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Braiding, Majorana Fermions and the Dirac Equation

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Abstract. This paper reviews a construction called *iterant algebra* that reconstructs Clifford algebra and matrix algebra from the point of view of the symmetric group acting on vectors as ordered lists of their components. This algebraic background is then used to discuss the braiding of Majorana Fermions, the structure of the Dirac equation and the form of the version of the Dirac equation due to Majorana.

1. Introduction

The simplest discrete system corresponds to the square root of minus one, when the square root of minus one is seen as an oscillation between plus and minus one. Thinking about the square root of minus one in this way, as an *iterant*, is explained below. Discrete systems can be embedded in non-commutative systems where discrete derivatives are replaced by commutators. This observation generalizes iterants and is explained at the end of Section 2 of the present paper. Iterant algebra generalizes matrix algebra and can be used to formulate the Clifford algebra for Majorana Fermions. This paper is a sequel to [8] and [5-8, 11-17] and it uses material from these papers. In this paper we give a very concise formulation of the basic concepts for our approach to iterants, Majorana Fermion operators and the Dirac equation. We have taken formulations in our previous papers and written them in the most condensed possible manner. These results in overlap with these papers, but the formulations are original to the present work. A further background of previous work of the author is given in the following references [1-4, 10, 19].

Section 2 is an introduction to the process algebra of iterants and how the square root of minus one arises from an alternating process. Section 3 discusses how Clifford algebras are fundamental to the structure of Fermions and how braiding is related to the Majorana Fermions. Section 4 discusses the structure of the Dirac equation and how the nilpotent and the Majorana operators arise naturally in this context. This section provides a link between our work and the work on nilpotent structures and the Dirac equation of Peter Rowlands [22]. We end this section with an expression in split quaternions for the Majorana Dirac equation in one dimension of time and three dimensions of space. The Majorana Dirac equation can be written as follows:

$$\left(\frac{\partial}{\partial t} + \hat{\eta}\eta \frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial y} + \hat{\varepsilon}\eta \frac{\partial}{\partial z} - \hat{\varepsilon}\hat{\eta}\eta m \right) \psi = 0$$

where η and ε are the simplest generators of iterant algebra with $\eta^2 = \varepsilon^2 = 1$ and $\eta\varepsilon + \varepsilon\eta = 0$, and $\hat{\varepsilon}$, $\hat{\eta}$ form a copy of this algebra that commutes with it. This combination of the simplest Clifford algebra



with itself is the underlying structure of Majorana Fermions, forming indeed the underlying structure of all Fermions.

2. Iterants

An iterant is a sum of elements of the form

$$[a_1, a_2, \dots, a_n] \sigma$$

where $[a_1, a_2, \dots, a_n]$ is a vector of scalars (usually real or complex numbers) and σ is a permutation on n letters. Such elements are themselves sums of elements of the form

$$[0, 0, \dots, 0, 1, 0, \dots, 0] \sigma = e_i \sigma$$

where the 1 is in the i -th place. The vectors e_i are the basic idempotents that generate the iterants with the help of the permutations.

If $a = [a_1, a_2, \dots, a_n]$, then let a^σ denote the vector with its elements permuted by the action of σ :

$$a^\sigma = [a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(n)}].$$

If a and b are vectors then their *elementary product* ab denotes the vector where $(ab)_i = a_i b_i$, and $a+b$ denotes the vector where $(a+b)_i = a_i + b_i$. Then products and sums are defined by the formulas below:

$$(a\sigma)(b\tau) = (ab^\sigma)\sigma\tau,$$

$$(ka)\sigma = k(a\sigma)$$

for a scalar k , and

$$(a+b)\sigma = a\sigma + b\sigma$$

where vectors are multiplied as above and we take the usual product of the permutations. All of matrix algebra and more is naturally represented in the iterant framework. See [17].

