FERMIONS ON THE LATTICE*

A.A. Slavnov

Steklov Mathematical Institute, Russian Academy of Sciences,
Vavilov st.42, GSP-1,117966, Moscow, Russia

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Problems in fermion models on the lattice are discussed.A general procedure which allows to remove fermion doubling preserving gauge invariance in anomaly free chiral models on the lattice is presented. A representation of fermion determinant as a path integral of bosonic effective action is constructed.

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1. Introduction

In this talk I discuss some problems which arise in fermion models on a lattice. Introduction of the space-time lattice for field theories serves two main goals. It replaces a field system by a discrete one opening new possibilities for nonperturbative calculations. Secondly, a lattice introduces a natural cut-off providing ultraviolet regularization. However when dealing with fermion models one meets difficultues. Fermions are described by anticommuting variables which prevents using standard methods of nonperturbative calculations by computer simulations. Moreover, a naive discretization of the Dirac action leads to spectrum doubling and therefore it does not provide a regularization of the original model, but substitutes it by another one. More sophisticated discretizations [1–6] (for detailed references see [7–9]) avoid the doubling problem but when applied to chiral models violate some of the basic physical requirements [10].

In the first part of my talk I willl discuss the problem of spectrum doubling in chiral fermion models. In the second part a bosonized formulation of fermion models will be given, which allows to avoid the problem of computation of integrals over Grassmanian variables.

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2. Gauge invariant formulation of anomaly free chiral models on a lattice

The origin of difficulties with gauge invariant formulation of chiral fermion models on the lattice is a singular behaviour of the fermion propagators and vertices near the edge of the Brillouin zone $p_\mu \sim \frac{\pi}{a}$. In particular in the Wilson model [1] the contribution of this region leads to gauge noninvariant counterterms. In the SLAC model [2–4] the same region produces nonlocal counterterms and spurious infrared divergencies [11]. In other words a lattice is not a good regularization for chiral fermions and to get an invariant calculational scheme one needs to supress the contribution of momenta of the order of the cut-off. In fact one meets the same problem which has to be solved if one wishes to construct an invariant regularization for the continuum chiral gauge models.

Recently we proposed a manifestly gauge invariant regularization for continuum anomaly free chiral models [12] (see also [13]). Our procedure is a generalization of the Pauli–Villars (PV) regularization which allows for an infinite number of auxilliary PV fields. It has been shown that when applied to anomaly free chiral models on the lattice this procedure leads to a gauge invariant continuum theory without fermion doubling both in the case of Wilson fermions [14] and in the Smit-Swift model [15].However both these formulations have some drawbacks. It is a lack of the gauge invariance for a finite lattice spacing in the first case and the necessity to introduce an additional Yukawa interaction in the second one. It seems that the most apropriate way to implement this regularization on the lattice is to use the SLAC discretization. Below I shall present the idea of the generalized PV regularization and show that using SLAC discretization for anomaly free chiral gauge models together with the generalized PV regularization one gets a manifestly gauge invariant model for undoubled lattice fermions which requires only local gauge invariant counterterms. When applied to anomaly free chiral Schwinger model it reproduces in the continuum limit the well known exact solution.

I shall present the general idea using as an example the continuum grand unified SO(10) model. It is worthwhile to emphasize that the model is not vector like as we consider the fermions in complex representation. All the arguments are applicable to the Standard Model or Weinberg–Salam model, provided the anomalies of quark and lepton sectors are compensated. We choose here the SO(10) model because all the equations in this case can be written in a compact form.

The Lagrangian of the SO(10) model can be written as follows

$$L = -\tfrac{1}{4}(F_{\mu\nu}^{ij})^2 + i\bar{\psi}_+^k \gamma_\mu(\partial_\mu - igA_\mu^{ij}\sigma^{ij})\psi_+^k \,. \tag{1}$$

The matrices σ^{ij} are the SO(10) generators: $\sigma^{ij} = 1/2[\Gamma^i, \Gamma^j]$, where Γ^i are Hermitian matrices which satisfy the Clifford algebra: $[\Gamma^i, \Gamma^j]_+ = 2\delta^{ij}$. The chiral SO(10) spinors $\psi_\pm = 1/2(1 \pm \Gamma_{11})\psi$, where $\Gamma_{11} = \Gamma_1\Gamma_2 \ldots \Gamma_{10}$, describe the 16 dimensional irreducible representation of SO(10) including quark and lepton fields. We assume also that the spinors $\psi_\pm$ are lefthanded $\psi_\pm = 1/2(1 \pm \gamma_5)\psi_\pm$. Index k numerates different generations.

