Cascades and Dissipative Anomalies in Relativistic Fluid Turbulence

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We develop a first-principles theory of relativistic fluid turbulence at high Reynolds and Péclet numbers. We follow an exact approach pioneered by Onsager, which we explain as a nonperturbative application of the principle of renormalization-group invariance. We obtain results very similar to those for nonrelativistic turbulence, with hydrodynamic fields in the inertial range described as distributional or "coarse-grained" solutions of the relativistic Euler equations. These solutions do not, however, satisfy the naive conservation laws of smooth Euler solutions but are afflicted with dissipative anomalies in the balance equations of internal energy and entropy. The anomalies are shown to be possible by exactly two mechanisms, local cascade and pressure-work defect. We derive "4/5th-law" type expressions for the anomalies, which allow us to characterize the singularities (structure-function scaling exponents) required for their not vanishing. We also investigate the Lorentz covariance of the inertial-range fluxes, which we find to be broken by our coarse-graining regularization but which is restored in the limit where the regularization is removed, similar to relativistic lattice quantum field theory. In the formal limit as speed of light goes to infinity, we recover the results of previous nonrelativistic theory. In particular, anomalous heat input to relativistic internal energy coincides in that limit with anomalous dissipation of nonrelativistic kinetic energy.

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I. INTRODUCTION

Relativistic hydrodynamics has a growing range of applications in current physics research, including energetic astrophysical objects such as gamma-ray bursts [1] and pulsars [2], high-energy physics of the early Universe and heavy-ion collisions [3], condensed matter physics of graphene [4,5] and strange metals [6,7], and black-hole gravitational physics via the fluid-gravity correspondence in AdS/CFT [8–11]. The ubiquity of relativistic hydrodynamics is natural, given that it represents a universal low-wave-number description of relativistic quantum field theories at scales much larger than the mean-free-path length. When the global length scales of such relativistic fluid systems are even larger, as measured by the dimensionless Reynolds number, then turbulent flow is likely. There is observational evidence for relativistic turbulence in high-energy astrophysical systems; e.g., gamma-ray bursts accelerate relativistic jets to Lorentz factors $\gamma \gtrsim 100$ and contain internal fluctuations with $\delta\gamma \sim 2$ [12]. Numerical simulations of relativistic fluid models have verified the occurrence of turbulence at high Reynolds numbers [13,14]. Relativistic turbulence is also observed in numerical solutions of conformal hydrodynamic models [15–17], and an analogous phenomenon is seen in their dual AdS black-hole solutions [18].

Despite the importance of relativistic fluid turbulence at high Reynolds number for many applications, there have been only a handful of theoretical efforts to elucidate the phenomenon [19–21]. Using a point-splitting approach, Fouxon and Oz [19] derived statistical relations for relativistic turbulence that, in the incompressible limit, reduce to the famous Kolmogorov "4/5th law" [22,23]. However, in the relativistic regime, their relations have nothing to do with energy of the fluid. This seems to suggest a profound difference between relativistic and nonrelativistic turbulence or, even more radically, an essential flaw in our current understanding of nonrelativistic turbulence. As concluded by Fouxon and Oz [19], "The interpretation of the Kolmogorov relation for the incompressible turbulence in terms of the energy cascade may be misleading."

We develop here the first-principles theory of relativistic fluid turbulence at high Reynolds and Péclet numbers,

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which reaches a very different conclusion. We establish the existence of a relativistic energy cascade in the traditional sense and an even more fundamental entropy cascade. The appearance of thermodynamic entropy is not surprising, considering its central role in the theory of dissipative relativistic hydrodynamics [8,10,24–26]. Our analysis follows a pioneering work of Onsager [27,28] on incompressible fluid turbulence, who proposed that turbulent flows at very high Reynolds numbers are described by singular or distributional solutions of the incompressible Euler equations. In 1945, Onsager derived the first example of a conservation-law anomaly, showing by a point-splitting argument that the zero-viscosity limit of Navier-Stokes solutions can dissipate fluid kinetic energy for a critical $1/3$ Hölder singularity of the fluid velocity [29]. Polyakov has pointed out the formal analogy of Kolmogorov's $4/5$th law and its point-splitting derivation to axial anomalies in quantum gauge field theories [30,31]. However, Onsager's analysis is deeper than the ensemble theory of Kolmogorov [22,32,33] or "K41" because it applies to individual flow realizations. It is also formally exact and requires no statistical hypotheses, such as isotropic or homogeneous ensembles, or mean-field arguments ignoring space-time intermittency. Onsager's proposals were not understood at the time, and he never published full proofs of his assertions. Thus, the theory went ignored until Onsager's $1/3$ Hölder condition for anomalous energy dissipation was rederived [34]. This triggered a stream of work in the mathematical PDE community that has improved upon the analysis, notably in Refs. [35–37]. More recently, concepts originating in the Nash-Kuiper theorem and Gromov's h-principle have been applied to mathematically construct dissipative Euler solutions of the type conjectured by Onsager [38,39]. This new circle of ideas has led to a proof that Onsager's $1/3$ criticality condition for energy dissipation is sharp [40].

Onsager's theory of dissipative Euler solutions and its application to fluid turbulence is still essentially unknown to the wider physics community, however. This is unfortunate because it is the most comprehensive theoretical framework for high Reynolds turbulence and generally applicable not only to kinetic-energy dissipation in incompressible fluid turbulence but also to cascades in magnetohydrodynamic turbulence [41,42], to dissipative anomalies of Lagrangian invariants such as circulations [43] and magnetic fluxes [44], and to cascades in compressible Navier-Stokes turbulence [45–48]. Furthermore, Onsager's analysis is based on very intuitive physical ideas. As we discussed in our earlier paper on nonrelativistic compressible Navier-Stokes turbulence [47] (hereafter, paper I), Onsager's argument is essentially a nonperturbative application of the principle of renormalization-group (RG) invariance [49–51] (see Ref. [52], Sec. IV, for a concise and insightful review of perturbative RG). High Reynolds turbulence is characterized by ultraviolet divergences of gradients of the velocity and other thermodynamic fields, referred to as a "violet catastrophe" by Onsager [27]. Regularizing these divergences introduces a new arbitrary length scale ℓ upon which objective physics cannot depend, and exploiting this invariance yields the main conclusions of the theory on fluid singularities, inertial range, local cascades, etc.

Onsager's unpublished work in 1945 employed a point-splitting approach [29], but here we exploit a more powerful coarse graining or "block-spin" regularization [35,53] for relativistic fluid turbulence. Many essential steps were already taken in paper I on nonrelativistic compressible turbulence, such as the identification of (neg)entropy as a key invariant and the development of appropriate nonperturbative tools of analysis, such as cumulant expansions for space-time coarse graining and mathematical distribution theory. Relativistic turbulence brings in some completely new difficulties, however. First, kinetic energy is usually given the central role in the theory of nonrelativistic energy cascade, but kinetic energy is a less natural quantity in relativity theory. We show here that internal energy is the appropriate basis for the theory of relativistic energy cascade. Another distinction of the relativistic theory is that our nonperturbative coarse-graining regularization preserves Galilean symmetry of nonrelativistic fluid models, but it breaks Lorentz symmetry. This is reminiscent of the lattice regularization of relativistic quantum field theories [54], which breaks Lorentz symmetry for finite lattice spacing a but recovers it in the continuum limit $a \to 0$. The situation here is similar, as we show that Lorentz symmetry is restored as our regularization parameter $\ell \to 0$, leading to a covariant description by relativistic Euler equations. Further differences exist, such as the unit normalization of relativistic velocity vectors, which leads to new terms in flux or anomaly formulas that do not appear nonrelativistically. An important caveat about the present work is that we consider only special-relativistic fluid turbulence in flat Minkowski space-time. General-relativistic (GR) fluids in curved space-times bring in additional technical difficulties. These seem tractable, but it makes sense to develop the theory first in Minkowski space-time, as the simplest setting possible. For remarks on full GR, conclusion Sec. VIII.

In this paper, we consider the $D = d + 1$-dimensional Minkowski space-time for any space dimension $d \geq 1$. This generality is motivated not only by the wider perspective it affords but also by fluid-gravity correspondence in AdS/CFT, which holds for general D [11,55]. We adopt the signature $- + \cdots +$ of the Minkowski metric $g^{\mu\nu}$. We follow standard relativistic notations, but we include explicit factors of speed of light c, e.g., space-time coordinates $x^\mu = (x^0, \mathbf{x}) = (ct, \mathbf{x})$, velocity vectors $V^\mu = \gamma(1, \mathbf{v}/c)$, etc., rather than use natural units with $c = 1$. This facilitates taking the limit $c \to \infty$ for comparison with the results of paper I.

II. RELATIVISTIC DISSIPATIVE FLUID MODELS

We consider here a relativistic fluid with conserved stress-energy tensor $T^{\mu\nu}$,

$$\partial_\nu T^{\mu\nu} = 0, \tag{1}$$

and with one conserved current J^μ,

$$\partial_\nu J^\nu = 0. \tag{2}$$

The latter may be interpreted as a particle number current (e.g., baryon number), and the fluid models that we consider reduce in the limit $c \to \infty$ and at zeroth order in gradients to the nonrelativistic compressible Euler equations. This choice allows us to compare our results here to those derived in paper I for nonrelativistic compressible turbulence. However, our analysis carries over straightforwardly to other fluid systems without the additional conserved current J^μ (e.g., zero chemical potential sectors and conformal fluids) and to multicomponent systems with more than one conserved current (e.g., two-fluid models of relativistic superfluids).

Even with the restrictions to Eqs. (1) and (2), there are many possible fluid models. Unlike the nonrelativistic case, where the compressible Navier-Stokes equations have a more canonical status and are employed almost universally in the fluid regime, there are still many dissipative relativistic fluid models competing as descriptions of the same physical system (e.g., see Ref. [56], Sec. 14, or Ref. [57], Chap. 6). We consider a broad class of dissipative relativistic fluid theories, which includes the traditional theories of Eckart [58] and Landau-Lifshitz (LL) [59] and the Israel-Stewart (IS) theory [24,25], in which the number current and stress tensor have the general form

$$J^\mu = nV^\mu + \sigma \hat{N}^\mu, \tag{3}$$

$$\begin{aligned} T^{\mu\nu} &= p\Delta^{\mu\nu} + \epsilon V^\mu V^\nu + \Pi^{\mu\nu}, \\ \Pi^{\mu\nu} &= \kappa(\hat{Q}^\mu V^\nu + \hat{Q}^\nu V^\mu) + \zeta\hat{\tau}\Delta^{\mu\nu} + 2\eta\hat{\tau}^{\mu\nu}. \end{aligned} \tag{4}$$

Here, n is number density, $p = p(\epsilon, n)$ the pressure, and $\epsilon = u + \rho c^2$ the total energy density, with $\rho = nm$ the rest-mass density for particle mass m and u the internal-energy density. The velocity vector V^μ, to be specified below, is future timelike and $V_\mu V^\mu = -1$. The quantity $N^\mu = \sigma\hat{N}^\mu$ is a dissipative number current, $Q^\mu = \kappa\hat{Q}^\mu$ a dissipative heat current, and $\Pi^{\mu\nu}_{\rm visc} = \tau\Delta^{\mu\nu} + \tau^{\mu\nu}$ a dissipative (viscous) stress tensor with $\tau = \zeta\hat{\tau}$ and $\tau^{\mu\nu} = 2\eta\hat{\tau}^{\mu\nu}$. Here, we have defined $\Delta^{\mu\nu} = g^{\mu\nu} + V^\mu V^\nu$ as the projection onto the space direction in the fluid rest frame, and the various dissipative terms satisfy

$$V_\mu \hat{N}^\mu = V_\mu \hat{Q}^\mu = V_\mu \hat{\tau}^{\mu\nu} = 0, \tag{5}$$

with $\hat{\tau}^{\mu\nu}$ also traceless and symmetric. We have made an unconventional choice to factor out the overall dependences on particle conductivity σ, thermal conductivity κ, bulk viscosity ζ, and shear viscosity η, in order to make clearer some of our arguments below. So-called particle- or Eckart-frame theories have $\hat{N}^\mu = 0$, so V^μ is the timelike unit vector in the J^μ direction and $n = -J^\mu V_\mu$. On the other hand, energy- or Landau-Lifshitz-frame theories have $\hat{Q}^\mu = 0$, so V^μ and ϵ are specified by the eigenvalue condition $T^{\mu\nu}V_\nu = -\epsilon V^\mu$, with a timelike unit eigenvector. In the class of models that we consider in detail, there is also an entropy current S^μ (discussed further below) that satisfies a balance equation of the form

$$\partial_\mu S^\mu = \sigma\frac{\hat{N}^\mu\hat{N}_\mu}{T^2} + \kappa\frac{\hat{Q}^\mu\hat{Q}_\mu}{T^2} + \zeta\frac{\hat{\tau}^2}{T} + 2\eta\frac{\hat{\tau}^{\mu\nu}\hat{\tau}_{\mu\nu}}{T}, \tag{6}$$

whose right-hand side, when all of the transport coefficients σ, κ, ζ, η are positive, is non-negative as required by the second law of thermodynamics. The specific assumptions made above are mostly to simplify our proof in the next section that effective coarse-grained equations obtained in the limit $\sigma, \kappa, \zeta, \eta \to 0$ correspond to distributional Euler solutions. With some appropriate corresponding assumptions, our analysis will apply to any dissipative fluid model consistent with the thermodynamic second law. In fact, our inertial-range analysis is completely general and applies to any distributional solution of the relativistic Euler equations, regardless of the dissipative model limits used to obtain the particular solution (or even to solutions constructed by other means).

Defining total internal-energy current (including the rest-mass contribution)

$$\mathcal{E}^\mu = -T^{\mu\nu}V_\nu = \epsilon V^\mu + \kappa\hat{Q}^\mu \tag{7}$$

and the internal-energy current

$$U^\mu = \mathcal{E}^\mu - mc^2 J^\mu = uV^\mu + \kappa\hat{Q}^\mu - (\sigma mc^2)\hat{N}^\mu, \tag{8}$$

it is straightforward to obtain from Eqs. (1) and (2), for all of the class of models we consider, the balance equations of the two internal-energy currents as

$$\begin{aligned} \partial_\mu\mathcal{E}^\mu &= \partial_\mu U^\mu \\ &= -(\partial_\mu V_\nu)T^{\mu\nu} = -p(\partial^\mu V_\mu) + \mathcal{Q}_{\rm diss} \end{aligned} \tag{9}$$

with the dissipative "heating" of the fluid given by

$$\mathcal{Q}_{\rm diss} \coloneqq -\kappa\hat{Q}^\mu A_\mu - \zeta\hat{\tau}\theta - 2\eta\hat{\tau}^{\mu\nu}\sigma_{\mu\nu}. \tag{10}$$

Here, $A^\mu = \mathcal{D}V^\mu$ is the acceleration vector, with $\mathcal{D} = V^\mu\partial_\mu$ the material derivative for an observer moving with the fluid,

$$\theta = \Delta^{\mu\nu}\partial_\nu V_\mu = \partial_\mu V^\mu \tag{11}$$

is the relativistic dilatation, and

$$\sigma_{\mu\nu} = \partial_{\langle\mu}V_{\nu\rangle} \equiv \partial^\perp_{(\mu}V_{\nu)} - \frac{\theta}{d}\Delta_{\mu\nu} = \partial_{(\mu}V_{\nu)} + A_{(\mu}V_{\nu)} - \frac{\theta}{d}\Delta_{\mu\nu} \tag{12}$$

is the relativistic strain, for $\partial^\perp_\mu = \Delta^\alpha_\mu\partial_\alpha$. Here, we use standard notations for relativistic fluids [56,57], in particular, with $C_{(\mu\nu)} = \frac{1}{2}(C_{\mu\nu} + C_{\nu\mu})$ the symmetrization on μ, ν, so $\sigma_{\mu\nu}$ is symmetric and traceless, and $\sigma_{\mu\nu}V^\nu = 0$.

The traditional theories of Eckart [58] and Landau-Lifshitz [59] have dissipative fluxes proportional to the following tensors:

$$\hat{N}^\mu = -T^2\partial^\mu_\perp\lambda, \tag{13}$$

$$\hat{Q}^\mu = -(\partial^\mu_\perp T + TA^\mu), \tag{14}$$

$$\hat{\tau} = -\theta, \qquad \hat{\tau}_{\mu\nu} = -\sigma_{\mu\nu}, \tag{15}$$

which are first order in gradients, with particle conductivity $\sigma = 0$ for Eckart and thermal conductivity $\kappa = 0$ for Landau-Lifshitz, so

$$\mathcal{Q}_{\rm diss} = \kappa A^\mu\partial_\mu T + \kappa A^\mu A_\mu + \zeta\theta^2 + 2\eta\sigma^{\mu\nu}\sigma_{\mu\nu}. \tag{16}$$

In particular, $\mathcal{Q}_{\rm diss} \geq 0$ for the Landau-Lifshitz theory. Above, we have used the standard relativistic thermodynamic potentials, the temperature T (or its inverse $\beta = 1/T$), and $\lambda = \mu/T$ for the chemical potential μ. For reviews of relativistic thermodynamics, see Ref. [60], and also Ref. [56], Sec. V, or Ref. [57], Sec. 2.3.7. Here, we note only that the relativistic chemical potential differs from its Newtonian counterpart μ_N by a rest-mass contribution, $\mu = \mu_N + mc^2$. The entropy current of the Eckart and Landau-Lifshitz theories is defined in terms of the entropy density per volume $s(\epsilon, n)$ and the thermodynamic potentials as

$$S^\mu = sV^\mu + \beta Q^\mu - \lambda N^\mu. \tag{17}$$

Using the thermodynamic second law $ds = \beta d\epsilon - \lambda dn$ and Eqs. (2) and (9), it is then easy to check that Eq. (6) holds. However, as is well known, the Eckart and Landau-Lifshitz theories are unstable and acausal in both linear [61] and nonlinear [62] regimes. In fact, these traditional parabolic theories can be formally ill posed for general initial data on noncharacteristic hypersurfaces, including surfaces of simultaneity in arbitrary inertial reference frames [63]. Although this problem might be avoided by specifying initial data only on characteristic hypersurfaces or by imposing suitable restrictions on initial data for general Cauchy surfaces, it is not currently understood how to use these theories as predictive evolutionary models of relativistic turbulent fluids [64].

The class of models that we consider also contain better-behaved models, however, such as the extended hydrodynamic theory of Israel-Stewart [24,25]. This is, itself, an entire class of models, each of which uses a different definition of the off-equilibrium fluid velocity. The particle-frame and energy-frame versions have both been shown to be stable, causal, and hyperbolic in the linear [65,66] and nonlinear [67–69] regimes, with somewhat better stability properties in the energy frame. In these models, the entropy current is not given by Eq. (17) but instead is modified by the addition of terms that are quadratic in the dissipative fluxes N^μ, Q^μ, τ, and $\tau^{\mu\nu}$. The form of the entropy current may be illustrated by the expression that holds in the energy-frame Israel-Stewart theory [68,69]:

$$S^\mu = sV^\mu - \lambda N^\mu - \frac{1}{2T}(\beta_0\tau^2 + \beta_1 N_\alpha N^\alpha + \beta_2\tau_{\alpha\beta}\tau^{\alpha\beta})V^\mu + \frac{\alpha_0}{T}\tau N^\mu + \frac{\alpha_1}{T}\tau^{\mu\nu}N_\nu. \tag{18}$$

The new term proportional to V^μ can be regarded as an off-equilibrium modification of the rest-frame entropy density s, and thus the coefficients β_i, $i = 1, 2, 3$ (not to be confused with $\beta = 1/T$), are required to be positive to ensure that nonvanishing gradients lower the entropy. The other two terms proportional to α_i, $i = 1, 2$, are purely spatial in the fluid rest frame and describe second-order contributions to dissipative entropy transport. All of the α and β coefficients are assumed to be smooth functions of ϵ, ρ. Imposing the second law of thermodynamics in the form of Eq. (6) constrains the dissipative fluxes [70]. For example, for the energy-frame Israel-Stewart theory, one finds

$$\tau = \zeta\hat{\tau} = -\zeta[\theta + \beta_0\mathcal{D}\tau + \cdots], \tag{19}$$

$$N^\mu = \sigma\hat{N}^\mu = -\sigma T[T\partial^\mu_\perp\lambda + \beta_1(\mathcal{D}N_\perp)^\mu + \cdots], \tag{20}$$

$$\tau^{\mu\nu} = 2\eta\hat{\tau}^{\mu\nu} = -2\eta[\sigma^{\mu\nu} + \beta_2(\mathcal{D}\tau)^{\langle\mu\nu\rangle} + \cdots], \tag{21}$$

with $(\mathcal{D}N_\perp)^\mu = \Delta^{\mu\nu}\mathcal{D}N_\nu$ and with $(\mathcal{D}\tau)^{\langle\mu\nu\rangle} = \mathcal{D}\tau^{\mu\nu} + \tau^{\mu\lambda}A_\lambda V^\nu + \tau^{\nu\lambda}A_\lambda V^\mu$ the part of $\mathcal{D}\tau^{\mu\nu}$ symmetric, traceless, and orthogonal to V^μ. Here, $(\cdots)$ indicates various terms that are second order in gradients, involving the fluxes N^μ, τ, $\tau^{\mu\nu}$, and the thermodynamic potentials. We note that, in the case of the particle-frame Israel-Stewart model, nearly identical equations hold but with $N^\mu \to Q^\mu$ and, in Eq. (20), $\sigma \to \kappa$ and $T\partial^\mu_\perp\lambda \to \partial^\mu_\perp T/T + A^\mu$. Unlike the original Eckart-Landau-Lifshitz theories, the relations (19)–(21) are not simple constitutive relations for the dissipative fluxes but are instead evolutionary equations that must be solved in time together with the conservation laws (1) and (2) in order to determine both the local thermodynamic variables and the dissipative fluxes.

It is a curious fact that in the IS theories, the "energy dissipation" $\mathcal{Q}_{\rm diss}$ in Eq. (9) may possibly be negative and thus may not act to heat the fluid. Indeed, out of the entire class of models that we consider in this paper, only the (ill-posed) Landau-Lifshitz theory guarantees that $\mathcal{Q}_{\rm diss} \geq 0$. It is generally argued that negative values of $\mathcal{Q}_{\rm diss}$ cannot be realized within the physical regime of validity of a fluid description. Since the dissipative fluxes in the energy-frame IS model differ from those in the LL theory only by terms second order in gradients, it is plausible that, for most circumstances, the dissipative fluxes obtained by solving the IS model will be nearly the same as those given by the LL constitutive relations, when evaluated with the IS model solutions. More generally, Geroch [72] and Lindblom [73] have argued that this close agreement with the Landau-Lifshitz or Eckart constitutive relations will hold in the energy or particle frame, respectively, for a wide set of extended dissipative relativistic fluid models that are hyperbolic, causal, and well posed. Thus, we typically expect to have $\mathcal{Q}_{\rm diss} \geq 0$ in energy-frame fluid models. Unfortunately, the arguments of Refs. [72,73] fail in the presence of shocks with near discontinuities extending down to lengths of the order of the mean free path. In fact, the IS fluid models and other broad classes of hyperbolic, causal, well-posed models of dissipative relativistic fluids do not even possess continuous solutions corresponding to strong shocks [68,74]. Thus, perhaps even more than for the nonrelativistic case, a better microscopic starting point for a theory of relativistic fluid turbulence might be relativistic kinetic theory or a relativistic quantum field theory rather than a dissipative fluid model. Fortunately, our principal results do not depend upon any particular model of dissipation but only require the general conservation laws (1) and (2), a fluid description with variables given by local thermodynamic equilibrium, and the second law of thermodynamics.

In this paper, we examine the hypothesis that the entropy production is anomalous in relativistic fluid turbulence. Thus, we assume, in the ideal limit σ, κ, η, $\zeta \to 0$, that distributional limits of the entropy production exist:

$$\Sigma = \mathcal{D}\text{-lim}_{\sigma,\kappa,\eta,\zeta\to 0}\left[\frac{\sigma\hat{N}^\mu\hat{N}_\mu}{T^2} + \frac{\kappa\hat{Q}^\mu\hat{Q}_\mu}{T^2} + \frac{\zeta\hat{\tau}^2}{T} + \frac{2\eta\hat{\tau}^{\mu\nu}\hat{\tau}_{\mu\nu}}{T}\right] = \Sigma_{\rm cond} + \Sigma_{\rm therm} + \Sigma_{\rm bulk} + \Sigma_{\rm shear} > 0. \tag{22}$$

We then show that any strong limits ϵ, ρ, V^μ of the local equilibrium fields are weak solutions of the relativistic Euler equations, under very mild additional assumptions. The anomalous entropy production of these Euler solutions is shown to occur by a nonlinear cascade mechanism, and we characterize the type of singularities required for nonvanishing entropy cascade. As in the nonrelativistic case, the ideal limit is really a limit of large Reynolds and Péclet numbers introduced by a nondimensionalization of the fluid equations. Because the fluid velocity V^μ is already nondimensional in natural units based on the speed of light c and is assumed to be of order unity, the Reynolds numbers are $\mathrm{Re}_\eta = \rho_0 c^2 L_0/\eta$ and $\mathrm{Re}_\zeta = \rho_0 c^2 L_0/\zeta$ as given by the shear and bulk viscosities [75], and the particle and thermal Péclet numbers are $\mathrm{Pe}_\sigma = \rho_0 L_0/\sigma T_0 (mc)^2$ and $\mathrm{Pe}_\kappa = \rho_0 c^3 L_0/\kappa T_0$. Here, $\rho_0 c^2$ is a typical energy density, L_0 a length characterizing the injection scale of the flow (as well as the turnover time L_0/c in natural units), and T_0 a temperature scale such as $T(\rho_0 c^2, \rho_0)$. There are additional dimensionless groups that multiply the terms of the dissipative fluxes that are second order in gradients, but no assumption needs to be made in our analysis about their magnitudes.

In addition to formulating a theory of the turbulent entropy balance, we also derive a turbulent internal-energy balance and describe, with precise formulas, the relativistic energy cascade. Although internal energy plays the primary role in this cascade, we show that a "relativistic kinetic energy" can also be defined and is dissipated by the turbulent cascade process. Conditions for the nonvanishing of the energy flux are very similar to those obtained in paper I for nonrelativistic flow, and the relativistic energy flux reduces in the limit $c \to \infty$ to the nonrelativistic kinetic-energy flux. An Onsager condition for the nonvanishing energy-dissipation anomaly is obtained, assuming positivity of the dissipative heating. Our main result on the entropy production anomaly requires no such additional assumption, and the proof requires only modest changes to that for nonrelativistic fluids, as we demonstrate in detail below.

