REVIEWS ON AXIOMATIC STUDY OF SYMMETRY BREAKING

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The work on broken symmetry within the axiomatic frame is by now rather extended and it is impossible to give a complete review here. To restore some balance I give a few references [1] concerning points not touched upon in the following. I shall limit myself on the notion of spontaneously broken symmetry. Most of the results are not very new. Earlier and more complete reviews are given in [2].

We use the Wightman framework of relativistic quantum field theory, at the end the framework has to be enlarged slightly to accommodate for supergauge transformations. $\mathcal{R}, \mathcal{H}, \Omega, \mathcal{U}(\Lambda, a)$ denote the algebra of unbounded strictly local fields, the Hilbert space, unique vacuum vector and unitary representation of the Poincaré group respectively.

I

A symmetry transformation is given by a group $\mathcal{G} \ni g$ and a representation by automorphisms α_g of $\mathcal{R}$. Here automorphism denotes a 1 - 1 map of $\mathcal{R}$ onto $\mathcal{R}$ preserving the algebraic structure, with $(\alpha(A))^* = \alpha(A^*)$ and being unitarily implemented on every fixed local subring of $\mathcal{R}$ (replacing the norm preservation in case of an algebra of bounded operators). Furthermore, it is assumed that α_g commutes with the 4 translations,

$$\alpha_g(A(x)) = \alpha_g(A)(x), \qquad A \in \mathcal{R} \tag{1}$$

where $A(x) = U(1,x)\, A\, U^{-1}(1,x)$. A field theory is specified by its vacuum functional $A \to \varphi_o(A) := (\Omega | A\Omega)$. Due to (1), $\varphi_g(A) = (\Omega| \alpha_g(A)\Omega)$ is again a translationally invariant functional on $\mathcal{R}$ and one has the two possibilities

(i) $\varphi_g(A) = \varphi_o(A)$ for all $A \in \mathcal{R}$: α_g is a conserved symmetry.

(ii) $\varphi_g(A) \neq \varphi_o(A)$ for some $A \in \mathcal{R}$: α_g is a spontaneously broken symmetry.

The symmetry transformation in both cases can be presented on $\mathcal{H}$ by a linear invertable operator V_g representing $\mathcal{G}$, commuting with $U(1,x)$, and with $\alpha_g(A) = VAV^{-1}$. V_g is defined by

$$A\Omega \to \alpha_g(A)\Omega := V_g A\Omega, \qquad \text{domain } \mathcal{D}_{V_g} = \mathcal{R}\Omega .$$

Lemma 1: In case (i) V_g is unitary, in case (ii) V_g is not closable (i.e. the adjoint is not densely defined), hence not unitary.

Case (ii) seems to allow for the possibility that two fields are linked by an automorphism such that, e.g., the 4 point functions are different, hence possibly the scattering amplitudes for the corresponding two kinds of particles. Up to now there is no concrete rigorous model theory exhibiting such features. Heuristic models like the σ-model show certain draw backs: One of the two kinds of particles is unstable [3].

II

For most conclusions one needs more specific structure: Guided by heuristic q.f.th. one assumes that $\mathcal{G}$ is composed of one parameter subgroups continuously connected to the identity, the infinitesimal transformations defined by integrals over densities

$$A \rightarrow \lim_{r\to\infty} i\,[Q_r, A] \quad , \quad A \in \mathcal{R}$$

$$Q_r = \int j_o(x)\, \vartheta_r(\vec{x})\, \eta(x^o)\, d^4x .$$

Here $j_\mu(x)$ is a real local field, $\mu = 0,1,2,3$, $\partial^\mu j_\mu(x) = 0$. For the following j_μ is assumed to be invariant under translations.+) The two cases then read

(i) $\lim_{r\to\infty} (\Omega | [Q_r, A]\, \Omega) = 0$ for all $A \in \mathcal{R}$: conserved symmetry.

(ii) $\lim_{r\to\infty} (\Omega | [Q_r, A]\, \Omega) \neq 0$ for some $A \in \mathcal{R}$: spontaneously broken symmetry.

One may now define a linear operator Q on $\mathcal{H}$ by $A\Omega \rightarrow \lim_{r\to\infty} [Q_r, A]\,\Omega = QA\Omega$, domain $\mathcal{D}_Q = \mathcal{R}\,\Omega$.

Lemma 1': In case (i) Q is hermitean, in case (ii) Q is not closable.*)

For further properties of Q in particular concerning selfadjointness see [4]. Famous result is the Goldstone theorem.