If η is the order two permutation of two elements, then $[a, b]^\eta = [b, a]$. We can define

$$i = [1, -1]\eta$$

and then

$$i^2 = [1, -1]\eta[1, -1]\eta = [1, -1][1, -1]^\eta \eta^2 = [1, -1][-1, 1] = [-1, -1] = -1.$$

In this way the complex numbers arise naturally from iterants. One can interpret $[1, -1]$ as an oscillation between $+1$ and -1 and η as denoting a temporal shift operator. The $i = [1, -1]\eta$ is a time sensitive element and its self-interaction has square minus one. In this way iterants can be interpreted as a formalization of elementary discrete processes.

A more general approach to discrete processes [18] includes this interpretation of iterants and the square root of negative unity. The more general approach is worth reprising in this context. Given a sequence of discrete algebraic elements $X_t (t=0, 1, \dots)$ (we take them to be associative but not necessarily commutative for this discussion), we define an *invertible shift operator* J so that

$$X_{t+1} = J^{-1} X_t J$$

Then one can define a discrete derivative ∇ by the equation

$$\nabla X_t = \frac{J(X_{t+1} - X_t)}{\Delta t}.$$

Note that, since $JX_{t+1} = X_t J$, this discrete derivative satisfies

$$\nabla X_t = \frac{X_t J - J X_t}{\Delta t} = \frac{[X_t, J]}{\Delta t}.$$

Thus ∇ is represented as a commutator and satisfies the Leibniz rule

$$\nabla(AB) = \nabla(A)B + A\nabla(B).$$

This means that we can shift the analysis of discrete systems to non-commutative worlds where all derivatives are represented by commutators. This is the subject of [18] and a number of our earlier papers.

3. Clifford Algebra, Majorana Fermions and Braiding

In Fermion algebra one has annihilation operators ψ and their conjugate creation operators ψ^\dagger . One has $\psi^2 = 0 = (\psi^\dagger)^2$ and a fundamental commutation relation

$$\psi\psi^\dagger + \psi^\dagger\psi = 1.$$

Pairs of such operators such as ψ and φ , anti-commute:

$$\psi\varphi = -\varphi\psi.$$

Majorana Fermion operators c that satisfy $c^\dagger = c$ so that they are their own anti-particles. There is a lot of interest in such operators in relation to quasi-particles. They are related to braiding and to topological quantum computing. See [21] where researchers found quasiparticle Majorana Fermions in edge effects in nano-wires. The Fibonacci model that we discuss in [6] is also based on Majorana particles, related to collective electronic excitations. If P is a Majorana Fermion particle, then P can interact with itself to either produce itself or to annihilate itself. This is the simple “fusion algebra” for this particle. One can write $P^2 = P + 1$ to denote the two possible self-interactions the particle P . The patterns of interaction and braiding of such a particle P give rise to the Fibonacci model.

Majorana operators are related to standard Fermion operators. The algebra for Majorana operators is $c^\dagger = c$ and $cc' = -c'c$ if c and c' are distinct Majorana Fermion operators with $c^2 = 1$ and $c'^2 = 1$. A standard Fermion operator is composed from two Majorana operators via

$$\psi = (c + ic')/2,$$

$$\psi^\dagger = (c - ic')/2.$$

Similarly one can make two Majorana operators from any single Fermion operator. If you take a set of Majoranas

$$\{c_1, c_2, c_3, \dots, c_n\}$$

then there are braiding operators that act on the vector space with these c_k as the basis. The operators are mediated by algebra elements

$$\tau_k = (1 + c_{k+1}c_k)/\sqrt{2},$$

$$\tau_k^{-1} = (1 - c_{k+1}c_k)/\sqrt{2}.$$

The braiding operators are

$$T_k : \text{Span}\{c_1, c_2, \dots, c_n\} \rightarrow \text{Span}\{c_1, c_2, \dots, c_n\}$$

