate, problem requires to define a spacelike hypersurface Σ_0' near to Σ_0 and control the various norms in the strip between these two hypersurfaces. This has been done in full detail in (10), subsection 4.1.3, and we do not repeat it here.

3.6.6 The Initial Data Condition in Terms of $\delta^{(3)}\mathbf{g}$ and $\delta^{(3)}\mathbf{k}$

All the initial data conditions can be reexpressed requiring that the metric stays near to the Kerr metric in a definite way. This is obtained requiring a condition similar to the “Strong asymptotic flat condition” defined in (6) and, moreover, that an L^2 integral, $\mathscr{J}$, over Σ_0/B_{R_0} (whose integrand is made by $\delta^{(3)}\mathbf{k}$, $\delta^{(3)}$Ricci and their derivatives) is bounded by ε. Its explicit expression is

$$\begin{aligned}\mathscr{J}(\delta^{(\mathbf{3})}\mathbf{g},\delta^{(\mathbf{3})}\mathbf{k}) = {} & R_0^{-\delta}\sup_{\Sigma_0/B_{R_0}}\left[|r^{3+\delta}\delta\Omega|+|r^{4+\delta}\delta X|+|r^{1+\delta}\delta\gamma|\right]\\ & +R_0^{-\delta}\left[\int_{\Sigma_0/B_{R_0}}\sum_{l=0}^{4}(1+d^2)^{(1+l)+\frac{5}{2}+\delta}|\nabla\!\!\!/^l\delta k|^2\right.\\ & \left.+\int_{\Sigma_0/B_{R_0}}\sum_{l=0}^{3}(1+d^2)^{(3+l)+\frac{5}{2}+\delta}|\nabla^l\delta B|^2\right]^{\frac{1}{2}}.\end{aligned} \tag{3.232}$$

To express the tensor quantities of the four-dimensional spacetime restricted to Σ_0 in terms of the three-dimensional quantities and their norm bounds in term of the corresponding three-dimensional ones, requires some work.This is a repetition, with some obvious modifications, of what has been done in (10), (mainly in Chapter 7) and in the original work (6), (mainly in Chapter 5) and we do not report it here; the interested reader can, nevertheless, look at the extended discussion in (5).

3.7 VI Step: The Extension of the Region V_*

As previously said once, we have proved that in the region V_* the estimates for the norms of the connection coefficients and for the Riemann tensor are better than those assumed in the “Bootstrap assumptions”; we have done the basic step to conclude that a region larger than V_* does exist where the previous quantity satisfies again the “Bootstrap assumptions”. To obtain this result, nevertheless, something more is needed. In fact the bootstrap argument requires two conditions to be satisfied. First, that a region with the assumed properties of the V_* region does exist, possibly a very small one and second it requires to prove that this region can be extended.

The first condition can be implemented by a local existence result in a small strip above Σ_0/B_{R_0}. In fact, of this solution we are interested only in the dependence region V associated with the annulus in Σ_0 with $r\in[R_0,R_0+\delta R]$, with δR

arbitrary small, and it is clear that starting from appropriate initial data we can satisfy in V all the "bootstrap assumptions".

The second condition to be satisfied requires to solve again an existence problem starting from the upper boundary of V_*. The strategy is not completely standard, but it has been already discussed in (10, Chapter 3, section 3.7.8) (see also (14)). Therefore, we sketch here only the more relevant points and the differences from the situation discussed there.

The region whose existence we have to prove is a "strip" above the last slice of the region V_* of arbitrary small width. Nevertheless, as there is no bound for the size of the region V_*, this existence problem is not a local problem in both directions; it is local in the "outgoing cones" direction, but not local in the "incoming cones" directions. Moreover this is a characteristic problem as its initial data are on the intersection of the portions of two cones, one outgoing starting at the intersection of the last slice of V_* with Σ_0, the other one being exactly the last slice. As the extension of the last slice cannot be controlled, to prove this result requires again a bootstrap argument. The difference is now that once we assume (again by a bootstrap assumption) that a portion of this strip does exist to prove that it can be extended, and that the whole strip does exist, requires to solve a characteristic local problem which is easy to manage. We do not give more details on it as this problem has already been treated as solved in two previous papers by the same authors, see (3; 4). Observe finally that the initial data in this case automatically satisfy the constraints for the characteristic problem as they are the restriction on these hypersurfaces of the Einstein solutions; the difference with respect to the discussion on (4) is that in this case the "initial data" are near to Kerr instead than to Minkowski, but this is not a problem.