III. RELATIVISTIC COARSE GRAINING

In our analysis, we employ a coarse-graining regularization very similar to that used in our nonrelativistic study in the companion paper I. Just as in the nonrelativistic case, nonvanishing dissipative anomalies as in Eq. (22) require that gradients $\partial_\mu^\perp V_\nu$, $\partial_\mu^\perp T$, $\partial_\mu^\perp \lambda$ must diverge as σ, κ, η, $\zeta \to 0$, and this makes it impossible to interpret the fluid dynamical equations in the naive sense in the ideal limit. As in the nonrelativistic problem, we can remove the ultraviolet divergences by space-time coarse graining. An essential difference, however, is that coarse graining with a spherically symmetric filter kernel guarantees invariance of turbulent fluxes in nonrelativistic flows under the full Galilean symmetry group, but there is no possible space-time coarse graining that can preserve Lorentz symmetry. For example, consider a general space-time filtering operation of the velocity field

$$\bar{V}^\mu(x) = \int d^D r \mathcal{G}_\ell(r) V^\mu(x+r), \tag{23}$$

with $\mathcal{G}_\ell(r) \coloneqq \ell^{-D}\mathcal{G}(r/\ell)$. Then, it is easy to check that Lorentz transformations $V^{\mu\prime}(x') = \Lambda^{\mu\nu}V_\nu(\Lambda^{-1}x')$ for $\Lambda \in SO(1,d)$ when applied to the coarse-grained field in Eq. (23) correspond to a coarse graining of the transformed field $V^{\mu\prime}(x')$ but with a different kernel,

$$\mathcal{G}'(r') = \mathcal{G}(\Lambda^{-1}r'). \tag{24}$$

The kernels in the two frames are the same if and only if the coarse-graining kernel satisfies, for all r in Minkowski space and all $\Lambda \in SO(1,d)$,

$$\mathcal{G}(\Lambda r) = \mathcal{G}(r). \tag{25}$$

This relation requires $\mathcal{G}(r)$ to depend upon the separation vector r^μ only through the relativistic proper-time interval $R^2 = -r_\mu r^\mu$. In that case, however, the space-time integral of the kernel must diverge,

$$\int d^D r\mathcal{G}(r) = \int_{-\infty}^{+\infty} dR^2\mathcal{G}(R^2)\int_{H_{R^2}} \frac{d^d\mathbf{r}}{|r^0|} = +\infty, \tag{26}$$

because of the noncompactness of the hyperboloids $H_{R^2} = \{r: -r_\mu r^\mu = R^2\}$. In contrast, in the nonrelativistic case, the orbits of the rotation group $SO(d)$ in its action on space are the spheres $S_\rho = \{\mathbf{r}: |\mathbf{r}| = \rho\}$, which are compact and have finite area. Because of the divergence in Eq. (26), it is impossible to define a coarse-graining operation that commutes with Lorentz transformations and whose kernel satisfies the properties of positivity $\mathcal{G}(r) \geq 0$ and normalization

$$\int d^D r\mathcal{G}(r) = 1. \tag{27}$$

Together with rapid decay and smoothness, these properties are necessary so that coarse graining is a regularizing operation that represents a local space-time averaging. As we see below, this leads to a breaking of Lorentz covariance of the coarse-grained fluid equations at finite ℓ and possible observer dependence of quantities such as turbulent cascade rates. However, we see that there is restoration of Lorentz symmetry in the limit $\ell \to 0$ (similar to the restoration of Lorentz invariance in lattice field theories in the limit of lattice spacing $a \to 0$).

The effect of Lorentz transformation on a filter kernel can be made more concrete by considering a pure boost in the 1-direction, with rapidity φ related to the relative velocity w by $w = c\tanh\varphi$. Using standard light-front coordinates $x^\pm = (x^0 \pm x^1)/\sqrt{2}$ in 0-1 planes [76], the boost transformation becomes

$$x'^\pm = e^{\pm\varphi}x^\pm, \tag{28}$$

with all other spatial variables $x^2, \ldots, x^d$ remaining unchanged. A filter kernel $\mathcal{G}_\ell$ is thus transformed into

$$\mathcal{G}'_\ell(r') = \mathcal{G}_\ell(e^{-\varphi}r'^+, e^{+\varphi}r'^-, r'^2, \ldots, r'^d). \tag{29}$$

Effectively, the coarse-graining scale is changed for the comoving observer to $\ell_+ = e^{+\varphi}\ell$ in the $+$ direction and to $\ell_- = e^{-\varphi}\ell$ in the $-$ direction, and it is unchanged in the remaining spatial directions $2, \ldots, d$. This discussion of the pure boost transformation underlines the fact that the notion of "scale" will be different for different observers.

While any filter kernel that is smooth and rapidly decaying in space-time can be adopted, it is also possible to use more singular kernels that will still regularize the equations of motion. For example, as in the nonrelativistic case, it is possible to filter only spatially at fixed instants of time [77]:

$$\mathcal{G}_\ell(r) = G_\ell(\mathbf{r})\delta(r^0), \tag{30}$$

where $G_\ell(\mathbf{r}) = \ell^{-d}G(\mathbf{r}/\ell)$ is a smooth kernel rapidly decaying in physical space. Such a coarse graining does not, of course, remain instantaneous in other reference frames. For example, for an observer moving with relative velocity w in the 1-direction, the kernel in Eq. (30) transforms into

$$\begin{aligned}\mathcal{G}'_\ell&(r'^0, r'^1, \ldots, r'^d) \\ &= \gamma^{-1}(w)G_\ell(r'^1/\gamma(w), r'^2, \ldots, r'^d)\delta(r'^0 + wr'^1/c),\end{aligned} \tag{31}$$

with Lorentz factor $\gamma(w) = (1 - w^2/c^2)^{-1/2}$ according to the general transformation formula (25). To the relatively moving observer, the filtering kernel has become noninstantaneous and, furthermore, is elongated along the 1-direction with modified spatial scale $\ell' = \gamma(w)\ell$ in that direction. Such elongation corresponds to the well-known fact that a stationary blob of fluid at an instant in the original frame is length contracted by the factor $1/\gamma(w)$ in the relatively moving frame but also sweeps through a distance larger by a factor $\gamma(w)$ as it moves in that frame. Once again, the notion of "scale" is seen to be different for different observers.

Another singular kernel of some interest is a spatially weighted average over the past light cone:

$$\mathcal{G}_\ell(r) = G_\ell(\mathbf{r})\delta(r^0 + |\mathbf{r}|). \tag{32}$$

This is a natural average that can be, in principle, computed at each point independently from incoming light signals [78]. It may also have some utility for numerical modeling of relativistic fluid turbulence by the large-eddy simulation (LES) methodology [79–81] since such averages can be computed on arbitrary spacelike Cauchy surfaces using only precomputed (past) values of simulated fields. For the observer moving with relative velocity w in the 1-direction, the light-cone average transforms into another light-cone average with a different spatial kernel:

$$\mathcal{G}'_\ell(r') = \delta(r'^0 + |\mathbf{r}'|) \times \begin{cases} fG_\ell(fr'^1, r'^2, \ldots, r'^d) & \text{for } r'^1 > 0 \\ f^{-1}G_\ell(f^{-1}r'^1, r'^2, \ldots, r'^d) & \text{for } r'^1 < 0, \end{cases} \tag{33}$$

for $f = \sqrt{(c-w)/(c+w)} = e^{-\varphi}$. In this particular case, the spatial kernel is elongated or contracted depending upon the relative signs of w and r'^1, and an initially reflection-symmetric kernel will not remain so in a boosted frame.

A property of the space-time coarse-graining operation (23) that must be kept in mind is that the coarse-grained fluid velocity vector $\bar{V}^\mu$, while it remains future timelike, is not generally a unit vector with respect to the Minkowski pseudometric. Under coarse graining,

$$V^\mu = \gamma(v)(1, \mathbf{v}/c) \Rightarrow \bar{V}^\mu = \overline{\gamma(v)}(1, \hat{\mathbf{v}}/c), \tag{34}$$

where we introduced the γ-weighted space-time average

$$\hat{\mathbf{v}} = \overline{\gamma(v)\mathbf{v}}/\overline{\gamma(v)}, \qquad \gamma(v) = (1 - v^2/c^2)^{-1/2}. \tag{35}$$

By convexity of the spatial Euclidean norm-square,

$$|\hat{\mathbf{v}}|^2 \leq \widehat{|\mathbf{v}|^2} < c^2. \tag{36}$$

Thus,

$$\bar{V}_\mu \bar{V}^\mu = -\overline{\gamma(v)}^2/\gamma(\hat{v})^2 < 0, \tag{37}$$

with $\hat{v} = |\hat{\mathbf{v}}|$, and $\bar{V}^\mu$ remains future timelike. However, generally $\overline{\gamma(v)} \neq \gamma(\hat{v})$ and thus $\bar{V}_\mu \bar{V}^\mu \neq -1$. Nonunit normalization of $\bar{V}^\mu$ introduces new terms into the coarse-grained equations of motion in the relativistic case, which have no counterpart nonrelativistically.

The most important feature of the space-time coarse graining is that, for a fixed scale ℓ, all of the dissipative transport terms in the coarse-grained conservation laws

$$\partial_\nu \bar{T}^{\mu\nu} = 0, \qquad \partial_\nu \bar{J}^\nu = 0 \tag{38}$$

become negligible in the ideal limit $\sigma, \kappa, \eta, \zeta \to 0$. As in the nonrelativistic case, this negligible direct effect of dissipation leads to the crucial concept of the inertial range of scales. Because of the key importance of this result, here we give a careful demonstration for the class of dissipative fluid theories treated in this paper. For simplicity, we consider only filter kernels that are entirely smooth in space-time, as more singular kernels (such as instantaneous or light-cone averages) would introduce additional purely technical complications. See Ref. [48] for further discussion. Furthermore, we assume that the kernel is C^∞, compactly supported in space-time, and it is thus a standard test function for space-time distributions, which further simplifies the proofs.

We illustrate the argument with the number conservation law, which contains the single dissipative term

$$\partial_\mu \overline{\sigma \hat{N}^\mu}(x) = -\frac{1}{\ell} \int d^D r (\partial_\mu \mathcal{G})_\ell(r) \sigma(x+r) \hat{N}^\mu(x+r), \tag{39}$$

where an integration by parts has been performed. Introducing, as a factor of unity, $T \cdot (1/T) = 1$, the Cauchy-Schwartz inequality gives

$$|\partial_\mu \overline{\sigma \hat{N}^\mu}(x)| \leq \frac{1}{\ell} \sqrt{\int_{\mathrm{supp}(\mathcal{G}_\ell)} d^D r (\sigma T^2)(x+r)} \times \sqrt{\int d^D r \frac{\sigma}{T^2}(x+r) |(\partial_\mu \mathcal{G})_\ell(r) \hat{N}^\mu(x+r)|^2}. \tag{40}$$

The first square-root factor vanishes in the ideal limit under mild assumptions (e.g., if σ goes to zero uniformly in space-time and if the temperature T remains locally square integrable). If we can show that the second square-root factor remains bounded in the ideal limit, then the product will also go to zero.

Because the projection tensor $\Delta^{\mu\nu}$ is symmetric and also non-negative (as seen by transforming into the fluid rest frame), it defines an inner product for which another application of the Cauchy-Schwartz inequality gives

$$|(\partial_\mu \mathcal{G})_\ell(r) \hat{N}^\mu(x+r)|^2 \leq (\partial^\perp_\mu G)_\ell (\partial^\mu_\perp G)_\ell(r) \cdot \hat{N}_\mu \hat{N}^\mu(x+r). \tag{41}$$

The integral inside the second square root in Eq. (40) is thus bounded by

$$\int d^D r (\partial^\perp_\mu G)_\ell(r) (\partial^\mu_\perp G)_\ell(r) \frac{\sigma \hat{N}_\mu \hat{N}^\mu}{T^2}(x+r), \tag{42}$$

and the second factor in the integrand in Eq. (42) is just the entropy production due to particle conductivity. Because this entropy production is assumed to converge distributionally to a nonvanishing measure Σ_{cond} in the ideal limit, this integral would remain bounded if the first factor were a smooth test function. Unfortunately, the last statement is generally false because $\partial^\mu_\perp G_\ell(r) = \Delta^{\mu\nu}(x+r)\partial_\nu G_\ell(r)$ acquires a "rough" dependence on space-time through the velocity $V^\mu(x+r)$ in the projection tensor. However, it is easy to show (see Appendix A) that

$$0 \leq (\partial^\perp_\mu G)_\ell(r)(\partial^\mu_\perp G)_\ell(r) \leq 2\gamma^2(v)|(\partial G)_\ell(r)|^2_E, \tag{43}$$

where we introduced the factor $\gamma(v) = V^0(x+r) > 0$ associated with the fluid velocity vector and also the Euclidean space-time norm

$$|(\partial G)_\ell(r)|_E^2 = \sum_{\mu=0}^{d} |(\partial_\mu G)_\ell(r)|^2. \tag{44}$$

The latter quantity no longer has any dependence on the fluid velocity vector $V^\mu(x+r)$, and it is a standard test function (C^∞ and compactly supported) when the kernel $\mathcal{G}(r)$ has the same properties. Thus, the integral inside the second square root of Eq. (40) is bounded by

$$2\|\gamma(v)\|_\infty^2 \int d^D r |(\partial^\mu G)_\ell(r)|_E^2 \frac{\sigma \hat{N}_\mu \hat{N}^\mu}{T^2}(x+r), \tag{45}$$

and it remains finite in the limit, if we assume that $\|\gamma(v)\|_\infty = \sup_x |V^0(x)| < \infty$. This requires an assumption that the fluid speed satisfies $v \leq c(1-\delta)$ for some fixed small $\delta \ll 1$. Such a δ will obviously be observer dependent. Under these conditions, we conclude that the coarse-grained dissipative number current term $\partial_\mu \overline{\sigma \hat{N}^\mu}(x) \to 0$, vanishing pointwise in the ideal limit.

The conclusion of this argument is that the coarse-grained particle conservation law for any fixed ℓ in the limit $\sigma, \kappa, \eta, \zeta \to 0$, becomes

$$\partial_\mu \bar{J}^\mu = \partial_\mu \overline{n V^\mu} = 0, \tag{46}$$

with the dissipative term tending to zero. We thus obtain the ideal particle conservation equation in a "coarse-grained sense." As shown in Ref. [48], the validity of the ideal fluid equations in this coarse-grained sense for all $\ell > 0$ is equivalent to their validity "weakly" or in the sense of distributions. We should emphasize the nontriviality of this result. Nonvanishing of the distributional limit $\Sigma_{\rm cond} = \mathcal{D}\text{-}\lim_{\sigma,\kappa,\eta,\zeta\to 0} \sigma \hat{N}_\mu \hat{N}^\mu / T^2$ requires that gradients of thermodynamic potentials must diverge, or $|\partial_\mu^\perp \lambda| \to +\infty$, if the Landau-Lifshitz contribution (13) to $\hat{N}^\mu$ is the dominant one. Nevertheless, even with such diverging gradients of fine-grained quantities (an "ultraviolet divergence"), the coarse-grained equations are regularized, and any limit fields n, V^μ as $\sigma, \kappa, \eta, \zeta \to 0$ will satisfy the ideal particle conservation law in the coarse-grained sense. For finite but very large values of the particle Péclet number Pe_σ, this means that there is a long inertial range of scales ℓ where the coarse-grained ideal equation is valid.

It is worth emphasizing that the dissipation length ℓ_σ where particle conductivity σ becomes non-negligible is presumably observer-dependent at finite Pe_σ, unlike the nonrelativistic case where all observers in different Galilean frames will agree on the dissipation lengths. Notice that the upper bound in Eq. (45) is not Lorentz invariant because the gamma factor $\gamma(v)$ and the Euclidean norm of the kernel gradient are both frame dependent. In fact, consider the example of a coarse-graining average over a Euclidean ball in space-time with radius ℓ, as calculated by a certain observer. When $\ell \gg \ell_\sigma$, then the dissipative contribution to the coarse-grained particle current will be negligible to this observer. However, this same coarse-grained particle current to a comoving observer with large relative velocity w in the 1-direction will correspond to a filter kernel (29) with dilated thickness $e^{+\varphi}\ell_\sigma$ in the + direction in the 0-1 plane and contracted thickness $e^{-\varphi}\ell$ in the − direction. As a consequence, ∂_- gradients of coarse-grained fields become large for this observer. When w is sufficiently close to c so that $e^{-\varphi}\ell \simeq \ell_\sigma$, then the comoving observer may find that dissipative particle transport is non-negligible for the coarse-grained current in his frame of reference. Of course, in the ideal limit $\sigma \to 0$ with the scale ℓ of the filter kernel fixed, every observer will agree that dissipative transport has vanished in the coarse-grained particle current because $\ell_\sigma \to 0$.

All of the conclusions derived above hold also for the coarse-grained equations of energy-momentum conservation, where at fixed ℓ in the limit $\sigma, \kappa, \eta, \zeta \to 0$, any limiting fields ϵ, ρ, and V^μ will satisfy

$$\partial_\nu \bar{T}^{\mu\nu} = \partial_\nu (\overline{\epsilon g^{\mu\nu} + p\Delta^{\mu\nu}}) = 0, \tag{47}$$

with all dissipation terms tending to zero. Just as for particle conservation, the range of ℓ over which these ideal equations are valid could be observer dependent at finite Reynolds and Péclet numbers. The proof of these statements is very similar to that given above for particle conservation, and we thus give complete details in Appendix A. From Eqs. (46) and (47), we can conclude that the relativistic Euler equations hold in the coarse-grained sense at fixed scale ℓ in the limit of infinite Reynolds and Péclet numbers; that is, ideal relativistic Euler equations hold distributionally.

There is one last remark on the coarse-graining regularization, which has fundamental importance in what follows. This coarse graining is a purely passive operation that is applied *a posteriori* to the fluid variables and that can effect no change whatsoever on any physical occurrence [47,82]. In prosaic terms, coarse graining corresponds to "removing one's spectacles" and observing the physical evolution at a reduced space-time resolution ℓ. The effective dynamical description is changed by regularization, of course, with coarse-grained variables satisfying much more complex equations than fine-grained fields. This is not unexpected because the coarse-grained variables are like block spins in the renormalization-group theory of critical phenomena [83,84], and such Wilson-Kadanoff RG procedures typically lead to very complicated effective descriptions. In fact, an implementation of the coarse graining by integrating out unresolved fields in a path-integral formulation yields an effective dynamics with higher-order nonlinearity, long-time memory, and induced stochasticity [85]. This is a manifestation of the "closure problem," in which coarse-grained variables like $\bar{V}^\mu(x)$ no longer satisfy simple closed equations of motion. As we see below, Onsager's method does not solve this problem but instead bypasses it by exploiting 4/5th-law type expressions for new, unclosed expressions. The essential idea is then to invoke the independence of the physics on the arbitrary coarse-graining scale ℓ. This simple invariance principle turns out to yield nontrivial consequences.

IV. ENERGY CASCADE

It is reasonable to expect that relativistic fluids at very high Reynolds and Péclet numbers should exhibit a turbulent energy cascade, just as nonrelativistic incompressible and compressible fluids do. However, the familiar notion of kinetic-energy cascade is not appropriate for relativistic turbulence because kinetic energy is not a very natural concept within relativity theory. On the other hand, we have seen in our discussion of nonrelativistic compressible fluids in paper I that energy cascade can be understood from the coarse-grained dynamics of the internal energy. Because the concept of internal energy remains valid in relativistic thermodynamics, it provides a good basis for the theory of relativistic energy cascade.

A resolved energy current is defined most simply as

$$\underline{\mathcal{E}}^\mu_\ell \coloneqq -\bar{T}^{\mu\nu}\bar{V}_\nu, \tag{48}$$

which (like resolved kinetic energy in nonrelativistic turbulence) is a nonlinear function of coarse-grained quantities. As we have shown in some detail in Appendix A, for length scales ℓ in the inertial range, or for all fixed ℓ in the ideal limit $\sigma, \kappa, \eta, \zeta \to 0$,

$$\begin{aligned}\bar{T}^{\mu\nu} &= \overline{\epsilon V^\mu V^\nu + p\Delta^{\mu\nu}} \\ &= \bar{p}g^{\mu\nu} + \overline{hV^\mu V^\nu},\end{aligned} \tag{49}$$

where $h = \epsilon + p$ is relativistic enthalpy. Notably, the quantity $\overline{hV^\mu V^\nu}$ is an exact formal analogue of the "Reynolds stress" of nonrelativistic fluid turbulence. If subsequent to the ideal limit $\sigma, \kappa, \eta, \zeta \to 0$, one considers the limit of regularization length scale $\ell \to 0$, one finds that

$$\mathcal{D}-\lim_{\ell\to 0}\underline{\mathcal{E}}^\mu_\ell = \epsilon V^\mu. \tag{50}$$

For this to hold, one needs only some modest regularity of the limiting variables ϵ, p, V^μ, such as finite (absolute) fourth-order moments in local space-time averages. Thus, the resolved energy current converges distributionally in the "continuum limit" $\ell \to 0$ to the fine-grained energy current of the Euler fluid. The naive energy balance obtained by setting $\mathcal{Q}_{\rm diss} = 0$ in Eq. (9) does not follow, however. To obtain the correct result, we can use the balance equation for the resolved energy current

$$\begin{aligned}\partial_\mu\underline{\mathcal{E}}^\mu_\ell &= \partial_\mu(-\bar{V}_\nu\bar{T}^{\mu\nu}) \\ &= -(\partial_\mu\bar{V}_\nu)\bar{T}^{\mu\nu} \\ &= -\bar{p}(\partial^\nu\bar{V}_\nu) - (\partial_\mu\bar{V}_\nu)\overline{hV^\mu V^\nu},\end{aligned} \tag{51}$$

obtained from $\partial_\mu\bar{T}^{\mu\nu} = 0$ and Eq. (49). The last term in Eq. (51) would not be present in the fine-grained internal-energy balance for a smooth Euler solution because of the orthogonality condition $(\partial_\mu V_\nu)V^\nu = 0$. It is analogous to the "deformation work" of nonrelativistic turbulent fluid, which appears, however, in the equation for the resolved kinetic energy rather than for resolved internal energy [e.g., see Eq. (paper I;43)]. This term is the source of possible energy-dissipation anomalies in relativistic fluid turbulence, and it gives the simplest representation of turbulent energy flux.

Despite the simplicity of the above formulation, here we follow an alternative approach based upon a relativistic Favre averaging, similar to that employed in paper I for nonrelativistic compressible turbulence. It should be emphasized that the entire theory presented below could be developed just as easily using Eq. (51) derived above [86]. However, the relativistic Favre-averaging approach is convenient to compare with results of paper I in the limit $c \to \infty$. The proper relativistic generalization of Favre averaging is motivated by the appearance of the enthalpy in Eq. (49). Note that the "null energy condition" $h \geq 0$ is a condition for stability of thermodynamic equilibrium [65], and in the strict form, $h > 0$ is required for causality of the relativistic Euler fluid [87]. We thus define the Favre-average coarse graining for a relativistic fluid by

$$\tilde{f} \coloneqq \overline{hf}/\bar{h} \tag{52}$$

with enthalpy weighting. With this definition, Eq. (49) becomes

$$\begin{aligned}\bar{T}^{\mu\nu} &= \bar{p}g^{\mu\nu} + \bar{h}\,\widetilde{V^\mu V^\nu} \\ &= \bar{p}g^{\mu\nu} + \bar{h}\tilde{V}^\mu\tilde{V}^\nu + \bar{h}\,\tilde{\tau}(V^\mu, V^\nu).\end{aligned} \tag{53}$$

As in the nonrelativistic theory, expanding in the pth-order cumulants $\tilde{\tau}(f_1, \ldots, f_p)$ of the Favre average produces only a single "unclosed" term in the coarse-grained stress-energy tensor, whereas expanding in pth-order cumulants $\bar{\tau}(f_1, \ldots, f_p)$ of the unweighted space-time coarse graining would produce more such unclosed terms. This is a significant advantage of the Favre average for potential applications to LES modeling of relativistic fluid turbulence. Within the Favre-averaging approach, it is convenient to define the resolved energy current by

$$\underset{\sim}{\mathcal{E}}^\mu \coloneqq -\tilde{V}_\nu\bar{T}^{\mu\nu} + \frac{\bar{p}}{\bar{h}}\tilde{\tau}(h, V^\mu) - \frac{1}{2}\tilde{\tau}(V_\nu, V^\nu)\overline{hV^\mu}. \tag{54}$$

Alternative expressions for this current follow from Eq. (49), the relation

$$\tilde{V}^\mu = \bar{V}^\mu + (1/\bar{h})\bar{\tau}(h, V^\mu), \tag{55}$$

and $V^\nu V_\nu = -1$, from which one can easily derive

$$-\tilde{V}_\nu \bar{T}^{\mu\nu} + \frac{\bar{p}}{\bar{h}}\bar{\tau}(h, V^\mu) = \overline{hV^\mu} + \overline{hV_\nu V^\nu V^\mu} - \widetilde{V^\mu V^\nu}\,\overline{hV_\nu} - \bar{p}\bar{V}^\mu. \tag{56}$$

Thus, by decomposing into Favre-average cumulants, one obtains that

$$\underset{\sim}{\mathcal{E}}^\mu = \overline{\epsilon V^\mu} + \bar{\tau}(p, V^\mu) + \bar{h}\left[\frac{1}{2}\tilde{\tau}(V_\nu, V^\nu)\tilde{V}^\mu + \tilde{\tau}(V_\nu, V^\mu)\tilde{V}^\nu + \tilde{\tau}(V_\nu, V^\nu, V^\mu)\right]$$
$$= \bar{\epsilon}\bar{V}^\mu + \bar{\tau}(h, V^\mu) + \bar{h}\left[\frac{1}{2}\tilde{\tau}(V_\nu, V^\nu)\tilde{V}^\mu + \tilde{\tau}(V_\nu, V^\mu)\tilde{V}^\nu + \tilde{\tau}(V_\nu, V^\nu, V^\mu)\right]. \tag{57}$$

Either from this expression or directly from the definition (54), one can see that

$$\mathcal{D}\text{-}\lim_{\ell\to 0}\underset{\sim}{\mathcal{E}}_\ell^\mu = \epsilon V^\mu, \tag{58}$$

under the same assumptions as Eq. (50). Once again, however, the naive energy balance (9) for the limiting current $\mathcal{E}^\mu = \epsilon V^\mu$ need not hold with $\mathcal{Q}_{\rm diss} = 0$ but instead may be modified by a turbulent dissipative anomaly if

$$\mathcal{Q}_{\rm diss} \coloneqq \mathcal{D}\text{-}\lim_{\eta,\zeta,\kappa,\sigma\to 0}\mathcal{Q}_{\rm diss}^{\eta,\zeta,\kappa,\sigma} \neq 0 \tag{59}$$

for $\mathcal{Q}_{\rm diss}^{\eta,\zeta,\kappa,\sigma}$ given by Eq. (10). We emphasize that the vanishing or not of the limit in Eq. (59) is an objective physical fact, which cannot depend upon any coarse graining.

To obtain the correct equation, one can use the inertial-range balance equation for the energy current defined in Eq. (54). Using $\partial_\mu \bar{T}^{\mu\nu} = 0$, one gets, after some straightforward calculations (see Ref. [88]),

$$\partial_\mu \underset{\sim}{\mathcal{E}}^\mu = -\bar{p}(\partial^\nu \bar{V}_\nu) + \mathcal{Q}_\ell^{\rm flux}, \tag{60}$$

with relativistic energy flux defined by

$$\mathcal{Q}_\ell^{\rm flux} \coloneqq \frac{1}{\bar{h}}(\partial_\nu \bar{p})\bar{\tau}(h, V^\nu) - \bar{h}(\partial_\mu \tilde{V}_\nu)\tilde{\tau}(V^\mu, V^\nu) - \frac{1}{2}\partial_\nu \overline{hV^\nu}\,\tilde{\tau}(V_\mu, V^\mu). \tag{61}$$

The energy flux $\mathcal{Q}_\ell^{\rm flux}$ can be interpreted as the "apparent dissipative heating" in the large scales based only on measurements resolved at that scale. The first two terms in the energy flux (61) are relativistic generalizations of the baropycnal work and the deformation work as defined by Aluie [45,46] for nonrelativistic compressible fluids, whereas the third term has no nonrelativistic analogue. The balance equation (60) is formally very similar to the nonrelativistic balance equation (paper I;63) for the "intrinsic large-scale internal energy," defined in Eq. (paper I;64). Not only does Eq. (60) resemble the nonrelativistic balance equation derived in paper I, but we show in Appendix B that it reduces to it in the formal limit $c \to \infty$. In particular, the relativistic energy flux that we defined in Eq. (61) converges as $c \to \infty$ to the nonrelativistic expressions in Refs. [45,46] and in Eq. (paper I;45).