via

$$T_k(x) = \tau_k x \tau_k^{-1}.$$

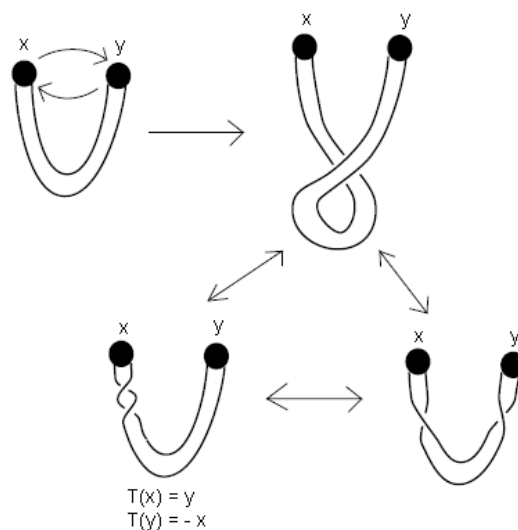


Figure 1: Braiding Action on a Pair of Fermions

The braiding is summarized by these formulas:

$$T_k(c_k) = c_{k+1},$$

$$T_k(c_{k+1}) = -c_k,$$

and T_k is the identity otherwise. This gives a very unitary representation of the Artin braid group, and it can be used to produce universal quantum computing in the presence of key local unitary operators. See Figure 1 for a depiction of this in relation to the topology of a belt that connects the Fermion particles. The relationship with the belt depends upon the fact that in quantum mechanics one must represent rotations of three dimensional space as unitary transformations. See [9]. In the Figure, the belt does not know which of the two Fermions to endow with the phase change, but the clever algebra above makes this decision. It is remarkable that fundamental topology occurs at the level of the Majorana Fermion operators.

Now we make contact between the iterants and the algebra of the Majorana Fermion operators. Let $e = [1, -1]$. Then $e^2 = [1, 1] = 1$ and $e\eta = [1, -1]\eta = [-1, 1]\eta = -e\eta$. Thus

$$e^2 = 1,$$

$$\eta^2 = 1,$$

and

$$e\eta = -\eta e.$$

We regard the basic iterants e and η as a fundamental pair of Majorana Fermions.

4. The Dirac Equation and Majorana Fermions

We now construct the Dirac equation. If the speed of light is equal to 1 (by convention), then energy E , momentum p and rest mass m are related by the (Einstein) equation

$$E^2 = p^2 + m^2.$$

Dirac constructed his equation by looking for an algebraic square root of $p^2 + m^2$ so that he could have a linear operator for E that would take the same role as the Hamiltonian in the Schrodinger equation. We will get to this operator by first taking the case where p is a scalar (we use one

dimension of space and one dimension of time.). Let $E = \alpha p + \beta m$ where α and β are elements of a possibly non-commutative, associative algebra. Then

$$E^2 = \alpha^2 p^2 + \beta^2 m^2 + pm(\alpha\beta + \beta\alpha).$$

Hence we satisfy $E^2 = p^2 + m^2$ if $\alpha^2 = \beta^2 = 1$ and $\alpha\beta + \beta\alpha = 0$. This is the Clifford algebra pattern and we can apply the iterant algebra generated by e and η . Then, with the quantum operator for momentum is $-i\frac{\partial}{\partial x}$ and the operator for energy is $i\frac{\partial}{\partial t}$, we have the Dirac equation

$$i\frac{\partial\psi}{\partial t} = -i\alpha\frac{\partial\psi}{\partial x} + \beta m\psi.$$

Let

$$O = i\frac{\partial}{\partial t} + i\alpha\frac{\partial}{\partial x} - \beta m$$

so that the Dirac equation takes the form

$$O\psi(x, t) = 0.$$

Now note that

$$Oe^{i(px-Et)} = (E - \alpha p - \beta m)e^{i(px-Et)}.$$

We let

$$\Delta = (E - \alpha p - \beta m)$$

and let

$$U = \Delta\beta\alpha = (E - \alpha p - \beta m)\beta\alpha = \beta\alpha E + \beta p - \alpha m,$$

then

$$U^2 = -E^2 + p^2 + m^2 = 0.$$

This nilpotent element leads to a (plane wave) solution to the Dirac equation. We have that

$$O\psi = \Delta\psi$$

for $\psi = e^{i(px-Et)}$. It follows that

$$O(\beta\alpha\Delta\beta\alpha\psi) = \Delta\beta\alpha\Delta\beta\alpha\psi = U^2\psi = 0,$$

from which it follows that

$$\psi = \beta\alpha U e^{i(px-Et)}$$

is a (plane wave) solution to the Dirac equation.