To regularize the model we add the analogous Lagrangian for the PV fields

$$\begin{aligned} L_R =& L + i\bar{\psi}_r\gamma_\mu(\partial_\mu - igA_\mu^{ij}\sigma^{ij})\psi_r \\ &- \frac{M_r}{2}\bar{\psi}_r C C_D \Gamma_{11}\bar{\psi}_r^T + (\psi_r \to \Phi_r) + \text{h.c.}. \end{aligned} \tag{2}$$

Here ψ_r are fermionic PV fields and $(\psi \to \Phi)$ denotes similar terms for bosonic PV fields Φ [12]. PV fields realize the reducible 32 dimensional representation of SO(10), C_D is the usual charge conjugation matrix and C is the SO(10) conjugation matrix $\sigma_{ij}^T C = -C\sigma_{ij}$.

The regularized Lagrangian (2) is obviously invariant with respect to SO(10) gauge transformations. However it includes the interaction of PV fields of both chiralities ψ_+^r, ψ_-^r, whereas the original Lagrangian (1) includes only ψ_+. Let us show that ψ_+ and ψ_- give identical contributions to the divergent diagrams. The spinorial loop with n external lines is proportional to $\mathrm{Tr}[(1 \pm \Gamma_{11})\sigma_{i_1 j_1} \ldots \sigma_{i_n j_n}]$. But

$$\mathrm{Tr}[\Gamma_{11}\sigma_{i_1 j_1} \ldots \sigma_{i_n j_n}] = 0, \quad \text{if} \quad n < 5\,. \tag{3}$$

Therefore, for divergent diagrams ($n \leq 4$) the difference between positive and negative chirality spinors disappear, and ψ_- spinors simply double the contribution of ψ_+ spinors.

Imposing the PV conditions

$$k + 2\sum_r(-1)^r c_r, \quad \sum_r c_r(-1)^r M_r^2 = 0\,, \tag{4}$$

where c_r are the numbers of the PV fields with the mass M_r, one can supress the leading asymptotics of the integrands in spinorial loops making the integrals convergent.

But if the number of generations is odd , for example $k = 1$, Eq. (4) cannot be satisfied for integer c_r. One can overcome this difficulty by allowing an infinite number of PV fields. Choosing $M_r = MR$ one gets for example the following expression for the polarization operator

$$\begin{aligned} \Pi_{\mu\nu} \sim \int d^4 l \sum_{r=-\infty}^{r=+\infty} (-1)^r \\ \times \frac{\mathrm{Tr}[(1+\gamma_5)\gamma_\mu(\hat{l} + M_r)\gamma_\nu(\hat{l - p} + M_r)]}{[l^2 + M_r^2][(l-p)^2 + M_r^2]}\,. \end{aligned} \tag{5}$$

Summation in the integrand can be done explicitely, giving for the leading Euclidean asymptotic the result

$$\sim \frac{\partial}{\partial R^2}\frac{\pi}{MR\sinh(\pi RM^{-1})},$$
$$R^2 = P_0^2(l) + P_1^2(l) + P_2^2(l) + P_3^2(l)\,. \qquad (6)$$

Analogous estimates can be given for next to the leading asymptotics and for other diagrams [12]. One sees that the integrands decrease exponentially providing the dezired supression of momenta of the order of the cut-off.