Finally, to complete the result we have to prove that, again, in the extended region, $V_* + \delta V$, the estimates $\delta\mathscr{R} \leq \varepsilon_0, \delta\mathscr{O} \leq \varepsilon_0$ hold; this requires some care. In fact from the "last slice initial data" one would be tempted to infer that the $\underline{\tilde{\mathscr{Q}}}$ norms, to be finite avoiding possible logarithmic divergences, would require a weight $|u|^{\tilde{\gamma}}$ with $\tilde{\gamma} < \gamma$, the analog of the loss of decay from the initial data to the solutions. The problem is in fact not present, as once we have a solution in $V_* + \delta V$ and a new double null canonical foliation, we can, exactly as before, estimate the $\underline{\tilde{\mathscr{Q}}}$ on the outgoing and incoming cones in terms of the initial norms on Σ_0 and repeat all the previous lemmas to reobtain the correct decay for the $\delta\mathscr{R}$ norms. Once this is done, we repeat exactly the same procedure done in V_* for the connection coefficients so that, finally, we have proved that in $V_* + \delta V$ all the bootstrap assumptions are still valid and, by contradiction, that V_* has to coincide with the global (external) spacetime.

4 The Final Result

We can state now with all the details our final result:

Theorem 41 *Assume that initial data are given on Σ_0 such that, outside of a ball centred in the origin of radius R_0, they are different from the "Kerr initial data of a Kerr spacetime with mass M satisfying*

$$\frac{M}{R_0} \ll 1, \quad J \leq M^2$$

for some metric corrections decaying faster than r^{-3} *toward spacelike infinity together with its derivatives up to an order* $q \geq 4$, *namely*[62]

$$g_{ij} = g_{ij}^{(\mathrm{Kerr})} + o_{q+1}(r^{-(3+\frac{\gamma}{2})}), \quad k_{ij} = k_{ij}^{(\mathrm{Kerr})} + o_q(r^{-(4+\frac{\gamma}{2})}) \tag{4.233}$$

where $\gamma > 0$. *Let us assume that the metric correction* δg_{ij}, *the second fundamental form correction* δk_{ij} *are sufficiently small, namely the function* $\mathscr{J}$ *equation* (3.232) *made by* L^2 *norms on* Σ_0 *of these quantities is small,*[63]

$$\mathscr{J}(\Sigma_0, R_0; \delta^{(3)}\mathbf{g}, \delta^{(3)}\mathbf{k}) \leq \varepsilon, \tag{4.234}$$

then this initial data set has a unique development, $\widetilde{\mathscr{M}}$, *defined outside the domain of influence of* B_{R_0} *with the following properties*:

(i) $\widetilde{\mathscr{M}} = \widetilde{\mathscr{M}}^+ \cup \widetilde{\mathscr{M}}^-$ *where* $\widetilde{\mathscr{M}}^+$ *consists of the part of* $\widetilde{\mathscr{M}}$ *which is in the future of* Σ / B_{R_0}, $\widetilde{\mathscr{M}}^-$ *the one to the past.*
(ii) $(\widetilde{\mathscr{M}}^+, g)$ *can be foliated by a canonical double null foliation* $\{C(u), \underline{C}(\underline{u})\}$ *whose outgoing leaves* $C(u)$ *are complete*[64] *for all* $|\lambda| \geq |u_0| = R_0$. *The boundary of* B_{R_0} *can be chosen to be the intersection of* $C(u_0)$ *with* Σ_0.
(iii) *The various null components of the Riemann tensor relative to the null frame associated with the double null canonical foliation, decay along the outgoing "cones" in agreement with the "Peeling Theorem".*

Remark **6** It is clear that, from the way this result has been obtained, the condition $\frac{M}{R_0} \ll 1$ has to be such that the development we prove is far from the event horizon we assume to exist in a spacetime, near to the Kerr spacetime, which is the boundary of the complete outer region. In fact, trying to go near to the event horizon or even to the "photosphere region", see (1) and references therein, we would immediately find serious problems trying to control the $\tilde{\mathscr{Q}}$ norms in terms of the initial data ones.

