EJTP **9, No. 26** (2012) 277–292 *Electronic Journal of Theoretical Physics*

Counting States in Spacetime

Lawrence B. Crowell*

Alpha Institute of Advanced Study, 11 Rutafa Street, H-1165 Budapest, Hungary
And
10600 Cibola Lp 311 NW Albuquerque, NM 87114 USA

Received 3 July 2011, Accepted 10 December 2011, Published 17 January 2012

Abstract: The recent derivation of an exact integer partition function (IPF) is related to a generating function for strings which compose the states of a black hole. Quantum gravity states have representations as modular functions for exception algebras. The SLOCC group for the accounting of these states is a quotient group on the Jordan matrix algebra. Quantum gravity is then described by a set of symmetries which act as quantum Golay codes. The emulation of quantum gravity in n-partite entanglements is then a technology for quantum computing and encryption.

Keywords: Stringy Black Holes; Integer Partition Function; Exceptional Groups; Jordan Matrix Algebra; Quantum Encryption
PACS (2010): 04.60.Cf ; 04.70.Dy; 03.67.Mn

1. The Quantum Coded Universe

This paper addresses quantum gravity as quantum encryption system. The microstates of a black hole are modeled by the integer partition function. The modular structure, or more explicitly the Dedekind η-function, suggests the most general symmetry for quantum cosmology and gravity is the Jordan matrix algebra.

The simulation of quantum black holes within this structure will lead to quantum computing and encryption-communication technology of a general variety. A black hole may be optically simulated, where entangled qubits may be integrated into a quantum optical computer. The first step will be with the optical simulation of black holes with 4-partite entanglement configurations with 8 states. A triality condition, or $\mathcal{O}^{\otimes 3}$, of three sets of these will then be a simulation of a black hole which obeys the proper integer partition of microstates.

* Email:lcrowell@swcp.com

2. Why Quantum Gravity?

Black holes are a theoretical laboratory for research into quantum gravity. Black holes are simpler to model than cosmologies and the models are comparatively straight forwards. They were also in the case of Hawking radiation the first general relativistic object to be considered in a quantum mechanical setting. Further, it is more plausible to build up a black hole from quantum states, or strings, than it is the entire universe. The black hole is defined by a set of simple parameters, mass, angular momentum and charge, which are macrostates that conceals a vast set of possible microstates. This fact was used to derive early thermodynamics and quantum mechanical properties of black holes.

The quantum mechanics of black holes could have had a much earlier start. At the 1930 Solvay conferences Niels Bohr and Albert Einstein debated the nature of quantum mechanics [1]. Einstein was convinced of reality and locality and argued staunchly for an incompleteness of quantum mechanics. Quantum theory could only be made complete if there are some hidden variables that underlay the probabilistic, nonlocal quirky aspects of quantum mechanics. Einstein proposed an interesting thought experiment. Einstein considered a device which consisted of a box with a door in one of its walls controlled by a clock. The box contains radiation, similar to a high-Q cavity in laser optics. The door opens for some brief period of time t, which is known to the experimenter. The loss of one photon with energy $E = \hbar\omega$ reduces the mass of the box-clock system by $m = E/c^2$, which is on a scale. Einstein argued that knowledge of t and the change in weight provides an arbitrarily accurate measurement of both energy and time which may violate the Heisenberg uncertainty principle $\Delta E\Delta t \simeq \hbar$ [1-2].

Bohr realized that the weight of the device is made by the displacement of a scale in spacetime. The clocks new position in the gravity field of the Earth, or any other mass, will change the clock rate by gravitational time dilation as measured from some distant point the experimenter is located. The temporal metric term for a spherical gravity field is $1 - 2GM/rc^2$, where a displacement by some δr means the change in the metric term is $\simeq (GM/c^2r^2)\delta r$. Hence the clocks time interval T is measured to change by a factor

$$T \rightarrow T\sqrt{(1 - 2GM\delta r/(rc)^2)} \simeq T(1 - GM\delta r/r^2c^2), \quad (1)$$

so the clock appears to tick slower. This changes the time span the clock keeps the door on the box open to release a photon. Assume that the uncertainty in the momentum is given by the $\Delta p \simeq \hbar/\Delta r < Tg\Delta m$, where $g = GM/r^2$. Similarly the uncertainty in time is found as $\Delta T = (Tg/c^2)\delta r$. From this $\Delta T > \hbar/\Delta mc^2$ is obtained and the Heisenberg uncertainty relation $\Delta T\Delta E > \hbar$. This demands a Fourier transformation between position and momentum, as well as time and energy.

This holds in some part to the quantum level with gravity, even if we do not fully understand quantum gravity. Consider the clock in Einsteins box as a black hole with mass m. The quantum periodicity of this black hole is given by some multiple of Planck masses. For a black hole of integer number n of Planck masses the time it takes a photon to travel across the event horizon is $t \sim Gm/c^3 = nT_p$, which are considered as the time intervals of the clock. The uncertainty in time the door to the box remains open is

$$\Delta T \simeq \frac{Tg}{c(\delta r - GM/c^2)}, \tag{2}$$

as measured by a distant observer. Similarly the change in the energy is given by $E_2/E_1 = \frac{1 - 2M/r_1}{1 - 2M/r_2} i$, which gives an energy uncertainty of

$$\Delta E \simeq \frac{\hbar g}{T_1 c^2(\delta r - GM/c^2)}. \tag{3}$$

Consequently the Heisenberg uncertainty principle still holds $\Delta E \Delta T \simeq \hbar$. Thus general relativity beyond the Newtonian limit preserves the Heisenberg uncertainty principle. It is interesting to note in the Newtonian limit this leads to a spread of frequencies $\Delta\omega \simeq \sqrt{c^5/G\hbar}$, which is the Planck frequency.

