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How vacuum fluctuations determine the properties of the vacuum

G. B. Mainland¹ and Bernard Mulligan²

¹ Department of Physics, The Ohio State University at Newark, 1179 University Dr., Newark, OH 43055, USA, ² Department of Physics, The Ohio State University, Columbus, OH 43210, USA

E-mail: mainland.1@osu.edu, mulligan.3@osu.edu

Abstract. Particle-antiparticle pairs are predicted by quantum field theory to appear as vacuum fluctuations. The model of the vacuum used here is postulated to have the following properties: To minimize the violation of conservation energy allowed by the Heisenberg uncertainty principle and to avoid violating conservation of angular momentum, vacuum fluctuations of charged particle-antiparticle pairs appear as bound states in the lowest energy level that has zero angular momentum. These transient atoms are polarized by electric fields somewhat similarly to the way that ordinary matter is polarized. As a consequence, the permittivity ϵ_0 of the vacuum can be calculated. Once the permittivity of the vacuum has been calculated, formulas for the speed of light c in the vacuum and the fine-structure constant α immediately follow. The values for ϵ_0 , c , and α calculated here agree with the accepted values to within a few percent. Only the leading terms in the formulas have been retained in the calculations. The absence of dispersion in the vacuum is discussed and explained.

1. Introduction: vacuum fluctuations – the vacuum as a dielectric

Values of the permittivity ϵ_0 of the vacuum and the speed c of light in the vacuum are properties of the vacuum. Initially it is not obvious that the value of the fine-structure constant α also is a property of the vacuum. However, as will be shown later in the Introduction, if one of the three quantities ϵ_0 , c , or α has been calculated, values for the other two immediately follow. Consequently, the value of α must also be a property of the vacuum.

Among the three quantities ϵ_0 , c , and α , physicists have had little idea how to calculate either c or α . But for almost a century they have known how to calculate the permittivity of a dielectric[1]. As a consequence, most of this article is devoted to calculating ϵ_0 , guided along the way by the well-established procedures for calculating the permittivity of dielectrics. Once ϵ_0 has been calculated, values for c and α immediately follow.

Vacuum fluctuations (VFs) are particle-antiparticle pairs that appear spontaneously in the vacuum as predicted by relativistic quantum field theory[2, 3, 4], violating conservation of energy to the extent allowed by the Heisenberg uncertainty principle. VFs are created on-mass-shell and are not associated with a Green's function or perturbation theory.

At this point a word of caution is required: Vacuum bubbles[3, 5] are a class of perturbation effects in quantum field theories that are sometimes confused with VFs because their Feynman diagrams look like bubbles that originate from and terminate in the vacuum. Vacuum bubbles are not VFs and, in fact, do not make a contribution to physical processes[3, 5].



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The idea that VFs play a role in determining the permittivity of the vacuum is very old: in 1934 Furry and Oppenheimer[6] wrote that VFs of charged particle-antiparticle pairs would affect the value of the dielectric constant of the vacuum; “Because of the polarizability of the nascent pairs, the dielectric constant of space into which no matter has been introduced differs from that of truly empty space.” In 1936 the idea of treating the vacuum as a medium with electric and magnetic polarizability was discussed by Weisskopf and Pauli[7, 8].

For VFs to be responsible for the permittivity of the vacuum, it is crucial that they possess elasticity (an effective spring constant). The possibility that a charged lepton-antilepton pair can form an atomic bound state, which possesses precisely this property, was discussed by Ruark[9]. At almost the same time, the formation of a charged lepton-antilepton bound state was elaborated on by Wheeler[10] when he calculated the decay rate of parapositronium, the spin-0, ground state of a bound electron and positron, into two photons. The experimental proof that such atoms exist was provided by Deutsch and Brown[11].

In 1957 Dicke[12] wrote about the possibility that the vacuum could be considered as a dielectric medium. More recently the possibility that the properties of the quantum vacuum determine, in the vacuum, the speed of light and the permittivity has been explored by a number of authors[13, 14, 15].

As will be shown, the VFs that primarily contribute to the permittivity of the vacuum are charged lepton-antilepton pairs. To conserve angular momentum and minimize the violation of conservation of energy, charged lepton-antilepton VFs must appear in the vacuum as atoms. The term “dielectric” can then be used in the usual sense: a photon passing through the vacuum is slowed by its interactions with transient atoms, a concept familiar from discussions of a physical dielectric[1, 16, 17].

In the discussion that follows, a quantum calculation of the the permittivity of the vacuum is presented that is similar both to a calculation of the permittivity of a physical dielectric[18] and to a previous quantum calculation of the the permittivity[19] of the vacuum by the authors. The result of the calculation is ¹

$$\epsilon_0 \cong \frac{6\mu_0}{\pi} \left(\frac{8e^2}{\hbar} \right)^2 = 9.10 \times 10^{-12} \frac{\text{C}}{\text{Vm}}. \quad (1)$$

In the above equation e is the magnitude of the charge on an electron, $\hbar$ is Planck’s constant divided by 2π , and μ_0 is the permeability of the vacuum with an assigned value $4\pi \times 10^{-7} \text{H/m}$. Thus in Equation (1), ϵ_0 is expressed in terms of two experimentally-determined quantities, e and $\hbar$. The experimental value for ϵ_0 is 2.8 % less than the calculated value. The calculation of ϵ_0 is simplified, and the numerical accuracy is reduced, by including (a) only contributions to lowest order in what turns out to be an expansion in powers of the fine-structure constant α and (b) only the interactions of photons with bound states of charged lepton-antilepton VFs.

Once ϵ_0 has been calculated, formulas for c and α immediately follow. Using $c = 1/\sqrt{\mu_0\epsilon_0}$ and the formula for ϵ_0 in Equation (1),

$$c \cong \sqrt{\frac{\pi}{6}} \frac{\hbar}{8e^2\mu_0} = 2.96 \times 10^8 \text{m/s}. \quad (2)$$

The accepted value is $c = 3.00 \times 10^8 \text{m/s}$, which is 1.3% more than the calculated value. The progress of a photon traveling through the vacuum is slowed when it interacts with and has a polarizing effect on a VF consisting of a charged lepton and antilepton bound into an atom, implying that the interaction of photons with vacuum fluctuations determines the speed of light in the vacuum.

¹ SI units are used throughout.

Any observer in an inertial frame of reference cannot detect relative motion of the vacuum and the inertial frame. Thus the observer would conclude that the vacuum is at rest with respect to the inertial frame. Leonhardt et al.[20] state, “In free space the quantum vacuum is Lorentz invariant, so a uniformly moving observer would not see any effect due to motion, but an accelerated observer would. This is known as the Unruh effect[21].” Since the vacuum determines the speed of light in the vacuum and the vacuum is at rest with respect to any inertial frame, it then follows that the speed of light in the vacuum is the same in any inertial frame. The previous statement is one of the two postulates in Einstein’s 1905 paper “On the Electrodynamics of Moving Bodies”[22] in which he introduced special relativity.