Now let us exploit the fact that a nonzero energy-dissipation anomaly in the ideal limit (59) cannot depend upon any particular choice of the regularization scale ℓ. Subsequent to the limit $\eta, \zeta, \kappa, \sigma \to 0$, one can thus consider the limit $\ell \to 0$ of the coarse-grained internal-energy balance (60) for the relativistic Euler fluid. It follows from Eq. (58) that the left-hand side converges distributionally to $\partial_\mu(\epsilon V^\mu)$ because the overall derivative ∂_μ can be transferred to a test function. We also define the distributional product of the dilatation $\theta = \partial_\mu V^\mu$ and the pressure p by a standard procedure [89]

$$p\circ\theta \coloneqq \mathcal{D}\text{-}\lim_{\ell\to 0}\bar{p}\cdot\bar{\theta}, \tag{62}$$

just as in the nonrelativistic case in paper I. Although all of the cumulant factors appearing in the energy flux (61) vanish as $\ell \to 0$, the flux $\mathcal{Q}_\ell^{\rm flux}$ itself need not vanish because the space-time gradients multiplying them diverge in the same limit. By taking the limit $\ell \to 0$ of Eq. (60), one thus obtains, for the relativistic Euler solutions, the distributional energy balance

$$\partial_\mu(\epsilon V^\mu) = -p\circ\theta + \mathcal{Q}_{\rm flux}, \tag{63}$$

with a possible anomaly due to energy cascade given by

$$\mathcal{Q}_{\rm flux} \coloneqq \mathcal{D}\text{-}\lim_{\ell\to 0}\mathcal{Q}_{{\rm flux},\ell} = \mathcal{D}\text{-}\lim_{\ell\to 0} - (\partial_\nu \bar{V}_\mu)\overline{hV^\mu V^\nu}. \tag{64}$$

Note that the second expression in the equation above arises from the corresponding $\ell \to 0$ limit of Eq. (51).

A condition for the nonvanishing of the anomaly $\mathcal{Q}^{\rm flux}$ can be obtained just as in the nonrelativistic case (see Ref. [46] and Sec. V of paper I), by deriving 4/5th-law type expressions for the turbulent energy flux. The key point is that the cumulants of fields with respect to the space-time coarse graining can be written instead as cumulants of their space-time increments with respect to an average over displacement vectors r^μ weighted by the filter kernel. In other words,

$$\bar{\tau}_\ell(f_1,...,f_p)(x+a) = \langle(\delta f_1)...(\delta f_p)\rangle^{\rm cum}_{\ell,a}, \quad (65)$$

where

$$\delta f_i(x;r) = f_i(x+r) - f_i(x) \quad (66)$$

are space-time increments and where, for any function $h(r)$,

$$\langle h\rangle_{\ell,a} = \int d^D r \mathcal{G}_\ell(r-a)h(r). \quad (67)$$

The superscript "cum" in Eq. (65) denotes the pth-order cumulant part of any pth-order moment. The details of the proof are given in Appendix B of Ref. [82], but the essential point is that cumulants are invariant under shifts of variables by constants and the increment $\delta f_i(x;r)$ is the shift of $f_i(x+r)$ by the quantity $-f_i(x)$ which is "constant," i.e., independent of r^μ. The translation by the spacetime vector a^μ in Eq. (65) is useful to derive expressions for all space-time gradients of coarse-graining cumulants in terms of increments, by differentiating with respect to a^μ and then setting $a^\mu = 0$. For example, for $p = 1$, one obtains with $\bar{f} = \tau(f)$ that

$$\partial_\mu \bar{f}(x) = -\frac{1}{\ell}\int d^D r(\partial_\mu \mathcal{G})_\ell(r)\delta f(x;r) \quad (68)$$

and analogous expressions for all $p > 1$ and all orders of derivatives (Ref. [82], Appendix B). Expanding the Favre-average cumulants into cumulants of the unweighted coarse graining, one thus obtains expressions for all of the contributions to the energy flux in terms of space-time increments of the thermodynamic fields.

From these expressions in terms of space-time increments, we can derive necessary conditions for turbulent energy-dissipation anomalies. Let us define scaling exponents of space-time structure functions by

$$\zeta^f_q := \liminf_{|r|_E\to 0} \frac{\log\|\delta f(r)\|^q_q}{\log|r|_E}, \quad (69)$$

where $\|\delta f(r)\|_q$ is the space-time L_q norm of the increment and $S^f_q(r) = \|\delta f(r)\|^q_q$ is thus the qth-order (absolute) structure function of f. From the expressions in Eqs. (65) and (68), one can see that the baropycnal work term in Eq. (61) vanishes as $\ell \to 0$, unless for every $q \geq 3$

$$\zeta^p_q + \zeta^h_q + \zeta^v_q \leq q. \quad (70)$$

Likewise, the deformation work and the third term in Eq. (61) vanish as $\ell \to 0$ unless for every $q \geq 3$ either

$$\zeta^h_q + 2\zeta^v_q \leq q \quad (71)$$

or

$$3\zeta^v_q \leq q. \quad (72)$$

The arguments here closely parallel those in paper I for the nonrelativistic case (see Ref. [88]). In deriving these results, we have assumed that the enthalpy h is bounded away from both zero and infinity. The inequalities (70)–(72) demonstrate that singularities of the fluid variables ϵ, ρ, and V^μ are required in the ideal limit $\sigma, \kappa, \eta, \zeta \to 0$ in order to obtain a nonvanishing energy-dissipation anomaly from turbulent cascade. This is a scale-local cascade process as long as all of the structure-function exponents satisfy $0 < \zeta^f_q < q$ for $f = p,\ h,\ v$ [90].

The internal-energy balance (63) of limiting Euler solutions can also be obtained from the fine-grained internal-energy balance (9) of the dissipative fluid model by directly taking the limit $\sigma, \kappa, \eta, \zeta \to 0$. In particle-frame fluid models, the dissipative heat current contribution $\partial_\mu(\kappa\hat{Q}^\mu)$ can be shown to vanish by arguments similar to those applied to the dissipative terms in the coarse-grained conservation laws. The details are presented in Appendix D. Defining $\mathcal{Q}_{\rm diss}$ as in Eq. (59) and also defining

$$p*\theta := \mathcal{D} - \lim_{\sigma,\kappa,\eta,\zeta\to 0} p\cdot\theta, \quad (73)$$

we then obtain the distributional balance equation

$$\partial_\mu(\epsilon V^\mu) = -p*\theta + \mathcal{Q}_{\rm diss}. \quad (74)$$

As in the nonrelativistic case discussed in paper I, one must expect that the limit $p*\theta$ in Eq. (73) is generally distinct from $p\circ\theta$ in Eq. (62); that is, the double limits of $\bar{p}_\ell\bar{\theta}_\ell$ for η, ζ, κ, $\sigma \to 0$ and for $\ell \to 0$ do not commute. In fact, the quantities $p*\theta$ and $\mathcal{Q}_{\rm diss}$ are presumably not completely universal and may depend upon the particular sequence η_k, ζ_k, κ_k, $\sigma_k \to 0$ used to reach infinite Reynolds and Péclet numbers. This is known to be true in the nonrelativistic limit, as verified in paper I. However, it is a consequence of Eq. (74) that the particular combination $-p*\theta + \mathcal{Q}_{\rm diss}$ depends only upon the limiting weak solution and not upon the particular sequence of transport coefficients used to obtain it.

A comparison of Eqs. (63) and (74) shows that the two balance equations can be simultaneously valid only if $-p\circ\theta + \mathcal{Q}_{\rm flux} = -p*\theta + \mathcal{Q}_{\rm diss}$. In that case, by introducing the relativistic pressure-work defect

$$\tau(p,\theta) := p*\theta - p\circ\theta, \quad (75)$$

we can then rewrite the inertial-range balance (63) as

$$\partial_\mu(\epsilon V^\mu) = -p*\theta + \mathcal{Q}_{\rm inert}, \quad (76)$$

where the total inertial energy dissipation is defined by

$$\mathcal{Q}_{\mathrm{inert}} \coloneqq \tau(p,\theta) + \mathcal{Q}_{\mathrm{flux}} = \mathcal{Q}_{\mathrm{diss}}. \tag{77}$$

As in the nonrelativistic case considered in paper I, the inertial-range energy dissipation can arise not only from energy cascade but also from the pressure-work defect. Relativistic shock solutions provide explicit examples with $\tau(p,\theta)\neq 0$ (Appendix E). Unlike the nonrelativistic case, it is not known rigorously that $\mathcal{Q}_{\mathrm{diss}}\geq 0$.

Another important distinction of the relativistic situation is that neither the energy flux $\mathcal{Q}_\ell^{\mathrm{flux}}$ nor the pressure work $\bar{p}_\ell\bar{\theta}_\ell$ at finite ℓ are Lorentz-invariant scalars, whereas the corresponding quantities are Galilei invariant in nonrelativistic compressible turbulence. Although $\bar{p}_\ell\bar{\theta}_\ell$ and the expression (61) for $\mathcal{Q}_\ell^{\mathrm{flux}}$ appear to define invariant scalars, they involve the kernel $\mathcal{G}_\ell(r)$, which is not frame invariant. Thus, the coarse-graining regularization breaks Lorentz symmetry, somewhat similar to lattice regularizations in relativistic quantum field theory with finite lattice constant a. In contrast, the fine-grained dissipation $\mathcal{Q}_{\mathrm{diss}}^{\eta,\zeta,\kappa,\sigma}$ and the fine-grained pressure work $p\cdot\theta$ are both Lorentz scalars, and thus their ideal limits $\mathcal{Q}_{\mathrm{diss}}$ and $p*\theta$ as $\eta,\zeta,\kappa,\sigma\to 0$ must be invariant as well. It may appear somewhat unsatisfactory that the energy flux $\mathcal{Q}_\ell^{\mathrm{flux}}$ and the resolved pressure work $\bar{p}_\ell\bar{\theta}_\ell$ at finite ℓ are observer dependent. However, Lorentz invariance is restored in the $\ell\to 0$ limit, as easily proved for the combinations $-p\circ\theta+\mathcal{Q}_{\mathrm{flux}}$ and, in particular, $\mathcal{Q}_{\mathrm{inert}}=\tau(p,\theta)+\mathcal{Q}_{\mathrm{flux}}$. The invariance of $-p\circ\theta+\mathcal{Q}_{\mathrm{flux}}$ can be seen from its equality with both $\partial_\mu(\epsilon V^\mu)$ and $-p*\theta+\mathcal{Q}_{\mathrm{diss}}$, which are Lorentz scalars. Likewise, $\mathcal{Q}_{\mathrm{inert}}=\mathcal{Q}_{\mathrm{diss}}$, which is an invariant scalar. It is reassuring that the net inertial-range dissipation is observer independent for the limit $\ell\to 0$.

This invariance must also hold, within some limits, for ℓ finite but very small, at large Reynolds and Péclet numbers. The reason is that the only effect of a change of inertial frame is to change the filter kernel from $\mathcal{G}_\ell$ to $\mathcal{G}'_\ell$ as in Eq. (24), but the $\ell\to 0$ limits of $\bar{p}_\ell\bar{\theta}_\ell$ and $\mathcal{Q}_\ell^{\mathrm{flux}}$ as distributions, when they exist at all, must be independent of the specific filter kernel adopted [89]. This argument implies that the two distributions $p\circ\theta$ and $\mathcal{Q}_{\mathrm{flux}}$ are, in fact, Lorentz-invariant scalars separately and not only in combination [91]. For sufficiently small ℓ inside a long inertial range at large Re and Pe, this invariance of the $\ell\to 0$ limiting distributions must hold approximately. On the other hand, some observer dependence presumably arises for ℓ small but nonzero. For example, two observers moving at sufficiently high relative velocities may disagree about the negligibility of the microscopic dissipation for the same coarse-grained fields. For one observer, $\mathcal{Q}_\ell^{\mathrm{inert}}$ may account for all of the dissipation of resolved fields, while for the other, the combination $\mathcal{Q}_\ell^{\mathrm{inert}\prime}+\underline{\mathcal{Q}}_{\mathrm{diss}}^{\eta,\zeta,\kappa,\sigma\prime}$ is necessary to account for all of the dissipation in resolved fields, where $\underline{\mathcal{Q}}_{\mathrm{diss}}^{\eta,\zeta,\kappa,\sigma}$ is the resolved viscous and conductive dissipation [92]. The observed flux contributions will then be distinct.

In this section, we have focused on the large-scale/resolved internal-energy balance, but there is also a complementary budget for the unresolved/subscale energy current. In the case of an unweighted space-time coarse graining, the unresolved current can be naturally defined by $\underline{K}^\mu \coloneqq -\bar{\tau}(T^{\mu\nu},V_\nu)$ so that its sum with the resolved current $\underline{\mathcal{E}}^\mu=-\bar{T}^{\mu\nu}\bar{V}_\nu$ accounts for the total energy current. Likewise, within the Favre-average coarse-graining approach, the subscale internal-energy current can be defined as $\underset{\sim}{K}^\mu \coloneqq \bar{\mathcal{E}}^\mu-\underset{\sim}{\mathcal{E}}^\mu$, which, with Eq. (54), gives

$$\underset{\sim}{K}^\mu = -\bar{\tau}(V_\nu,T^{\mu\nu}) + \widetilde{V^\mu V^\nu}\,\bar{\tau}(h,V_\nu) + \frac{1}{2}\tilde{\tau}(V_\nu,V^\nu)\overline{hV^\mu}. \tag{78}$$

From the separate balance equations for $\bar{\mathcal{E}}^\mu$ and $\underset{\sim}{\mathcal{E}}^\mu$, it easily follows that

$$\partial_\mu \underset{\sim}{K}^\mu = -\bar{\tau}(p,\theta) + \bar{\mathcal{Q}}_{\mathrm{diss}} - \mathcal{Q}_{\mathrm{flux},\ell}. \tag{79}$$

The source term on the right-hand side is the difference between the true dissipative heating $\bar{\mathcal{Q}}_{\mathrm{diss}}$ and the "apparent dissipation" $\mathcal{Q}_{\mathrm{flux},\ell}$ based on measurements at scales $>\ell$, together with the pressure-work defect $\bar{\tau}(p,\theta)$ which represents the difference between the true pressure work $\overline{p*\theta}$ and the apparent pressure work $\bar{p}\cdot\bar{\theta}$ based on fields resolved also down to scales ℓ. Using the expression (57) for $\underset{\sim}{\mathcal{E}}^\mu$, the subscale internal-energy current can be rewritten in terms of relativistic Favre-average cumulants of the velocity. In particular, its negative becomes

$$-\underset{\sim}{K}^\mu = \bar{h}\left(\frac{1}{2}\tilde{\tau}(V_\nu,V^\nu)\tilde{V}^\mu + \tilde{\tau}(V_\nu,V^\mu)\tilde{V}^\nu + \tilde{\tau}(V_\nu,V^\nu,V^\mu)\right) + \bar{\tau}(p,V^\mu). \tag{80}$$

Substituting this expression into Eq. (79) yields a balance equation very similar in form to the nonrelativistic subscale kinetic-energy balance obtained in Eq. (paper I;66) and, in fact, formally reducing to the latter in the limit $c\to\infty$ (Appendix B). This identity will prove very important for the discussion in the following section.

Finally, note the remarkable fact that our theoretical analysis of relativistic energy cascade has entirely bypassed kinetic energy, which plays a central role in nonrelativistic turbulence theory. In fact, there does not seem to be a generally accepted definition of kinetic energy for a relativistic fluid. Any reasonable definition must be frame dependent since the standard kinetic energy of a nonrelativistic fluid itself changes under Galilean transformations. This would make relativistic kinetic energy a poor choice as a basis for a theory of turbulence, as all "cascade terms" would become frame dependent and Lorentz invariance would not be restored even in the limit $\ell \to 0$. In some applications, however, it may be useful to refer to a kinetic energy and its cascade, at least in a specific reference frame. For example, in gamma-ray bursts, one expects that the motion or kinetic energy of the jet will be dissipated by turbulent cascade into internal energy (and, ultimately, radiation). We show in Appendix C that our approach predicts such a kinetic-energy cascade. To obtain this result, we define the relativistic "kinetic-energy current" $\mathcal{T}^\mu$ so that $T^{0\mu} = \mathcal{T}^\mu + \mathcal{E}^\mu$ holds, i.e., so that the total energy current $T^{0\mu}$ in the fixed frame is decomposed as a sum of kinetic- and (total) internal-energy current. With this definition, we show that the cascade of relativistic internal energy as the primary process leads, as a secondary phenomenon, to the cascade of relativistic kinetic energy. This is formally similar to nonrelativistic turbulence but where the cascade of kinetic energy is instead the primary process and the cascade of "intrinsic resolved internal energy" is secondary [see Eq. (paper I;63)]. To properly motivate our definition of kinetic-energy current, we must first consider the nonrelativistic limit in Appendix B. Thus, the discussion of relativistic kinetic-energy cascade, although important in applications, is best delayed to Appendix C.

V. ENTROPY CASCADE

Hydrodynamic turbulence, like any other macroscopic irreversible process, must be consistent with the second law of thermodynamics. In the relativistic case, in particular, positive entropy production is a primary constraint on dissipative fluid models [8,10,24–26]. For nonrelativistic compressible turbulence, we have argued in paper I that there is a cascade of (neg)entropy, which is in addition to energy cascade and which is even more fundamental. All of these arguments carry over to relativistic fluid turbulence. The resolved pressure work in the balance equations (51) or (60) for the large-scale internal-energy current is a space-time structured source of internal energy. In relativistic thermodynamics, as in the nonrelativistic case, the entropy per volume $s(\epsilon,\rho)$ is a concave function of ϵ and ρ, so the creation of large-scale structure in ϵ corresponds to a decrease of entropy at large scales. To balance this destruction, one can then expect that there will be an inverse cascade of the entropy, which is injected by microscopic dissipation/entropy production. As in the nonrelativistic case, we may define a "resolved entropy"

$$\underline{s} := s(\bar{\epsilon},\bar{\rho}) \tag{81}$$

and an "unresolved/subscale entropy"

$$\triangle s := \overline{s(\epsilon,\rho)} - s(\bar{\epsilon},\bar{\rho}) \le 0, \tag{82}$$

whose nonpositivity follows from the concavity of the entropy. It is somewhat more natural to consider the negentropy or information density $\imath(\epsilon,\rho) = -s(\epsilon,\rho)$, which is convex and whose unresolved/subscale contribution $\Delta\imath = -\triangle s$ is non-negative. In this equivalent picture, the pressure work injects negentropy at large scales, which should cascade forward to small scales where it can be efficiently destroyed by dissipative transport. In order to formalize such notions, one must derive a balance equation for the large-scale entropy.

This balance is straightforward to derive after taking the limit $\eta,\zeta,\sigma,\kappa \to 0$ for fixed positive ℓ. Using the first law of thermodynamics $ds = \beta d\epsilon - \lambda dn$ and $\bar{\mathcal{D}} = \bar{V}_\mu \partial^\mu$, one gets

$$\bar{\mathcal{D}}\,\underline{s} = \underline{\beta}\,\bar{\mathcal{D}}\,\bar{\epsilon} - \underline{\lambda}\,\bar{\mathcal{D}}\,\bar{n}, \tag{83}$$

where we employ the notation $\underline{\phi} = \phi(\bar{\epsilon},\bar{\rho})$ for arbitrary smooth functions ϕ of ϵ, ρ. The equations

$$\bar{\mathcal{D}}\,\bar{n} = -\bar{n}\,\bar{\theta} - \partial_\mu \bar{\tau}(n, V^\mu), \tag{84}$$

$$\bar{\mathcal{D}}\,\bar{\epsilon} = -\bar{\epsilon}\,\bar{\theta} - \overline{p * \theta} - \partial_\mu \bar{\tau}(\epsilon, V^\mu) + \bar{\mathcal{Q}}_{\rm diss} \tag{85}$$

are direct consequences of Eqs. (46) and (74). Using the Gibbs homogeneous relation $(\epsilon+p)/T = s + \lambda n$, one obtains, after some straightforward calculations, a balance equation of the following form:

$$\partial_\mu \underline{S}^\mu = \Sigma_\ell^{\rm inert}. \tag{86}$$

The vector whose divergence appears on the left,

$$\underline{S}^\mu := \underline{s}\bar{V}^\mu + \underline{\beta}\,\bar{\tau}(\epsilon, V^\mu) - \underline{\lambda}\,\bar{\tau}(n, V^\mu), \tag{87}$$

is a natural expression for the resolved entropy current, with $\underline{s}\bar{V}^\mu$ describing the entropy transport by large-scale advection, $\underline{\beta}\,\bar{\tau}(\epsilon, V^\mu)$ the entropy transport due to subscale internal-energy current, and $\underline{\lambda}\,\bar{\tau}(n, V^\mu)$ the entropy transport due to subscale number current. It should be noted that entropy current due to such turbulent subscale transport will not generally be orthogonal to $\bar{V}^\mu$ in the Minkowski pseudometric and thus not purely spatial in the rest frame of the coarse-grained fluid velocity.

The source on the right-hand side of Eq. (86) is the inertial-range entropy production

$$\Sigma_\ell^{\rm inert} = -I_\ell^{\rm mech} + \underline{\beta}\bar{\mathcal{Q}}^{\rm diss} + \Sigma_\ell^{\rm flux}, \tag{88}$$

where anomalous input of negentropy from pressure work is defined by

$$I_\ell^{\rm mech} := \underline{\beta}(\overline{p * \theta} - \underline{p}\,\bar{\theta}) \tag{89}$$

and (forward) negentropy flux is defined by

$$\Sigma_\ell^{\rm flux} := (\partial_\mu \underline{\beta})\bar{\tau}(\epsilon, V^\mu) - (\partial_\mu \underline{\lambda})\bar{\tau}(n, V^\mu). \tag{90}$$

The latter expression is also natural, as it represents entropy production due to subscale transport of internal energy and particle number acting against large-scale gradients of the (entropically) conjugate thermodynamic potentials. In particular, $\Sigma_\ell^{\rm flux} > 0$ when the subscale transport vectors are "downgradient" or opposite to the gradients of $\underline{T}$ and $\underline{\lambda}$. Finally, note that one can further decompose the anomalous negentropy input as

$$I_\ell^{\rm mech} = \underline{\beta}\,\bar{\tau}(p, \theta) + I_\ell^{\rm flux}, \tag{91}$$

where the first term is the contribution from the pressure-dilatation defect and the second term,

$$I_\ell^{\rm flux} := \underline{\beta}(\bar{p} - \underline{p})\bar{\theta}, \tag{92}$$

is fluxlike, representing work of subscale pressure fluctuations against large-scale dilatation. These expressions are exactly analogous to those derived in Sec. VI of paper I for the turbulent entropy balance of nonrelativistic compressible fluid flows. In fact, as we show in Appendix B, the formal limit $c \to \infty$ recovers the previously derived nonrelativistic expressions.

Now consider the case where there is a nonvanishing entropy production anomaly as in Eq. (22). If such an anomaly exists, it cannot depend upon the arbitrary coarse-graining scale ℓ. Thus, for ideal turbulence at infinite Reynolds and Péclet numbers, we may consider the subsequent limit $\ell \to 0$ of the inertial-range entropy balance, with the coarse-graining regularization removed. This yields a fine-grained entropy balance for the relevant weak solutions of the relativistic Euler equations:

$$\partial_\mu (sV^\mu) = \Sigma_{\rm inert} \tag{93}$$

Because all coarse-graining cumulants vanish distributionally as $\ell \to 0$, the resolved entropy current in Eq. (87) must converge in the sense of distributions to sV^μ under relatively mild assumptions (e.g., when ϵ and ρ are bounded in space-time). The limit $\Sigma_{\rm inert}$ of the source (88) is $\Sigma_{\rm inert} = -I_{\rm mech} + \Sigma_{\rm flux} + \beta \circ \mathcal{Q}_{\rm diss}$, where $I_{\rm mech} = I_{\rm flux} + \beta \circ \tau(p, \theta)$ with

$$\beta \circ \tau(p, \theta) := \mathcal{D}\text{-}\lim_{\ell \to 0} \underline{\beta}\,\bar{\tau}(p, \theta) \tag{94}$$

and where

$$\beta \circ \mathcal{Q}_{\rm diss} := \mathcal{D}\text{-}\lim_{\ell \to 0} \underline{\beta}\,\bar{\mathcal{Q}}_{\rm diss} \tag{95}$$

The limit source need not vanish. Although entropy is conserved for smooth solutions of relativistic Euler equations, there may be anomalous entropy production for weak solutions. Relativistic shock solutions with discontinuities in the fluid variables are, of course, a well-known example of such dissipative weak solutions (Appendix E). We see below, however, that even continuous solutions may exhibit anomalous entropy production.

Precisely the same balance equation can be obtained by taking the limit $\eta, \zeta, \sigma, \kappa \to 0$ of the fine-grained entropy balance (6) for the dissipative fluid model. The limit of the dissipative entropy production is, of course, obtained directly from our fundamental hypothesis (22). The fine-grained entropy current for the dissipative fluid model also converges to sV^μ in the limit $\eta, \zeta, \sigma, \kappa \to 0$. This can be verified without great difficulty for models of the Israel-Stewart class. Recall that, in such models, the entropy current does not have the naive form (17) that it assumes in the Eckart-Landau-Lifshitz models, but it is instead modified as in Eq. (18) by terms that are second order in gradients. Taking the latter energy-frame expression as a concrete example, we factor out the dependence upon the transport coefficients η, ζ, σ and introduce the rescaled variables $\hat{\tau}^{\mu\nu}$, $\hat{\tau}$, $\hat{N}^\mu$. This yields the representation

$$S^\mu = sV^\mu - \sigma\lambda\hat{N}^\mu - \frac{1}{2}(\zeta\beta_0\Sigma_\zeta + \sigma\beta_1\Sigma_\sigma + 2\eta\beta_2\Sigma_\eta)V^\mu + \zeta\sigma\frac{\alpha_0}{T}\hat{\tau}\hat{N}^\mu + \eta\sigma\frac{\alpha_1}{T}\hat{\tau}^{\mu\nu}\hat{N}_\nu. \tag{96}$$

Here, we have denoted as Σ_ζ, Σ_σ, Σ_η the three terms in the fine-grained entropy production (6) that are proportional to ζ, σ, η, respectively. According to our fundamental hypothesis (22), these converge to positive distributions $\Sigma_{\rm bulk}$, $\Sigma_{\rm cond}$, $\Sigma_{\rm shear}$ in the limit η, ζ, $\sigma \to 0$. Because of the remaining factors of ζ, σ, η appearing in Eq. (96), however, one should expect that the β terms will all vanish in that limit. Likewise, the α terms should vanish because they are quadratic in the transport coefficients ζ, σ, η. These arguments are not rigorous because the factors involving ϵ, ρ, V^μ in those terms do not remain smooth in the limit. Nevertheless, it is possible to show, by simple inequalities, that these terms do vanish in the sense of distributions and, thus, $\mathcal{D}\text{-}\lim_{\eta,\zeta,\sigma,\kappa\to 0} S^\mu_{\eta,\zeta,\sigma,\kappa} = sV^\mu$. For details, see Appendix D. One thus finally obtains the entropy balance for the limiting Euler solution

$$\partial_\mu (sV^\mu) = \Sigma_{\rm diss}, \tag{97}$$

with $\Sigma_{\rm diss} > 0$ given by the limit in Eq. (22). The equality

$$\Sigma_{\rm inert} = \Sigma_{\rm diss} \tag{98}$$

is demanded by consistency with the inertial-range limiting balance (93), just as in the nonrelativistic theory.