The uncertainty $\Delta E \simeq \hbar/\Delta t$ larger than the Planck mass gives an event horizon. The horizon has a radius $R \simeq 2G\Delta E/c^4$, which is the uncertainty in the radial position $R = \Delta r$ associated with the energy fluctuation. Putting this together with the Planckian uncertainty in the Einstein box we then have

$$\Delta r \Delta t \simeq \frac{2G\hbar}{c^4} = \ell^2_{Planck}/c. \tag{4}$$

So this argument can be pushed to understand the nature of noncommutative coordinates in quantum gravity.

By this simple argument a new type of quantization appears. The parallel translation of vectors in spacetime defines curvature, but where this can manifest itself in a quantized form. This could have been realized far earlier than when quantum gravity started to gain interest in the late 1960s. The uncertainty above defines a unit of area, such as a quantal unit of area on a black hole.

3. Black holes as Degenerate States and the Partition function

The entropy of a black hole is a measure of the number of microstates, where for N degenerate microstates the entropy is $S = k\, log(N)$, which is associated with gravity [3]. The entropy for large N is determined by the area of the event horizon $S = kA/4L_p^2$, where for the Schwarzschild black hole $A = 16\pi M^2$. The black hole is a system which holds a set of states with energy $E = M$ in a degeneracy $g(E) = exp(4\pi E^2)$ and the partition function is [4]

$$Z(\beta) = \sum_E e^{4\pi E^2} e^{-\beta E}. \tag{5}$$

This partition function is divergent for $E \rightarrow \infty$. The statistics for the number of degenerate microstates for a black hole is unbounded, and thus the partition function diverges. Black hole entropy is a coarse graining microstate states, which has been accomplished in string theory for large N. The horizon area is a summation of these quantum numbers

$$A = 16\pi\alpha \sum_{i=1}^{N} n_i, \tag{6}$$

for α_p a Planck area. The quantum numbers n_i determine an element of the horizon area. The energy is then counted as $E_n = \alpha E_p \sqrt{n}$, for $n = \sum_{i=1}^{N} n_i$[4]

The degeneracy for E_n is the number of ways $n > 0$ is a sum of N or less positive integers n_i. It is the cardinality of the set of elements $\{n_1, n_2, , \ldots, n_m\}$, such that $n = \sum_{i=1}^{m} n_i$ for $1 \leq m < N$. The number of ways a positive integer m may be written as a sum of m positive integers is the same problem as computing the number of ways of arranging n balls in m cells in a row. The result is a degeneracy for the energy E_n

$$g(E_n) = \sum_{m=1}^{N} \binom{n-1}{m-1}, \tag{7}$$

for $N \leq n$. We also have that $m \leq n$, which cuts the degeneracy further in

$$g'(E_n) = \sum_{m=1}^{n} \binom{n-1}{m-1}. \tag{8}$$

The partition function is a summation of the two degenerate sets,

$$Z(\beta) = \sum_{n=1}^{N}\sum_{m=1}^{n} \binom{n-1}{m-1} e^{-E_p\alpha\sqrt{n}} + \sum_{n=M+1}^{\infty}\sum_{m=1}^{N} \binom{n-1}{m-1} e^{-E_p\alpha\sqrt{n}}. \tag{9}$$

The two portions of the partition functions play a role at n small and $n >> N$, and may be computed independently [4]. The convergence occurs for $n >> N$ with

$$Z(\beta) \simeq \sum_{n=N+1}^{\infty} (n-1)^{N-1} e^{-\beta E_p \alpha \sqrt{n}}. \tag{10}$$

This is a convergent partition function. Conversely for a low black hole temperature $n << N$, the degeneracy from the binomial theorem is $g'(E_n) \simeq 2^{n-1}$ and the black hole entropy is $S = k\ ln(2^{n-1}) = (n-1)ln2$. The area $A = 16\pi\alpha^2 n$ permits us to set $\alpha = (1/2)\sqrt{ln2/\pi}$. This gives the entropy of the black hole $S = A/4$.

The degeneracy is extended to consider the energy level as due to a spin with an elementary energy $E_j = \sqrt{j(j+1)}$. The spin is $SU(2)$, which is a double covering symmetry of the spherical gravity field for a Schwarzschild black hole, with $j = \frac{1}{2}$. There are two states $\{\frac{1}{2}, -\frac{1}{2}\}$ and for m spins there are 2^m possible spin configurations. Hence the degeneracy for energy n is

$$g(E_n) = \sum_{m=1}^{N} \binom{n-1}{m-1} 2^m = 3^{n-1}. \tag{11}$$

The entropy of the black hole is now $S = (n-1)ln\ 3$ and $\alpha = (1/2)\sqrt{ln\ 3/\pi}$. This is then in agreement with the result by Hod [5] that quasi-normal modes are

$$\omega_n = \alpha^2 = \frac{ln\ 3}{4\pi} + 2\pi i(n + \frac{1}{2}) + \ldots. \tag{12}$$

