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ELECTROPRODUCTION OF NUCLEON RESONANCES*

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ABSTRACT

We compute the differential cross section for the process

$$e + p \rightarrow e + p_R$$

where p_R is a nucleon resonance characterized by parity, spin J, and mass M_R. The two inelastic form factors describing this cross section are expressed in terms of three amplitudes characterizing the (p, p_R) electromagnetic vertex. The kinematic and analytic structure of these three amplitudes as a function of q^2 are discussed. The case of the 33 resonance is discussed in some detail.

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I. INTRODUCTION

The study of inelastic electron-proton scattering is likely to increase in importance as a probe of the structure of the nucleon as higher energies and momentum transfers become available. Furthermore, from both the experimental and theoretical standpoint, excitation of resonant states (isobars) of the nucleon will be of particular interest. We here review and extend (1,2,3) the phenomenological description of such a process, giving the general vertex and differential cross section for $e^- + p \rightarrow e + p_R$, where p_R is a nucleon resonance characterized by parity π_R, spin J, and mass M. We find the analogue of the description of elastic e-p scattering by the Rosenbluth formula; our main result is the following expression for the cross section, where only the final electron is detected, and where the initial particles are unpolarized

$$\left.\frac{d\sigma}{d\Omega}\right|_{lab} = \frac{\alpha^2 \cos^2\frac{\theta}{2}}{4\epsilon^2 \sin^4\frac{\theta}{2}\left[1+\frac{2\epsilon}{m}\sin^2\frac{\theta}{2}\right]}\left\{\frac{q^4}{q^{*4}}\left|f_c\right|^2 + \left(\frac{q^2}{2q^{*2}} + \frac{M^2}{m^2}\tan^2\frac{\theta}{2}\right)\left[\left|f_+\right|^2 + \left|f_-\right|^2\right]\right\} \tag{1.1}$$

Here ϵ and θ are incident electron energy and scattering angle, and M and m are isobar and nucleon masses, respectively. $q^2 = 4\epsilon\epsilon' \sin^2\frac{\theta}{2}$ is the invariant four-momentum transfer, while q^{*2} is the magnitude of the three-momentum transfer from the electron in the isobar rest frame:

$$q^{*2} = q^2 + \frac{1}{4M^2}\left[M^2 - m^2 - q^2\right]^2 \tag{1.2}$$

this frame is especially convenient for the kinematical analysis of the process. The form factors f_c and $f_\pm$ are the analogues of G_E and G_M for elastic scattering, and are functions of q^2.

For $q^2 = 0$ the transverse form factors $f_\pm$ are related to the photo-absorption cross section integrated over the resonance

$$\underset{\text{Lab; over resonance}}{\int} \sigma_\gamma(\omega)\, d\omega = \frac{4\pi^2\alpha}{M^2-m^2}\, \frac{M^2}{m} \left[|f_+|^2 + |f_-|^2 \right]_{q^2=0} \tag{1.3}$$

Detailed properties of the form factors f_c, $f_\pm$ are highly model dependent. However, in the limit $q^* \to 0$ (which implies $q_o \to M-m$) the form factors have simple threshold behaviors:

1) Normal parity[1] transitions $\frac{1}{2}^+ \to \frac{3}{2}^-, \frac{5}{2}^+, \ldots.$

$$\begin{aligned} f_c &\sim q^{*J-\frac{1}{2}} \\ f_\pm &\sim q^{*J-\frac{3}{2}} \end{aligned} \tag{1.4}$$

2) Abnormal parity transitions $\frac{1}{2}^+ \to \frac{1}{2}^-, \frac{3}{2}^+, \frac{5}{2}^-, \ldots.$

$$\begin{aligned} f_c &\sim q^{*J+\frac{1}{2}} \\ f_\pm &\sim q^{*J-\frac{1}{2}} \end{aligned} \tag{1.5}$$

Unfortunately, these threshold properties may not be of use, because only spacelike momentum transfers are available experimentally, and it is not clear whether the threshold behavior still persists there. For example, if we use the nonrelativistic reduction to nuclear physics (m, $M \to \infty$) then we know from the Bessel functions involved

$$f_c \sim (q^*R)^L \text{ for } q^*R < L \tag{1.6}$$

where $L = J \pm \frac{1}{2}$ depending on the parity and R is the radius of the target. On the other hand, kinematics [Eq. (1.2)] tells us

$$q^* \geq \frac{m+M}{2M} \; (M-m) \tag{1.7}$$

Taking $(M-m) \sim 2\mu L$ as is roughly the case experimentally (μ is the pion mass), we find

$$R \lesssim \frac{1}{2\mu} \tag{1.8}$$

in order that threshold behavior still persist for spacelike q^2. This is a rather small interaction radius.

For the normal parity transitions, we find an additional relation between f_c and $f_\pm$ valid near threshold, which is an additional test on the spin-parity assignment and the assumption of dominance of the threshold behavior. This relation

$$\frac{|f_c|^2}{|f_+|^2 + |f_-|^2} \approx \frac{2J+1}{2J-1} \; \frac{q_o^2}{q^{*2}} \tag{1.9}$$

is well-known in nuclear physics. In particular it is the relation which allows one to get photon lifetimes for electric transitions from Coulomb excitation.

The rest of this paper is devoted to the details of deriving the cross section and to discussion, as best as we can on general grounds alone, of what behavior might be expected of the form factors.

In Section II, we write down the general electromagnetic vertex function between isobar and the nucleon. Two representations are used. The first is the Jacob-Wick (4) helicity representation, written down in the isobar rest frame. In the second we explicitly describe the isobar by a generalized spinor wave function (5)

$$\psi_r^{\mu_1 \cdots \mu_{J-\frac{1}{2}}} \quad (\mu_i = 1,\ldots,4;\ r = 1,\ldots 4) \tag{1.10}$$

In each instance three form factors are involved.

In Section III, we square the matrix element and sum over spins to obtain the cross section, using again both the helicity and spinor wave function methods. The connection between the two approaches is then established.

In Section IV, the threshold behavior of the form factors is derived. In Section V, we review the analytic properties of the form factors expected from Feynman diagram considerations. Unlike elastic scattering, complex singularities appear and the form factors need not be real. Indeed in certain circumstances the phase of the form factor is determined by the Watson final-state theorem, and we discuss how this comes about. Finally, in Section VI, we summarize briefly implications of some models for the $\frac{3}{2}^+$ resonance, in particular that of Fubini, Nambu and Wataghin (6) and of Gourdin and Salin (7).

II. STRUCTURE OF THE VERTEX

The general process of electron excitation of the nucleon is illustrated in Fig. 1. We are primarily interested in the case where the final nucleon state is a nucleon isobar characterized by a spin J, parity π_R, and mass M. The problem then is to study the nucleon-isobar electromagnetic vertex

$$< P'|J_\mu(0)|P >$$

where $|P>$ is the Heisenberg state vector of the initial nucleon and $|P'>$ that of the final nucleon system (they are both eigenstates of the four momentum operator P_μ). $J_\mu(0)$ is the electromagnetic current operator of the strongly interacting system evaluated at the space-time point $x_\mu = 0$.

In analyzing the nucleon-isobar electromagnetic vertex, it is most convenient to work in the rest frame of the final isobar. In this frame, the final state is an eigenstate of angular momentum J, the spin of the isobar. We shall use two methods to analyze the vertex. The first is the helicity analysis of Jacob and Wick (4) and the second is an explicit construction of the wave function of the isobar in the spirit of Rarita and Schwinger (8). We present both methods since, although they lead to the same result, they tend to emphasize different aspects of the problem and hence give one some additional insight. We start with the helicity analysis (9).

Helicity Analysis

In the rest frame of the isobar the quantity we want to study is

$$< \pi_R JM \left| J_\mu(0) \right| \underset{\sim}{q}^* \lambda >$$

where λ is the helicity of the initial nucleon and $\underset{\sim}{q}^* = \underset{\sim}{q} = \underset{\sim}{P}$ is its momentum. We now want to use angular momentum conservation and the fact that $\underset{\sim}{J}(0)$ transforms as a vector under rotation while $J_o(0)$ transforms as a scalar. Our final state is already an eigenstate of angular momentum. The problem is therefore to expand the initial nucleon state in eigenstates of angular momentum. From the work of Jacob and Wick, one knows immediately how to do this. The basic theorem is that

$$\left| q\ jm\lambda > = \left(\frac{2j+1}{4\pi}\right)^{\frac{1}{2}} \int d\Omega_q \mathcal{D}^{j}_{m\lambda}(-\varphi_q - \theta_q \varphi_q)^* \right| \underset{\sim}{q}, \lambda > \qquad (2.1)$$

One gets the appropriate energy-angular momentum-helicity eigenstate by integrating the momentum eigenstate over solid angles and using as a weighting function the rotation matrices $\mathcal{D}^{j\,*}_{m\lambda}$. (We follow the angular momentum notation of Edmonds (10).) The final φ_q in the argument of the $\mathcal{D}^{j}_{m\lambda}$ functions is merely a definition of the overall phase of the states ($\sim e^{i\lambda\varphi_q}$) as discussed in Jacob and Wick. The completeness properties of the $\mathcal{D}^{j}_{m\lambda}$ allow us to invert this relation

$$\left| \underset{\sim}{q}\lambda > = \sum_{jm} \left(\frac{2j+1}{4\pi}\right)^{\frac{1}{2}} \mathcal{D}^{j}_{m\lambda}(-\varphi_q - \theta_q \varphi_q) \right| qjm\lambda > \qquad (2.2)$$

and give us the needed result. That is

$$< \pi_R JM \big| J_\mu(0) \big| \underset{\sim}{q}\lambda > = \sum_{jm} \left(\frac{2j+1}{4\pi}\right)^{\frac{1}{2}} \mathcal{D}^{j}_{m\lambda}(-\varphi_q - \theta_q \varphi_q) < \pi_R JM \big| J_\mu(0) \big| qjm\lambda > \tag{2.3}$$

It is convenient at this point to introduce eigenstates of parity. From the work of Jacob and Wick the parity operator $\hat{\pi}$ acting on the nucleon state gives

$$\hat{\pi} \big| qjm\lambda > = (-1)^{j-\frac{1}{2}} \big| qjm - \lambda > \quad . \tag{2.4}$$

We can therefore introduce the parity eigenstates

$$\big| q\pi jm > = \frac{1}{\sqrt{2}} \left[\big| qjm\lambda > \pm (-1)^{j-\frac{1}{2}} \big| qjm - \lambda > \right] \tag{2.5}$$

The problem is therefore reduced to studying the matrix elements

$$< \pi_R JM \big| J_\mu(0) \big| q\pi jm >$$

where only the appropriate value of π can contribute. We can now use the Wigner-Eckart Theorem to extract the M dependence of the matrix elements.[2]

$$< \pi_R JM \big| \underset{\sim}{J}_\rho(0) \big| q\pi jm > = (-1)^{J-M} \begin{pmatrix} J & 1 & j \\ -M & \rho & m \end{pmatrix} < \pi_R J \big\| \underset{\sim}{J}(0) \big\| q\pi j > \tag{2.6}$$