Thm 2: Denote by E_o the projection on the mass zero vectors in $\mathcal{H}$. Then

$$\lim_{r\to\infty} (\Omega | [Q_r, A]\,\Omega) = \lim_{r\to\infty} (\Omega | Q_r E_o A \Omega) - (\Omega | A E_o Q_r \Omega) \tag{2}$$

$$= 2 \lim_{r\to\infty} (\Omega | Q_r E_o A \Omega). \tag{3}$$

A clean proof of (2) was first presented in [5], of (3) in [6].

Useful for the proof of this and other assertions is

Lemma 3: If $\|Q_r \Omega\| \leq c < \infty$ or $\|E_o Q_r \Omega\| \leq c < \infty$ for $r \to \infty$, then $\lim_{r\to\infty} (\Omega | [Q_r, A] \Omega) = 0$. To our knowledge, this conclusion is not invertable, not even if $\eta(x^o)$ is chosen dependent on r [8].

Vectors from $E_o \mathcal{H}$ contributing in (2) are called Goldstone states. Their quantum numbers depend on Q_r. In case j_μ is a covariant vector field they have helicity 0. In case of space time dimension 4 and 3 one knows that this situation indeed may occur [2]. In case of dimension 2 it was first pointed out in [9] that the situation is different.

Thm 4: In case of space time dimension 2 and j_μ a covariant vector field, $\| Q_r \Omega \|$ is bounded, hence $\lim_{r\to\infty} (\Omega | [Q_r, A] \Omega) = 0$ by lemma 3.

The only zero mass particles observed in nature are the photon and the neutrinos with helicity 1 and 1/2. Therefore one has to investigate currents of more complicate transformation properties. Let $j_\mu(x) = t_{\mu m}(x)$ where m stands for a collection of vector and spinor indices and

$$\mathcal{U}(\Lambda, a)\, t_{\mu m}(x)\, \mathcal{U}^{-1}(\Lambda, a) = \Lambda^{-1}{}_{\mu}{}^{\mu'} D_m{}^{m'}(\Lambda^{-1})\, t_{\mu' m'}(\Lambda x + a)$$

with a finite representation of the homogeneous Lorentz group (h. L. g.).

Consider first the case that D is a one valued representation. Then we have [10]

Thm 5: Decompose $t_{\mu m}$ to get irreducible representations (decomposition of $\Lambda \otimes D$) If no covariant vector current occurs, then $\lim_{r\to\infty} (\Omega | [Q_{rm}, A] \Omega) = 0$ for all $A \in \mathcal{R}$.

Hence there are no Goldstone states with helicity $\neq 0$. However, it has to be noted that the Wightman framework is assumed for the proof. In case of an indefinite metric the situation may be different: In the Gupta Bleuler frame the photon states may be interpreted as Goldstone states of certain gauge transformation [10, 11].

In case D is a two valued representation slightly less can be proved [10].

Thm 6: Decompose $t_{\mu m}$ into irreducible parts (decomposition of $\Lambda \otimes D$). If one gets only representations of type (b+h, b) with $2b+h \geq 2$, then $\lim_{r\to\infty} (\Omega | [Q_r, A] \Omega) = 0$ for all A.

The remaining cases $(\frac{1}{2}, 0)$, $(\frac{3}{2}, 0)$ occur for a conserved current transforming according to $\Lambda \otimes$ (spin 1/2-representation) as expected for spin 1/2 Goldstone states.

There are, of course, problems arising if one deals with spin 1/2 currents which should be local relative to $\mathcal{R}$ when $\mathcal{R}$ contains the currents themselves.

In the framework of super gauge transformations [12] these difficulties can be avoided and within this scheme indeed Goldstone states with spin 1/2 may occur. However, the frame of local quantum field theory has to be enlarged by adding a certain underworld involving elements of a Graßmann algebra and in consequence an indefinite metric. I would like to demonstrate this by presenting a simple example.