This calculation suggests that we multiply the operator O by $\beta\alpha$ on the right, obtaining the operator

$$D = O\beta\alpha = i\beta\alpha\frac{\partial}{\partial t} + i\beta\frac{\partial}{\partial x} - \alpha m,$$

and the equivalent Dirac equation

$$D\psi = 0.$$

For the specific ψ above we have $D(Ue^{i(px-Et)}) = U^2e^{i(px-Et)} = 0$. Rowlands [22] does this trick in the context of quaternion algebra. Here we use the split quaternions. We regard this method of construction via the split quaternions as fundamental since it connects the Dirac equation to the process algebra and iterants by which we have begun this development. Note that the solution to the Dirac equation that we have found is expressed in Clifford algebra or iterant algebra form. It can be articulated into specific vector solutions by using an iterant or matrix representation of the algebra.

We see that the algebra element $U = \beta\alpha E + \beta p - \alpha m$ with $U^2 = 0$ is the core of this plane wave solution to the Dirac equation. A natural non-commutative algebra arises directly from articulation of

discrete process and can be regarded as essential information in a Fermion. It is natural to compare this algebra structure with algebra of creation and annihilation operators that occur in quantum field theory.

If we let

$$\tilde{\psi} = e^{i(px+Et)}$$

(reversing time), then

$$D\tilde{\psi} = (-\beta\alpha E + \beta p - \alpha m)\psi = U^\dagger \tilde{\psi},$$

This gives a definition of U^\dagger corresponding to the anti-particle for $U\psi$.

We have

$$U = \beta\alpha E + \beta p - \alpha m$$

and

$$U^\dagger = -\beta\alpha E + \beta p - \alpha m$$

Note that

$$(U + U^\dagger)^2 = (2\beta p + \alpha m)^2 = 4(p^2 + m^2) = 4E^2,$$

and

$$(U - U^\dagger)^2 = -(2\beta\alpha E)^2 = -4E^2.$$

Thus

$$U^2 = (U^\dagger)^2 = 0$$

and

$$UU^\dagger + U^\dagger U = 4E^2.$$

This is a direct appearance of the Fermion algebra corresponding to the Fermion plane wave solutions to the Dirac equation. Furthermore, the decomposition of U and U^\dagger into the Majorana Fermion operators corresponds to $E^2 = p^2 + m^2$. Normalizing by dividing by $2E$ we have

$$A = \frac{\beta p + \alpha m}{E}$$

and

$$B = i\beta\alpha.$$

so that

$$A^2 = B^2 = 1$$

and

$$AB + BA = 0.$$

then

$$U = (A + Bi)E$$

and

$$U^\dagger = (A - Bi)E,$$

articulating the Fermion operators in terms of the simpler Clifford algebra of Majorana operators (split quaternions once again).

4.1 Writing in the Full Dirac Algebra

We have written the Dirac equation in one dimension of space and one dimension of time. We here translate the formalism to three dimensions of space. Take an independent Clifford algebra generated by $\sigma_1, \sigma_2, \sigma_3$ with $\sigma_i^2 = 1$ for $i=1,2,3$ and $\sigma_i\sigma_j = -\sigma_j\sigma_i$ for $i \neq j$. Assume that α and β as used above generate an independent Clifford algebra commuting with the algebra of the σ_i . Replace scalar

momentum p by a 3-vector momentum $p = (p_1, p_2, p_3)$ and let $p \bullet \sigma = p_1 \sigma_1 + p_2 \sigma_2 + p_3 \sigma_3$. Replace

$$\frac{\partial}{\partial x} \text{ with } \nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) \text{ and } \frac{\partial p}{\partial x} \text{ with } \nabla \bullet p.$$

The Dirac equation takes the form:

$$i \frac{\partial \psi}{\partial t} = -i \alpha \nabla \bullet \sigma \psi + \beta m \psi.$$