Since in this case $j = J, J\pm1$ there are three independent reduced matrix elements which are each functions of $|\underset{\sim}{q}| = q^*$. We also have for the fourth component

$$< \pi_R JM \big| J_o(0) \big| q\pi jm > = (-1)^{J-M} \begin{pmatrix} J & 0 & j \\ -M & 0 & m \end{pmatrix} < \pi_R J \big\| J_o(0) \big\| q\pi j > \tag{2.7}$$

and only get a contribution here if $J = j$. There is, however, still one relation between these four reduced matrix elements which comes from the continuity equation for the nucleon electromagnetic current

$$q_\mu < \pi_R JM \left| J_\mu(0) \right| \underset{\sim}{q}\lambda > = 0 \qquad (2.8)$$

Using

$$\hat{\underset{\sim}{q}} \cdot \underset{\sim}{J}(0) = \sum_\lambda \mathcal{D}^{1}_{\lambda o} (-\varphi_q - \theta_q \varphi_q) \, \underset{\sim}{J}_\lambda(0) \qquad (2.9)$$

and the formulae of Edmonds for combining two $\mathcal{D}$ functions one finds after a little algebra that the continuity equation says

$$\sum_j \left(\frac{2j+1}{2J+1}\right)^{\frac{1}{2}} (j\tfrac{1}{2}10 \left| j1J\tfrac{1}{2}\right.) < \pi_R J \left|\left| \underset{\sim}{J}(0) \right|\right| q^* \pi j > = \frac{q_o}{q^*} < \pi_R J \left|\left| J_o(0) \right|\right| q^* \pi J > \qquad (2.10)$$

and therefore only three of the reduced matrix elements are independent and the electromagnetic nucleon-isobar (spin J) vertex is characterized by three independent form factors. This result is well-known in nuclear physics, for electron excitation $\frac{1}{2}^+ \rightarrow J^{\pi_R}$ can take place by one Coulomb, one transverse electric and one transverse magnetic multipole. The result was first given in covariant form by Durand, de Celles, and Marr (1).

It is convenient for our later work to define the linear combination

$$f_\rho \equiv \left(\frac{m}{8\pi M}\right)^{\frac{1}{2}} \sum_j \left(\frac{2j+1}{2J+1}\right)^{\frac{1}{2}} (j\tfrac{1}{2}1\rho \left| j1J\tfrac{1}{2}+\rho\right.) < \pi_R J \left|\left| \underset{\sim}{J}(0) \right|\right| q^* \pi j > \left(\frac{EE'\Omega^2}{mM}\right)^{\frac{1}{2}} \qquad (2.11)$$

with $\rho = \pm 1, 0$ and

$$f_c \equiv \left(\frac{m}{8\pi M}\right)^{\frac{1}{2}} < \pi_R J \left|\left| J_o(0) \right|\right| q^* \pi J > \left(\frac{EE'\Omega^2}{mM}\right)^{\frac{1}{2}} \qquad (2.12)$$

The continuity equation then simply eliminates f_o

$$f_o = \frac{q_o}{q^*} f_c \tag{2.13}$$

and the transformation has the further property that

$$\sum_{\rho} |f_\rho|^2 = \left(\frac{m}{8\pi M}\right) \sum_{j} \left| < \pi_R J \left| \left| \underline{J}(0) \right| \right| q^* \pi j > \right|^2 \left(\frac{EE'\Omega^2}{mM}\right) \tag{2.14}$$

Spinor Wave Function Analysis

As an alternative to the helicity description given above, we may write the vertex function in terms of a general spinor wave function describing a particle of spin J. This wave function

$$\Psi_r^{\mu_1 \cdots \mu_{J-\frac{1}{2}}} (P,\lambda) \qquad (r = 1, \ldots, 4;\ \mu_\alpha = 1, 2, 3, 4)$$

is the generalization of the Dirac spinor for $J = \frac{1}{2}$ (where there are no indices μ) and the Rarita-Schwinger wave function for spin 3/2. It is[3]

1) Symmetric under any permutation of indices μ_α.

2) Zero upon contraction of any pair of indices μ_α.

3) A solution of the Dirac equation

$$(i\gamma \cdot P + m)_{rs} \Psi_s^{\mu_1 \cdots \mu_{J-\frac{1}{2}}} = 0 \tag{2.15}$$

4) "Orthogonal" to γ_μ and P_μ;

$$P_{\mu_1} \Psi^{\mu_1 \cdots \mu_{J-\frac{1}{2}}} = \gamma_{\mu_1} \Psi^{\mu_1 \cdots \mu_{J-\frac{1}{2}}} = 0 \tag{2.16}$$

5) Normalized

$$\bar{\Psi}^{\mu_1 \cdots \mu_{J-\frac{1}{2}}} (P,\lambda)\, \Psi^{\mu_1 \cdots \mu_{J-\frac{1}{2}}} (P,\lambda) = 1 \tag{2.17}$$

Our task is to reconstruct the electromagnetic vertex function in this language in analogy to the conventional (11) relativistic treatment of, say, the proton vertex. From the helicity description we know that there will be three independent form factors; our choice of spinor covariants will be motivated *ex post facto*. The choice we do make has the convenient property that cross terms between the different form factors vanish when the amplitude is squared and spins are summed in constructing the cross section. So without further ado, we write for the general form of the vertex function for normal parity transitions $1/2^+ \to 3/2^-$, $5/2^+$, ...

$$< P'\lambda' | J_\mu \epsilon_\mu | P\lambda > = \frac{1}{\Omega} \left(\frac{mM}{E'E} \right)^{\frac{1}{2}} \bar{\Psi}^{\mu_1 \cdots \mu_{J-\frac{1}{2}}}_{(P',\lambda')} q_{\mu_2} \cdots q_{\mu_{J-\frac{1}{2}}} \times \left[\begin{array}{l} q_{\mu_1} (\epsilon \cdot q P \cdot q - \epsilon \cdot P\, q^2)\, g_1(q^2) \\ + 2i\epsilon_{\mu_1 \alpha\beta\gamma} P_\alpha q_\beta S_\gamma\, g_2(q^2) \\ + M q_{\mu_1} \gamma \cdot S \gamma_5 \left(g_2(q^2) + g_3(q^2) \right) \end{array} \right] u(P,\lambda) \tag{2.18}$$

where the pseudovector S_μ is defined by

$$S_\mu = i\epsilon_{\mu\nu\lambda\sigma} P_\nu q_\lambda \epsilon_\sigma \quad . \tag{2.19}$$

For abnormal parity transitions the only change that need be made is the replacement

$$u(P,\lambda) \to \gamma_5\, u(P,\lambda) \tag{2.20}$$

For spin 1/2 final states the spinor Ψ has no indices μ_α and consequently the coefficient of g_2 does not make sense. In this case there are only the two form factors g_1 and g_3. Indeed $q^2 g_1$ and g_3 correspond to the electric and magnetic form factors G_E and G_M used in the proton elastic vertex function.

To proceed toward obtaining a cross section in terms of these form factors it is convenient to specialize to the rest frame of the isobar. From property (5) the lower components ($r = 3,4$) as well as the time components ($\mu_\alpha = 4$) of the isobar spinor vanish in this frame, and it is easy to go to a two-component "Pauli spinor" formalism for both isobar and proton. Upon carrying out this reduction we obtain,[4] for normal parity transitions,

$$< P'\lambda' \left| J_\mu \epsilon_\mu \right| P\lambda > = \frac{M^2}{\Omega} \left(\frac{(E+m) q^{*2J+1}}{2E} \right)^{\frac{1}{2}} \chi^\dagger_{\alpha_1 \ldots \alpha_{J-\frac{1}{2}}}(\lambda') \hat{\underset{\sim}{q}}_{\alpha_2} \ldots \hat{\underset{\sim}{q}}_{\alpha_{J-\frac{1}{2}}}$$

$$\times \left[\begin{array}{l} - \hat{\underset{\sim}{q}}_{\alpha_1} \left[\dfrac{\underset{\sim}{\epsilon} \cdot \hat{\underset{\sim}{q}}\, q_0 - \epsilon_0 q^*}{M} \right] g_1(q^2) \\ + \left[(\underset{\sim}{\epsilon} \times \hat{\underset{\sim}{q}}) \times \hat{\underset{\sim}{q}} \right]_{\alpha_1} 2 g_2(q^2) \\ + \hat{\underset{\sim}{q}}_{\alpha_1} \left[i \underset{\sim}{\sigma} \cdot \underset{\sim}{\epsilon} \times \hat{\underset{\sim}{q}} \right] \left[g_2(q^2) + g_3(q^2) \right] \end{array} \right] \chi(\lambda) \tag{2.21}$$

This will be the most useful form for calculating the spin sums, which are most easily done in the isobar rest frame. We caution the reader not to infer too hastily the "threshold behavior," i.e., the behavior for low q^*, from this formula; this is discussed in more detail in Section IV.

For abnormal parity transitions we make the replacement in Eq. (2.21)

$$\chi(\lambda) \rightarrow \left(\frac{q^*}{E + m} \right) \underset{\sim}{\sigma} \cdot \hat{\underset{\sim}{q}}\, \chi(\lambda) \tag{2.22}$$

III. CROSS SECTIONS

The cross sections for the process illustrated in Fig. 1 can be written in standard fashion as (12)

$$d\sigma = 2\alpha^2 \frac{d^3p'}{2\epsilon'} \frac{1}{q^4} W_{\mu\nu} N_{\mu\nu} \frac{1}{\left[(p\cdot P)^2 - m_\ell^2 m^2\right]^{\frac{1}{2}}} \tag{3.1}$$

where

$$\begin{aligned} N_{\mu\nu} &= -\frac{1}{2} \mathrm{Tr}\, \gamma_\mu (m_\ell - i\gamma\cdot p)\, \gamma_\nu (m_\ell - i\gamma\cdot p') \\ &= 2\left[p_\mu p'_\nu + p_\nu p'_\mu - \delta_{\mu\nu}(p\cdot p' + m^2)\right] \end{aligned} \tag{3.2}$$

p and p' are the initial and final lepton four-momenta, m is the nucleon mass and m_ℓ is the lepton mass. The covariant tensor $W_{\mu\nu}$ is given by

$$W_{\mu\nu} = (2\pi)^3 \Omega \overline{\sum_{\text{initial}}} \sum_{\text{final}} \delta^{(4)}(p-p'-q) < P\left|J_\nu(o)\right|P'> < P'\left|J_\mu(o)\right|P > (E) \tag{3.3}$$

where Ω is the normalization volume, E is the energy of the target, $\overline{\sum}_{\text{initial}}$ indicates an average over nuclear orientations. From general considerations of covariance, parity conservation, and current conservation,

$$q_\mu W_{\mu\nu} = W_{\mu\nu} q_\nu = 0 \ , \tag{3.4}$$

the tensor $W_{\mu\nu}$ is known to have the form (13)

$$W_{\mu\nu} = W_1(q^2, q\cdot P)\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + W_2(q^2, q\cdot P)\frac{1}{m^2}\left(P_\mu - \frac{P\cdot q}{q^2} q_\mu\right)\left(P_\nu - \frac{P\cdot q}{q^2} q_\nu\right) \tag{3.5}$$

The cross section in the laboratory frame is given by

$$\frac{d^2\sigma}{d\Omega' dp'} = \frac{2\alpha^2}{q^4}\left(\frac{|\underset{\sim}{p}'|^2}{\epsilon'|\underset{\sim}{p}|}\right)\frac{1}{m}\Big[2\,\epsilon\epsilon' - |\underset{\sim}{p}||\underset{\sim}{p}'|\cos\theta - 2m_\ell^2\;W_1$$

$$+\;\epsilon\epsilon' + |\underset{\sim}{p}||\underset{\sim}{p}'|\cos\theta + m_\ell^2\;W_2\Big] \tag{3.6}$$

If we neglect the lepton mass this formula simplifies

$$\frac{d^2\sigma}{d\Omega' dE'} = \frac{4\alpha^2}{q^4}\frac{\epsilon'^2}{m}\cos^2\frac{\theta}{2}\left[W_2(q^2,q\cdot P) + 2W_1(q^2,q\cdot P)\tan^2\frac{\theta}{2}\right] \tag{3.7}$$

$$q^2 = 4\epsilon'\epsilon\sin^2\frac{\theta}{2}$$

Knowledge of $W_{\mu\nu}$ also gives us the total photoabsorption cross section

$$\sigma_\gamma = \frac{1}{2}\sum_{pol}\frac{(2\pi)^2\alpha}{|k\cdot P|}e_\mu W_{\mu\nu} e_\nu = \frac{(2\pi)^2\alpha}{|k\cdot P|}W_1(0,\,-k\cdot P) \tag{3.8}$$

where the kinematics are illustrated in Fig. 2.