This is equivalent to a redefinition $\alpha^2 \to \alpha^2 + ln(j(j+1))/4\pi$. In appropriate units the normal modes for a black hole of mass M are

$$\omega_n = \frac{ln\ 3}{8\pi M} + \frac{i}{4M}(n + \frac{1}{2}) + \ldots, \tag{13}$$

so the partition function of the black hole is

$$Z[\beta] \;=\; \sum_n e^{4\pi\omega_n} e^{-\beta\alpha\sqrt{n}}. \tag{14}$$

The string states in 26 dimensional is described by the Virasoro algebra. The generating function $Tr(q^N)$ is the density of string states for $q \;=\; e^{2\pi iz}$ and $N \;=\; \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n$ the string number operator. The trace $Tr(q^N)$ over 24 transverse string oscillations is

$$Tr(q^N) \;=\; \prod_{n=1}^{\infty} Tr(q^n) \;=\; \prod_{n=1}^{\infty}(1 \;-\; q^n)^{-24}, \tag{15}$$

where $\prod_{n=1}^{\infty}(1 \;-\; q^n)$ is the partition function[6]. q^n counts the number of way to write $n \;=\; a_1 \;+\; 2a_2 + \ldots\; ma_m\; \ldots$, for a_i a natural number. This partition function is related to the Dedekind η-function

$$\eta \;=\; q^{\tau/2}\prod_{n=1}^{\infty}(1 \;-\; q^{n\tau}), \tag{16}$$

for $Tr(\rho^N) \;=\; q^{24\tau}\eta^{-24}$. The generating function written as

$$Tr(q^N) \;=\; f(q) \;=\; exp\Big(\sum_{n=1}^{\infty} ln(1 \;-\; q^n)\Big) \;=\; \sum_{n=0}^{\infty} p(n)q^n,) \tag{17}$$

which is the $p(n)$ partition of the integers This function $p(n)$ is approximated with the Hardy-Ramanujan approximate formula for the partition of the integers. The quantum statistics of states on a black hole are similarly computed, and such generating functions should be modular. These may be Ramanujan mock Θ functions, as suggested by Dyson [7].

In both the case of black holes and the string generating function, the fundamental issue involves the partitioning of states as partitions of integers. The last equation for $Tr(q^N)$ contains the complex term $e^{2\pi inz}$, for $nz \;=\; 4\omega_n \;-\; ln\; 3/4\pi$. The remainder is an approximation for the integer partition function

$$p(n) \;\simeq\; 3e^{-\beta\alpha\sqrt{n}} \tag{18}$$

By using the Hardy-Ramanujan approximation for the integer partition the result is

$$p(n) \;\simeq\; \frac{1}{4n\sqrt{3}}e^{4\pi\sqrt{n}}, \tag{19}$$

which is remarkably similar formula to that obtained for the black hole.

This suggests the solution to the partition function is a partitioning of the integers, and the statistics are accounted for by an IPF. Recently a partition function has been derived as the finite sum and an algebraic number. This partition function $p(n)$ computes the partitions of an integer n, which is the sequence of integers:

$$1,\ 1,\ 2,\ 3,\ 5,\ 7,\ 11,\ 15,\ 22,\ 30,\ 42,\ 56,\ 77,\ 101,\ 135,\ 176,\ 231,$$
$$297,\ 385,\ 490,\ 627,\ 792,\ 1002,\ \ldots,\ p(100) = 190569292,\ \ldots \tag{20}$$

The general form for $p(n)$ has been a major problem in number theory. The circle method of Hardy and Ramanujan produced an asymptotic approximation function

$$p(n) \simeq \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{2n/3}}. \tag{21}$$

Bruinier, Folsom, Kent and Ono [8-9] derived a formula for $p(n)$ as a finite sum of algebraic numbers using the Dedekind η function $\eta = q^{1/24}\prod_{n=1}^{\infty}(1 - q^n)$ and the Eisenstein series

$$E_2(z) = 1 - 24\sum_{n=1}^{\infty}\sum_{d|n} dp^n. \tag{22}$$

A merimorphic form $F(z)$ with weight -2 by the discrete group $\Gamma_0(6)$ is defined according to Eisenstein series

$$\begin{aligned} F(z) &= \frac{1}{2}\frac{E_2(z) - 2E_2(2z) - 3E_2(3z) + 5E_2(6z)}{(\eta(z)\eta(2z)\eta(3z)\eta(6z))^2} \\ &= q^{-1} - 10 - 29q - 104q^3 - 273q^3 - \dots, \end{aligned} \tag{23}$$

which is eigenvalued on the upper half of the complex plane $\Delta E_k(q) = k(k - 1)E_k(q)$, and is a Maass form.

$$P(z) = -\Big(\frac{1}{2\pi i}\frac{d}{dz} + \frac{1}{2\pi y}\Big)F(z) \tag{24}$$

with eigenvalue -2 with respect to the hyperbolic Laplacian $\Delta = -y^2(\partial_x^2 + \partial_y^2)$.

The group $\Gamma_0(6)$ acts on these modular forms to construct a $\Gamma_0(6)$ equivalence classes $\mathcal{Q}_n$. The anti-holomorphic form $P(z)$ on this class at discrete point in the hyperbolic half-plane are summed over in a trace formula

$$tr(n) = \sum_{Q\in\mathcal{Q}_n} P(z_Q) \tag{25}$$

where $x_Q \in Q(x, y)$. This then gives a formula for the integer partition $p(n) = \frac{1}{2n+1}Tr(n)$

The integer partition function is constructed in the same manner partition functions are derived in statistical mechanics. The trace formula for the string states is a form of statistical mechanical device, and this mathematics gives a precise way in which these states may be partitioned to form an over all Boltzmann-like distribution, or e^{-iS} in a path integral.

4. Quantum Gravity as $\mathcal{O}^{\otimes 3}$

The application of the partition of integers to quantum states of gravity means they have representations as modular functions. The exceptional groups have Jacobi θ-function representations. The largest of the exceptional groups is the E_8 group, which does not have a complex representation, but subgroups do, such as $E_6 \times SU(3)$. Further, this may be overcome by extending the heterotic group into the Jordan matrix algebra.