This procedure may be transferred in a straightforward way to lattice models. In particular for Wilson fermions one can use a standard gauge invariant discretization of the action (2) and add to it the Wilson mass terms for the original fields ψ_+ and the PV fields ψ_+^r, ψ_-^r. These terms break explicit gauge invariance, however due to supression of the momenta $\sim \frac{p_\mu}{a}$ in this model the gauge invariance is automatically restored in the continuum limit [14]. To preserve gauge invariance for finite lattice spacing one can introduce the Wilson mass terms in a gauge invariant way via Yukawa interaction with Higgs fields as in the Smit–Swift model [5, 6]. It was shown that in distinction with the original Smit-Swift model the PV fields make the effective Yukawa interaction weak and one can develop a manifestly gauge invariant perturbation expansion for this model [15]. As we have already mentioned generalized PV regularization is combined most naturally with the SLAC discretization procedure [16, 17]. In this case the regularized action looks as follows

$$\begin{aligned} I = &\sum_{k,\mu,x,y} \bar{\psi}_+^k(x)\gamma_\mu i D_\mu(x-y) P\exp\left\{iga\sum_{z_\mu=x_\mu}^{y_\mu} A_\mu(z)\right\}\psi_+^k(y) \\ &+ \sum_{r,\mu,x,y} \bar{\psi}^r(x)\gamma_\mu i D_\mu(x-y) P\exp\left\{iga\sum_{z_\mu=x_\mu}^{y_\mu} A_\mu(z)\right\}\psi^r(y) \\ &- \frac{M_r}{2}\bar{\psi}_r(x) C_D C \Gamma_{11}\bar{\psi}_r^T + \sum_{r,\mu,x,y}\bar{\Phi}^r(x)\gamma_\mu i D_\mu(x-y) \\ &\times P\exp\left\{iga\sum_{z_\mu=x_\mu}^{y_\mu} A_\mu(z)\right\}\Phi^r(y) - \frac{M_r}{2}\bar{\Phi}_r(x) C_D C\bar{\Phi}_r^T + \text{h.c}\,. \end{aligned} \qquad (7)$$

Here D_μ is the SLAC derivative

$$D_\mu(x) = \int_{\frac{-\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4} i k_\mu \exp\{ikx\}\,. \qquad (8)$$

Other notations are as above. This action is manifestly gauge invariant and does not produce fermion doubling, but the interaction is nonlocal. For example the three point vertex looks as follows

$$\Gamma^3 = g\gamma_\mu \frac{P_\mu(p) - P_\mu(q)}{K_\mu(k)}\sigma_{ij} , \tag{9}$$

with

$$K_\mu(k) = \frac{\exp[iak_\mu - 1]}{ia} . \tag{10}$$

Here P_μ is the sawtooth function

$$P_\mu(p) = p_\mu - 2m\frac{2\pi}{a}; \frac{\pi}{a} < p_\mu < (2m+1)\frac{\pi}{a} . \tag{11}$$

If $p < \frac{\pi}{a}$ but $q > \frac{\pi}{a}$ the interaction vertex becomes $\sim \frac{a^{-1}}{K_\mu(k)}$ which in the absence of PV fields results in the appearance of the nonlocal divergent terms [11]. When the PV fields are introduced the contribution of the region $p \sim a^{-1}$ is supressed and nonlocal divergencies do not appear. We shall illustrate it again by the vacuum polarization diagrams generated by the vertex (9). The typical terms look as follows

$$\begin{aligned} \Pi^{(\pm)r}_{\mu\nu} = &- \frac{g^2}{K_\mu K_\nu} \int_{\frac{-\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 l}{(2\pi)^4} \mathrm{Tr}[\sigma_{ij}(1 \pm \Gamma_{11})\sigma_{kl}] \\ &\times \mathrm{Tr}[(1+\gamma_5)\gamma_\mu(\hat{P}(l) + M_r)\gamma_\nu(\hat{P}(l+p) + M_r)] \\ &\times \frac{[P_\mu(l) - P_\mu(l+p)][P_\nu(l+p) - P_\nu(l)]}{[P^2(l) + M_r^2][P^2(l+p) + M_r^2]} . \end{aligned} \tag{12}$$

The contribution of the original fermion is given by $\Pi^{+(0)}_{\mu\nu}(M_r = 0)$. Contribution of the bosonic fields differs by sign. By the same reasons as above ψ_r^+ and ψ_r^- give the same contributions. As we have already seen the masses M_r and the coefficients c_r can be choosen in such a way to supress the asymptotics of the integrand for $p_\mu \sim \frac{\pi}{a}$. (In the case of odd number of generations an infinite number of PV fields is needed). Therefore, the integral over the region $\frac{\pi}{a} - |p| \leq |l| \leq \frac{\pi}{a}$ can be done vanishing in the limit $a \to 0$. In the remaining integral over $|l| < \frac{\pi}{a} - |p|$

$$P_\mu(l) - P_\mu(l+p) \sim p_\mu . \tag{13}$$

The nonlocal factors K_μ are compensated and in the continuum limit one gets a manifestly gauge invariant expression which requires only local counterterms. The diagrams with three and four lines are analyzed in the same way as a polarization operator. In the continuum limit they reduce to the usual gauge invariant expressions and no nonlocal counterterms appear.