Given that anomalous entropy production is possible for weak solutions, what degree of singularity of the fluid variables is required for a nonvanishing anomaly? To answer this question, we can prove an Onsager-type singularity theorem, which gives necessary conditions for an anomaly. The basic idea is the same as in the nonrelativistic case [48] and is easy to explain. We first rewrite the resolved entropy balance (86) as

$$\partial_\mu \underline{S}^\mu = \underline{\beta}(\bar{\mathcal{Q}}_{\rm diss} - \bar{\tau}(p,\theta)) - I_\ell^{\rm flux} + \Sigma_\ell^{\rm flux}. \tag{99}$$

The flux terms $I_\ell^{\rm flux}$ and $\Sigma_\ell^{\rm flux}$ may be readily expressed in terms of space-time increments of the fluid variables, using the cumulant-expansion methods described in Sec. IV. The term that is difficult to estimate directly is the one involving $\bar{\mathcal{Q}}_{\rm diss} - \bar{\tau}(p,\theta)$. Note that these two quantities separately may be nonuniversal and may depend upon the particular sequence ζ_k, σ_k, $\eta_k \to 0$ used to obtain the limiting Euler solution. Fortunately, exactly the same combination appears in the balance equation (79) for the subscale internal-energy current $\underset{\sim}{K}^\mu$. Thus, one can define an intrinsic resolved entropy current in the Favre-averaging approach as

$$\begin{aligned}\underset{\sim}{\mathcal{S}}^{*\mu} &\coloneqq \underline{S}^\mu - \underline{\beta}\,\underset{\sim}{K}^\mu \\ &= \underline{s}\bar{V}^\mu + \underline{\beta}\,\bar{\tau}(h,V^\mu) - \underline{\lambda}\,\bar{\tau}(n,V^\mu) \\ &\quad + \underline{\beta}\,\bar{h}\left(\frac{1}{2}\tilde{\tau}(V_\nu,V^\nu)\tilde{V}^\mu + \tilde{\tau}(V_\nu,V^\mu)\tilde{V}^\nu \right.\\ &\quad \left. + \tilde{\tau}(V_\nu,V^\nu,V^\mu)\right),\end{aligned} \tag{100}$$

where the second equality uses Eq. (80). It follows from the two balance equations (79) and (99) that this intrinsic entropy current satisfies the following balance:

$$\partial_\mu \underset{\sim}{\mathcal{S}}^{*\mu} = \Sigma_\ell^{\rm inert*}, \tag{101}$$

where net inertial-range entropy production is defined by

$$\Sigma_\ell^{\rm inert*} = -I_\ell^{\rm flux} + \Sigma_\ell^{\rm flux*}, \tag{102}$$

with the intrinsic negentropy flux

$$\begin{aligned}\Sigma_\ell^{\rm flux*} &\coloneqq \Sigma_\ell^{\rm flux} - (\partial_\mu\underline{\beta})\underset{\sim}{K}^\mu + \underline{\beta}\mathcal{Q}_\ell^{\rm flux} \\ &= (\partial_\mu\underline{\beta})\bar{\tau}(h,V^\mu) - (\partial_\mu\underline{\lambda})\bar{\tau}(n,V^\mu) + \underline{\beta}\mathcal{Q}_\ell^{\rm flux} \\ &\quad + \bar{h}(\partial_\mu\underline{\beta})\left(\frac{1}{2}\tilde{\tau}(V_\nu,V^\nu)\tilde{V}^\mu + \tilde{\tau}(V_\nu,V^\mu)\tilde{V}^\nu \right.\\ &\quad \left. + \tilde{\tau}(V_\nu,V^\nu,V^\mu)\right).\end{aligned} \tag{103}$$

Just as for the naive version of the resolved entropy current, $\mathcal{D}\text{-}\lim_{\ell\to 0}\underset{\sim}{\mathcal{S}}^{*\mu} = sV^\mu$, since all of the additional cumulant terms vanish in the limit. Furthermore, and crucially, all source terms on the right-hand side of Eq. (101) are fluxlike and are products of subscale cumulant terms and gradients of resolved fields, which allows us to express them in terms of space-time increments. There is a rough analogy of our entropy current modification with the Israel-Stewart correction, in that our current modification is a higher-order moment of the coarse-graining average: Whereas the naive entropy current in the second line of Eq. (100) involves, at most, second-order moments of ϵ, n, and V^μ, the correction on the third and fourth lines involves third-order moments. Note, however, that our correction term does not have to be small relative to the naive term.

A fundamental observation is that all individual terms in the intrinsic entropy balance (101) depend only upon the limiting Euler solution and not on the sequence used to obtain it. In fact, the same equation can be obtained from the distributional Euler solution directly, without considering the underlying microscopic model (dissipative fluid dynamics, kinetic equation, quantum field theory, etc.). To see this, one can use the homogeneous Gibbs relation $\underline{s} = \underline{\beta}(\bar{\epsilon} + \underline{p}) - \underline{\lambda}\,\bar{n}$ and the definition $\underset{\sim}{K}^\mu = \bar{\mathcal{E}}^\mu - \underset{\sim}{\mathcal{E}}^\mu$ to rewrite the intrinsic entropy current as

$$\underset{\sim}{\mathcal{S}}^{*\mu} = \underline{\beta}\,\underset{\sim}{\mathcal{E}}^\mu + \underline{\beta}\,\underline{p}\,\bar{V}^\mu - \underline{\lambda}\bar{N}^\mu. \tag{104}$$

One can then derive the intrinsic entropy balance (101) directly from the inertial-range balance equation (60) for $\underset{\sim}{\mathcal{E}}^\mu$, the particle conservation equation $\partial_\mu\bar{N}^\mu = 0$, and thermodynamic relation $\partial_\mu(\underline{\beta}\,\underline{p}) = \bar{n}(\partial_\mu\underline{\lambda}) - \bar{\epsilon}(\partial_\mu\underline{\beta})$. This crucial observation implies that our results for anomalous entropy production are universal and apply to all distributional solutions of the relativistic Euler equations, not only those obtained as ideal limits of Israel-Stewart-type dissipative fluid models.

The necessary conditions for anomalous entropy production follow directly from the intrinsic entropy balance (101), exactly as for the nonrelativistic case considered in Ref. [48]. The conclusion is that the entropy anomaly can be nonzero only if, for every $q \geq 3$, at least one of the following three conditions is satisfied by the structure-function scaling exponents defined in Eq. (69):

$$2\min\{\zeta_q^\epsilon, \zeta_q^\rho\} + \zeta_q^v \leq q, \tag{105}$$

$$\min\{\zeta_q^\epsilon, \zeta_q^\rho\} + 2\zeta_q^v \leq q, \tag{106}$$

$$3\zeta_q^v \leq q. \tag{107}$$

The first inequality (105) is implied by (and thus replaces) the inequality (70), which was shown earlier to be necessary for nonvanishing of the baropycnal work as $\ell \to 0$, while the inequalities (106) and (107) replace (71) and (72), which were shown to be necessary for nonvanishing of the other two contributions to energy flux. The above inequalities would be equalities for a K41-dimensional scaling determined by mean energy flux, and the departure from the upper bound is a measure of the space-time intermittency of the solution fields [23,93]. These upper bounds, even if they hold as equalities, imply that ϵ, ρ, V^μ must be nonsmooth or singular in spacetime for the ideal limit. Roughly speaking, limit solutions with anomalous entropy production can have, at most, $1/3$ of a derivative in a space-time L^q sense.

For nonrelativistic fluids, the conditions analogous to Eqs. (105)–(107) are known to be necessary also for an energy-dissipation anomaly [48]. While $\mathcal{Q}_{\rm flux} = 0$ if none of those conditions holds, it is, in principle, still possible that $\tau(p,\theta) = \mathcal{Q}_{\rm diss} > 0$. When the balance equation (60) for resolved internal energy is rewritten as

$$\partial_\mu \underset{\sim}{\mathcal{E}}^\mu = -\overline{p * \theta} + \mathcal{Q}_\ell^{\rm inert} \tag{108}$$

with

$$\mathcal{Q}_\ell^{\rm inert} := \bar{\tau}(p,\theta) + \mathcal{Q}_\ell^{\rm flux}, \tag{109}$$

then it differs strikingly from the balance equation (101) for intrinsic resolved entropy because the terms $\bar{\tau}(p,\theta)$ and $\mathcal{Q}_\ell^{\rm inert}$ are not determined uniquely as $\ell \to 0$ by the limiting weak Euler solution. Those terms, in fact, generally depend upon the underlying dissipative fluid model sequence, as seen, for example, for the nonrelativistic limit of shock solutions where a Prandtl-number dependence remains. In Ref. [48], the vanishing energy-dissipation anomaly is instead derived from the vanishing entropy production anomaly. That proof carries over to relativistic fluids whenever the dissipative fluid model satisfies the bounds

$$\Sigma_{\rm diss}^{\zeta,\eta,\kappa,\sigma} \geq \mathcal{Q}_{\rm diss}^{\zeta,\eta,\kappa,\sigma}/T \geq 0. \tag{110}$$

Amusingly, the only dissipative relativistic model in the class that we consider that guarantees Eq. (110) is the classical energy-frame Landau-Lifshitz theory [94], which is formally ill posed and acausal. The result will be true, nevertheless, if the viscous transport fields τ, $\tau^{\mu\nu}$ in the relativistic fluid model are sufficiently well approximated by the constitutive relations of the Landau-Lifshitz theory. Such results have been proved [72,73] but need to be extended to solutions with shocks or other milder turbulent singularities to show that conditions (105)–(107) are necessary for anomalous energy dissipation.

VI. RELATIONS TO OTHER APPROACHES

We now briefly discuss the relation of our analysis with other approaches to relativistic fluid turbulence that have been proposed in the literature.

A. Barotropic fluid models

In paper I, we have criticized nonrelativistic barotropic models as being physically inapplicable to fluid turbulence since this is a strongly dissipative process. The same criticisms carry over to relativistic barotropic models if those are defined as in Refs. [56,95], for example. These authors take $\epsilon = \epsilon(\rho)$ as the condition for barotropicity, which implies that $p = p(\epsilon,\rho) = p(\rho)$. As in the nonrelativistic case, the internal energy per rest mass $e = u/\rho$ can be obtained from the integral

$$e = \int \frac{p d\rho}{\rho^2} \tag{111}$$

if and only if the fluid is isentropic with entropy per mass $s_m = s/\rho$ constant in space-time (see Ref. [57], Sec. 2.4.10). This is inconsistent with the irreversible production of entropy by turbulence unless the fluid is somehow strongly coupled to another physical field which very efficiently carries off the generated entropy. Furthermore, one obtains from Eq. (111) and $\partial_\mu J^\mu = 0$ that

$$\partial_\mu (u V^\mu) = -p\theta, \tag{112}$$

which omits viscous heating, so the "heat reservoir" to which such a hypothetical fluid is coupled must also rapidly absorb this excess energy. Barotropic equations of state, together with formula (111) for internal energy, are thus physically inconsistent, as soon as one includes dissipative terms in J^μ and $T^{\mu\nu}$, and are unsuitable as models of turbulence in fluids that are isolated or only weakly coupled to additional fields. These remarks apply to the special case of polytropic equations of state with $p(\rho) = K\rho^\Gamma$ for exponent Γ, whenever the internal-energy density is determined from the relation $u = p/(\Gamma - 1)$, as is very standard in numerical simulations with relativistic polytropic models. Such models cannot correctly represent the time-irreversible physics of relativistic fluid turbulence, which is created by the spectrum of singularities that develop in the solutions. Note that barotropic fluid models in the sense of Refs. [56,95] are already known to be physically inadequate to describe the irreversible evolution of relativistic shocks (Ref. [57], Sec. 2.4.10).

These criticisms do not apply to relativistic barotropic equations of state if those are defined instead by the alternative condition $p = p(\epsilon)$, e.g., as in Ref. [57]. Note that such a formulation of barotropicity is more general because it makes sense even when the constituent particles of the fluid have zero rest mass and $\rho \equiv 0$. There is no physical inconsistency of such an equation of state with irreversible entropy production by microscopic dissipation. For example, ultrarelativistic fluids with vanishingly small coldness $mc^2/k_B T \ll 1$ (Ref. [57], Sec. 2.4.4) and models of hot, optically thick, radiation-pressure-dominated plasmas (Ref. [57], Sec. 2.4.8) both satisfy $p = \frac{1}{3}\epsilon$ and are thus barotropic in this second sense. Both of these models have a nonconstant thermodynamic entropy, which can be made consistent with the second law of thermodynamics by addition of suitable dissipative terms to the ideal fluid equations. More generally, conformally invariant fluid models that describe low-wave-number dynamics of conformal quantum-field theories [8] and nonconformal fluid models in the zero charge-density sector [26] satisfy both $p = p(\epsilon)$ and dissipative second-order hydrodynamical equations similar to the Israel-Stewart models consistent with the second law of thermodynamics. In fact, the exact shock solutions considered in Appendix E are for conformal fluids [21]. All of our conclusions apply to such models, with the simplification that hydrodynamics now reduces to the equation $\partial_\nu T^{\mu\nu} = 0$ for the stress-energy tensor alone.

B. Point splitting and statistical states

In paper I, we argued that point-splitting regularizations are inadequate for nonrelativistic compressible fluid turbulence, and the same arguments hold for relativistic fluid turbulence. Previously, Fouxon and Oz [19] used a point-splitting technique in the setting of an externally forced relativistic fluid satisfying

$$\partial_\nu T^{\mu\nu} = F^\mu \tag{113}$$

for a Minkowski force F^μ. Assuming that a statistically homogeneous and stationary state exists, those authors derived an exact statistical relation

$$\langle T_{0\mu}(\mathbf{0},t) T_{i\mu}(\mathbf{r},t)\rangle = \frac{1}{D}\mathcal{P}_\mu r_i \quad \text{(no sum on } \mu), \tag{114}$$

with $\langle\cdot\rangle$ denoting the ensemble average and with $\mathcal{P}_\mu = \langle T_{0\mu}(\mathbf{0},t)F_\mu(\mathbf{0},t)\rangle$ a "power input." In the formal nonrelativistic limit $c\to\infty$, this relation reduces, in conformal models with sound speed $c_s = c/\sqrt{d}$ [96], to the classical "$12/d(d+2)$th law" for d-dimensional incompressible fluid turbulence (e.g., Ref. [97]), but for finite speeds of light, the relation (114) has nothing to do with energy of the fluid. As noted earlier, Fouxon and Oz [19] made the following conclusion: "Our analysis indicates that the interpretation of the Kolmogorov relation for the incompressible turbulence in terms of the energy cascade may be misleading."

Needless to say, our analysis contradicts this conclusion. We have already discussed the limitations of point-splitting regularizations in paper I, and we shall not repeat that discussion here. We only point out that the anomalies obtained by the point-splitting arguments of Ref. [19] are for quantities such as $T^2_{0\mu}(x)$, which are not conserved quantities even for smooth solutions of relativistic Euler equations and which have no obvious physical significance. The specific quantities are chosen in Ref. [19] simply so that a point-splitting regularization applies. One cannot conclude that the energy cascade and energy-dissipation anomaly must be absent in relativistic turbulence because a certain regularization method is insufficient to derive them. The alternative coarse-graining regularization employed by us here shows that cascades and dissipative anomalies for both energy and entropy naturally arise in relativistic fluid turbulence. Furthermore, in conformal fluid models with $c_s = c/\sqrt{d}$, the relativistic energy flux $\mathcal{Q}^{\rm flux}_\ell$ considered by us reduces, in the nonrelativistic limit, to the standard kinetic-energy flux for an incompressible fluid,

$$\lim_{c\to\infty} c\,\mathcal{Q}^{\rm flux}_\ell = -\rho_0 \nabla\bar{\mathbf{v}}{:}\tau(\mathbf{v},\mathbf{v}), \tag{115}$$

with constant mass density ρ_0, following the arguments in Appendix B. Aluie (private communication) has shown that the standard 4/5th law of Kolmogorov, which is ordinarily derived by point splitting, can also be obtained from Eq. (115) for incompressible Navier-Stokes [98]. Thus, there is no unique way to extend the incompressible 4/5th law to relativistic turbulence, but our extension describes energy cascade in the relativistic regime.

To underscore this point, we briefly discuss here the energy balance for forced statistical steady states of relativistic fluid turbulence. This is a rather artificial setting, quite distinct from most real-world relativistic turbulence, e.g., in astrophysics, in which there is no local stirring and no ensembles. Therefore, in this paper, we focus on freely evolving turbulence and individual flow realizations. However, our considerations carry over directly to forced, steady-state ensembles. Note that the Minkowski force can quite generally be composed as

$$F^\mu = \frac{1}{c^2} h A^\mu_{\rm ext} - \frac{1}{c} Q_{\rm cool} V^\mu, \tag{116}$$

with $V_\mu A^\mu_{\rm ext} = 0$. Here, $A^\mu_{\rm ext}$ is an external acceleration field with units of $(\text{length})/(\text{time})^2$, and $Q_{\rm cool}$ is a cooling rate density with units of (energy)/(volume)(time). As usual, we include factors of c to facilitate discussion of the nonrelativistic limit. The internal-energy balance in the presence of a Minkowski force becomes

$$\partial_\mu \mathcal{E}^\mu = -p\theta + \mathcal{Q}_{\rm diss} - \frac{1}{c} Q_{\rm cool}. \tag{117}$$

It follows that, in a statistically homogeneous and stationary state, one has the fine-grained balance

$$\frac{1}{c}\langle Q_{\rm cool}\rangle = \langle \mathcal{Q}_{\rm trans}\rangle + \langle \mathcal{Q}_{\rm diss}\rangle, \tag{118}$$

where $\mathcal{Q}_{\rm trans} = -p\theta$ is the mechanical production of internal energy by pressure work. Our inertial-range internal-energy balance (60), with the addition of the Minkowski force, becomes

$$\partial_\mu \underset{\sim}{\mathcal{E}}{}^\mu_\ell = -\bar{p}_\ell \bar{\theta}_\ell + \mathcal{Q}^{\rm flux}_\ell - \tilde{V}_{\ell,\mu} \bar{F}^\mu_{{\rm ext},\ell}, \tag{119}$$

which now includes the coarse-graining length scale ℓ explicitly. One thus has

$$\langle \tilde{V}_{\ell,\mu}\bar{F}^{\mu}_{\mathrm{ext},\ell}\rangle = -\langle \bar{p}_\ell\bar{\theta}_\ell\rangle + \langle \mathcal{Q}^{\mathrm{flux}}_\ell\rangle = \langle \mathcal{Q}_{\mathrm{trans}}\rangle + \langle \mathcal{Q}^{\mathrm{inert}}_\ell\rangle, \tag{120}$$

where $\mathcal{Q}^{\mathrm{inert}}_\ell = \mathcal{Q}^{\mathrm{flux}}_\ell + \bar{\tau}_\ell(p,\theta)$ is the total inertial-range effective dissipation from both energy cascade and the pressure-work defect and $\mathcal{Q}_{\mathrm{trans}} = -p * \theta$. At length scales much smaller than the scale L of the Minkowski force, $\langle \tilde{V}_{\ell,\mu}\bar{F}^{\mu}_{\mathrm{ext},\ell}\rangle \simeq (1/c)\langle Q_{\mathrm{cool}}\rangle$, and

$$\langle \mathcal{Q}^{\mathrm{inert}}_\ell\rangle \simeq \frac{1}{c}\langle Q_{\mathrm{cool}}\rangle - \langle \mathcal{Q}_{\mathrm{trans}}\rangle = \langle \mathcal{Q}_{\mathrm{diss}}\rangle, \qquad \ell \ll L. \tag{121}$$

We thus find that the ideal dissipation rate has a constant ensemble average for scales ℓ in the inertial range, which equals the energy-dissipation rate of the microscopic fluid model. This is formally identical to the statistical energy-balance relation that we obtained in the nonrelativistic case, and it reduces to this relation in the limit $c \to \infty$.

It is more traditional to expect that the effective energy-dissipation rate at inertial-range lengths ℓ is set by the external input of kinetic energy by the large-scale forcing, but, of course, kinetic energy is not a natural relativistic quantity. Analogous constraints arise relativistically from the conditions

$$\langle F^\mu\rangle = 0, \tag{122}$$

which are necessary if a statistically homogeneous and stationary state is to exist for the forced fluid described by Eq. (113). The $\mu = 0$ condition gives

$$\langle Q_{\mathrm{cool}}\gamma\rangle = \frac{1}{c}\langle hA^0_{\mathrm{ext}}\rangle. \tag{123}$$

In the limit $c \to \infty$, this becomes

$$\langle Q_{\mathrm{cool}}\rangle \simeq c\langle \rho A^0_{\mathrm{ext}}\rangle = \langle \rho\mathbf{v}\cdot\mathbf{A}_{\mathrm{ext}}\rangle. \tag{124}$$

Here, we use the orthogonality condition $A^0_{\mathrm{ext}} = \mathbf{v}\cdot\mathbf{A}_{\mathrm{ext}}/c$. Since the equation of motion projected orthogonal to V^μ takes the form $\mathcal{D}V^\mu = (1/c^2)A^\mu_{\mathrm{ext}} + \cdots$ in the presence of a Minkowski force, the limit of the spatial components as $c \to \infty$ becomes $D\mathbf{v} = \mathbf{A}_{\mathrm{ext}} + \cdots$. Thus, Eq. (124) is equivalent to the usual nonrelativistic relation that $\langle Q_{\mathrm{cool}}\rangle = \langle Q_{\mathrm{in}}\rangle$, where $Q_{\mathrm{in}} = \rho\mathbf{v}\cdot\mathbf{A}_{\mathrm{ext}}$ is the kinetic-energy injection rate per volume by the external forcing. We note, in passing, that the constraints $\langle F^i\rangle = 0$ from the spatial components similarly reduce, in the nonrelativistic limit $c \to \infty$, to the condition $\langle \rho\mathbf{A}_{\mathrm{ext}}\rangle = 0$, or no net momentum injection by the external forcing.

In addition to energy balance, there must also be an entropy balance for homogeneous and stationary ensembles. In the presence of a Minkowski force, the fine-grained entropy balance (6) is found using Eq. (117) to be modified to

$$\partial_\mu S^\mu = \Sigma_{\mathrm{diss}} - \frac{1}{c}\beta Q_{\mathrm{cool}}. \tag{125}$$

Thus, for a homogeneous and stationary ensemble,

$$\langle \Sigma_{\mathrm{diss}}\rangle = \frac{1}{c}\langle \beta Q_{\mathrm{cool}}\rangle, \tag{126}$$

and microscopic entropy production is balanced by entropy removal by cooling. The inertial-range entropy balance (101) is likewise modified by a Minkowski force, with the divergence of Eq. (104) using Eq. (119) given by

$$\partial_\mu \underset{\sim}{S}^{*\mu} = \Sigma^{\mathrm{inert}*}_\ell - \underline{\beta}\tilde{V}_\mu\bar{F}^\mu. \tag{127}$$

When the Minkowski force is supported mainly at the large scale L, one obtains the inertial-range mean balance

$$\langle \Sigma^{\mathrm{inert}*}_\ell\rangle = \langle \underline{\beta}\tilde{V}_\mu\bar{F}^\mu\rangle \simeq \frac{1}{c}\langle \beta Q_{\mathrm{cool}}\rangle, \qquad \ell \ll L. \tag{128}$$

This mean entropy balance is formally the same as Eq. (paper I;107) for the nonrelativistic case and reduces to it in the limit $c \to \infty$. The physical picture is also the same as for nonrelativistic compressible turbulence, with entropy produced at small scales inverse cascading through the inertial range up to scales $\ell \simeq L$, where external cooling can remove the excess entropy. Equivalently (and perhaps more naturally), the negentropy injected by a large-scale cooling will forward cascade to small scales where irreversible microscopic transport can destroy it. If one makes the distinction in Eq. (102) between negentropy flux and anomalous negentropy input, then one can also write

$$\langle \Sigma^{\mathrm{flux}*}_\ell\rangle \simeq \frac{1}{c}\langle \beta Q_{\mathrm{cool}}\rangle + \langle I^{\mathrm{flux}}_\ell\rangle, \qquad \ell \ll L, \tag{129}$$

where the negentropy flux proper is equal, on average, to the total negentropy input at large scales, both from external cooling and from anomalous negentropy input.

C. Linear wave-mode decompositions

In paper I, we have also called into question the validity of representing turbulent solutions by decompositions into linear wave modes. This is a very popular approach in nonrelativistic plasma astrophysics and has recently been developed for Poynting-dominated relativistic MHD turbulence [99]. We do not consider charged plasmas in the present paper but only fluids of electrically neutral particles. Here, we just briefly discuss the issues with decompositions into linear wave modes. A basic problem is that thermodynamic relations such as $p = p(\epsilon,\rho)$ and $s = s(\epsilon,\rho)$ impose nonlinear constraints on solutions of the fluid equations, which thus live in nonlinear submanifolds of function space. Wave modes ϵ', ρ' obtained by linearization of the fluid equations around a uniform equilibrium background ϵ_0, ρ_0 only satisfy these thermodynamic constraints to the linearized level. This may be an adequate representation when fluctuations are relatively small, satisfying $\epsilon'/\epsilon_0,\ \rho'/\rho_0 \ll 1$. However, turbulence generally produces fluctuations much larger than the means, where this linear approximation to the thermodynamic relations is inadequate. Decomposition into linear wave modes is thus clearly an approximation, with an unknown range of validity. We note that in conformal fluids with AdS gravity duals, the linear wave-mode decomposition corresponds, on the gravity side, to the expansion in quasinormal modes about the uniform AdS black hole. Expansion in such quasinormal modes has recently been independently argued [16] to be inapplicable to the turbulent regime.

VII. EMPIRICAL PREDICTIONS AND EVIDENCE

High-energy astrophysical plasma flows are probably the best candidates in nature to exhibit relativistic fluid turbulence, but remote observations of such systems poorly constrain theory. In order to confront theory with precise evidence, the only recourse at the moment is numerical simulation of turbulence for relativistic kinetic equations or dissipative fluid models. Here, we briefly discuss the relations of our work to the existing body of numerical simulations. Confining our attention to electrically neutral fluids, as considered in the present work, the most relevant numerical studies have been motivated either by astrophysics [13,14] or by the fluid-gravity correspondence [15–17]. Numerical codes exist for simulating the particle-frame Israel-Stewart model [100], but we are aware of no turbulence simulations so far that exploit such codes. (The only exception is the study of Ref. [16] for a very similar second-order dissipative model of conformal fluids in $2+1$ space-time, discussed further below.) Instead, most studies have solved the relativistic Euler fluid equations using dissipative numerical schemes to remove the energy cascaded to small scales rather than a physical viscosity.

We first discuss the astrophysically motivated simulations in $3+1$ space-times with topology $T^3 \times R$. Zrake and MacFadyen [13] solved the stress-energy equation (113) and Eq. (2) for conserved particle number. They employed a relativistic ideal-gas equation of state $p = (\Gamma - 1)u$ for $\Gamma = 4/3$ and adopted a Minkowski force

$$F^\mu = \rho A^\mu - \rho(u/u_0)^4 V^\mu, \tag{130}$$

with terms representing mechanical stirring and radiative cooling, respectively. The space resolutions of their simulations were 256^3, 512^3, 1024^3, 2048^3, and they had a mean relativistic Mach number of about $\mathrm{Ma} = 2.67$. Radice and Rezzolla [14] instead solved only the stress-energy equation (113) for a radiation-pressure-dominated fluid with $p = (1/3)\epsilon$ and with a Minkowski force

$$F^\mu = F_0(t)(0, f^i) \tag{131}$$

for f^i a zero space-average, solenoidal, random vector supported at low wave numbers. They performed four runs with $F_0(t) = 1,\ 2,\ 5,\ 10 + (t/2)$ with space resolutions $128^3, 256^3, 512^3, 1024^3$, and with relativistic Mach numbers $\mathrm{Ma} = 0.362,\ 0.543,\ 1.003,\ 1.759$. The simulations of both groups are consistent with a forward energy cascade, although they had, at their disposal, no concrete formula such as our Eq. (61) in order to make a precise measurement of relativistic energy flux.