In the early universe when the temperature was sufficiently high that it was difficult for charged lepton-antilepton VFs to bind into charged lepton-antilepton atoms, the number density of charged lepton-antilepton atoms that are VFs would have been less than today. With fewer bound, charged lepton-antilepton VFs per unit volume to slow the progress of photons, the speed of light would have been greater. Moffat[23] has pointed out that inflation[24] with the speed of light much greater in the early universe than it is now allows “all regions in the universe to be causally connected”, solving the horizon problem. A sufficiently high superluminary speed also, “leads to a mechanism of monopole suppression in cosmology and can resolve the flatness problem.”

The fine-structure constant α , which is defined by

$$\alpha \equiv \frac{e^2}{4\pi\epsilon_0\hbar c}, \quad (3)$$

has historically attracted theoretical interest because it is dimensionless. The fine-structure constant was introduced by Sommerfeld in 1916[25] to quantify relativistic corrections in the Bohr theory of the hydrogen atom and is a measure of the strength of the electromagnetic interaction. However, Sommerfeld was not the first to recognize the importance of the combined symbols used to define α : in the period from 1905-1910[26] both Planck[27] and Einstein[28] discussed the fact that Planck’s constant h and $e^2/(4\pi\epsilon_0 c)$ had the same dimensions. α grows logarithmically with energy scale, and equals $1/137.036\dots$ at zero energy scale. Using the formula for ϵ_0 in Equation (1) and the formula for c in Equation (2), Equation (3) becomes

$$\frac{1}{\alpha} \cong 8^2 \sqrt{3\pi/2} = 138.93\dots \quad (4)$$

The experimental value for $1/\alpha$ is 1.4% less than the value calculated in Equation (4).

This article is organized as follows: In §2 examples of attempts to calculate the fine-structure constant are briefly discussed. In §3 properties of charged lepton-antilepton VFs are discussed, including the energy source for their creation and a proof of their existence. In §4 the role that the Heisenberg uncertainty principle plays in describing VFs is discussed, yielding an ansatz for the number density of VFs. §5 discusses how VFs act as harmonic oscillators. In §6 dielectric properties of the vacuum are calculated and discussed in five subsections: In §6.1 the interaction Hamiltonian describing the interaction of photons with VFs is constructed, in §6.2 a quantum calculation of the polarization of VFs is performed, and in §6.3 a general formula for the permittivity ϵ_0 of the vacuum is derived. In §6.4 and §6.5 the contributions to ϵ_0 from charged lepton-antilepton vacuum fluctuations and quark-antiquark VFs are, respectively, calculated and discussed. Finally, in §6.6 the permittivity ϵ_0 of the vacuum is calculated. The process that primarily determines the value of ϵ_0 , photon capture by atoms consisting of charged lepton-antilepton vacuum fluctuations and the subsequent decay of these atoms, is discussed.

2. Attempts to calculate the fine-structure constant

In 1988 Richard Feynman wrote as follows about the fine-structure constant: “It has been a mystery ever since it was discovered more than fifty years ago, and all good theoretical physicists

put this number up on their wall and worry about it.... It's one of the greatest damn mysteries of physics: a magic number that comes to us with no understanding by man. You might say the 'hand of God' wrote that number, and 'we don't know how He pushed his pencil.' "[29]

Kragh [30] writes that there are three reasons that the fine-structure constant is not thought to be as important today as it was in the early decades after its discovery: (a) α is only one of four coupling constants, the other three being the gravitational, weak, and strong. (b) As mentioned previously, α grows logarithmically with energy scale, and equals $1/137.036...$ only at zero energy scale. (c) There is evidence that the fine-structure constant is not constant over time[31]. On the other hand, in Kinoshita's review article[32] written 80 years after Sommerfeld's introduction of α , he points out that α "characterizes the whole range of physics including elementary particle, atomic, macroscopic, and microscopic systems."

In the past there have been two general theoretical methods for attempting to calculate the fine-structure constant. (a) physics nonsense: a physics-based discussion that makes no sense, but nevertheless yields a numerical expression for α . (b) numerical coincidence (numerology): a number is found that closely approximates the experimental value for α .

In the first category, the most famous physics nonsense determination of α was by Eddington in about 1930. At that time there were only two known "elementary" particles, the electron and the proton. Dirac had incorrectly assumed that a proton is a hole in the, generally filled, sea of negative energy states of the electron. Eddington started with Dirac's theory that uses a 4×4 matrix to describe an electron. Eddington also used a 4×4 matrix to describe the proton. To describe both particles requires a 16×16 matrix, which, if symmetric, would have 136 independent elements. Later, when $1/\alpha$ was found to be closer to 137, Eddington stated that the orbital motion of the electron about the proton added one more degree of freedom, bringing the total to 137. The physics itself is incorrect, and even if it were correct, there is no reason that the number of degrees of freedom should equal the reciprocal of α . A summary of Eddington's approach was given by Birge[33] in 1934 in which the value Eddington calculates for α is characterized as simply being a result of numerology.

Hans Bethe wrote, "Beck, Riesler and I were at Cambridge on fellowships and had listened to Eddington's unbelievable talk about the number 137". To make fun of Eddington's "derivation", they submitted the article *On the Quantum theory of the Temperature of Absolute Zero*[34]. They had noticed the numerical coincidence that $-273 = -(2 \times 137 - 1)$ so they proposed the following equation relating the temperature T_0 of absolute zero to α :

$$T_0 = -\left(\frac{2}{\alpha} - 1\right) \text{ degrees Celsius.} \quad (5)$$

They then wrote, "Putting $T_0 = -273^\circ$, we obtain for $1/\alpha$ the value 137 in perfect agreement within the limits of accuracy with the value obtained by totally independent methods." Of course, the value of absolute zero depends on the temperature scale while α is independent of units. The editor of *Die Naturwissenschaften* approved the article, and it was published in January, 1931. In March, 1931, a "Correction" was published in *Die Naturwissenschaften* stating that the article "was intended to characterize a certain class of papers in theoretical physics of recent years which are purely speculative and based on spurious numerical agreements."

Apparently neither the criticism by Bethe et al. nor Birge had a significant affect on Eddington's self-assurance that he had theoretically calculated the value of α . Eddington incorporated the above argument into his 1936 book [35] and continued to "clarify" his method for calculating his value for α in his final book[36] published after his death in 1944. In 1994 Kilmister[37] published an analysis of Eddington's work, noting that Eddington always used the term *fine-structure constant* for $1/\alpha$, rather than for α .

The second group of calculations rely on a numerical coincidence. For example, Allen[38]

noted that the mass m_e of the electron divided by 1 atomic mass unit (u) satisfies

$$\frac{m_e}{u} \cong 10\alpha^2. \quad (6)$$

Other examples consist of formulas involving integers, π , square roots, cube roots, etc. that yield a number close to the experimental value for α . Such an example by Wyler[39], which is associated with the symmetric space of the group of Maxwell's equations, is very close to the accepted value, $1/\alpha = 137.03599\dots$

$$\frac{1}{\alpha} \cong \frac{16\pi^3}{9} \sqrt[4]{\frac{5!}{\pi}} \cong 137.03608. \quad (7)$$

So many examples similar to Equation (7) are now known that it is likely that no physical insight into the value of α is to be gained from them.