5 Conclusions

As mentioned in the introduction the global existence proof is separated from the "peeling result". The global existence near Kerr spacetime required, in a broad sense, to subtract the Kerr part. This is done concerning the Riemann components looking for the estimates for the "time derivative" of the Riemann tensor, which eliminates the contribution of the Kerr spacetime, or more in general of any stationary spacetime. The subtraction for the connection coefficients is vice

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[62] The components of the metric tensor written in dimensional coordinates.

[63] This will also imply a slightly stronger condition on the decay of the metric and second fundamental form components, basically that $\int_{R_0}^{\infty} \mathrm{d}r r^{5+\gamma} |\delta g_{ij}|^2 < \infty$, $\int_{R_0}^{\infty} \mathrm{d}r r^{7+\gamma} |\delta k_{ij}|^2 < \infty$.

[64] By this we mean that the null geodesics generating $C(u)$ can be indefinitely extended toward the future.

versa made in a more general way as the subtracted part has not to be time-independent.[65] This is in some sense the more original part as, in a more external region, the peeling decay has been already proved in (15).

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6 Appendix

6.1 Proof of Various Equations, Inequalities and Lemmas

In the proofs of various Lemmas, for the metric components we can use directly the difference between a quantity and its Kerr counterpart, Therefore, for the metric components with $\hat{}$ we denote the Kerr part, $\hat{O}_{(0)} = O^{(\mathrm{Kerr})}_{(0)}$, while their meaning is different for the connection coefficients and for the null components of the Riemann tensor.

Proof of inequality (3.109). This result is just one of the standard estimates used in Chapter 5 of (10) to show how from the control of the $\mathcal{Q}$ norms one can obtain the control of the sup norms of the null Riemann components. The only difference is that here we are considering the $\tilde{R}$ Weyl field instead of the Riemann tensor R and that the $\mathcal{Q}$ norms are substituted by the $\tilde{\mathcal{Q}}$ norms. The detailed proof in this specific case is given in the Appendix of (5).

Proof of inequalities (3.123). Looking at the explicit expressions (3.122) and (3.125) it is immediate to realize that inequalities 3.125 are immediately derived by the bootstrap assumptions (2.48) while for the right-hand side terms in (3.122) we only need to control $\mathbf{D}_3\delta X, \mathbf{D}_4\delta X$ and $\nabla\!\!\!/\,\delta X$. The control of the first two terms arises from the explicit expression of $\mathbf{D}_3\delta X$ (see (3.211), Lemma 311). The control of $\nabla\!\!\!/\,\delta X$ requires to derive equation (3.211) and obtain from this equation an estimate for $\nabla\!\!\!/\,\delta X$ proceeding as in Lemma 311. Observe that the loss of derivatives due to the presence on the right end side of $\nabla\!\!\!/\,\zeta$ is only apparent

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[65] In principle one could use the same strategy used for the Riemann components, namely to define "time derivatives" of the connection coefficients, write for them the structure equations, estimate their norms and recover by a time integration the connection coefficients, this method, although less general, should give the same result we have obtained.

as in the estimate $\not\nabla\zeta$ there is not any loss of derivatives (see the discussion in subsection 3.4.5 of (5)).

Proof of Lemma 311. $\{\delta\Omega\}$**:** $\delta\Omega$ satisfies the equation

$$\partial_{\underline{N}}(\delta\Omega) = -2(\Omega)^2\delta\underline{\omega} + (\delta\Omega)\left(\frac{(\Omega+\Omega^{(\mathrm{Kerr})})}{\Omega^{(\mathrm{Kerr})^2}}\right)\partial_{\underline{N}}\Omega^{(\mathrm{Kerr})}. \tag{6.235}$$