Let

$$O = i \frac{\partial}{\partial t} + i \alpha \nabla \bullet \sigma - \beta m$$

so the Dirac equation takes the form

$$O \psi(x, t) = 0.$$

By our previous discussion we let

$$\psi(x, t) = e^{i(p \bullet x - Et)}$$

and construct solutions by applying the Dirac operator to this ψ . The two Clifford algebras interact to generalize the nilpotent solutions and Fermion algebra that we have detailed for one spatial dimension to this three dimensional case. The modified Dirac operator is

$$D = i \beta \alpha \frac{\partial}{\partial t} + \beta \nabla \bullet \sigma - \alpha m.$$

We have that

$$D \psi = U \psi$$

where

$$U = \beta \alpha E + \beta p \bullet \sigma - \alpha m.$$

Then $U^2 = 0$ and $U \psi$ is a solution to the modified Dirac Equation, as before. We articulate the structure of the Fermion operators and locate the corresponding Majorana Fermion operators. Details are left to the reader.

4.2 Majorana Fermions

We end with a brief discussion making a Dirac algebra that is distinct from the one generated by $\alpha, \beta, \sigma_1, \sigma_2, \sigma_3$. From the new algebra we obtain an equation that can have real solutions. This was the strategy Majorana [20] followed to construct his Majorana Fermions. A real equation can have solutions that are invariant under complex conjugation. The solutions can correspond to particles that are their own anti-particles. We describe this Majorana algebra in terms of the split quaternions ε and η . For convenience we use the matrix representation given below. The reader of this paper can substitute the corresponding iterants.

$$\varepsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $\hat{\varepsilon}$ and $\hat{\eta}$ generate another, independent algebra of split quaternions, commuting with the first algebra generated by ε and η . A totally real Majorana Dirac equation can be written as follows:

$$\left(\frac{\partial}{\partial t} + \hat{\eta} \eta \frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial y} + \hat{\varepsilon} \eta \frac{\partial}{\partial z} - \hat{\varepsilon} \hat{\eta} \eta m \right) \psi = 0.$$

To see that this is a correct Dirac equation, note that

$$\hat{E} = \alpha_x \hat{p}_x + \alpha_y \hat{p}_y + \alpha_z \hat{p}_z + \beta m$$

(Here the ‘‘hats’’ denote the quantum differential operators corresponding to the energy and momentum.) will satisfy

$$\hat{E}^2 = \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2 + m^2$$

if the algebra generated by $\alpha_x, \alpha_y, \alpha_z, \beta$ has each generator of square one and each distinct pair of generators anti-commuting. We obtain the general Dirac equation by replacing \hat{E} by $i\frac{\partial}{\partial t}$, and \hat{p}_x with $-i\frac{\partial}{\partial x}$ (and same for y, z).

$$\left(i\frac{\partial}{\partial t} + i\alpha_x \frac{\partial}{\partial x} + i\alpha_y \frac{\partial}{\partial y} + i\alpha_z \frac{\partial}{\partial z} - \beta m \right) \psi = 0.$$

This is equivalent to

$$\left(\frac{\partial}{\partial t} + \alpha_x \frac{\partial}{\partial x} + \alpha_y \frac{\partial}{\partial y} + \alpha_z \frac{\partial}{\partial z} + i\beta m \right) \psi = 0.$$

We take

$$\alpha_x = \hat{\eta}\eta, \alpha_y = \varepsilon, \alpha_z = \hat{\varepsilon}\eta, \beta = i\hat{\varepsilon}\hat{\eta}\eta,$$

These elements satisfy the requirements for the Dirac algebra. Note how we have a significant interaction between the commuting square root of minus one (i) and the element $\hat{\varepsilon}\hat{\eta}$ of square minus one in the split quaternions. This return to our original considerations about the source of the square root of minus one. Both viewpoints combine in the element $\beta = i\hat{\varepsilon}\hat{\eta}\eta$ that makes this Majorana algebra work. Since the algebra appearing in the Majorana Dirac operator is constructed entirely from two commuting copies of the split quaternions, there is no appearance of the complex numbers, and when written out in 2×2 matrices we obtain coupled real differential equations to be solved. This ending is actually a beginning of a new study of Majorana Fermions. That will commence in a sequel to the present paper.