3. Vacuum fluctuations of charged lepton-antilepton pairs

The appearance of VFs is a stochastic process: as such, either VFs appear on mass shell or they don't appear at all. As discussed previously, a charged lepton-antilepton VF that results from a fluctuation of the Dirac field will appear in the vacuum as a transient atom. During the time while such an atom exists, it can interact with a photon. The fundamental interaction of a photon with a VF is the capture of the photon by the charged lepton-antilepton VF.

Field theory provides a simple explanation for the source of the energy available for the creation of a VF and a proof that VFs must exist. The structure of an atom that is a VF does not play a significant role in the discussion of the energy required to produce the VF. (Of course, the structure of the VF, which results from the electromagnetic interaction of the charged lepton-antilepton pair, is important for the calculation of the decay rate of the atom that is a VF.) Thus an atom consisting of a charged lepton and antilepton in its ground state with zero angular momentum can, as far as its creation is concerned, be approximately represented by a free, neutral, spin-0, Klein-Gordon field $\phi(x)$, as first suggested by Pauli and Weisskopf[8] and elaborated on by Wentzel[40]. Using a field to describe a particle with internal degrees of freedom is discussed in Ref.[2]; representing a particle by a quantum field is essential to understanding the mathematical structure of a VF of a quantum field. Indeed, it is the quanta of a free field that behave as free particles[4].

The energy available for the creation of VFs is the energy in the vacuum, called the zero-point energy, and is given by the vacuum expectation value $\langle 0|H|0\rangle$ of the Hamiltonian H for a free, neutral Klein-Gordon field[5]. If there is no cutoff in momentum, the vacuum expectation $\langle 0|H|0\rangle$ is infinite; if there is a cutoff, the vacuum energy is finite, but still large. As Peskin and Schroeder point out, since experiments measure only energy differences, the energy associated with VFs cannot be measured directly in elementary particle experiments[5].

The *presence* of Klein-Gordon VFs, however, is easily established. To show that vacuum fluctuations of charged lepton-antilepton pairs must exist, first note that for a free field $\phi(x)$, the vacuum expectation value $\langle 0|\phi(x)|0\rangle = 0$. In contrast, the expectation value of the product of the free field at two different locations x and x' is [2, 3] $\langle 0|\phi(x)\phi(x')|0\rangle \neq 0$. Accordingly, the vacuum expectation of the square of the field $\phi(x)$ deviates from the square of the vacuum expectation value of the field $\phi(x)$. This demonstrates that the free field $\phi(x)$ in the vacuum is nonzero and means that field theory predicts that fluctuations of a free (noninteracting) field occur in the vacuum.

The purpose of the present paper is to demonstrate a measurable effect of the presence of charged lepton-antilepton VFs, namely, the importance of these VFs in establishing the value of ϵ_0 . This calculation does not depend on an absolute value for the zero-point energy. As Peskin

and Schroeder point out, a potential method for establishing (measuring) an absolute value for the zero-point energy in the vacuum lies in understanding the coupling of that energy to gravity via the cosmological constant in Einsteins equations. Such a direct comparison seems at present unlikely, since the calculation and measurement appear to differ by a factor of around 10^{120} [5].

4. Heisenberg uncertainty principle and charged lepton-antilepton vacuum fluctuations

There are three types of charged lepton-antilepton vacuum fluctuations: (a) electron-positron (b) muon-antimuon (c) tau-antitau. The electron-positron VF in the lowest energy level that has zero angular momentum is parapositronium, which is a singlet spin state[11, 41, 42]. Initially attention will be restricted to parapositronium since the corresponding calculations for muon-antimuon and tau-antitau VFs can be obtained by replacing the mass of the electron with the mass of the muon or tau, respectively.

For parapositronium that is a VF, the Heisenberg uncertainty principle is

$$\Delta E_{p-Ps} \Delta t_{p-Ps} \geq \frac{\hbar}{2}. \quad (8)$$

Denoting the mass of an electron (or positron) by m_e , ΔE_{p-Ps} is the energy $2m_e c^2$ for the production of parapositronium that is a VF². The minimum uncertainty in time is the average time Δt_{p-Ps} that a parapositronium VF exists. Then Equation (8) yields

$$\Delta t_{p-Ps} = \frac{\hbar}{4m_e c^2}. \quad (9)$$

During the time Δt_{p-Ps} , a beam of light travels a distance L_{p-Ps} given by

$$L_{p-Ps} = c \Delta t_{p-Ps} = \frac{\hbar}{4m_e c}. \quad (10)$$

Since a parapositronium VF appears from the vacuum at essentially a single location and since nothing can travel faster than the speed of light, while they exist the maximum displacement between the electron and positron in parapositronium is L_{p-Ps} : for a time $\Delta t_{p-Ps}/2$ the electron and positron can move apart, and for a time $\Delta t_{p-Ps}/2$ they must move back toward each other in order to annihilate on average at the time Δt_{p-Ps} . Since an energy $2m_e c^2$ has already been borrowed from the volume L_{p-Ps}^3 of the vacuum for the creation of a parapositronium atom, it is reasonable to assume that another parapositronium VF is unlikely to form in the same volume. Thus for the model being discussed,

$$\text{number of parapositronium atoms/volume} = \frac{1}{L_{p-Ps}^3}. \quad (11)$$

The result Equation (11) can immediately be generalized to other charged lepton-antilepton VFs and quark-antiquark VFs. The number density of charged lepton-antilepton VFs ranges from $1.12 \times 10^{39}/\text{m}^3$ for electron-positron VFs to $4.70 \times 10^{49}/\text{m}^3$ for tau-anti-tau VFs. In a 6,000 W, CO₂ cutting laser with a beam diameter of 0.32 mm, the number density of photons is on the order of 10^{22} photons/ m^3 . Even in such an intense laser beam, the number density of charged lepton-antilepton VFs is much greater than the number density of photons. Thus if a charged lepton-antilepton VF interacts with a photon at all, it essentially always interacts with only one photon.

² The binding energy of parapositronium, which is small in comparison with $2m_e c^2$, is being neglected.

The number density of atoms or molecules of an ideal gas at STP is $2.68 \times 10^{25}/\text{m}^3$. It is possible for the number density of VFs to be many orders of magnitude greater than the number density of atoms or molecules in a gas because a VF cannot exert a force. Consider first the electromagnetic force: if a VF has not already absorbed radiation, it cannot spontaneously emit radiation. If it did, the radiated photons would exist after the VF has disappeared back into the vacuum, permanently violating conservation of energy. If a VF has interacted with a photon, when the VF vanishes back into the vacuum the VF must emit a photon identical to the incident photon in order to conserve energy, momentum, and angular momentum. Since a VF cannot “permanently” exchange a photon with either a VF or a physical quantum, it cannot exert a force on either. Similar arguments establish that VFs cannot exert a force of any type.