III

Consider a free Dirac spinor field ψ (x) of mass zero on a Fock space $\mathcal{H}$ and a Graßmann algebra $\Lambda(E_4)$ with elements e_B over a 4 dimensional vector space E_4. B denotes a subset of $\{1,2,3,4\}$. The elements of $\Lambda(E_4)$ are used besides of the complex numbers to multiply operators and states. This we do by a direct product. On $\mathcal{H}$ there is an operator I being 1 on integer spin states, -1 on half integer spin states. Consider the conserved currents

$$j^{\mu}_{\beta}(x) = e_{(\beta)} \otimes I\, \gamma^{0}_{(\beta)\sigma} \gamma^{\mu}_{\sigma\varrho} \psi_{\varrho} + \hat{\psi}_{\sigma} \gamma^{\mu}_{\sigma(\beta)} I \otimes e_{(\beta)}$$

$$J^{\mu}_{\beta}(x) = \frac{1}{i}\left[e_{(\beta)} \otimes I\, \gamma^{0}_{(\beta)\sigma} \gamma^{\mu}_{\sigma\varrho} \psi_{\varrho} - h.c.\right]$$

with summation over repeated indices unless in brackets ($\beta = 1,2,3,4$). The I is to get relative locality with respect to ψ , the e_{β} accomplish j^{μ}_{β}, J^{μ}_{β} to obey local commutativity, the latter being needed for iterated commutators and finite transformations. With $q_{r\beta} = \int j^{0}_{\beta}(x)\, \vartheta_r(\vec{x})\, \eta(x^0)\, d^4x$, $Q_{r\beta} = \int J^{0}_{\beta}(x)\, \vartheta_r(\vec{x})\, \eta(x^0)\, d^4x$ one easily computes

$$[q_{r\beta}, \psi_{\lambda}(y)] = - e_{(\beta)}\, I\, \delta_{\lambda(\beta)}$$

$$[q_{r\beta}, \hat{\psi}_{\lambda}(y)] = e_{(\beta)}\, I\, \gamma^{0}_{(\beta)\lambda}$$

$$[Q_{r\beta}, \psi_{\lambda}(y)] = - i\, e_{(\beta)} I\, \delta_{\lambda(\beta)}$$

$$[Q_{r\beta}, \hat{\psi}_{\lambda}(y)] = - i\, e_{(\beta)} I\, \gamma^{0}_{(\beta)\lambda}$$

for sufficiently large r. Since all iterated commutators vanish for large r one gets immediately the finite transformations

$$\psi_\lambda \to \psi_\lambda - i\tau\,\delta_{\lambda(\beta)}\, e_{(\beta)}\, I \qquad \text{for} \quad q_{\gamma\beta}$$

$$\psi_\lambda \to \psi_\lambda + \sigma\,\delta_{\lambda(\beta)}\, e_\beta\, I \qquad \text{for} \quad Q_{\gamma\beta}$$

($\tau, \sigma \in \mathbb{R}'$) defining automorphisms on $\mathcal{A} = \Lambda(E_4) \otimes (\mathcal{R}, I)$, which replaces $\mathcal{R}$ in the enlarged framework. It can be shown that these transformations cannot be unitarily implemented. Therefore one has the situation of a spontaneously broken (super gauge) symmetry. This is also true in case of two dimensional space time.

More details of the example can be found in [14]. In particular it turns out that the enlarged state space has to carry an indefinite metric if one wants an involution to be defined on $\Lambda(E_4)$ which coincides with the adjoint operation with respect to the scalar product. (In theorem 6 only one simple commutator is considered. For that one could also replace the $e_{(\beta)}$ by c numbers thus avoiding additional substructure and indefinite metric).

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Footnotes:

+) ϑ, η are real test functions, $\vartheta_r(\vec{x}) = \vartheta(\frac{|\vec{x}|}{r})$, $\vartheta(s) = 1$ for $s < 1$, 0 for $s \geq 2$. η has compact support and $\int (x^o)dx^o = 1$.

*) Case (ii) does not correspond to a not closable derivation as considered in the talk by D. Robinson.

Discussion

Swieca: Do you know the appropriate axiomatic setting to cover the case of gauge invariance of the second kind?

Reeh: I should say no. There are papers in which parts of the results are generalized (Y. Dothan, E. Gal-Ezer, Nuovo Cim. 12A, 465 (1972); R. Ferrari, Nuovo Cim. 14A, 386 (1973)).

Swieca: In the case of supersymmetries, do you have automorphisms or generalizations of them?

Reeh: Yes, the transformations are locally unitarily implemented. Details can be found in [14].

Schroer: There are cases which were not discussed in your spontaneous symmetry frame work: Tensor currents and spinor currents which have an explicit x-dependence, i.e. formally $[P_\mu, Q] \neq 0$. Can you comment on the possibility of having spontaneously broken symmetries of such currents.

Reeh: I do not have an answer ready for the case of spinor currents. I would conjecture that the general situation should be essentially the same, i.e. there should be a Goldstone-theorem too and examples of a spontaneous break down, just as one knows, e.g., in the case of dilatation and conformal currents (compare, e.g. my Haifa lecture, ref. [2]).