It follows

$$\partial_{\underline{N}}|\delta\Omega| \le F|\delta\Omega| + 2\Omega^2|\delta\underline{\omega}| \tag{6.236}$$

where

$$F = \left|(\Omega+\Omega^{(\mathrm{Kerr})})\Omega^{(\mathrm{Kerr})^{-2}}\partial_{\underline{N}}\Omega^{(\mathrm{Kerr})}\right| = O\left(\frac{M}{r^2}\right). \tag{6.237}$$

and integrating on $\underline{C}(\underline{u};[u_0,u])$ we obtain

$$|\delta\Omega|(\underline{u},u) \le c\left(|\delta\Omega|_{\underline{C}(\underline{u})\cap\Sigma_0} + \int_{u_0}^{u} du'|2\Omega^2\delta\underline{\omega}|(\underline{u},u')\right). \tag{6.238}$$

As we have already proved that the estimates for $\delta\mathscr{O}^{(1)}$ are better than the bootstrap assumptions, then for $\delta\underline{\omega}$ the following estimate holds:

$$|r^2|u|^{2+\delta}\delta\underline{\omega}| \le \frac{\varepsilon_0}{N_0}$$

and we can obtain from the previous inequality the following one:

$$\begin{aligned}
|r^2|u|\delta\Omega|(\underline{u},u) &\le |r^{3+\delta}\delta\Omega|_{\underline{C}(\underline{u})\cap\Sigma_0} + \int_{u_0}^{u} du'|2\Omega^2\delta\underline{\omega}|(\underline{u},u')r(u',\underline{u})^2|u'| \\
&\le c\left(|r^3\delta\Omega|_{\underline{C}(\underline{u})\cap\Sigma_0} + \frac{\varepsilon_0}{N_0}\int_{u_0}^{u} du'\frac{1}{|u'|^{1+\delta}}\right) \\
&\le c\left(|r^3\delta\Omega|_{\underline{C}(\underline{u})\cap\Sigma_0} + \frac{\varepsilon_0}{N_0|u|^{\delta}}\right).
\end{aligned} \tag{6.239}$$

From it

$$|r^2|u|^{1+\delta}\delta\Omega|(\underline{u},u) \le c\left(|r^3\delta\Omega|_{\underline{C}(\underline{u})\cap\Sigma_0} + \frac{\varepsilon_0}{N_0}\right) \le \frac{\varepsilon_0}{N_1}, \tag{6.240}$$

with $2^{-1} > N_1^{-1} > N_0^{-1}$, choosing the initial data sufficiently small and N_0 sufficiently large.

$\{\delta X\}$**:** δX satisfies the following equation:

$$\partial_{e_3}\delta X^a - (\partial_c X^a_{(\mathrm{Kerr})})\delta X^c = -\left(\frac{2}{\Omega^{(\mathrm{Kerr})}}\frac{Q\Delta}{\Sigma R^2}\frac{\partial}{\partial r_b}X^a_{(\mathrm{Kerr})}\right)\delta\Omega + 4\Omega^2\delta\zeta^a \tag{6.241}$$

where we have immediately, with $\sigma > 0$,

$$|\partial_c X^a_{(\mathrm{Kerr})}| = |(\partial_c \omega_B)\delta^a_\phi| = \left|\left(\partial_c \frac{2Mar_b}{\Sigma R^2}\right)\right| \leq \frac{M^4}{R_0^{4-\sigma}} \frac{1}{r^{1+\sigma}} \tag{6.242}$$

and

$$\left|\left(\frac{2}{\Omega^{(\mathrm{Kerr})}} \frac{Q\Delta}{\Sigma R^2} \frac{\partial}{\partial r_b} X^a_{(\mathrm{Kerr})}\right)\right| \leq \frac{M^2}{R_0^2} \frac{1}{r^2}. \tag{6.243}$$

Finally, using the improved estimates for the $\delta\mathscr{O}^{(1)}$ norms we have

$$|\delta\zeta^a| \leq c \frac{\varepsilon_0}{N_0} \frac{1}{r^3|u|^{2+\delta}}. \tag{6.244}$$

