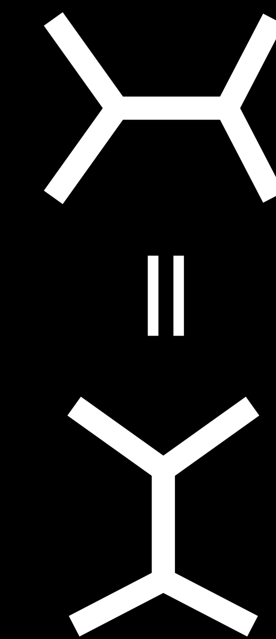
# Spinning conformal blocks and applications

Rejon Barrera, F.G.

# Spinning Conformal Blocks and Applications



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Spinning Conformal Blocks and Applications Fernando G. Rejón Barrera

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# Spinning Conformal Blocks and Applications

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# Publications

This thesis is based on the following publications:

[1] F. Rejon-Barrera and D. Robbins,
*"Scalar-Vector Bootstrap"*,
*JHEP* **01** (2016) 139 `arXiv:1508.02676 [hep-th]`.

Presented in Chapter **2**.

[2] D. M. Hofman, D. Li, D. Meltzer, D. Poland, and F. Rejon-Barrera,
*"A Proof of the Conformal Collider Bounds"*,
*JHEP* **06** (2016) 111, `arXiv:1603.03771 [hep-th]`.

Presented in Chapter **4**.

[3] A. Castro, E. Llabrés, and F. Rejon-Barrera,
*"Geodesic Diagrams, Gravitational Interactions & OPE Structures"*,
*JHEP* **06** (2017) 099, `arXiv:1702.06128 [hep-th]`.

Presented in Chapter **5**.

[4] P. Kravchuk and F. Rejon-Barrera,
*Unpublished*,
2017.

Presented in Chapter **3**.

Other publications by the author:


[5] W. Bietenholz, M. Bögli, F. Niedermayer, M. Pepe, F. G. Rejon-Barrera, and U. J. Wiese,
*"Topological Lattice Actions for the 2d XY Model"*,
*JHEP* **03** (2013) 141, `arXiv:1212.0579 [hep-lat]`.

[6] W. Bietenholz, U. Gerber, and F. G. Rejón-Barrera,
*"Berezinskii–Kosterlitz–Thouless transition with a constraint lattice action"*,
*J. Stat. Mech.* **1312** (2013) P12009, `arXiv:1307.0485 [hep-lat]`.

Contribution of the author to the publications:

The author participated to all the discussions in all the publications. In [1] the author computed the spinning conformal blocks from scalar ones and derived the particular simplification of the mixed-symmetric projector which led to the main result. In [2] the author computed the large spin light-cone limit approximation of the relevant spinning correlators resulting in the calculation of the anomalous dimensions as well as OPE coefficients. In addition the author contributed in corroborating the results from causality arguments. In [3] the author derived the bulk differential operators and computed the cubic gravitational interactions generated by these, in the context of geodesic diagrams. In [4] the author derived the three-row weight-shifting operators, and computed all the $6j$ symbols, coefficients and differential bases, to write the recursion relation for seed spinning partial waves.

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# SUMMARY

Conformal invariance is one concept that is common to many areas of theoretical physics; from critical phenomena in statistical and condensed matter systems, through the description of particle physics at long distances, to quantum gravity via the holographic principle. Conceptually, *conformal symmetry* is characterized by transformations that preserves angles, which include rotations, translations, as well as coordinate-dependent dilatations.

In modern times, it is believed that generic quantum field theories (QFT) are parametrized by coupling constants that vary continuously along an *RG flow*, starting and ending at fixed points. At both fixed points, the symmetries of the QFT are upgraded to angle preserving transformations, thus becoming a *Conformal Field Theory* (CFT). An alternate approach for studying QFTs is then to start from a CFT corresponding to either fixed point and flowing back to the non-conformal regime. The reason this is a viable strategy is that conformal symmetry fixes all the observables of the theory up to a set of numerical coefficients called *CFT data*, and furthermore imposes consistency conditions that, if solved, completely characterize the space of valid CFTs. The quest for solving these consistency conditions is called *the conformal bootstrap program*, and was first proposed in the 1970s [7, 8]. Moreover, CFTs can be defined mathematically in a rigorous way which is a considerable improvement when compared to the usual Lagrangian description of QFTs.

In two dimensions, the group of conformal transformations is enhanced by an infinite amount of symmetries which allowed the complete solution of a large class of theories—the minimal models—in the 1980s [9]. However, for arbitrary dimensions, it took more than 20 years for explicit solutions to the *bootstrap equations* to appear. The main reasons for this delay are the lack of mathematical results describing the kinematic functions involved in the bootstrap equations—*conformal partial waves*—as well as the absence of appropriate numerical and analytical technology for solving these equations.

Since then, the literature has exploded with many new developments and results on CFTs using the bootstrap philosophy. Nonetheless, the bootstrap program is far from complete. In particular the conformal partial waves describing spinning fields are not fully understood yet. Furthermore, the application of current techniques to discover possible new theories in higher dimensions, as well as uncovering universal properties and their connection to quantum gravity, are yet to be explored. Finally, in order to achieve a complete classification of CFTs in arbitrary dimensions, more efficient and powerful techniques for solving the bootstrap equations will need to be developed.

In this thesis we discuss two improvements in the computation of spinning partial waves in closed form for arbitrary space-time dimensions. Then applying two analytical bootstrap techniques for constraining spinning four point functions, we compute some universal constraints on the CFT data related to the conformal stress-tensor. In the context of holography this result shows the attractive nature of the gravitational interactions in AdS. Finally, we apply CFT techniques to provide a dictionary between spinning partial waves and geodesic objects living in AdS.

# Samenvatting

Conforme invariantie komt voor in een groot aantal deelgebieden van de theoretische natuurkunde; van kritische systemen in statistische fysica en gecondenseerde materie, via de beschrijving van deeltjesfysica op grote lengteschalen, tot quantumzwaartekracht door middel van het holografisch principe. Conceptueel gezien is een *conforme symmetrie* een transformatie die hoeken behoudt, zoals rotaties, translaties en coördinaatafhankelijke dilataties.

Vandaag de dag denkt men dat algemene quantumveldentheorieën (QFT) geparametriseerd worden door koppelingsconstantes die continu variëren langs een *RG-flow*, die begint en eindigt op een stationair punt. Op beide stationaire punten worden de symmetrieën van de QFT uitgebreid tot hoekgetrouwe transformaties en wordt de theorie een *conforme veldentheorie* (CFT). Een alternatieve methode om QFT's te bestuderen bestaat er dan uit om te beginnen vanuit een CFT, die correspondeert met één van de stationaire punten, en langs de RG-flow terug te 'stromen' naar het niet-conforme regime. Deze methode heeft kans van slagen doordat conforme symmetrie alle observabelen van de theorie vastlegt op een verzameling numerieke coëfficiënten na (die de *CFT-data* genoemd worden) en bovendien consistentievereisten oplegt die, als ze opgelost kunnen worden, de ruimte van valide CFT's volledig vastlegt. De zoektocht naar de oplossing van deze consistentievereisten staat bekend als de *conforme bootstrap* en werd voor het eerst geopperd in de jaren zeventig [7, 8]. Bovendien kunnen CFTs wiskundig rigoureus gedefinieerd worden, een flinke verbetering ten opzichte van de gebruikelijke Lagrangiaanse beschrijving van QFTs.

De groep van conforme transformaties wordt in twee dimensies uitgebreid met een oneindige hoeveelheid symmetrieën Dit maakt het mogelijk dat in de jaren tachtig een grote klasse van theorieën—de *minimal models*—volledig werd opgelost [9]. Voor willekeurige dimensies duurde het echter meer dan twintig jaar voordat expliciete oplossingen van de *bootstrapvergelijkingen* werden gevonden. De voornaamste redenen voor deze vertraging zijn het gebrek aan wiskundige resultaten over de kinematische functies die een rol spelen in de bootstrapvergelijkingen—*conforme partiële golven*—en het gebrek aan geschikte numerieke en analytische technieken om deze vergelijkingen op te lossen.

Sindsdien heeft er een explosie van nieuwe ontwikkelingen en resultaten over CFT's in de literatuur plaatsgevonden, gevoed door de bootstrap-filosofie. Desalniettemin is het bootstrap-programma verre van compleet. De conforme partiële golven die velden met spin beschrijven zijn bijvoorbeeld nog niet volledig begrepen. Bovendien zijn de huidige technieken nog niet toegepast op zowel de zoektocht naar nieuwe theorieën in hogere dimensies als het ontdekken van universele eigenschappen en hun verband met quantumzwaartekracht. Tenslotte moeten efficiëntere en krachtigere technieken om de bootstrapvergelijkingen op te lossen worden ontwikkeld om een volledige classificatie van CFT's in willekeurige dimensies te kunnen geven.

In dit proefschrift behandelen we twee verbeteringen in de berekening van partiële golven met spin in gesloten vorm voor willekeurige ruimtetijddimensie. Door twee analytische bootstrap-methodes toe te passen om vierpuntsfuncties met spin te begrenzen kunnen we bepaalde universele begrenzingen op de CFT-data gerelateerd aan de conforme stresstensor berekenen. In de context van holografie toont dit resultaat de aantrekkende werking van de zwaartekrachtsinteracties in AdS. Tenslotte passen we CFT-technieken toe om een woordenboek tussen partiële golven met spin en geodetische objecten in AdS te geven.

# 1 Introduction: Conformal field theories in $d > 2$

On techniques for studying and solving the space of consistent conformal field theories in general dimensions

The objective of this chapter is to introduce the relevant concepts for this thesis, in a detailed and self-consistent way. The information that we present ranges from basic concepts in conformal field theory (CFT) to state-of-the-art techniques from the modern literature on higher dimensional CFTs.

## 1.1 Review of basic concepts

The first section concerns basic information regarding conformal field theories in arbitrary dimensions including symmetry generators, conformal representations, the conformal algebra, etc. This is commonly found in books and lecture notes, e.g. [10–13], hence readers familiar with this material may skip it. The discussion of modern CFT concepts starts in section 1.2.

### 1.1.1 Symmetries, Ward identities, charge generators, and correlation functions

Consider a Lie group $G$. The action of an element $g \in G$ on the coordinates $x^\mu \in \mathbb{R}^d$, for arbitrary $d$ can be written as

$$gx^\mu = e^{\theta^a T_a} x^\mu = x^\mu + \theta^a T_a x^\mu + O(\theta^2). \tag{1.1}$$

For a field $\phi$ in a representation $\rho$ of $G$ we can define a similar action:

$$(g\phi)(x) = e^{\theta^a \rho(T_a)}\phi(x) = \phi(x) + \theta^a \rho(T_a)\phi(x) + O(\theta^2). \tag{1.2}$$

One can also define a "full" transformation, where both the field and coordinates are transformed at once:

$$(g\phi)(gx) = e^{\theta^a \Delta(T_a)}\phi(x) = \phi(x) + \theta^a \Delta(T_a)\phi(x) + O(\theta^2). \tag{1.3}$$

This can be connected to the previous definitions by noticing

$$\phi(x) = \phi(g^{-1}gx) = \phi(gx) - \theta^a (T_a x^\mu)\partial_\mu \phi(gx) + O(\theta^2), \tag{1.4}$$

which implies that, at first order,

$$\begin{aligned}(g\phi)(gx) - \phi(gx) &= \theta^a \Delta(T_a)\phi(x) - \theta^a (T_a x^\mu)\partial_\mu \phi(gx) + O(\theta^2) \\ &= \theta^a \Delta(T_a)\phi(gx) - \theta^a (T_a x^\mu)\partial_\mu \phi(gx) + O(\theta^2),\end{aligned} \tag{1.5}$$

where in the second line we used the fact that $\theta^a \Delta(T_a)[\phi(gx) - \phi(x)] = O(\theta^2)$. Thus, comparing with (1.2) leads to

$$\rho(T_a)\phi(x) = \Delta(T_a)\phi(x) - (T_a x^\mu)\partial_\mu \phi(x) \tag{1.6}$$

Let us now compute how the action $S$,

$$S = \int d^d x \mathcal{L}(\phi(x), \partial_\mu \phi(x)), \tag{1.7}$$

transforms under $g$. For simplicity of notation we define

$$x'^\mu \equiv g x^\mu, \quad \phi'(x') \equiv (g\phi)(gx), \quad \mathcal{F}(\phi(x)) \equiv e^{\theta^a \Delta(T_a)}\phi(x). \tag{1.8}$$

Hence, under $g$, we have

$$\begin{aligned}S' = \int d^d x' \mathcal{L}(\phi'(x'), \partial'_\mu \phi'(x')) &= \int d^d x' \mathcal{L}(\mathcal{F}(\phi(x)), \partial'_\mu \mathcal{F}(\phi(x))) \\ &= \int d^d x \left|\frac{\partial x'}{\partial x}\right| \mathcal{L}\left(\mathcal{F}(\phi(x)), \frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu \mathcal{F}(\phi(x))\right).\end{aligned} \tag{1.9}$$

Now at first order, the determinant of the Jacobian can be approximated by

$$\det(1+M) \approx 1 + \operatorname{Tr} M \Rightarrow \left|\frac{\partial x'}{\partial x}\right| = 1 + \partial_\mu \left(\theta^a T_a x^\mu\right) + O(\theta^2). \tag{1.10}$$

Therefore, at first order in $\theta$ (assuming $\theta$ depends on $x$),

$$\begin{aligned}S' = \int d^d x \, &(1 + \partial_\mu \left(\theta^a T_a x^\mu\right)) \\ &\times \mathcal{L}\left(\phi(x) + \theta^a \Delta(T_a)\phi(x), \left(\delta^\nu_\mu - \partial_\mu \left(\theta^a T_a x^\nu\right)\right)\partial_\nu \left(\phi(x) + \theta^a \Delta(T_a)\phi(x)\right)\right) \\ &= \mathcal{L} - j^\mu_a \partial_\mu \theta^a + \{\cdots\}\theta^a,\end{aligned} \tag{1.11}$$

where

$$j^\mu_a = \left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\partial_\nu \phi - \delta^\mu_\nu \mathcal{L}\right] T_a x^\nu - \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\Delta(T_a)\phi(x). \tag{1.12}$$

Assuming the action is symmetric under rigid transformations $g$, then terms multiplying $\theta^a$ add up to zero . Meaning that if the action obeys the equations of motion, then the change in the action must be

$$0 = S' - S = -\int d^d x j^\mu_a \partial_\mu \theta^a = \int d^d x \theta^a \partial_\mu j^\mu_a \tag{1.13}$$

for arbitrary $\theta^a(x)$. In other words, a continuous symmetry implies a conserved current $j$

$$\partial_\mu j^\mu_a = 0. \tag{1.14}$$

In terms of correlators, symmetries can be expressed in terms of the so-called Ward identities. Consider the correlator

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle = \frac{1}{Z}\int \mathcal{D}\phi\ \phi(x_1)\cdots\phi(x_n)e^{-S[\phi]}. \tag{1.15}$$

Applying an infinitesimal transformation to the field

$$\phi(x) \to \phi'(x) = \phi(x) + \theta^a \rho(T_a)\phi(x) \tag{1.16}$$

implies

$$\begin{aligned}
\langle \phi(x_1)\cdots\phi(x_n)\rangle &= \frac{1}{Z}\int \mathcal{D}\phi'\ \phi'(x_1)\ldots\phi'(x_n)e^{-S[\phi']} \\
= \frac{1}{Z}\int \mathcal{D}\phi\{\phi(x_1)&+\theta^a\rho(T_a)\Phi(x_1)\}\cdots\{\phi(x_n)+\theta^a\rho(T_a)\phi(x_n)\}e^{-S[\phi]-\int d^d x\theta^a(x)\partial_\mu j^\mu_a} \\
&\approx \langle \phi(x_1)\cdots\phi(x_n)\rangle + \sum_i \theta^a(x_i)\langle\phi(x_1)\cdots\rho(T_a)\phi(x_i)\ldots\phi(x_n)\rangle \\
&\quad - \int d^d x \theta^a(x)\partial_\mu\langle j^\mu_a \phi(x_1)\cdots\phi(x_n)\rangle.
\end{aligned} \tag{1.17}$$

In the first identity we relabeled the field variable, in the second identity we used (1.13) and assumed the measure is invariant $\mathcal{D}\phi' = \mathcal{D}\phi$. Expressing the summation as an integral

$$\begin{aligned}
\sum_i \theta^a(x_i)\langle\phi(x_1)\cdots\rho(T_a)\phi(x_i)\cdots\phi(x_n)\rangle = \\
\int d^d x \theta^a(x)\sum_i\langle\phi(x_1)\cdots\rho(T_a)\phi(x_i)\cdots\phi(x_n)\rangle\delta(x-x_i),
\end{aligned} \tag{1.18}$$

leads to the Ward identity

$$\partial_\mu \langle j_a^\mu \phi(x_1) \cdots \phi(x_n) \rangle = \sum_i \langle \phi(x_1) \cdots \rho(T_a)\phi(x_i) \cdots \phi(x_n) \rangle \delta(x - x_i). \quad (1.19)$$

Integrating this expression provides a way to compute how quantum field operators transform under symmetry generators, i.e. the quantum version of (1.3). Let $V$ be a volume containing one and only one field of the correlator, say $\phi(x_1)$, then

$$\begin{aligned} \int_V d^d x \partial_\mu \langle j_a^\mu \phi(x_1) \cdots \phi(x_n) \rangle &= \langle Q_a[\partial V] \phi(x_1) \cdots \phi(x_n) \rangle \\ &= \langle (\rho(T_a)\phi(x_1)) \cdots \phi(x_n) \rangle, \quad (1.20) \end{aligned}$$

where we defined

$$Q_a[\partial V] = \int_{\partial V} ds_\mu j_a^\mu. \quad (1.21)$$

As long as $\partial V$ does not cross other points, we are free to deform it. Let us take $\partial V$ to be a thin box bounded by

$$t_- < t_1 < t_+, \quad t_1 = x_1^0 \quad (1.22)$$

and spatial infinity. We can then write the surface as the difference of two "time" slices $\partial V = \partial V_+ - \partial V_-$. Writing the correlator as a time-ordered expectation value

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \langle 0 | T\{\phi(t_1, x_1) \cdots \phi(t_n, x_n)\} | 0 \rangle, \quad (1.23)$$

implies that the difference $Q_a[\partial V_+] - Q_a[\partial V_-]$ becomes a commutator $[Q_a, \phi(x_1)]$. More precisely,

$$\begin{aligned} \langle 0 | T\{[Q_a, \phi(t_1, x_1)] \cdots \phi(t_n, x_n)\} | 0 \rangle &= \langle (Q_a[\partial V_+] - Q_a[\partial V_-]) \phi(x_1) \cdots \phi(x_n) \rangle \\ &= \langle (\rho(T_a)\phi(x_1)) \cdots \phi(x_n) \rangle \\ &= \langle 0 | T\{(\rho(T_a)\phi(t_1, x_1)) \cdots \phi(t_n, x_n)\} | 0 \rangle, \quad (1.24) \end{aligned}$$

where in the second line we used (1.20). Therefore

$$[Q_a, \phi(x)] = \rho(T_a)\phi(x). \quad (1.25)$$

Note that this result is independent of our choice of quantization (1.22) since we will always end up enclosing $x_1$, regardless of which coordinate we pick as time. Exponentiating this result, shows how the field transformation (1.2) translates into operator valued fields:

$$(g\phi)(x) = e^{\theta^a Q_a} \phi(x) e^{-\theta^a Q_a} = e^{\theta^a \rho(T_a)} \phi(x). \quad (1.26)$$

To be more explicit about the relation (1.25), let us write for a moment the $Q_a$ as $Q(\rho(T_a))$. Then, via the Jacobi identity and (1.25), we have a relation between commutators of $Q$ and commutators of $\rho(T)$: [1]

$$[Q(\rho(T_a^1)), Q(\rho(T_b^2))] = Q(-[\rho(T_a^1), \rho(T_b^2)]). \tag{1.27}$$

Another way of integrating the Ward identities, gives differential equations for correlators. Integrating over the whole volume, the left hand side of (1.19) vanishes by Stokes' theorem, assuming the $\langle j_a^\mu \cdots \rangle$ goes to zero at infinity. This implies then that

$$\sum_i \langle \phi(x_1) \cdots \rho(T_a)\phi(x_i) \cdots \phi(x_n)\rangle = 0. \tag{1.28}$$

Not that this is the infinitesimal version of

$$\begin{aligned} \langle \phi(x'_1) \cdots \phi(x'_n)\rangle &= \frac{1}{Z}\int \mathcal{D}\phi' \; \phi'(x'_1)\dots\phi'(x'_n)e^{-S[\phi']} \\ &= \frac{1}{Z}\int \mathcal{D}\phi \; \mathcal{F}(\phi(x_1))\dots\mathcal{F}(\phi(x_n))e^{-S[\phi]} \\ &= \langle \mathcal{F}(\phi(x_1)) \cdots \mathcal{F}(\phi(x_n))\rangle, \end{aligned} \tag{1.29}$$

where in the first identity we relabeled the field variable, in the second identity we assumed $\mathcal{D}\phi' = \mathcal{D}\phi$ and that the action is symmetric.

### 1.1.2 Conformal transformations and the algebra of generators

Consider $d$-dimensional coordinates $x^\mu$, $\mu = 1, \dots, d$, with metric tensor $\eta^{\mu\nu}$. A conformal transformation is a diffeomorphism

$$x^\mu \to x'^\mu = e^{\theta^a T_a} x^\mu, \tag{1.30}$$

which rescales the metric by a position dependent factor

$$\eta_{\mu\nu} \to \Lambda(x)\eta_{\mu\nu}. \tag{1.31}$$

At first order in $\theta$, the coordinate transformation (1.30) induces the following change in the metric:

$$\eta_{\mu\nu} \to \eta'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\eta_{\rho\sigma} = \eta_{\mu\nu} - (\eta_{\rho\nu}\partial_\mu + \eta_{\rho\mu}\partial_\nu)\,\theta^a T_a x^\rho + O(\theta^2). \tag{1.32}$$

[1] Note that the repeated action of $Q$, reverses the order of $\rho(T)$; $[Q(\rho(T_a^1)), [Q(T_b^2), \phi(x)]] = \rho(T_b^2)\rho(T_a^1)\phi(x)$. This is the reason why there is a negative sign on the right hand side of (1.27).

It is easy to check that (1.31) then implies

$$\left(\eta_{\rho\nu}\partial_\mu + \eta_{\rho\mu}\partial_\nu - \frac{2}{d}\eta_{\mu\nu}\partial_\rho\right)\theta^a T_a x^\rho = 0. \tag{1.33}$$

The most generic solution to this equation is given by

$$\theta^a T_a x^\mu = \alpha^\mu + \omega^\mu{}_\nu x^\nu + \gamma x^\mu + 2(\beta\cdot x)x^\mu - \beta^\mu x^2, \tag{1.34}$$

where the parameters $\alpha$, $\omega$, $\gamma$, and $\beta$ correspond to translations, rotations, dilatations, and special conformal transformations (SCT), respectively. Comparing this expression with (1.1) we find the form of $\theta^a$ and $T_a$: [2]

$$\theta^a = \{\alpha^\mu, \frac{1}{2}\omega^{\mu\nu}, \gamma, \beta^\mu\}, \tag{1.35}$$

$$T_a = \{p_\mu, m_{\mu\nu}, d, k_\mu\}. \tag{1.36}$$

where

$$\begin{aligned} p_\mu &= \partial_\mu, \quad m_{\mu\nu} = x_\nu\partial_\mu - x_\mu\partial_\nu, \\ d &= x^\mu\partial_\mu, \quad k_\mu = 2x_\mu x\cdot\partial - x^2\partial_\mu. \end{aligned} \tag{1.37}$$

Exponentiating (1.34) for each different parameter produces the finite version of the conformal transformations:

$$\begin{aligned} \text{translation}: &\quad x^\mu \to x^\mu + a^\mu \\ \text{rotation}: &\quad x^\mu \to \Lambda^\mu{}_\nu x^\nu \\ \text{dilatation}: &\quad x^\mu \to \lambda x^\mu \\ \text{SCT}: &\quad x^\mu \to \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2x^2}. \end{aligned} \tag{1.38}$$

Given the form of $T_a$ we can compute $\rho(T_a)$ using (1.6). For simplicity let us assume that the field does not change under the full transformation (1.3), i.e. $(g\phi)(gx) = \phi(x)$, then

$$\rho(T_a) = \{P_\mu, M_{\mu\nu}, D, K_\mu\}, \tag{1.39}$$

where

$$\begin{aligned} P_\mu &= -\partial_\mu, \quad M_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu, \\ D &= -x^\mu\partial_\mu, \quad K_\mu = -2x_\mu x\cdot\partial + x^2\partial_\mu. \end{aligned} \tag{1.40}$$

[2] Note that the $\frac{1}{2}$ factor next to $\omega$ is to compensate for the double counting when contracting both indices.