The Jordan algebra is a 3×3 matrix comprised of three scalars and $\mathcal{O}_i$, for $i = 1, 2, 3$. The 16 supersymmetric partners are the two defined in two sets of spinor fields which are the superpartner to 8 vector fields which define three octionionic valued fields or E_8s. These exist in a Jordan exceptional algebra [10] of the 3×3 matrix with diagonal scalar elements and octonionic diagonal elements

$$J^3(\mathcal{O}) \;=\; \begin{pmatrix} x & \mathcal{O}_1 & \mathcal{O}_2^* \\ \mathcal{O}_1^* & y & \mathcal{O}_3 \\ \mathcal{O}_2 & \mathcal{O}_3^* & z \end{pmatrix}, \tag{26}$$

which has $27 \;=\; 24 + 3$ variables or dimensions. The F_4 group is the set of automorphisms of the $J^3(\mathcal{O})$, with the three polynomials in these 27 dimensions, These polynomials are the result of the trace of the Jordan matrix, the Jordan product $J^2 \;=\; J \circ J$, for $A \circ B \;=\; \frac{1}{2}(AB \;+\; BA)$, the Freudenthal product

$$J * J \;=\; J^2 \;\; tr(J)J \;+\; tr(J \cdot J)I, \sigma(J) \;=\; tr(J \cdot J),$$

and the determinant.$det(J) \;=\; (1/3)tr(J \cdot J) \circ J$. The polynomial invariants are

$$Tr(J) \;=\; x \;+\; y \;+\; z \tag{27}$$

$$s(J) \;=\; xy \;+\; yz \;+\; xz \;\; \mathcal{O}_1\mathcal{O}_1^* \;\; \mathcal{O}_2\mathcal{O}_2^* \;\; \mathcal{O}_3\mathcal{O}_3^* \tag{28}$$

$$det(J) \;=\; xyz \;+\; (\mathcal{O}_1\mathcal{O}_3)\mathcal{O}_2 \;+\; \mathcal{O}_2^*(\mathcal{O}_1^*\mathcal{O}_3^*)z|\mathcal{O}_1|^2 y|\mathcal{O}_2|^2 x|\mathcal{O}_x|^2.$$

Consider a theory with the Jacobi matrix algebra, with its θ-function realization. The 3 scalars of the 27 dimensional $J^3(\mathcal{O})$ on a light cone reduce to a two dimensional space $\sim\; R^2$ and the three octonions form ($\mathcal{O}_v,\; \mathcal{O}_s,\; \mathcal{O}_{s'}$ are vector plus spinor elements, related to each other in $N \;=\; 8$ supersymmetry, where the two spinor elements are conjugates. The relevant elements are the 24 elements of the Jordan matrix which form the transverse oscillators. The Leech lattice θ-function series is $\Theta_{\Lambda_{24}}(z) \;=\; \Theta_{E_8}(z)^3 \;-\; 720\Delta_{24}(z)$, where $\Delta_{24}(z) \;=\; \sum_{n=0}^{\infty} \tau(n)q^n$ and $\tau(n)$ are Ramanujan numbers [11]. $\Delta_{24} \;=\; q\prod_n(1 \;-\; q^n)^{24}$, for $q \;=\; e^{2\pi iz}$, which gives the Dedekind η-function $\eta(q) \;=\; (\Delta_{24})^{1/24}$.

A quotient which can set up this structure is the moduli space by defining the monad and octad elements of the $\mathcal{C}_{24}$, the C-set preserved by the Mathieu group M_{24}. $\mathcal{C}^{24}$ has weight distribution $0^1,\; 8^{759},\; 12^{2576}, 16^{759},\; 24^1$[11]. Let the 1 monad and 759 octads define $\mathcal{C}_{24}^{0,8}$. Now construct the moduli space

$$\mathcal{M}_{\Lambda_{24}} \;\sim\; \frac{\mathcal{C}_{24}^{0,8}}{E_8^{\otimes 3} \times \mathcal{O}P^2}, \tag{29}$$

for $\mathcal{O}P^2$ the 16 dimensional projective Cayley plane. The Mathieu group is a permutation group on $\mathcal{C}_{24}$, and this moduli restricts the action of the E_8 roots. This moduli space has $760 \;-\; (720 \;+\; 16) =\; 24$ dimensions. This 24 dimensional moduli space may be decomposed as

$$\mathcal{M}_{\Lambda_{24}} \;\rightarrow\; \frac{SO(24)}{SO(16) \times \mathbf{128}R^4}. \tag{30}$$

The $SO(24)$ is the multiplet of physical states for the open string. The gravitational massless state is the symmetric part of $\Omega^{\mu\nu} \;=\; \alpha_{-1}^{\mu}\tilde{\alpha}_{-1}^{\nu}|0\rangle$ in the closed string. The symmetric part of the state $(1/2)|\Omega^{\mu\nu}\rangle$ is a spin 2 field, which is the graviton. The antisymmetric portion is a second rank tensor. The reduction from R^8 to R^4 reflects the fact the space, here a Euclideanized 8 dimensional spacetime, is reduced to 4 dimension, as $\mathbf{28}/R^4$..

This may be further decomposed on the algebra level with

$$so(24) \ \rightarrow \ so(8)^{\oplus 3} \oplus (\mathbf{8},\ \mathbf{8},\ \mathbf{8}) \tag{31}$$

Further $\mathbf{128} \ \rightarrow \ (\mathbf{2},\ \mathbf{2},\ \mathbf{2},\ \mathbf{2},\ \mathbf{2},\ \mathbf{2},\ \mathbf{2})$. And $so(16) \ \rightarrow \ so(8) \ + \ so(8) + (\mathbf{8},\ \mathbf{8})$. This produces the reduced moduli space

$$\mathcal{M}_{\Lambda_{24}} \ \rightarrow \ \frac{SO(4,4)}{\mathbf{2},\ \mathbf{2},\ \mathbf{2},\ \mathbf{2}}. \tag{32}$$

With $SO(4,4) \ = \ [SL(2,\ \mathcal{R})]^4 \otimes (\mathbf{2},\ \mathbf{2},\ \mathbf{2},\ \mathbf{2},)$. The moduli space under this reduction is $SL(2,\mathcal{R})^4$, which corresponds to an extremal black hole.