Using these estimates from the definition

$$|\delta X| = \sqrt{\sum_a |\delta X^a|^2}$$

we write, defining

$$\begin{aligned} G &= -\left(\frac{2}{\Omega^{(\mathrm{Kerr})}} \frac{Q\Delta}{\Sigma R^2} \frac{\partial}{\partial r_b} X^a_{(\mathrm{Kerr})}\right), \qquad (6.245) \\ \partial_{\underline{N}}|\delta X|^2 &= 2|\delta X|\partial_{\underline{N}}|\delta X| = 2\Omega \sum_a \delta X^a \partial_{e_3} \delta X^a \\ &= 2\Omega \sum_a \delta X^a \left[(\partial_c X^a_{(\mathrm{Kerr})})\delta X^c + G\delta\Omega + 4\Omega^2 \delta\zeta^a\right] \\ &\leq 4\Omega \left(\sum_a |(\partial_c X^a_{(\mathrm{Kerr})})|\right) |\delta X|^2 + |G||\delta\Omega||\delta X| + 4\Omega^2 \left(\sum_a |\delta\zeta^a|\right) |\delta X| \end{aligned} \tag{6.246}$$

and immediately

$$\begin{aligned} \partial_{\underline{N}}|\delta X| &\leq 4\Omega \left(\sum_a |(\partial_c X^a_{(\mathrm{Kerr})})|\right) |\delta X| + |G||\delta\Omega| + 4\Omega^2 \left(\sum_a |\delta\zeta^a|\right) \\ &\leq c \frac{M^4}{R_0^{4-\sigma}} \frac{1}{r^{1+\sigma}} |\delta X| + \left[\frac{M^2}{R_0^2} \frac{\varepsilon_0}{r^4|u|^{1+\delta}} + \frac{\varepsilon_0/N_0}{r^3|u|^{2+\delta}}\right] \\ &\leq c \frac{M^4}{R_0^{4-\sigma}} \frac{1}{r^{1+\sigma}} |\delta X| + \frac{\varepsilon_0}{r^3|u|^{2+\delta}} \left(\frac{1}{N_0} + c\frac{M^2}{R_0^2}\right). \end{aligned} \tag{6.247}$$

Applying the Gronwall Lemma we obtain

$$\begin{aligned} |\delta X| &\leq \exp\left\{c \frac{M^4}{R_0^{4-\sigma}} \int_{u_0}^u \frac{1}{r^{1+\sigma}}\right\} \left[c\varepsilon_0 \left(\frac{1}{N_0} + \frac{M^2}{R_0^2}\right) \int_{u_0}^u \frac{1}{r^3|u'|^{2+\delta}} \mathrm{d}u'\right] \\ &\leq \exp\left\{c \frac{M^4}{R_0^4}\right\} c\varepsilon_0 \left(\frac{1}{N_0} + \frac{M^2}{R_0^2}\right) \frac{1}{r^3|u|^{1+\delta}} \leq \frac{\varepsilon_0}{2} \frac{1}{r^3|u|^{1+\delta}} \end{aligned} \tag{6.248}$$

which proves the result.

$\{\delta\gamma\}$**:** The Eq. (3.203) satisfied by $\delta\gamma_{ab}$,

$$\begin{aligned}\partial_{\underline{N}}(\delta\gamma_{ab}) - \Omega\mathrm{tr}\underline{\chi}(\delta\gamma_{ab}) = -\left[\frac{\partial X^c_{(\mathrm{Kerr})}}{\partial\omega^a}(\delta\gamma_{cb}) + \frac{\partial X^c_{(\mathrm{Kerr})}}{\partial\omega^b}(\delta\gamma_{ac})\right] \\ + \left[\Omega\gamma^{(\mathrm{Kerr})}_{ab}\delta\mathrm{tr}\underline{\chi} + \delta\Omega\gamma^{(\mathrm{Kerr})}_{ab}\mathrm{tr}\underline{\chi}^{(\mathrm{Kerr})} + 2\Omega^{(\mathrm{Kerr})}\delta\hat{\underline{\chi}}_{ab} + 2\delta\Omega\hat{\underline{\chi}}^{(\mathrm{Kerr})}_{ab}\right]\end{aligned}$$