With these results we obtain the conformal algebra:

$$\begin{aligned} [D, P_\mu] &= P_\mu, \quad [D, K_\mu] = -K_\mu, \\ [K_\mu, P_\nu] &= 2\eta_{\mu\nu} D - 2M_{\mu\nu}, \\ [P_\rho, M_{\mu\nu}] &= \eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu, \quad [K_\rho, M_{\mu\nu}] = \eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu, \\ [M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} + \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\sigma} M_{\mu\rho}. \end{aligned} \tag{1.41}$$

Note that different conventions of the conformal algebra correspond to different conventions for the transformation (1.2). For example, the conventions of [12] are recovered by sending the generators $\rho(T_a)$ defined here to $-i\rho(T_a)$.

The field generators (1.40) can be mapped to the generators $J_{AB}$, $A = 1, \cdots, d, d+1, d+2$ of $SO(d+1, 1)$ via

$$\begin{aligned} M_{\mu\nu} &= J_{\mu\nu}, \quad P_\mu = -J_{d+1\mu} + J_{d+2\mu}, \\ K_\mu &= -J_{d+1\mu} - J_{d+2\mu}, \quad D = J_{d+1d+2}, \end{aligned} \tag{1.42}$$

satisfying

$$[J_{AB}, J_{CD}] = \eta_{BC} J_{AD} - \eta_{AC} J_{BD} + \eta_{AD} J_{BC} - \eta_{BD} J_{AC}, \tag{1.43}$$

with metric $\eta_{AB} = \text{diag}(1, 1, \ldots, 1, -1)$. This map will be very useful later.

### 1.1.3 Conformal representations and correlators

Recall that the symmetry generators in (1.40) are valid for fields satisfying $(g\phi)(gx) = \phi(x)$. However, for more generic fields, these results are modified according to (1.6). Our aim is to compute (1.25) for each conformal generator.[3] Let us write the charge $Q_a$, associated to $\rho(T_a)$, with the same symbol but with a hat on top, e.g. $\rho(T_a) = M_{\mu\nu} \to Q_a = \hat{M}_{\mu\nu}$. A trick to compute $[Q_a, \phi(x)]$ is to study the case $[Q_a, \phi(0)]$ and then use $\hat{P}$ to translate back to an arbitrary position. More precisely, using (1.26), it is easy to check that

$$\begin{aligned} [Q_a, \phi(x)] &= [Q_a, e^{-x^\mu \hat{P}_\mu} \phi(0) e^{x^\mu \hat{P}_\mu}] = e^{-x^\mu \hat{P}_\mu} [e^{x^\mu \hat{P}_\mu} Q_a e^{-x^\mu \hat{P}_\mu}, \phi(0)] e^{x^\mu \hat{P}_\mu} \\ &= e^{-x\cdot\hat{P}} \left[ Q_a + [x\cdot\hat{P}, Q_a] + \frac{1}{2}[x\cdot\hat{P}, [x\cdot\hat{P}, Q_a]] + \cdots, \phi(0) \right] e^{x\cdot\hat{P}}, \end{aligned} \tag{1.44}$$

where in the second line we used Hausdorff formula

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \cdots. \tag{1.45}$$

[3] Note that in the literature, one finds that the commutator notation $[Q, \phi]$ is often exchanged by $Q\phi$. One can understand the latter as surrounding $\phi$ with the charge $Q$ in the path integral, which is equivalent to the commutator notation as discussed in subsection 1.1.1.

Recall that the commutators between $Q$ are given in (1.27).

For rotations, let us suppose $\phi$ is in an irreducible representation of $SO(d)$ with matrix generators $S_{\mu\nu}$, then

$$[\hat{M}_{\mu\nu}, \phi(0)] = -S_{\mu\nu}\phi(0), \tag{1.46}$$

where the form of $S$ depends on the representation of $\phi$; for example for a spin $\ell$ tensor

$$\begin{aligned}(S_{\mu\nu})^{b_1,\cdots,b_\ell}_{a_1,\cdots,a_\ell} = (-\eta_{\mu a_1}\delta^{b_1}_{\nu} + \eta_{\nu a_1}\delta^{b_1}_{\mu})\delta^{b_2}_{a_2}\dots\delta^{b_\ell}_{a_\ell} + \dots \\ + \delta^{b_1}_{a_1}\dots\delta^{b_{\ell-1}}_{a_{\ell-1}}(-\eta_{\mu a_\ell}\delta^{b_\ell}_{\nu} + \eta_{\nu a_\ell}\delta^{b_\ell}_{\mu}).\end{aligned} \tag{1.47}$$

For dilatations and SCT, let us suppose[4]

$$[\hat{D}, \phi(0)] = -\Delta\phi(0), \quad [\hat{K}_\mu, \phi(0)] = -\mathcal{K}_\mu\phi(0), \tag{1.48}$$

respectively. The algebra between $-\Delta$ , $-\mathcal{K}_\mu$, $-S_{\mu\nu}$ must be the same as the full algebra (1.41) with the translation generator removed. In particular

$$[\Delta, S_{\mu\nu}] = 0, \quad [\Delta, \mathcal{K}_\mu] = \mathcal{K}_\mu \tag{1.49}$$

Because $\phi$ is in an irreducible representation of $SO(d)$ and $\Delta$ commutes with $S_{\mu\nu}$, then $\Delta$ must be proportional to the identity. Furthermore, the commutator with $\mathcal{K}_\mu$ implies $\mathcal{K}_\mu = 0$. This representation is called conformal primary and is labeled by the $SO(d)$ Dynkin labels and the conformal dimension $\Delta$.

Let us assume for a moment that $\phi(0)$ is not a primary, i.e. $[\hat{K}_\mu, \phi(0)] \neq 0$. This implies that

$$[\hat{D}, [\hat{K}_\mu, \phi(0)]] = [[\hat{D}, \hat{K}_\mu], \phi(0)] + [\hat{K}_\mu, [\hat{D}, \phi(0)]] = -(-1 + \Delta)[\hat{K}_\mu, \phi(0)]. \tag{1.50}$$

Hence $K$ acts like a lowering operator for the conformal dimension. Given that physical theories have bounded conformal dimensions (more on that later), acting repeatedly with $K$ must end when we hit a primary operator. On the other hand, by a similar calculation,

$$[\hat{D}, [\hat{P}_\mu, \phi(0)]] = -(1 + \Delta)[\hat{P}_\mu, \phi(0)], \tag{1.51}$$

which implies that $P$ rises the dimension. Operators of the form $P_{\mu_1}P_{\mu_2}\cdots\phi(0)$ are called descendants of $\phi(0)$. A conformal multiplet is then the collection of a primary with all its descendants.

[4] The negative sign on these commutators is because, as seen in (1.40), the action on the fields is inverse to that of the coordinates (1.37).

Putting the previous results together, leads to the action of the quantum generators on primaries $\phi(x)$:

$$[\hat{P}_\mu, \phi(x)] = -\partial_\mu \phi(x), \tag{1.52}$$
$$[\hat{M}_{\mu\nu}, \phi(x)] = (x_\mu \partial_\nu - x_\nu \partial_\mu)\phi(x) - S_{\mu\nu}\phi(x), \tag{1.53}$$
$$[\hat{D}, \phi(x)] = -x \cdot \partial\phi(x) - \Delta\phi(x), \tag{1.54}$$
$$[\hat{K}_\mu, \phi(x)] = (-2x_\mu x^\nu + x^2 \delta_\mu^\nu)\partial_\nu \phi(x) + 2x^\nu(-\Delta\delta_{\mu\nu} + S_{\mu\nu})\phi(x). \tag{1.55}$$

Note that the right hand side of these expressions gives the general version of $\rho(T_a)$ in (1.40), c.f. (1.25). Using (1.6), we extract the action of $\Delta(T_a) = \{\tilde{P}_\mu, \tilde{M}_{\mu\nu}, \tilde{D}, \tilde{K}_\mu\}$ on the fields:

$$\begin{aligned} \tilde{P}_\mu = 0, \quad \tilde{M}_{\mu\nu} = -S_{\mu\nu}, \\ \tilde{D} = -\Delta, \quad \tilde{K}_\mu = 2x^\nu(-\Delta\delta_{\mu\nu} + S_{\mu\nu}). \end{aligned} \tag{1.56}$$

Using these on (1.3) along with the finite coordinate transformations (1.38) allows us to write the finite version of the full field transformation:

$$(g\phi)(gx) = \Omega(x)^{-\Delta}\mathcal{R}(\Sigma(x)^{-1})\phi(x), \tag{1.57}$$

where

$$\frac{\partial g x^\mu}{\partial x^\nu} = \Omega(x)\Sigma^\mu{}_\nu(x), \quad \Sigma^T\Sigma = 1, \quad \Sigma^\mu{}_\nu(x) \in SO(d), \tag{1.58}$$

and $\mathcal{R}$ is the $SO(d)$ representation acting on the field indices. For completeness, the scale and rotation factors for each transformation are

$$\Omega(x) = \begin{cases} 1 & \text{translation} \\ 1 & \text{rotation} \\ \lambda & \text{dilatation} \\ \frac{1}{1-2b\cdot x + b^2x^2} & \text{SCT} \end{cases} \tag{1.59}$$

$$\Sigma^\mu{}_\nu(x) = \begin{cases} \delta^\mu_\nu & \text{translation} \\ \Lambda^\mu{}_\nu & \text{rotation} \\ \delta^\mu_\nu & \text{dilatation} \\ \delta^\mu_\nu - 2b^\mu x_\nu + 2\Omega(x)(x^\mu - x^2b^\mu)(b_\nu - b^2x_\nu) & \text{SCT} \end{cases} \tag{1.60}$$

Using the previous results for $\rho(T_a)$ in (1.28), leads to a set of differential equations for the conformal correlators of primaries:

$$\begin{aligned} \sum_i \frac{\partial}{\partial x_i^\mu} \langle \phi(x_1) \cdots \phi(x_n) \rangle = 0, \\ \sum_i \left( x_{i\mu} \frac{\partial}{\partial x_i^\nu} - x_{i\nu} \frac{\partial}{\partial x_i^\mu} - S^i_{\mu\nu} \right) \langle \phi(x_1) \cdots \phi(x_n) \rangle = 0, \\ \sum_i \left( -x_i^\mu \frac{\partial}{\partial x_i^\mu} - \Delta_i \right) \langle \phi(x_1) \cdots \phi(x_n) \rangle = 0, \\ \sum_i \left( (-2x_{i\mu} x_i{}^\nu + x_i^2 \delta_\mu^\nu) \frac{\partial}{\partial x_i^\nu} + 2x_i^\nu (-\Delta_i \delta_{\mu\nu} + S^i_{\mu\nu}) \right) \langle \phi(x_1) \cdots \phi(x_n) \rangle = 0, \end{aligned} \tag{1.61}$$

whereas the finite constraints are given by (1.29) and (1.57)

$$\langle \phi(x_1') \cdots \phi(x_n') \rangle = \left( \prod_i \Omega(x_i)^{-\Delta_i} \mathcal{R}_i \left( \Sigma(x_i)^{-1} \right) \right) \langle \phi(x_1) \cdots \phi(x_n) \rangle \tag{1.62}$$

### 1.1.4 The stress tensor

Using (1.12) we can compute the currents associated to conformal transformations via (1.37) and (1.56):

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\nu \phi(x) - \delta^\mu_\nu \mathcal{L}, \tag{1.63}$$

$$(M_{\mu\nu})^\rho = x_\nu T^\rho_\mu - x_\mu T^\rho_\nu + \frac{\partial \mathcal{L}}{\partial \partial_\rho \phi} S_{\mu\nu} \phi(x), \tag{1.64}$$

$$(D)^\rho = x^\nu T^\rho_\nu + \frac{\partial \mathcal{L}}{\partial \partial_\rho \phi} \Delta \phi(x), \tag{1.65}$$

$$(K_\mu)^\rho = 2x_\mu x^\nu T^\rho_\nu - x^2 T^\rho_\mu + 2x^\nu \frac{\partial \mathcal{L}}{\partial \partial_\rho \phi} \left( \Delta \delta_{\mu\nu} - S_{\mu\nu} \right) \phi(x). \tag{1.66}$$

The last term in $(M_{\mu\nu})^\rho$ can be removed in a consistent way,[5] which makes the stress tensor symmetric under the exchange of indices. In some cases, the last term $(D)^\rho$ can also be removed in a similar way [12], implying that conformal invariance is a consequence of scale invariance and Poincare symmetry. As shown later, removing the second term of $(D)^\rho$ makes the stress tensor traceless, which

[5]This is done by adding the divergence of the so-called Belifante tensor $B$ to the definition of the stress-tensor: $T^{\mu\nu} \to T^{\mu\nu} + \partial_\rho B^{\rho\mu\nu}$, where $B$ is antisymmetric in the first two indices. The reason we can do this is that both the classical conservation law and the Ward identity are unchanged. Then it is easy to find a suitable expression for $B$ which cancels the last term of $(M_{\mu\nu})^\rho$. See [12] for a full derivation.

in turn implies conformal invariance. This is easily seen from equation (1.13) and (1.33):

$$\int d^dxT^{\mu\nu}\partial_\mu\theta_\nu = \frac{1}{2}\int d^dxT^{\mu\nu}(\partial_\mu\theta_\nu + \partial_\nu\theta_\mu) = \frac{1}{d}\int d^dxT^\mu_\mu\partial\cdot\theta = 0. \tag{1.67}$$

See [14–21] for recent developments on the topic of how scale invariance is related to conformal invariance.

For the moment, let us assume that

$$(M_{\mu\nu})^\rho = x_\nu T^\rho_\mu - x_\mu T^\rho_\nu, \tag{1.68}$$

$$(D)^\rho = x^\nu T^\rho_\nu. \tag{1.69}$$

Then, from the point of view of the Ward identity (1.19), classical conservation, symmetry, and tracelessness conditions for the stress tensor are promoted to

$$\frac{\partial}{\partial x^\mu}\langle T^\mu_\nu\cdots\rangle = -\sum_i \delta(x-x_i)\frac{\partial}{\partial x^\nu_i}\langle\cdots\rangle, \tag{1.70}$$

$$\frac{\partial}{\partial x^\rho}\langle\left(x_\nu T^\rho_\mu - x_\mu T^\rho_\nu\right)\cdots\rangle = \langle(T_{\mu\nu}-T_{\nu\mu})\cdots\rangle = -\sum_i\delta(x-x_i)S^i_{\mu\nu}\langle\cdots\rangle, \tag{1.71}$$

$$\frac{\partial}{\partial x^\rho}\langle x^\nu T^\rho_\nu\cdots\rangle = \langle T^\mu_\mu\cdots\rangle = -\sum_i\delta(x-x_i)\Delta_i\langle\cdots\rangle, \tag{1.72}$$

where $\cdots$ represents $\phi(x_1)\cdots\phi(x_n)$.

## 1.2 Embedding space and polynomial encoding

The map (1.42) suggests that there exists a coordinate map (embedding) where the conformal group acts linearly, as $SO(d+1,1)$, in $d+2$ dimensional space. Thus when computing conformal transformations, instead of applying each type of transfomration in (1.38) separately, one applies a single $SO(d+1,1)$ rotation followed by a simple coordinate constraint, as explained later. As we will see below, this is useful when calculating correlators given that the conformal constraints (1.62) simplify greately when the symmetry group is $SO(d+1,1)$. Furthermore, contracting coordinate indices with arbitrary polarization vectors results in a simpler framework for manipulating tensorial quantities (note that this last technique is not unique to embedding formalism, but the combination results in a very efficient framework for doing conformal field theory).

The main idea of the embedding formalism, first proposed by [22], is to map the $\mathbb{R}^d$ coordinates $x^\mu$, $\mu=1,\ldots,d$ into the coordinates $P^A$, $A=1,\ldots,d,d+1,d+2$

living in a subspace $\mathcal{M}^{d+1,1} \subset \mathbb{R}^{d+1,1}$. In order for this to work, we need to impose that the action of $SO(d+1,1)$ on $\mathcal{M}^{d+1,1}$ is closed. A subspace of $R^{d+1,1}$ that satisfies such condition is the light-cone $P^2 = 0$. However, this is still $d+1$ dimensional. To reduce one dimension further, we can look at a fixed section of the cone $P^+ = f(P^\mu)$, where we defined light-cone coordinates

$$P^\pm = P^{d+2} \pm P^{d+1}. \tag{1.73}$$

But clearly, $SO(d+1,1)$ transformations will send points within that section to a different one in general. Thus the correct subspace $\mathcal{M}^{d+1,1}$ is the space of null rays

$$P^2 = 0, \quad P \sim \lambda P, \tag{1.74}$$

and the map is

$$x^\mu = P^\mu, \quad x^2 = P^+ P^-. \tag{1.75}$$

To check that the action of $SO(d+1,1)$ on $\mathcal{M}^{d+1,1}$, indeed corresponds to a conformal transformation in $\mathbb{R}^d$, we can use a nice trick from [11]: First notice that the metric on a fixed section is, by (1.75),

$$dP^A dP_A = dP^\mu dP_\mu - dP^+ dP^- \Big|_{P^\mu = x^\mu, P^+ = f(x), P^- = x^2/P^+}. \tag{1.76}$$

This is invariant under the action of $SO(d+1,1)$. However, to stay in $P^+ = f(x)$ we need to apply a position dependent scaling $P \to \lambda(P)P$. The metric then changes to

$$dP^A dP_A \to \frac{\partial \lambda(P) P^A}{\partial P^B} \frac{\partial \lambda(P) P_A}{\partial P^C} dP^B dP^C = \lambda(P)^2 dP^A dP_A, \tag{1.77}$$

provided $P^2 = 0$ (and consequently $P \cdot dP = 0$). If $f(P) = \text{const}$ then $dP^2 = dx^2$, and so the result above is just the definition of a conformal transformation (1.31). Without loss of generality, we fix the section to $f(P) = 1$, so that

$$P^\mu = x^\mu, \quad P^{d+1} = \frac{1 - x^2}{2}, \quad P^{d+2} = \frac{1 + x^2}{2}. \tag{1.78}$$

Now consider lifting the field $\phi(x)$ to $\mathcal{M}^{d+1,1}$ via projection $\phi(x) = \mathcal{P}(x)\Phi(P)$.[6] Then $\phi(x)$ is a conformal field if

$$[\hat{J}_{AB}, \Phi(P)] = (P_A \partial_B - P_B \partial_A)\Phi(P) - \tilde{S}_{AB}\Phi(P), \tag{1.79}$$

[6] The embeddig of both bosonic and fermionic fields has been described in many papers [23–36].

corresponds to $\phi(x)$ satisfying (1.52)–(1.55). The map between $\hat{J}$ and the conformal generators is given in (1.42). With this definition of $\phi$, the commutators are

$$\begin{aligned}
[\hat{P}_\mu, \phi(x)] &= \mathcal{P}(-\partial_\mu + \tilde{S}_{d+1\mu} - \tilde{S}_{d+2\mu})\mathcal{P}^{-1}\phi(x), && (1.80)\\
[\hat{M}_{\mu\nu}, \phi(x)] &= \mathcal{P}(x_\mu\partial_\nu - x_\nu\partial_\mu - \tilde{S}_{\mu\nu})\mathcal{P}^{-1}\phi(x), && (1.81)\\
[\hat{D}, \phi(x)] &= \mathcal{P}(-x\cdot\partial - \tilde{\Delta} - \tilde{S}_{d+1d+2})\mathcal{P}^{-1}\phi(x), && (1.82)\\
[\hat{K}_\mu, \phi(x)] &= \mathcal{P}\left(-2x_\mu x\cdot\partial + x^2\partial_\mu - 2\tilde{\Delta}x_\mu + \tilde{S}_{d+1\mu} + \tilde{S}_{d+2\mu}\right)\mathcal{P}^{-1}\phi(x), && (1.83)
\end{aligned}$$

where we used

$$\begin{aligned}
\frac{\partial\Phi(P)}{\partial x^\mu} &= \frac{\partial\Phi(P)}{\partial P^\mu} - P_\mu\left(\frac{\partial}{\partial P^{d+1}} - \frac{\partial}{\partial P^{d+2}}\right)\Phi(P), && (1.84)\\
P^A\frac{\partial\Phi(P)}{\partial P^A} &= -\tilde{\Delta}\Phi(P). && (1.85)
\end{aligned}$$

Matching these with (1.52)–(1.55) leads to the constraint

$$P\cdot\partial\Phi(P) = -\Delta\Phi(P), \tag{1.86}$$

as well as a set of differential equations for the projector $\mathcal{P}(x)$:[7]

$$\begin{aligned}
\partial_\mu\mathcal{P} + \mathcal{P}\left(\tilde{S}_{d+1\mu} - \tilde{S}_{d+2\mu}\right) &= 0, && (1.87)\\
(x_\nu\partial_\mu - x_\mu\partial_\nu + S_{\mu\nu})\mathcal{P} - \mathcal{P}\tilde{S}_{\mu\nu} &= 0, && (1.88)\\
x\cdot\partial\mathcal{P} - \mathcal{P}\tilde{S}_{d+1d+2} &= 0, && (1.89)\\
(2x_\mu x\cdot\partial - x^2\partial_\mu - 2x^\nu S_{\mu\nu})\mathcal{P} + \mathcal{P}(\tilde{S}_{d+1\mu} + \tilde{S}_{d+2\mu}) &= 0. && (1.90)
\end{aligned}$$

For a field in the $\ell$-tensor representation, with $\tilde{S}$ given by the straightforward extension of (1.47) to embedding indices, the projector satisfying the previous equations is

$$\mathcal{P} = \frac{\partial P^{A_1}}{\partial x^{\mu_1}}\cdots\frac{\partial P^{A_\ell}}{\partial x^{\mu_\ell}}, \quad \text{where} \quad \frac{\partial P^A}{\partial x^\mu} = (\delta^\alpha_\mu, -x_\mu, x_\mu), \tag{1.91}$$

with the caveat that (1.90) is (assuming $\ell = 1$ for simplicity)

$$\begin{aligned}
&((2x_\mu x\cdot\partial - x^2\partial_\mu)\delta^\rho_\sigma - 2x^\nu(S_{\mu\nu})^\rho_{\ \sigma})\mathcal{P}^A_\rho \\
&\qquad + \mathcal{P}^B_\sigma((\tilde{S}_{d+1\mu})^A_B + (\tilde{S}_{d+2\mu})^A_B) = 2\delta_{\sigma\mu}P^A, \qquad (1.92)
\end{aligned}$$

therefore we must impose that $\Phi(P)$ is transverse on each index as well

$$P^{A_r}\Phi(P)_{A_1\cdots A_\ell} = 0, \quad \forall r = 1,\ldots,\ell. \tag{1.93}$$

[7] These equations are true when acting on $\Phi(P)$.

The elements in the projector $\mathcal{P}$ satisfy the following properties

$$\frac{\partial P^A}{\partial x^\mu} P_A = 0,$$
$$\frac{\partial P^A}{\partial x^\mu} \frac{\partial P^B}{\partial x_\mu} = \eta^{AB} + P^A \bar{P}^B + P^B \bar{P}^A, \quad \text{where} \quad \bar{P}^A = (0, -1, 1). \tag{1.94}$$

This means that the embedding of fields is not unique; tensors differing by terms proportional to $P^A$ project to the same field on $\mathbb{R}^d$. Also, the tracelessness and symmetric properties of the indices of tensors on $\mathcal{M}^{d+1,1}$ are carried over to $\mathbb{R}^d$.