The decomposition $E_8 \ \rightarrow \ SO(16) \times \mathbf{128}$ contains the spinors of $SO^*(16) \ \sim spin(16)$ which contain the 28 monopole charge plus the dual 28 magnetic monopole charges, or NUT charges, 2 mass scalars and 70 scalar of $\mathcal{N} \ = \ 8$ SUGRA. The complex spinor realization indicates the split form $E_{8(8)}$ is decomposed into $spin(16) \otimes \mathbf{128}$ and is equivalent to $SO(16, C)$, which is the corresponding group by the Konstant-Sekiguchi (KS) [12] which acts on $\mathbf{128}$. The $spin(16) \ \sim \ SO(16, C)$ is the corresponding n-partite entanglement SLOCC group for the maximal compact subgroup decomposition of the exceptional group. The Leech lattice restricted to action on monads and octads under maximal decomposition is $SL(2,\ \mathcal{C})^4$. The KS correspondence with Λ_{24} is constructed as $\mathcal{O}^3$ and the E_8 lattice has a complex construction by Eisenstein or Gaussian integers. The Λ_{12} with Eisenstein integers is a complex realization of the Leech lattice. The Barnes-Wall lattice Λ_{16} fixes a 16-dimesional lattice subspace in R^{24} of the Leech lattice. This is a complex realization of the E_8 lattice, where the 240 roots of E_8 have an 18-fold representation with Gaussian integers.

This is an introduction to SLOCC groups for the Jordan matrix algebra. It is likely there exists a range of possible quotient space structures based on octonion triples in the Jordan matrix algebra. This above is to illustrate the simplest one which conforms to the integer partition, the Dedekind η-function and the Δ^{24}. This appears to be a wide open field of investigation in mathematical physics.

5. F_4 Diagonalization of $J^3(\mathcal{O})$ and Quantum Foundations

The F_4 is the exceptional Lie group as an automorphism with respect to the commutative polynomials, or equivalently is the automorphism of the Jordan algebra. The minimal automorphism group of the E_8 is the G_2 group and F_4 and G_2 are centralizers in E_8. The relationship between the two groups is a triality condition. This triality condition is found within the F_4 as well. The determinant of the Jordan matrix determines a cubic Lagrangian. Which for the scalar elements gauge valued momenta $p_j \ = \ -i\partial_j \ + \ iA_j$

is this is a Chern-Simons Lagrangian. The cubic Lagrangian or Chern-Simons term is the $\underline{\omega}_3(\Gamma)$ which determines the renormalization group flow of the metric to the Einstein field equation for the AdS spacetime. The exceptional group F_4 is the maximal group of the 24-cell, which has a representation as B_4, or as a set of 16 rectified tetrahedral cells and one set of 8 octahedral cells. The other representation is according to D_4, which are three sets of 8 rectified tetrahedra. This representation is a homomorphism induced by the quotient group F_4/D_4, or F_4 splits over $D_4 \sim SO(8)$. The Dynkin diagram for F_4 is

$$\odot - - - \odot === o - - - o$$

and D_4 is

```
o   o
 \ /
  o
  |
  o
```

where the circle-dot represents short roots. The quotient group will then have to roots remaining as o—o which is the S^3 or $SU(3)$ group. Let the roots of the D_4 be represented by $(x_1,\ x_2,\ x_3,\ x_4)$, where x_4 is the central root at the hub. Under the action of $SU(3)$ with roots $(y,\ z)$ is then

$$(x_1,\ x_2,\ x_3,\ x_4)^y\ =\ (x_2,\ x_1,\ x_3,\ x_4),\ (x_1,\ x_2,\ x_3,\ x_4)^z\ =\ (x_1,\ x_3,\ x_2,\ x_4) \qquad (33)$$

where F_4 is then the split extension of D_3 by the S^3 or $F_4\ = D_4S^3$. This is a triality condition of D_4 in F_4, where there are three copies of $SO(8)$ determined by the automorphism of E_8.

The Jordan matrix algebra with the scalar light cone condition is a 26 dimensional Lorentzian spacetime, which in general has z = 1. Thus the degrees of freedom lost in the breaking of the restricted Lorentz symmetry in few dimensions are preserved. They are preserved in the more general cubic Lagrangian of the Jordan matrix algebra. The degrees of freedom are preserved in the nonlinear σ-model and the high energy theory at $z\ =\ 3$ plus the degrees of freedom lost in the breaking of the Lorentz symmetry are embedded in the F_4/B_4 model. The theory is then ultimately contained in the $J^3(\mathcal{O})$ group F_4 acts as an automorphism over.

The F_4 is a representation of the 24-cell according the Hurwitz quaternions. A recent paper Waegell and Aravind [13] illustrates a proof of the Kochen-Specker theorem with respect to the 24-cell. The Kochen-Specker theorem is smiilar to the Bell inequality violation theorem that is a no-go result for hidden variable theories [14]. The 24-cell is a mathematical structure that comes up in the context of systems of qubits. Asher Peres [15] gave a proof, which based on the symmetry of the root system of the exceptional Lie algebra F_4. The proof employs 48 vectors in 4-space which are isomorphic to the vertices of a 24-cell and its dual. These vectors are root vectors of F_4, which under multiplication by any set of scalars defines a set of lines in 4-space. We identify each of these vectors with a quantum state $|\psi_i\rangle$ $i\ =\ 1,\ \ldots,\ 24$, and a projection operator $P_i\ =\ |\psi_i\rangle\langle\psi_i|$. These have three eigenvalues 0 and one of 1. This means one can compute 72 sets of mutually orthogonal lines, where this is four-fold redundancy, and there are only 18 independent lines, which correspond to entangled pairs of 9 lines.