In addition to absorbing and emitting photons, VFs can also interact through the annihilation of a physical particle and a VF or a physical antiparticle and a VF. For example, a physical electron can annihilate with a positron that is part of a VF. The electron that was also part of the VF then becomes a physical electron with a location different from that of the original, physical electron, giving rise to zitterbewegung[2].

5. Charged lepton-antilepton vacuum fluctuations as harmonic oscillators

Here attention is again restricted to parapositronium. In the center-of-mass rest frame, the relative position of the positron and electron are given by $\vec{r} = \vec{r}_+ - \vec{r}_-$, where $\vec{r}_+$ and $\vec{r}_-$ are, respectively, the positions of the positron and electron. Note that $\vec{r}$ points in the direction of the electric dipole moment of parapositronium. For the hydrogen atom $\vec{r}$ usually points from the positive nucleus to the electron. However, the Schrödinger equation for parapositronium, which in the center-of-mass reference frame has a spherically symmetric wave function, is identical to the Schrödinger equation describing states of the hydrogen atom with spherically symmetric wave functions except that the reduced mass μ is different: for hydrogen the reduced mass is approximately m_e , where m_e is the mass of an electron. For parapositronium the reduced mass is $m_e/2$. The non-relativistic binding energy for parapositronium, E_{p-Ps} , is obtained from the $n = 1$ binding energy of hydrogen[43] just by changing the reduced mass:

$$E_{p-Ps} = -\frac{(m_e/2)e^4}{2(4\pi\epsilon_0)^2\hbar^2} = -\frac{m_e\alpha^2c^2}{4}. \quad (12)$$

From both the classical[1, 16, 17] and quantum[18] calculations of the permittivity of physical matter consisting of atoms (or molecules), it follows that for the matter to possess permittivity, the atoms must be able to oscillate when interacting with an electric field (or photons). Taking the electric field to point in the x -direction, since the atom oscillates along the direction of the electric field, only the oscillatory properties of the atom in the x -direction are of significance in the interaction. Thus, when calculating the permittivity, the interaction of the electromagnetic field with the atom can be described by a one-dimensional potential.

Similarly to many other systems in physics that can be described by a one-dimensional potential $U(x)$, the parapositronium atom can be expected to oscillate if the potential has a minimum at x_e and can be expanded in a Taylor series about that minimum:

$$U(x) = U(x_e) + \frac{dU(x)}{dx}\bigg|_{x=x_e}(x-x_e) + \frac{1}{2!}\frac{d^2U(x)}{dx^2}\bigg|_{x=x_e}(x-x_e)^2 + \dots \quad (13a)$$

In the above formula $x = x_+ - x_-$. At x_e there is a relative minimum of the potential so $\frac{dU(x)}{dx}$ is zero at x_e . Choosing the origin of the x -axis at the equilibrium position x_e so that $x_e = 0$, Equation (13a) can be rewritten as

$$U(x) \cong U(x_e = 0) + \frac{1}{2}Kx^2. \quad (13b)$$

Equation (13b) is the equation for a one-dimensional harmonic oscillator potential with a spring constant $K = \frac{d^2U(x)}{dx^2}|_{x=x_e}$. The constant term $U(x_e = 0)$ just shifts all energy levels by the same amount.

As pointed out by Feynman[44], when interacting with an electric field, an atom in its ground state interacts with the electric field as if it were a harmonic oscillator with the first two energy levels separated by the binding energy of the atom. Since adjacent harmonic oscillator energy levels are separated by an energy $\hbar\omega^0$, from the expression Equation (12) for the binding energy E_{p-Ps} of parapositronium,

$$\omega^0 = \frac{|E_{p-Ps}|}{\hbar} = \frac{m_e\alpha^2c^2}{4\hbar}. \quad (14)$$

The spring constant K of the harmonic oscillator corresponding to parapositronium is

$$K = \mu(\omega^0)^2 = \frac{m_e}{2} \left(\frac{m_e\alpha^2c^2}{4\hbar} \right)^2. \quad (15)$$

Using the formula for the energy[43] of a harmonic oscillator in one dimension, $E = \hbar\omega^0(n + 1/2)$, $n = 0, 1, 2, \dots$, the energy eigenvalues E_n^0 of parapositronium are

$$E_n^0 = U(x_e = 0) + \hbar\omega^0\left(n + \frac{1}{2}\right). \quad (16)$$

The harmonic oscillator matrix element that appears in the calculation of the permittivity does not depend on the constant term $U(x_e = 0)$ in the potential, so there is no need to determine a specific value.

In the center-of-mass rest frame of the parapositronium VF, the parapositronium VF is described by the harmonic oscillator Hamiltonian H^0 ,

$$H^0 = \frac{1}{2}\mu \left(\frac{dx}{dt} \right)^2 + U(x_e = 0) + \frac{1}{2}\mu(\omega^0)^2x^2, \quad (17)$$

where ω^0 is defined in Equation (14).

6. Calculation of the permittivity of the vacuum

6.1. Interaction Hamiltonian for photons interacting with vacuum fluctuations

The one-dimensional harmonic oscillator described by the Hamiltonian Equation (17) can undergo an electric dipole transition between its ground state and first excited state because the initial and final states have opposite parity. The interaction Hamiltonian describing the dipole interaction of a photon with a VF follows:

The dipole moment (operator) p_x of an atom in the presence an electromagnetic wave with its electric field in the x -direction is $p_x = e(x_+ - x_-) = ex$, where e is the magnitude of the charge on an an electron. The Hamiltonian H^1 describing the interaction of the electric dipole of the atom with the electromagnetic wave is

$$H^1 = -\mathbf{p} \cdot \mathbf{E}(\mathbf{t}) = -exE_0 \cos \omega t. \quad (18)$$

The two Hamiltonian equations for the Hamiltonian $H = H^0 + H^1$ can be combined to yield

$$-Kx + eE_0 \cos \omega t = \mu \frac{d^2x(t)}{dt^2}, \quad (19)$$

which is just Newton's second law. The classical calculation of the permittivity of ordinary matter is based on Equation (19) with one exception: for ordinary matter there is also

a phenomenological term proportional to velocity that describes damping that results from radiation and collisions. For gases damping is often very small and can be neglected. While neglecting damping is an approximation for physical particles in a dielectric, it is exactly true for VFs. VFs can neither radiate energy nor lose energy in collisions with other quanta. If they did, after they vanished they would permanently leave behind energy, violating the principle of conservation of energy.

To describe the interaction of VFs with an electric field, one change has to be made to H^1 . If an atom is continually interacting with photons from an electric field $E_0 \cos \omega t \hat{\mathbf{x}}$, then the interaction H^1 of the atom with the field is given by Equation (18). As discussed in §4, even in an intense laser beam the number density of photons is much less than the number density of VFs. Thus the probability that at a given time a VF interacts with more than one photon is very small. That is, if a VF interacts with an electromagnetic wave, it almost always does so with only one photon.