We now proceed to calculate the covariant tensor $W_{\mu\nu}$ and then the cross sections. We must evaluate

$$W_{\mu\nu} = mM\frac{d^3P'}{E'}\delta^4(P-P'-q)\,\frac{1}{2}\sum_\lambda\sum_M <\pi_R JM|J_\mu(0)|q\lambda><q\lambda|J_\nu(0)|\pi_R JM>\left(\frac{EE'\Omega^2}{mM}\right) \tag{3.9}$$

We see that we can take out $\frac{d^3P'}{E'}\delta^4(P-P'-q)m^2$ as a factor if the final state is an isobar and we define

$$W_{\mu\nu} = \frac{d^3P'}{E'}\delta^4(P-P'-q)\;m^2T_{\mu\nu} \tag{3.10}$$

We can now use the general form of $W_{\mu\nu}$ or $T_{\mu\nu}$ to simplify the calculation. Since there are only two general form factors T_1 and T_2 we need only two relations to determine these quantities. We can therefore compute

$$T_{44} = \left(\frac{M}{m}\right)\left\{-\frac{1}{8\pi}\left| < \pi_R J \left\| J_o(0) \right\| q\pi J > \right|^2 \left(\frac{EE'\Omega}{mM}\right)\right\} = - \left|f_c(q^*)\right|^2 \left(\frac{M}{m}\right)^2 \quad (3.11)$$

and

$$\sum_{\lambda=1}^{3} T_{\lambda\lambda} = \left(\frac{M}{m}\right)\left\{\frac{1}{8\pi} \sum_{i} \left| < \pi_R J \left\| \underset{\sim}{J}(0) \right\| q^*\pi j > \right|^2 \left(\frac{EE'\Omega}{mM}\right)\right\}$$

$$= \left\{\left|f_+(q^*)\right|^2 + \left|f_-(q^*)\right|^2 + \frac{q_o^2}{q^{*2}} \left|f_c(q^*)\right|^2\right\} \left(\frac{M}{m}\right)^2 \quad (3.12)$$

From the general form of $T_{\mu\nu}$

$$T_{\mu\nu} = T_1(q^2)\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + T_2(q^2) \frac{1}{m^2} \; P_\mu - \frac{P \cdot q}{q^2} q_\mu \; \left(P_\mu - \frac{P \cdot q}{q^2} q_\mu\right) \quad (3.13)$$

it follows that in our special frame we can write

$$T_{\mu\mu} = 3T_1 - \frac{M^2}{m^2} \frac{q^{*2}}{q^2} T_2 = \left[\left|f_+\right|^2 + \left|f_-\right|^2 - \frac{q^2}{q^{*2}} \left|f_c\right|^2\right] \frac{M^2}{m^2} \quad (3.14)$$

and

$$\frac{1}{M^2} P'_\mu \, T_{\mu\nu} P'_\nu = - T_{44} = \left|f_c\right|^2 \left(\frac{M^2}{m^2}\right) = \frac{q^{*2}}{q^2} \left\{- T_1 + \frac{M^2}{m^2} \frac{q^{*2}}{q^2} T_2\right\} \quad (3.15)$$

These equations give one the two necessary relations and we can solve to get

$$T_1 = \frac{1}{2}\left(|f_+|^2 + |f_-|^2\right) \frac{M^2}{m^2} \tag{3.16}$$

$$T_2 = \frac{1}{2}\left(\frac{2q^4}{q^{*4}} |f_c|^2 + \frac{q^2}{q^{*2}} \left(|f_+|^2 + |f_-|^2\right)\right) \tag{3.17}$$

Inserting our expressions for W_1 and W_2 [c.f. Eqs. (3.5), (3.10) and (3.13)] in the formula for the cross section, Eq. (3.7), we find

$$\frac{d\sigma}{d\Omega}\bigg|_{lab} = \left(\frac{\alpha^2}{4\epsilon^2 \sin^4\frac{\theta}{2}}\right)\left(\cos^2\frac{\theta}{2}\right)\left(\frac{1}{1 + \frac{2\epsilon \sin^2\frac{\theta}{2}}{m}}\right)$$
$$\times\left[\frac{q^4}{q^{*4}} |f_c|^2 + \left(\frac{q^2}{2q^{*2}} + \frac{M^2}{m^2}\tan^2\frac{\theta}{2}\right)\left(|f_+|^2 + |f_-|^2\right)\right] \tag{3.18}$$

This formula is our main result.[5] We see that one can only measure the combination $\left(|f_+|^2 + |f_-|^2\right)$ in experiments where only the final electron is detected. The Coulomb and transverse form factors can be separated experimentally by doing experiments at fixed q^2 and varying θ or by looking at $\theta = 180^\circ$ where only the transverse form factors contribute. Taking the limit $m \to \infty$, $M/m \to 1$ reduces the above formulae to that usually used in analyzing electron excitation of nuclei. From Eq. (3.8) we have for the photoabsorption cross section integrated over the isobar resonance

$$\int_{\text{over resonance}} \sigma_\gamma(\omega)d\omega = \frac{4\pi^2\alpha}{M^2 - m^2} \frac{M^2}{m} \left(\left|f_+\right|^2 + \left|f_-\right|^2\right)_{q^2=0} \tag{3.19}$$

This expression gives us the transverse form factors evaluated at one momentum transfer, namely $q^2 = 0$.

This last relation can be used to give an approximate formula for the inelastic electron scattering cross section at small q^2, the Weizäcker-Williams approximation. Keeping terms of order q^2 in the last bracket in Eq. (3.18) we find

$$\left.\frac{d\sigma}{d\Omega}\right|_{\substack{\text{lab}\\ q^2\to 0}} = \frac{\alpha}{2\pi^2 \sin^2\frac{\theta}{2}} \left(\frac{\epsilon'}{\epsilon}\right)\left[1 + \frac{(M^2 - m^2)^2}{8m^2\epsilon\epsilon'}\right] \frac{m}{M^2 - m^2} \int_{\text{over resonance}} \sigma_\gamma(\omega)d\omega \tag{3.20}$$

Wave Function Description

We may also compute W_1 and W_2 in terms of the form factors g_1, g_2 and g_3 and obtain a connection between them and the magnitudes of f_c and $f_\pm$. We return to Eqs. (3.9) and (3.10) for the expression for $T_{\mu\nu}$:

$$\epsilon'_\nu \epsilon_\mu \quad T_{\mu\nu} = \frac{\Omega^2 EE'}{2m^2} \sum_{\lambda\lambda'} < P\lambda \left|J_\nu(0)\right| P'\lambda' > < P'\lambda' \left|J_\mu(0)\right| P\lambda > \tag{3.21}$$

We choose to evaluate $T_{\mu\nu}$ in the isobar rest frame to facilitate doing the spin sums. We first construct explicitly the state $\chi_{\alpha_1 \ldots \alpha_{J-\frac{1}{2}}}(\lambda')$ for the case of the spin aligned along some axis s. Were the z-axis to lie along s, we would clearly have

$$\chi_{\alpha_1 \ldots \alpha_{J-\frac{1}{2}}} = \hat{\omega}_{\alpha_1} \ldots \hat{\omega}_{\alpha_{J-\frac{1}{2}}} \chi(\lambda = \tfrac{1}{2}) \tag{3.22}$$

where

$$\hat{\omega}_\alpha = - \frac{1}{\sqrt{2}} (1, + i, 0) \tag{3.23}$$

and $\chi = \binom{1}{0}$. The sum over λ' may be replaced by

$$\sum_{\lambda'} \rightarrow (2J+1) \int \frac{d\Omega_s}{4\pi}$$

i.e., an average over orientations of $\underset{\sim}{s}$. Upon inserting Eqs. (3.22), (3.23), and (2.21) into (3.21) and summing out the spins of initial proton, we find, for normal parity transitions

$$\epsilon_\mu T_{\mu\nu} \epsilon'_\nu = \frac{M^5 q^{*\,2J+1}(E+m)}{4m^2} (2J+1) \int \frac{d\Omega_s}{4\pi} \mathrm{Tr} \left(\frac{1+\underset{\sim}{\sigma}\cdot \underset{\sim}{s}}{2}\right) \left[(\hat{\underset{\sim}{q}}\cdot\hat{\underset{\sim}{\omega}}^*)(\hat{\underset{\sim}{q}}\cdot\hat{\underset{\sim}{\omega}})\right]^{J-3/2}$$

$$\times \left[-(\hat{\underset{\sim}{q}}\cdot\hat{\underset{\sim}{\omega}}^*) \left[\frac{\underset{\sim}{\epsilon}\cdot\hat{\underset{\sim}{q}}\; q_0 - \epsilon_0 q^*}{M}\right] g_1 + (\hat{\underset{\sim}{\omega}}^* \times \hat{\underset{\sim}{q}}) \cdot (\hat{\underset{\sim}{q}} \times \underset{\sim}{\epsilon})\; 2g_2 + i\; (\hat{\underset{\sim}{\omega}}^*\cdot\hat{\underset{\sim}{q}})\underset{\sim}{\sigma}\cdot\underset{\sim}{\epsilon} \times \hat{\underset{\sim}{q}}(g_2+g_3)\right]$$

$$\times \left[-\hat{\underset{\sim}{q}}\cdot\hat{\underset{\sim}{\omega}} \left[\frac{\underset{\sim}{\epsilon}'\cdot\hat{\underset{\sim}{q}}\; q_1 - \epsilon'_0 q^*}{M}\right] g_1^* + (\hat{\underset{\sim}{\omega}} \times \hat{\underset{\sim}{q}}) \cdot (\hat{\underset{\sim}{q}} \times \underset{\sim}{\epsilon}')\; 2g_2^* - i\; (\hat{\underset{\sim}{\omega}}\cdot\hat{\underset{\sim}{q}})\underset{\sim}{\sigma}\cdot\underset{\sim}{\epsilon}' \times \hat{\underset{\sim}{q}}(g_2^*+g_3^*)\right]$$