### 1.2.1 $SO(d)$ representation conventions

Before moving forward to the polynomial encoding of the tensor indices, let us define our conventions for $SO(d)$ representations. Lie algebra representations are labeled by weight vectors $\lambda$ with dimension given by the rank of the algebra. A convenient basis for $\lambda$ is the so-called Dynkin basis

$$\lambda = [a_1, \ldots, a_n], \tag{1.95}$$

where the Dynkin labels $a_i$ are integers with $a_i \geq 0$ and $n$ is the rank of the algebra. The rank-$n$ algebras related to the $SO(d)$ group are

$$B_n \leftrightarrow SO(2n+1), \tag{1.96}$$
$$D_n \leftrightarrow SO(2n). \tag{1.97}$$

The fundamental weights are given by $\bar{a}_i = [0, \ldots, 0, 1, 0, \ldots, 0]$, i.e. only $a_i = 1$ and the rest are zero. These correspond to antisymmetric tensors of $i$ indices, with the following exceptions: for $B_n$, $\bar{a}_n$ is the fundamental spinor representation, and for $D_n$, the two chiral spinors are $\bar{a}_{n-1}$ and $\bar{a}_n$. For a weight $\lambda$, we define the signature $Y(\lambda)$, defined as $Y(\lambda) = (y_1, \ldots, y_n)$, where

$$y_i = \sum_{j=i}^{n} \lambda_j - \begin{cases} \frac{\lambda_n}{2} & \text{for } B_n \\ \frac{\lambda_{n-1} + (1+2\delta_{in})\lambda_n}{2} & \text{for } D_n \end{cases} \tag{1.98}$$

and $y_i \in \mathbb{Z} \ \forall \ i$ for bosonic representations, or $y_i \in \mathbb{Z} + \frac{1}{2} \ \forall \ i$ for fermionic. In this work we are concerned with bosonic representations only. Therefore we can graphically represent $Y(\lambda)$ with a Young diagram, where $y_i$ is the length of the

$i$-th row from top to bottom.[8] The relation between rows $y_i$ is

$$0 \leq y_n \leq y_{n-1} \leq \cdots \leq y_1, \quad \text{for } B_n, \tag{1.99}$$
$$0 \leq |y_n| \leq y_{n-1} \leq \cdots \leq y_1, \quad \text{for } D_n. \tag{1.100}$$

### 1.2.2 Encoding tensors with polynomials

In order to simplify calculations with tensors, it is useful to contract all indices with polarization vectors $z$. For symmetric traceless tensors it is enough to use one polarization vector

$$\phi_{\mu_1\cdots\mu_\ell}(x) \to \phi(x,z) \equiv z^{\mu_1}\cdots z^{\mu_\ell}\phi_{\mu_1\cdots\mu_\ell}(x). \tag{1.101}$$

However, for mixed-symmetric tensors, one can either introduce Grassmann polarisations to implement the antisymmetrisation between rows of the Young diagram [29], or use different vectors $z^i$ and implement the antisymmetrisation manually [37]. Here we follow the latter approach. For a field in the representation $\rho = (\ell_1, \ldots, \ell_n)$, we define

$$\begin{aligned}\phi_{\mu_1\cdots\mu_{\ell_1}\mu_{\ell_1+1}\cdots\mu_{|\rho|}}(x) &\to \phi(x, z^1, z^2, \ldots z^n) \\ &\equiv (z^1)^{\mu_1}\cdots(z^1)^{\mu_{\ell_1}}(z^2)^{\mu_{\ell_1+1}}\cdots(z^n)^{\mu_{|\rho|}}\phi_{\mu_1\cdots\mu_{\ell_1}\mu_{\ell_1+1}\cdots\mu_{|\rho|}}(x),\end{aligned} \tag{1.102}$$

where $|\rho| = \sum_i \ell_i$. Now irreducible representations must be traceless, which implies

$$\delta^{\alpha\beta}\frac{\partial^2}{\partial(z^i)^\alpha\partial(z^j)^\beta}\phi(x,z^1,\ldots,z^n) \propto \text{trace of } \phi = 0, \quad \forall\ i,j. \tag{1.103}$$

Therefore we can ignore terms $O(z^i \cdot z^j)$ in $\phi(x, z^1, \ldots)$, or equivalently, set $z^i \cdot z^j = 0$.

Now the mixed-symmetric properties of $\rho$ imply that symmetrizing any of the indices of row $j$ with all indices of row $i$ vanishes, whenever $i < j$ [29,37]. In other words,

$$z^i \cdot \frac{\partial}{\partial z^j}\phi(x, z^1, \ldots) = 0, \quad \forall\ i < j. \tag{1.104}$$

[8] Note that diagrams with a number of rows equal to $n$ may or may not be irreducible. For example, in $D_n$, diagrams of height $n$ split into self-dual and anti-self-dual representations ($y_n > 0$ and $y_n < 0$ respectively). Unless otherwise specified, we will assume that $d$ is sufficiently large so that Young diagrams are irreducible.

In embedding space, we promote $z^i_\mu \to Z^i_A$, so that $\phi(x, z^1, \cdots) \to \Phi(P, Z^1, \cdots)$. Recall that terms proportional to $P^A$ project to zero, therefore

$$Z^i \cdot P = 0, \quad \forall i. \tag{1.105}$$

Note that we can project an encoded polynomial on $\mathcal{M}^{d+1,1}$ to the corresponding encoded polynomial on $\mathbb{R}^d$, by choosing a particular value of $Z^A$. More explicitly, from the projector (1.91), we have [9]

$$Z^A = z^\mu \frac{\partial P^A}{\partial x^\mu} = (z^\alpha, -z \cdot x, z \cdot x), \tag{1.106}$$

which implies

$$\phi(x, z^1, \dots) = \Phi(P, Z^1, \dots)\big|_{P^A = \left(x^\alpha, \frac{1-x^2}{2}, \frac{1+x^2}{2}\right), (Z^i)^A = ((z^i)^\alpha, -z^i \cdot x, z^i \cdot x)}. \tag{1.107}$$

It is easy to check that for this definition of $Z^A$, the traceless condition carries over

$$Z^i \cdot Z^j = 0. \tag{1.108}$$

In summary, a field $\phi(x, z^1, \dots, z^n)$ in the conformal representation $[\Delta, \rho = (\ell_1, \dots, \ell_n)]$ is lifted to a field $\Phi(P, Z^1, \dots, Z^n)$, where

$$P^2 = Z^i \cdot P = Z^i \cdot Z^j = 0, \quad \forall i, j, \tag{1.109}$$

and the field satisfies the following differential equations

$$P \cdot \frac{\partial}{\partial P} \Phi = -\Delta \Phi, \tag{1.110}$$

$$Z^i \cdot \frac{\partial}{\partial Z^i} \Phi = \ell_i \Phi, \quad \forall i, \tag{1.111}$$

$$P \cdot \frac{\partial}{\partial Z^i} \Phi = 0, \quad \forall i, \tag{1.112}$$

$$Z^i \cdot \frac{\partial}{\partial Z^j} \Phi = 0, \quad \forall i < j. \tag{1.113}$$

Note that these are equivalent to the following field transformation rule

$$\begin{aligned} \Phi(\lambda P, \alpha^1 Z^1 + \beta^1 P, \alpha^2 Z^2 + \beta^2 P + \gamma^{2,1} Z^1, \alpha^3 Z^3 + \beta^3 P + \gamma^{3,1} Z^1 + \gamma^{3,2} Z^2, \dots) \\ = \lambda^{-\Delta} \left( \prod_{i=1}^{n} (\alpha^i)^{\ell_i} \right) \Phi(P, Z^1, Z^2, \dots). \end{aligned} \tag{1.114}$$

[9] Here the coordinates are labeled $X^A = (X^\alpha, X^{d+1}, X^{d+2})$.

## 1.3 Correlation functions in embedding space

Recall that general correlation functions transform as (1.29), where the specific case for conformal symmetry is given in (1.62). As seen in the previous section, conformal primaries are encoded by $SO(d+1,1)$ tensors transforming as (1.114). Therefore, the embedding version of (1.62) for primaries $[\Delta_i, \rho_i]$, $i = 1, \ldots, m$ is

$$G\left(\lambda_i P_i; \alpha_i^j Z_i^j + \beta_i^j P_i + \gamma_i^{j,k} Z_i^k\big|_{0<k<j}\right) = \left(\prod_{i=1}^{m} \lambda_i^{-\Delta_i} \prod_{j=1}^{h(\rho_i)} (\alpha_i^j)^{(\rho_i)_j}\right) G(P_i; Z_i^j), \tag{1.115}$$

where $h(\rho_i)$ is the number of rows in the Young digram $\rho_i$, $(\rho_i)_j$ represents the number of boxes in the $j$-th row of $\rho_i$, and

$$\begin{aligned} G(P_i; Z_i^j) &\equiv \langle \Phi(P_1, Z_1^j)\Phi(P_2, Z_2^j)\cdots\Phi(P_m, Z_m^j)\rangle, \\ \Phi(P_i, Z_i^j) &\equiv \Phi(P_i, Z_i^1, Z_i^2, \ldots Z_i^{h(\rho_i)}). \end{aligned} \tag{1.116}$$

### 1.3.1 Scalar constraints

Let us suppose for a moment that all fields in $G$ are scalar $[\Delta_i, (0)]$. Then $G$ must only be made out of terms $P_{ij} \equiv -2P_i \cdot P_j$,[10] with $i \neq j$. Then (1.115) has the following solutions for $m = 2, 3, 4$:

$$c_{\Phi_1}\delta_{\Phi_1,\Phi_2}\mathcal{K}_2^{\Delta_1}(P_1, P_2), \quad \mathcal{K}_2^{\Delta_1}(P_1, P_2) = \frac{1}{P_{12}^{\Delta_1}}, \tag{1.117}$$

$$\lambda_{\Phi_1,\Phi_2,\Phi_3}\mathcal{K}_3^{\Delta_i}(P_i), \quad \mathcal{K}_3^{\Delta_i}(P_i) = \frac{1}{P_{12}^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}} P_{13}^{\frac{\Delta_1+\Delta_3-\Delta_2}{2}} P_{23}^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}}, \tag{1.118}$$

$$\mathcal{K}_4^{\Delta_i}(P_i) f(U,V), \quad \mathcal{K}_4^{\Delta_i}(P_i) = \frac{\left(\frac{P_{24}}{P_{14}}\right)^{\frac{\Delta_{12}}{2}} \left(\frac{P_{14}}{P_{13}}\right)^{\frac{\Delta_{34}}{2}}}{P_{12}^{\frac{\Delta_1+\Delta_2}{2}} P_{34}^{\frac{\Delta_3+\Delta_4}{2}}}, \; U = \frac{P_{12}P_{34}}{P_{13}P_{24}}, \; V = \frac{P_{14}P_{23}}{P_{13}P_{24}}, \tag{1.119}$$

where $\Delta_{ij} \equiv \Delta_i - \Delta_j$, $c$ and $\lambda$ are arbitrary constants, and $f$ is an arbitrary function.

[10] It will be clear why we use this normalization later.

### 1.3.2 Spinning constraints

Now for generic spinning fields $[\Delta_i, \rho_i]$, $i = 1, \ldots, m$, we can combine the previous scalar results into an ansatz

$$G(P_i; Z_i^j) = \mathcal{K}_m^{\Delta_i}(P_i) Q(P_i, Z_i^j), \tag{1.120}$$

with $Q$ a scale invariant function: $Q(\lambda_i P_i; Z_i^j) = Q(P_i, Z_i^j)$, satisfying the spinning constraints

$$Q\left(P_i; \alpha_i^j Z_i^j + \beta_i^j P_i + \gamma_i^{j,k} Z_i^k\big|_{0<k<j}\right) = \left(\prod_{i=1}^{m} \prod_{j=1}^{h(\rho_i)} (\alpha_i^j)^{(\rho_i)_j}\right) Q(P_i; Z_i^j). \tag{1.121}$$

Note that, in general, $Q$ can have more than one solution. Each solution is called a tensor structure. The form of $Q$ can be constructed out basic building blocks, by contracting the exterior products of $P_i$ with $Z_i^j$ [4],

$$B_i^n(P_i, Z_i^j) \equiv P_i \wedge Z_i^1 \wedge Z_i^2 \wedge \cdots \wedge Z_i^n, \tag{1.122}$$

with an appropriate normalization so that each building block—and hence $Q$—is scale invariant. This is easy to understand, since

$$B_i^n\left(P_i, \alpha_i^j Z_i^j + \beta_i^j P_i + \gamma_i^{j,k} Z_i^k\big|_{0<k<j}\right) = \left(\prod_{j=1}^{n} (\alpha_i^j)\right) B_i^n(P_i, Z_i^j). \tag{1.123}$$

So in order to get the correct exponents $(\rho_i)_j$, we must include several $B_i^n$, for particular values of $n$. Let us now see some particular examples. The simplest building block corresponds to the case with one spin-1 field. That is

$$V_{i,jk}^{(Z_i^1)} \equiv 2\frac{B_j^0 \cdot B_i^1 \cdot B_k^0}{\sqrt{P_{ij} P_{ik} P_{jk}}} = \frac{P_{ik} Z_i^1 \cdot P_j - P_{ij} Z_i^1 \cdot P_k}{\sqrt{P_{ij} P_{ik} P_{jk}}}, \tag{1.124}$$

where $V_{i,jk}^{(Z_i^1)} = -V_{i,kj}^{(Z_i^1)}$. For two spin-1 fields,

$$H_{ij}^{(Z_i^1, Z_j^1)} \equiv -\frac{B_i^1 \cdot B_j^1}{P_{ij}} = Z_i^1 \cdot Z_j^1 + 2\frac{Z_i^1 \cdot P_j Z_j^1 \cdot P_i}{P_{ij}}, \tag{1.125}$$

where $H_{ij}^{(Z_i^1, Z_j^1)} = H_{ji}^{(Z_j^1, Z_i^1)}$. One can check that any contraction of $B$'s to form a structure of $n$ spin-1's, with $n \geq 3$, can be written in terms of $H$ and $V$ [31].

For higher representations, one can produce more complicated contractions of $B$, or, alternatively, use the basis of building blocks $\{V, H\}$ with a manual antisimetrization. The simplest case is that of one antisymmetric spin-2 field and one vector:

$$-\frac{B_j^1 \cdot B_i^2 \cdot B_k^0}{\sqrt{P_{ij}P_{ik}P_{jk}}} = V_{i,jk}^{([Z_i^1)} H_{ij}^{(Z_i^2],Z_j^1)}, \tag{1.126}$$

where we the antisymmetrization is only among polarizations of the same coordinate $Z_i$, and it is defined without a numerical prefactor:
$T^{[\mu_1\cdots\mu_l]} = \sum_{\sigma\in S_l} \mathrm{sgn}(\sigma)\delta_{\nu_1}^{\mu_{\sigma(1)}}\cdots\delta_{\nu_l}^{\mu_{\sigma(l)}}T^{\nu_1\cdots\nu_l}$. For two spin-2 antisymmetric fields, there are two options:

$$\begin{aligned} -\frac{2}{3}\frac{B_i^2 \cdot B_j^2}{P_{ij}} &= H_{ij}^{([Z_i^1,Z_j^1)} H_{ij}^{(Z_i^2],Z_j^2)}, \\ -4\frac{(B_k^0 \cdot B_i^2 \cdot B_j^0)\cdot(B_i^0 \cdot B_j^2 \cdot B_k^0)}{P_{ij}P_{ik}P_{jk}} &= V_{i,jk}^{([Z_i^1)} V_{j,ik}^{([Z_j^1)} H_{ij}^{(Z_i^2],Z_i^2])}. \end{aligned} \tag{1.127}$$

As an example, let us take a term of the form

$$Q = \left(H_{12}^{(Z_1^1,Z_2^1)}\right)^{\ell_1-\ell_2}\left(H_{12}^{([Z_1^1,Z_2^1)} H_{12}^{(Z_1^2],Z_2^2)}\right)^{\ell_2}. \tag{1.128}$$

Then, by (1.123), it transforms as

$$Q \to \left(\prod_{i=1}^{2}\prod_{j=1}^{2}(\alpha_i^j)^{\ell_j}\right) Q, \tag{1.129}$$

which corresponds to a tensor structure of two fields in the $\rho = (\ell_1, \ell_2)$ representation of $SO(d)$. From the point of view of the basis $\{V, H\}$, we can understand this as follows; the term $(\dots)^{\ell_2}$ means that $H$ joins the first $\ell_2$ boxes on the first and second rows of $\rho_1$ with those of $\rho_2$, while $(\cdots)^{\ell_1-\ell_2}$ joins of the last $\ell_1 - \ell_2$ boxes in the first row of $\rho_1$ with those of $\rho_2$.

For general representations, one can write all the tensor structures in $Q$ via contractions of $B$'s,[11] but we will use the basis of $H$ and $V$ here.

### Building and counting tensor structures

Let us describe how to build and count tensor structures out of $V$'s and $H$'s, by looking at the Young diagrams of the field representations. We follow an argument similar to that in [29], but we will arrive at the more formal counting formula from [38].[12]

[11] Here we are ignoring parity odd structures, which are contracted with the antisymmetric $\epsilon$-tensor. See [31, 38] for more details.

[12] In that work, the conformal frame formalism was used to arrive at the formula.

Consider a correlator of fields $\Phi_i$ in representations $[\Delta_i, \rho_i]$. In the basis of building blocks $V$ and $H$, $V_i$ fills a box of the Young diagram $\rho_i$, while $H_{ij}$ connects two boxes between diagrams $\rho_i$ and $\rho_j$. Therefore each tensor structure appearing in $Q(P_i, Z_i^j)$, corresponds to a linearly independent filling/connection of all boxes between all $\rho_i$.

Of course this process has to be done in a consistent way, as to respect the Young symmetry of each $\rho_i$. For example, there cannot be two identical $V_i$'s in the same column of the Young diagram. Hence, for a fixed row $j$, there are $(\rho_i)_j - (\rho_i)_{j+1} \geq 0$ boxes that admit identical $V_i$'s, with coordinates $(j, (\rho_i)_{j+1} + 1), (j, (\rho_i)_{j+1} + 2), \ldots, (j, (\rho_i)_j)$. Given that each row is symmetrized, let us assume that we start filling the row's boxes from right to left. Then, for each filling configuration, the empty boxes constitute a valid and independent Young diagram $\lambda$, with following property

$$(\rho_i)_1 \geq \lambda_1 \geq (\rho_i)_2 \geq \lambda_2 \geq (\rho_i)_3 \geq \cdots . \tag{1.130}$$

But this is the condition that the representations in $\mathrm{Res}^{SO(d)}_{SO(d-1)} \rho_i$ must satisfy. More precisely, the branching rules for dimensional reduction are given by

$$\mathrm{Res}^{B_n}_{D_n} \rho = \bigotimes_{\rho_1 \geq \lambda_1 \geq \rho_2 \geq \lambda_2 \geq \cdots \geq \rho_n \geq |\lambda_n|} \lambda, \quad \text{i.e. } SO(2n+1) \to SO(2n), \tag{1.131}$$

$$\mathrm{Res}^{D_n}_{B_{n-1}} \rho = \bigotimes_{\rho_1 \geq \lambda_1 \geq \rho_2 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq |\rho_n|} \lambda, \quad \text{i.e. } SO(2n) \to SO(2n-1). \tag{1.132}$$

Here, we are assuming that the dimension $d$ is large enough so that both branching rules are the same (e.g. if $\rho_n = 0$).

So far we have looked at the filling of $\rho_i$ with identical $V_i$'s. However, in $(n \geq 4)$-point correlators, there are $n-2$ independent $V_i$ [31] (e.g. $V_{1,23}$ and $V_{1,24}$ in a four-point correlator[13]). We proceed by considering the possible fillings of the remaining empty diagrams $\mathrm{Res}^{SO(d)}_{SO(d-1)} \rho_i$ with the next $V_i'$, and so on. Thus, the possible ways we can fill a diagram $\rho_i$ with $n-2$ independent $V_i$'s, is in one to one correspondence with

$$\mathrm{Res}^{SO(d)}_{SO(d+2-m)} \rho_i, \quad m = \min(d+2, n). \tag{1.133}$$

Finally, the remaining empty box diagrams (if any) are connected in a consistent way, via the building blocks $H$. If we think of each diagram as a tensor, then each independent configuration of box connections, corresponds to the different ways of contracting the tensor indices into a scalar. The number of possible

[13]Note that $V_{1,34}$ is a linear combination of the other two by (1.184).

contractions is then the multiplicity of the singlet in the tensor product of representations. Therefore the number of linearly independent tensor structures in an $n$-point function is

$$\left( \bigotimes_{i=1}^{n} \text{Res}^{SO(d)}_{SO(d+2-m)} \rho_i \right)^{SO(d+2-m)}, \tag{1.134}$$

where $(\rho)^G$ projects out the singlets in $G$. This is the same formula that was found in [38], which is consistent with that of [29].

### 1.3.3 Projecting correlators to $\mathbb{R}^d$

To project embedded polynomial expressions into $\mathbb{R}^d$, we use (1.106) and (1.78) to derive the following results

$$P_{ij} \equiv -2P_i \cdot P_j = x_{ij}^2, \quad Z_i^k \cdot P_j = -z_i^k \cdot x_{ij}, \quad Z_i^k \cdot Z_j^l = z_i^k \cdot z_j^l, \tag{1.135}$$

where $x_{ij} \equiv x_i - x_j$. Then the building blocks $V$ and $H$ project to

$$V_{i,jk}^{(Z_i^l)} = (z_i^l)^\mu k_\mu^{(ijk)}, \tag{1.136}$$

$$H_{ij}^{(Z_i^k, Z_j^l)} = (z_i^k)^\mu m_{\mu\nu}^{(ij)} (z_j^l)^\nu, \tag{1.137}$$

where

$$k_\mu^{(ijk)} = \frac{x_{ij}^2 (x_{ik})_\mu - x_{ik}^2 (x_{ij})_\mu}{\sqrt{x_{ij}^2 x_{ik}^2 x_{jk}^2}},$$

$$m_{\mu\nu}^{(ij)} = \delta^{\mu\nu} - 2\frac{(x_{ij})_\mu (x_{ij})_\nu}{x_{ij}^2}. \tag{1.138}$$

## 1.4 Two- and three-point functions

### 1.4.1 Two-point function

From the discussion of the previous section, a tensor structure of two fields $\Phi_i$, $\Phi_j$ can only contain building blocks $H_{ij}$. Therefore their two-point function must be non-zero, only when their $SO(d)$ representations are the same. Then the two-point function of a field $\Phi$ in the representation $[\Delta, (\ell_1, \ell_2, \ldots, \ell_n)]$ is

$$\langle \Phi(P_1)\Phi(P_2) \rangle = c_\Phi \mathcal{K}_2^\Delta \left( \prod_{i=1}^{n-1} (\mathbf{H}_{12}^i)^{\ell_i - \ell_{i+1}} \right) (\mathbf{H}_{12}^n)^{\ell_n} \tag{1.139}$$

where we defined

$$\mathbf{H}_{ij}^{\ell} \equiv H_{ij}^{([Z_i^1, Z_j^1)} \cdots H_{ij}^{(Z_i^\ell], Z_j^\ell)}. \tag{1.140}$$

This corresponds to two Young diagrams $(1^\ell)$ (using standard notation $(k^n) = (\overbrace{k, k, \ldots, k}^{n \text{ times}})$), joined by $H_{ij}$'s.

For example, for operators $\mathcal{O}$ in the symmetric-traceless representation $(\ell)$, we have

$$\langle \mathcal{O}(P_1)\mathcal{O}(P_2)\rangle = c_\Phi \mathcal{K}_2^\Delta \left(H_{12}^{(Z_1,Z_2)}\right)^\ell \tag{1.141}$$

### 1.4.2 Three-point functions

For three representations, no general formula is known for the solution of (1.121). But we can construct the tensor structures case by case, with the algorithm described in 1.3.2. Let us define the analogues of (1.140):

$$\mathbf{V}_{ij,k}^{1,\ell} \equiv V_{i,jk}^{([Z_i^1)} H_{ij}^{(Z_i^2, Z_j^1)} \cdots H_{ij}^{(Z_i^\ell], Z_j^{\ell-1})}, \tag{1.142}$$

$$\mathbf{V}_{ij,k}^{2,\ell} \equiv V_{i,jk}^{([Z_i^1)} V_{j,ik}^{([Z_j^1)} H_{ij}^{(Z_i^2, Z_j^2)} \cdots H_{ij}^{(Z_i^\ell], Z_j^\ell])}. \tag{1.143}$$

Here, $\mathbf{V}^{1,\ell}$ corresponds to a Young diagram $\rho_i = (1^\ell)$, where one of the boxes is filled with $V_{i,jk}$ and the other $\ell - 1$ boxes are joined to the boxes of $\rho_j = (1^{\ell-1})$ with $H_{ij}$. While $\mathbf{V}^{2,\ell}$ corresponds to two diagrams $(1^\ell)$ with $\ell - 1$ boxes joined with $H_{ij}$, and one box of $\rho_i$ ($\rho_j$) filled with $V_{i,jk}$ ($V_{j,ik}$) respectively.

In the rest of this subsection, we write the relevant three-point functions that will be useful later in this work. Their generic form is, by (1.120),

$$\langle \Phi_1(P_1, Z_1^i)\Phi_2(P_2, Z_2^i)\Phi_3(P_3, Z_3^i)\rangle = \mathcal{K}_3^{\Delta_i} \sum_l \lambda^l Q^l(P_i, Z_i^j), \tag{1.144}$$

where the sum is over all independent solutions of (1.121), $\lambda$ are called three-point function constants, and the fields $\Phi_i$ are in representations $[\Delta_i, \rho_i]$. We specify the $SO(d)$ representations $\rho_1, \rho_2, \rho_3$ before each expression.