If a VF absorbs a photon at time t_i , the photon vanishes at that instant. The electric field at the moment of interaction is $E(t_i) = E_0 \cos \omega t_i \equiv \mathbb{E}_0$, implying that the VF interacts with the electric field $\mathbb{E}_0 \hat{\mathbf{x}}$. Thus, for VFs Equation (18) becomes

$$H^{1VF} = -exE_0 \cos \omega t_i \equiv -ex\mathbb{E}_0. \quad (20)$$

As a part of the capture process, a parapositronium atom will, to some extent, be excited by $\mathbb{E}_0$ to resonate at its characteristic frequency ω^0 , as discussed in the following section.

6.2. Quantum calculation of the polarization of vacuum fluctuations

Because photons interact with atoms that are VFs somewhat similarly to the way that they interact with ordinary atoms, it is possible to calculate the permittivity ϵ_0 of the vacuum using techniques similar to those employed for calculating the permittivity ϵ of a physical dielectric. In the vacuum the interactions that primarily contribute to the value of ϵ_0 are photon capture by charged lepton-antilepton VFs bound into the lowest energy state that has zero angular momentum. The charged lepton-antilepton pair quickly annihilate, emitting a photon identical to the incident, captured photon.

Let $\psi_n^0(x, t)$ be solutions to the unperturbed Schrödinger equation,

$$i\hbar \frac{\partial \psi_n^0(x, t)}{\partial t} = H^0 \psi_n^0(x, t) = E_n^0 \psi_n^0(x, t), \quad (21)$$

where E_n^0 and H^0 are given in Equation (16) and Equation (17), respectively. From Equation (21) it follows that the energy dependence of $\psi_n^0(x, t)$ can be factored,

$$\psi_n^0(x, t) = e^{-E_n^0 t/\hbar} \psi_n^0(x). \quad (22)$$

The “exact” wave function $\psi(x, t)$ satisfies the Schrödinger equation,

$$i\hbar \frac{\partial \psi(x, t)}{\partial t} = (H^0 + H^{1VF}) \psi(x, t). \quad (23)$$

Perturbation theory is now used to calculate $\psi(x, t)$ to first order in the perturbation H^{1VF} : the “exact” wave function $\psi(x, t)$ is written as

$$\begin{aligned} \psi(x, t) &= a_0(t) \psi_0^0(x, t) + a_1(t) \psi_1^0(x, t), \\ &= a_0(t) e^{-iE_0^0 t/\hbar} \psi_0^0(x) + a_1(t) e^{-iE_1^0 t/\hbar} \psi_1^0(x). \end{aligned} \quad (24)$$

Substituting Equation (24) into Equation (23),

$$\begin{aligned} i\hbar \frac{da_0(t)}{dt} e^{-iE_0^0 t/\hbar} \psi_0^0(x) + i\hbar \frac{da_1(t)}{dt} e^{-iE_1^0 t/\hbar} \psi_1^0(x) \\ = H^{1VF} a_0(t) e^{-iE_0^0 t/\hbar} \psi_0^0(x) + H^{1VF} a_1(t) e^{-iE_1^0 t/\hbar} \psi_1^0(x). \end{aligned} \quad (25)$$

Using the orthogonality relations,

$$\int_{-\infty}^{\infty} dx \psi_{n'}^{0*}(x) \psi_n^0(x) = \delta_{n',n}, \quad (26)$$

and noting that $\psi_0^0(x)$ and $\psi_1^0(x)$ have opposite parity while $H^{1VF} \equiv -ex\mathbb{E}_0$ has negative parity, equations for $\frac{da_0(t)}{dt}$ and $\frac{da_1(t)}{dt}$ are immediately obtained from Equation (25):

$$\frac{da_0(t)}{dt} = a_1(t) \frac{ie\mathbb{E}_0}{\hbar} \langle x \rangle_{0,1} e^{-i\omega^0 t}, \quad (27a)$$

$$\frac{da_1(t)}{dt} = a_0(t) \frac{ie\mathbb{E}_0}{\hbar} \langle x \rangle_{1,0} e^{i\omega^0 t}. \quad (27b)$$

In the above equation

$$\langle x \rangle_{n',n} \equiv \int_{-\infty}^{\infty} dx \psi_{n'}^{0*}(x) x \psi_n^0(x) \quad (28)$$

and, as discussed in §5,

$$\omega^0 \equiv \frac{1}{\hbar} (E_1^0 - E_0^0). \quad (29)$$

Letting the time t_i that the photon interacts with the parapositronium VF be infinitesimally greater than $t = 0$ implies that

$$a_0(t=0) = 1 \quad \text{and} \quad a_1(t=0) = 0, \quad (30)$$

because the parapositronium VF is initially in its ground state, which, when interacting with a photon, is represented by the ground state $\psi_0^0(x, t)$ of the harmonic oscillator. Using Equation (30), Equation (27b) becomes

$$\frac{da_1(t)}{dt} = \frac{ie\mathbb{E}_0}{\hbar} \langle x \rangle_{1,0} e^{i\omega^0 t}, \quad (31)$$

which can immediately be integrated to yield

$$a_1(t) = \frac{e\mathbb{E}_0}{\hbar\omega^0} \langle x \rangle_{1,0} e^{i\omega^0 t}. \quad (32)$$

Note that Equation (32) satisfies Equation (30) because the interaction has not yet occurred at time $t_i = 0$. If the formula for $a_1(t)$ in Equation (32) is substituted into Equation (27a), a correction to $a_0(t)$ is obtained that is on the order of $(e\mathbb{E}_0)^2$. Thus from Equation (30) and Equation (32), to order $e\mathbb{E}_0$ the “exact” wave function $\psi(x, t)$ is given by

$$\begin{aligned} \psi(x, t) &= e^{-iE_0^0 t/\hbar} \psi_0^0(x) + \frac{e\mathbb{E}_0}{\hbar\omega^0} \langle x \rangle_{1,0} e^{i\omega^0 t} e^{-iE_1^0 t/\hbar} \psi_1^0(x), \\ &= e^{-iE_0^0 t/\hbar} [\psi_0^0(x) + \frac{e\mathbb{E}_0}{\hbar\omega^0} \langle x \rangle_{1,0} \psi_1^0(x)]. \end{aligned} \quad (33)$$

From Equation (18) and Equation (20) the expectation value $\langle p^{VF} \rangle$ of the electric dipole moment in the state characterized by $\psi(x, t)$ is

$$\begin{aligned}
 \langle p^{VF} \rangle &= \int_{-\infty}^{\infty} dx \psi^*(x, t) (ex) \psi(x, t), \\
 &= \int_{-\infty}^{\infty} dx e^{iE_0^0 t/\hbar} [\psi_0^{0*}(x) + \frac{e\mathbb{E}_0}{\hbar\omega^0} \langle x \rangle_{1,0} \psi_1^{0*}(x)] (ex) e^{-iE_0^0 t/\hbar} [\psi_0^0(x) + \frac{e\mathbb{E}_0}{\hbar\omega^0} \langle x \rangle_{1,0} \psi_1^0(x)], \\
 &\cong \int_{-\infty}^{\infty} dx [\psi_0^{0*}(x) (ex) \frac{e\mathbb{E}_0}{\hbar\omega^0} \langle x \rangle_{1,0} \psi_1^0(x) + \frac{e\mathbb{E}_0}{\hbar\omega^0} \langle x \rangle_{1,0} \psi_1^{0*}(x) (ex) \psi_0^0(x)], \\
 &= \frac{2e^2\mathbb{E}_0}{\hbar\omega^0} \langle x \rangle_{1,0}^2.
 \end{aligned} \tag{34}$$