Suppose there were some hidden variable which accounts for this system. This would give an exact value to each of the 18 operators. The 9 must assume the value 1 in each of the 9 sets of pairs, an odd number, However, there is an even number of 1 with the pairs, and an underlying theory which determines the values of the 18 operators would require an even number also be odd. This is an informal proof of the Kochen-Specker theory in four dimensions.

F_4 is the isometry group of the projective plane over the octonions. The exceptional group G_2 is the automorphism on $\mathcal{O}$, or equivalently that $F_4 \times G_2$ defines a centralizer on E_8. The quotient between the 52 dimensional F_4 and the 36 dimensional $so(9) \sim B_4$ defines the short exact sequence

$$F_4/B_4 : 1 \rightarrow spin(9) \rightarrow F_{52\backslash 16} \rightarrow \mathcal{O}P^2 - - > 1, \tag{34}$$

where $F_{52\backslash 16}$ means F_4 restricted to 36 dimensions, which are the kernel of the map to the 16 dimensional Moufang or Cayley plane $\mathcal{O}P^2$. Geometrically the F_4 define the symmetry of the 24-cell according to Hurwitz quaternions. The B_4 defines a more restricted symmetry on the 24-cell according to 16 tetrahedral cells and 8 octahedral cells. The 16 tetrahedral cells are mapped to $\mathcal{O}P^2$ and the 8 octrahedral cells define $so(8) \in J^3(\mathcal{O})$. The D_4 representations is three sets of 8 tetrahedral cells and define the $so(8)S^3 \in J^3(\mathcal{O})$. On the algebraic level $f_4 \simeq so(8) \oplus V \oplus \theta_1 \oplus \theta_2$, which explicitly describes the triality condition the three octonions with the $so(8)$. More generally according to octonions $f_4 \sim so(\mathcal{O}) \oplus \mathcal{O}^3$, and f_4 diagonalizes the Jordan cubic matrix. The 36 sets of 4 mutually orthogonal rays is contained in $F_{52\backslash 16}$ above. The short exact sequence defines the $f_4 \sim so(9) \oplus S^9$ in the short exact sequence above. This means the K-S theorem is a consequence of qubit structure which has the Kostant-Sekiguchi isomorphism with black holes.

The 24-cell has the largest group representation F_4 in 52 dimensions, of which the $SO(9)$ in 36 dimension defines a short exact sequence between $spin(9)$ and the Moufang plane $\mathcal{O}P^2$. The $B_4 \sim SO(9)$ symmetry of the 24-cell by 16 tetrahedral and 8 octahedral cells. The elements of the exceptional Jordan matrix is composed of elements V_{ab} which are accompanies by 16 superpartners θ_{ab}, $\bar{\theta}_{ab}$, where the indices a and b indicate internal elements which transform these elements to $N \times N$ matrices in $SU(N)$. This obtains for a single D-brane, in particular here a $D0$-brane, where for $N > 1$ this gauge group is $SU(8)^N$, or the embedding group $SU(8N)$. The Lagrangian assumes the form [16]

$$L = (1/2)(tr(\partial_\mu V_i)^2 \ \ (1/2g)tr[V_i, V_j]^2 - \bar{\theta}_i\gamma_j[\theta^i, V^j]), \tag{35}$$

where integer the indices i, j denote the matrix indices.. Here the superpartners to the vectors V transform as spinors under the $SO(9)$ transverse rotations, and the matrices V_{ab}, θ_{ab} (vectors and spinors in $J^3(\mathcal{O})$) are components in a 10 dimensional super Yang-Mills space. This Lagrangian is applied as the $SO(9)$ theory in the $BFSS$ matrix theory [16].

The F_4 diagonalizes the Jordan algebra and is the automorphism of the Jordan product [17]. The Freudenthal product system

$$A{*}B \;=\; A{\circ}B - AoB - (1/2)(ATr(B) + BTr(A)) + \frac{1}{2}Tr(A)Tr(B) - Tr(A{\circ}B) \quad (36)$$

is preserved under F_4, as well as the determinant

$$det(A) \;=\; (1/3)[Tr(A * A)oA] \quad (37)$$

The determinant of the Jordan Matrix M is the cubic characteristic equation

$$det(M \;-\; \lambda I) \;=\; -\lambda^3 \;+\; (Tr(M)\lambda^2 \;-\; \sigma(M) \;+\; det(M)I \;=\; 0 \quad (38)$$

for $\sigma(M) \;=\; (1/2)(Tr(M)^2 \;-\; Tr(M^2))$ and with real eigenvalues λ. The matrix M as a form $\mathcal{O}^3$ may be written as a 24×24 matrix. The projector matrices for F_4 $P_i \;=\; |\psi_i\rangle\langle\psi_i|$ or as $v_i v_i^\dagger$ diagonalize M with

$$Mvv^\dagger \;=\; \lambda vv^\dagger,\; vv^\dagger M \;=\; \lambda vv^\dagger, \quad (39)$$

or $Mvv^\dagger \;+\; vv^\dagger M \;=\; 2\lambda vv^\dagger$. Multiplication on the right with v and $v^\dagger v \;=\; 1$ gives $Mv \;+\; vv^\dagger Mv \;=\; 2\lambda v$ and under the trace $Tr(vv^\dagger) \;=\; 1$ this gives $Mv \;=\; \lambda v$. The determinant of $M \;-\; \lambda I$

$$det(M \;-\; \lambda I) \;=\; (M \;-\; \lambda I)o((M \;-\; \lambda I) * (M \;-\; \lambda I)), \quad (40)$$

for $X * X \;=\; X^2 \;-\; (tr\; X)X \;+\; \sigma(X)I$, defines $(M \;-\; \lambda I) * (M \;-\; \lambda I)$ as the projector matrix which diagonalizes the $X \;=\; M \;-\; \lambda I$. Thus $X * X \;=\; vv^\dagger$ when normalized is a projector in the Cayley plane. Given two projectors with different eigenvalues λ and λ' have the determinant