Both of these groups also measured the scaling exponents $\zeta_p^{\|v}$ of longitudinal velocity structure functions using the ESS procedure [101], and Ref. [13] also measured the exponents ζ_p^v for an absolute Minkowski-norm velocity structure function. Both of these studies found $\zeta_p^{\|v} \le p/3$ and $\zeta_p^v \le p/3$ for $p \ge 3$, consistent with our theoretical predictions. The phenomenological model of She-Lévêque [102] was found to be a reasonable approximation to the ESS results for $\zeta_p^{\|v}$, but not for ζ_p^v in Ref. [13], which took on smaller values than $\zeta_p^{\|v}$ associated with greater space-time intermittency. When $p < 3$, our analysis makes no theoretical predictions for $\zeta_p^{\|v}$ or ζ_p^v, aside from the reasonable inference by concavity that $\zeta_p > p/3$. The direct (non-ESS) measurements of Ref. [13] yielded $\zeta_2^{\|v} \doteq \zeta_2^v \doteq 1$ (Burgers-like), whereas Ref. [14] claimed consistency with $\zeta_2^{\|v} \doteq 2/3$ (K41). This discrepancy could be due to the larger Mach number in the simulations of Ref. [13] (see their Fig. 1, which shows clear evidence of shocks). On the other hand, the spectra in Fig. 2 of Ref. [14] at low wave numbers are consistent with $\zeta_2^{\|v} > 2/3$, and the higher wave numbers are plausibly contaminated by bottleneck effects. In our opinion, neither of the simulations [13,14] achieved a long enough inertial range to yield quantitatively reliable results for scaling exponents.

Motivated by black-hole gravitational physics through the fluid-gravity correspondence [8,9,11], there have also been simulations of relativistic fluid turbulence in $2+1$ space-time dimensions, both for free-decaying [15,16] and externally forced [17] cases. Here, the evolution of low-wave-number perturbations to black holes in a $D+1$-dimensional, asymptotically AdS space-time is expected to be equivalent to a relativistic hydrodynamics on the $D = d + 1$-dimensional conformal boundary of AdS space. Thus, $3+1$-dimensional black holes correspond to relativistic hydrodynamics in $2+1$ dimensions. All of our considerations are independent of the space dimension d and thus apply for $d = 2$, but this case is likely to be substantially more complex than $d > 2$. Even for incompressible fluid turbulence, $d = 2$ is a much richer problem than $d > 2$. For example, freely decaying and externally forced incompressible turbulence appear substantially similar for $d > 2$, with both exhibiting an energy-dissipation anomaly. However, the enstrophy-dissipation anomaly predicted for $d = 2$ incompressible turbulence [103,104] appears only in forced turbulence, whereas there is no enstrophy anomaly for freely decaying turbulence unless the initial data are very singular [105,106]. Viscous energy dissipation always tends to zero in $d = 2$ incompressible turbulence, but the energy accumulates in large scales by quite different mechanisms in the two cases: "vortex merger" [28,107] for freely decaying turbulence and "inverse energy cascade" [103] for forced turbulence. The previously mentioned simulations of $2 + 1$ relativistic turbulence also seem to indicate that there is no energy-dissipation anomaly there and that vortex-merger and inverse-cascade processes occur. It should be kept in mind, however, that all of the discussed simulations are at low relativistic Mach numbers. At higher Mach numbers, shocks will surely proliferate, leading to irreversible energy dissipation and entropy production. Such behavior was observed in Refs. [108,109] for simulations of $d = 2$ nonrelativistic compressible turbulence, motivated by large-scale dynamics of galactic accretion disks. We thus believe that the phenomenology of $2 + 1$ relativistic turbulence will be quite nonuniversal, depending upon the relativistic Mach number, free decay vs forced, precise details of the initial data, etc.

The simulations cited above already largely support the present work, but our theory makes a rich array of further predictions for relativistic fluid turbulence that are easily subject to empirical tests. Chief among these predictions are the following: (1) anomalous energy dissipation both by local energy cascade and by the pressure-work defect, (2) anomalous input of negentropy into the inertial range by pressure work, in addition to any external input by large-scale cooling mechanisms, (3) negentropy cascade to small scales through a flux of intrinsic inertial-range entropy, and (4) singularity or "roughness" of fluid fields to sustain cascades of energy and entropy so that at least one of the exponent inequalities (105)–(107) must hold. The explicit formulas (61) for energy flux and Eq. (103) for intrinsic entropy flux provide quantitative measures of cascade rates in relativistic turbulence. Furthermore, in order to provide mean fluxes of the predicted signs, the expressions (61) and (103) require specific space-time correlations to develop, e.g., "downgradient turbulent transport" with $\bar{\tau}(h, V^\mu)$, $\bar{\tau}(n, V^\mu)$ anticorrelated with the thermodynamic gradients $\partial_\mu \underline{T}$, $\partial_\mu \underline{\lambda}$, respectively. These many predictions provide an ample field of study for future numerical investigation.

VIII. SUMMARY AND FUTURE DIRECTIONS

The theory developed in this paper is based upon the hypothesis that relativistic fluid turbulence should exhibit dissipative anomalies of energy and entropy, similar to those observed for incompressible fluids. From this hypothesis alone, we have shown that the high Reynolds- and Péclet-number limits should be governed by distributional or coarse-grained solutions of the relativistic Euler equations. We have also demonstrated that precisely characterized singularities or "roughness" of the fluid fields is required to permit dissipative anomalies. The argument closely follows that of Onsager [27,28] for incompressible fluids, which we have explained as a nonperturbative application of the principle of renormalization-group invariance [52].

One of the key open questions is certainly the extension of the present special-relativistic theory to GR turbulence. There is reason to believe that much of the present theory will carry over straightforwardly to GR since curved Lorentzian manifolds are locally diffeomorphic to Minkowski space. However, new effects may arise if curvature scales become comparable to inertial-range turbulence scales. The main technical problem in extending our theory to GR is the development of suitable coarse graining in curved space-times in order to regularize turbulent ultraviolet divergences. Coarse-graining operations in GR have also attracted recent interest because of problems in cosmology and in the interpretation of cosmological observations, and much of this parallel work [110,111] should carry over to general-relativistic turbulence. Here, we may note that an Onsager singularity theorem has already been proved for incompressible fluid turbulence on general compact Riemannian manifolds by exploiting a coarse-graining regularization defined with a heat kernel smoothing [112].

Even in Minkowski space, our work opens important new directions of study. Our quantitative formulas (61) for energy flux and (103) for entropy flux allow for an exploration of the physical mechanisms of relativistic turbulent cascades [113,114]. The vortex-stretching mechanism of Taylor [115] is widely believed to drive the $d = 3$ incompressible energy cascade, but it is unclear whether such physics carries over to relativistic fluids. The equations of motion with the coarse-grained tensor (53) derived in this paper also provide the mathematical foundations for LES modeling of relativistic turbulence in Minkowski space [79–81]. Such LES holds promise to be an important tool in numerical investigation of local turbulence in high-energy astrophysical events, such as gamma-ray bursts. Finally, there are interesting implications of the present work for black hole physics because the fluid-gravity duality connects relativistic fluid dynamics in $d + 1$ Minkowski space-time to Einstein's equations in a Poincaré patch of a $D + 1$-dimensional AdS black hole solution. Thus, when high-Reynolds-number turbulence develops in a relativistic fluid in Minkowski space, our Onsager singularity theorem implies not only that the fluid fields must become rough but also that rough metrics must develop in the turbulent solutions of the Einstein equations in the dual gravitational description.

The roughness or Hölder singularity of the turbulent velocity $V^\mu(x)$, in particular, has profound implications for relativistic fluid turbulence. It was pointed out in a landmark work of Bernard *et al.* [116] on nonrelativistic incompressible turbulence that fluid velocities with Hölder exponent $h < 1$ have nonunique Lagrangian particle trajectories. It was shown by those authors in a synthetic model of turbulence that the Lagrangian trajectories become "spontaneously stochastic" in the high Reynolds-number limit, with randomness of trajectories persisting even when the initial particle location and the advecting velocity become deterministic and perfectly specified. The physical mechanism is explosive Richardson-type turbulent dispersion of particle pairs, which is also expected to hold (with some modifications) in relativistic fluid turbulence [117]. It has subsequently been shown that such "spontaneous stochasticity" of Lagrangian particle trajectories holds at Burgers shocks [118] and is necessary in incompressible Navier-Stokes turbulence for anomalous dissipation of passive scalars [119,120]. These considerations carry over directly to relativistic fluid worldlines $X^\mu(X_0, \tau)$ defined by the equations

$$dX^\mu/d\tau = V^\mu(X(\tau), \tau), \qquad X^\mu(0) = X_0^\mu. \tag{132}$$

Because of the Hölder singularities of the turbulent velocity vector predicted by our analysis, the fluid worldlines must become spontaneously stochastic, with a random ensemble of worldlines passing through each fixed event X_0. This implies a turbulent breakdown of Lagrangian conservation laws that hold for smooth solutions of the relativistic Euler equations, such as the Kelvin theorem [121] (Ref. [57], Sec. 3.7.5). Likewise, in relativistic astrophysical plasmas, the Alfvén theorem on magnetic flux conservation for ideal MHD solutions [122,123] must be fundamentally altered by spontaneous stochasticity effects. In nonrelativistic theory, this fact leads to fast turbulent magnetic reconnection independent of collisional resistivity [44,124,125], and our present work implies that the same turbulent mechanisms can act in relativistic magnetic reconnection.

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APPENDIX A: DERIVATION OF COARSE-GRAINED RELATIVISTIC EULER EQUATIONS

In this appendix, we give key details of the proof of validity of the relativistic Euler equations in the coarse-grained or weak sense for any ideal limits of thermodynamic fields ϵ, ρ, V^μ as Pe_σ, Pe_κ, Re_η, $\mathrm{Re}_\zeta \to \infty$.

Most of the argument for the particle-conservation equation has been given in Sec. III. One final estimate was left unproved, involving the Lorentz-invariant norm defined by

$$\Delta^{\mu\nu} A_\mu A_\nu,$$

where $\Delta^{\mu\nu} = g^{\mu\nu} + V^\mu V^\nu$ projects perpendicular to the relativistic fluid velocity vector V^μ with respect to the Minkowski pseudometric. Lorentz transforming into the fluid rest frame $A_\mu \to A'_\mu$,

$$\Delta^{\mu\nu} A_\mu A_\nu = |\mathbf{A}'|^2,$$

coinciding with the standard Euclidean norm of the spatial part of the vector. The above norm is, in fact, only a seminorm because $\Delta^{\mu\nu} V_\mu V_\nu = 0$. In deriving a bound on the dissipative terms in the coarse-grained conservation laws in Sec. III, we needed an estimate on this seminorm above in terms of the Euclidean norm.

To obtain this, we note that

$$\Delta^{\mu\nu} A_\mu A_\nu = \mathbf{a}^\top \Delta \mathbf{a},$$

where $\mathbf{a}$ is the D-dimensional vector with components A_μ of the covariant vector and Δ is the $D \times D$-dimensional matrix with entries $\Delta^{\mu\nu}$ of the contravariant tensor. We then use the standard bound

$$|\mathbf{a}^\top \Delta \mathbf{a}| \leq \|\Delta\|_2 \|\mathbf{a}\|_2^2, \tag{A1}$$

where $\|\Delta\|_2 = \sqrt{\rho(\Delta^\top \Delta)}$ and $\rho(\mathbf{M})$ is the spectral radius of the $D \times D$-dimensional matrix $\mathbf{M}$ (Ref. [126], Sec. 2.3). Since Δ is real and symmetric, one furthermore has $\|\Delta\|_2 = \rho(\Delta)$. We thus must compute the eigenvalues of Δ. This is simply done by an orthogonal transformation, which rotates the spatial part of the vector $V^\mu = \gamma(v)(1, \mathbf{v}/c)$ into the 1-direction. Note that such a purely spatial rotation changes neither the Minkowski pseudonorm nor the Euclidean norm of A_μ. After this rotation, the matrix Δ becomes block diagonal, with a lower block that is the $(d-1) \times (d-1)$ identity matrix and an upper block that is the 2×2 matrix,

$$\Delta_2 = \frac{1}{1-\beta_v^2} \begin{pmatrix} \beta_v^2 & \beta_v \\ \beta_v & 1 \end{pmatrix}, \qquad \beta_v = v/c.$$

The matrix Δ_2 has an eigenvalue 0 with eigenvector $(-1, \beta_v)^\top$ (whose components are obviously those of the covariant vector $V_\mu = g_{\mu\nu} V^\nu$) and an eigenvalue $(1+\beta_v^2)/(1-\beta_v^2)$ greater than 1 with eigenvector $(\beta_v, 1)^\top$. It follows that $\rho(\Delta) = [(1+\beta_v^2)/(1-\beta_v^2)]$. Finally, noting that $\|\mathbf{a}\|_2^2 = |A|_E^2$, we obtain the bound

$$|\Delta^{\mu\nu}A_\mu A_\nu| \leq \frac{1+\beta_v^2}{1-\beta_v^2}|A|_E^2.$$

This upper estimate is optimal, in that it can actually be achieved for a suitable spacelike vector A_μ corresponding to the second eigenvector above. Since $1+\beta_v^2 \leq 2$, we obtain the bound stated in Eq. (43) in the main text.

The dissipative terms in the coarse-grained energy-momentum conservation equation are estimated in a very similar fashion. Here, we sketch briefly the bound for the shear-viscosity term, which can be written as

$$c_\mu \partial_\nu(\overline{2\eta\hat{\tau}^{\mu\nu}}(x)) = -\frac{1}{\ell}\int d^D r c_\mu (\partial_\nu \mathcal{G})_\ell(r)\cdot 2\eta(x+r)\hat{\tau}^{\mu\nu}(x+r), \quad \text{(A2)}$$

and we have introduced a constant vector c^μ that can be set to 1 for any particular component of the equation and zero for the others, in order to select that component. The Cauchy-Schwartz inequality applied to this term gives

$$|c_\mu \partial_\nu(\overline{2\eta\hat{\tau}^{\mu\nu}}(x))| \leq \frac{1}{\ell}\sqrt{\int_{\mathrm{supp}(\mathcal{G}_\ell)} d^D r (2\eta T^2)(x+r)} \times \sqrt{\int d^D r \frac{2\eta}{T^2}(x+r)|c_\mu(\partial_\nu\mathcal{G})_\ell(r)\cdot\hat{\tau}^{\mu\nu}(x+r)|^2}. \quad \text{(A3)}$$

The first square-root factor goes to zero in the ideal limit under mild assumptions on η and T, as long as the second square-root factor remains bounded. To estimate the second term, we note that, for any second-rank covariant tensors $A_{\mu\nu}$, $B_{\mu\nu}$, the quantity

$$\Delta^{\mu\alpha}\Delta^{\nu\beta}A_{\mu\nu}B_{\alpha\beta} = \sum_{ij} A_{ij}{}' B_{ij}{}'$$

when the tensors are transformed to $A_{\mu\nu}{}'$, $B_{\mu\nu}{}'$ in the rest frame of the fluid. The expression on the right is the standard Frobenius inner product of $d\times d$ matrices, and thus the expression on the left is a degenerate inner product (vanishing whenever either tensor is a product of the form $V_\mu C_\nu$ or $C_\mu V_\nu$). Employing the Cauchy-Schwartz inequality for this degenerate inner product gives

$$|c_\mu(\partial_\nu\mathcal{G})_\ell(r)\cdot\hat{\tau}^{\mu\nu}(x+r)|^2 \leq (c_\mu^\perp c_\perp^\mu)\cdot(\partial_\nu^\perp G)_\ell(\partial_\perp^\nu G)_\ell(r)\cdot\hat{\tau}_{\mu\nu}\hat{\tau}^{\mu\nu}(x+r). \quad \text{(A4)}$$

The above inequality yields the following upper bound for the integral under the second square root in Eq. (A3):

$$(c_\mu^\perp c_\perp^\mu)\int d^D r(\partial_\nu^\perp G)_\ell(r)(\partial_\perp^\nu G)_\ell(r)\cdot\frac{2\eta\hat{\tau}_{\mu\nu}\hat{\tau}^{\mu\nu}}{T^2}(x+r). \quad \text{(A5)}$$

Now, using Eq. (43) in the main text and the similar inequality

$$0 \leq (c_\mu^\perp c_\perp^\mu) \leq 2\gamma^2(v)|c|_E^2, \quad \text{(A6)}$$

we obtain our final estimate for the integral under the second square root,

$$4|c|_E^2\|\gamma(v)\|_\infty^4 \int d^D r|(\partial G)_\ell(r)|_E^2\cdot\frac{2\eta\hat{\tau}_{\mu\nu}\hat{\tau}^{\mu\nu}}{T^2}(x+r). \quad \text{(A7)}$$

This upper estimate converges in the ideal limit to

$$4|c|_E^2\|\gamma(v)\|_\infty^4 \int d^D r|(\partial G)_\ell(r)|_E^2\Sigma_\eta(x+r) \quad \text{(A8)}$$

and thus remains bounded. We conclude that the shear-viscosity term in the coarse-grained energy-momentum equation vanishes in the ideal limit.

Similar results are obtained for the bulk-viscosity term in the coarse-grained energy-momentum equation using the identity

$$c_\mu\partial_\nu(\overline{\zeta\hat{\tau}\Delta^{\mu\nu}}(x)) = -\frac{1}{\ell}\int d^D r c_\mu^\perp(\partial_\perp^\mu\mathcal{G})_\ell(r)\cdot\zeta(x+r)\hat{\tau}(x+r) \quad \text{(A9)}$$

and, for the thermal-conductivity term, using

$$c_\mu\partial_\nu(\overline{\kappa\hat{Q}^\mu V^\nu + \kappa\hat{Q}^\nu V^\mu}(x)) = -\frac{1}{\ell}\int d^D r\kappa(x+r)c_\mu^\perp\hat{Q}^\mu(x+r)\cdot(\partial_\nu\mathcal{G})_\ell(r)V^\nu(x+r) - \frac{1}{\ell}\int d^D r c_\mu V^\mu(x+r)\cdot\kappa(x+r)(\partial_\nu^\perp\mathcal{G})_\ell(r)\hat{Q}^\nu(x+r). \quad \text{(A10)}$$

The bulk-viscosity term is treated very similarly to the shear-viscosity term. For the thermal-conductivity term, we need to use the standard Cauchy-Schwartz inequality $|c_\mu V^\mu| \leq |c|_E|V|_E$ and the following estimate for the Euclidean norm of the fluid velocity vector:

$$|V|_E^2 = \gamma^2(v)(1+v^2/c^2) \leq 2\gamma^2(v). \quad \text{(A11)}$$

The details are straightforward and left to the reader.

APPENDIX B: NONRELATIVISTIC LIMIT

Space-time coarse-graining kernels in relativistic theory $\mathcal{G}(r) = \mathcal{G}(r^0, \mathbf{r})$ and in nonrelativistic (Newtonian) theory

$\mathcal{G}_N(\mathbf{r},\tau)$ are related by a simple change of dimensions through scaling with c:

$$\mathcal{G}(r^0,\mathbf{r}) = (1/c)\mathcal{G}_N(\mathbf{r}, r^0/c).$$

Thus,

$$\begin{aligned}\bar{f}(x) &= \int d^D r \mathcal{G}_\ell(r) f(x+r)\\ &= \int d^d\mathbf{r}\int d\tau \mathcal{G}_{N,\ell}(\mathbf{r},\tau) f(\mathbf{x}+\mathbf{r}, t+\tau)\\ &= \bar{f}^N(\mathbf{x},t),\end{aligned}\tag{B1}$$

and there is no need to distinguish between $\bar{f}$ and $\bar{f}^N$ as $c\to\infty$. This is not true, in general, for more singular coarse graining in space-time. Consider as an example the backward light-cone average with

$$\mathcal{G}(r) = G(\mathbf{r})\delta(r^0 + |\mathbf{r}|).$$

In that case,

$$\bar{f}(x) = \int d^d\mathbf{r} G_\ell(\mathbf{r}) f(\mathbf{x}+\mathbf{r}, t - |\mathbf{r}|/c).$$

Then, in the limit $c\to\infty$,

$$\bar{f} = \bar{f}^N - \frac{1}{c}\int d^d\mathbf{r} G_\ell(\mathbf{r})|\mathbf{r}|\dot{f}(\mathbf{x}+\mathbf{r},t) + O\left(\frac{1}{c^2}\right),$$

where

$$\bar{f}^N(\mathbf{x},t) = \frac{1}{c}\int d^d\mathbf{r} G_\ell(\mathbf{r}) f(\mathbf{x}+\mathbf{r},t)$$

is the nonrelativistic instantaneous spatial coarse graining. In this case, $\bar{f}$ and $\bar{f}^N$ are distinct. We assume hereafter a smooth space-time coarse graining.

Even with smooth space-time coarse graining, the relativistic and nonrelativistic Favre averages are distinct because, respectively,

$$\tilde{f} := \overline{hf}/\bar{h}, \qquad \tilde{f}^N := \overline{\rho f}/\bar{\rho},$$

where the first is weighted by $h = \rho c^2 + h_N$, with $h_N = u + p$ the nonrelativistic (Newtonian) enthalpy and the second weighted by ρ. Straightforward Taylor expansion in $1/c^2$ gives

$$\tilde{f} = \tilde{f}^N + \frac{1}{c^2\bar{\rho}^2}(\overline{fh_N}\,\bar{\rho} - \overline{f\rho}\,\overline{h_N}) + O\left(\frac{1}{c^4}\right).$$

While the relativistic and nonrelativistic Favre averages are distinct, they do agree to leading order in $1/c^2$.

With these preliminaries, we now consider the formal nonrelativistic limit of $c\to\infty$. We note the standard relations

$$\partial_\mu = \left(\frac{1}{c}\partial_t, \boldsymbol{\nabla}\right),\tag{B2}$$

$$V^\mu = \left(1 + \frac{1}{2c^2}|\mathbf{v}|^2 + O\left(\frac{1}{c^4}\right), \frac{1}{c}\mathbf{v} + O\left(\frac{1}{c^3}\right)\right),\tag{B3}$$

$$\mathcal{D} = V^\mu\partial_\mu = \frac{1}{c}D + O\left(\frac{1}{c^3}\right),\tag{B4}$$

with $D = \partial_t + \mathbf{v}\cdot\boldsymbol{\nabla}$, and

$$\theta = \partial_\mu V^\mu = \frac{1}{c}\Theta + O\left(\frac{1}{c^3}\right),\tag{B5}$$

with $\Theta = \boldsymbol{\nabla}\cdot\mathbf{v}$, or, more generally,

$$\partial_\mu(fV^\mu) = \frac{1}{c}[\partial_t f + \boldsymbol{\nabla}\cdot(f\mathbf{v})] + O\left(\frac{1}{c^3}\right).\tag{B6}$$

Furthermore, because cumulants of constants vanish, we have relations such as

$$\bar{\tau}(V^0, f_2, f_3, \ldots, f_n) = \frac{1}{2c^2}\bar{\tau}(|\mathbf{v}|^2, f_2, f_3, \ldots, f_n),\tag{B7}$$

$$\bar{\tau}(V^0, V^0, f_3, \ldots, f_n) = \left(\frac{1}{2c^2}\right)^2\bar{\tau}(|\mathbf{v}|^2, |\mathbf{v}|^2, f_3, \ldots, f_n),\tag{B8}$$

and so forth. The same relations hold also for Favre cumulants, just replacing $\bar{\tau}$ by $\tilde{\tau}$.

1. Inertial-range energy balance

We consider first the energy balance (60) or

$$\partial_\mu \underset{\sim}{\mathcal{E}}^\mu = -\bar{p}\,\bar{\theta} + \mathcal{Q}_\ell^{\text{flux}}.\tag{B9}$$

Note that $\epsilon V^\mu = c^2 J^\mu + uV^\mu$, so

$$\partial_\mu\overline{\epsilon V^\mu} = \partial_\mu\overline{uV^\mu} = \frac{1}{c}[\partial_t\bar{u} + \boldsymbol{\nabla}\cdot(\bar{u}\,\bar{\mathbf{v}} + \bar{\tau}(u,\mathbf{v}))] + O\left(\frac{1}{c^3}\right).\tag{B10}$$

By the results (B3), (B7), and (B8),

$$\bar{\tau}(p, V^\mu) = \left(O\left(\frac{1}{c^2}\right), \frac{1}{c}\bar{\tau}(p,\mathbf{v}) + O\left(\frac{1}{c^3}\right)\right),\tag{B11}$$

$$\frac{1}{2}\tilde{\tau}(V_\nu, V^\nu)\tilde{V}^\mu = \left(\frac{1}{2c^2}\tilde{\tau}^N(v_i, v_i) + O\left(\frac{1}{c^4}\right), \frac{1}{2c^3}\tilde{\tau}^N(v_i, v_i)\mathbf{v} + O\left(\frac{1}{c^5}\right)\right), \quad (B12)$$

$$\tilde{\tau}(V_\nu, V^\nu, V^\mu) = \left(O\left(\frac{1}{c^4}\right), \frac{1}{c^3}\tilde{\tau}^N(v_i, v_i, \mathbf{v}) + O\left(\frac{1}{c^5}\right)\right), \quad (B13)$$

and

$$\begin{aligned}\tilde{\tau}(V_\nu, V^\mu)\tilde{V}^\nu &= \left(O\left(\frac{1}{c^4}\right), -\frac{1}{2c^3}\tilde{\tau}^N(|\mathbf{v}|^2, \mathbf{v})\cdot 1 + \frac{1}{c^3}\tilde{\tau}^N(v_i, \mathbf{v})\tilde{v}_i^N + O\left(\frac{1}{c^5}\right)\right)\\ &= \left(O\left(\frac{1}{c^4}\right), -\frac{1}{2c^3}\tilde{\tau}^N(v_i, v_i, \mathbf{v}) + O\left(\frac{1}{c^5}\right)\right). \quad (B14)\end{aligned}$$

Putting all of these results together with the formula (57) for $\underset{\sim}{\mathcal{E}}^\mu$ and $\bar{h} = \bar{\rho}c^2 + \bar{h}_N$ gives

$$\partial_\mu \underset{\sim}{\mathcal{E}}^\mu \simeq \frac{1}{c}\partial_t\left(\bar{u} + \frac{1}{2}\bar{\rho}\tilde{\tau}^N(v_i, v_i)\right) + \frac{1}{c}\nabla\cdot\left(\bar{u}\,\bar{\mathbf{v}} + \bar{\tau}(h, \mathbf{v}) + \frac{1}{2}\bar{\rho}\tilde{\tau}^N(v_i, v_i)\tilde{\mathbf{v}}^N + \frac{1}{2}\bar{\rho}\tilde{\tau}^N(v_i, v_i, \mathbf{v})\right). \quad (B15)$$

Now consider relativistic energy flux given by

$$Q_\ell^{\rm flux} = \frac{1}{\bar{h}}(\partial_\nu\bar{p})\bar{\tau}(h, V^\nu) - \bar{h}(\partial_\mu\tilde{V}_\nu)\tilde{\tau}(V^\mu, V^\nu) - \frac{1}{2}\partial_\nu\overline{hV^\nu}\,\tilde{\tau}(V_\mu, V^\mu). \quad (B16)$$

Easily from previous estimates, we get

$$\frac{1}{\bar{h}}(\partial_\nu\bar{p})\bar{\tau}(h, V^\nu) \simeq \frac{1}{c\bar{\rho}}\nabla\bar{p}\cdot\bar{\tau}(\rho, \mathbf{v}). \quad (B17)$$

Next, observe that

$$\tilde{\tau}(V^\mu, V^\nu) = \left[\begin{array}{c|c} O(\frac{1}{c^4}) & O(\frac{1}{c^3}) \\ \hline O(\frac{1}{c^3}) & \frac{1}{c^2}\tilde{\tau}^N(\mathbf{v}, \mathbf{v})\end{array}\right] \quad (B18)$$

and

$$\partial_\mu\tilde{V}_\nu = \left[\begin{array}{c|c} O(\frac{1}{c^3}) & O(\frac{1}{c^3}) \\ \hline O(\frac{1}{c^3}) & \frac{1}{c}\nabla\tilde{\mathbf{v}}^N\end{array}\right], \quad (B19)$$

so

$$\bar{h}(\partial_\mu\tilde{V}_\nu)\tilde{\tau}(V^\mu, V^\nu) \simeq \frac{1}{c}\bar{\rho}\nabla\tilde{\mathbf{v}}^N : \tilde{\tau}^N(\mathbf{v}, \mathbf{v}). \quad (B20)$$

For the last term, use $hV^\nu = c^2J^\nu + h_NV^\nu$ to obtain

$$\partial_\nu\overline{hV^\nu} \simeq \frac{1}{c}[\partial_t h_N + \nabla\cdot(h_N\mathbf{v})]. \quad (B21)$$

Since, in addition,

$$\frac{1}{2}\tilde{\tau}(V_\nu, V^\nu) \simeq \frac{1}{2c^2}\tilde{\tau}^N(v_i, v_i), \quad (B22)$$

we thus find

$$\frac{1}{2}\partial_\nu\overline{hV^\nu}\,\tilde{\tau}(V_\mu, V^\mu) = O\left(\frac{1}{c^3}\right). \quad (B23)$$

In conclusion,

$$Q_\ell^{\rm flux} \simeq \frac{1}{c\bar{\rho}}\nabla\bar{p}\cdot\bar{\tau}(\rho, \mathbf{v}) - \frac{1}{c}\bar{\rho}\nabla\tilde{\mathbf{v}}^N : \tilde{\tau}^N(\mathbf{v}, \mathbf{v}) = \frac{1}{c}Q_\ell^{\rm flux}, \quad (B24)$$

where $Q_\ell^{\rm flux}$ is the nonrelativistic energy flux.