(3.24)

The integrations and traces are now straightforward; cross terms between different g_i cancel out, and after some calculation we find[6]

$$\epsilon_\mu T_{\mu\nu}\epsilon'_\nu = \frac{M^5(E+m)(q^*)^{2J+1}\rho^2(2J+1)!!}{4m^2\, 2^{J-\frac{1}{2}}\,(2J)!!}\left[\frac{(\underline{\epsilon}\cdot\hat{\underline{q}}q_o - \epsilon_o q^*)(\underline{\epsilon}'\cdot\hat{\underline{q}} - \epsilon'_o q^*)}{M^2}\;|g_1|^2\right.$$

$$\left. + (\underline{\epsilon}\times\hat{\underline{q}})\cdot(\underline{\epsilon}'\times\hat{\underline{q}})\left\{\frac{(2J+3)}{(2J-1)}\;|g_2|^2 + |g_3|^2\right\}\right] \qquad (3.25)$$

We insert the factor ρ^2 to take into account abnormal parity transitions, in which case the expressions Eqs. (3.24) and (3.25) contain an extra factor due to the γ_5:

$$\rho = \begin{cases} 1 & \frac{1}{2}^+ \to \frac{3}{2}^-,\ \frac{5}{2}^+,\ \ldots \\ \frac{q^*}{E+m} & \frac{1}{2}^+ \to \frac{3}{2}^+,\ \frac{5}{2}^-,\ \ldots \end{cases} \qquad (3.26)$$

To liberate Eq. (3.25) from the isobar rest frame, we compare it with Eq. (3.13) and identify

$$T_1 = \frac{M^5(E+m)(q^*)^{2J+1}\rho^2}{m^2\,2^{J+3/2}}\;\frac{(2J+1)!!}{2J!!}\;\left\{\frac{2J+3}{2J-1}\;|g_2|^2 + |g_3|^2\right\} \qquad (3.27)$$

$$T_2 = \frac{m^2 q^2}{M^2 q^{*2}}\,T_1 + \frac{M(E+m)(q^*)^{2J-1}\;q^4\rho^2}{2^{J+3/2}}\;\frac{(2J+1)!!}{2J!!}\;|g_1|^2 \qquad (3.28)$$

In terms of the f's, we thus find

$$\frac{1}{2}\left[\left|f_+\right|^2 + \left|f_-\right|^2\right] = \frac{M^3(E+m)\ (q^*)^{2J+1}\rho^2}{2^{J+3/2}}\ \frac{(2J+1)!!}{2J!!}\left\{\frac{(2J+3)}{(2J-1)}\left|g_2\right|^2 + \left|g_3\right|^2\right\} \tag{3.29}$$

$$\left|f_c\right|^2 = \frac{M(E+m)\ (q^*)^{2J+3}\rho^2}{2^{J+3/2}}\ \frac{(2J+1)!!}{(2J)!!}\ \left|g_1\right|^2 \tag{3.30}$$

We again caution the reader not to read off the threshold (low q^*) behavior of the cross section from these formulae; detailed discussion of this point is in Section IV.

IV. THRESHOLD BEHAVIOR

From our general discussion of the nucleon-isobar electromagnetic vertex we can extract the dependence on q^* as $q^* \to 0$ and $q_o \to M-m$. This dependence is very useful in nuclear physics as it allows one to identify the multipolarity of the transitions involved. It also allows one to predict which transitions will become important as the momentum transfer to the nucleon is increased. In the case of the nucleon however, the applicability to physical situations is not so straightforward, as discussed in the introduction.

Spinor Wave Function Method

In the limit $q^* \to 0$, we search for those spinor covariants, linear combinations of the ones appearing in Eq. (2.18), which contain the fewest possible powers of q^*. Considering first the case of normal parity, we find that the amplitude

$$< P'\lambda' \left| J_\mu \epsilon_\mu \right| P\lambda >_{\text{threshold}} = \frac{1}{\Omega}\left(\frac{mM}{EE'}\right)^{\frac{1}{2}} \bar{\Psi}(P',\lambda')^{\mu_1 \cdots \mu_{J-\frac{1}{2}}} q_{\mu_2} \cdots q_{\mu_{J-\frac{1}{2}}}$$
$$\times\, g_o M^2(\epsilon_{\mu_1}\, q\cdot P - q_{\mu_1}\, \epsilon\cdot P)u(P,\lambda) \qquad \left(\frac{1}{2}^+ \to \frac{3}{2}^-, \ \ldots\right) \tag{4.1}$$

is the only (gauge invariant) vertex which behaves as $q^{*J-3/2}$ as $q^* \to 0$. Thus, provided $g_o \neq 0$ for $q^* = 0$, this amplitude dominates all others for sufficiently small q^*. Upon reducing this expression to the isobar rest frame and then re-expressing the amplitude in terms of form factors g_1, g_2, g_3 we find

$$g_1 = - g_o \left(\frac{M}{q^*}\right)^2 \tag{4.2}$$

$$g_2 = - g_3 = + \frac{g_o M q_o}{2(q^*)^2} \tag{4.3}$$

and in terms of f_c and $f_\pm$ we find the threshold behavior

$$|f_c|^2 \sim (q^*)^{2J-1} \tag{4.4}$$

$$|f_+|^2 + |f_-|^2 \sim (q^*)^{2J-3} \qquad \left(\frac{1}{2}^+ \to \frac{3}{2}^-, \frac{5}{2}^+ \ldots\right) \tag{4.5}$$

as well as the important relation

$$\frac{|f_+|^2 + |f_-|^2}{|f_c|^2} \to \left(\frac{2J+1}{2J-1}\right)\left(\frac{q_o}{q^*}\right)^2 \tag{4.6}$$

which follows from Eqs. (4.2), (4.3), (3.29), and (3.30).

If this threshold behavior persists in the physical region of space-like momentum transfers (i.e., this amplitude is still the dominant one), we may relate forward and backward scattering at the same momentum transfer q^2, and test the spin-parity assignment of the resonance as well as the hypothesis that the threshold behavior dominates.

Returning to Eq. (1.1),

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{"threshold"}} \cong \frac{\alpha^2 \cos^2\frac{\theta}{2}\, |f_c|^2}{4\epsilon^2 \sin^4\frac{\theta}{2}\left[1 + \frac{2\epsilon}{m}\sin^2\frac{\theta}{2}\right]} \left(\frac{q_o}{q^*}\right)^2$$
$$\times\left[\frac{q^2}{q_o^2} - \frac{(2J-3)}{2(2J-1)}\frac{q^2}{2q^{*2}} + \frac{M^2}{m^2}\left(\frac{2J+1}{2J-1}\right)\tan^2\frac{\theta}{2}\right] \qquad \left(\frac{1}{2}^+ \to \frac{3}{2}^-, \frac{5}{2}^+ \ldots\right) \tag{4.7}$$

The assumption going into this result is that the "threshold amplitude" Eq. (4.1) is dominant for physically accessible q^2. Notice that any q^2 dependence in $g_o(q^2)$ cancels out in the ratio in Eq. (4.6).

For abnormal parity transitions, it is not possible to find a threshold amplitude such as Eq. (4.1) which behaves as $(q^*)^{J-3/2}$. This comes about because the γ_5, coupling large and small components of the spinors, gives a factor $q^*\underset{\sim}{\sigma}\cdot\underset{\sim}{\hat{q}}$. There are, however, two independent amplitudes which have the threshold behavior $(q^*)^{J-1/2}$. The first is Eq. (4.1) with $u(P,\lambda) \to \gamma_5 u(P,\lambda)$. A second may be taken to be

$$< J_\mu \epsilon_\mu > \sim \overline{\Psi}(P',\lambda')^{\mu_1 \cdots \mu_{J-1/2}} q_{\mu_2} \cdots q_{\mu_{J-1/2}} \left[\epsilon_{\mu_1} \gamma\cdot q - q_{\mu_1} \gamma\cdot\epsilon\right] \gamma_5 u(P,\lambda) \tag{4.8}$$

This means, in particular, that there will be no constraint relating f_c to $f_\pm$ near threshold such as Eq. (4.6). The threshold behaviors we do obtain for this case are evidently

$$\left|f_c\right|^2 \sim (q^*)^{2J+1} \tag{4.9}$$

$$\left|f_\pm\right|^2 \sim (q^*)^{2J-1} \quad \left(\tfrac{1}{2}^+ \to \tfrac{3}{2}^+, \tfrac{5}{2}^- \cdots\right) \tag{4.10}$$

For the case of $\frac{1}{2}^+ \to \frac{1}{2}^\pm$ transitions, only two form factors occur (g_2 must vanish) and similar considerations as before lead to threshold behaviors

$$\left.\begin{aligned} \left|f_c\right|^2 &\sim (q^*)^4 \\ \left|f_\pm\right|^2 &\sim (q^*)^2 \end{aligned}\right\} \qquad \tfrac{1}{2}^+ \to \tfrac{1}{2}^+ \tag{4.11}$$

$$\left.\begin{aligned} \left|f_c\right|^2 &\sim (q^*)^2 \\ \left|f_\pm\right|^2 &\sim 1 \end{aligned}\right\} \qquad \tfrac{1}{2}^+ \to \tfrac{1}{2}^- \tag{4.12}$$

Helicity Representation:

The entire q^* dependence of the reduced matrix elements is contained in the nucleon state vectors $|q^*\lambda>$. One can give an explicit construction of this state by using the fact that the state vectors form a basis for an infinite dimensional unitary representation of the Poincaré group. That is, there is a unitary operator $e^{\Omega\hat{K}_z}$ which "boosts" us from the rest frame of the particle to a momentum q^* in the z direction. We write

$$|q^*\lambda> = e^{\Omega\hat{K}_z}|0\lambda> \tag{4.13}$$

where

$$\Omega = \tanh^{-1} \frac{q^*}{(q^{*2} + m^2)^{\frac{1}{2}}} \xrightarrow[q^* \to 0]{} \frac{q^*}{m} \tag{4.14}$$

The operator $\hat{K}_z$ which generates our Lorentz transformation in the z direction ($\hat{K}_z = \hat{M}_{34}$ where $M_{\mu\nu}$ is the covariant angular momentum tensor), is a polar vector operator under spatial rotations and reflections. If we now go back to the fundamental theorem

$$|q^*jm\lambda> = \left(\frac{2j+1}{4\pi}\right)^{\frac{1}{2}} \int d\Omega_q \mathcal{D}^{j}_{m\lambda} (-\varphi_q - \theta_q\varphi_q)^* \hat{R}_{-\varphi_q-\theta_q\varphi_q} e^{\Omega\hat{K}_z} |0\lambda> \tag{4.15}$$

where

$$\hat{R}_{\alpha\beta\gamma} = e^{i\hat{J}_z\alpha} e^{i\hat{J}_y\beta} e^{i\hat{J}_z\gamma} \tag{4.16}$$

is the finite rotation operator and think of letting $q^* \to 0$ we see that we must let $\hat{K}_z$ act enough times in the expansion of the exponential so that we can get a basis for the j^{th} representation of the rotation group. If it doesn't appear enough times, the integration over $\mathcal{D}^{j*}_{m\lambda}$ will give zero. Furthermore, since $\hat{K}_z$ is a polar vector it must act enough times to give us a state of the correct parity, which is what we eventually want. Since each time a $\hat{K}_z$ acts, it carries with it a power of $\left(\frac{q^*}{M}\right)$ we can read off the q^* dependence in the various cases. We find (1)

a) Normal Parity Transitions: $1/2^+ \to 3/2^-, 5/2^+, 7/2^- \ldots$ etc.