$$P_\lambda \circ (M \circ P_{\lambda'}) \;=\; (P_\lambda \circ M) \circ P_{\lambda'}, \;\; \rightarrow \;\lambda' P_\lambda \circ P_{\lambda'} \lambda (P_\lambda \circ P_{\lambda'}, \quad (41)$$

and for the two eigenvalues $\lambda \;\neq\; \lambda'$. If we consider the two projectors nonzero this means they are orthogonal. If we assume three eigenvalues with $Tr(P_\lambda) \;=\; \sigma(M \;-\; \lambda I)$ the determinant formula above is derived. The matrix $M \;=\; \sum_{i=1}^3 \lambda_i P_{\lambda_i}$, and composed of projectors.

The Cayley plane is F_4/B_4, and the short exact sequence defines the projective octonionic plane. The quotient defines the 36 dimensional space which contains two copies of the 18 lines with eigenstates $\{0,\; 0,\; 0, 1\}$. This may be interpreted as a type of quantization. Further, the $\mathcal{O}P^2$ plane is a quotient of the E_6 group with the parabolic subgroup P_1. This is a Borel subgroup of matrices which defines the Heisenberg group.

6. Counting States and Quantum Error Correction Codes

The integer partition function is the accounting system for black hole microstates requires the quantum states transform by exception groups. These groups act as quantum error correction codes which conserve quantum information. The equivalency between quantum entanglement groups and the moduli space for black holes is a form of qubit conservation, for the semi-simple and nilpotent orbit configuration on the moduli space are Noetherian conservation laws. A complete partition function for black hole microstates extends this system to the exceptional group $J^3(\mathcal{O})$.

This quantum error correction code model of quantum gravity means that quantum information in the entire universe is conserved. This should then be applied to the universe as a whole. Spacetime cosmologies should be organized similarly with a correspondence between entangled quantum states and the symmetries of the cosmology. The cosmological horizon in a deSitter spacetime defines the entropy of the spacetime, and this may be defined by determinants or hyperdeterminants that correspond to entangled states. These states may be entangled with the bulk or with other cosmologies in the multi-verse.

This theoretical prospect for quantum gravity has some possible applications, though of course not directly with black holes. The Leech lattice is a representation of the Mathieu group M_{24}, which is the automorphism of a Golay code. The Steiner system of the octads $S(5,\ 8,\ 24)$ defines a binary Golay code in a vector space spanned by the octads. The connection to the Cayley plane or $\mathcal{O}P$ with F_4 indicates this could be run on a quantum computer. The simulation of this quantum gravity model would amount to a quantum Golay code encryption or error correction system. A computerized modeling of black holes with this exceptional group realization is an encryption system of states, which can be used in a real world quantum computerized system of communications.

The overlap of a state and its time development is

$$|\langle\psi|\psi\ +\ \delta\psi\rangle|^2\ =\ |\langle\psi|\psi\rangle|^2\ -\ (\langle\ H^2\ \rangle\ -\ \langle\ H\ \rangle)\delta t, \tag{42}$$

which gives the Berry phase of a state

$$\phi\ =\ \int dt\sqrt{\langle\ H^2\ \rangle\ -\ \langle\ H\ \rangle}, \tag{43}$$

where this is the Fubini-Study metric. This is the phase angle for a principal bundle $\pi : \mathcal{H}\ \rightarrow\ P\mathcal{H}$. This principal bundle may be Lie algebra valued. Photons under certain conditions may emulate quantum field in curved spacetime [18]. Consider the energy eigenvalues of the state space be $E_i\ =\ \hbar\omega$ which are functions of a one dimensional parameter r, which is a function of time $r\ =\ r(t)$. The spectrum becomes a continuum and frequencies a continuous function of this parameter. This dependency is a Doppler shift with frequency spectrum $\omega'\ =\ (1\ -\ nv/c)\omega$, where n is an index of refraction, $v\ =\ dr/dt$ a velocity and c the speed of light. The index of refraction along the one dimensional space is then assumed to vary according to $n\ =\ n_0\ +\ \delta n$. The Doppler equation defines a retarded time $1\ -\ nv/c =\ \omega/\omega'\ =\ \nu\tau$ for $\tau\ =\ t\ -\ r/v$. The effective frequency ν' is then

$$\nu'\ =\ \frac{v}{c}\frac{\partial n}{\partial\tau}\ =\ \frac{v^2}{c}\frac{\partial\delta n}{\partial\tau}. \tag{44}$$

The Hawking temperature is $kT = \hbar\alpha/2\pi$ for

$$\alpha = -\frac{1}{\delta n}\frac{\partial \delta n}{\partial \tau} \tag{45}$$

Now consider the photons in entangled states. In general the SLOCC states in an entanglement permit the teleportation of states [12]. Two states are SLOCC related by a teleportation if they can be inter-converted to each other in a reversible manner with some probability of success. This uses group theory, where the group G_{SLOCC} for this process is an N-partite system of qubits with some group $GL(2, \mathcal{C})$. The states further transform as a $(\mathbf{2},\ \mathbf{2},\ \ldots,\ \mathbf{2})$.

$$G_{SLOCC} = SL(2,\ \mathcal{C})_1 \otimes SL(2,\ \mathcal{C})_2 \otimes \ldots \otimes SL(2, (\mathcal{C})_N, \tag{46}$$

where the composite state

$$|\psi_{12\ldots N}\rangle = SL(2,\ \mathcal{C})_1 \otimes SL(2,\ \mathcal{C})_2 \otimes \ldots \otimes SL(2,\ \mathcal{C})_N|\phi_{12\ldots N}\rangle. \tag{47}$$

This is then an N-partite quantum information system where the entanglements are determined by the group element G_{SLOCC} and polynomials of this group. This is the moduli space for black holes composed of qubits and the U-duality group.