From the results (B15), (B24), and $\bar{p}\,\bar{\theta} \simeq (1/c)\bar{p}\,\bar{\Theta}$, we thus obtain, as the nonrelativistic limit of the inertial-range internal-energy balance for the relativistic Euler equation,

$$\begin{aligned}&\partial_t\left(\bar{u} + \frac{1}{2}\bar{\rho}\tilde{\tau}^N(v_i, v_i)\right) + \nabla\cdot\left(\bar{u}\,\bar{\mathbf{v}} + \bar{\tau}(h, \mathbf{v}) + \frac{1}{2}\bar{\rho}\tilde{\tau}^N(v_i, v_i)\tilde{\mathbf{v}}^N + \frac{1}{2}\bar{\rho}\tilde{\tau}^N(v_i, v_i, \mathbf{v})\right)\\ &= -\bar{p}\,\bar{\Theta} + Q_\ell^{\rm flux}. \quad (B25)\end{aligned}$$

This is nothing other than the nonrelativistic balance equation for intrinsic large-scale internal energy, obtained in Eq. (I;63) of paper I.

2. Inertial-range entropy balance

We now consider the intrinsic inertial-range entropy current in the Favre formulation, $\underset{\sim}{S}^{*\mu} = \underline{S}^\mu - \underline{\beta}\underset{\sim}{K}^\mu$, and its balance equation

$$\partial_\mu\underset{\sim}{S}^{*\mu} = -I_\ell^{\rm flux} + \Sigma_\ell^{\rm flux*} \quad (B26)$$

with the intrinsic negentropy flux

$$\Sigma_\ell^{\rm flux*} = \Sigma_\ell^{\rm flux} - (\partial_\mu \underline{\beta})\underset{\sim}{K}^\mu + \underline{\beta}\mathcal{Q}_\ell^{\rm flux}. \qquad \text{(B27)}$$

First, we note a standard difference between relativistic and Newtonian thermodynamics due to the distinction between rest mass and energy in the latter:

$$\epsilon = \rho c^2 + u, \qquad \lambda = \beta m c^2 + \lambda_N. \qquad \text{(B28)}$$

See Ref. [60] or [57], Sec. 2.3.6. Using these relations, one easily finds that

$$\begin{aligned}\underset{\sim}{S}^\mu &= \underline{s}\bar{V}^\mu + \underline{\beta}\,\bar{\tau}(u, V^\mu) - \underline{\lambda}_N\bar{\tau}(n, V^\mu)\\ &= \left(\underline{s} + O\left(\frac{1}{c^2}\right), \frac{1}{c}[\underline{s}\,\bar{\mathbf{v}} + \underline{\beta}\,\bar{\tau}(u,\mathbf{v})\right.\\ &\quad \left. - \underline{\lambda}_N\bar{\tau}(n,\mathbf{v})] + O\left(\frac{1}{c^3}\right)\right). \qquad \text{(B29)}\end{aligned}$$

On the other hand, it follows directly from the formula (80) for $\underset{\sim}{K}^\mu$ and the estimates in the previous subsection that

$$\begin{aligned}-\underset{\sim}{K}^\mu = &\left(\frac{1}{2}\bar{\rho}\tilde{\tau}^N(v_i, v_i) + O\left(\frac{1}{c^2}\right),\right.\\ &\frac{1}{2c}\bar{\rho}\tilde{\tau}^N(v_i, v_i)\tilde{\mathbf{v}}^N + \frac{1}{2c}\bar{\rho}\tilde{\tau}^N(v_i, v_i, \mathbf{v})\\ &\left. + \frac{1}{c}\bar{\tau}(p, \mathbf{v}) + O\left(\frac{1}{c^3}\right)\right). \qquad \text{(B30)}\end{aligned}$$

As an aside, we note that this last result implies that the balance equation (79) for $\underset{\sim}{K}^\mu$ reduces in the limit $c \to \infty$ to the nonrelativistic balance equation (paper I;64) for the subscale kinetic energy. We finally obtain that

$$\partial_\mu \underset{\sim}{S}^{*\mu} \simeq \frac{1}{c}[\partial_t \underline{s}^* + \nabla \cdot \underline{\mathbf{s}}^*], \qquad \text{(B31)}$$

where

$$\underline{s}^* = \underline{s} + (1/2)\underline{\beta}\,\bar{\rho}\,\tilde{\tau}^N(v_i, v_i) \qquad \text{(B32)}$$

is the nonrelativistic intrinsic inertial-range entropy and

$$\begin{aligned}\underline{\mathbf{s}}^* = &\,\underline{s}\,\bar{\mathbf{v}} + \underline{\beta}\,\bar{\tau}(h_N, \mathbf{v}) - \underline{\lambda}_N\bar{\tau}(n, \mathbf{v})\\ &+ \underline{\beta}\left[\frac{1}{2}\bar{\rho}\tilde{\tau}^N(v_i, v_i)\tilde{\mathbf{v}}^N + \frac{1}{2}\bar{\rho}\tilde{\tau}^N(v_i, v_i, \mathbf{v})\right] \qquad \text{(B33)}\end{aligned}$$

is the associated spatial current. See Eqs. (paper I;97) and (paper I;99).

On the other hand, using again the relation (B28) between relativistic and Newtonian thermodynamic quantities, one finds that Eq. (90) yields

$$\begin{aligned}\Sigma_\ell^{\rm flux} &= (\partial_\mu \underline{\beta})\bar{\tau}(u, V^\mu) - (\partial_\mu \underline{\lambda}_N)\bar{\tau}(n, V^\mu)\\ &\simeq \frac{1}{c}[\nabla\underline{\beta}\cdot\bar{\tau}(u,\mathbf{v}) - \nabla\underline{\lambda}_N\cdot\bar{\tau}(n,\mathbf{v})]\\ &= \frac{1}{c}\Sigma_\ell^{{\rm flux},N}, \qquad \text{(B34)}\end{aligned}$$

where $\Sigma_\ell^{{\rm flux},N}$ is the (naive) entropy flux in nonrelativistic compressible turbulence. Directly from Eq. (B30) and the asymptotics for $\mathcal{Q}_\ell^{\rm flux}$ in the previous subsection, one finds that

$$\begin{aligned}&\underline{\beta}\mathcal{Q}_\ell^{\rm flux} - (\partial_\mu\underline{\beta})\underset{\sim}{K}^\mu\\ &\simeq \frac{1}{c}\underline{\beta}\mathcal{Q}_\ell^{\rm flux} + \frac{1}{2c}(\partial_t\underline{\beta})\bar{\rho}\tilde{\tau}^N(v_i, v_i)\\ &\quad + \frac{1}{c}\nabla\underline{\beta}\cdot\left[\frac{1}{2}\bar{\rho}\tilde{\tau}^N(v_i, v_i)\tilde{\mathbf{v}}^N + \frac{1}{2}\bar{\rho}\tilde{\tau}^N(v_i, v_i, \mathbf{v}) + \bar{\tau}(p, \mathbf{v})\right]. \qquad \text{(B35)}\end{aligned}$$

This corresponds exactly to Eq. (paper I;93) in the nonrelativistic theory. Finally, the very simple equality

$$I_\ell^{\rm flux} = \underline{\beta}(\bar{p} - \underline{p})\bar{\theta} \simeq \frac{1}{c}\underline{\beta}(\bar{p} - \underline{p})\bar{\Theta} = \frac{1}{c}I_\ell^{{\rm flux},N} \qquad \text{(B36)}$$

shows that the relativistic inertial-range entropy balance (101) reduces in the limit $c \to \infty$ to the balance (paper I;98) of nonrelativistic intrinsic inertial-range entropy.

APPENDIX C: RELATIVISTIC FLUID KINETIC ENERGY AND ITS TURBULENT CASCADE

As discussed in the main text, we define kinetic-energy current, rather naturally, as

$$\mathcal{T}^\mu \coloneqq T^{0\mu} - \mathcal{E}^\mu. \qquad \text{(C1)}$$

Clearly, the "current" $\mathcal{T}^\mu$ does not transform as a vector under Lorentz transformations. However, using Eq. (9) for $\mathcal{E}^\mu$, it satisfies

$$\partial_\mu \mathcal{T}^\mu = p(\partial_\mu V^\mu) - \mathcal{Q}_{\rm diss}, \qquad \text{(C2)}$$

which is a Lorentz scalar and the equation that is reasonably expected to hold for a kinetic-energy current. It is worth noting that the combination $\mathcal{T}^\mu + \mathcal{U}^\mu$ is a locally conserved quantity, satisfying

$$\partial_\mu(\mathcal{T}^\mu + \mathcal{U}^\mu) = 0, \qquad \text{(C3)}$$

since

$$\mathcal{T}^\mu + \mathcal{U}^\mu = T^{0\mu} - c^2 J^\mu. \qquad \text{(C4)}$$

It thus represents the total energy current in the fixed frame minus the energy current of rest mass.

To further emphasize the naturalness of the proposed definition (C1), consider the case of a relativistic Euler fluid. Although there are generally dissipative contributions to $\mathcal{T}^\mu$, its components for an ideal fluid are simply

$$\mathcal{T}^0 = h\gamma_v^2 - p - \epsilon\gamma_v, \qquad \mathcal{T}^i = (h\gamma_v^2 - \epsilon\gamma_v)\frac{v^i}{c}. \tag{C5}$$

The kinetic-energy density in Eq. (C5) is straightforwardly rewritten as

$$\mathcal{T}^0 = h(\gamma_v - 1)^2 + (\epsilon + 2p)(\gamma_v - 1). \tag{C6}$$

It follows directly from Eq. (C6) that $\mathcal{T}^0 = 0$ when $\gamma_v = 1$, $\mathcal{T}^0$ is a convex function of γ_v when $h \geq 0$, and $\mathcal{T}^0$ is non-negative when, in addition, $\epsilon + 2p \geq 0$. If instead $\epsilon + 2p < 0$, then $\mathcal{T}^0 < 0$ for $\gamma_v \gtrsim 1$. Non-negativity and convexity are guaranteed by the strong energy condition on the stress-energy tensor [127]:

$$\left(T^{\mu\nu} - \frac{T}{D-2}g^{\mu\nu}\right)W_\mu W_\nu \geq 0, \qquad W^\mu \text{ timelike}. \tag{C7}$$

Indeed, for a perfect fluid, the strong energy condition is equivalent to

$$h \geq 0, \qquad (D-3)\epsilon + (D-1)p \geq 0, \tag{C8}$$

which for $3 \leq D \leq 5$ implies both $h \geq 0$ and $\epsilon + 2p \geq 0$. For most reasonable equations of state and for sufficiently low space-time dimensions, the proposed kinetic-energy density thus has the properties naturally expected.

The $c \to \infty$ limit of the kinetic-energy current defined in Eq. (C1) also recovers the familiar nonrelativistic expressions. From Eq. (C5) and $h = \rho c^2 + h_N$, it follows straightforwardly that the ideal part of the kinetic-energy current becomes

$$\begin{aligned} \mathcal{T}^0 &\sim \frac{1}{2}\rho|\mathbf{v}|^2 + O\left(\frac{1}{c^2}\right), \\ \mathcal{T}^i &\sim \left(\frac{1}{2}\rho|\mathbf{v}|^2 + p\right)\frac{v^i}{c} + O\left(\frac{1}{c^3}\right) \end{aligned} \tag{C9}$$

and dissipative contributions to $\mathcal{T}^0$ vanish in the limit $c \to \infty$. Using the orthogonality condition $\Pi^{0i} = \Pi^{ij}v_j/c$, Eq. (C2), in the limit, becomes

$$\begin{aligned} &\partial_t\left(\frac{1}{2}\rho|\mathbf{v}|^2\right) + \boldsymbol{\nabla}\cdot\left[\left(\frac{1}{2}\rho|\mathbf{v}|^2 + p\right)\mathbf{v} + \boldsymbol{\Pi}_N\cdot\mathbf{v}\right] \\ &\quad = p(\boldsymbol{\nabla}\cdot\mathbf{v}) - Q_{\rm diss}, \end{aligned} \tag{C10}$$

where $\boldsymbol{\Pi}_N$ is the nonrelativistic limit of the viscous stress and $Q_{\rm diss} = -\boldsymbol{\Pi}_N : \nabla\mathbf{v}$ is the nonrelativistic viscous dissipation [74]. Note the cancellation of terms proportional to κ from Eqs. (4) and (7) in $T^{0\mu} - \mathcal{E}^\mu$ as $c \to \infty$. In the parabolic Eckart and Landau-Lifshitz theories, $\boldsymbol{\Pi}_N = -2\eta_N\mathbf{S} - \zeta_N\Theta\mathbf{I}$, and Eq. (C10) is the standard kinetic-energy balance for the nonrelativistic compressible Navier-Stokes equation. In the better-behaved Israel-Stewart models, Eq. (C10) still holds as $c \to \infty$, but now the viscous stress is determined by an independent dynamical equation $D_t\Pi_N = \cdots$. Thus, the kinetic-energy balance is that of a nonrelativistic extended irreversible thermodynamic (EIT) model. We note that Eq. (C3) likewise reduces, in the limit $c \to \infty$, to conservation of total nonrelativistic fluid energy:

$$\begin{aligned} &\partial_t\left(\frac{1}{2}\rho|\mathbf{v}|^2 + u\right) \\ &\quad + \boldsymbol{\nabla}\cdot\left[\left(\frac{1}{2}\rho|\mathbf{v}|^2 + u + p\right)\mathbf{v} + \boldsymbol{\Pi}_N\cdot\mathbf{v} + \mathbf{q}\right] = 0, \end{aligned} \tag{C11}$$

where $\mathbf{q}$ is the nonrelativistic heat flux. This is given by Fourier's law $\mathbf{q} = -\kappa_N\boldsymbol{\nabla}T$ in the Eckart and Landau-Lifshitz theories (Ref. [128], Sec. 136), and in the Israel-Stewart theories, it is determined by a separate equation $D_t\mathbf{q} = \ldots$. The recovery of these standard nonrelativistic results is another measure of appropriateness of definition (C1).

In a turbulent flow, we expect a dissipative anomaly of kinetic energy in the infinite Reynolds-number limit. Indeed, the energy dissipation $\mathcal{Q}_{\rm diss}$ in the kinetic-energy balance (C2) is the same quantity (16) that appears in the internal-energy balance but with the opposite sign. As seen from Eq. (16), the nonvanishing of $\mathcal{Q}_{\rm diss}$ in the limit $\eta, \zeta, \sigma, \kappa \to 0$ requires diverging gradients, $\sigma^{\mu\nu}$, θ, $A^\mu \to \infty$. In order to eliminate these divergences and to obtain a well-defined set of fluid equations in the infinite Reynolds and Péclet limit, we have employed a coarse-graining regularization, which yields finite inertial-range balance relations for resolved internal energy and resolved entropy. To study the inertial-range cascade of kinetic energy, one must introduce a similar resolved kinetic energy, which we take to be

$$\underset{\sim}{\mathcal{T}}^\mu := \bar{T}^{0\mu} - \underset{\sim}{\mathcal{E}}^\mu. \tag{C12}$$

Notice that our procedure here is exactly the opposite of that for the nonrelativistic case in paper I, where the resolved kinetic energy $(1/2)\bar{\rho}|\tilde{\mathbf{v}}|^2$ was the primary quantity and the "intrinsic" resolved internal energy was introduced as a secondary quantity $\bar{u}^* = \bar{E} - (1/2)\bar{\rho}|\tilde{\mathbf{v}}|^2$, with $E = (1/2)\rho|\mathbf{v}|^2 + u$ the (fine-grained) total energy density; see Eq. (paper I;64). For reasons already discussed, here we instead treat the resolved internal-energy current $\underset{\sim}{\mathcal{E}}^\mu$ defined in Eq. (54) as the primary quantity and the resolved kinetic-energy density $\underset{\sim}{\mathcal{T}}^\mu$ in Eq. (C12) above as the secondary one. The definition (C12) is a suitable regularization of $\mathcal{T}^\mu$ since the estimates derived in Appendix A show that all nonideal contributions to it vanish as η, ζ, σ, $\kappa \to 0$ with ℓ fixed and clearly $\mathcal{D}-\lim_{\ell\to 0}\underset{\sim}{\mathcal{T}}^\mu = \mathcal{T}^\mu$. Furthermore, using Eq. (60) for $\underset{\sim}{\mathcal{E}}^\mu$,

$$\partial_\mu \underset{\sim}{\mathcal{T}}^\mu = \bar{p}\,\bar{\theta} - \mathcal{Q}_\ell^{\rm flux}, \tag{C13}$$

where the limit $\eta, \zeta, \sigma, \kappa \to 0$ has been taken and $\mathcal{Q}_\ell^{\rm flux}$ is the inertial-range energy flux defined in Eq. (61). This energy flux thus appears not only as a "source" of resolved internal energy but also as a "sink" of resolved kinetic energy due to loss by turbulent cascade.

The previous analysis yields the dissipative anomaly in the kinetic-energy balance at infinite Reynolds and Péclet numbers. If the limit $\ell \to 0$ is taken subsequent to the limit η, ζ, σ, $\kappa \to 0$, then Eq. (C13) becomes

$$\partial_\mu \mathcal{T}^\mu = p\circ\theta - \mathcal{Q}_{\rm flux}, \tag{C14}$$

with definitions as in Eqs. (62) and (64). Alternatively, the balance equation for kinetic-energy current at infinite Reynolds and Péclet numbers may be obtained by directly taking the limit as $\eta, \zeta, \sigma, \kappa \to 0$ of the fine-grained balance (C2), which gives

$$\partial_\mu \mathcal{T}^\mu = p * \theta - \mathcal{Q}_{\rm diss}, \tag{C15}$$

with the definitions (59) and (73). Comparing the previous two expressions, one finds that $\mathcal{Q}_{\rm diss} = \mathcal{Q}_{\rm inert}$ with $\mathcal{Q}_{\rm inert} = \tau(p,\theta) + \mathcal{Q}_{\rm flux}$ and $\tau(p,\theta)$ in Eq. (75). Thus, from the inertial-range perspective, the dissipative anomaly of relativistic kinetic energy can be produced both by turbulent cascade and by the pressure-work defect.

In the limit $c \to \infty$, the above theory reproduces the results of paper I for nonrelativistic kinetic-energy cascade. This is remarkable, given the secondary role of kinetic energy in relativistic fluid turbulence and its primary role in nonrelativistic turbulence. To show this, it is helpful to use mass conservation $\partial_\mu \bar{J}^\mu = 0$ to write

$$\partial_\mu \underset{\sim}{\mathcal{T}}^\mu = \partial_\mu \overline{\mathcal{T}^\mu + \mathcal{U}^\mu} - \partial_\mu \underset{\sim}{\mathcal{E}}^\mu. \tag{C16}$$

In the limit $c \to \infty$, the inertial-range balance $\partial_\mu \overline{\mathcal{T}^\mu + \mathcal{U}^\mu} = 0$ becomes

$$\partial_t \overline{\frac{1}{2}\rho|\mathbf{v}|^2 + u} + \boldsymbol{\nabla}\cdot \overline{\left(\frac{1}{2}\rho|\mathbf{v}|^2 + u + p\right)\mathbf{v}} = 0. \tag{C17}$$

If one subtracts the nonrelativistic limit (B25) of the balance for $\underset{\sim}{\mathcal{E}}^\mu$ from the latter equation (C17), then one obtains from Eq. (C13), as $c \to \infty$, that

$$\partial_t\left(\frac{1}{2}\bar{\rho}|\tilde{\mathbf{v}}^N|^2\right) + \boldsymbol{\nabla}\cdot\left[\left(\bar{p} + \frac{1}{2}\bar{\rho}|\tilde{\mathbf{v}}^N|^2\right)\tilde{\mathbf{v}}^N + \bar{\rho}\tilde{\tau}^N(\mathbf{v},\mathbf{v})\cdot\tilde{\mathbf{v}}^N - \frac{\bar{p}}{\bar{\rho}}\bar{\tau}(\rho,\mathbf{v})\right] = \bar{p}\,\bar{\Theta} - Q_\ell^{\rm flux}. \tag{C18}$$

In this manner, the inertial-range kinetic-energy balance equation (paper I; 43) of nonrelativistic turbulence can be recovered as $c \to \infty$ from the relativistic balance (C13).

APPENDIX D: FINE-GRAINED BALANCES OF INTERNAL ENERGY AND ENTROPY IN THE IDEAL LIMIT

In this appendix, we derive the balances of internal energy and entropy for the relativistic Euler solutions by considering directly the ideal limit of the fine-grained balances from the dissipative fluid model.

We begin by considering $\partial_\mu \mathcal{E}^\mu$ with the particle-frame energy current in Eq. (7), or $\mathcal{E}^\mu = \epsilon V^\mu + \kappa \hat{Q}^\mu$. We must show that the contribution of the second term vanishes distributionally in the limit $\kappa, \eta, \zeta \to 0$. After smearing with a general test function φ, a straightforward estimate by a Cauchy-Schwartz inequality gives

$$\begin{aligned}\left|\int d^D x(\partial_\mu\varphi)\kappa\hat{Q}^\mu\right| &\le \sqrt{\int_{\mathrm{supp}(\varphi)} d^D x \kappa T^2 \int d^D x(\partial^\perp_\mu\varphi\partial^\mu_\perp\varphi)\frac{\kappa\hat{Q}_\mu\hat{Q}^\mu}{T^2}},\\ &\le \sqrt{2\|\gamma(v)\|^2_\infty \int_{\mathrm{supp}(\varphi)} d^D x \kappa T^2 \int d^D x |\partial\varphi|^2_E \Sigma_\kappa},\end{aligned} \tag{D1}$$

using Eq. (A6) and the definition $\Sigma_\kappa = \kappa\hat{Q}_\mu\hat{Q}^\mu/T^2$ to obtain the last estimate. The second integral inside the square root is bounded when $\Sigma_{\rm therm} = \mathcal{D}-\lim_{\eta,\zeta,\kappa\to 0}\Sigma_\kappa$ exists, while the first integral vanishes in the limit. We conclude that $\partial_\mu(\kappa\hat{Q}^\mu) \overset{\mathcal{D}}{\to} 0$ as κ, η, $\zeta \to 0$.

We next consider $\partial_\mu S^\mu$ with the entropy current given by the energy-frame Israel-Stewart formula Eq. (96). We must show that only the term $\partial_\mu(sV^\mu)$ survives in the ideal limit and that all of the direct dissipative contributions vanish distributionally. The easiest one to treat is the $\lambda\sigma\hat{N}^\mu$ term in S^μ, which gives a vanishing contribution by the same argument used above for $\kappa\hat{Q}^\mu$.

The terms $\eta\beta_2\Sigma_\eta V^\mu$, $(1/2)\zeta\beta_0\Sigma_\zeta V^\mu$, $(1/2)\sigma\beta_1\Sigma_\sigma V^\mu$ all give contributions to $\partial_\mu S^\mu$ that are bounded in the same manner. We thus consider only the first one. After smearing by a test function φ, its contribution is bounded by

$$\left|\int d^D x(\partial_\mu\varphi)V^\mu\eta\beta_2\Sigma_\eta\right|$$
$$\leq \sqrt{2}\|\gamma(v)\|_\infty \int d^D x|\partial\varphi|_E\eta\beta_2\Sigma_\eta$$
$$\leq \sqrt{2}\|\gamma(v)\|_\infty \sqrt{\int_{\mathrm{supp}(\varphi)} d^D x\eta^2\beta_2^2\Sigma_\eta}$$
$$\times\sqrt{\int d^D x|\partial\varphi|_E^2\Sigma_\eta}, \tag{D2}$$

using $|\partial_\mu\varphi V^\mu| \leq |\partial\varphi|_E|V|_E$ and Eq. (A11) for the first inequality, and Cauchy-Schwartz for the second. Since $\Sigma_{\mathrm{shear}} = \mathcal{D}\text{-}\lim_{\eta,\zeta,\sigma\to 0}\Sigma_\eta$, the second square-root factor is bounded. For the first square-root factor, note that

$$\int_{\mathrm{supp}(\varphi)} d^D x\eta^2\beta_2^2\Sigma_\eta \leq \|\eta\beta_2\|^2_{L^\infty(\mathrm{supp}(\varphi))}\Sigma_\eta(\mathrm{supp}(\varphi)), \tag{D3}$$

with $\Sigma_\eta(K) = \int_K d^D x\Sigma_\eta$. For the compact set $\mathrm{supp}(\varphi)$, take $\psi\in C_c^\infty$ with $\psi\geq 0$ and $\psi|_{\mathrm{supp}(\varphi)} = 1$. Then, $\Sigma_\eta(\mathrm{supp}(\varphi)) \leq \int d^D x\psi\Sigma_\eta$, so

$$\lim_{\sigma,\eta,\zeta\to 0}\sup \Sigma_\eta(\mathrm{supp}(\varphi)) \leq \int d^D x\psi\Sigma_{\mathrm{shear}} \tag{D4}$$

follows also from $\Sigma_{\mathrm{shear}} = \mathcal{D}\text{-}\lim_{\eta,\zeta,\sigma\to 0}\Sigma_\eta$. Finally, since $\|\eta\beta_2\|_{L^\infty(\mathrm{supp}(\varphi))}\to 0$ as σ, η, $\zeta\to 0$, the upper bounds (D2)–(D4) show that the entire contribution vanishes in the ideal limit.

The terms $\eta\sigma(\alpha_1/T)\hat{\tau}^{\mu\nu}\hat{N}_\nu$, $\zeta\sigma(\alpha_0/T)\hat{\tau}\hat{N}^\mu$ also give contributions that are both bounded similarly, and we consider only the first. After smearing with a test function,

$$\left|\int d^D x(\partial_\mu\varphi)\eta\sigma\frac{\alpha_1}{T}\hat{\tau}^{\mu\nu}\hat{N}_\nu\right|$$
$$\leq \sqrt{\int_{\mathrm{supp}(\varphi)} d^D x\frac{1}{2}\sigma\eta\alpha_1^2 T\Sigma_\eta \int d^D x(\partial_\mu^\perp\varphi\partial_\perp^\mu\varphi)\Sigma_\sigma}, \tag{D5}$$

by the Cauchy-Schwartz inequality and the definitions of Σ_η, Σ_σ. Then,

$$\int d^D x(\partial_\mu^\perp\varphi\partial_\perp^\mu\varphi)\Sigma_\sigma \leq 2\|\gamma(v)\|^2_\infty \int d^D x|\partial\varphi_E|^2\Sigma_\sigma \tag{D6}$$

using Eq. (A6) and

$$\int_{\mathrm{supp}(\varphi)} d^D x\sigma\eta\alpha_1^2 T\Sigma_\eta \leq \|\sigma\eta\alpha_1^2 T\|_{L^\infty(\mathrm{supp}(\varphi))}\Sigma_\eta(\mathrm{supp}(\varphi)). \tag{D7}$$

The term $\Sigma_\eta(\mathrm{supp}(\varphi))$ is bounded as in Eq. (D4). In the ideal limit, $\|\sigma\eta\alpha_1^2 T\|_{L^\infty(\mathrm{supp}(\varphi))}\to 0$, and thus the bounds (D5)–(D7) imply that the contribution to $\partial_\mu S^\mu$ vanishes distributionally as σ, η, $\zeta\to 0$.