$(\ell_2, \ell_3), (0), (\ell_1, \ell_2, \ell_3)$

$$\lambda \mathcal{K}_3^{\Delta_i} \left(\mathbf{V}_{31,2}^{1,1}\right)^{\ell_1 - \ell_2} \left(\mathbf{V}_{31,2}^{1,2}\right)^{\ell_2 - \ell_3} \left(\mathbf{V}_{31,2}^{1,3}\right)^{\ell_3}, \tag{1.145}$$

$(\ell_2+1,0),(0),(\ell_1,\ell_2,0)$

$$\mathcal{K}_3^{\Delta_i}\left(\lambda^1\mathbf{V}_{31,2}^{2,1}+\lambda^2\mathbf{H}_{31}^1\right)\left(\mathbf{V}_{31,2}^{1,1}\right)^{\ell_1-\ell_2-1}\left(\mathbf{V}_{31,2}^{1,2}\right)^{\ell_2}, \tag{1.146}$$

$(\ell_2,\ell_3+1),(0),(\ell_1,\ell_2,\ell_3)$

$$\mathcal{K}_3^{\Delta_i}\left(\lambda^1\mathbf{V}_{31,2}^{2,2}+\lambda^2\mathbf{H}_{31}^2\right)\left(\mathbf{V}_{31,2}^{1,1}\right)^{\ell_1-\ell_2}\left(\mathbf{V}_{31,2}^{1,2}\right)^{\ell_2-\ell_3-1}\left(\mathbf{V}_{31,2}^{1,3}\right)^{\ell_3}, \tag{1.147}$$

$(\ell_2+1,\ell_3+1),(0),(\ell_1,\ell_2,\ell_3)$

$$\begin{aligned}\mathcal{K}_3^{\Delta_i}&\left(\lambda^1\mathbf{V}_{31,2}^{2,2}\mathbf{V}_{31,2}^{2,1}+\lambda^2\mathbf{V}_{31,2}^{2,2}\mathbf{H}_{31}^1+\lambda^3\mathbf{H}_{31}^2\mathbf{V}_{31,2}^{2,1}+\lambda^4\mathbf{H}_{31}^2\mathbf{H}_{31}^1\right)\\&\times\left(\mathbf{V}_{31,2}^{1,1}\right)^{\ell_1-\ell_2-1}\left(\mathbf{V}_{31,2}^{1,2}\right)^{\ell_2-\ell_3-1}\left(\mathbf{V}_{31,2}^{1,3}\right)^{\ell_3},\end{aligned} \tag{1.148}$$

$(\ell_2+2,0),(0),(\ell_1,\ell_2,0)$

$$\mathcal{K}_3^{\Delta_i}\left(\lambda^1\left(\mathbf{V}_{31,2}^{2,1}\right)^2+\lambda^2\mathbf{H}_{31}^1\mathbf{V}_{31,2}^{2,1}+\lambda^3\left(\mathbf{H}_{31}^1\right)^2\right)\left(\mathbf{V}_{31,2}^{1,1}\right)^{\ell_1-\ell_2-2}\left(\mathbf{V}_{31,2}^{1,2}\right)^{\ell_2}, \tag{1.149}$$

$(\ell_2,\ell_3+2),(0),(\ell_1,\ell_2,\ell_3)$

$$\begin{aligned}\mathcal{K}_3^{\Delta_i}&\left(\lambda^1\left(\mathbf{V}_{31,2}^{2,2}\right)^2+\lambda^2\mathbf{H}_{31}^2\mathbf{V}_{31,2}^{2,2}+\lambda^3\left(\mathbf{H}_{31}^2\right)^2\right)\\&\times\left(\mathbf{V}_{31,2}^{1,1}\right)^{\ell_1-\ell_2}\left(\mathbf{V}_{31,2}^{1,2}\right)^{\ell_2-\ell_3-2}\left(\mathbf{V}_{31,2}^{1,3}\right)^{\ell_3},\end{aligned} \tag{1.150}$$

$(\ell_2,\ell_3+1,1),(0),(\ell_1,\ell_2,\ell_3)$

$$\begin{aligned}\mathcal{K}_3^{\Delta_i}&\left(\lambda^1\mathbf{V}_{31,2}^{2,3}\mathbf{V}_{31,2}^{2,2}+\lambda^2\mathbf{V}_{31,2}^{2,3}\mathbf{H}_{31}^2+\lambda^3\mathbf{H}_{31}^3\mathbf{V}_{31,2}^{2,2}+\lambda^4\mathbf{H}_{31}^3\mathbf{H}_{31}^2\right)\\&\times\left(\mathbf{V}_{31,2}^{1,1}\right)^{\ell_1-\ell_2}\left(\mathbf{V}_{31,2}^{1,2}\right)^{\ell_2-\ell_3-1}\left(\mathbf{V}_{31,2}^{1,3}\right)^{\ell_3-1}.\end{aligned} \tag{1.151}$$

For the cases with one trivial representation (0), it is easy to check that the number of structures matches (1.134), for generic values of $\ell_i$. Consider the representations $\rho=(\rho_1,\rho_2,\rho_3)$ and $\sigma=(\sigma_1,\sigma_2,\sigma_3)$. Given that the tensor product of two identical representations gives a scalar, then we need to count number of diagrams that are shared between $\mathrm{Res}^{SO(d)}_{SO(d-1)}\rho$ and $\mathrm{Res}^{SO(d)}_{SO(d-1)}\sigma$. In other words, we need to find $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ such that

$$\rho_1 \geq \lambda_1 \geq \rho_2 \geq \lambda_2 \geq \rho_3 \geq \lambda_3 \geq 0 \ \wedge \ \sigma_1 \geq \lambda_1 \geq \sigma_2 \geq \lambda_2 \geq \sigma_3 \geq \lambda_3 \geq 0. \quad (1.152)$$

The number of solutions is then given by

$$(\min(\rho_3, \sigma_3) + 1)(\min(\rho_2, \sigma_2) - \max(\rho_3, \sigma_3) + 1)(\min(\rho_1, \sigma_1) - \max(\rho_2, \sigma_2) + 1). \quad (1.153)$$

In particular, taking $\rho_1 = \ell_2 + i, \rho_2 = \ell_3 + j, \rho_3 = k$ and $\sigma_1 = \ell_1, \sigma_2 = \ell_2, \sigma_3 = \ell_3$, gives

$$(i+1)(j+1)(k+1) \quad (1.154)$$

tensor structures, where we assumed $k \leq \ell_3, \ \ \ell_3 + j \leq \ell_2, \ \ \ell_2 + i \leq \ell_1$. This is consistent with the number of terms in the expressions above.

Note that there is a class of spinning three-point functions with the minimum number of tensor structures, i.e. one, which we will call 'seed-like three-point functions'. From the previous discussion it is easy to see that these have $SO(d)$ representations $(\ell_1, \ell_2, \ldots, \ell_n)$, $(\ell_2, \ldots, \ell_n)$, $(0)$.

For the case where all three operators $\mathcal{O}_i$ are in the symmetric-traceless representation $(\ell_i)$, a solution to (1.121) can be written as follows [31]

$$\begin{aligned} &Q(P_i, Z_i) \\ &= \sum_{n_i \geq 0} C_{n_1,n_2,n_3} \times (V_{1,23})^{\ell_1 - n_2 - n_3}(V_{2,31})^{\ell_2 - n_3 - n_1}(V_{3,21})^{\ell_3 - n_1 - n_2} H_{12}^{n_1} H_{13}^{n_3} H_{23}^{n_2} \ , \\ &\qquad\qquad\qquad l_i - n_j - n_k \geq 0, \ \forall \ i, j, k, \end{aligned} \quad (1.155)$$

where we omitted the explicit dependence on the polarization $Z$, and $C$ are the three-point function constants. Note that, as shown in [31], in $d = 3$ not all of these terms are independent. For higher dimensions, the number of structures is given by

$$N(\ell_1, \ell_2, \ell_3) = \frac{1}{6}(\ell_1+1)(\ell_1+2)(3\ell_2-\ell_1+3) - \frac{1}{24}p(p+2)(2p+5) - \frac{1}{16}(1-(-1)^p) \ , \quad (1.156)$$

with $\ell_1 \leq \ell_2 \leq \ell_3$ and $p = \max(0, \ell_1 + \ell_2 - \ell_3)$. One can check that this expression matches (1.134). For example, for $\ell_1 = 0$ the formula reduces to $N(0, \ell_2, \ell_3) = \ell_2 + 1$, which is consistent with (1.154).

### 1.4.3 Conserved operators

A fundamental property of a unitary theory is that all states in its Hilbert space have positive norm. For the case of conformal field theories, the spectrum of states splits into multiplets, each of which is a collection of a primary with all its descendants (recall from subsection 1.1.3 that descendants are constructed by applying the translation operator $P$ onto the primary). Imposing positvity on the descendant states then restricts the possible conformal dimensions $\Delta$ of the primaries [39–42]. More precisely,

$$\Delta \geq \frac{d-2}{2}, \tag{1.157}$$

for scalar operators, whereas

$$\Delta \geq (\rho)_1 - h(\rho) + d - 1 \tag{1.158}$$

for operators in the $\rho \in SO(d)$ representation, where $(\rho)_i$ is the number of boxes in the $i$-th row of $\rho$ and $h(\rho)$ is the number of total rows. It turns out that this bound is saturated by conserved currents. To see this consider a tensor $g_{\mu_1 \dots \mu_{|\rho|}}$ in representation $\rho$ of $SO(d)$. This can be recovered by a transverse tensor $G_{A_1 \dots A_{|\rho}}$ in embedding space by contracting its indices $A_i$ with the projector $\mathcal{P}$ defined in (1.91). Therefore the divergence of $g$ is given by

$$\partial_x^{\mu_{|\rho|}} g_{\mu_1 \dots \mu_{|\rho|}} = \partial_x^{\mu_{|\rho|}} \mathcal{P}^{A_1 \dots A_{|\rho|}}_{\mu_1 \dots \mu_{|\rho|}} G_{A_1 \dots A_{|\rho|}} \tag{1.159}$$

which includes a term where the derivative hits $\mathcal{P}$ and a term where it acts on $G$. From (1.91) and (1.94) it is easy to check that (see [29] for the full derivation)

$$0 = \partial_x^{\mu_{|\rho|}} g_{\mu_1 \dots \mu_{|\rho|}} = \mathcal{P}^{A_1 \dots A_{|\rho|-1}}_{\mu_1 \dots \mu_{|\rho|-1}} R_{A_1 \dots A_{|\rho|-1}}, \tag{1.160}$$

$$R_{A_1 \dots A_{|\rho|-1}} = \left( \partial_P^{A_{|\rho|}} - \frac{1}{P \cdot \bar{P}} \left[ (\bar{P} \cdot \partial_P) P^{A_{|\rho|}} + ((\rho)_1 - h(\rho) + d - \Delta - 1) \, \bar{P}^{A_{|\rho|}} \right] \right) G_{A_1 \dots A_{|\rho|}}, \tag{1.161}$$

where the second term vanishes since $G$ is transverse,[14] and last term must be zero given that the contraction with $\bar{P}$ is not $SO(d+1,1)$ invariant. Thus (1.158) is saturated as promised.

Moreover, from the point of view of the Ward identities (1.19), having a conserved operator in a correlator, gives additional constraints. In general, these

[14]Strictly speaking the contraction with $P$ may not vanish exactly, but up to a term $O(P^2)$. Then the derivative $\bar{P} \cdot \partial_P$ gives $O(\bar{P} \cdot P)$ which cancels the denominator, thus preserving $SO(d+1,1)$ invariance.

constraints will produce relations between the different tensor structures $Q$ (c.f. (1.120)), thereby reducing the 'degrees of freedom' of the correlator. Further recent developments on properties of correlation functions for conserved currents can be found in [38, 43, 44]. Let us now see an example for three-point functions. Consider a three-point function of two traceless-symmetric spin-2 tensors and a scalar. By equation (1.149) this is

$$G_{\Delta_1,\Delta_2,\Delta_3|2,2,0} = \left(\alpha H_{12}^2 + \beta H_{12}V_{1,23}V_{2,31} + \gamma V_{1,23}^2 V_{2,31}^2\right)\mathcal{K}_3^{\Delta_1,\Delta_2,\Delta_3} . \tag{1.162}$$

Now assuming the stress tensors are conserved, this implies that $\Delta_1 = \Delta_2 = d$, and (in the language of encoded polynomials)

$$(\partial_P \cdot D)G_{\Delta_1,\Delta_2,\Delta_3|2,2,0} = 0, \tag{1.163}$$

where $D$ is defined in (5.5). Applying the derivative then results in the following relations

$$\begin{aligned}\alpha &= \frac{4h(h-1)(2h+1) - 4\Delta_3 h(2h-1) + \Delta_3^2(2h-1)}{2\Delta_3(\Delta_3+2)(h-1)}\gamma \, , \\ \beta &= -\frac{2 + 4h^2 + \Delta_3 - 2h(\Delta_3+1)}{(h-1)(\Delta_3+2)}\gamma \, ,\end{aligned} \tag{1.164}$$

where $h = d/2$. Therefore in this case, $G$ has only one independent tensor structure, labeled by a single coefficient $\gamma$.

## 1.5 Weight shifting operators

Now we discuss a formalism that allows us to transform the representations of conformal fields. The first instance of this idea was found in [45] in terms of three-point functions, but it was later understood more formally in [46]. In this section, we follow the exposition of [46].

The basic idea is that tensoring a finite-dimensional representation $\Omega$ of $SO(d+1,1)$ with a conformal primary $\phi$, decomposes into irreducible representations different to that of $\phi$. More explicitly, a field in the tensor product space can be written as

$$\tilde{\phi}(x) = e^A \otimes \mathcal{D}_A(x)\phi(x), \tag{1.165}$$

where $e^A$, $A = 1, \ldots, \dim\Omega$ are basis vectors of $\Omega$. Then by choosing a specific form of $\mathcal{D}_A$, we can 'transform' the conformal representation of $\phi$ to a different one. However, note that $\mathcal{D}_A(x)\phi(x)$ is a conformal primary with an extra index, associated to the finite-dimensions representation $\Omega$.

For a given $SO(d+1,1)$ representation $\Omega$ and a field $\phi$ in $[\Delta, \rho]$, the possible choices of $\mathcal{D}$ are in one to one correspondence with the irreducible components of $\Omega \otimes [\Delta, \rho]$. To compute this tensor product, we first need to know what conformal representations correspond to $\Omega$, given by its decomposition to the subalgebra $SO(1,1) \times SO(d)$:

$$\Omega \to \bigotimes_{i=-(\Omega)_1,\dots,(\Omega)_1} [i, \omega_i], \tag{1.166}$$

which is a sum over $2(\Omega)_1 + 1$ terms (recall $(\Omega)_1$ is the number of boxes in the first row of $\Omega$). Then the tensor product is

$$\Omega \otimes [\Delta, \rho] = \bigotimes_{i=-(\Omega)_1,\dots,(\Omega)_1} \bigotimes_{\lambda \in \omega_i \otimes \rho} [\Delta + i, \lambda]. \tag{1.167}$$

Let us work out a simple example with $\Omega = (1) \to [-1,(0)] \oplus [0,(1)] \oplus [1,(0)]$ and $[\Delta, \rho] = [\Delta, (0)]$. The formula (1.167) then predicts the existence of three representations: $[\Delta - 1, (0)]$, $[\Delta, (1)]$, $[\Delta + 1, (0)]$. We can find these primaries from (1.165), by imposing $[1 \otimes K_\mu + K_\mu \otimes 1, \tilde{\phi}] = 0$. These are

$$\begin{aligned}
O^-(0) &= e^A \otimes \psi_A(0)\phi(0), \\
O_\mu(0) &= e^A \otimes \left(\Delta_\phi \partial_\mu \psi_A(0) + \psi_A(0)\partial_\mu\right)\phi(0), \\
O^+(0) &= e^A \otimes \big(\Delta_\phi(2\Delta_\phi - d + 2)\partial^2\psi_A(0) \\
&\quad + d(2\Delta_\phi - d + 2)\partial_\mu\psi_A(0)\partial^\mu + d\psi_A(0)\partial^2\big)\phi(0),
\end{aligned} \tag{1.168}$$

where the dimension of $\psi$ is $-1$. Note that here $\psi$ is not a physical field but just a mathematical construct. In the next subsection we define its value in the embedding formalism.

In summary, for a given finite dimensional representation $\Omega$ of $SO(d+1,1)$, there is a set of associated weight-shifting operators $\mathcal{D} : [\Delta, \rho] \to [\Delta + i, \lambda] \in \Omega \otimes [\Delta, \rho]$.

### 1.5.1 Embedding space construction

As seen in the previous example, $\mathcal{D}_A$ is constructed in such a way that when applied to $\phi$, it produces a primary in a different representation. Here we describe how this is done in embedding space, introduced in section 1.2. This subsection is based on [4] and generalizes the construction of [46] to arbitrary bosonic $SO(d)$ representations.

Consider an embedded field $\Phi(P; Z^j)$ in representation $[\Delta, \rho]$, satisfying (1.110)–(1.113):

$$(P \cdot \partial_P + \Delta)\,\Phi = 0, \quad \big(Z^i \cdot \partial_{Z^i} - (\rho)_i\big)\,\Phi = 0, \quad P \cdot \partial_{Z^i}\Phi = Z^i \cdot \partial_{Z^j}\Phi\big|_{i<j} = 0. \tag{1.169}$$

Then in order for $\mathcal{D}_A\Phi$ to be in the representation $[\Delta + i, \lambda]$, we must impose

$$\begin{gathered}([P \cdot \partial_P, \mathcal{D}_A] + i\mathcal{D}_A)\,\Phi = 0, \quad \big([Z^i \cdot \partial_{Z^i}, \mathcal{D}_A] + \{(\rho)_i - (\lambda)_i\}\,\mathcal{D}_A\big)\,\Phi = 0, \\ [P \cdot \partial_{Z^i}, \mathcal{D}_A]\Phi = [Z^i \cdot \partial_{Z^j}, \mathcal{D}_A]\Phi\big|_{i<j} = 0. \end{gathered} \tag{1.170}$$

Furthermore, $\mathcal{D}_A$ must be interior on the subspace $P^2 = Z^i \cdot P = Z^i \cdot Z^j = 0$. In other words, for every independent test function $t(P; Z^j) = O(P^2, Z^i \cdot P, Z^i \cdot Z^j)$, in the same representation as $\Phi$, we demand $\mathcal{D}_A t(P; Z^j) = O(P^2, Z^i \cdot P, Z^i \cdot Z^j)$. The test functions can be written in terms of the basis $B^m$, defined in (1.122), as

$$t(P; Z^j) = (B^m \cdot B^n)\, f^{mn}(P; Z^j))\big|_{0 \le m \le n}, \tag{1.171}$$

where $B^m \cdot B^n = O(P^2, Z^i \cdot P, Z^i \cdot Z^j)$ is in the representation $[-2, (2^m, 1^{n-m})]$, and consequently, $f^{mn}$ is in $[\Delta + 2, \rho - (2^m, 1^{n-m})]$. Note that we do not need to know $f^{mn}$ explicitly, as the interior condition can be re-written as

$$[\mathcal{D}_A, B^m \cdot B^n] f^{mn}(P; Z^j)\big|_{0 \le m \le n} = O(P^2, Z^i \cdot P, Z^i \cdot Z^j). \tag{1.172}$$

We will write the explicit form of the operators $\mathcal{D}_A$ for $SO(d)$ representations with $h(\rho) \le 3$ in chapter 3. For now, notice that in the example (1.168), the embedded operator associated to $\psi_A$ is simply $P_A$ (it trivially satisfies (1.170) and (1.172)).

### 1.5.2 Three-point function differential basis

Now we describe the implications of computing correlation functions that include weight-shifting operators $\mathcal{D}_A$. Consider weight-shifting operators $\mathcal{D}_i^{(a)} : [\Delta_i', \rho_i'] \to [\Delta_i, \rho_i]$, where the upper index $(a)$ labels which operator it is from the decomposition (1.167), and the lower index means it depends on $P_i$. Inserting this operator into a correlator, produces a quantity that transforms as a conformal correlator with an extra index associated to a finite-dimensional representation

$$e^A \otimes \mathcal{D}_{iA}^{(a)} \langle \Phi_1 \cdots \Phi_i' \cdots \Phi_j \cdots \Phi_n \rangle. \tag{1.173}$$

Notice that $e^A \otimes \mathcal{D}_{iA}^{(a)} \Phi_i' \in [\Delta, \rho]$ by definition. Therefore the quantity above has the same transformation properties as the correlator $\langle \Phi_1 \cdots \Phi_i \cdots \Phi_j \cdots \Phi_n \rangle$.

Alternatively, we could have chosen a different operator

$$e^A \otimes \mathcal{D}^{(b)}_{jA}\langle \Phi_1 \cdots \Phi_i \cdots \Phi'_j \cdots \Phi_n \rangle, \tag{1.174}$$

in such a way that the resulting representations are the same to those of (1.173). Hence there must be a linear relation between the two bases:

$$\mathcal{D}^{(a)}_{iA}\langle \Phi_1 \cdots \Phi'_i \cdots \Phi_j \cdots \Phi_n \rangle = \sum_{\Phi'_j, b} \alpha_{\Phi'_j, b} \mathcal{D}^{(b)}_{jA}\langle \Phi_1 \cdots \Phi_i \cdots \Phi'_j \cdots \Phi_n \rangle, \tag{1.175}$$

where $\Phi'_j \in \Omega \otimes \Phi_j$. Specializing to three-point functions this then implies

$$\mathcal{D}^{(b)}_{kA}\langle \Phi_i \Phi_j \Phi'_k \rangle^{(a)} = \sum_{\Phi'_i \in \Omega \otimes \Phi_i, m, n} \begin{Bmatrix} \Phi_i & \Phi_j & \Phi'_i \\ \Phi_k & \Omega & \Phi'_k \end{Bmatrix}^{ab}_{mn} \mathcal{D}^{(n)}_{iA}\langle \Phi'_i \Phi_j \Phi_k \rangle^{(m)}, \tag{1.176}$$

where the coefficients $\{\cdots\}$ are called $6j$ symbols, and we included an index in the correlator to label tensor structures, i.e.,

$$\langle \Phi_1 \Phi_2 \Phi_3 \rangle = \sum_a \lambda_a \langle \Phi_1 \Phi_2 \Phi_3 \rangle^{(a)}. \tag{1.177}$$

Now notice that the weight-shifting operator $\mathcal{D}^{(a)}_{iA}\mathcal{D}^{(b)A}_{i} : \Phi'_i \to \Phi''_i$ is associated to a scalar representation of $SO(d+1,1)$. Therefore, according to (1.167), this operator must be proportional to the identity:

$$\mathcal{D}^{(a)}_{iA}\mathcal{D}^{(b)A}_{i} = \begin{pmatrix} \Phi'_i \\ \Phi_i \ \Omega \end{pmatrix}^{ab} \delta_{\Phi'_i, \Phi''_i} \tag{1.178}$$

Thus, contracting (1.176) with $\mathcal{D}^{(c)A}_{i}$ on both sides gives

$$\mathcal{D}^{(c)}_{i} \cdot \mathcal{D}^{(b)}_{k}\langle \Phi_i \Phi_j \Phi'_k \rangle^{(a)} = \sum_{m,n} \begin{Bmatrix} \Phi_i & \Phi_j & \Phi'_i \\ \Phi_k & \Omega & \Phi'_k \end{Bmatrix}^{ab}_{mn} \begin{pmatrix} \Phi'_i \\ \Phi_i \ \Omega \end{pmatrix}^{cn} \langle \Phi'_i \Phi_j \Phi_k \rangle^{(m)}. \tag{1.179}$$

This is a very powerful relation because it maps different three-point function representations by acting with differential operators on two points. This will be exploited later on.

The particular case where $\Phi_i$ and $\Phi'_k$ are scalars, and $\Phi'_i$ and $\Phi_k$ are symmetric traceless tensors was first studied in [45]. Given the large amount of literature that uses this differential basis, we will use it instead of re-writing everything in terms of the new basis (1.179). In [45] they define a basis of 5 operators:[15]

$$\begin{aligned} D_{1\,ij} \equiv & -\frac{1}{2}P_{ij}\left(Z_i\cdot\frac{\partial}{\partial P_j}\right) - (Z_i\cdot P_j)\left(P_i\cdot\frac{\partial}{\partial P_j}\right) \\ & \qquad - (Z_i\cdot Z_j)\left(P_i\cdot\frac{\partial}{\partial Z_j}\right) + (Z_j\cdot P_i)\left(Z_i\cdot\frac{\partial}{\partial Z_j}\right)\,, \\ D_{2\,ij} \equiv & -\frac{1}{2}P_{ij}\left(Z_i\cdot\frac{\partial}{\partial P_i}\right) - (Z_i\cdot P_j)\left(P_i\cdot\frac{\partial}{\partial P_i}\right) + (Z_i\cdot P_j)\left(Z_i\cdot\frac{\partial}{\partial Z_i}\right)\,, \\ & H_{ij}^{(Z_i,Z_j)} \text{ as in (1.125).} \end{aligned} \tag{1.180}$$

The operator $D_{1\,ij}$ increases the spin at position $i$ by one and decreases the dimension by one at position $i$; $D_{2\,ij}$ increases the spin at position $i$ by one and decreases the dimension by one at position $j$. $H_{ij}$ increases the spin by one at both $i$ and $j$ and leaves the conformal dimensions unchanged.