To obtain the final equality, phases were chosen so that $\psi_0^0(x)$ and $\psi_1^0(x)$ are both real. Using the one-dimensional harmonic oscillator result,

$$\langle x \rangle_{1,0} = \sqrt{\frac{\hbar}{2\mu\omega^0}}, \tag{35}$$

it immediately follows from Equation (34) that

$$\langle p^{VF} \rangle_{0,0} = \frac{(e^2/\mu)\mathbb{E}_0}{(\omega^0)^2}. \tag{36}$$

Describing an oscillator as a harmonic oscillator, the quantum formula for the expectation value of the electric dipole of a VF agrees with the classical formula for the electric dipole of an atom of ordinary matter with two exceptions that have already been discussed: (1) Since VFs can neither radiate energy nor lose energy in a collision, there is no damping term. (2) Because VFs essentially always only interact with a single photon, only the value of the electric field at the instant of interaction is relevant so there is no dependence on the angular frequency ω of the incident photon.

To allow for the possibility that there is more than one type of atom made from particle-antiparticle VFs, an index j is added: accordingly, the charge $e \rightarrow q_j$, the reduced mass $\mu \rightarrow \mu_j$, and the resonant frequency $\omega^0 \rightarrow \omega_j^0$.

$$\langle p_j^{VF} \rangle_{0,0} = \frac{(q_j^2/\mu_j)\mathbb{E}_0}{(\omega_j^0)^2}. \tag{37}$$

6.3. General formula for the permittivity ϵ_0 of the vacuum

In a dielectric[45] the electric displacement $D(t)$ satisfies

$$D(t) = \epsilon E(t) = \epsilon_0 E(t) + P(t), \tag{38}$$

where ϵ is the permittivity of the dielectric, $E(t)$ is the electric field, and $P(t)$ is the electric polarization (dipole moment per unit volume). $P(t)$ can be expressed in terms of the individual dipole moments $p_j(t)$:

$$P(t) = \sum_j N_j p_j(t). \tag{39}$$

In Equation (39) N_j is the number of oscillators per unit volume of the j^{th} variety that are available to interact. In a uniform, classical electric field, the field is everywhere $E_0 \cos \omega t$.

Since the electric polarization is proportional to the electric field, the electric field cancels out of Equation (39).

The electric polarization $P(t)$ is responsible for the increase from $\epsilon_0 E(t)$ to $\epsilon E(t)$ because of photons interacting with oscillators in the material dielectric and results entirely from polarization of the atoms, molecules or both in the dielectric. It then follows that in the vacuum $\epsilon_0 E(t)$ must result entirely from the electric polarization $P^{VF}(t)$ of atoms, molecules, or both that are VFs. Thus,

$$\epsilon_0 E(t) = P^{VF}(t), \quad (40a)$$

or

$$\epsilon_0 = \frac{P^{VF}(t)}{E(t)}. \quad (40b)$$

The measurement of the permittivity ϵ_0 of the vacuum occurs over a time interval Δt . For any time t_i in the interval Δt for which a photon-VF interaction occurs, the electric field at that instant is $E(t_i) = E_0 \cos \omega t_i \equiv \mathbb{E}_0$. As shown in Equation (37), the polarization density is proportional to $\mathbb{E}_0$, implying that the instantaneous value of the electric field will cancel out in Equation (40). From Equation (40b), Equation (39) and Equation (37)

$$\epsilon_0 = \sum_j N_j^{VF} \frac{p_j^{VF}(t_i)}{\mathbb{E}_0} = \sum_j N_j^{VF} \frac{\langle p_j^{VF} \rangle_{0,0}}{\mathbb{E}_0} = \sum_j N_j^{VF} \frac{(q_j^2/\mu_j)}{(\omega_j^0)^2}. \quad (41)$$

Since the value of the electric field has canceled out of Equation (41), ϵ_0 is independent of the polarizing electric field, including its frequency. As a result, there is no dispersion in the vacuum.

Contributions from various types of VFs must now be summed over in Equation (41) to obtain the formula for ϵ_0 . The three types of VFs first considered are atomic, bound states of a charged lepton and antilepton, namely, parapositronium, muon-antimuon bound states, and tau-antitau bound states. Quark-antiquark states will be discussed later. Initially attention is restricted to a parapositronium VF.

6.4. Contribution to ϵ_0 from charged lepton-antilepton vacuum fluctuations

The progress of a photon traveling through the vacuum is slowed when it interacts with and has a polarizing effect on a VF consisting of a charged lepton and antilepton bound into an atom in its ground state. The fundamental interaction of a photon with a VF is the capture of the photon by the charged lepton-antilepton VF. Labeling the initial (incident) and final (emitted) photons, respectively, by γ_i and γ_f , to lowest order the two Feynman diagrams that contribute to the process $\gamma_i + \text{charged lepton-antilepton VF} \rightarrow \gamma_f$ are shown in Fig. 1. In the diagrams p_-, p_+, k_i , and k_f are, respectively, the four-momenta of the lepton, antilepton, initial photon, and final photon.

A quantum electrodynamics calculation[19] determines the decay rate Γ of this quasi-stationary atom. Using Γ , the effect of this interaction on the permittivity of the vacuum can be calculated. Coherence between the initial and final state is maintained. As required by conservation of energy and momentum, when the photon-excited, quasi-stationary state annihilates, the energy and momentum that were originally borrowed from the vacuum are returned to the vacuum. An isolated, ordinary atom consisting of a charged lepton-antilepton pair is kinematically forbidden from capturing a photon, annihilating, and then emitting a single photon; however, because the kinematics is different for the capture and release of a photon by a VF, the process is kinematically allowed for charged lepton-antilepton VFs.

The electromagnetic decay rate Γ_{p-Ps} for a parapositronium VF after it has interacted with the incident photon to form a quasi-stationary state is[19]