We conclude that $\partial_\mu S^\mu \xrightarrow{\mathcal{D}} \partial_\mu(sV^\mu)$ as σ, η, $\zeta\to 0$ when S^μ is given by the energy-frame formula Eq. (96). The argument for the entropy current of the particle-frame Israel-Stewart theory is identical, with the replacements $\sigma\to\kappa$, $\hat{N}^\mu\to\hat{Q}^\mu$.

APPENDIX E: RELATIVISTIC SHOCK SOLUTIONS

1. Reduced conformal model and shock solution

We consider here an exact family of shock solutions for dissipative relativistic fluid models in $1+1$ space-time dimensions, which were obtained in the previous work of Liu and Oz [21]. The $1+1$ fluid models considered by those authors are reduced conformal fluids (RCFs) obtained from a $D=(d+1)$-dimensional conformal fluid (note that our D is instead denoted 2σ in Ref. [21]) and have corresponding dimensionally reduced gravity duals [129]. We recall that the equation of state for the pressure in D-dimensional conformal fluids is given by a power of the temperature

$$p = \alpha T^D, \tag{E1}$$

with a dimensionless constant α. The tracelessness of the stress-energy tensor requires an energy density

$$\epsilon = (D-1)p = \alpha(D-1)T^D. \tag{E2}$$

There is no additional conserved current J^μ in the RCFs considered by Ref. [21], and consequently, $\lambda = 0$. The resulting first law of thermodynamics $d\epsilon = Tds$, as well as the homogeneous Gibbs relation $h = \epsilon + p = sT$, implies that the entropy density is

$$s = \alpha D T^{D-1} = D\alpha^{1/D}p^{(D-1)/D}. \tag{E3}$$

In the energy-frame description, the nonideal part of the stress tensor (4) is transverse to the velocity, $V_\mu\Pi^{\mu\nu} = 0$. As in Ref. [21], we consider only first-order terms in the gradient expansion. Since bulk viscosity $\zeta = 0$ for conformal fluids, the only transport coefficient at this order is shear viscosity η with $\Pi^{\mu\nu} = -2\eta\sigma^{\mu\nu}$. Upon reduction to $1+1$ dimensions, this appears as an effective bulk viscosity, so

$$\Pi^{\mu\nu} = -\zeta\theta\Delta^{\mu\nu}, \tag{E4}$$

with $\zeta = (1/2\pi)(D-2/D-1)s$. However, just as in Ref. [21], we take $\zeta := \zeta(T)$ to be an arbitrary function since none of our results depends upon any particular choice.

Representing the two-velocity as $V^\mu = \gamma_v(1, \beta_v)$, any stationary solution of the 1 + 1 viscous model satisfies

$$\frac{d}{dx}[(Dp - \zeta\theta)\gamma_v^2\beta_v] = 0, \tag{E5}$$

$$\frac{d}{dx}[p(1 + D\gamma_v^2\beta_v^2) - \zeta\theta\gamma_v^2] = 0. \tag{E6}$$

Equations (E5) and (E6) follow from $\nabla_\mu T^{\mu\nu} = 0$, setting $\nu = 0, 1$ and they imply

$$f_e = (Dp - \zeta\theta)\gamma_v^2\beta_v, \tag{E7}$$

$$f_p = p(1 + D\gamma_v^2\beta_v^2) - \zeta\theta\gamma_v^2, \tag{E8}$$

where $f_e \equiv T^{01}$ and $f_p \equiv T^{11}$ are constant energy and momentum fluxes. Using Eqs. (E7) and (E8), Ref. [21] obtained smooth viscous shock solutions by quadrature. We do not employ these integral expressions but only use the following important consequences of Eqs. (E7) and (E8):

$$\epsilon = (D - 1)p = f_e/\beta_v - f_p, \tag{E9}$$

$$p - \zeta\theta = f_p - f_e\beta_v. \tag{E10}$$

In particular, the representation (E9) of the pressure in terms of the velocity is analogous to the Bernoulli-type relation exploited by Becker to study shock solutions of the nonrelativistic compressible Navier-Stokes equations for $Pr = 3/4$ [130]. Together, Eqs. (E9) and (E10) completely determine $\zeta\theta$ in terms of the velocity, yielding identical results for any choice of viscosity $\zeta(T)$.

The viscous model solutions of interest converge in the infinite Reynolds-number limit to stationary shock solutions of the relativistic Euler equations. These are piecewise constant, with a preshock velocity β_0 to the left and a postshock value β_1 to the right. The possible values are obtained by equating the two expressions for the pressure from Eqs. (E9) and (E10) with $\zeta = 0$:

$$f_e/\beta_v - f_p = (D - 1)p = (D - 1)[f_p - f_e\beta_v]. \tag{E11}$$

This yields a quadratic polynomial in β_v with coefficients depending upon D and $R \coloneqq f_p/f_e$. The condition for two distinct real roots is $|R| > 2(D - 1)^{1/2}/D$. The product of the roots is given by

$$\beta_0\beta_1 = 1/(D - 1) \coloneqq \beta_s^2, \tag{E12}$$

where $\beta_s = c_s/c$ and $c_s = c/\sqrt{D - 1}$ is the sound speed. The condition $h = \epsilon + p = D \cdot p > 0$ requires that both sides of Eq. (E11) are positive. Using the quadratic formula for the roots, it is easy to check that this holds if and only if $|R| < 1$. The simultaneous conditions

$$1 > |R| > 2(D - 1)^{1/2}/D \tag{E13}$$

require $D > 2$ in order for inviscid shock solutions to exist. A relation between pressures p_0, p_1 or temperatures T_0, T_1 on both sides of the shock can be obtained by using Eq. (E7) for $\zeta = 0$, which gives

$$p_0/p_1 = (T_0/T_1)^D = (\beta_1\gamma_1^2/\beta_0\gamma_0^2). \tag{E14}$$

Equations (E12) and (E14) imply that the fluid on one side of the shock has supersonic velocity and lower temperature, whereas the other side is subsonic with higher temperature. As noted in Ref. [21], positive entropy production requires that colder supersonic fluid flows into the shock front and hotter subsonic fluid flows out.

We derive here all of the source terms that appear in the internal-energy and entropy balances for these shock solutions, both those in the fine-grained (dissipation-range) balances as $\zeta \to 0$ and those in the coarse-grained (inertial-range) balances as $\ell \to 0$. It should be pointed out that first-order dissipative relativistic fluid models of the type considered are acausal and have unstable solutions even at global equilibrium [61]. Thus, the viscous shock solutions obtained by Ref. [21] are expected to be unstable to small perturbations. However, they are exact stationary solutions that, as $\zeta \to 0$, converge in L^p norms for any $p \in [1, \infty)$ to stationary shock solutions of relativistic Euler equations and thus provide an example for our general mathematical framework. We emphasize that the viscous model solutions are employed only to evaluate dissipation-range quantities, whereas all of our inertial-range limit results hold with complete generality for all relativistic Euler shocks with the equation of state (E1). Inviscid solution fields are all discontinuous step functions

$$f(x) = \begin{cases} f_0 & x < 0 \\ f_1 & x > 0 \end{cases} = f_0 + (\Delta f)\theta(x), \tag{E15}$$

where $\Delta f = f_1 - f_0$ and $\theta(x)$ is the Heaviside step function. We also use the notation $f_{av} = \frac{1}{2}(f_0 + f_1)$ for the average value on both sides of the shock. A fact that we use frequently for ideal step-function fields is

$$\bar{f}(x) = f_0 + (\Delta f)\bar{\theta}(x), \qquad \bar{g}(x) = g_0 + (\Delta g)\bar{\theta}(x) \tag{E16}$$

and thus

$$\bar{g} = g_0 + \frac{\Delta g}{\Delta f}(\bar{f} - f_0), \qquad \partial_x\bar{g} = \frac{\Delta g}{\Delta f}\partial_x\bar{f}. \tag{E17}$$

Furthermore,

$$\partial_x\bar{f}(x) = (\Delta f)\bar{\delta}(x). \tag{E18}$$

The coarse graining that is employed here is purely spatial, with a kernel G. Because the solutions are stationary in the rest frame of the shock, there is no need for temporal coarse graining.

2. Energy balance

a. Dissipation range

It can be easily shown for stationary shocks of these RCFs that $\mathcal{Q}_{\rm diss}$ and $p*\theta$ exist as distributions separately, not just in combination. The fine-grained energy balance equation (9) in the $\zeta\to 0$ limit thus reads

$$\partial_x(\epsilon\gamma_v\beta_v) = \mathcal{Q}_{\rm diss} - p*\theta. \tag{E19}$$

We now calculate the two distributions $\mathcal{Q}_{\rm diss}$ and $p*\theta$ appearing above as sources or sinks of the energy density.

Viscous pressure work $p*\theta$: Direct differentiation yields the dilatation factor

$$\theta := \partial_x(\gamma_v\beta_v) = \gamma_v^3\partial_x\beta_v, \tag{E20}$$

and making use of the Bernoulli relation (E9) for the pressure, one obtains

$$(D-1)p\theta = \left(\frac{f_e}{\beta_v} - f_p\right)\gamma_v^3\partial_x\beta_v. \tag{E21}$$

It is straightforward to check that the right-hand side of Eq. (E21) can be expressed as a total x derivative:

$$(D-1)p\theta = \frac{d}{dx}\left[f_e\ln\left(\frac{\beta_v}{1+\sqrt{1-\beta_v^2}}\right) + (D-1)\gamma_v\beta_v p\right].$$

The distributional limit as $\zeta\to 0$ is thus found to be

$$\begin{aligned} p*\theta &= \mathcal{D}\text{-}\lim_{\zeta\to 0} p\theta \\ &= \left\{\frac{f_e}{D-1}\ln\left(\frac{\beta_1}{\beta_0}\frac{1+\sqrt{1-\beta_0^2}}{1+\sqrt{1-\beta_1^2}}\right) + \Delta[\gamma_v\beta_v p]\right\}\delta(x). \end{aligned} \tag{E22}$$

Viscous dissipation $\mathcal{Q}_{\rm diss}$: The simplest approach to derive $\mathcal{Q}_{\rm diss}$ is to use the fine-grained energy balance

$$\zeta\theta^2 - p\theta = \partial_x(\epsilon\gamma_v\beta_v) \tag{E23}$$

to obtain that, as $\zeta\to 0$,

$$\mathcal{Q}_{\rm diss} - p*\theta = \Delta[\gamma_v\beta_v\epsilon]\delta(x). \tag{E24}$$

From Eqs. (E22) and (E24), we get

$$\begin{aligned} \mathcal{Q}_{\rm diss} &= \mathcal{D}\text{-}\lim_{\zeta\to 0}\zeta\theta^2 \\ &= \left\{\frac{f_e}{D-1}\ln\left(\frac{\beta_1}{\beta_0}\frac{1+\sqrt{1-\beta_0^2}}{1+\sqrt{1-\beta_1^2}}\right) + D\Delta[\gamma_v\beta_v p]\right\}\delta(x). \end{aligned} \tag{E25}$$

b. Inertial range

The resolved energy in the limit $\zeta\to 0$ satisfies

$$\partial_x(\overline{\gamma_v\beta_v}\,\bar\epsilon) = \mathcal{Q}_\ell^{\rm flux} - \bar p\,\bar\theta. \tag{E26}$$

We now calculate the distributional limit as $\ell\to 0$ of the two terms appearing above as sources or sinks.

Inertial pressure work $p\circ\theta$: Since γ_v, β_v, and p are all step functions in the ideal limit, Eq. (E17) gives

$$\bar p\partial_x\overline{\gamma_v\beta_v} = \frac{\Delta[\gamma_v\beta_v]}{\Delta p}\partial_x\left(\frac{1}{2}\bar p^2\right).$$

It follows that

$$p\circ\theta = \mathcal{D}\text{-}\lim_{\ell\to 0}\bar p\partial_x\overline{\gamma_v\beta_v} = p_{av}\Delta[\gamma_v\beta_v]\delta(x). \tag{E27}$$

Note that, as required, this result is completely independent of the choice of the filter kernel G.

Energy flux $\mathcal{Q}_{\rm flux}$: By the definition in Eq. (64),

$$\begin{aligned} \mathcal{Q}_{\rm flux} &= -\mathcal{D}\text{-}\lim_{\ell\to 0}\overline{(hu^\mu u^\nu)}\nabla_\mu\bar u_\nu \\ &= \mathcal{D}-\lim_{\ell\to 0}(\overline{(h\gamma_v^2\beta_v)}\partial_x\overline{\gamma_v} - \overline{(h\gamma_v^2\beta_v)\beta_v}\partial_x\overline{\gamma_v\beta_v}). \end{aligned}$$

Enthalpy can be replaced with pressure using $h = D\cdot p$. The balance (E7) with $\zeta = 0$ for both terms then gives, in the limit $\ell\to 0$,

$$\begin{aligned} D\overline{(p\gamma_v^2\beta_v)}\partial_x\overline{\gamma_v} &= f_e\partial_x\overline{\gamma_v} \xrightarrow{\mathcal{D}} f_e\Delta\gamma_v\delta(x), \\ D\overline{(p\gamma_v^2\beta_v)\beta_v}\partial_x\overline{\gamma_v\beta_v} &= f_e\overline{\beta_v}\partial_x\overline{\gamma_v\beta_v} \\ &= f_e\frac{\Delta(\overline{\gamma_v\beta_v})}{\Delta\beta_v}\partial_x(\overline{\beta_v}^2) \\ &\xrightarrow{\mathcal{D}} f_e\Delta[\gamma_v\beta_v]\beta_v^{av}\delta(x), \end{aligned}$$

where Eq. (E17) was used for the second term. Together, these yield

$$\mathcal{Q}_{\rm flux} = f_e\{\Delta\gamma_v - \beta_v^{av}\Delta[\gamma_v\beta_v]\}\delta(x). \tag{E28}$$

We see again that the limiting inertial range result is independent of the choice of filter kernel G. To compare this term with those previously calculated, we note that, for any ideal shock solution, Eq. (E9) implies

$$\Delta(\epsilon\gamma_v\beta_v) = f_e\Delta\gamma_v - f_p\Delta(\gamma_v\beta_v), \tag{E29}$$

and Eq. (E10) with $\zeta = 0$ implies that

$$p_{av} = f_p - f_e\beta_{av}. \tag{E30}$$

These relations can be used to rewrite the formula (E28) for $\mathcal{Q}_{\rm flux}$ as

$$\mathcal{Q}_{\rm flux} = \{\Delta[\gamma_v\beta_v\epsilon] + p_{av}\Delta[\gamma_v\beta_v]\}\delta(x). \tag{E31}$$

Equations (E27) and (E31) immediately show that

$$\mathcal{Q}_{\rm flux} - p\circ\theta = \Delta[\gamma_v\beta_v\epsilon]\delta(x), \tag{E32}$$

as required by the limit of the balance (E26).

The relation (E28) has a further interesting implication that $\mathcal{Q}_{\rm flux} < 0$ for relativistic Euler shocks with the equation of state (E1). Using the relation (E12) for the product $\beta_0\beta_1$, it is easy to show that

$$J(\beta_0, D) \coloneqq \beta_{av}\frac{\Delta(\gamma_v\beta_v)}{\Delta\gamma_v} = \frac{1}{2}\left(\frac{1}{\gamma_0\gamma_1} + \frac{D}{D-1}\right), \tag{E33}$$

which may be regarded as a function of just one of the two velocities (say, β_0) and D. Using the above definition and Eq. (E28),

$$\mathcal{Q}_{\rm flux} = f_e\Delta\gamma_v[1 - J(\beta_0, D)]\delta(x). \tag{E34}$$

As noted earlier, positive entropy production at the shock requires that $\Delta\gamma_v < 0$, so $\mathcal{Q}_{\rm flux} < 0$ if the second factor in Eq. (E34) is positive over the range $\beta_s < \beta_0 < 1$. Direct calculation of the derivative gives

$$\frac{\partial}{\partial\beta_0}J(\beta_0, D) = -(\beta_0^4 - \beta_s^4)\frac{\gamma_0\gamma_1}{\beta_0^3} < 0, \tag{E35}$$

while

$$J(\beta_s, D) = 1, \qquad J(1, D) = \frac{D}{2(D-1)} > \frac{1}{2}. \tag{E36}$$

Thus, $1/2 < J(\beta_0, D) < 1$ over the permitted range of β_0, so the second factor in Eq. (E34) remains positive and $\mathcal{Q}_{\rm flux} < 0$. This is a more extreme version of what occurs for shocks in a nonrelativistic, compressible Navier-Stokes fluid, where $Q_{\rm flux} = 0$ (Appendix A of paper I). In both cases, irreversible shock heating is not due to energy cascade, and in the relativistic case, inverse energy cascade even contributes cooling rather than heating.

Pressure-dilatation defect: By subtracting Eq. (E27) from Eq. (E22), we find

$$\begin{aligned}\tau(p,\theta) &\equiv p*\theta - p\circ\theta \\ &= \left\{\Delta[\gamma_v\beta_v p] - p_{av}\Delta[\gamma_v\beta_v] \right.\\ &\quad \left. + \frac{f_e}{D-1}\ln\left(\frac{\beta_1}{\beta_0}\frac{1+\sqrt{1-\beta_0^2}}{1+\sqrt{1-\beta_1^2}}\right)\right\}\delta(x).\end{aligned} \tag{E37}$$

Together with Eqs. (E24) and (E31), this yields

$$\mathcal{Q}_{\rm diss} = \mathcal{Q}_{\rm flux} + \tau(p,\theta). \tag{E38}$$

The latter equality can also be obtained by comparing the relations (E24) and (E32), corroborating the general result (77). Because $\mathcal{Q}_{\rm diss} > 0$ whereas $\mathcal{Q}_{\rm flux} < 0$, it follows that $\tau(p,\theta) > 0$. Just as for the nonrelativistic shocks discussed in paper I, the pressure-dilation defect is responsible for the net irreversible heating at the shock.

3. Entropy balance

a. Dissipation range

The fine-grained entropy balance for stationary solutions is given simply by

$$\partial_x(s\gamma_v\beta_v) = \frac{\zeta\theta^2}{T}. \tag{E39}$$

Viscous entropy production: It follow immediately from the above that, for discontinuous shock solutions,

$$\Sigma_{\rm diss} \coloneqq \mathcal{D}\text{-}\lim_{\zeta\to 0}\frac{\zeta\theta^2}{T} = \Delta[\gamma_v\beta_v s]\delta(x). \tag{E40}$$

The entropy production anomaly is thus completely independent of the details of the molecular dissipation and, obviously, $\Sigma_{\rm diss} \geq 0$. As already noted in Ref. [21], this positivity is equivalent to the condition that

$$1 < \frac{s_1\gamma_1\beta_1}{s_0\gamma_0\beta_0} = \left(\frac{\beta_1\gamma_0^{D-2}}{\beta_0\gamma_1^{D-2}}\right)^{1/D}, \tag{E41}$$

where Eq. (E14) has been used to obtain the second expression. This ratio is 1 for $\beta_0 = \beta_1 = \beta_s$ and, considered as a function of β_0 and D, it is shown, by a straightforward calculation, to have a positive β_0-derivative for $\beta_0 \neq \beta_s$. This implies that $\beta_0 > \beta_s > \beta_1$ is required for positive entropy production, as claimed earlier.

b. Inertial range

The resolved entropy equation for stationary solutions of the RCF models is

$$\begin{aligned}\partial_x(\underline{s}\,\overline{\gamma_v\beta_v} + \underline{\beta}\,\bar{\tau}(\epsilon,\gamma_v\beta_v)) \\ = \Sigma_\ell^{\rm flux} + \underline{\beta}(\overline{\mathcal{Q}}_{\rm diss} - \bar{\tau}(p,\theta)),\end{aligned} \tag{E42}$$

where $\Sigma_\ell^{\rm flux} = \partial_x \underline{\beta}\,\bar{\tau}(\epsilon, \gamma_v\beta_v)$. This entropy evolution equation is considerably simpler than the general Eq. (86) since $\lambda = 0$ and because the pressure is proportional to the energy density so that $I_\ell^{\rm flux} \equiv 0$.

Inertial-range viscous heating $\beta\circ\mathcal{Q}_{\rm diss}$: From Eq, (E25), $\mathcal{Q}_{\rm diss} = q_*\delta(x)$, so

$$\bar{\mathcal{Q}}_{\rm diss} = q_*\bar{\delta}(x). \tag{E43}$$

From the formula (E2), we see that the inverse temperature $\beta = 1/T$ satisfies $\beta = \alpha^{1/D}p^{-1/D}$, and thus

$$\underline{\beta} = \alpha^{1/D}\left(\frac{\bar{\epsilon}}{D+1}\right)^{-1/D} = \alpha^{1/D}\bar{p}^{-1/D}. \tag{E44}$$

Using Eq. (E18) to write $\bar{\delta} = \frac{\partial_x\bar{p}}{\Delta p}$, we get

$$\underline{\beta}\bar{\mathcal{Q}}_{\rm diss} = \alpha^{1/D}\frac{Dq_*}{D-1}\frac{1}{\Delta p}\frac{d}{dx}[\bar{p}^{(D-1)/D}] = \frac{q_*}{\Delta\epsilon}\frac{d\underline{s}}{dx}, \tag{E45}$$

and therefore, as $\ell \to 0$,

$$\beta\circ\mathcal{Q}_{\rm diss} = q_*\frac{\Delta s}{\Delta\epsilon}\delta(x). \tag{E46}$$

Pressure-dilatation defect $\beta\circ\tau(p,\theta)$: Our earlier result in Eq. (E22), that $p*\theta = q_{PV}\delta(x)$, yields, by the same argument,

$$\mathcal{D}-\lim_{\ell\to 0}\underline{\beta}\,\overline{p*\theta} = q_{PV}\frac{\Delta s}{\Delta\epsilon}\delta(x). \tag{E47}$$

On the other hand, using Eq. (E44), we have

$$\underline{\beta}\,\bar{p}\,\bar{\theta} = \alpha^{1/D}\bar{p}^{\frac{D-1}{D}}\partial_x(\overline{\gamma_v\beta_v}) = \frac{D\alpha^{1/D}}{2D-1}\frac{\Delta[\gamma_v\beta_v]}{\Delta p}\partial_x[\bar{p}^{\frac{2D-1}{D}}]. \tag{E48}$$

Thus,

$$\mathcal{D}\text{-}\lim_{\ell\to 0}\underline{\beta}\,\bar{p}\,\bar{\theta} = \frac{D\alpha^{1/D}}{2D-1}\frac{\Delta[\gamma_v\beta_v]}{\Delta p}\Delta[p^{\frac{2D-1}{D}}]\delta(x). \tag{E49}$$

The following relations are useful and follow directly from Eqs. (E2) and (E3):

$$\Delta\left(\frac{1}{T}\right) = \alpha^{1/D}\Delta[p^{-1/D}] = \frac{(D-1)(s_1\epsilon_0 - s_0\epsilon_1)}{D\epsilon_0\epsilon_1}, \tag{E50}$$

$$\Delta\left(\frac{p^2}{T}\right) = \alpha^{1/D}\Delta[p^{\frac{2D-1}{D}}] = \frac{\Delta[s\epsilon]}{D(D-1)}. \tag{E51}$$

With these, we have that $I_{\rm mech} = \beta\circ\tau(p,\theta)$ is given by

$$\beta\circ\tau(p,\theta) = \frac{1}{\Delta\epsilon}\left[q_{PV}\Delta s - \frac{\Delta[\gamma_v\beta_v]\Delta[s\epsilon]}{2D-1}\right]\delta(x). \tag{E52}$$

Combined contribution $\beta\circ\mathcal{Q}_{\rm diss} - \beta\circ\tau(p,\theta)$: From Eq. (E24), we obtain

$$q_* - q_{PV} = \Delta[\gamma_v\beta_v\epsilon]. \tag{E53}$$

Thus, the combined contribution of these terms is simply

$$\beta\circ\mathcal{Q}_{\rm diss} - \beta\circ\tau(p,\theta) = \frac{1}{\Delta\epsilon}\left[\Delta s\Delta[\gamma_v\beta_v\epsilon] + \frac{\Delta[\gamma_v\beta_v]\Delta[s\epsilon]}{2D-1}\right]\delta(x). \tag{E54}$$

Negentropy flux $\Sigma_\ell^{\rm flux}$: First, we consider the contribution $(\partial_x\underline{\beta})\overline{\epsilon\gamma_v\beta_v}$. From Eq. (E44), we have

$$\partial_x\underline{\beta} = -\frac{\alpha^{1/D}}{D}\bar{p}^{-\frac{D+1}{D}}\partial_x\bar{p}. \tag{E55}$$

Using Eq. (E16) to write

$$\overline{\epsilon\gamma_v\beta_v} = \epsilon_0\gamma_0\beta_0 + \frac{\Delta(\epsilon\gamma_v\beta_v)}{\Delta p}(\bar{p} - p_0), \tag{E56}$$

a straightforward calculation shows

$$(\partial_x\underline{\beta})\overline{\epsilon\gamma_v\beta_v} \overset{\mathcal{D}}{\to} -\frac{1}{D}\frac{\Delta s\Delta[\gamma_v\beta_v\epsilon]}{\Delta\epsilon}\delta(x) - \left(\frac{\Delta[\gamma_v\beta_v\epsilon] - \gamma_0\beta_0\Delta\epsilon}{\Delta\epsilon}\right)\epsilon_0\alpha^{-1/D}\Delta[p^{-1/D}]\delta(x). \tag{E57}$$

The other term is computed likewise using

$$(\partial_x\underline{\beta})\bar{\epsilon}\,\overline{\gamma_v\beta_v} = -\alpha^{1/D}\frac{D-1}{D}\bar{p}^{-\frac{1}{D}}\partial_x\bar{p}\left(\gamma_0\beta_0 + \frac{\Delta[\gamma\beta]}{\Delta p}(\bar{p} - p_0)\right), \tag{E58}$$

where, after some calculation, one has

$$(\partial_x\underline{\beta})\bar{\epsilon}\,\overline{\gamma_v\beta_v} \overset{\mathcal{D}}{\to} -\alpha^{1/D}\frac{(D-1)^2}{2D-1}\frac{\Delta[\gamma_v\beta_v]}{\Delta\epsilon}\Delta[p^{(2D-1)/D}]\delta(x) + \frac{1}{D}\left(\frac{\Delta[\gamma_v\beta_v]\epsilon_0 - \gamma_0\beta_0\Delta\epsilon}{\Delta\epsilon}\right)\Delta s\delta(x). \tag{E59}$$

Therefore, Eqs. (E57) and (E59) in combination show

$$\begin{aligned}\Sigma_{\rm flux} &:= \mathcal{D}\text{-}\lim_{\ell\to 0}\partial_x \underline{\beta}\,\bar{\tau}(\epsilon,\gamma_v\beta_v)\\ &= \alpha^{1/D}\frac{(D-1)^2}{2D-1}\frac{\Delta[\gamma_v\beta_v]}{\Delta\epsilon}\Delta[p^{(2D-1)/D}]\delta(x)\\ &\quad -\frac{1}{D}(\Delta[\gamma_v\beta_v]\epsilon_0-\gamma_0\beta_0\Delta\epsilon+\Delta[\gamma_v\beta_v\epsilon])\frac{\Delta s}{\Delta\epsilon}\delta(x)\\ &\quad -\alpha^{1/D}\left(\frac{\Delta[\gamma_v\beta_v\epsilon]-\gamma_0\beta_0\Delta\epsilon}{\Delta\epsilon}\right)\epsilon_0\Delta[p^{-1/D}]\delta(x).\end{aligned} \tag{E60}$$

The relations (E50) and (E51) can then be employed to simplify the expression for the flux to

$$\Sigma_{\rm flux} = \frac{1}{\Delta\epsilon}\left\{(\epsilon_1 s_0-\epsilon_0 s_1)\Delta[\gamma_v\beta_v]-\frac{\Delta[\gamma_v\beta_v]\Delta[s\epsilon]}{2D-1}\right\}\delta(x). \tag{E61}$$

Adding together the formulas (E54) and (E61), one has, after minor manipulation, that

$$\beta\circ\mathcal{Q}_{\rm diss}-\beta\circ\tau(p,\theta)+\Sigma_{\rm flux}=\Delta[\gamma_v\beta_v s]\delta(x), \tag{E62}$$

in agreement with Eq. (E42) and the dissipation-range result (E40), as demanded by the general equality Eq. (98).