can be written as

$$\begin{aligned}&\partial_{\underline{N}}(\delta\gamma_{ab}) - \overline{\Omega\mathrm{tr}\underline{\chi}}(\delta\gamma_{ab}) \\ &\quad = (\Omega\mathrm{tr}\underline{\chi} - \overline{\Omega\mathrm{tr}\underline{\chi}})(\delta\gamma_{ab}) + (G^c_a(\delta\gamma_{cb}) + G^c_b(\delta\gamma_{ca})) \\ &\qquad + F(O^{(\mathrm{Kerr})}, \delta\Omega, \delta\mathrm{tr}\underline{\chi}, \delta\underline{\chi})\end{aligned} \tag{6.249}$$

where

$$\begin{aligned}&F(O^{(\mathrm{Kerr})}, \delta\Omega, \delta\mathrm{tr}\underline{\chi}, \delta\underline{\chi}) \\ &\quad = \left[\Omega\gamma^{(\mathrm{Kerr})}_{ab}\delta\mathrm{tr}\underline{\chi} + \delta\Omega\gamma^{(\mathrm{Kerr})}_{ab}\mathrm{tr}\underline{\chi}^{(\mathrm{Kerr})} + 2\Omega^{(\mathrm{Kerr})}\delta\hat{\underline{\chi}}_{ab} + 2\delta\Omega\hat{\underline{\chi}}^{(\mathrm{Kerr})}_{ab}\right].\end{aligned} \tag{6.250}$$

As

$$\partial_{\underline{N}} r(u,\underline{u}) = \frac{\partial}{\partial u} r(u,\underline{u}) = \frac{r(u,\underline{u})}{2}\overline{\Omega\mathrm{tr}\underline{\chi}} \tag{6.251}$$

it follows

$$\begin{aligned}\partial_{\underline{N}}\frac{(\delta\gamma_{ab})}{r^2} &= \frac{1}{r^2}\partial_{\underline{N}}(\delta\gamma_{ab}) - \frac{2}{r^3}\frac{r}{2}\overline{\Omega\mathrm{tr}\underline{\chi}}(\delta\gamma_{ab}) \\ &= \frac{1}{r^2}\left[\partial_{\underline{N}}(\delta\gamma_{ab}) - \overline{\Omega\mathrm{tr}\underline{\chi}}(\delta\gamma_{ab})\right] \\ &= \frac{1}{r^2}\Big[(\Omega\mathrm{tr}\underline{\chi} - \overline{\Omega\mathrm{tr}\underline{\chi}})(\delta\gamma_{ab}) + (G^c_a(\delta\gamma_{cb}) + G^c_b(\delta\gamma_{ca})) \\ &\quad + F(O^{(\mathrm{Kerr})}, \delta\Omega, \delta\mathrm{tr}\underline{\chi}, \delta\underline{\chi})\Big]\end{aligned}$$

Therefore,

$$\begin{aligned}\partial_{\underline{N}}(r^{-2}\delta\gamma_{ab}) &= (\Omega\mathrm{tr}\underline{\chi} - \overline{\Omega\mathrm{tr}\underline{\chi}})(r^{-2}\delta\gamma_{ab}) \\ &\quad + \left(G^c_a(r^{-2}\delta\gamma_{cb}) + G^c_b(r^{-2}\delta\gamma_{ca})\right) + \frac{F}{r^2}\end{aligned} \tag{6.252}$$

Defining

$$|\delta\gamma| \equiv \sum_{ab}|\delta\gamma_{ab}|$$

we have the following inequality:

$$\begin{aligned}
\partial_{\underline{N}}\frac{|\delta\gamma|^2}{r^4} &= \frac{2|\delta\gamma|}{r^2}\partial_{\underline{N}}\frac{|\delta\gamma|}{r^2} = \frac{1}{r^4}\left[2|\delta\gamma|\partial_{\underline{N}}|\delta\gamma| - \frac{4}{r}\frac{r\overline{\Omega\mathrm{tr}\underline{\chi}}}{2}|\delta\gamma|\right]\\
&= \sum_{ab}(\delta\gamma_{ab})\left[\frac{1}{r^4}\left(-\overline{\Omega\mathrm{tr}\underline{\chi}}(\delta\gamma_{ab}) + \partial_{\underline{N}}(\delta\gamma_{ab})\right)\right]\\
&= \sum_{ab}\frac{(\delta\gamma_{ab})}{r^2}\left[(\Omega\mathrm{tr}\underline{\chi} - \overline{\Omega\mathrm{tr}\underline{\chi}})\frac{(\delta\gamma_{ab})}{r^2} + \left(G_a^c\frac{(\delta\gamma_{cb})}{r^2} + G_b^c\frac{(\delta\gamma_{ca})}{r^2}\right)\right.\\
&\quad\left. + \frac{F(O^{(\mathrm{Kerr})}, \delta\Omega, \delta\mathrm{tr}\underline{\chi}, \delta\underline{\chi})}{r^2}\right]\\
&\le \frac{|\delta\gamma|}{r^2}\left[|(\Omega\mathrm{tr}\underline{\chi} - \overline{\Omega\mathrm{tr}\underline{\chi}})|\frac{|\delta\gamma|}{r^2} + 2|G|\frac{|\delta\gamma|}{r^2} + \frac{|F|}{r^2}\right]
\end{aligned}$$

which implies

$$\partial_{\underline{N}}\left(\frac{|\delta\gamma|}{r^2}\right) \le \left[|(\Omega\mathrm{tr}\underline{\chi} - \overline{\Omega\mathrm{tr}\underline{\chi}}) + 2|G|\right]\left(\frac{|\delta\gamma|}{r^2}\right) + \frac{|F|}{r^2}. \tag{6.253}$$

As before we have for $|G|$ and $|(\Omega\mathrm{tr}\underline{\chi} - \overline{\Omega\mathrm{tr}\underline{\chi}})|$ the following estimates, with $\sigma > 0$

$$|G| = \sum_{ac}|G_c^a| \le c\frac{M^4}{R_0^{4-\sigma}}\frac{1}{r^{1+\sigma}}, \quad |(\Omega\mathrm{tr}\underline{\chi} - \overline{\Omega\mathrm{tr}\underline{\chi}})| \le \frac{c}{r^2}, \tag{6.254}$$

therefore, applying Gronwall Lemma,

$$\begin{aligned}
\left(\frac{|\delta\gamma|}{r^2}\right)(u,\underline{u}) &\le \left(\frac{|\delta\gamma|}{r^2}\right)(u_0,\underline{u}) + \exp\left\{c\frac{M^4}{R_0^{4-\sigma}}\int_{u_0}^{u}\frac{1}{r^{1+\sigma}}\right\}\int_{u_0}^{u}\frac{|F|}{r^2}(u',\underline{u})\mathrm{d}u'\\
&\le c\int_{u_0}^{u}\frac{|F|}{r^2}(u',\underline{u})\mathrm{d}u'
\end{aligned} \tag{6.255}$$

Observe now that the following inequality holds:

$$\begin{aligned}
\frac{|F|}{r^2} &\le \frac{1}{r^2}\left|\Omega\gamma_{ab}^{(\mathrm{Kerr})}\delta\mathrm{tr}\underline{\chi} + \delta\Omega\gamma_{ab}^{(\mathrm{Kerr})}\mathrm{tr}\underline{\chi}^{(\mathrm{Kerr})} + 2\Omega^{(\mathrm{Kerr})}\delta\hat{\underline{\chi}}_{ab} + 2\delta\Omega\hat{\underline{\chi}}_{ab}^{(\mathrm{Kerr})}\right|\\
&\le \frac{1}{r^2}\left[|\Omega|r^2|\delta\mathrm{tr}\underline{\chi}| + |\delta\Omega|r^2|\mathrm{tr}\underline{\chi}^{(\mathrm{Kerr})}| + 2|\Omega^{(\mathrm{Kerr})}||\delta\hat{\underline{\chi}}| + 2|\delta\Omega||\hat{\underline{\chi}}^{(\mathrm{Kerr})}|\right]\\
&\le c\frac{\varepsilon_0}{r^2|u|^{2+\delta}}.
\end{aligned} \tag{6.256}$$

Integrating the final result is

$$||u|^{1+\delta}\delta\gamma| \le c\varepsilon_0\left(\frac{1}{N_0} + \frac{1}{N_1}\right) \le \frac{\varepsilon_0}{2}, \tag{6.257}$$

proving the result.

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