$$\Gamma_{p-Ps} = \frac{\alpha^5 m_e c^2}{\hbar}. \quad (42)$$

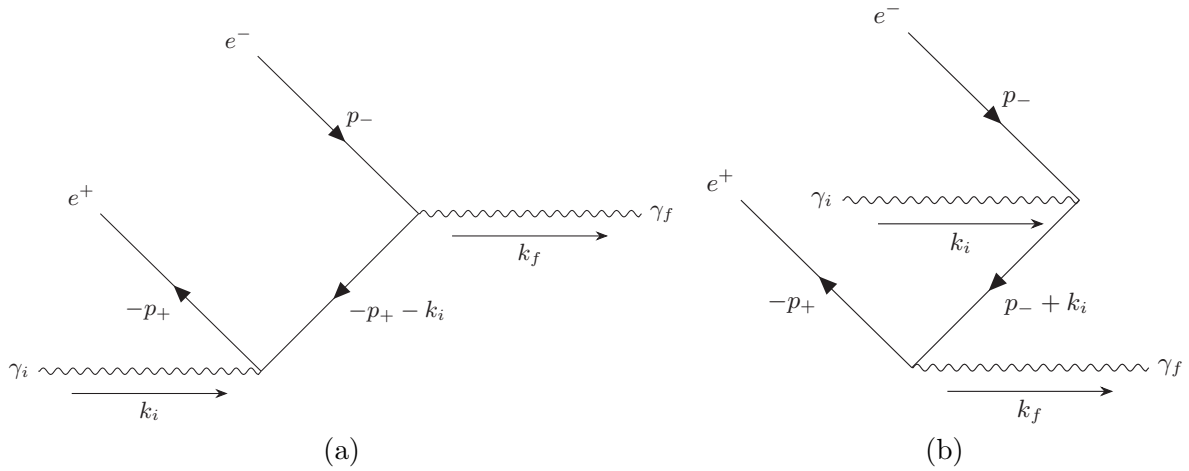


Figure 1. (a) Photon γ_i interacts with an antilepton that then annihilates with a lepton, emitting photon γ_f . (b) Photon γ_i interacts with a lepton that then annihilates with an antilepton, emitting photon γ_f .

The above rate is twice the decay rate of ordinary parapositronium into two photons[10, 46].

The probability that an excited parapositronium VF has not decayed during a time t is $e^{-\Gamma_{p-Ps}t}$, and the probability that it has decayed electromagnetically is $1 - e^{-\Gamma_{p-Ps}t}$. The quantity N_j^{VF} in Equation (41) for a parapositronium VF is the number of parapositronium VFs per unit volume with which a photon actually interacts. At equilibrium the average rate at which a parapositronium VF absorbs a photon equals the average rate for a parapositronium VF to annihilate and emit a photon. As a consequence, the average probability that parapositronium absorbs a photon during a time t is $1 - e^{-\Gamma_{p-Ps}t}$.

For a parapositronium VF the quantity N_j^{VF} , denoted by N_{p-Ps} , is the number density of parapositronium VFs multiplied by the probability that a parapositronium VF will absorb an incoming photon during the lifetime Δt_{p-Ps} of the parapositronium VF:

$$N_{p-Ps}^{VF} \cong \frac{1}{L_{p-Ps}^3} \times (1 - e^{-\Gamma_{p-Ps} \Delta t_{p-Ps}}). \quad (43)$$

Since $\Gamma_{p-Ps} \Delta t_{p-Ps} \ll 1$, the term $1 - e^{-\Gamma_{p-Ps} \Delta t_{p-Ps}}$ is very nearly equal to $\Gamma_{p-Ps} \Delta t_{p-Ps}$. Thus

$$N_{p-Ps}^{VF} \cong \frac{1}{L_{p-Ps}^3} \times \Gamma_{p-Ps} \Delta t_{p-Ps} = \frac{\alpha^5}{4} \left(\frac{4m_e c}{\hbar} \right)^3. \quad (44)$$

To better understand Equation (44), for a particular atom (or molecule) let N be the number of atoms per unit volume, let Γ be the decay rate of a photon-excited atom into the atom in its ground state plus a photon, and let t be the average time that the atom exists. For ordinary matter N_j in Equation (39) is N , but for a vacuum fluctuation it is $N\Gamma t$ as given in Equation (44). How does this difference arise? Making no assumptions about the magnitude of Γt and using the logic that led to Equation (43), $N_j = N(1 - e^{-\Gamma t})$. If the atom is stable, the lifetime t is infinite so Γt is infinite, and $N_j = N$ as expected. On the other hand, if the lifetime is sufficiently small that $\Gamma t \ll 1$, as is the case for a parapositronium VF, $N_j^{VF} \cong N\Gamma t$ as given in Equation (44) ³.

³ As pointed out by A. Bohm[47], a physical state prepared in a scattering experiment contains background information about the reaction in which it was created. The above description of a single photon interacting with

Substituting Equation (14) and Equation (44) into Equation (41),

$$\epsilon_0 \cong \sum_j \frac{8^3 \alpha e^2}{\hbar c}. \quad (45)$$

Note that the mass of the electron has cancelled from the expression for ϵ_0 , implying that bound muon-antimuon and tau-antitau VFs each contribute the same amount to the value of ϵ_0 as parapositronium VFs contribute. Including the contributions from the three types of charged, bound lepton-antilepton VFs yields

$$\epsilon_0 \cong 3 \frac{8^3 \alpha e^2}{\hbar c} + \left(\text{any contribution from} \right. \\ \left. \text{quark-antiquark VFs.} \right) \quad (46)$$

6.5. Contribution to ϵ_0 from quark-antiquark vacuum fluctuations

The only other charged particle-antiparticle VFs that might exhibit elastic behavior are quark-antiquark VFs; however, as will now be shown, contributions to ϵ_0 from quark-antiquark VFs is substantially reduced in comparison with those from charged lepton-antilepton VFs.

First consider the heavy quarks $Q = c, b$, or t , where it is appropriate to think in terms of static quark potentials for $Q\bar{Q}$ bound states. Let $m_{Q\bar{Q}}$ be the mass of the least massive $Q\bar{Q}$ bound state that has zero angular momentum. From the Heisenberg uncertainty principle, the $Q\bar{Q}$ bound state will have an average lifetime $\Delta t_{Q\bar{Q}} = \hbar/(2M_{Q\bar{Q}}c^2)$. During the time $\Delta t_{Q\bar{Q}}$ light will travel a distance $L_{Q\bar{Q}} = c\Delta t_{Q\bar{Q}}$. As previously noted, the decay rate of a photon-excited parapositronium VF into a photon is twice the decay rate of ordinary parapositronium into two photons. Since the decay rate of a photon-excited $Q\bar{Q}$ VF into a photon is not known, it is approximated by twice the decay rate $\Gamma_{Q\bar{Q} \rightarrow \gamma + \gamma}$ of an ordinary $Q\bar{Q}$ bound state into two photons. From Equation (44) it then follows that

$$N_{Q\bar{Q}}^{VF} \sim \frac{1}{L_{Q\bar{Q}}^3} \times 2\Gamma_{Q\bar{Q} \rightarrow \gamma + \gamma} \Delta t_{Q\bar{Q}} \sim 8c \left(\frac{M_{Q\bar{Q}}}{\hbar} \right)^2 \Gamma_{Q\bar{Q} \rightarrow \gamma + \gamma}. \quad (47)$$

Letting m_Q and q_Q denote, respectively, the mass and charge of the heavy quark Q and letting $\omega_{Q\bar{Q}}^0$ denote the resonant angular frequency of the $Q\bar{Q}$ VF, from Equation (41) it follows that the contribution of the $Q\bar{Q}$ VF to ϵ_0 is

$$\epsilon_0 \left(\begin{array}{c} \text{contribution} \\ \text{from } Q\bar{Q} \text{ VF} \end{array} \right) \sim 8c \left(\frac{M_{Q\bar{Q}}}{\hbar} \right)^2 \Gamma_{Q\bar{Q} \rightarrow \gamma + \gamma} \frac{q_Q^2/(m_Q/2)}{(\omega_{Q\bar{Q}}^0)^2}. \quad (48)$$