References

- [1] Kauffman L 1985 Sign and Space, In Religious Experience and Scientific Paradigms *Proceedings of the 1982 IASWR Conference, (Stony Brook, New York: Institute of Advanced Study of World Religions)* pp 118-164
- [2] Kauffman L 1987 Self-reference and recursive forms *Journal of Social and Biological Structures* pp 53-72
- [3] Kauffman L 1987 Special relativity and a calculus of distinctions *Proceedings of the 9th Annual Intl. Meeting of ANPA, Cambridge, England (Pub. by ANPA West)* pp 290-311
- [4] Kauffman L 1987 Imaginary values in mathematical logic *Proceedings of the Seventeenth International Conference on Multiple Valued Logic, May 26-28 Boston MA (IEEE Computer Society Press)* pp 282-289
- [5] Kauffman L H 1994 *Knot Logic In Knots and Applications* ed L Kauffman (World Scientific Pub. Co.) pp 1-110
- [6] Kauffman L H 2016 *Knot logic and topological quantum computing with Majorana Fermions. In "Logic and algebraic structures in quantum computing and information"* (Lecture Notes in Logic) eds J Chubb, J Chubb, Ali Eskandarian and V Harizanov (Cambridge University Press)
- [7] Kauffman L H and Lomonaco S J 2018 Braiding, Majorana Fermions and Topological Quantum Computing *Special Issue of QIP on Topological Quantum Computing* (to appear)
- [8] Kauffman L H 2015 Iterants, Fermions and Majorana Operators *Unified Field Mechanics - Natural Science Beyond the Veil of Spacetime* eds R Amoroso, L H Kauffman and P Rowlands (World Scientific Pub. Co.) pp 1-32
- [9] Kauffman L H 1991, 1994, 2001, 2012 *Knots and Physics* (World Scientific Pub)

- [10] Kauffman L H 2002 Time imaginary value, paradox sign and space *Computing Anticipatory Systems CASYS AIP Conference Proceedings Fifth International Conference (Liege, Belgium) – 2001* ed D Dubois **627**
- [11] Kauffman L H and H Noyes Pierre 1996 Discrete Physics and the Derivation of Electromagnetism from the formalism of Quantum Mechanics *Proc. of the Royal Soc. Lond. A* **452** pp 81-95
- [12] Kauffman L H and H Noyes Pierre 1996 Discrete Physics and the Dirac Equation *Physics Letters A* **218** pp 139-146
- [13] Kauffman L H 1998 Noncommutativity and discrete physics *Physica D* **120** pp 125-138
- [14] Kauffman L H 1998 Space and time in discrete physics *Intl. J. Gen. Syst.* **27** pp 1-3, 241-273
- [15] Kauffman L H 2004 Non-commutative worlds *New Journal of Physics* **6** pp 2-46
- [16] Kauffman L H 2007 Differential geometry in non-commutative worlds *Quantum Gravity - Mathematical Models and Experimental Bounds* eds B Fauser, J Tolksdorf and E Zeidler, (Birkhauser) pp 61-75
- [17] Kauffman L H 2017 Iterant algebra. *Entropy* **19** 7 pp 94A17
- [18] Kauffman L H 2018 Non-Commutative Worlds and Classical Constraints *Entropy* **20** 483
- [19] Kauffman L H and Samuel J. Lomonaco Jr 2007 q-deformed spin networks, knot polynomials and anyonic topological quantum computation. *J. Knot Theory Ramifications* **16** 3 pp 267–332
- [20] Majorana E 1937 A symmetric theory of electrons and positrons *I Nuovo Cimento* **14** pp 171-184
- [21] Mourik V, Zuo K, Frolov S M, Plissard S R, Bakkers E P A M and Kouwenhoven L P 2012 Signatures of Majorana Fermions in hybrid superconductor-semiconductor devices (*Preprint arXiv:1204.2792*)
- [22] Rowlands P 2007 Zero to Infinity - The Foundations of Physics *Series on Knots and Everything* (World Scientific Publishing Co.) **41**

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