$$f_c \sim (q^*)^{J-1/2}$$
$$f_\pm \sim (q^*)^{J-3/2} \tag{4.17}$$

b) Abnormal Parity Transitions: $1/2^+ \to 1/2^-, 3/2^+, 5/2^-, 7/2^+ \ldots$ etc.

$$f_c \sim (q^*)^{J+1/2}$$
$$f_\pm \sim (q^*)^{J-1/2} \tag{4.18}$$

A little care is necessary in the case of $1/2^+ \to 1/2^\pi$ transitions for if $J = 1/2$, then

$$f_+ \equiv 0 \qquad J = 1/2 \tag{4.19}$$

and there are only two form factors. Furthermore in the case of $1/2^+ \to 1/2^+$ transitions, the $q^* \to 0$ limit of the electric monopole operator which is just 1, the total charge, cannot cause transitions;

therefore the threshold behavior in this case

$$\begin{aligned} f_c &\sim q^{*2} \\ f_- &\sim q^* \end{aligned} \qquad \frac{1}{2}^+ \rightarrow \frac{1}{2}^+ \tag{4.20}$$

One might be tempted to conclude from these arguments that the expansion parameter for the threshold behavior is $\frac{q^*}{m}$. However, this argument is invalid since, at least in nuclear physics, it is known that the relevant quantity is the size of the target and not the reciprocal of its mass.

In the case of normal parity transitions the relation (4.6) between the transverse and coulomb form factors follows from the continuity equation, for if we write

$$< \pi_R J \left\| J_o(0) \right\| q^* \pi J > = a_c q^{*J-\frac{1}{2}} \qquad J > \frac{1}{2} \tag{4.21}$$

$$< \pi_R J \left\| \underline{J}(0) \right\| q^* \pi J - 1 > = a_t q^{*J-\frac{3}{2}} \qquad q^* \rightarrow 0 \tag{4.22}$$

then the continuity equation (2.10) gives

$$a_t = q_o a_c \left(\frac{2J+1}{2J-1}\right)^{\frac{1}{2}} \frac{1}{\left(J-1\ \frac{1}{2}\ 1\ 0 \middle| J-1\ 1\ J\ \frac{1}{2}\right)} \tag{4.23}$$

and one finds from Eqs. (2.11) and (2.12)

$$\frac{|f_\pm|^2}{|f_c|^2} = \left(\frac{q_o}{q^*}\right)^2 \left| \frac{\left(J-1\ \frac{1}{2}\ 1\ \pm 1 \middle| J-1\ 1\ J\ \frac{1}{2} \pm 1\right)}{\left(J-1\ \frac{1}{2}\ 1\ 0 \middle| J-1\ 1\ J\ \frac{1}{2}\right)} \right|^2 \tag{4.24}$$

or

$$\frac{|f_\pm|^2 + |f_-|^2}{|f_c|^2} \xrightarrow[q^* \rightarrow 0]{} \left(\frac{q_o}{q^*}\right)^2 \left(\frac{J+\frac{1}{2}}{J-\frac{1}{2}}\right) \qquad \text{Normal parity transitions } J > \frac{1}{2} \tag{4.25}$$

In the abnormal parity case $< \pi_R J \,\|\, \underset{\sim}{J}(0) \,\|\, q^* \,\pi\, J-1 >$ and $< \pi_R J \,\|\, \underset{\sim}{J}(0) \,\|\, q^* \,\pi\, J >$ both go as $q^{*J-1/2}$ and the continuity equation therefore does not determine $|f_+|^2 + |f_-|^2$ in terms of $|f_c|^2$.

V. ANALYTIC PROPERTIES

The analytic properties of the form factors $g_i(q^2)$ are more complex than the form factors $F_1(q^2)$ and $F_2(q^2)$ of elastic electron-proton scattering. Not only is there the cut for timelike $q^2 < -4\mu^2 (\mu \equiv m_\pi)$ but there are complex singularities which appear for isobar masses large enough to cause instability. The form factors then become complex. The complete analytic properties of an arbitrary vertex graph with an unstable particle on one leg are unknown; however, it is known that there are no singularities in the upper half q^2 plane. In addition the triangle diagram has been extensively studied; we here review the situation for the kinematics appropriate to this problem. We consider the diagram shown in Fig. 3 as a function of q^2 for real M^2. The Landau singularities (14) are the normal thresholds $q^2 = -4\mu^2$, $M^2 = (m+\mu)^2$, and the "anomalous" threshold given by

$$\det k_i \cdot k_j = \left\| \begin{matrix} -\mu^2 & -\mu^2 - \frac{q^2}{2} & +\frac{\mu^2}{2} \\ -\mu^2 - \frac{q^2}{2} & -\mu^2 & \frac{m^2+\mu^2-M^2}{2} \\ +\frac{\mu^2}{2} & \frac{m^2+\mu^2-M^2}{2} & -m^2 \end{matrix} \right\| = 0 \tag{5.1}$$

The solution of this equation is

$$q^2 = -2\mu^2 - \frac{\mu^2}{2m^2}(M^2-m^2-\mu^2) \pm \frac{\mu}{m}\sqrt{\left(1-\frac{\mu^2}{4m^2}\right)[M^2-(m-\mu)^2][(m+\mu)^2-M^2]}$$

$$\cong -2\mu^2\left\{1+\frac{(M-m)}{2m} \pm \sqrt{1-\frac{(M-m)^2}{\mu^2}}\right\} \qquad \left(\frac{\mu^2}{M^2} \ll 1\right) \tag{5.2}$$

On general grounds, from the Nambu representation (14), we know that for $M^2 = m^2$ there is no anomalous threshold, and the singularities consist only of a cut in the q^2 plane from $-4\mu^2$ to $-\infty$. As M^2 is increased the anomalous singularity emerges into the physical sheet (we take the minus sign in Eq. 5.2) and moves down to $q^2 \approx -2\mu^2$ as $M \to m + \mu$. For $M > m + \mu$ the singularity moves into the lower half q^2 plane as shown in Fig. 4, and again this is dictated by the Nambu representation. If we give the particle a width, $M^2 \to (M-i\Gamma)^2$, the singularity moves to the right.

We conclude from this behavior that the form factors will, for kinematical reasons, be complex and that they possess complex singularities. It is, of course, an open question of how important these are. For the 33 resonance, this diagram (the "pion current" term) is relatively unimportant. However, for higher resonances such effects may have to be considered.

The Final State Theorem and Complex Singularities:

We have seen in the previous discussion that the triangle diagrams develop complex singularities when the mass of the isobar is such that it is unstable against pion decay. We know, however, from the general properties of unitarity and invariance under time reversal that partial wave transition amplitudes for a weak process (in this case electromagnetic) leading to a strongly interacting pair of particles in the final state has a phase equal to the scattering phase shift of those two particles at the appropriate energy. Since the complex singularities are intimately related to the phase of the production amplitude in the physical region it is interesting to see how this connection can come about. To see this connection we make a very simple model of the scattering and production process which is unitary, invariant under time reversal, and contains a triangle diagram. We consider scalar photons, pions, and nucleons since the spin complexities are not really relevant to the points under discussion. We consider the pion production mechanism to be that of the photon interacting with the pion current and ejecting a pion as illustrated in Fig. 5. This is often referred to as the "retardation term". The pion can then rescatter off the nucleon and we take a point two-pion-two-nucleon coupling of strength λ to describe this. We then consider s-wave electropion production. Now the s-wave pion-nucleon scattering amplitude in this model is given by the sum of graphs illustrated in Fig. 6. Let us define the basic bubble

$$B(W^2) = -i \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2+\mu^2-i\eta} \frac{1}{(P-q-\ell)^2+M^2-i\eta} \tag{5.3}$$

where $(P-q)^2 = -W^2$, the square of the total energy in the center-of-momentum system. Then we know from the analytic properties of Feynman diagrams that $B(W^2)$ has a spectral representation

$$B(W^2) = \frac{1}{\pi} \int_{(M+\mu)^2}^{\infty} \frac{\rho(\sigma^2)\, d\sigma^2}{\sigma^2 - W^2 - i\eta} \tag{5.4}$$

We also know from unitarity that

$$\text{Im } B(W^2) = \rho(\sigma^2) = +\pi^2\, 2 \int \frac{d^4\ell}{(2\pi)^4} \delta(\ell^2+\mu^2)\, \theta(\ell_o)\delta\left((p-q-\ell)^2+M^2\right)\theta(p_o-q_o-\ell_o) \tag{5.5}$$

which corresponds to putting the intermediate particles on the mass shell. The scattering amplitude $T(W^2)$ in this model is now given by

$$T(W^2) = \lambda + \lambda^2 B(W^2) + \lambda^3 B^2(W^2) + \dots = \frac{\lambda}{1-\lambda B(W^2)}$$

$$= \frac{\lambda}{1 - \frac{\lambda}{\pi} \int_{(M+\mu)^2}^{\infty} \frac{\rho(\sigma^2)\, d\sigma^2}{\sigma^2 - W^2 - i\eta}} \tag{5.6}$$

and since

$$\mathrm{Im}\, \frac{1}{T(W^2)} = -\rho(W^2) \tag{5.7}$$

we can write

$$T(W^2) = \frac{1}{\rho(W^2)}\, e^{i\delta} \sin\delta \tag{5.8}$$

Let us now look at the electroproduction amplitude. If we first study the Born term $F_o(q^2,W^2)$ we have

$$F_o(q^2,W^2) = \int \frac{d\Omega_k}{4\pi} \frac{1}{t^2 + \mu^2} \tag{5.9}$$

where $t = P - Q = q + k$ and $(P-q)^2 = (Q+k)^2 = -W^2$. This function is real for space-like q^2 ($q^2 > 0$) and has the analytic properties in q^2 indicated in Fig. 7 for $W \cong M + \mu$.