Finally, note that (1.179) is not limited to increasing spin, it can also be used for lowering it. See [46] for more details and applications.

## 1.6 Four-point functions and conformal blocks

Consider the four-point function $\langle\Phi_1\Phi_2\Phi_3\Phi_4\rangle$, with fields $\Phi_i(P_i;Z_i^j)$ in representations $[\Delta_i,\rho_i]$. From equations (1.119) and (1.120), this is given by

$$\langle\Phi_1\Phi_2\Phi_3\Phi_4\rangle = \mathcal{K}_4^{\Delta_i}(P_i)\sum_k Q_k(P_i,Z_i^j)f_k(U,V), \tag{1.181}$$

where sum runs over all independent tensor structures which includes a total of

$$\left(\bigotimes_{i=1}^{4}\mathrm{Res}^{SO(d)}_{SO(d-2)}\rho_i\right)^{SO(d-2)} \tag{1.182}$$

elements, assuming $d\geq 3$. Similar to the three-point function case, we will build the functions $Q_k$ by antisymmetrizing the basis $\{H,V\}$ from (1.124) and (1.125). Note that, although we can write three different $V$'s for each position $i$

$$\{V_{i,jk},V_{i,jl},V_{i,kl}\}, \tag{1.183}$$

only two are linearly independent, due to

$$\sqrt{P_{il}P_{jk}}V_{i,jk}^{(Z_i^n)} + \sqrt{P_{ij}P_{kl}}V_{i,kl}^{(Z_i^n)} - \sqrt{P_{ik}P_{jl}}V_{i,jl}^{(Z_i^n)} = 0. \tag{1.184}$$

[15] Here we use a slightly different notation that emphasizes what each operator is doing. In [45] they call $D_{ii}$ and $D_{ij}$ what we call $D_{1\,ij}$ and $D_{2,\,ij}$ respectively.

A conformal partial wave $W_{\mathcal{O}}$ is defined as the contribution of a primary operator $\mathcal{O}$ (and its descendants) to the four-point function. In other words,

$$\langle \Phi_1 \Phi_2 \Phi_3 \Phi_4 \rangle = \sum_{\mathcal{O}} \Lambda_{\mathcal{O}} W_{\mathcal{O}}(P_i), \tag{1.185}$$

where $\Lambda_{\mathcal{O}}$ are expansion constants. On the other hand, from (1.181), we have

$$W_{\mathcal{O}}(P_i) = \mathcal{K}_4^{\Delta_i}(P_i) \sum_k Q_k(P_i, Z_i^j) g_{\mathcal{O}k}(U, V), \tag{1.186}$$

where we defined

$$f_k(U, V) = \sum_{\mathcal{O}} \Lambda_{\mathcal{O}} g_{\mathcal{O}k}(U, V). \tag{1.187}$$

The functions $g_{\mathcal{O}}$ are called conformal blocks. Given the simple relation between these and the partial waves (1.186), sometimes the terms are interchanged in the literature. However, the main difference is that the conformal blocks only depend on the invariant cross-ratios $U, V$, while the partial waves carry the four-point tensor structures as well as the kinematic factor $\mathcal{K}$.

In the following subsections we will discuss several techniques for computing conformal partial waves (conformal blocks), known in the literature. However, this by no means an exhaustive list. See [30, 34–37, 47–61] and references therein for other developments in conformal block technology.

### 1.6.1 Conformal partial waves I: The OPE

Consider two fields $\phi_1(x_1)$, $\phi_2(x_2)$ in a configuration where $x_1$ and $x_2$ are close to each other. One can approximate the product of $\phi_1 \phi_2$ (by product we mean a pair of operators inserted in a correlator) as a sum over all local operators $\mathcal{U}$ at $x_2$:

$$\phi_1(x_1) \times \phi_2(x_2) = \sum_{\mathcal{U}} f_{12\mathcal{U}}(x_{12}) \mathcal{U}(x_2). \tag{1.188}$$

Now, because of conformal symmetry, we can reorder the sum over $\mathcal{U}$ in terms of conformal primaries and descendants

$$\sum_{\mathcal{U}} \to \sum_{\mathcal{O} \text{ primaries}} \sum_{\alpha = \mathcal{O}, P\mathcal{O}, PP\mathcal{O}, \ldots}, \tag{1.189}$$

where $P \cdots P\mathcal{O}$ represents the action of the translation generator (1.52). From the integrated Ward identity (1.20), this is equivalent to applying derivatives to the correlator, so we can write

$$\phi_1(x_1) \times \phi_2(x_2) = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}} F_{\mathcal{O}}(x_{12}, \partial_{x_2}) \mathcal{O}(x_2), \tag{1.190}$$

where the function $F$ generates the sum over the descendants. This is the conformal operator product expansion (OPE) and has a finite radius of convergence. More precisely, it converges as long the closest operator to $\phi_1(x_1)$ is $\phi_2(x_2)$ [62,63].

The explicit form of $F$ can be obtained by imposing consistency when inserted in correlators. For example, inserting (1.190) in $\langle\phi_1\phi_2\mathcal{O}\rangle$ implies

$$\langle\phi_1(x_1)\phi_2(x_2)\mathcal{O}(x_3)\rangle = \lambda_{12\mathcal{O}}F_{\mathcal{O}}(x_{12}, \partial_{x_2})\langle\mathcal{O}(x_2)\mathcal{O}(x_3)\rangle, \tag{1.191}$$

for all primaries $\mathcal{O}$ appearing in the OPE $\phi_1 \times \phi_2 \ni \mathcal{O}$. Then using the explicit form of the two- and three-point functions (1.139), (1.144), one can fix the form of $F$. Note that if the coefficient in the two-point function $c_{\mathcal{O}}$ is normalized to one, then the constant $\lambda$ is the same as the one appearing inside the three-point function. Hence the term three-point function constant is interchangeable with OPE coefficient.

For non-scalar operators, the generalization of (1.190) is straightforward:

$$\phi_{1\ \mu_1\cdots\mu_{\ell_1}}(x_1)\phi_{2\ \nu_1\cdots\nu_{\ell_2}}(x_2) = \sum_{\mathcal{O}}\sum_{k}\lambda^k_{12\mathcal{O}}F^k_{\mathcal{O}\ \mu_1\cdots\mu_{\ell_1};\nu_1\cdots\nu_{\ell_2}}(x_{12}, \partial_{x_2})\cdot\mathcal{O}(x_2), \tag{1.192}$$

where $k$ counts the different tensor structures in $\langle\phi_1\phi_2\mathcal{O}\rangle$, and the indices of $\mathcal{O}$ are contracted with $F$.

Now consider the four-point function $\langle\phi_1\phi_2\phi_3\phi_4\rangle$ and replace the pairs $\phi_1\phi_2$, $\phi_3\phi_4$ by their OPE expansions (1.192), to obtain

$$\langle\phi_1\phi_2\phi_3\phi_4\rangle = \sum_{\mathcal{O}}\sum_{r,s}\lambda^r_{12\mathcal{O}}\lambda^s_{34\mathcal{O}}W^{rs}_{\mathcal{O}}(x_i), \tag{1.193}$$

where

$$W^{rs}_{\mathcal{O}}(x_i) = C^r_{\mathcal{O}}(x_{12}, \partial_{x_2})C^s_{\mathcal{O}}(x_{34}, \partial_{x_4})\langle\mathcal{O}(x_2)\mathcal{O}(x_4)\rangle \tag{1.194}$$

is the conformal partial wave contribution from the primary $\mathcal{O}$ in the (12)(34) channel. Inserting the OPE in different pairs of operators will produce different, yet consistent, expansions for the four-point function. However, note that only two channels will have intersecting regions of convergence. For example, for (12)(34) and (14)(23), we can choose the position of $\Phi_2$ and $\Phi_3$ such that $\Phi_2$ is closest to $\Phi_1$, and $\Phi_3$ is closest to $\Phi_2$.

Using this idea, conformal partial waves can be computed in principle [64–66]. However, in practice we use more efficient techniques that do not require the explicit knowledge of the OPE, as we review in the next subsections.

### 1.6.2 Conformal partial waves II: The Casimir

Consider the three-point function $\langle\Phi_1\Phi_2\Phi_3\rangle$ and apply (1.28) for the embedding generator $J$ (1.42)

$$(J_{1AB}+J_{2AB})\langle\Phi_1(P_1)\Phi_2(P_2)\Phi_3(P_3)\rangle = -J_{3AB}\langle\Phi_1(P_1)\Phi_2(P_2)\Phi_3(P_3)\rangle. \quad (1.195)$$

Then, defining the Casimir differential operator $J_{12}^2 \equiv -\frac{1}{2}(J_1{}^{AB}+J_2{}^{AB})(J_{1AB}+J_{2AB})$, we have

$$\begin{aligned} J_{12}^2\langle\Phi_1(P_1)\Phi_2(P_2)\Phi_3(P_3)\rangle &= -\frac{1}{2}J_3{}^{AB}J_{3AB}\langle\Phi_1(P_1)\Phi_2(P_2)\Phi_3(P_3)\rangle \\ &= \mathcal{C}_{\Phi_3}\langle\Phi_1(P_1)\Phi_2(P_2)\Phi_3(P_3)\rangle, \end{aligned} \quad (1.196)$$

where in the last identity we used the property of the Casimir operator $C$, $[C,\Phi(x)] = \mathcal{C}_\Phi\Phi(x)$, with

$$C = -\frac{1}{2}J_{AB}J^{AB} = D(D-d) - P_\mu K^\mu - \frac{1}{2}M_{\mu\nu}M^{\mu\nu}, \quad (1.197)$$

$$\mathcal{C} = \Delta(\Delta-d) + S_{\mu\nu}S^{\mu\nu}. \quad (1.198)$$

For a tensor field $\Phi$ in the representation $[\Delta,(\ell_1,\ell_2,\cdots)]$, the eigenvalue $S_{\mu\nu}S^{\mu\nu}$ is given by [52,67],

$$S_{\mu\nu}S^{\mu\nu} = \sum_{i=1}^{\lfloor d/2\rfloor}\ell_i(\ell_i+d-2i). \quad (1.199)$$

See also appendix F of [37] for a derivation of this result.

Another way of understanding (1.197) is by inserting the OPE of $\Phi_1\Phi_2$ in the three-point function. This implies that the action of the Casimir differential operator $J_{12}^2$ on $\Phi_1\times\Phi_2$ can be replaced by the Casimir eigenvalue of $\Phi_3\in\Phi_1\times\Phi_2$. Therefore applying $J_{12}^2$ on (1.194) leads to

$$\left(J_{12}^2-\mathcal{C}_\mathcal{O}\right)W_\mathcal{O}^{rs}(P_i) = 0. \quad (1.200)$$

Then by the relation (1.186), this translates into a coupled system of differential equations for the conformal blocks $g_{\mathcal{O}k}^{rs}(U,V)$. Given that the Casimir differential operator is second order, proper boundary conditions must be imposed on $g_\mathcal{O}$. These are obtained by taking the asymptotic limit of (1.194) when $x_{12},x_{34}\to 0$.

The Casimir differential equation for $g_\mathcal{O}$ was solved in closed form for even dimensions $d$ and $\Phi_i = [\Delta_i,(0)]$, in [68]. For operators in arbitrary representations, recursion relations from the Casimir equations were obtained in [56]. As a reference, higher order Casimir operators can also be used for studying conformal blocks. See for example [47,69].

### 1.6.3 Conformal partial waves III: Shadow projectors

Writing the four-point function as an expectation value, we have

$$\langle\Phi_1\Phi_2\Phi_3\Phi_4\rangle = \sum_\Psi \frac{\langle 0|\,\Phi_1\Phi_2\,|\Psi\rangle\,\langle\Psi|\,\Phi_3\Phi_4\,|0\rangle}{\langle\Psi|\Psi\rangle}, \tag{1.201}$$

where we have inserted a complete set of states $1 = \sum_\Psi \frac{|\Psi\rangle\langle\Psi|}{\langle\Psi|\Psi\rangle}$. Now, using conformal symmetry, we can reorder the sum over $\Psi$ in terms of conformal primaries and descendants

$$\langle\Phi_1\Phi_2\Phi_3\Phi_4\rangle = \sum_{\mathcal{O}\ \text{primaries}} \sum_{\alpha=\mathcal{O},P\mathcal{O},PP\mathcal{O},\ldots} \frac{\langle\Phi_1\Phi_2\alpha\rangle\langle\alpha\Phi_3\Phi_4\rangle}{\langle\alpha\alpha\rangle}, \tag{1.202}$$

just like we did for the OPE in subsection 1.6.1. The sum over $\alpha$ could also be generated via derivatives, but that is no better than (1.194). Instead, to obtain a more useful expression note that, the commutation properties of the Casimir imply that the three-point functions $\langle\cdots\alpha\rangle$ have the same eigenvalue $\mathcal{C}_\mathcal{O}$. Furthermore, we see that (1.196) holds for arbitrary values of $P_3$. Therefore both

$$\sum_{\alpha=\mathcal{O},P\mathcal{O},PP\mathcal{O},\ldots} \frac{\langle\Phi_1\Phi_2\alpha\rangle\langle\alpha\Phi_3\Phi_4\rangle}{\langle\alpha\alpha\rangle} \tag{1.203}$$

and

$$\mathcal{N}_\mathcal{O}\int dPdP'\langle\Phi_1\Phi_2\mathcal{O}(P)\rangle\cdot\left(\langle\mathcal{O}(P)\mathcal{O}(P')\rangle|_{\Delta_\mathcal{O}\to\tilde{\Delta}_\mathcal{O}}\right)\cdot\langle\mathcal{O}(P')\Phi_3\Phi_4\rangle, \tag{1.204}$$

satisfy the Casimir differential equation with the same eigenvalue. The function inside the integral was chosen so that they have the same transformation properties under the conformal group. Here we defined $\tilde{\Delta} = d - \Delta$, and the dots $\cdot$ imply that the indices of $\mathcal{O}(P)$ in the three-point function are contracted with those of $\mathcal{O}(P)$ in the two-point function (the same for $\mathcal{O}(P')$ ).

The non-local operator with dimension $\tilde{\Delta}$ is called the shadow operator [28, 70–72], and is defined by

$$\widetilde{\mathcal{O}}(P) = \int dP'\left(\langle\mathcal{O}(P)\mathcal{O}(P')\rangle|_{\Delta_\mathcal{O}\to\tilde{\Delta}_\mathcal{O}}\right)\cdot\mathcal{O}(P'). \tag{1.205}$$

However, the expressions (1.203) and (1.204) are still not equivalent. This is because we have not yet imposed the correct boundary conditions on (1.204). An elegant way of doing this, from [28], is to demand that under the monodromy $\mathcal{M}$ : $P_{12}\to e^{4\pi i}P_{12}$, the integral picks up a factor $e^{2\pi i\Delta_\mathcal{O}}$ (as opposed to $e^{2\pi i(d-\Delta_\mathcal{O})}$, which is the other solution). The reason for this is that in the $x_{12} \to 0$ expansion, (1.194) has a common factor $(x_{12}^2)^{\frac{\Delta_{\mathcal{O}}}{2}}$.

In summary, we have found two equivalent expressions for the projector $P_{\mathcal{O}}$ in

$$1 = \sum_{\mathcal{O} \text{ primaries}} P_{\mathcal{O}}. \tag{1.206}$$

These are

$$P_{\mathcal{O}} = \sum_{\alpha=\mathcal{O},P\mathcal{O},PP\mathcal{O},\ldots} \frac{|\alpha\rangle\langle\alpha|}{\langle\alpha\alpha\rangle} = \mathcal{N}_{\mathcal{O}} \int dP \, |\mathcal{O}(P)\rangle \cdot \langle\widetilde{\mathcal{O}}(P)| \,, \tag{1.207}$$

where it is understood that when inserting the integral in a four-point function, a monodromy projection must be applied. Note that the coefficient $\mathcal{N}_{\mathcal{O}}$ is fixed by demanding

$$\langle \mathcal{O} P_{\mathcal{O}} \cdots \rangle = \langle \mathcal{O} \cdots \rangle. \tag{1.208}$$

The partial waves are then given by the insertion of $P_{\mathcal{O}}$ in the four-point function:

$$\langle \Phi_1\Phi_2\Phi_3\Phi_4 \rangle = \sum_{\mathcal{O}} \langle \Phi_1\Phi_2 P_{\mathcal{O}} \Phi_3\Phi_4\rangle = \sum_{\mathcal{O},r,s} \lambda^r_{12\mathcal{O}} \lambda^s_{34\mathcal{O}} W^{rs}_{\mathcal{O}}(P_i), \tag{1.209}$$

where

$$W^{rs}_{\mathcal{O}}(P_i) = \sum_t (M_{34\mathcal{O}})^{st} \mathcal{N}_{\mathcal{O}} \int dP \langle \Phi_1\Phi_2\mathcal{O}(P)\rangle^{(r)} \cdot \langle \widetilde{\mathcal{O}}(P)\Phi_3\Phi_4\rangle^{(t)} \Big|_{\mathcal{M}=e^{2\pi i \Delta_{\mathcal{O}}}}, \tag{1.210}$$

and the mixing matrices $(M_{34\mathcal{O}})^{st}$ relate the OPE coefficients $\lambda_{34\mathcal{O}}$ with $\lambda_{34\widetilde{\mathcal{O}}}$:

$$\sum_s \lambda^s_{34\mathcal{O}} (M_{34\mathcal{O}})^{st} = \lambda^t_{34\widetilde{\mathcal{O}}}. \tag{1.211}$$

Using (1.205), the mixing matrices can be computed via

$$\begin{aligned}\sum_p \lambda^p_{12\widetilde{\mathcal{O}}} \langle \Phi_1\Phi_2\widetilde{\mathcal{O}}(P)\rangle^{(p)} &= \sum_{r,p} \lambda^r_{12\mathcal{O}} (M_{12\mathcal{O}})^{rp} \langle \Phi_1\Phi_2\widetilde{\mathcal{O}}(P)\rangle^{(p)} \\ &= \sum_r \lambda^r_{12\mathcal{O}} \int dP' \left( \langle \mathcal{O}(P)\mathcal{O}(P')\rangle|_{\Delta_{\mathcal{O}}\to\tilde{\Delta}_{\mathcal{O}}} \right) \cdot \langle \Phi_1\Phi_2\mathcal{O}(P')\rangle^{(r)}. \end{aligned} \tag{1.212}$$

### 1.6.4 Spinning partial waves via weight-shifting operators

Another approach for computing conformal blocks, in particular for the case where the four-point functions include fields in tensor representations of $SO(d)$ (also known as spinning correlators), was first demonstrated in [45]. It exploits the relation (1.179) (however they used the basis (1.180)) for writing the spinning conformal blocks by applying differential operators on scalar conformal blocks (i.e. those in the expansion of four-point functions with scalar fields). However, as discussed below, this is not the complete story. This technique only relates conformal blocks whose exchanged operators are in the same representation. For example, starting with scalar blocks, the class of spinning conformal blocks derived in this way is associated to the exchange of traceless symmetric operators. Whereas in general, spinning four-point functions may contain contributions from traceless operators with different index symmetry.[16]

Nevertheless, it turns out that, exploiting other properties of the weight-shifting operators from section 1.5, results in an algorithm for—in principle—generating all spinning conformal blocks via the application of differential operators [46]. We reproduce this algorithm in the rest of this subsection.

Let us introduce the following notation for the shadow projector (1.207),

$$P_{\mathcal{O}} \equiv |\mathcal{O}\rangle \bowtie \langle\mathcal{O}| \,, \tag{1.213}$$

which makes its action more explicit. Then the partial wave (1.210), corresponding to the exchange of the primary $\mathcal{O}$, is

$$W_{\mathcal{O}}^{rs} = \langle\Phi_1\Phi_2\mathcal{O}\rangle^{(r)} \bowtie {}^{(s)}\langle\mathcal{O}\Phi_3\Phi_4\rangle. \tag{1.214}$$

Inverting the relation (1.179) for a seed-like three-point function $\langle\Phi_i\Phi_j\mathcal{O}\rangle$ (c.f. subsection 1.4.2), results in an expression of the form

$$\langle\Phi_i'\Phi_j'\mathcal{O}\rangle^{(a)} = \mathscr{D}_{\Phi_i\Phi_j}^{(a)\Phi_i'\Phi_j'}\langle\Phi_i\Phi_j\mathcal{O}\rangle, \tag{1.215}$$

where

$$\mathscr{D}_{\Phi_i\Phi_j}^{(a)\Phi_i'\Phi_j'} : \Phi_i\Phi_j \to \Phi_i'\Phi_j'. \tag{1.216}$$

Therefore we can write

$$\langle\Phi_1'\Phi_2'\mathcal{O}\rangle^{(r)} \bowtie {}^{(s)}\langle\mathcal{O}\Phi_3'\Phi_4'\rangle = \mathscr{D}_{\Phi_1\Phi_2}^{(r)\Phi_1'\Phi_2'}\mathscr{D}_{\Phi_3\Phi_4}^{(s)\Phi_3'\Phi_4'}\langle\Phi_1\Phi_2\mathcal{O}\rangle \bowtie \langle\mathcal{O}\Phi_3\Phi_4\rangle, \tag{1.217}$$

where the partial wave on the right hand side is called 'seed partial wave' (note that it does not carry any extra indices because only one tensor structure appears in each three-point function). For $\mathcal{O}$ in the symmetric traceless representation

[16]This can be understood easily by looking at all possible non-zero three-point functions related to the OPE. For example (1.145) is non-zero for generic values of $\ell_i$, implying that the OPE $[\Delta_1,(\ell_2,\ell_3)] \times [\Delta_2,(0)]$ contains an operator in the representation $[\Delta,(\ell_1,\ell_2,\ell_3)]$.

(STT), $\mathcal{O} = [\Delta, (\ell)]$, and $\Phi_i = [\Delta_i, (0)]$ a scalar, the expression above reproduces the result in [45], however in the new basis of weight-shifting operators. In this case, the right hand side becomes the scalar partial wave studied in [64, 68, 69]. Note that (1.217) does not capture all possible spinning partial waves, since it does not explain how to compute the seed blocks $\langle \Phi_1 \Phi_2 \mathcal{O} \rangle \bowtie \langle \mathcal{O} \Phi_3 \Phi_4 \rangle$, for non-symmetric traceless $\mathcal{O}$. One cannot simply apply an operator of the type $\mathscr{D}^{(a)\Phi_i' \mathcal{O}'}_{\Phi_i \mathcal{O}}$ that changes the representation of $\mathcal{O}$, because the operation $\bowtie$ integrates over all positions of $\mathcal{O}$. Hence a different approach must be used, which we describe now.