These arguments can be generalized to arbitrary diagrams [16]. The analogous construction for the anomaly free chiral Shwinger model reproduces correctly the exact solution [16, 17].

3. Bosonization of fermion determinants

Now we discuss the problem of bosonized description of fermion models. The discusion will be restricted to the case of vectorial interaction. We shall not deal explicitely with the problem of spectrum doubling. All the reasonings are automatically transferred to the models improved for example by adding the Wilson term. The problem of bosonization of fermionic theories in dimensions $D > 2$ was studied by several authors (see *e.g.* [18–22], but no quite satisfactory solution was known.

Recently Lusher [23] proposed the algorithm for the approximate inversion of the QCD fermion determinant replacing it by an infinite series of bosonic determinants. In this talk I will describe an alternative approach which allows to write the exact expression for the fermion determinant as a path integral of the exponent of a local bosonic action [24]. In my approach a four dimensional fermionic system is replaced by a five dimensional constrained bosonic one.

Having in mind application to QCD I consider the interaction of fermions with Yang–Mills field. First of all we present the determinant of the Dirac operator as the determinant of the Hermitean operator by using the identity

$$\det(\hat{D}+m) = \det[\gamma_5(\hat{D}+m)]\,, \quad \hat{D} = \gamma_\mu D_\mu\,. \tag{14}$$

In Eq. (14) D_μ is the lattice covariant derivative

$$D_\mu \psi(x) = \tfrac{1}{2a}[U_\mu^+(x)\psi(x+a_\mu) - U_\mu(x)\psi(x-a_\mu]\,. \tag{15}$$

U_μ is a lattice gauge field. We consider a finite lattice with periodic boundary conditions. To provide the positivity of the determinant we consider the case of two degenerate fermion flavours interacting vectorialy with the Yang-Mills field.

It is convinient to present the fermion determinant in the following form

$$\det[\gamma_5(\hat{D}+m)]^2 = \int \exp\left\{a^4 \sum_x \bar{\psi}(x)(\hat{D}^2 - m^2)\psi(x)\right\} d\bar{\psi}d\psi\,. \tag{16}$$

We shall prove that the integral over fermionic fields ψ can be replaced by an integral of a five dimensional lattice bosonic action. The spatial components x are defined as above. The fifth component t to be defined on the one dimensional lattice of the length L with the lattice spacing b:

$$L = 2Nb, \quad -N < n \leq N\,. \tag{17}$$

We choose b in such a way that $b << a$ and in the continuum limit

$$2Nb^2 = Lb \to 0\,. \tag{18}$$

We are going to prove the following equality

$$\begin{aligned}
&\int \exp\left\{a^4 \sum_x \bar{\psi}(x)(\hat{D}^2 - m^2)\psi(x)\right\} d\bar{\psi} d\psi \\
&= \lim_{\lambda\to 0, b\to 0} \int \exp\left\{a^4 b \sum_{n=-N+1}^{N} \sum_x \left[\lambda \frac{\phi^*_{n+1}(x) - \phi^*_n(x)}{b}\phi_n(x)\right.\right. \\
&\left.\left. +\phi^*_n(x)(\hat{D}^2 - m^2)\phi_n(x) - \frac{i}{\sqrt{L}}(\phi^*_n(x)\chi(x) + \chi^*(x)\phi_n(x))\right]\right\} d\phi^*_n d\phi_n d\chi^* d\chi.
\end{aligned} \tag{19}$$

Here $\phi_n(x)$ and $\chi(x)$ are bosonic fields which carry the same spinorial and colour indices as the fields $\psi(x)$. The fields $\phi_n(x)$ satisfy free boundary conditions *i.e.*

$$\phi_n = 0, \quad n \leq -N, \quad n > N\,. \tag{20}$$

Note that the lattice derivative with respect to the fifth coordinate t is choosen in the form of a triangular matrix.As we shall see it provides the triviality of the determinant which arises after integration over fields ϕ. This is a crucial ingredient of our construction.