If the isobar is just unstable, $W > M + \mu$, there is a complex branch cut running between $\pm 2i\sqrt{2}\, \frac{W - (M+\mu)}{\mu}$. As W goes below $M+\mu$, the branch cut moves onto the real axis as indicated. We shall also need the properties of the triangle diagram which we define as

$$\Delta(g^2,W^2) = -i \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell+q)^2 + \mu^2 - i\eta} \frac{1}{(p-q-\ell)^2 + M^2 - i\eta} \tag{5.10}$$

This diagram has already been studied and has the analytic properties indicated in Fig. 4. $\Delta(q^2,W^2)$ is complex for spacelike q^2. We can calculate Im $\Delta(q^2,W^2)$ by again noting that for $q^2 > 0$, the only possible real intermediate state is where the final pion and nucleon are on the mass shell

$$\mathrm{Im}\Delta(q^2,W^2)_{q^2>0} = +2\pi^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell+q)^2+\mu^2} \delta(\ell^2+\mu^2)\theta(\ell_o)\delta\Big((p-q-\ell)^2+M^2\Big)\theta(p_o-q_o-\ell_o). \tag{5.11}$$

Now since $\underline{P}-\underline{q} = 0$ in the center-of-momentum system we can write this as

$$\begin{aligned}\mathrm{Im}\Delta(q^2,W^2)_{q^2>0} &= +\pi^2 F_o(q^2,W^2) \int \frac{d^4\ell}{(2\pi)^4} \frac{\delta(\ell^2+\mu^2)\delta\Big((p-q-\ell)^2+M^2\Big)}{(\ell+q)^2+\mu^2} \theta(\ell_o)\theta(p_o-q_o-\ell_o) \\ &= F_o(q^2,W^2)\, \rho(W^2)\end{aligned} \tag{5.12}$$

We can now write the electroproduction amplitude in this model

$$\begin{aligned}F(q^2,W^2) &= F_o(q^2,W^2) + \lambda\Delta(q^2,W^2)\Big[1 + \lambda B(W^2) + \lambda^2 B^2(W^2) + \ldots\Big] \\ &= F_o(q^2,W^2) + \frac{\lambda\Delta(q^2,W^2)}{1-\lambda B(W^2)}\end{aligned} \tag{5.13}$$

or

$$F(q^2,W^2) = \frac{F_o(q^2,W^2)\left[1 - \frac{\lambda}{\pi}\int\limits_{(M+\mu)^2}^{\infty} \frac{\rho(\sigma^2)d\sigma^2}{\sigma^2-W^2-i\eta}\right] + \lambda\Delta(q^2,W^2)}{1 - \frac{\lambda}{\pi}\int\limits_{(M+\mu)^2}^{\infty} \frac{\rho(\sigma^2)d\sigma^2}{\sigma^2-W^2-i\eta}} \tag{5.14}$$

We now note the important property that for $q^2 > 0$, the imaginary part of the numerator vanishes for we find

$$\operatorname{Im} T(W^2)^{-1}F(q^2,W^2)_{q^2>0} = F_o(q^2,W^2)\,\rho(W^2) - \operatorname{Im}\Delta(q^2,W^2) = 0 \tag{5.15}$$

Therefore we conclude that for $q^2 > 0$

$$F(q^2,W^2)_{q^2>0} = \left|F(q^2,W^2)\right| e^{i\delta(W^2)} \tag{5.16}$$

which is just the final state theorem. We also see that as far as the analytic properties in q^2 are concerned, $F(q^2,W^2)$ has the complex singularities of both $F_o(q^2,W^2)$ and $\Delta(q^2,W^2)$.

VI. APPLICATIONS

As an example of an application of the preceding formalism, we discuss briefly electron excitation of the first excited state of the nucleon, the $T = 3/2$, $J = 3/2$ resonance. We first consider the model of Fubini, Nambu, and Wataghin,[6] in which the photon is absorbed on a nucleon via an isovector magnetic dipole interaction. The term so obtained has the same structure $\tau_3 \underset{\sim}{\sigma} \cdot \underset{\sim}{\epsilon} \times \underset{\sim}{q} G_M^{(v)}(q^2)$ as the Born amplitude for absorption of a neutral pion of momentum $\underset{\sim}{\epsilon} \times \underset{\sim}{q}$. The full photoproduction amplitude of the 33 is then proportional to the amplitude for $\pi^0 p \to N^*$, the proportionality factor being the ratio of Born terms. Because this ratio includes the factor $G_M^{(v)}(q^2)$, we find that the inelastic form factor becomes proportional to $G_M^{(v)}(q^2)$. This is a model-dependent result and may not necessarily be the case for higher resonances, where the production mechanism is more peripheral and momentum dependent.

To make a more detailed comparison, we need the Rarita-Schwinger wave function for a pion-nucleon system with $J = 3/2$. This is found to be (in the static limit)

$$\underset{\sim}{\Psi} = \sqrt{\frac{3}{2}} \left\{ \hat{\underset{\sim}{\ell}} - \frac{1}{3} \underset{\sim}{\sigma}(\underset{\sim}{\sigma} \cdot \hat{\underset{\sim}{\ell}}) \right\} \chi \tag{6.1}$$

where $\underset{\sim}{\ell}$ is the momentum of the final pion, and χ is the proton spinor. In this notation, we find from Eq. (15) in the paper of Fubini, Nambu, and Wataghin

$$< p'\lambda' \left| J_\mu e_\mu \right| p\lambda > \propto e^{i\delta_{33}} \sin \delta_{33} \, \underset{\sim}{\Psi}^\dagger \cdot (\underset{\sim}{q} \times \underset{\sim}{\epsilon}) \chi \, G_M^{(v)}(q^2) \tag{6.2}$$

To establish contact with the covariants multiplying our $g_1(q^2)$, we use the following identity $(\underset{\sim}{B} = \underset{\sim}{\epsilon} \times \underset{\sim}{\hat{q}})$

$$\underset{\sim}{\psi}^\dagger \cdot \underset{\sim}{B} \chi = \frac{1}{2} \underset{\sim}{\psi}^\dagger (\underset{\sim}{\sigma} \cdot \underset{\sim}{B}) \cdot \underset{\sim}{\sigma} \chi = \underset{\sim}{\psi}^\dagger (\underset{\sim}{\sigma} \cdot \underset{\sim}{B}) \cdot \left[\underset{\sim}{\hat{q}} (\underset{\sim}{\sigma} \cdot \underset{\sim}{\hat{q}}) - \frac{1}{2} (\underset{\sim}{\sigma} \cdot \underset{\sim}{\hat{q}}) \underset{\sim}{\sigma} (\underset{\sim}{\sigma} \cdot \underset{\sim}{\hat{q}}) \right] \chi$$
$$= - i \underset{\sim}{\psi}^\dagger \cdot [\underset{\sim}{B} \times \underset{\sim}{q} + i \underset{\sim}{\hat{q}} \, \underset{\sim}{\sigma} \cdot \underset{\sim}{B}] (\underset{\sim}{\sigma} \cdot \underset{\sim}{q}) \chi \tag{6.3}$$

Comparing with Eq. (2.21), we find that

$$g_1 = 0 \quad q^{*2} g_2 = q^{*2} g_3 \propto G_V(q^2) \tag{6.4}$$

For the normalization of g_2 and g_3, we may use the covariant treatment given by Gourdin and Salin.[7] They treat the isobar as a discrete state, but use a different set of invariants. They write

$$\left(\frac{EE'\Omega^2}{MM}\right)^{\frac{1}{2}} < P - q, \lambda' \left| J_\rho(0) \right| P\lambda > = \bar{\Psi}_\nu (P-q, \lambda') \gamma_5 \left\{ \frac{iC_3}{\mu} (q_\nu \gamma_\rho - q \cdot \gamma \delta_{\nu\rho}) \right.$$
$$\left. - \frac{C_4}{\mu^2} \left(P'_\rho q_\nu - (P' \cdot q) \delta_{\nu\rho} \right) - \frac{C_5}{\mu^2} \left(P_\rho q_\nu - (P \cdot q) \, \delta_{\nu\rho} \right) \right\} u(P, \lambda) \tag{6.5}$$

where $P' = P - q$, and $\mu \equiv m_\pi$. (For real photon processes the last two invariants are identical since $q^2 \to 0$ and $\epsilon_\mu q_\mu = 0$ where ϵ_μ is the photon polarization.) From an analysis of photoproduction they give the form factors at $q^2 = 0$ as

$$C_3(0) = 0.37$$
$$C_4(0) = C_5(0) = -0.0043 \tag{6.6}$$

The relation between these two sets of invariants is merely a matter of algebra and judicious use of the Dirac equation and the subsidiary condition. The resulting relations are

$$(2M^2q^{*2})g_1 = \frac{2M}{\mu} C_3 + \frac{C_4}{\mu^2}\left[(M^2-m^2) - q^2\right] + \frac{1}{\mu^2}(C_4+C_5)\left[(M^2+m^2) + q^2\right]$$

$$(2M^2q^{*2})g_2 = -\left\{\frac{M+m}{\mu} C_3 - \frac{q^2}{\mu^2} C_4 + \frac{1}{2\mu^2}(C_4+C_5)\left[(M^2-m^2) + q^2\right]\right\} \qquad (6.7)$$

$$(2M^2q^{*2})\,(g_3-g_2) = \frac{C_3}{\mu}\left(\frac{M^2-m^2-q^2}{M}\right) - \frac{2q^2}{\mu^2} C_4 + \frac{1}{\mu^2}(C_4+C_5)\left[(M^2-m^2) + q^2\right] .$$

Some kinematic relations are useful in dealing with these quantities. They are:

$$q^{*2} = \left[\frac{(M+m)^2 + q^2}{2M}\right]\left[\frac{(M-m)^2 + q^2}{2M}\right] \qquad (6.8)$$

$$E + m = \frac{(M+m)^2 + q^2}{2M} \qquad q_0 = \frac{-1}{2M}\left[M^2-m^2-q^2\right] .$$

We should emphasize at this point that the easiest way of going from a relativistic form of the vertex to a cross section is not to square and introduce a projection operator for the spin J particle but to simply use the expression above to get the helicity amplitudes $f_\pm, f_c$ (or equivalently $g_{1,2,3}$) and the cross section has already been given in terms of these quantities. This saves a tremendous amount of labor.

It is also useful to have an explicit relation between the helicity amplitudes which characterize the vertex and the g's. To get this we go to the rest frame of the isobar. To compare with the helicity representation we first construct nuclear wave functions of definite helicity. These are indicated in Table I.

Table I.

The Components of the Dirac Helicity Wave Functions

$\sqrt{2}u(P,\uparrow)$	$\sqrt{2}u(P,\downarrow)$
$a(r + s)$	$-b^*(r + s)$
$b(r + s)$	$a^*(r + s)$
$a(r - s)$	$b^*(r - s)$
$b(r - s)$	$-a^*(r - s)$

$$a = \cos\frac{\theta}{2}\, e^{-\frac{i\varphi}{2}}$$

$$b = \sin\frac{\theta}{2}\, e^{+\frac{i\varphi}{2}}$$

$$r = \cos \chi/2$$

$$s = \sin \chi/2$$

$$\text{ctn}\ \chi/2 = q^*/m$$

(They are still normalized to $u^\dagger u = 1$.) We can now extract the helicity amplitudes by looking at the coefficients of the appropriate $\mathcal{D}^{j}_{m\lambda}$ functions. The result is simply

$$
\begin{aligned}
f_c &= \left(\frac{2}{3}\right)^{\frac{1}{2}} \frac{q^*}{M} \left[\left(\frac{M}{2(E+m)}\right)^{\frac{1}{2}} Mq^{*3}\right] g_1 \\
f_+ &= 2\left[\left(\frac{M}{2(E+m)}\right)^{\frac{1}{2}} Mq^{*3}\right] g_2 \qquad (6.9) \\
f_- &= -\left(\frac{4}{3}\right)^{\frac{1}{2}} \left[\left(\frac{M}{2(E+m)}\right)^{\frac{1}{2}} Mq^{*3}\right] g_3 \quad .
\end{aligned}
$$

These amplitudes, when squared, give us the results we obtained previously by comparing the cross sections.