Taking (1.176) with $\Phi_j = \mathbf{1}$ and $\Phi_i = \Phi_k' = \mathcal{O}$, $\Phi_k = \Phi_i' = \mathcal{O}'$ gives a relation between two bases of $\Omega \otimes \Phi_i$ for two-point functions:

$$\mathcal{D}^{(b)}_{2A} \langle \mathcal{O}(P_1, Z_1^j) \mathcal{O}(P_2, Z_2^j) \rangle = \sum_a \begin{Bmatrix} \mathcal{O} & \mathbf{1} & \mathcal{O}' \\ \mathcal{O}' & \Omega & \mathcal{O} \end{Bmatrix}^{\bullet b}_{\bullet a} \mathcal{D}^{(a)}_{1A} \langle \mathcal{O}'(P_1, Z_1^j) \mathcal{O}'(P_2, Z_2^j) \rangle, \tag{1.218}$$

where $\mathcal{D}^{(b)} : \mathcal{O} \to \mathcal{O}'$ and $\mathcal{D}^{(n)} : \mathcal{O}' \to \mathcal{O}$. Then using the properties of the shadow projector

$$\begin{aligned} \langle \mathcal{O} | &= \langle \mathcal{O} \mathcal{O} \rangle \bowtie \langle \mathcal{O} | \,, \\ | \mathcal{O}' \rangle &= | \mathcal{O}' \rangle \bowtie \langle \mathcal{O}' \mathcal{O}' \rangle, \end{aligned} \tag{1.219}$$

in the previous expression, implies

$$\langle \mathcal{O} \mathcal{D}^{(b)}_A \mathcal{O} \rangle \bowtie \langle \mathcal{O}' \mathcal{O}' \rangle = \sum_a \begin{Bmatrix} \mathcal{O} & \mathbf{1} & \mathcal{O}' \\ \mathcal{O}' & \Omega & \mathcal{O} \end{Bmatrix}^{\bullet b}_{\bullet a} \langle \mathcal{O} \mathcal{O} \rangle \bowtie \langle \mathcal{D}^{(a)}_A \mathcal{O}' \mathcal{O}' \rangle. \tag{1.220}$$

Therefore we obtain an identity that exchanges the position of the weight-shifting operator within the projector $\bowtie$:

$$| \mathcal{D}^{(b)}_A \mathcal{O} \rangle \bowtie \langle \mathcal{O}' | = \sum_a \begin{Bmatrix} \mathcal{O} & \mathbf{1} & \mathcal{O}' \\ \mathcal{O}' & \Omega & \mathcal{O} \end{Bmatrix}^{\bullet b}_{\bullet a} | \mathcal{O} \rangle \bowtie \langle \mathcal{D}^{(a)}_A \mathcal{O}' | \,. \tag{1.221}$$

Furthermore, combining (1.221) with (1.178) gives a relation that changes the representation of $\mathcal{O}$ in $\bowtie$ to $\mathcal{O}'$:

$$\begin{aligned} | \mathcal{D}^{(m)}_A \mathcal{O} \rangle \bowtie \langle \mathcal{D}^{(n)A} \mathcal{O} | &= \sum_a \begin{Bmatrix} \mathcal{O} & \mathbf{1} & \mathcal{O}' \\ \mathcal{O}' & \Omega & \mathcal{O} \end{Bmatrix}^{\bullet m}_{\bullet a} | \mathcal{O} \rangle \bowtie \langle \mathcal{D}^{(a)}_A \mathcal{D}^{(n)A} \mathcal{O} | \\ &= N_{mn} \, | \mathcal{O} \rangle \bowtie \langle \mathcal{O} | \,, \end{aligned} \tag{1.222}$$

where

$$N_{mn} = \sum_a \begin{Bmatrix} \mathcal{O} & \mathbf{1} & \mathcal{O}' \\ \mathcal{O}' & \Omega & \mathcal{O} \end{Bmatrix}^{\bullet m}_{\bullet a} \begin{pmatrix} \mathcal{O} \\ \mathcal{O}' \; \Omega \end{pmatrix}^{an} . \tag{1.223}$$

Then using this result, the partial wave associated with $\mathcal{O}$ can be written as

$$\begin{aligned}\langle\Phi_1\Phi_2\mathcal{O}\rangle^{(a)} \bowtie {}^{(b)}\langle\mathcal{O}\Phi_3\Phi_4\rangle &= N_{mn}^{-1}\langle\Phi_1\Phi_2\mathcal{D}_A^{(m)}\mathcal{O}\rangle^{(a)} \bowtie {}^{(b)}\langle\mathcal{D}^{(n)A}\mathcal{O}\Phi_3\Phi_4\rangle \\ &= N_{mn}^{-1}\sum_{\Phi_1'\in\Omega\otimes\Phi_1,\Phi_4'\in\Omega\otimes\Phi_4}\sum_{r,s,t,u}\begin{Bmatrix}\Phi_1 & \Phi_2 & \Phi_1' \\ \mathcal{O}' & \Omega & \mathcal{O}\end{Bmatrix}_{rs}^{am}\begin{Bmatrix}\Phi_4 & \Phi_3 & \Phi_4' \\ \mathcal{O}' & \Omega & \mathcal{O}\end{Bmatrix}_{tu}^{bn} \\ &\qquad \times \mathcal{D}_1^{(s)}\cdot\mathcal{D}_4^{(u)}\langle\Phi_1'\Phi_2\mathcal{O}'\rangle^{(r)} \bowtie {}^{(t)}\langle\mathcal{O}'\Phi_3\Phi_4'\rangle. \quad (1.224)\end{aligned}$$

where we used (1.176) in the second line. Then $\langle\Phi_1'\Phi_2\mathcal{O}'\rangle^{(r)} \bowtie {}^{(t)}\langle\mathcal{O}'\Phi_3\Phi_4'\rangle$ can be written in terms of a seed partial wave $W_{\mathcal{O}'}$ via (1.217). In particular, if $\mathcal{O}'$ is a symmetric traceless tensor, and we pick $\Phi_i$ such that $\langle\Phi_1\Phi_2\mathcal{O}\rangle^{(a)} \bowtie {}^{(b)}\langle\mathcal{O}\Phi_3\Phi_4\rangle$ is a seed, then (1.224) provides a way to write any seed partial wave in terms of scalar partial waves.

## 1.7 Bootstrap

Earlier in 1.6.1, it was elucidated that the four-point function can be reduced to a combination of two-point functions via the insertion of the OPE. More generally, inserting one OPE in an $n$-point function, leads to an expression that depends on the representations of the operators exchanged in the OPE $[\Delta_i, \rho_i]$, the OPE coefficients $\lambda_j$, and $(n-1)$-point functions. By repeating this procedure, the $(n-1)$-point functions are reduced to two-point functions, which are completely determined by conformal symmetry. Therefore, knowledge of the 'CFT data' $\{[\Delta_i, \rho_i], \lambda_j\}$ associated to a theory, is enough to completely determine any $n$-point function.

Inserting the OPE in different orders between different pairs of operators leads a different expression of the same $n$-point function. If two or more expansions have a common region of convergence, then they must agree—this is called OPE associativity—, leading to non-trivial constraints for the CFT data in general. It is then important to understand the properties of the OPE when inserted into correlators, which we describe in this section.

### 1.7.1 OPE coefficient relations

Recall from subsection 1.6.1 that, up to a two-point function normalization, the coefficients that appear in the OPE are the same as the three-point function constants. Hence, by associativity, different insertions of the OPE in a three-point function leads to relations between OPE coefficients. More precisely, consider the three-point function $\langle \Phi_1 \Phi_2 \Phi_3 \rangle$. In our notation, the coefficient associated to the operator $\Phi_3$ appearing in the $\Phi_1 \times \Phi_2$ OPE is, $\lambda_{\Phi_1\Phi_2\Phi_3}$. By associativity, inserting the OPE in (12) must be consistent with inserting it in (13) and (23). This then gives linear relations between the OPE coefficients $\lambda_{\Phi_1\Phi_2\Phi_3}$, $\lambda_{\Phi_1\Phi_3\Phi_2}$, and $\lambda_{\Phi_2\Phi_3\Phi_1}$. Moreover, the OPE is symmetric under $\Phi_1 \times \Phi_2 \to \Phi_2 \times \Phi_1$, so that $\lambda_{\Phi_{\sigma(1)}\Phi_{\sigma(2)}\Phi_{\sigma(3)}}$ is related to $\lambda_{\Phi_1\Phi_2\Phi_3}$, with $\sigma$ any permutation of the three points. From (1.144) we have

$$\begin{aligned}\sum_l \lambda^l_{\Phi_1\Phi_2\Phi_3} Q^l_{\Phi_1\Phi_2\Phi_3}(P_i, Z_i^j) &= \sum_{l'} \lambda^{l'}_{\Phi_{\sigma(1)}\Phi_{\sigma(2)}\Phi_{\sigma(3)}} Q^{l'}_{\Phi_{\sigma(1)}\Phi_{\sigma(2)}\Phi_{\sigma(3)}}(P_{\sigma(i)}, Z^j_{\sigma(i)}) \\ &= \sum_{l',l} \lambda^{l'}_{\Phi_{\sigma(1)}\Phi_{\sigma(2)}\Phi_{\sigma(3)}} (\mathcal{M}^{\Phi_i}_{\Phi_{\sigma(i)}})^{l'l} Q^l_{\Phi_1\Phi_2\Phi_3}(P_i, Z_i^j), \end{aligned} \tag{1.225}$$

or

$$\lambda^l_{\Phi_1\Phi_2\Phi_3} = \sum_{l'} \lambda^{l'}_{\Phi_{\sigma(1)}\Phi_{\sigma(2)}\Phi_{\sigma(3)}} (\mathcal{M}^{\Phi_i}_{\Phi_{\sigma(i)}})^{l'l}, \tag{1.226}$$

where we used the fact that $\mathcal{K}_3^{\Delta_1\Delta_2\Delta_3}(P_1, P_2, P_3)$ is invariant under the simultaneous permutation of two $P$ and two $\Delta$. Note that the form of the matrix $\mathcal{M}$ depends on how we define the tensor structures $Q$ for the permuted case.

### 1.7.2 Bootstrap equations

So far, OPE associativity in three-point functions resulted in somewhat trivial constraints for OPE coefficients. This is because, not only is the kinematic factor $\mathcal{K}_3$ invariant under permutations, but all the OPE coefficients depend on the same triplet of operators. For four-point functions, however, the partial wave expansion involves all the OPE coefficients of operators appearing in each channel, and additionally, the conformal blocks depend on the operator dimensions and spins in a non-trivial way [64, 68, 69]. This suggests that OPE associativity produces non-trivial constraints for the CFT data, and furthermore, solving them provides a way to classify the space of consistent CFTs. This is called the conformal bootstrap program, and it has already produced numerous successful studies in both numerical and analytical fronts. For a concise review on current results from the conformal bootstrap see [73]. A more detailed overview of the state-of-the-art developments in bootstrap technology has recently appeared in [74].

For four-point functions, there are three independent partial wave expansions (or channels): (12)(34) or s channel, (14)(23) or t channel, (13)(24) or u channel. However, convergence of the OPE expansion $(ij)(kl)$ is guaranteed only if there is a sphere containing the points $i, j$, which does not intersect a sphere containing $k, l$.

Thus for a given configuration of points $i, j, k, l$, only two of the three channels converge. Nonetheless, it turns out that partial waves in a given channel can be analytically continued to a larger region of regularity, by applying conformal transformations which do not change the cross-ratios $U$, $V$ [48, 63].

Note that for fermionic fields, different channels might be equal up to a sign due to the Grassmann nature of the fields [34]. Here we focus on bosonic quantities only. Imposing OPE associativity—also known as crossing symmetry in the context of four-point functions—on the partial waves (1.185) then implies the so-called bootstrap equations

$$\begin{aligned}\sum_{\mathcal{O}} &\Lambda^{12;34}_{\mathcal{O}} W^{\Phi_1\Phi_2\Phi_3\Phi_4}_{\mathcal{O}}(P_1, P_2, P_3, P_4; Z^j_1, Z^j_2, Z^j_3, Z^j_4) \\ &= \sum_{\mathcal{O}'} \Lambda^{14;23}_{\mathcal{O}'} W^{\Phi_1\Phi_4\Phi_2\Phi_3}_{\mathcal{O}'}(P_1, P_4, P_2, P_3; Z^j_1, Z^j_4, Z^j_2, Z^j_3) \\ &= \sum_{\mathcal{O}''} \Lambda^{13;24}_{\mathcal{O}''} W^{\Phi_1\Phi_3\Phi_2\Phi_4}_{\mathcal{O}''}(P_1, P_3, P_2, P_4; Z^j_1, Z^j_3, Z^j_2, Z^j_4), \end{aligned} \quad (1.227)$$

where $\Lambda^{ij;kl}_{\mathcal{O}}$ is the matrix of OPE coefficients $\lambda^r_{ij\mathcal{O}}\lambda^s_{kl\mathcal{O}}$ as in (1.193). These equations can be re-casted in terms of conformal blocks via (1.186). Note that in order completely 'solve a CFT' one needs to solve (1.227) for all possible fields $\Phi_i$. Generically, including more correlators, results in stronger constraints. Compare for example [53, 75] with [50, 76]. At this point is important to realize that in order to study the implications of the bootstrap equations (1.227) in their current form, it is essential to have enough knowledge and control over the partial waves $W$ in coordinate representation. Alternative approaches also exist [57–59, 77–87], where the partial waves are mapped into a different space via, e.g. integral transforms.

Similar to the three-point function case, we can also look at the constraints arising from exchanging the order of the OPE $\Phi_i \times \Phi_j \to \Phi_j \times \Phi_i$, so-called exchange symmetry. However, from the discussion of the previous subsection, and the partial wave expansion (1.193), it is easy to see that these constraints are solved term by term in $\mathcal{O}$, leading to 'trivial' restrictions on the form of $W_{\mathcal{O}}$. Nonetheless, these relations are useful when computing new spinning partial waves, as they provide sanity checks.

### 1.7.3 Light-cone limit

Solving the bootstrap equations (1.227) is a difficult task because it is an infinite system of equations in an infinite number of variables. Nevertheless, approaches for successfully extracting useful information do exist, as reviewed in [73, 74]. In particular, it is possible to study the bootstrap equations analytically in certain interesting limits that still capture different universal features of the theory. These include the large dimension limit [63], the light-cone limit [2, 88–106], as well as the Regge limit [80, 107–112].

Here we focus on the light-cone limit. This is defined by taking the following limit of the cross-ratios[17]

$$u \ll v < 1, \tag{1.228}$$

and re-organizing the sum over primaries in (1.227)—parametrized by dimensions $\Delta$ and spins $\ell$—as a sum parametrized by twists $\tau = \Delta - \ell$ and spins $\ell$ [88, 89]. The result is that the minimal twist contributions from the s-channel are only reproduced by double-trace operators of large spin in the t-channel. Let us see an example in detail.

Let us focus on the scalar correlator case $\langle \Phi_i \Phi_j \Phi_k \Phi_l \rangle$, where the bootstrap equation (for $s = t$) is

$$\sum_{\mathcal{O}} \lambda_{ij\mathcal{O}} \lambda_{kl\mathcal{O}} g_{\mathcal{O}}^{\Delta_{ij},\Delta_{kl}}(u,v) = u^{\frac{\Delta_k+\Delta_l}{2}} v^{-\frac{\Delta_j+\Delta_k}{2}} \sum_{\mathcal{O}} \lambda_{il\mathcal{O}} \lambda_{kj\mathcal{O}} g_{\mathcal{O}}^{\Delta_{il},\Delta_{kj}}(v,u). \tag{1.229}$$

It is useful to introduce a different set of variables

$$u = z\bar{z}, \quad v = (1-z)(1-\bar{z}). \tag{1.230}$$

Now the light-cone limit is reproduced by taking small $z$, so that $\bar{z} = (1-v) + O(z)$ and $u = z(1-v) + O(z^2)$. When matching the bootstrap equation (1.229), we work in the variables $z$ and $v$.

In the s-channel, the scalar conformal blocks have the following behavior around $u = 0$ [64]

$$g_{\tau,\ell}^{\Delta_{ij},\Delta_{kl}}(u,v) = u^{\tau/2} f_{\tau,\ell}^{ij,kl}(1-v)(1+O(u)), \tag{1.231}$$

$$f_{\tau,\ell}^{ij,kl}(1-v) = (1-v)^{\ell}\, {}_2F_1\Big(\frac{\tau}{2} + \ell - \frac{\Delta_{ij}}{2}, \frac{\tau}{2} + \ell + \frac{\Delta_{kl}}{2}, \tau + 2\ell; 1-v\Big). \tag{1.232}$$

From this expression, it is clear that the leading order contributions in $u$ will be given by operators with minimal twist $\tau_m$.

In the t-channel the large spin scalar blocks factorize into [88, 99]

$$g_{\tau,\ell}^{\Delta_{il},\Delta_{kj}}(v,u) \overset{\substack{\ell \gg 1,\ z\ell^2 \lesssim O(1) \\ u \ll v < 1}}{=} f_1^{\Delta_{il},\Delta_{kj}}(\ell, z) f_2^{\Delta_{il},\Delta_{kj}}(\tau, v)(1 + O(1/\sqrt{\ell}, \sqrt{z})), \tag{1.233}$$

[17] There are just the projection of the embedded cross ratios $U$ and $V$ from (1.119) to $\mathbb{R}^d$.

where $u \approx z(1-v)$ and

$$f_1^{\Delta_{il},\Delta_{kj}}(\ell,z) = \pi^{-\frac{1}{2}} 2^{2\ell} \ell^{\frac{1}{2}} z^{\frac{\Delta_{il}-\Delta_{kj}}{4}} K_{\frac{\Delta_{kj}-\Delta_{il}}{2}}(2\ell\sqrt{z}), \tag{1.234}$$

$$\begin{aligned} &f_2^{\Delta_{il},\Delta_{kj}}(\tau,v) \\ &= \frac{2^\tau v^{\frac{\tau}{2}}}{(1-v)^{d/2-1}} {}_2F_1\left(\frac{\tau-d+2-\Delta_{il}}{2}, \frac{\tau-d+2+\Delta_{kj}}{2}, \tau-d+2; v\right). \end{aligned} \tag{1.235}$$

This limit holds for even dimensions $d \geq 2$ as long as the sum over $\ell$ only receives contributions in the region where the product $\ell^2 z$ is kept fixed. Note that sometimes, different normalizations of the conformal block are used. For example, by multiplying $f_{\tau,\ell}^{ij,kl}$ and $f_1^{\Delta_{il},\Delta_{kj}}(\ell,z)$ with a factor of $(-2)^{-\ell}$. However, this will only change the normalization of the OPE coefficients for the large spin operators in the t-channel.

Using these results we can demand that both sides of (1.229) match order by order in $z$ and $v$, and fix the spectrum and OPE coefficients of the large spin operators in the t-channel.

### s-channel

Expanding the functions $f_{\tau,\ell}$ around $v=0$ gives the following

$$\begin{aligned} \sum_{\mathcal{O}} \lambda_{ij\mathcal{O}} \lambda_{kl\mathcal{O}} g_{\mathcal{O}}^{\Delta_{ij},\Delta_{kl}}(u,v) &\overset{|u|\ll|v|<1}{\approx} \delta_{\mathbb{I}\in\phi_i\times\phi_j} + \sum_{\tau_m,\ell_m} \lambda_{ij(\tau_m,\ell_m)} \lambda_{kl(\tau_m,\ell_m)} \\ \times \frac{z^{\frac{\tau_m}{2}}(1-v)^{\frac{\tau_m}{2}+\ell_m}\Gamma(\tau_m+2\ell_m)}{\Gamma(\frac{\tau_m}{2}+\ell_m-\frac{\Delta_{ij}}{2})\Gamma(\frac{\tau_m}{2}+\ell_m+\frac{\Delta_{ij}}{2})} &\sum_{k=0}^{\infty} \frac{(\frac{\tau_m}{2}+\ell_m-\frac{\Delta_{ij}}{2})_k(\frac{\tau_m}{2}+\ell_m+\frac{\Delta_{ij}}{2})_k}{(k!)^2} v^k \\ \times [2\psi(k+1) - \psi(\frac{\tau_m}{2}+\ell_m+k-\frac{\Delta_{ij}}{2}) &- \psi(\frac{\tau_m}{2}+\ell_m+k+\frac{\Delta_{ij}}{2}) - \ln(v)] \end{aligned} \tag{1.236}$$

provided $|v| < 1$, $\Delta_{kl} - \Delta_{ij} = 0$,[18] and we used $u \approx z(1-v)$. The term $\delta_{\mathbb{I}\in\phi_i\times\phi_j}$ gives a 1 if the identity operator can be exchanged (i.e. if $\phi_i \times \phi_j = 1 + \dots$) and 0 otherwise.

[18] There are further expansions of this type when $\Delta_{kl} - \Delta_{ij}$ is a positive integer, but for our purposes we will not assume this. As far as I know there are no closed form expansions for arbitrary $\Delta_{kl} - \Delta_{ij}$.

### t-channel

For the t-channel we get

$$\frac{u^{\frac{\Delta_k+\Delta_l}{2}}}{v^{\frac{\Delta_j+\Delta_k}{2}}}\sum_{\mathcal{O}}\lambda_{il\mathcal{O}}\lambda_{kj\mathcal{O}}g_{\mathcal{O}}^{\Delta_{il},\Delta_{kj}}(v,u) \overset{\substack{\ell\gg1,\ z\ell^2\lesssim O(1)\\ u\ll v<1}}{\approx} \frac{z^{\frac{\Delta_k+\Delta_l}{2}}(1-v)^{\frac{\Delta_k+\Delta_l}{2}}}{v^{\frac{\Delta_j+\Delta_k}{2}}}\delta_{\mathbb{I}\in\phi_i\times\phi_l} + \frac{z^{\frac{\Delta_k+\Delta_l}{2}}(1-v)^{\frac{\Delta_k+\Delta_l}{2}}}{v^{\frac{\Delta_j+\Delta_k}{2}}}\sum_{\tau}\left(\sum_{\ell}\lambda_{il(\tau,\ell)}\lambda_{kj(\tau,\ell)}f_1^{\Delta_{il},\Delta_{kj}}(\ell,z)\right)f_2^{\Delta_{il},\Delta_{kj}}(\tau,v), \tag{1.237}$$

where the delta is now 1 when $\phi_i\times\phi_l=1+\ldots$. Notice that the OPE coefficients $\lambda_{il(\tau,\ell)}\lambda_{kj(\tau,\ell)}$ need to, at least, cancel the terms in $f_1(\ell,z)$ that contribute to the $\ell^2 z\gg1$ region of the sum over $\ell$,[19] namely the $2^{2\ell}$ term. The factor in parenthesis is then a finite expression depending on $z$ only, which must match the powers of $z$ in (1.236).

However, at this point, $f_2$ has a power series expansion for small $v$, that does not reproduce the $\ln(v)$ term in (1.236). To do it, we assume that the allowed values of $\tau$ have a particular pattern, say $\tau\approx\tau(n)+\gamma(n,\ell)$,[20] where $\gamma(n,\ell)$ is a small correction—an anomalous dimension—that depends on $n$ and $\ell$. Then we replace the sum over $\tau$ by a sum over $n$ and expand the terms that depend on $\tau$ around small $\gamma(n,\ell)$. Clearly, $\ln(v)$ will arise from terms of the form $v^{g(\tau)}$.

Here we write the expansion of the OPE coefficients $\lambda_{il(\tau,\ell)}\lambda_{kj(\tau,\ell)}$ around $\gamma(n,\ell)=0$, as

$$\lambda_{il(\tau,\ell)}\lambda_{kj(\tau,\ell)}=\lambda_{il(\tau(n),\ell)}\lambda_{kj(\tau(n),\ell)}+\lambda_{il(\tau(n),\ell)}\lambda_{kj(\tau(n),\ell)}C^{il,kj}(n,\ell), \tag{1.238}$$

$$\lambda_{il(\tau(n),\ell)}\lambda_{kj(\tau(n),\ell)}C^{il,kj}(n,\ell)\equiv\gamma(n,\ell)[\partial_\gamma\lambda_{il(\tau(n)+\gamma,\ell)}\lambda_{kj(\tau(n)+\gamma,\ell)}]_{\gamma=0}. \tag{1.239}$$

Putting these results into (1.237) gives

$$\frac{u^{\frac{\Delta_k+\Delta_l}{2}}}{v^{\frac{\Delta_j+\Delta_k}{2}}}\sum_{\mathcal{O}}\lambda_{il\mathcal{O}}\lambda_{kj\mathcal{O}}g_{\mathcal{O}}^{\Delta_{il},\Delta_{kj}}(v,u) \overset{\substack{\ell\gg1,\ z\ell^2\lesssim O(1)\\ u\ll v<1}}{\approx} \frac{z^{\frac{\Delta_k+\Delta_l}{2}}(1-v)^{\frac{\Delta_k+\Delta_l}{2}}}{v^{\frac{\Delta_j+\Delta_k}{2}}}\delta_{\mathbb{I}\in\phi_i\times\phi_l} + \frac{z^{\frac{\Delta_k+\Delta_l}{2}}(1-v)^{\frac{\Delta_k+\Delta_l}{2}}}{v^{\frac{\Delta_j+\Delta_k}{2}}}\sum_{n}\left(\sum_{\ell}\lambda_{il(\tau(n),\ell)}\lambda_{kj(\tau(n),\ell)}f_1^{\Delta_{il},\Delta_{kj}}(\ell,z)\times\left[1+C^{il,kj}(n,\ell)+\frac{\gamma(n,\ell)}{\tau'(n)}\frac{d}{dn}\right]\right)f_2^{\Delta_{il},\Delta_{kj}}(\tau(n),v). \tag{1.240}$$

[19] This is because $f_1(\ell,z)$ is only valid for fixed $z\ell^2$, and thus we want the sum to be dominated by that region. The term $2^{2\ell}$ blows up for large $\ell$ making the $\ell^2 z\gg1$ sector significant.

[20] This is equivalent to assuming that for each $\ell$ there is only one exchanged operator with twist $\tau(n)+\gamma(n,\ell)$. Otherwise we would need to parametrize the twist pattern by an additional index.

Notice that acting $\frac{d}{dn}$ on $v^n$ produces $v^n \ln(v)$ as expected.