The least massive $c\bar{c}$ bound state, $\eta_c(1S)$, has $J = 0$ and has positive charge conjugation parity[48], implying that a photon excited $\eta_c(1S)$ VF can decay into a single photon. To obtain an order-of-magnitude estimate of the maximum contribution that an $\eta_c(1S)$ VF could make to ϵ_0 , the experimental decay rate $\Gamma_{\eta_c(1S) \rightarrow \gamma + \gamma}$ of $\eta_c(1S)$ into two photons is $7.69 \times 10^{18}/s$ [48]. It is not obvious which energy should be used to calculate the angular frequency $\omega_{Q\bar{Q}}^0$, but the minimum possible energy $E_{\min}$ yields a minimum possible value for $\omega_{Q\bar{Q}}^0$ and a maximum possible value for the contribution of an $\eta_c(1S)$ VFs to ϵ_0 . $E_{\min}$ is the difference between the mass $m_{\eta_c(1S)}$ of $\eta_c(1S)$ and the masses m_c of the charm and anti-charm quarks when they are

a VF clearly depicts such a circumstance despite the fact that the details of the discussion involve a decay rate Γ . Usually, a decay rate such as $\Gamma_{p \rightarrow Ps}$ is associated only with a stochastic process. In each individual interaction of a photon with a VF, the coherence of the final photon with the initial photon is assured, as promised at the beginning of §6.4.

weakly bound: $E_{\min} = m_{\eta_c(1S)} - 2m_c \sim 2.98 \text{ GeV} - 2 \times 1.27 \text{ GeV} = 0.44 \text{ GeV}$ [48]. From Equation (48) the maximum possible contribution to ϵ_0 from $\eta_c(1S)$ VFs is then calculated to be

$$\epsilon_0 \left(\begin{array}{c} \text{maximum possible} \\ \text{contribution from } \eta_c(1S) \text{ VFs} \end{array} \right) \sim 1.3 \times 10^{-3} \frac{e^2}{\hbar c}. \quad (49)$$

Comparing Equation (49) with Equation (46), the maximum contribution to ϵ_0 from $\eta_c(1S)$ VFs is about 10^{-4} times smaller than the combined contribution from the three charged lepton-antilepton VFs.

The the least massive $b\bar{b}$ bound state, $\eta_b(1S)$, has $J = 0$ and has positive charge conjugation parity[48]. Its mass is known experimentally, but the decay rate into two photons is not[48]. There are, however, theoretical calculations of the decay rate that range from 0.22 keV to 0.45 keV[49, 50, 51, 52]. To determine the maximum contribution to ϵ_0 from oscillations of $\eta_b(1S)$ VFs, the maximum value for the decay rate and the minimum value of energy associated with the state are used: $E_{\min}$ is the difference between the mass $m_{\eta_b(1S)}$ of $\eta_b(1S)$ and the masses m_b of the bottom and anti-bottom quarks when they are weakly bound: $E_{\min} = m_{\eta_b(1S)} - 2m_b \sim 9.40 \text{ GeV} - 2 \times 4.3 \text{ GeV} = 0.8 \text{ GeV}$ [48]. From Equation (48) The maximum contribution to ϵ_0 from oscillations of $\eta_b(1S)$ VFs is then calculated to be

$$\epsilon_0 \left(\begin{array}{c} \text{maximum possible} \\ \text{contribution from } \eta_b(1S) \text{ VFs} \end{array} \right) \sim 2.6 \times 10^{-5} \frac{e^2}{\hbar c}, \quad (50)$$

which is about 10^{-6} times smaller than the contribution from the three charged lepton-antilepton VFs.

For the heavy quarks c and b , as the mass increases from m_c to m_b , the minimum possible angular frequency increases, and the decay rate of $Q\bar{Q}$ into two photons decreases. Both effects decrease the maximum possible contribution to ϵ_0 and suggest the contribution to ϵ_0 from the top quark t would be smaller than that from either c or b . On the other hand, the square of the charge for t is four times larger than that for b . Although there is no experimental information about $\eta_t(1S)$ [48], from the above discussion the contribution of $\eta_t(1S)$ VFs to ϵ_0 is expected to be small compared with the contribution from the three charged lepton-antilepton VFs.

For the light quarks $q = u, d$, or s , the π^0 , η , and η' are the least massive $J = 0$ combinations of $q\bar{q}$ bound states that decay into two photons. Here, however, the comparison with an oscillator is less appropriate; a Bethe-Salpeter approach to $q\bar{q}$ bound states fails completely[53]. As discussed in Ref. [54], a completely relativistic approach is required, an approach that shows no indication that a $q\bar{q}$ pair can be characterized by an oscillator potential energy. Moreover, since the strong interactions are primarily responsible for the binding of these relativistic states, $q\bar{q}$ bound state VFs would have much higher natural frequencies than the electromagnetically bound, charged lepton-antilepton VFs that are much more weakly bound. Accordingly $q\bar{q}$ bound state VFs contribute little to ϵ_0 and to lowest order need not be considered.

6.6. Calculation of ϵ_0

Ignoring the small contributions to ϵ_0 from quark-antiquark VFs, an approximate formula for ϵ_0 is immediately obtained from Equation (46) using the defining formula $\alpha = e^2/(4\pi\epsilon_0\hbar c)$ to eliminate α and then using $c = 1/\sqrt{\mu_0\epsilon_0}$:

$$\epsilon_0 \cong \frac{6\mu_0}{\pi} \left(\frac{8e^2}{\hbar} \right)^2 = 9.10 \times 10^{-12} \frac{\text{C}}{\text{Vm}}. \quad (51)$$

The calculation of ϵ_0 in Equation (51) is strictly a quantum calculation: (1) The existence of on-mass-shell, charged lepton-antilepton VFs are predicted by quantum field theory. (2) The appearance of charged lepton-antilepton VFs as bound states in the lowest energy level with $J = 0$ is predicted by the Heisenberg uncertainty principle and conservation of angular momentum. (3) The energy levels of the bound state VFs are calculated using non-relativistic quantum mechanics. (4) The decay rates of the charged lepton-antilepton bound state VFs are calculated using quantum electrodynamics. (5) The calculation of ϵ_0 from (1)-(4) is a quantum derivation. Once (1)-(4) are known, however, it is possible to obtain the formula Equation (51) using a classical derivation similar to that used to obtain the permittivity of a physical dielectric[55].

From Equation (51) equations for c and α immediately follow as shown in the Introduction. Only the lowest-order terms in α have been retained in calculating ϵ_0 . The binding energy of parapositronium was neglected when calculating Δt_{p-P_s} in Equation (9), and only the leading term was retained when calculating ω_j^0 and Γ_{p-P_s} . The formulas for ϵ_0 , c , and α exhibit the point of view that e and $\hbar$ are fundamental constants. (In SI units μ_0 is a parameter with assigned a value.) The calculation of ϵ_0 also describes how the properties of the vacuum determine the speed of light in the vacuum and the fine-structure constant.

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