The operator $-\hat{D}^2 + m^2$ is Hermitean and can be diagonalized by a unitary transformation. As it does not depend on t this transformation will make the exponent in the r.h.s. of Eq. (19) diagonal with respect to all variables except for t. After this transformation the r.h.s. of Eq. (19) acquires the form

$$\begin{aligned}
I &= \lim_{\lambda\to 0, b\to 0} I(\lambda, b)\,, \\
I(\lambda, b) &= \int \exp\left\{b \sum_{n=-N+1}^{N} \sum_\alpha \left[\lambda \frac{\phi^{*\alpha}_{n+1} - \phi^{*\alpha}_n}{b}\phi^\alpha_n - \phi^{*\alpha}_n B^\alpha \phi^\alpha_n\right.\right. \\
&\left.\left. - \frac{i}{\sqrt{L}}(\phi^{*\alpha}_n \chi^\alpha + \chi^{*\alpha}_n \phi^\alpha_n)\right]\right\} d\phi^*_n d\phi_n d\chi^* d\chi; \quad \phi^{*\alpha}_{N+1} = 0\,.
\end{aligned} \tag{21}$$

Index α refers now to the eigenstates of the operator $-\hat{D}^2 + m^2$, B_α being the corresponding eigenvalues. Obviously $B_\alpha > 0$.

The integral (21) is convergent as the real part of the action in the exponent is positive. It is easy to see by rewriting the action in terms of Fourier components:

$$S = \frac{1}{2\pi} \int_{-\frac{\pi}{b}}^{\frac{\pi}{b}} \left\{ \tilde{\phi}^{*\alpha}(k)[-\lambda(e^{-ikb} - 1)b^{-1} + B^\alpha]\tilde{\phi}^\alpha(k) \right.$$
$$\left. + \frac{i}{2\pi\sqrt{L}}(e^{-iknb}\tilde{\phi}^{*\alpha}(k)\chi^\alpha + e^{iknb}\chi^{*\alpha}\tilde{\phi}^\alpha(k)) \right\} dk \,. \tag{22}$$

The real part is

$$\mathrm{Re}\, S = \frac{1}{2\pi} \int_{-\frac{\pi}{b}}^{\frac{\pi}{b}} \tilde{\phi}^{*\alpha}(k)[-\lambda(\cos kb - 1)b^{-1} + B^\alpha]\tilde{\phi}^\alpha(k) dk \,. \tag{23}$$

As $\lambda < 0$ and $B^\alpha > 0$, $ReS > 0$.

Performing in the Eq. (21) the integration over ϕ_n one gets

$$I(\lambda, b) = \det(C^{-1}) \int \exp\left\{ \frac{-b}{L} \sum_\alpha \sum_{n,m=-N+1}^{N} \chi^{*\alpha}(C^\alpha)^{-1}_{mn}\chi^\alpha \right\} d\chi^* d\chi \,, \tag{24}$$

where $C_\alpha(k)$ is the kernel of the quadratic form in the Eq. (21). In the coordinate space C_α is a triangular matrix with the diagonal elements

$$\sim (\lambda + B_\alpha b) \,. \tag{25}$$

Therefore

$$\det(C) = \exp\left\{ \sum_\alpha \ln(\lambda + B_\alpha b)^{2N} \right\} \,. \tag{26}$$

Separating the constant term one gets

$$\det(C) = \exp\left\{ 2Nb\lambda^{-1} \sum_\alpha B_\alpha + O(Lb) \right\}$$
$$= \exp\{\lambda^{-1} L \,\mathrm{Tr}[-\hat{D}^2 + m^2] + O(Lb)\} \,. \tag{27}$$

Using the explicit form of the operator D one can easily verify that $\mathrm{Tr}[-\hat{D}^2 + m^2]$ is a nonessential constant. It follows also from the fact that the trace of a local operator is local. It has to be gauge invariant and a polynomial of the second order in the fields A_μ. The only possible solution is a constant. So we can include $\det(C^{-1})$ into normalization constant.