### Matching the bootstrap equation

By matching (1.236) and (1.240), one can compute the large $\ell$ spectrum as well as the behavior of the OPE coefficients in the t-channel. We assume the OPE coefficients factorize at large $\ell$ into $\lambda_{il(\tau(n),\ell)}\lambda_{kj(\tau(n),\ell)} \overset{\ell\gg 1}{\approx} N_{il,kj}(n)L_{il,kj}(\ell)$, and perform the sum over $\ell$ at zero-th order in $\gamma$. This is done by approximating the sum by an integral, and using [88, 96]

$$\int_0^\infty dl\ l^\alpha K_\nu(2l\sqrt{z}) = \frac{z^{-(\alpha+1)/2}}{4}\Gamma(\frac{1+\alpha-\nu}{2})\Gamma(\frac{1+\alpha+\nu}{2}), \tag{1.241}$$

where $1+\alpha-\nu > 0$ and $1+\alpha+\nu > 0$. The correct value of $\alpha$ (which in turn, fixes $L_{il,kj}(\ell)$) is obtained by matching leading order of (1.236) in $z$ (i.e. the 1).[21] With the $z$ dependence out of the way, we fix $N_{il,kj}(n)$ and $\tau(n)$ by checking that the $v$ dependence agrees on both sides. In other words, performing the sum over $n$ in (1.240) must match the identity in (1.236).

Matching the subleading terms $z^{\tau_m/2}$,[22] fixes the anomalous dimensions $\gamma(n,\ell)$, as well as the OPE corrections $C^{il,kj}(n,\ell)$ in terms of the OPE coefficients of the minimal twist operator $\lambda_{ij(\tau_m,\ell_m)}\lambda_{kl(\tau_m,\ell_m)}$. In particular, let us concentrate on the $\ln(v)$ terms. Writing the form of the anomalous dimensions as

$$\gamma(n,\ell) = \frac{\gamma_n}{\ell^{p(\tau_m)}}, \tag{1.242}$$

and matching the $z^{\tau_m}$ term, fixes the exponent $p(\tau_m)$. Whereas matching the factors $v^k$ multiplying $\ln(v)$, fixes the form of $\gamma_n$ (one has to pick $\gamma_n$ such that performing the full sum over $n$ reproduces (1.236), order by order in $v$). Following the same procedure for the non-log terms fixes the OPE corrections $C^{il,kj}(n,\ell)$.

## 1.8 Lorentzian CFT and causality

So far, we have focused on conformal field theories in Euclidean space $\mathbb{R}^d$ because, from the point of view of physical observables (correlators), it is known that a Lorentzian theory that is unitary, causal, and Lorentz invariant is in one-to-one

[21] Note that the first term on the right hand side of (1.240) is subleading in z for unitary theories $\Delta_i > 0$, c.f. subsection 1.4.3, so we can ignore it.

[22] Here we have to assume that the first term on the right hand side of (1.240) is subleading in z with respect to $z^{\tau_m/2}$. In other words $\Delta_m - \ell_m < \Delta_k + \Delta_l$. For example if $\Delta_k = \Delta_l = \Delta$ and the minimal twist operator is the stress tensor, then we have $d-2 < 2\Delta$, which is just the unitarity bound for scalars.

correspondence with a Euclidean theory satisfying reflection positivity, crossing symmetry, Euclidean invariance, and certain growth conditions [113–118]. However, this does not explain what exactly happens to the Euclidean theory when causality is violated. It turns out that by studying this question, combined with the bootstrap philosophy, one gains more insight into the space of consistent CFT data [94].

### 1.8.1 Causality

Causality is the statement that local field operators may only have an effect on others within their light-cones. In other words, in Lorentzian signature $x_i = (t_i, x_i^1, \ldots, x_i^{d-1}) \in \mathbb{R}^{1,d-1}$,

$$[\phi_1(x_1), \phi_2(x_2)]\big|_{x_{12}^2>0} = 0. \tag{1.243}$$

Recall from section 1.3 that conformal correlators develop singularities whenever $x_{ij}^2 = 0$. In the Euclidean theory, with coordinates $x_i = (x_i^0, x_i^1, \ldots, x_i^{d-1}) \in \mathbb{R}^d$, this only happens when $x_i = x_j$. Hence we have crossing symmetry $[\phi_i(x_i), \phi_j(x_j)] = 0$ as long as the points are not coincident. On the other hand, in Lorentzian signature, singularities appear when touching the light-cone of other operators $(t_i - t_j)^2 = \sum_k (x_i^k - x_j^k)^2$.

One can try mapping the Euclidean correators to Minkowski space, by analytically continuing one arbitrary direction, say $x_i^0 \to e^{i\alpha} x_i^0$, at the expense of making the correlators multivalued. Imposing causality in this setting, then means that for timelike separated operators, their commutator must be given by the discontinuity across the branch cut of the light-cone singularity. It is easier to understand this statement with an example. Consider the two-point function $\langle \phi(0)\phi(x)\rangle = (x^2)^{-\Delta}$, with $x = (x^0, x^1)$. After analytic continuation $x^0 \to e^{i\alpha} x^0$, it becomes $e^{-2i\alpha\Delta}((x^0)^2 - (x^1)^2)^{-\Delta}$, with $e^{-2i\alpha} = -1$. For timelike operators $x^0 > x^1$, there are many $\alpha$ that send $x^0$ above the singularity $x^0 = ix^1$. One option is following the path $\alpha \in [0, \pi/2]$, hitting the imaginary axis from the right. Another option is to send $x^0 \to -x^0$ and then follow the path $\alpha \in [0, -\pi/2]$, hitting the imaginary axis from the left. Note that the branch cut (whose direction we have to set in order for the analytic continuation to make sense) has to be crossed by one of the two paths. The difference between the two paths is then

$$\langle \phi(0)\phi(x)\rangle|_{\text{first}} - \langle \phi(0)\phi(x)\rangle|_{\text{second}} = \left(e^{-i\pi\Delta} - e^{i\pi\Delta}\right)((x^0)^2 - (x^1)^2)^{-\Delta}. \tag{1.244}$$

If we set our interpretation such that the first path corresponds to a time-ordered correlator, while the second path is anti-time-ordered, then we arrive at the expected result:

$$\langle[\phi(0), \phi(x)]\rangle\big|_{x^2<0} = \left(e^{-i\pi\Delta} - e^{i\pi\Delta}\right)((x^0)^2 - (x^1)^2)^{-\Delta} \neq 0 \tag{1.245}$$

On the other hand, if the operators are spacelike $x^0 < x^1$, then none of the paths crosses the branch cut, which means that all analytic continuations live in the same sheet of the multivalued correlator. In other words,

$$\langle[\phi(0), \phi(x)]\rangle\big|_{x^2>0} = 0. \tag{1.246}$$

For a correlator with more operators, the same idea applies; different time orderings correspond to different paths around light-cone singularities.

Another way of making sense of the analytic continuation is the the $i\epsilon$ recipe. A correlator in the following time order

$$\langle\phi_1(t_1, x_1)\cdots\phi_n(t_n, x_n)\rangle, \tag{1.247}$$

is computed by the following euclidean correlator

$$\lim_{\epsilon_j\to 0}\langle\phi_1(t_1 - i\epsilon_1, x_1)\cdots\phi_n(t_n - i\epsilon_n, x_n)\rangle, \tag{1.248}$$

where we take the metric to be $\mathbb{R}^{1,d-1}$ and $\epsilon_1 > \cdots > \epsilon_n > 0$. This has the effect of moving the position of the light-cone singularities out of the imaginary axis in a particular way, which is equivalent to the previous discussion in terms of continuation paths.

In summary, whenever the branch points of the correlator coincide with the light-cone singularities, causality is guaranteed. In the first sheet of the Lorentzian correlator (i.e. the one that embeds the Euclidean region), causality then follows from the form of the Euclidean correlator (the fact that it is singular at coincident points). However, after crossing a branch cut into another sheet, it is not obvious that the singularities will remain at the same place. Therefore a way to test for causality, is to check that correlators in the second sheet are not singular away from light-cones. Combining this idea with bootstrap techniques, it was shown in [94] that causality implies non-trivial constraints on OPE coefficients of light operators in scalar correlators. This was later generalized to spinning correlators in [2, 95].

### 1.8.2 Constraints from causality

Now we review the argument of [94] for constraining the CFT data in the four-point functions of scalars:[23]

$$G(z, \bar{z}) = \langle\phi(0)O(z, \bar{z})O(1)\phi(\infty)\rangle. \tag{1.249}$$

[23] Here $\phi(\infty)$ implies $\lim_{x\to\infty} x^{2\Delta_\phi}\phi(x)$.

Here the light-cone coordinates $z, \bar{z}$ are defined in (1.230). More precisely, plugging the coordinates

$$x_1 = (0,0), \quad x_2 = (x_2^0, x_2^1), \quad x_3 = (0,1), \quad x_4 = (0,\infty), \tag{1.250}$$

into the cross-ratios results in

$$u = z\bar{z}, \quad v = (1-z)(1-\bar{z}), \quad z = ix_2^0 + x_2^1, \quad \bar{z} = -ix_2^0 + x_2^1. \tag{1.251}$$

Analytically continuing $x_2^0 \to it_2$ then implies $z = -t_2 + x_2^1, \quad \bar{z} = t_2 + x_2^1$. Note that $\bar{z} = z^*$ in the Euclidean case, while in Lorentzian signature they are independent. We take this continuation as the first sheet of $G(z,\bar{z})$. As discussed in the previous subsection, the other analytic continuation (which crosses the branch cut of $\phi(0)$), differs by a clockwise rotation around the light-cone $z = 0$: $z \to ze^{-2\pi i}$. Hence the second sheet is $G(ze^{-2\pi i}, \bar{z})$.

The idea now is to check that the correlator is analytic in a region close to the light-cone of $O(1)$, $(z,\bar{z}) \sim (1,1)$, for both sheets (and hence causal). Following, [94] we define

$$z = 1 + \sigma, \tag{1.252}$$

$$\bar{z} = 1 + \eta\sigma, \tag{1.253}$$

where $\sigma$ is complex with $\mathrm{Im}(\sigma) \geq 0$ and $|\sigma| \leq R$, while $\eta$ is real and satisfies $0 < \eta \ll R \ll 1$. On the $\sigma$ plane, this corresponds to a half disc above $\sigma = 0$. We refer to this as region $D$.

Now we define the normalized four-point functions on the first and the second sheet as

$$G_\eta(\sigma) \equiv \frac{\langle \phi(0) O(z,\bar{z}) O(1) \phi(\infty)\rangle}{\langle \phi(0)\phi(\infty)\rangle \langle O(z,\bar{z}) O(1)\rangle} = (\eta\sigma^2)^{\Delta_O} G(1+\sigma, 1+\eta\sigma), \tag{1.254}$$

$$\widehat{G}_\eta(\sigma) \equiv \frac{\langle \phi(0) O(ze^{-2\pi i},\bar{z}) O(1) \phi(\infty)\rangle}{\langle \phi(0)\phi(\infty)\rangle \langle O(z,\bar{z}) O(1)\rangle} = (\eta\sigma^2)^{\Delta_O} G((1+\sigma)e^{-2\pi i}, 1+\eta\sigma). \tag{1.255}$$

In what follows, we show that both of these are analytic in $D$, and finite at $\sigma = 0$. As seen in 1.7.2, we can expand $G(z,\bar{z})$ in the s, t, and u channels respectively:

$$G(z,\bar{z}) = (z\bar{z})^{-\frac{1}{2}(\Delta_O + \Delta_\phi)} \sum_{\mathcal{O}} \lambda_{\phi O \mathcal{O}} \lambda_{O\phi\mathcal{O}} g^{\Delta_{\phi O}, -\Delta_{\phi O}}_{\Delta_{\mathcal{O}}, \ell_{\mathcal{O}}}(z,\bar{z}), \tag{1.256}$$

$$G(z,\bar{z}) = [(1-z)(1-\bar{z})]^{-\Delta_O} \sum_{\mathcal{O}} \lambda_{OO\mathcal{O}} \lambda_{\phi\phi\mathcal{O}} g^{0,0}_{\Delta_{\mathcal{O}}, \ell_{\mathcal{O}}}(1-z, 1-\bar{z}), \tag{1.257}$$

$$G(z,\bar{z}) = (z\bar{z})^{\frac{1}{2}(\Delta_\phi + \Delta_O)} \sum_{\mathcal{O}} \lambda_{\phi O \mathcal{O}} \lambda_{O\phi\mathcal{O}} g^{\Delta_{\phi O}, -\Delta_{\phi O}}_{\Delta_{\mathcal{O}}, \ell_{\mathcal{O}}}(1/z, 1/\bar{z}). \tag{1.258}$$

The form of the prefactors indicate that these expansions converge for $|z| < 1$, $|1-z| < 1$, and $|z| > 1$ respectively. The analyticity of $G_\eta(\sigma)$ in $D$ is then a direct consequence of the convergence of the t-channel expansion. Furthermore, in the limit $z, \bar{z} \to 1$ with $(1-z)/(1-\bar{z})$ fixed (equivalently, $\sigma \to 0$ with $\eta$ fixed), the t-channel has an expansion in conformal dimension [68, 69]

$$G_\eta(\sigma) = 1 + \sum_{\Delta,\ell} a_{\Delta,\ell}\, \eta^{\frac{1}{2}(\Delta-\ell)} \sigma^{\Delta}, \tag{1.259}$$

where the sum is over all primaries and descendants appearing in $\phi\times\phi$. For unitary theories, $\Delta > 0$ (c.f. subsection 1.4.3), this implies that $G_\eta(0) = 1$ is finite.

On the second sheet, the transformation $z \to z e^{-2\pi i}$ can have a nontrivial effect on the t-channel sum. Thus analyticity of $\widehat{G}$ is not guaranteed from the convergence of (1.257). For example, in the light-cone limit $\bar{z} \to 1$ with $z$ fixed (equivalently, $\eta \to 0$ and $\sigma$ finite), the t-channel expansion is, c.f. (1.236),

$$G(z,\bar{z}) = [(1-z)(1-\bar{z})]^{-\Delta_O} \left(1 + \sum_{\mathcal{O}_m} \lambda_m (1-\bar{z})^{\frac{1}{2}(\Delta_m - \ell_m)} \tilde{g}_{\Delta_m,\ell_m}(1-z) + \dots \right), \tag{1.260}$$

where $\lambda_m = \lambda_{OO\mathcal{O}_m}\lambda_{\phi\phi\mathcal{O}_m}$ is the coefficient of the contributions of minimal twist operators $\mathcal{O}_m$. From (1.236),[24] one can check that $\tilde{g}_{\Delta_m,\ell_m}(1-z)$ is regular on the first sheet around $z = 1$, but develops singularities on the second sheet due to the log term. More precisely, applying $z \to z e^{-2\pi i}$ to the corresponding version of (1.236) and then taking the $z \to 1$ limit, gives the following expression for $\widehat{G}_\eta(\sigma)$ in $\eta \ll |\sigma| \ll 1$:

$$\widehat{G}_\eta(\sigma) = 1 - i\hat{\lambda}_m \frac{\eta^{\frac{1}{2}(\Delta_m - \ell_m)}}{\sigma^{\ell_m - 1}} + \dots, \tag{1.261}$$

where $\mathcal{O}_m$ is the minimal twist operator of largest spin and

$$\hat{\lambda}_m = \lambda_m \times \frac{2^{1-\ell_m}\pi\Gamma(\Delta_m + \ell_m)^2}{(\Delta_m + \ell_m - 1)\Gamma(\frac{1}{2}(\Delta_m + \ell_m))^4}. \tag{1.262}$$

Due to these singularities, the convergence on the first sheet does not imply convergence on the second sheet. However, this does not mean that $\widehat{G}_\eta(0)$ is divergent, as we will see below.

[24] Note that (1.236) is an s-channel expansion, so we need to apply $z \to 1-\bar{z}$, $\bar{z} \to 1-z$ to make it compatible with (1.260).

**Positivity to the rescue**

Reflection positivity in radial quantization (see e.g. [10,11]) states that $\langle f|\, f\rangle \geq 0$, where

$$|f\rangle \equiv \int_0^1 dr_1 \int_0^{2\pi} d\theta f(r_1,\theta_1) O(r_1 e^{i\theta_1}, r_1 e^{-i\theta_1})\phi(0)\,|0\rangle\,, \tag{1.263}$$

for arbitrary $f(r,\theta)$. In fact, following [28, 49, 88], this can be further refined by inserting the projector $P_{\mathcal{O}}$ from (1.207), and still have positivity:

$$\langle f|P_{\mathcal{O}}|f\rangle \geq 0. \tag{1.264}$$

It was shown in [94] that $\langle f|\, f\rangle \geq 0$ implies the s-channel expansion can be written as

$$G(z,\bar{z}) = (z\bar{z})^{-\frac{1}{2}(\Delta_O+\Delta_\phi)} \sum_{h,\bar{h}>0} a_{h,\bar{h}} z^h \bar{z}^{\bar{h}}, \qquad a_{h,\bar{h}} \geq 0. \tag{1.265}$$

Additionally, this can be checked explicitly in $d = 4$, using the closed form conformal blocks of [64]. The refined condition $\langle f|P_{\mathcal{O}}|f\rangle \geq 0$ implies that each partial wave is also a positive coefficient expansion:

$$\lambda_{O\phi\mathcal{O}}\lambda_{\phi O\mathcal{O}} g_{\mathcal{O}}^{\Delta_{\phi O},-\Delta_{\phi O}}(z,\bar{z}) = z^{-a}\bar{z}^{-b} \sum_{p,q\in\mathbb{Z}_+} b_{p,q} z^p \bar{z}^q, \qquad b_{p,q} \geq 0, \tag{1.266}$$

where the powers $a, b$ depend on the scaling dimensions.

Recall that in Euclidean signature ($\bar{z} = z^*$), the s-channel expansion converges for $|z| < 1$. Therefore the positivity of $a_{h,\bar{h}}$ implies that in Lorentzian signature, the sum still converges for independent complex variables $|z|, |\bar{z}| < 1$. More precisely,

$$\left|(z\bar{z})^{\frac{1}{2}(\Delta_O+\Delta_\phi)} G(z,\bar{z})\right| = \left|\sum_{h,\bar{h}>0} a_{h,\bar{h}} z^h \bar{z}^{\bar{h}}\right| \leq \sum_{h,\bar{h}>0} a_{h,\bar{h}} |z|^h |\bar{z}|^{\bar{h}}, \quad |z|, |\bar{z}| < 1. \tag{1.267}$$

Thus $\widehat{G}_\eta(\sigma)$ is analytic in the region $D \cap \{|z|, |\bar{z}| < 1\}$. Actually, using radial coordinates [48], it is shown in [94], that the region of convergence for the s-channel expansion can be expanded to the whole complex plane excluding $[1, +\infty)$. In other words, $\widehat{G}_\eta(\sigma)$ is analytic in $D/[0, R]$.

The same line of arguments hold for the u-channel expansion in terms of $1/z$ and $1/\bar{z}$. This results in the analyticity of $\widehat{G}_\eta(\sigma)$ in $D/[-R, 0]$. Hence $\widehat{G}_\eta(\sigma)$ is analytic in $D$.[25]

[25] The branch cuts of $G(z,\bar{z})$ with respect to $z$ are chosen as follows. The first branch cut originates from $z = 1$ and lies in the lower half plane. The second branch cut originates from $z = 0$ and lies in the negative real axis. The branch cuts on the $\bar{z}$ plane are chosen in the same way.

Restricting to $\sigma \in \mathbb{R}$, leads to the inequality $|\widehat{G}_\eta(\sigma)| \leq G_\eta(\sigma)$, or equivalently

$$\mathrm{Re}(G_\eta(\sigma) - \widehat{G}_\eta(\sigma)) \geq 0, \qquad \sigma \in [-R, R]. \tag{1.268}$$

### Bounds from analyticity

Analyticity of both $G_\eta(\sigma)$ and $\widehat{G}_\eta(\sigma)$ in the region $D$ implies that no poles are picked up by the contour integral around $D$:

$$\oint_{\partial D} d\sigma \sigma^k G_\eta(\sigma) = \oint_{\partial D} d\sigma \sigma^k \widehat{G}_\eta(\sigma) = 0, \tag{1.269}$$

for any $k > -1$. The contour is a sum of the half circle $S$ and the real line segment $[-R, R]$. In particular, for $\ell_m \geq 2$ we can write the sum rule

$$\mathrm{Re} \oint_{\partial D} d\sigma \sigma^{\ell_m - 2} (\widehat{G}_\eta(\sigma) - G_\eta(\sigma)) = 0. \tag{1.270}$$

However, from (1.259) and (1.261), the part of the integral along the semicircle picks up the residue of the pole in $\sigma$:

$$\mathrm{Re} \int_S d\sigma \sigma^{\ell_m - 2} (\widehat{G}_\eta(\sigma) - G_\eta(\sigma)) = \pi \hat{\lambda}_m \eta^{\frac{1}{2}(\Delta_m - \ell_m)} + O(R^{\ell_m - 1}), \tag{1.271}$$

where we used the identity

$$\mathrm{Re}\, i \int_S d\sigma \sigma^n = -\pi \delta_{n,-1}. \tag{1.272}$$

Finally, from the sum rule (1.270) and the positivity property (1.268), we obtain for $\ell_m > 1$[26]

$$\hat{\lambda}_m = \frac{1}{\pi} \lim_{R \to 0} \lim_{\eta \to 0} \eta^{-\frac{1}{2}(\Delta_m - \ell_m)} \int_{-R}^{R} d\sigma \sigma^{\ell_m - 2} \mathrm{Re}(G_\eta(\sigma) - \widehat{G}_\eta(\sigma)) \geq 0, \tag{1.273}$$

where $\hat{\lambda}_m \propto \lambda_{OO\mathcal{O}_m} \lambda_{\phi\phi\mathcal{O}_m}$.

Note that if $\mathcal{O}_m$ is the stress tensor, then (1.273) gives a trivial constraint the OPE coefficients $\lambda_{OOT}\lambda_{\phi\phi T}$, as it can also be derived using the Ward identity (1.20). However, as we will see in chapter 4, reproducing this argument for spinning correlators leads to nontrivial constraints.

[26] Otherwise the dots in (1.261) dominate and the integral may not be well defined.

## 1.9 Preview of the main results

In this introduction we illustrated the concepts and techniques that constitute the backbone of the thesis. We started from the basic properties that characterize conformal field theories (CFTs), and then moved to a formalism which encodes the conformal symmetry into the linear action of the orthogonal group. Using this language, we discussed how the two- and three-point correlation functions of the theory are fixed due to the symmetries of the CFT. Related to this, we reviewed a useful technique for transforming the representations of the fields inside correlators via the action of derivatives with respect to the coordinates. Then we moved to four-point functions and described several techniques to expand these objects in terms of conformal partial waves and the numerical parameters that determine each theory—the CFT data. Next using the associativity properties of this expansion we reviewed how this can be translated into self-consistency conditions—the bootstrap equations—that every well-behaved CFT must satisfy. Finally, we showed two analytical techniques for extracting universal information regarding the CFT data: one of them is based on solving the bootstrap equations in a particular kinematic limit (the light-cone limit), and the other exploits the positivity and analyticity of the partial wave expansions to derive constraints for causal theories.

Moving forward, the thesis is structured as follows

- in chapter 2 we explicitly construct all conformal blocks required for the four-point function of two scalars and two vectors, in closed form. In particular this includes the conformal block associated to the exchange of an operator with mixed symmetry, in terms of the usual conformal blocks for traceless-symmetric operators.
- in chapter 3 we use the weight-shifting operator formalism to construct a recursion for computing all the spinning seed partial waves that can appear in the four-point function of traceless-symmetric tensors in closed form. The recursion connects these mixed-symmetric seed partial waves to the usual traceless-symmetric ones in a finite number of steps, given by the number of boxes in the second and third rows of the mixed-symmetric Young diagram.
- in chapter 4 we apply both the light-cone bootstrap, as well as causality arguments to constrain the following spinning four-point functions: two scalars and two conserved spin-1 currents ($J$), two scalars and two vectors ($V$), and two scalars and two conserved spin-2 tensors ($T$). For each case, solving the light-cone bootstrap determines the spectrum of the infinite towers of large spin double-trace operators, as well as their anomalous dimensions and

OPE coefficients in terms of the stress-tensor CFT data. From causality, we derive constraints for the OPE coefficients $\lambda_{JJT}$, $\lambda_{VVT}$, and $\lambda_{TTT}$. Then combining both results shows that the anomalous dimensions of the large spin double-trace operators have a definite sign.

- finally in chapter 5 we show how spinning partial waves appear in AdS gravity. In particular we demonstrate how spinning Witten diagrams decompose in terms of spinning partial waves, and we develop a formalism for generating gravitational bulk interactions directly from the different OPE tensor structures. Given the nature of the duality this also provides a technique for computing spinning partial waves using holography.