For a 2 Q-bit system this construction is apparent. You have a stat of the form $\sum_{ij} a_{ij}|i, j\rangle$ for i and j running form 0 to 1. The elements a_{ij} transform as $(\mathbf{2},\ \mathbf{2})$ of the G_{SLOCC}. The invariant element is the determinant of these matrices so $det(a_{ij})$ transformed under the G_{SLOCC} into

$$det(a_{ij}) \rightarrow det(a'_{ij}) = det(U_{i'i}a_{ij}U'^{*}_{j'j}) = det(a_{ij}) \tag{48}$$

with the obvious result on the determinant of a product that the transformation elements have unit determinant. The entanglement entropy is given by this measure so $S_{ij} = 4|det(a_{ij})|^2$. For multipartite systems the same rule generally applies, but the matrix interpretation is different. For an N-partite system the entanglement entropy is given by a $2 \times 2 \times \ldots \times 2$ (N times) set of elements. This leads to the entangled states $|00\rangle + |11\rangle$ and $|01\rangle + |10\rangle$, without normalization, for singlet and triplet entangled states. The isomorphism $SO(2,2) \simeq SL(2,\mathcal{C}) \times SL(2,\mathcal{C})$ connects the SLOCC state by the KS theorem to the local Lorentz symmetry induced optically.

This may be extended to 3 and 4 qubit systems, such as the W state or cat state, and the GHZ state. The algebraic structure is established by the entanglement employed. The exceptional E_8 group is decomposed as $E_8 \rightarrow SL(2,\ \mathcal{R})^{\otimes 8}$. The entanglement SLOCC group is the associated product $SL(2,\ \mathcal{C})^{\otimes 8}$. There are 8 qubits, where 7 of them are spin-valued elements on the Fano-plane and the remainder is a unit element. There are 7 triplets of elements which may be ordered, with the additional unit element, and 7 four-way multiplications which do not include the unit. Then with 8 qubits there are 14 independent basis vectors composed of 4 elements for a state built up accordingly. This may then be extended further into the Jordan matrix algebra.

This emulation of quantum gravity entanglement with various n-partite configurations will lead to the implementation of quantum Golay coding. At this time the largest entangled state is an $N = 5$ Noon state. Clearly progress is needed in the preparation of states. The emulation of quantum gravity states means that the optical black hole will process qubits in ways which are encrypted in a complicated manner. An eavesdropper who wishes to read a qubit stream needs to have access to a large data set, which can be deprived to him.

References

[1] A. D. Aczel, *Entanglement*, PLUME, Penguin Publishing, New York, NY (2003)

[2] N. Bohr, "Discussion With Einstein On the Epistemological Problems in Atomic Physics," *Albert Einstein: Philosopher-Scientist*, **1**, ed. P. A. Schilpp, Open Court Classics (1949)

[3] E. P. Verlinde, "On the Origin of Gravity and the Laws of Newton," http://arxiv.org/abs/1001.0785

[4] J. Mkel, "Notes Concerning "On the Origin of Gravity and the Laws of Newton" by E. Verlinde (arXiv:1001.0785)," http://arxiv.org/abs/1001.3808

[5] S. Hod, "Bohr's Correspondence Principle and The Area Spectrum of Quantum Black Holes," *Phys. Rev. Lett.* **81** 4293 (1998) http://arxiv.org/abs/gr-qc/9812002

[6] M. B. Green, J. H. Schwarz, E. Witten, *Superstring Theory*, p 117, Cambridge Univeristy Press (1987)

[7] F. Dyson, "Birds and Frogs", Notices of the AMS (February 2009) 212233 http://www.ams.org/notices/200902/rtx090200212p.pdf

[8] J. H. Bruinier, K. Ono, "An Algebraic Formula For the Partition Function," http://www.aimath.org/news/partition/brunier-ono.pdf

[9] A. Folsom, Z. A. Kent, K. Ono, "ℓ-adic Properties of the Partition Function," http://www.aimath.org/news/partition/folsom-kent-ono.pdf

[10] J. C. Baez, "The Octonions," *Bull. Amer. Math. Soc.* **39**, 145-205 (2002). Errata in *Bull Amer. Math. Soc.* **42**, 213 (2005)

[11] J. H Conway, N. J. A. Sloane, *Sphere Packing, Lattices, and Groups*, Springer (1991)

[12] L. Borsten, D. Dahanayake, M. J. Duff, A. Marrani, W. Rubens, "Four-qubit entanglement from string theory," *Phys.Rev.Lett.* **105**:100507 (2010) http://arxiv.org/abs/1005.4915

[13] M. Waegell, P.K.Aravind, "Parity proofs of the Kochen-Specker theorem based on the 24 rays of Peres," http://arxiv.org/abs/1103.6058

[14] S. Kochen and E.P. Specker, "The problem of hidden variables in quantum mechanics", *Journal of Mathematics and Mechanics* **17**, 5987 (1967).

[15] A. Peres: *J. Phys.* **A 24**, L175 (1991)

[16] T. Banks, W. Fischler, S.H. Shenker, L. Susskind, "M Theory As A Matrix Model: A Conjecture," *Phys.Rev.* **D55** 5112-5128 (1997)

[17] T. Dray, C. A. Manogue, "The Octonionic Eigenvalue Problem," *Adv. Appl. Clifford Algebras* **8**, 341-364 (1998)

[18] T. G. Philbin, C. Kuklewicz, S. Robertson, S. Hill, F. König, and U. Leonhardt, *Science* **319**, 1367 (2008) http://arxiv.org/abs/0711.4796