From the integrated photoabsorption cross section we can get a value for the transverse form factors at $q^2 = 0$. The relation is Eq. (3.19)

$$\int_{\text{Lab}} \sigma_\gamma(\omega) d\omega = 2.81 \times 10^{-25} \text{cm}^2 \text{MeV} \left[|f_+|^2 + |f_-|^2 \right]_{q^2=0} \qquad (6.10)$$

if we use the following numbers for the 3-3 resonance

$$
\begin{aligned}
M &= 1238 \text{ MeV} = 8.86\mu \\
m &= 938 \text{ MeV} = 6.72\mu \\
\mu &= m_{\pi+} = 139.6 \text{ MeV} \quad .
\end{aligned}
$$

From the numbers for C_3, C_4, C_5 given by Gourdin and Salin in their fit to photoproduction and using the formulae above one finds

$$\left[|f_+|^2 + |f_-|^2 \right]_{q^2=0} = 0.55 \qquad \text{(Gourdin and Salin)} \qquad (6.11)$$

This gives

$$\int_{\text{Lab}} \sigma_\gamma\ (\omega)d\omega \cong 1.55 \times 10^{-25}\text{cm}^2\ \text{MeV}\ . \tag{6.12}$$

This seems a little larger than the value obtained from the experimental data (15)[8]

$$\int_{\text{Lab}} \sigma_\gamma\ (\omega)d\omega \cong 0.96 \times 10^{-25}\text{cm}^2\ \text{MeV}\ . \tag{6.13}$$

In the case of the 33 resonance, which SU(6) (and its generalizations) classify in the same representation as the nucleon, there exist symmetry arguments relating theoretically the inelastic form factors to elastic form factors. For example, the static SU(6) theory predicts

$$< N^{*+}\ \left|\mu_z\right|\ p > = \frac{2\sqrt{2}}{3}\,\mu_p \tag{6.14}$$

where μ_z is the z-component of the magnetic moment operator and $\mu_p = 2.78$ is the magnetic moment of the proton in nuclear magnetons. We note also the prediction of Salam, Delbourgo and Strathdee (17) on the basis of U(6,6) that

$$g_1 = 0 \qquad g_2 = g_3 \tag{6.15}$$

and

$$\left(\frac{2Mq^{*2}}{E + m}\right)\ g_2 = \frac{2}{<m>^2\sqrt{3}}\ G^V(q^2)\mu_p \tag{6.16}$$

which reduces to (6.14) in the static limit $q^2 \to 0$ and gives

$$\left|f_+\right|^2 + \left|f_-\right|^2 \approx 0.48\left(\frac{M}{\langle m \rangle}\right)^4 \tag{6.17}$$

This agrees with experiment for a reasonable choice of $\langle m \rangle$, the mean mass of the 56-plet of SU(6). That g_1 is small and that $\left|f_+\right|^2 + \left|f_-\right|^2 \sim \left|G^{(v)}\right|^2$ is already known from electroproduction experiments, so that the U(6,6) prediction is in qualitative agreement with the facts.

As a final application of our general discussion let us consider the transition to the $J^\pi = 3/2^-$ state at 1512 MeV. Cone *et al*. (3) have seen this transition. They attempt to analyze it as an E1 and on the basis of purely transverse excitation. They find a contradiction in that the transverse E1 form factors go as $(q^*)^{\frac{1}{2}-\frac{1}{2}} = \text{const}$ whereas their experimental data increases with q^{*2}. One possibility is that the threshold behavior is not valid in the physical region for this process. Another possibility, however, is seen from our general "threshold formula" for normal parity transitions which for $J = 3/2$ takes the very simple form

$$\frac{d\sigma}{d\Omega}(3/2^- \leftarrow 1/2^+)\ \text{"threshold"} \cong \frac{\alpha^2 \cos^2\frac{\theta}{2}}{4\epsilon^2 \sin^4\frac{\theta}{2}} \; \frac{\left|f_c\right|^2}{\left(1+\frac{2\epsilon}{m}\sin^2\frac{\theta}{2}\right)} \left\{\frac{q^2}{q^{*2}} + 2\frac{M^2}{m^2}\frac{q_o^2}{q^{*2}}\tan^2\frac{\theta}{2}\right\} . \tag{6.18}$$

We note that Coulomb excitation is actually the dominant process until one gets to the very backward angles and $\left|f_c\right|^2 \sim (q^*)^2$ for this transition. The analysis of $1/2^+ \rightarrow 5/2^+$ (1688 MeV) can also be carried out using our threshold formulae. One would conclude the cross section should go like $|f_c|^2 \sim (q^*)^4$ for this transition. Cone *et al.* have attempted to interpret their data in terms of either a transverse E2, $|f_\pm|^2 \sim (q^*)^2$ or a transverse M2, $|f_\pm|^2 \sim (q^*)^4$. The data is consistent with either interpretation in this case, and hence also with our threshold formula for normal parity transitions.

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LIST OF FOOTNOTES

1. For the special case $\frac{1}{2}^+ \to \frac{1}{2}^+$, $f_c \sim q^{*2}$ and $f_- \sim q^*$;

$$f_+ = 0 \text{ for } J = \frac{1}{2} \; .$$

2. The spherical components of $\underset{\sim}{J}$ are $J_{\pm 1} = \mp \frac{1}{\sqrt{2}} (J_x \pm iJ_y)$, $J_o = J_z$.

3. One of us uses a metric such that $v_\mu = (\underset{\sim}{v}, iv_o)$, $a \cdot b = \underset{\sim}{a} \cdot \underset{\sim}{b} - a_o b_o$. The γ matrices are hermitian and satisfy $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. The Dirac equation is $(i\gamma \cdot P + m)\, u(P,\lambda) = 0$ and we take our spinors to be normalized to $\bar{u}(P,\lambda')\, u(P,\lambda) = \delta_{\lambda\lambda'}$. Also $\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4$.

4. Notice that in this frame $S_4 = 0$; $\underset{\sim}{S} = M(\underset{\sim}{\epsilon} \times \underset{\sim}{q})$. Also

$$\chi^+(\lambda)\, \chi(\lambda) = 1 \; .$$

5. We can immediately generalize this result to include the contribution of states of other spins and parities. For fixed final electron energy ϵ' the mass of the final nuclear state $M_f^2 = -(q - P)^2$ is fixed. Thus if we include all states for fixed M_f^2 in the sum Eq. (3.3), we can always write

$$W_{1,2}\,(q^2, M_f) = \sum_{J^\pi} W_{1,2}^{J^\pi}\,(q^2, M_f^2)$$

and the cross section is simply

$$\left(\frac{d\sigma}{d\Omega' d\epsilon'}\right)_{lab} = \frac{\alpha^2}{4\epsilon^2 \sin \frac{4\theta}{2}} \sum_{J^\pi} \left[W_2^{J^\pi}\,(q^2, M_f^2) + 2W_1^{J^\pi}\,(q^2, M_f^2) \tan^2 \frac{\theta}{2} \right] \; .$$

The kinematical factor $\left[1 + \frac{2\epsilon}{m} \sin^2 \frac{\theta}{2}\right]^{-1}$ which comes from integrating the δ-function associated with a discrete intermediate state

$$\delta\left[(P-q)^2 + M^2\right] = \int \frac{d^3P'}{2E'}\ \delta^{(4)}\ (P'-P+q)$$

over final electron energies ϵ' no longer appears.

6. Note the identity $2\underset{\sim}{\hat{\omega}}^{*}\cdot\ \underset{\sim}{a}\ \ \underset{\sim}{\hat{\omega}}\cdot\underset{\sim}{b} = \underset{\sim}{a}\cdot\underset{\sim}{b} - \underset{\sim}{a}\cdot\underset{\sim}{s}\ \ \underset{\sim}{b}\cdot\underset{\sim}{s} + i(\underset{\sim}{a}\times\underset{\sim}{b})\cdot\underset{\sim}{s}$.

7. The helicity amplitudes are defined with an extra phase $e^{i\lambda\varphi}$ in the wave functions as mentioned in the text.

8. If we use the revised values of the coupling constant given by Mathews (16)

$$C_3\ (0) = 0.298$$

$$C_4\ (0) + C_5\ (0) = 0.0336$$

we get

$$|f_+|^2 + |f_-|^2 = 0.322$$

$$\int_{\text{Lab}} \sigma_\gamma(\omega)d\omega = 0.91 \times 10^{-25}\text{cm}^2\ \text{MeV}$$

in satisfactory agreement with the experimental value quoted in (6.13). Mathews' numbers came from using an experimental value

$$\int_{\text{Lab}} \sigma_\gamma(\omega)d\omega = 0.92 \times 10^{-25}\text{cm}^2\ \text{MeV}\quad \text{[Experiment - Mathews]}$$

which thus provides an excellent check on our calculations.