To get an explicit form of the remaining terms it is sufficient to find the stationary point of the exponent in the Eq. (21). The corresponding classical equations look as follows

$$\begin{aligned} -\lambda(\phi^{*\alpha}_{n+1} - \phi^{*\alpha}_n)b^{-1} + B^\alpha\phi^{*\alpha}_n + iL^{-\frac{1}{2}}\chi^{*\alpha} &= 0; \quad n \neq N\,, \\ \lambda(\phi^\alpha_n - \phi^\alpha_{n-1})b^{-1} + B^\alpha\phi^\alpha_n + iL^{-\frac{1}{2}}\chi^\alpha &= 0; \quad n \neq -N+1\,, \qquad (28) \\ -\phi^{*\alpha}_N b^{-1}\lambda - \phi^{*\alpha}_N B^\alpha - iL^{-\frac{1}{2}}\chi^{*\alpha} &= 0\,, \\ \phi^\alpha_{-N+1}b^{-1}\lambda + \phi^\alpha_{-N+1}B^\alpha + iL^{-\frac{1}{2}}\chi^\alpha &= 0 \qquad (29) \end{aligned}$$

for small b Eqs (28) may be approximated by the differential equations

$$\begin{aligned} -\lambda\partial_t\phi^{*\alpha} + B^\alpha\phi^{*\alpha} + iL^{-\frac{1}{2}}\chi^{*\alpha} &= 0\,, \\ \lambda\partial_t\phi^\alpha + B^\alpha\phi^\alpha + iL^{-\frac{1}{2}}\chi^\alpha &= 0\,, \end{aligned} \qquad (30)$$

whereas Eqs (29) play the role of boundary conditions:

$$\phi^{*\alpha}\Big(\frac{L}{2}\Big) = 0; \quad \phi^\alpha\Big(-\frac{L}{2}\Big) = 0\,. \qquad (31)$$

The solution of these equations is

$$\begin{aligned} \phi^{*\alpha}(t) &= -\frac{i}{\sqrt{LB^\alpha}}\chi^{*\alpha}\Big(1 - \exp\Big\{B^\alpha\lambda^{-1}\Big(t - \frac{L}{2}\Big)\Big\}\Big)\,, \\ \phi^\alpha(t) &= -\frac{i}{\sqrt{LB^\alpha}}\chi^\alpha\Big(1 - \exp\Big\{-B^\alpha\lambda^{-1}\Big(t + \frac{L}{2}\Big)\Big\}\Big)\,. \end{aligned} \qquad (32)$$

Substituting these solutions to the Eq. (21) we get in the limit $b \to 0$

$$\begin{aligned} \lim_{b\to 0} I(\lambda, b) &= \int \exp\Big\{-\int_{-\frac{L}{2}}^{\frac{L}{2}} \sum_\alpha \frac{\chi^{*\alpha}\chi^\alpha}{B^\alpha L}\Big[1 - \exp\Big\{\frac{B^\alpha}{\lambda}\Big(t - \frac{L}{2}\Big)\Big\}\Big]dt\Big\}d\chi^* d\chi \\ &= \int \exp\Big\{-\sum_\alpha \chi^{*\alpha}(B^\alpha)^{-1}\chi^\alpha\Big[1 - \frac{\lambda}{B^\alpha L}[1 + \exp\{-B^\alpha\lambda^{-1}L\}]\Big]\Big\}d\chi^* d\chi\,. \end{aligned} \qquad (33)$$

In the limit $\lambda \to 0$ this equation reduces to

$$I = \lim_{\lambda \to 0, b \to 0} I(\lambda, b) = \exp\left\{ -\sum_\alpha \chi^{*\alpha} (B^\alpha)^{-1} \chi^\alpha \right\} d\chi^{*\alpha} d\chi^\alpha$$
$$= \det(-\hat{D}^2 + m^2) \,. \tag{34}$$

The equality (19) is proven. Therefore we showed that a four dimensional fermion determinant can be written as a path integral of the exponent of a five dimensional local bosonic action. In the same way one can present a two dimensional fermion determinant as a path integral for three-dimensional bosonic theory. In the case of lattice models this procedure leads to a well defined bosonic path integral. No numerical simulations in this approach have been tried so far and it would be very important to see how the method works in practical calculations.

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