A further implication of the formula (E61) for entropy flux is that $\Sigma_{\rm flux}>0$ at these relativistic Euler shocks. Although not presented here, arguments like those applied to $\mathcal{Q}_{\rm flux}$ show this result and are confirmed by numerically plotting Eq. (E61) as a function of R for each $D>2$. It is interesting that $\Sigma_{\rm flux}>0$ was also found for planar shock solutions of nonrelativistic compressible Euler equations in paper I. In both cases, there is a forward cascade of negentropy at the shock, even though the energy flux is vanishing or negative.

[1] R. Narayan and P. Kumar, *A Turbulent Model of Gamma-Ray Burst Variability*, Mon. Not. R. Astron. Soc. **394**, L117 (2009).
[2] N. Bucciantini, *Review of the Theory of Pulsar-Wind Nebulae*, Astron. Nachr. **335**, 234 (2014).
[3] R. Derradi De Souza, T. Koide, and T. Kodama, *Hydrodynamic Approaches in Relativistic Heavy Ion Reactions*, Prog. Part. Nucl. Phys. **86**, 35 (2016).
[4] L. Fritz, J. Schmalian, M. Müller, and S. Sachdev, *Quantum Critical Transport in Clean Graphene*, Phys. Rev. B **78**, 085416 (2008).
[5] A. Lucas, J. Crossno, K. C. Fong, P. Kim, and S. Sachdev, *Transport in Inhomogeneous Quantum Critical Fluids and in the Dirac Fluid in Graphene*, Phys. Rev. B **93**, 075426 (2016).
[6] C. Hoyos, B. S. Kim, and Y. Oz, *Lifshitz Hydrodynamics*, J. High Energy Phys. 11 (2013) 145.
[7] R. A. Davison, K. Schalm, and J. Zaanen, *Holographic Duality and the Resistivity of Strange Metals*, Phys. Rev. B **89**, 245116 (2014).
[8] R. Baier, P. Romatschke, D. T. Son, A. O. Starinets, and M. A. Stephanov, *Relativistic Viscous Hydrodynamics, Conformal Invariance, and Holography*, J. High Energy Phys. 04 (2008) 100.
[9] S. Bhattacharyya, S. Minwalla, V. E. Hubeny, and M. Rangamani, *Nonlinear Fluid Dynamics from Gravity*, J. High Energy Phys.02 (**2008**) 045.
[10] S. Bhattacharyya, V. E. Hubeny, R. Loganayagam, G. Mandal, S. Minwalla, T. Morita, M. Rangamani, and H. S. Reall, *Local Fluid Dynamical Entropy from Gravity*, J. High Energy Phys. 06(2008) 055.
[11] S. Bhattacharyya, R. Loganayagam, I. Mandal, S. Minwalla, and A. Sharma, *Conformal Nonlinear Fluid Dynamics from Gravity in Arbitrary Dimensions*, J. High Energy Phys. 12 (2008) 116.
[12] T. Piran, *The Physics of Gamma-Ray Bursts*, Rev. Mod. Phys. **76**, 1143 (2005).
[13] J. Zrake and A. I. MacFadyen, *Spectral and Intermittency Properties of Relativistic Turbulence*, Astrophys. J. Lett. **763**, L12 (2013).
[14] D. Radice and L. Rezzolla, *Universality and Intermittency in Relativistic Turbulent Flows of a Hot Plasma*, Astrophys. J. Lett. **766**, L10 (2013).
[15] F. Carrasco, L. Lehner, R. C. Myers, O. Reula, and A. Singh, *Turbulent Flows for Relativistic Conformal Fluids in* $2+1$ *Dimensions*, Phys. Rev. D **86**, 126006 (2012).
[16] S. R. Green, F. Carrasco, and L. Lehner, *Holographic Path to the Turbulent Side of Gravity*, Phys. Rev. X **4**, 011001 (2014).
[17] J. R. Westernacher-Schneider, L. Lehner, and Y. Oz, *Scaling Relations in Two-Dimensional Relativistic Hydrodynamic Turbulence*, J. High Energy Phys. 12 (2015) 067.
[18] A. Adams, P. M. Chesler, and H. Liu, *Holographic Turbulence*, Phys. Rev. Lett. **112**, 151602 (2014).
[19] I. Fouxon and Y. Oz, *Exact Scaling Relations in Relativistic Hydrodynamic Turbulence*, Phys. Lett. B **694**, 261 (2010).
[20] C. Eling, I. Fouxon, and Y. Oz, *Gravity and a Geometrisation of Turbulence: An Intriguing Correspondence*, Contemp. Phys. **52**, 43 (2011).
[21] X. Liu and Y. Oz, *Shocks and Universal Statistics in* $(1+1)$*-Dimensional Relativistic Turbulence*, J. High Energy Phys. 03 (2011) 6.
[22] A. N. Kolmogorov, *Dissipation of Energy in Locally Isotropic Turbulence*, Dokl. Akad. Nauk SSSR **32**, 16 (1941).
[23] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, England, 1995).
[24] W. Israel and J. M. Stewart, *Transient Relativistic Thermodynamics and Kinetic Theory*, Ann. Phys. (N.Y.) **118**, 341 (1979).
[25] W. Israel and J. M. Stewart, *On Transient Relativistic Thermodynamics and Kinetic Theory. II*, Proc. R. Soc. A **365**, 43 (1979).
[26] P. Romatschke, *New Developments in Relativistic Viscous Hydrodynamics*, Int. J. Mod. Phys. E **19**, 1 (2010).

[27] L. Onsager, *The Distribution of Energy in Turbulence*, in *Minutes of the Meeting of the Metropolitan Section held at Columbia University, New York, 1945* [*Phys. Rev.* 68, 281 (1945)].

[28] L. Onsager, *Statistical Hydrodynamics*, Nuovo Cim. **6**, 279 (1949).

[29] G. L. Eyink and K. R. Sreenivasan, *Onsager and the Theory of Hydrodynamic Turbulence*, Rev. Mod. Phys. **78**, 87 (2006).

[30] A Polyakov, *Conformal Turbulence*, arXiv:hep-th/9209046.

[31] A. M. Polyakov, *The Theory of Turbulence in Two Dimensions*, Nucl. Phys. **B396**, 367 (1993).

[32] A. N. Kolmogorov, *The Local Structure of Turbulence in Incompressible Viscous Fluid for Very Large Reynolds Numbers*, Dokl. Akad. Nauk SSSR **30**, 9 (1941).

[33] A. N. Kolmogorov, *On the Degeneration of Isotropic Turbulence in an Incompressible Viscous Fluid*, Dokl. Akad. Nauk SSSR **31**, 538 (1941).

[34] G. L. Eyink, *Energy Dissipation without Viscosity in Ideal Hydrodynamics I. Fourier Analysis and Local Energy Transfer*, Physica D (Amsterdam) **78**, 222 (1994).

[35] P. Constantin, W. E., and E. Titi, *Onsager's Conjecture on the Energy Conservation for Solutions of Euler's Equation*, Commun. Math. Phys. **165**, 207 (1994).

[36] J. Duchon and R. Robert, *Inertial Energy Dissipation for Weak Solutions of Incompressible Euler and Navier-Stokes Equations*, Nonlinearity **13**, 249 (2000).

[37] A. Cheskidov, P. Constantin, S. Friedlander, and R. Shvydkoy, *Energy Conservation and Onsager's Conjecture for the Euler Equations*, Nonlinearity **21**, 1233 (2008).

[38] C. De Lellis and L. Székelyhidi, Jr., *The h-Principle and the Equations of Fluid Dynamics*, Bull. Am. Math. Soc. **49**, 347 (2012).

[39] C. De Lellis and L. Székelyhidi, Jr., *Continuous Dissipative Euler Flows and a Conjecture of Onsager*, in *European Congress of Mathematics, Kraków, 2012*, edited by R. Latała, A. Ruciński, P. Strzelecki, J. Świątkowski, and D. Wrzosek (European Mathematical Society, Zürich, 2013), pp. 13–30.

[40] P. Isett, *A Proof of Onsager's Conjecture*, arXiv:1608.08301.

[41] R. E. Caflisch, I. Klapper, and G. Steele, *Remarks on Singularities, Dimension and Energy Dissipation for Ideal Hydrodynamics and MHD*, Commun. Math. Phys. **184**, 443 (1997).

[42] H. Aluie, *Coarse-Grained Incompressible Magnetohydrodynamics: Analyzing the Turbulent Cascades*, New J. Phys. **19**, 025008 (2017).

[43] G. L. Eyink, *Turbulent Cascade of Circulations*, C.R. Phys. **7**, 449 (2006).

[44] G. L. Eyink and H. Aluie, *The Breakdown of Alfvén's Theorem in Ideal Plasma Flows: Necessary Conditions and Physical Conjectures*, Physica D (Amsterdam) **223**, 82 (2006).

[45] H. Aluie, *Compressible Turbulence: The Cascade and Its Locality*, Phys. Rev. Lett. **106**, 174502 (2011).

[46] H. Aluie, *Scale Decomposition in Compressible Turbulence*, Physica D (Amsterdam) **247**, 54 (2013).

[47] G. L. Eyink and T. D. Drivas, preceding paper, *Cascades and Dissipative Anomalies in Compressible Fluid Turbulence*, Phys. Rev. X **8**, 011022 (2018).

[48] T. D. Drivas and G. L. Eyink, *An Onsager Singularity Theorem for Turbulent Solutions of Compressible Euler Equations*, Commun. Math. Phys., DOI: 10.1007/s00220-017-3078-4 (2017).

[49] E. C. G. Stueckelberg and A. Petermann, *La Normalisation des Constantes Dans la Théorie des Quanta*, Helv. Phys. Acta **26**, 499 (1953).

[50] M. Gell-Mann and F. E. Low, *Quantum Electrodynamics at Small Distances*, Phys. Rev. **95**, 1300 (1954).

[51] N. N. Bogolyubov and D. V. Shirkov, *Group of Charge Renormalization in Quantum Field Theory*, Dokl. Akad. Nauk SSSR **103**, 391 (1955).

[52] D. J. Gross, *Applications of the Renormalization Group to High-Energy Physics*, in *Methods in Field Theory, Les Houches 1975, Session XVIII*, edited by R. Balian and J. Zinn-Justin (North-Holland Publishing, Amsterdam, 1976), pp. 141–250.

[53] G. L. Eyink, *Local Energy Flux and the Refined Similarity Hypothesis*, J. Stat. Phys. **78**, 335 (1995).

[54] K. G. Wilson, *Confinement of Quarks*, Phys. Rev. D **10**, 2445 (1974).

[55] M. Haack and A. Yarom, *Nonlinear Viscous Hydrodynamics in Various Dimensions Using* AdS/CFT, J. High Energy Phys. 10 (**2008**) 063.

[56] N. Andersson and G. L. Comer, *Relativistic Fluid Dynamics: Physics for Many Different Scales*, Living Rev. Relativity **10**, 1 (2007).

[57] L. Rezzolla and O. Zanotti, *Relativistic Hydrodynamics* (Oxford University Press, New York, 2013).

[58] C. Eckart, *The Thermodynamics of Irreversible Processes. III. Relativistic Theory of the Simple Fluid*, Phys. Rev. **58**, 919 (1940).

[59] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon Press, New York, 1959).

[60] W. Israel, *Covariant Fluid Mechanics and Thermodynamics: An Introduction*, in *Relativistic Fluid Dynamics*, Lecture Notes in Mathematics, edited by A. M. Anile and Y. Choquet-Bruhat (Springer, Berlin, Heidelberg, 1987), pp. 152–210.

[61] W. A. Hiscock and L. Lindblom, *Generic Instabilities in First-Order Dissipative Relativistic Fluid Theories*, Phys. Rev. D **31**, 725 (1985).

[62] W. A. Hiscock and L. Lindblom, *Nonlinear Pathologies in Relativistic Heat-Conducting Fluid Theories*, Phys. Lett. A **131**, 509 (1988).

[63] P. Kostädt and M. Liu, *Causality and Stability of the Relativistic Diffusion Equation*, Phys. Rev. D **62**, 023003 (2000).

[64] Unfortunately, it was shown in Ref. [63] that characteristic hypersurfaces can only exist for $D > 2$ when the kinetic vorticity or twist 2-form vanishes identically, $\omega^{\mu\nu} = \Delta^{\mu}_{\alpha}\Delta^{\nu}_{\beta}\partial^{[\beta}V^{\alpha]} = 0$. Since turbulent flow is highly vortical, this condition is likely never satisfied in the flows of interest. For general Cauchy surfaces, restrictions of initial data to those with bounded and sufficiently smooth solutions of the parabolic equations might possibly select the physically relevant solutions. However, the postselection of initial data based on properties of resultant solutions is not directly practical for numerical simulation studies, which would require some more sophisticated implementation.

[65] W. A. Hiscock and L. Lindblom, *Stability and Causality in Dissipative Relativistic Fluids*, Ann. Phys. (N.Y.) **151**, 466 (1983).

[66] W. A. Hiscock and L. Lindblom, *Stability in Dissipative Relativistic Fluid Theories*, Contemp. Math. **71**, 181 (1988).

[67] W. A. Hiscock and T. S. Olson, *Effects of Frame Choice on Nonlinear Dynamics in Relativistic Heat-Conducting Fluid Theories*, Phys. Lett. A **141**, 125 (1989).

[68] T. S. Olson, Ph.D. thesis, Montana State University, 1990 (unpublished).

[69] T. S. Olson, *Stability and Causality in the Israel-Stewart Energy Frame Theory*, Ann. Phys. (N.Y.) **199**, 18 (1990).

[70] The requirement imposed by the entropy balance (6) does not fully characterize the dissipative fluxes, by far. As discussed by Israel and Stewart [24] and Hiscock and Lindblom [65] (note added in proof), many additional second-order terms are possible in addition to those included in the standard Israel-Stewart model. Even additional first-order terms can be included under weakened symmetry assumptions and may be required by microscopic physics (e.g., Ref. [71]).

[71] D. T. Son and P. Surowka, *Hydrodynamics with Triangle Anomalies*, Phys. Rev. Lett. **103**, 191601 (2009).

[72] R. Geroch, *Relativistic Theories of Dissipative Fluids*, J. Math. Phys. (N.Y.) **36**, 4226 (1995).

[73] L. Lindblom, *The Relaxation Effect in Dissipative Relativistic Fluid Theories*, Ann. Phys. (N.Y.) **247**, 1 (1996).

[74] R. Geroch and L. Lindblom, *Causal Theories of Dissipative Relativistic Fluids*, Ann. Phys. (N.Y.) **207**, 394 (1991).

[75] Note that the relativistic viscosities η, ζ as defined in our paper are c times their nonrelativistic counterparts η_N, ζ_N because $\theta, \sigma^{\mu\nu}$ as $c \to \infty$ are $1/c$ times their nonrelativistic analogues $\Theta, \mathbf{S}$ in paper I.

[76] P. A. M. Dirac, *Forms of Relativistic Dynamics*, Rev. Mod. Phys. **21**, 392 (1949).

[77] The fact that spatial coarse graining alone regularizes time derivatives is due to the fact that the fields in question satisfy equations of motion that are (at least) first order in time.

[78] J. Dunkel, P. Hänggi, and S. Hilbert, *Non-local Observables and Lightcone-Averaging in Relativistic Thermodynamics*, Nat. Phys. **5**, 741 (2009).

[79] C. Meneveau and J. Katz, *Scale-Invariance and Turbulence Models for Large-Eddy Simulation*, Annu. Rev. Fluid Mech. **32**, 1 (2000).

[80] W. Schmidt, *Large Eddy Simulations in Astrophysics*, Living Rev. Comp. Astrophys. **1**, 2 (2015).

[81] D. Radice, *General-Relativistic Large-Eddy Simulations of Binary Neutron Star Mergers*, Astrophys. J. Lett. **838**, L2 (2017).

[82] G. L. Eyink, *Turbulent General Magnetic Reconnection*, Astrophys. J. **807**, 137 (2015).

[83] L. P. Kadanoff, *Scaling Laws for Ising Models Near T_c*, Physics **2**, 263 (1966).

[84] K. G. Wilson, *Renormalization Group and Critical Phenomena. I. Renormalization Group and the Kadanoff Scaling Picture*, Phys. Rev. B **4**, 3174 (1971).

[85] G. L. Eyink, *Turbulence Noise*, J. Stat. Phys. **83**, 955 (1996).

[86] The resolved energy current defined in Eq. (48) has an expression analogous to Eq. (57), which is given by $\underline{\mathcal{E}}^\mu = \bar{\epsilon}\bar{V}^\mu + \bar{\tau}(h, V^\mu) + \bar{h}\,\widetilde{\tau}(V^\nu, V_\nu)\widetilde{V}^\mu + \widetilde{V}^\mu\widetilde{V}^\nu\bar{\tau}(h, V_\nu) - \bar{h}\,\widetilde{\tau}(V^\mu, V^\nu)\bar{V}_\nu$. An intrinsic resolved energy current is defined by $\underline{\mathcal{E}}_*^\mu \coloneqq \underline{\mathcal{E}}^\mu - (1/2)\overline{hV^\mu}\,\bar{\tau}(V_\nu, V^\nu)$, so the balance equation like Eq. (60) holds with an energy flux $\mathcal{Q}_\ell^{*\text{flux}} = -\widetilde{V}^\nu\partial_\mu\bar{V}_\nu\bar{\tau}(h, V^\nu) - \bar{h}\partial_\mu\bar{V}_\nu\widetilde{\tau}(V^\mu, V^\nu) - (1/2)\partial_\mu(\overline{hV^\mu})\bar{\tau}(V_\nu, V^\nu)$, which is purely inertial range. The inequalities (70)–(72) follow. Entropy cascade may also be analyzed in this manner, defining an intrinsic entropy current $\underline{S}_*^\mu$ as in Eq. (104) but with $\underline{\mathcal{E}}_*^\mu$ replacing the intrinsic energy current appearing there. This is equivalent to defining $\underline{S}_*^\mu$ as in the first line of Eq. (100) but with subscale internal-energy current $\underline{K}_*^\mu \coloneqq \bar{\mathcal{E}}^\mu - \underline{\mathcal{E}}_*^\mu$.

[87] R. Geroch and L. Lindblom, *Dissipative Relativistic Fluid Theories of Divergence Type*, Phys. Rev. D **41**, 1855 (1990).

[88] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevX.8.011023 for more mathematical details.

[89] M. Oberguggenberger, *Multiplication of Distributions and Applications to Partial Differential Equations*, Pitman Research Notes in Mathematics Series, Vol. 259 (Longman Scientific & Technical, Harlow, Essex, 1992).

[90] G. L. Eyink, *Locality of Turbulent Cascades*, Physica D (Amsterdam) **207**, 91 (2005).

[91] In somewhat more detail, the pressure-work term transformed to a new Lorentz frame is $((p\star\mathcal{G}_\ell)(\theta\star\mathcal{G}_\ell))' = (p'\star\mathcal{G}'_\ell)(\theta'\star\mathcal{G}'_\ell)$, where "$\star$" denotes space-time convolution. Because of the independence of the distributional product on $\mathcal{G}$, one recovers $(p\circ\theta)' = p'\circ\theta'$ in the limit $\ell \to 0$. The distributional limit $\mathcal{Q}_{\text{flux}}$ is then also independent of the filter kernel and a Lorentz scalar because $p\circ\theta$ and $\partial_\mu(\epsilon V^\mu)$ separately possess those properties.

[92] Precisely, $\underline{\mathcal{Q}}_{\text{diss}}^{\eta,\zeta,\kappa,\sigma} = -(\partial_\mu\bar{V}_\nu)\overline{\kappa\hat{Q}^{(\mu}V^{\nu)} + \zeta\hat{\tau}\Delta^{\mu\nu} + 2\eta\hat{\tau}^{\mu\nu}}$.

[93] G. L. Eyink, *Besov Spaces and the Multifractal Hypothesis*, J. Stat. Phys. **78**, 353 (1995).

[94] Not even the classical particle-frame Eckart theory satisfies Eq. (110) because of the presence of the sign-indefinite term $Q^\mu A_\mu$ in $\mathcal{Q}_{\text{diss}}^{\zeta,\eta,\kappa}$ for that model.

[95] E. Gourgoulhon, *An Introduction to Relativistic Hydrodynamics*, in *Stellar Fluid Dynamics and Numerical Simulations: From the Sun to Neutron Stars*, edited by M. Rieutord and B. Dubrulle, EAS Publications Series Vol. 21 (EDP Sciences, Les Ulis, 2006), 43–79.

[96] I. Fouxon and Y. Oz, *Conformal Field Theory as Microscopic Dynamics of Incompressible Euler and Navier-Stokes Equations*, Phys. Rev. Lett. **101**, 261602 (2008).

[97] K. Gawędzki, *Easy Turbulence*, in *Theoretical Physics at the End of the Twentieth Century: Lecture Notes of the CRM Summer School, Banff, Alberta*, CRM Series in Mathematical Physics, edited by Y. Saint-Aubin and L. Vinet (Springer, New York, 2002).

[98] H. Aluie, *Generalizing the 4/5-th Law to Compressible Turbulence* (unpublished).

[99] M. Takamoto and A. Lazarian, *Compressible Relativistic Magnetohydrodynamic Turbulence in Magnetically*

Dominated Plasmas and Implications for a Strong-Coupling Regime, Astrophys. J. Lett. **831**, L11 (2016).

[100] M. Takamoto and S. Inutsuka, *A Fast Numerical Scheme for Causal Relativistic Hydrodynamics with Dissipation*, J. Comput. Phys. **230**, 7002 (2011).

[101] R. Benzi, S. Ciliberto, R. Tripiccione, C. Baudet, F. Massaioli, and S. Succi, *Extended Self-Similarity in Turbulent Flows*, Phys. Rev. E **48**, R29 (1993).

[102] Z.-S. She and E. Leveque, *Universal Scaling Laws in Fully Developed Turbulence*, Phys. Rev. Lett. **72**, 336 (1994).

[103] R. H. Kraichnan, *Inertial Ranges in Two-Dimensional Turbulence*, Phys. Fluids **10**, 1417 (1967).

[104] G. K. Batchelor, *Computation of the Energy Spectrum in Homogeneous Two-Dimensional Turbulence*, Phys. Fluids **12**, II-233 (1969).

[105] G. L. Eyink, *Dissipation in Turbulent Solutions of 2D Euler Equations*, Nonlinearity **14**, 787 (2001).

[106] C. V. Tran and D. G. Dritschel, *Vanishing Enstrophy Dissipation in Two-Dimensional Navier–Stokes Turbulence in the Inviscid Limit*, J. Fluid Mech. **559**, 107 (2006).

[107] J. C. Mcwilliams, *The Emergence of Isolated Coherent Vortices in Turbulent Flow*, J. Fluid Mech. **146**, 21 (1984).

[108] A. G. Kritsuk, M. L. Norman, B. Sjögreen, D. Kotov, and H. Yee, *Scaling in Two-Dimensional Compressible Turbulence 16th European Turbulence Conference, Stockholm, Sweden* (NASA Publications, 2017), p. 255, http://digitalcommons.unl.edu/nasapub/255.

[109] G. Falkovich and A. G. Kritsuk, *How Vortices and Shocks Provide for a Flux Loop in Two-Dimensional Compressible Turbulence*, Phys. Rev. Fluids **2**, 092603 (2017).

[110] M. Korzyński, *Covariant Coarse Graining of Inhomogeneous Dust Flow in General Relativity*, Classical Quantum Gravity **27**, 105015 (2010).

[111] J. Brannlund, R. van den Hoogen, and A. Coley, *Averaging Geometrical Objects on a Differentiable Manifold*, Int. J. Mod. Phys. D **19**, 1915 (2010).

[112] P. Isett and S.-J. Oh, *A Heat Flow Approach to Onsager's Conjecture for the Euler Equations on Manifolds*, Trans. Am. Math. Soc. **368**, 6519 (2016).

[113] G. L. Eyink, *Multi-scale Gradient Expansion of the Turbulent Stress Tensor*, J. Fluid Mech. **549**, 159 (2006).

[114] G. L. Eyink, *A Turbulent Constitutive Law for the Two-Dimensional Inverse Energy Cascade*, J. Fluid Mech. **549**, 191 (2006).

[115] G. I. Taylor, *The Statistical Theory of Isotropic Turbulence*, J Aeronaut Sci **4**, 311 (1937).

[116] D. Bernard, K. Gawędzki, and A. Kupiainen, *Slow Modes in Passive Advection*, J. Stat. Phys. **90**, 519 (1998).

[117] C. P. Dettmann and N. E. Frankel, *Stochastic Dynamics of Relativistic Turbulence*, Phys. Rev. E **53**, 5502 (1996).

[118] G. L. Eyink and T. D. Drivas, *Spontaneous Stochasticity and Anomalous Dissipation for Burgers Equation*, J. Stat. Phys. **158**, 386 (2015).

[119] T. D. Drivas and G. L. Eyink, *A Lagrangian Fluctuation-Dissipation Relation for Scalar Turbulence, I. Flows with No Bounding Walls*, J. Fluid Mech. **829**, 153 (2017).

[120] T. D. Drivas and G. L. Eyink, *A Lagrangian Fluctuation-Dissipation Relation for Scalar Turbulence, II. Wall-Bounded Flows*, J. Fluid Mech. **829**, 236 (2017).

[121] P. J. Greenberg, *The General Theory of Space-like Congruences with an Application to Vorticity in Relativistic Hydrodynamics*, J. Math. Anal. Appl. **30**, 128 (1970).

[122] A. Lichnerowicz, *Relativistic Hydrodynamics and Magnetohydrodynamics*, Southwest Center for Advanced Studies, and The Mathematical Physics Monograph Series, Vol. 35 (W.A. Benjamin, New York, 1967).

[123] J. D. Bekenstein and E. Oron, *New Conservation Laws in General-Relativistic Magnetohydrodynamics*, Phys. Rev. D **18**, 1809 (1978).

[124] A. Lazarian and E. T. Vishniac, *Reconnection in a Weakly Stochastic Field*, Astrophys. J. **517**, 700 (1999).

[125] G. Eyink, E. Vishniac, C. Lalescu, H. Aluie, K. Kanov, K. Bürger, R. Burns, C. Meneveau, and A. Szalay, *Flux-Freezing Breakdown in High-Conductivity Magnetohydrodynamic Turbulence*, Nature (London) **497**, 466 (2013).

[126] G. H. Golub and C. F. Van Loan, *Matrix Computations, Johns Hopkins Studies in the Mathematical Sciences* (Johns Hopkins University Press, Baltimore, MD, 1996).

[127] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time*, *Cambridge Monographs on Mathematical Physics* (Cambridge University Press, Cambridge, England, 1975).

[128] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, 2nd ed. (Pergamon Press, New York, 1987).

[129] I. Kanitscheider and K. Skenderis, *Universal Hydrodynamics of Non-conformal Branes*, J. High Energy Phys. 04 (2009) 062.

[130] R. Becker, *Stosswelle und Detonation*, Z. Phys. **8**, 321 (1922); [*Impact Waves and Detonation*, National Advisory Committee for Aeronautics Technical Memorandum, NACA-TM-505 and NACA-TM-506 (1929)].