FIGURE CAPTIONS

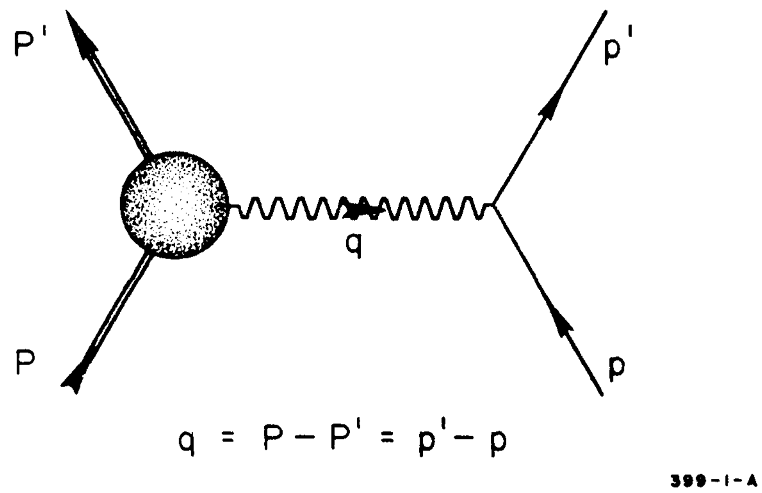

FIG.1 - KINEMATICS FOR INELASTIC SCATTERING

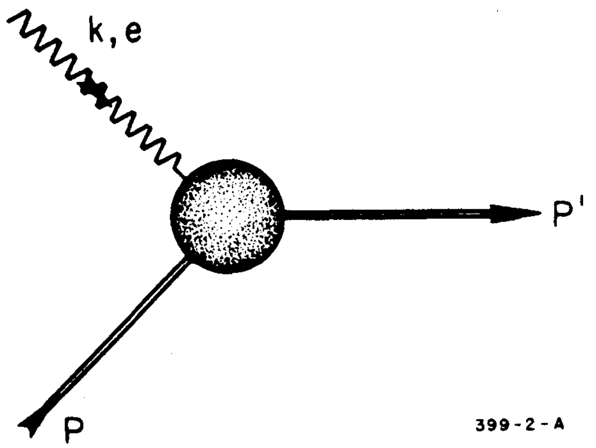

FIG. 2 - ELECTROMAGNETIC VERTEX FOR PHOTOPRODUCTION OF AN ISOBAR

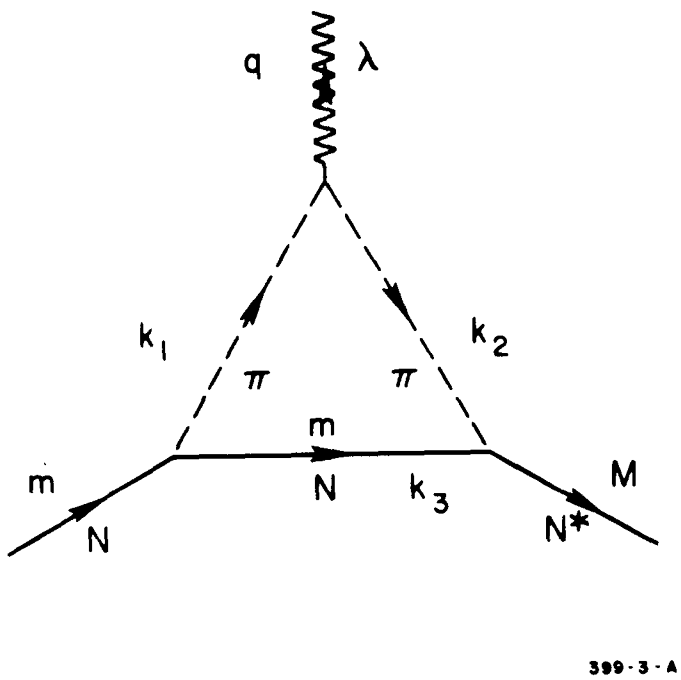

FIG. 3 - TRIANGLE DIAGRAM

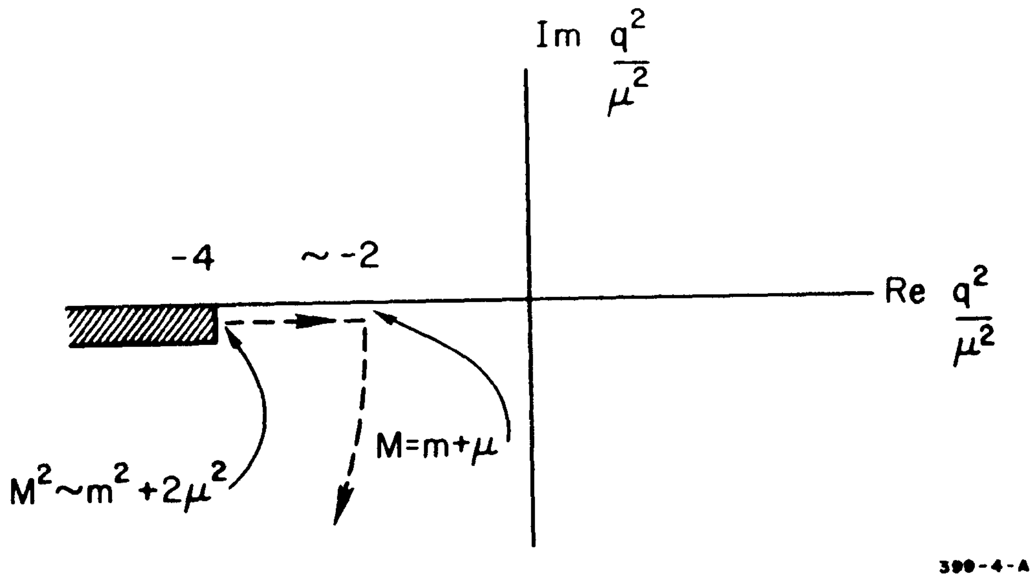

FIG. 4 - SINGULARITIES OF THE TRIANGLE DIAGRAM

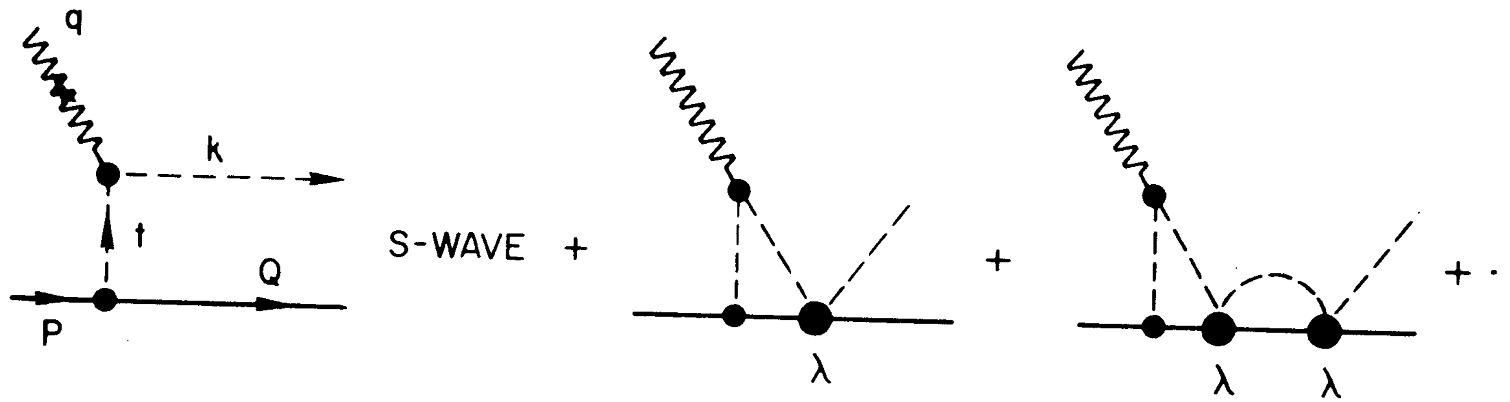

FIG. 5- MODEL OF ELECTROPRODUCTION AMPLITUDE

399-5-A

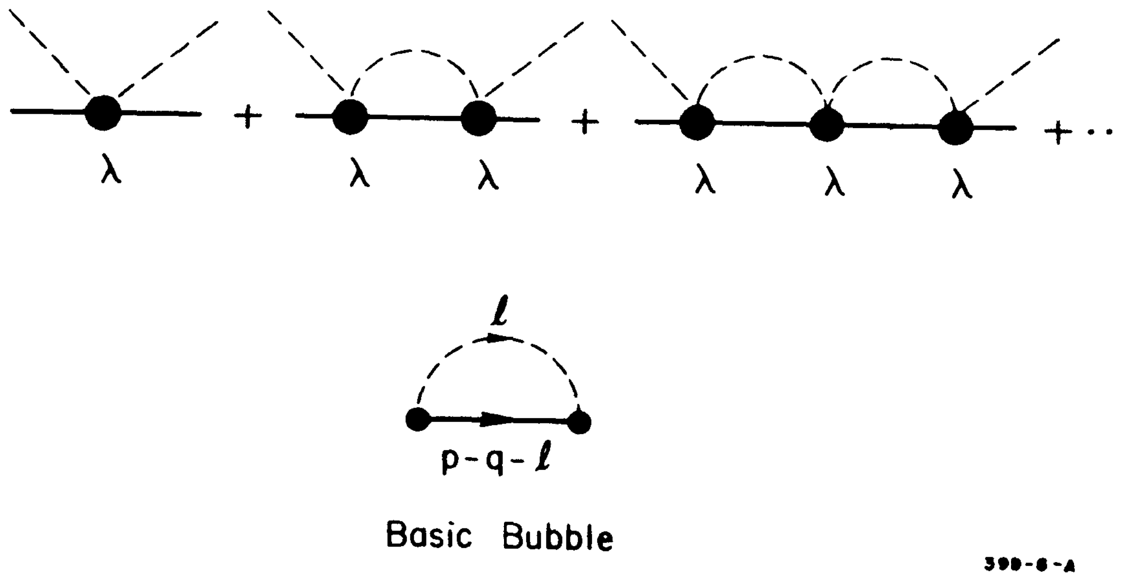

FIG. 6-MODEL OF S-WAVE MESON-NUCLEON SCATTERING AMPLITUDE

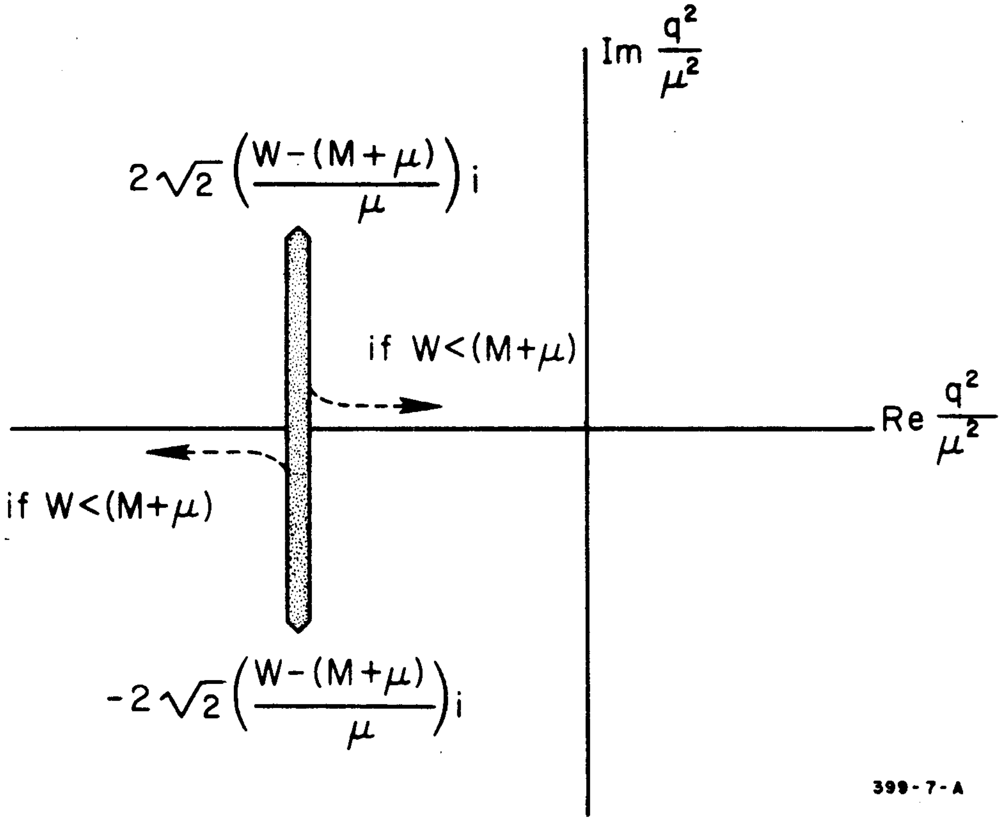

FIG. 7 - SINGULARITIES OF THE BORN TERM