# 2 Scalar-Vector conformal blocks

On conformal blocks for the four-point function of two scalars and two vectors

Conformal blocks are an essential ingredient for studying conformal field theories. They are universal in the sense that the they provide a basis for expanding the four-point functions of any theory, in terms of CFT data. Therefore they must be generic enough to encode the kinematic dependence of four points in any possible interacting conformal theory, in any space-time dimension. This makes the computation of conformal blocks in closed form quite difficult. In this chapter, based on [1], we provide explicit expressions for all spinning conformal blocks in the four-point function of two scalars and two vectors.

## 2.1 Introduction

As discussed in section 1.6 there exist several techniques for computing conformal blocks, each with their advantages and disadvantages. The first instance of an explicit closed form solution was given in [68, 68, 69]. In those articles, the authors obtained closed form expressions for conformal blocks of four scalars (also referred to as scalar blocks) in even dimensions $d$. Furthermore, they showed that the OPE, Casimir, and shadow projector techniques (see subsections 1.6.1, 1.6.2 and 1.6.3 respectively, for details) give consistent results. Later, as mentioned in subsection 1.6.4, the spinning conformal blocks associated to symmetric-traceless (STT) operators $\mathcal{O} = [\Delta_{\mathcal{O}}, (\ell)]$ were first derived using the differential operators defined in (1.180).

However, for spinning four-point functions, operators in representations other than STT can be exchanged. In [28, 29] it was shown, using the shadow operator method, that conformal blocks associated to the exchange of $\mathcal{A} = [\Delta_{\mathcal{A}}, (k+1,1)]$ (with fixed $k$) can be calculated as a (finite) sum of scalar blocks evaluated at zero spin. In principle, this can be implemented on a computer but the numerical evaluation of these blocks is quite resource intensive, due to the fact that the number of terms in the sum increases rapidly with the spin of $\mathcal{A}$. In numerical computations one might get away with if the maximum spin of $\mathcal{A}$ is not too large, but this approach is hopeless in the analytic bootstrap, where one needs to have control over the conformal blocks at very high spin (c.f. subsection 1.7.3). Later in [37], recursion relations for spinning conformal blocks for mixed-symmetric exchanges were given, by exploiting the recursive properties of the integrand in the shadow integral (1.210). However, these are recursions in the length of the first row of the exchanged representation.

The main objective of this chapter is to use the shadow projector technique to find an explicit and closed form expression for the conformal block associated to the mixed-symmetric operator $\mathcal{A}$ (for arbitrary $k$), in the four-point function of two scalars and two vectors, and whose 'complexity' does not increase with $k$. Note, from the discussion of 1.6.4, that this corresponds to a seed spinning block, and thus can be used to generate spinning conformal blocks of other correlators which exchange $\mathcal{A}$. This result is achieved by writing the contractions appearing in the shadow integrand in a way that resembles conformal blocks associated to the symmetric traceless representation. In passing we re-derive expressions for these symmetric traceless blocks by applying differential operators acting on scalar blocks, as in subsection 1.5.2.

This chapter is organized as follows. In section 2.3 we write explicit expressions for the relevant three-point functions that appear in the conformal partial waves, when written in the shadow formalism. Using these we then compute the mixing matrices and numerical normalization of the shadow projector. Section 2.4 presents the main tool for expressing the conformal block associated to the the mixed-symmetric representation $\mathcal{A}$, in terms of conformal blocks in the symmetric representation $\mathcal{O}$. In section 2.5 we put all the results together and compute the symmetric traceless conformal blocks for the four-point functions $\langle\phi\phi\phi\phi\rangle$, $\langle\phi v\phi\phi\rangle$, $\langle\phi\phi\phi v\rangle$, and $\langle\phi v\phi v\rangle$. Finally, we present the main result of this chapter, which is a closed form expression for the conformal blocks $\langle\phi v\phi v\rangle$ in the mixed-symmetric representation $\mathcal{A}$, in terms of the symmetric ones. We conclude in section 2.6. Details, identities, definitions, and extended computations are written in appendices A, B, C, D, E, F, G, and H.

## 2.2 Overview and conventions

In this chapter we will use so-called physical coordinates $x^\mu \in \mathbb{R}^d$ rather than the embedding formalism from section 1.2. Conformal correlators in physical space are obtained from the embedded version via the projections in subsection 1.3.3. Notice that these expressions are still contracted with polarization vectors $z_i^j$. The full expressions in terms of open indices are obtained by applying a proper differential operator that implements the $SO(d)$ symmetry. For symmetric traceless representation this is given by [31, 119]

$$D_a = \left(\frac{d}{2} - 1 + z \cdot \frac{\partial}{\partial z}\right) \frac{\partial}{\partial z^a} - \frac{1}{2} z_a \frac{\partial^2}{\partial z \cdot \partial z}. \tag{2.1}$$

Then the non-contracted tensor is

$$f_{a_1,\ldots,a_\ell} = \frac{1}{\ell!(d/2-1)_\ell} D_{a_1} \cdots D_{a_\ell} f(z). \tag{2.2}$$

For the representation $(\ell, 1)$, the relevant differential operator was computed in [37]. However, here we will construct the tensor structures 'by hand' by projecting the indices of the tensor structures to the relevant $SO(d)$ representations, and using the physical space building blocks $\{m, k\}$ from (1.138):

$$k_a^{(ijk)} = \frac{x_{ij}^2 \left(x_{ik}\right)_a - x_{ik}^2 \left(x_{ij}\right)_a}{\left(x_{ij}^2 x_{ik}^2 x_{jk}^2\right)^{1/2}}, \tag{2.3}$$

$$m_{ab}^{(ij)} = \delta_{ab} - \frac{2}{x_{ij}^2} \left(x_{ij}\right)_a \left(x_{ij}\right)_b. \tag{2.4}$$

In appendix A, we write several useful formulae and identities for these building blocks.

The main object of interest is the four-point function $\langle \phi v \phi v \rangle$. From (1.181) and the algorithm described in subsection 1.3.2, this is given by

$$\begin{aligned}
&\langle \phi_1(x_1) v_{2\,a}(x_2) \phi_3(x_3) v_{4\,b}(x_4) \rangle \\
&\quad = \mathcal{K}_4^{\Delta_i} \Big[ q_0(u,v) m_{ab}^{(24)} + q_{11}(u,v) k_a^{(214)} k_b^{(412)} + q_{12}(u,v) k_a^{(214)} k_b^{(432)} \\
&\qquad\qquad + q_{21}(u,v) k_a^{(234)} k_b^{(412)} + q_{22}(u,v) k_a^{(234)} k_b^{(432)} \Big].
\end{aligned} \tag{2.5}$$

More precisely, the two Young diagrams $\square$ can be either connected with $m$, or filled with $k$. Given that there are two types of $k$ for each box, we arrive at the tensor structures above.

The partial waves for $f_{ab}$ are, by (1.210),[1]

$$W_O^{rs} = \sum_t (M_{34O})^{st} \mathcal{N}_O \times \int d^d x_0 \, \langle \phi_1(x_1) v_{2\,a}(x_2) O_{b_1\cdots b_\ell}(x_0) \rangle^{(r)} \left\langle \phi_3(x_3) v_{4\,b}(x_4) \widetilde{O}^{b_1\cdots b_\ell}(x_0) \right\rangle^{(t)} \bigg|_{\mathcal{M}=e^{2\pi i \Delta_O}} , \tag{2.6}$$

and the conformal blocks can be read off from (1.186). It is easy to see from the counting formula (1.134), that there are two classes of non-zero three-point functions involving a scalar and a vector. In particular, one of the three-point functions is given by (1.146) with $\ell_2 = 0$, and the other is (1.145) with $\ell_2 = 1$, $\ell_3 = 0$. This means that $O$ can be of two types. One is the the totally symmetric traceless tensor of spin $\ell$, $\mathcal{O}_{a_1\cdots a_\ell}(x)$, which includes scalars and vectors as special cases. The other type is a mixed symmetry tensor $\mathcal{A}_{a_1 a_2 b_1\cdots b_k}(x)$ which is completely traceless, is antisymmetric in $a_1$ and $a_2$, is totally symmetric in the $b_i$, and vanishes when antisymmetrized over any three indices. In terms of $SO(d)$ representations, $\mathcal{O}_{a_1\cdots a_\ell} \in (\ell)$, while $\mathcal{A}_{a_1 a_2 b_1\cdots b_k} \in (k+1, 1)$ . For each of these cases we construct projectors onto the given representation in Appendix B. For $\mathcal{O}_{a_1\cdots a_\ell}$ and $\mathcal{A}_{a_1 a_2 b_1\cdots b_k}$ we use projectors

$$\Pi^{(\ell)\, b_1\cdots b_\ell}_{a_1\cdots a_\ell}, \qquad \text{and} \qquad \widetilde{\Pi}^{(k)\, c_1 c_2 d_1\cdots d_k}_{a_1 a_2 b_1\cdots b_k}, \tag{2.7}$$

respectively.

Our plan of action for finding a closed form for $W_{\mathcal{A}}$ in terms of partial waves $W_{\mathcal{O}}$, is the following:

- calculate the mixing matrices $M$ and normalization factors $\mathcal{N}$ in (2.6),
- compute the integrand of (2.6) for $O = \mathcal{A}$, and find a way to relate it to integrands of $O = \mathcal{O}$,
- write the partial waves $W_{\mathcal{O}}$ in terms of conformal blocks $g_{\mathcal{O}}$ and use the previous result to give an expression for $g_{\mathcal{A}}$ in terms of $g_{\mathcal{O}}$.

[1] Recall from the discussion of subsection 1.6.3 that the integral must pick up a factor $e^{2\pi i \Delta_{\mathcal{O}}}$ under the monodromy $\mathcal{M} : x_{12}^2 \to e^{4\pi i} x_{12}^2$, in order for the partial wave to have the desired boundary conditions.

## 2.3 Mixing matrices and normalization factors

### 2.3.1 Three-point functions

For the sake of clarity, we reproduce the relevant three-point functions from subsection 1.4.2 in physical space. The symbols $\phi$, $v$, $\mathcal{O}$, and $\mathcal{A}$ represent scalar, vector, symmetric traceless, and mixed-symmetric traceless fields respectively. We also write the explicit form of (1.179) for each case, which allows us to write $\langle \phi v \mathcal{O} \rangle$ in terms of $\langle \phi\phi\mathcal{O} \rangle$.

**Case $\langle \phi\phi\mathcal{O} \rangle$:**

$$\langle \phi_1(x_1)\phi_2(x_2)\mathcal{O}_{a_1\cdots a_\ell}(x_3)\rangle = \lambda_{12\mathcal{O}} S^{\lambda}_{a_1\cdots a_\ell}(x_i;\Delta_i), \tag{2.8}$$

where we defined

$$S^{\lambda}_{a_1\cdots a_\ell}(x_i;\Delta_i) = \left(x_{12}^2\right)^{\frac{1}{2}(-\Delta_1-\Delta_2+\Delta_{\mathcal{O}})} \left(x_{13}^2\right)^{\frac{1}{2}(-\Delta_1+\Delta_2-\Delta_{\mathcal{O}})} \left(x_{23}^2\right)^{\frac{1}{2}(\Delta_1-\Delta_2-\Delta_{\mathcal{O}})} \\ \times \Pi^{(\ell)\, b_1\cdots b_\ell}_{a_1\cdots a_\ell} k^{(312)}_{b_1}\cdots k^{(312)}_{b_\ell}. \tag{2.9}$$

**Case $\langle \phi v\mathcal{O} \rangle$:**

$$\langle \phi_1(x_1)v_a(x_2)\mathcal{O}_{b_1\cdots b_\ell}(x_3)\rangle = \alpha_{12\mathcal{O}} S^{\alpha}_{a\, b_1\cdots b_\ell}(x_i;\Delta_i) + \beta_{12\mathcal{O}} S^{\beta}_{a\, b_1\cdots b_\ell}(x_i;\Delta_i), \tag{2.10}$$

where

$$\begin{aligned} & S^{\alpha}_{a\, b_1\cdots b_\ell}(x_i;\Delta_\phi,\Delta_v,\Delta_{\mathcal{O}}) \\ &\qquad \equiv \left(x_{12}^2\right)^{\frac{1}{2}(-\Delta_\phi-\Delta_v+\Delta_{\mathcal{O}})} \left(x_{13}^2\right)^{\frac{1}{2}(-\Delta_\phi+\Delta_v-\Delta_{\mathcal{O}})} \left(x_{23}^2\right)^{\frac{1}{2}(\Delta_\phi-\Delta_v-\Delta_{\mathcal{O}})} \\ &\qquad\qquad \times \Pi^{(\ell)\, c_1\cdots c_\ell}_{b_1\cdots b_\ell} k^{(213)}_{a} k^{(312)}_{c_1}\cdots k^{(312)}_{c_\ell} \\ &= \frac{1}{2\,(1-\Delta_{\mathcal{O}})} \left[ m^{(12)}{}_a{}^c \left(\frac{\partial}{\partial x_1^c} + 2\,(\Delta_\phi - 1)\,\frac{(x_{12})_c}{x_{12}^2}\right) S^{\lambda}_{b_1\cdots b_\ell}(x_i;\Delta_\phi-1,\Delta_v,\Delta_{\mathcal{O}}) \right. \\ &\qquad \left. + \left(\frac{\partial}{\partial x_2^a} - 2\,(\Delta_v-1)\,\frac{(x_{12})_a}{x_{12}^2}\right) S^{\lambda}_{b_1\cdots b_\ell}(x_i;\Delta_\phi,\Delta_v-1,\Delta_{\mathcal{O}}) \right], \end{aligned} \tag{2.11}$$

and

$$
\begin{aligned}
&S^{\beta}_{a\,b_1\cdots b_\ell}(x_i;\Delta_\phi,\Delta_v,\Delta_{\mathcal{O}}) \\
&\quad= \left(x_{12}^2\right)^{\frac{1}{2}(-\Delta_\phi-\Delta_v+\Delta_{\mathcal{O}})}\left(x_{13}^2\right)^{\frac{1}{2}(-\Delta_\phi+\Delta_v-\Delta_{\mathcal{O}})}\left(x_{23}^2\right)^{\frac{1}{2}(\Delta_\phi-\Delta_v-\Delta_{\mathcal{O}})} \\
&\qquad\qquad \times \Pi^{(\ell)\,c_1\cdots c_\ell}_{b_1\cdots b_\ell} m^{(23)}_{ac_1} k^{(312)}_{c_2}\cdots k^{(312)}_{c_\ell} \\
&\quad= \frac{\Delta_\phi-\Delta_v-\Delta_{\mathcal{O}}+\ell+1}{\ell} S^{\alpha}_{a\,b_1\cdots b_\ell}(x_i;\Delta_\phi,\Delta_v,\Delta_{\mathcal{O}}) \\
&\qquad -\frac{1}{\ell}\left(\frac{\partial}{\partial x_2^a}-2\left(\Delta_v-1\right)\frac{(x_{12})_a}{x_{12}^2}\right) S^{\lambda}_{b_1\cdots b_\ell}(x_i;\Delta_\phi,\Delta_v-1,\Delta_{\mathcal{O}}), \qquad (2.12)
\end{aligned}
$$

as can be verified by explicit computation. Note that if $\ell = 0$, then we only have the first term in (2.10).

**Case $\langle \phi v \mathcal{A}\rangle$:**

$$
\begin{aligned}
\langle \phi(x_1) v_a(x_2) \mathcal{A}_{b_1 b_2 c_1\cdots c_k}(x_3)\rangle &= \gamma_{\phi v\mathcal{A}}\left(x_{12}^2\right)^{\frac{1}{2}(-\Delta_\phi-\Delta_v+\Delta_{\mathcal{A}})}\left(x_{13}^2\right)^{\frac{1}{2}(-\Delta_\phi+\Delta_v-\Delta_{\mathcal{A}})} \\
&\times \left(x_{23}^2\right)^{\frac{1}{2}(\Delta_\phi-\Delta_v-\Delta_{\mathcal{A}})} \widetilde{\Pi}^{(k)\,d_1 d_2 e_1\cdots e_k}_{b_1 b_2 c_1\cdots c_k} m^{(23)}_{ad_1} k^{(312)}_{d_2} k^{(312)}_{e_1}\cdots k^{(312)}_{e_k}. \qquad (2.13)
\end{aligned}
$$

### 2.3.2 Normalization factors

According to (1.207) and (1.208), we compute the normalization factor $\mathcal{N}_{\mathcal{O}}$ by demanding

$$
\langle \mathcal{O}_{a_1\cdots a_\ell}(x_3)\varphi_1(x_1)\varphi_2(x_2)\rangle = \langle \mathcal{O}_{a_1\cdots a_\ell}(x_3) P_{\mathcal{O}}\varphi_1(x_1)\varphi_2(x_2)\rangle\,, \qquad (2.14)
$$

which leads to (see appendix E)

$$
\mathcal{N}_{\mathcal{O}} = \pi^{-d}\frac{(\Delta_{\mathcal{O}}+\ell-1)\,(d-\Delta_{\mathcal{O}}+\ell-1)\,\Gamma(\Delta_{\mathcal{O}}-1)\Gamma(d-\Delta_{\mathcal{O}}-1)}{\Gamma(\Delta_{\mathcal{O}}-\frac{d}{2})\Gamma(\frac{d}{2}-\Delta_{\mathcal{O}})}. \qquad (2.15)
$$

Similarly, for the mixed symmetry case we obtain

$$
\mathcal{N}_{\mathcal{A}} = \pi^{-d}\frac{(\Delta_{\mathcal{A}}+k)\,(d-\Delta_{\mathcal{A}}+k)\,\Gamma(\Delta_{\mathcal{A}})\Gamma(d-\Delta_{\mathcal{A}})}{(\Delta_{\mathcal{A}}-2)\,(d-\Delta_{\mathcal{A}}-2)\,\Gamma(\Delta_{\mathcal{A}}-\frac{d}{2})\Gamma(\frac{d}{2}-\Delta_{\mathcal{A}})}. \qquad (2.16)
$$

Note that $\mathcal{N}$ is independent of $\Delta_1$ and $\Delta_2$, as it should be.

### 2.3.3 Computation of mixing matrices

In physical space, the definition of the shadow operators (1.205) for $\mathcal{O}$ and $\mathcal{A}$ are

$$
\widetilde{\mathcal{O}}_{a_1\cdots a_\ell}(x_1) = \Pi^{(\ell)\,b_1\cdots b_\ell}_{a_1\cdots a_\ell}\int \frac{d^d x_0}{\left(x_{01}^2\right)^{d-\Delta_{\mathcal{O}}}} m^{(01)}{}_{b_1}{}^{c_1}\cdots m^{(01)}{}_{b_\ell}{}^{c_\ell}\mathcal{O}_{c_1\cdots c_\ell}(x_0), \qquad (2.17)
$$

$$\widetilde{\mathcal{A}}_{b_1 b_2 c_1 \cdots c_k}(x_3) = \widetilde{\Pi}^{(k)\, d_1 d_2 e_1 \cdots e_k}_{b_1 b_2 c_1 \cdots c_k} \times \int \frac{d^d x_0}{(x_{03}^2)^{d-\Delta_\mathcal{A}}} m^{(03)}{}_{d_1}{}^{f_1} m^{(03)}{}_{d_2}{}^{f_2} m^{(03)}{}_{e_1}{}^{g_1} \cdots m^{(03)}{}_{e_k}{}^{g_k} \mathcal{A}_{f_1 f_2 g_1 \cdots g_k}(x_0). \quad (2.18)$$

Then to compute the mixing matrices $M$ in

$$\sum_s \lambda^s_\mathcal{O} (M_\mathcal{O})^{st} = \lambda^t_{\widetilde{\mathcal{O}}}, \quad (2.19)$$

we will use equation (1.212).

**Case** $\langle \phi\phi\widetilde{\mathcal{O}}\rangle$**:**

Consider first the case where $\mathcal{O}_{a_1 \cdots a_\ell}$ is symmetric traceless, and the other two operators are scalars $\phi_1$ and $\phi_2$. The three-point function with $\widetilde{\mathcal{O}}$ is, by (2.8)

$$\left\langle \phi_1(x_1)\phi_2(x_2)\widetilde{\mathcal{O}}_{a_1\cdots a_\ell}(x_3)\right\rangle = \lambda_{\widetilde{\mathcal{O}}} \left(x_{12}^2\right)^{\frac{1}{2}\left(-\Delta_1-\Delta_2+\Delta_{\widetilde{\mathcal{O}}}\right)} \left(x_{13}^2\right)^{\frac{1}{2}\left(-\Delta_1+\Delta_2-\Delta_{\widetilde{\mathcal{O}}}\right)} \times \left(x_{23}^2\right)^{\frac{1}{2}\left(\Delta_1-\Delta_2-\Delta_{\widetilde{\mathcal{O}}}\right)} \Pi^{(\ell)\, b_1\cdots b_\ell}_{a_1\cdots a_\ell} k^{(312)}_{b_1} \cdots k^{(312)}_{b_\ell}, \quad (2.20)$$

where $\Delta_{\widetilde{\mathcal{O}}} = d - \Delta_\mathcal{O}$. Inserting the definition of the shadow operator (2.17) and performing the integral leads to [2]

$$\lambda_{\widetilde{\mathcal{O}}} = \pi^{d/2} \frac{\Gamma(\Delta_\mathcal{O} - \frac{d}{2})\Gamma(\Delta_\mathcal{O} + \ell - 1)}{\Gamma(\Delta_\mathcal{O} - 1)\Gamma(d - \Delta_\mathcal{O} + \ell)} \times \frac{\Gamma(\frac{1}{2}(d + \Delta_1 - \Delta_2 - \Delta_\mathcal{O} + \ell))\Gamma(\frac{1}{2}(d - \Delta_1 + \Delta_2 - \Delta_\mathcal{O} + \ell))}{\Gamma(\frac{1}{2}(\Delta_1 - \Delta_2 + \Delta_\mathcal{O} + \ell))\Gamma(\frac{1}{2}(-\Delta_1 + \Delta_2 + \Delta_\mathcal{O} + \ell))} \lambda_\mathcal{O}, \quad (2.21)$$

where we are taking $\lambda_\mathcal{O} = \lambda_{12\mathcal{O}}$. The mixing matrix is then just a number given by (2.19).

**Case** $\langle \phi v\widetilde{\mathcal{O}}\rangle$**:**

Next, we consider symmetric traceless $\mathcal{O}(x_3)$, but in a three-point function with a scalar $\phi(x_1)$ and a vector $v_a(x_2)$. In this case, the OPE coefficients of (2.10) get

[2] Details on the computation of these integrals are given in appendix E

mixed:

$$\begin{aligned}
&\alpha_{\widetilde{\mathcal{O}}} = \pi^{d/2} \\
&\times \frac{\Gamma(\frac{1}{2}(d+\Delta_\phi-\Delta_v-\Delta_{\mathcal{O}}+\ell+1))\Gamma(\frac{1}{2}(d-\Delta_\phi+\Delta_v-\Delta_{\mathcal{O}}+\ell-1))\Gamma(\Delta_{\mathcal{O}}-\frac{d}{2})}{\Gamma(\frac{1}{2}(\Delta_\phi-\Delta_v+\Delta_{\mathcal{O}}+\ell+1))\Gamma(\frac{1}{2}(-\Delta_\phi+\Delta_v+\Delta_{\mathcal{O}}+\ell+1))\Gamma(\Delta_{\mathcal{O}})} \\
&\quad\times \frac{\Gamma(\Delta_{\mathcal{O}}+\ell-1)}{\Gamma(d-\Delta_{\mathcal{O}}+\ell)}\left[\frac{1}{2}\left(\left(\Delta_{\mathcal{O}}+\ell-1\right)\left(d-\Delta_{\mathcal{O}}-1\right)-\left(\Delta_{\mathcal{O}}-1\right)\left(\Delta_\phi-\Delta_v\right)\right)\alpha_{\mathcal{O}}\right. \\
&\qquad\qquad\left.-\left(\Delta_{\mathcal{O}}-\frac{d}{2}\right)\left(\Delta_\phi-\Delta_v+\Delta_{\mathcal{O}}+\ell-1\right)\beta_{\mathcal{O}}\right], \qquad (2.22)
\end{aligned}$$

$$\begin{aligned}
&\beta_{\widetilde{\mathcal{O}}} = \pi^{d/2} \\
&\times \frac{\Gamma(\frac{1}{2}(d+\Delta_\phi-\Delta_v-\Delta_{\mathcal{O}}+\ell-1))\Gamma(\frac{1}{2}(d-\Delta_\phi+\Delta_v-\Delta_{\mathcal{O}}+\ell-1))\Gamma(\Delta_{\mathcal{O}}-\frac{d}{2})}{\Gamma(\frac{1}{2}(\Delta_\phi-\Delta_v+\Delta_{\mathcal{O}}+\ell+1))\Gamma(\frac{1}{2}(-\Delta_\phi+\Delta_v+\Delta_{\mathcal{O}}+\ell+1))\Gamma(\Delta_{\mathcal{O}})} \\
&\quad\times \frac{\Gamma(\Delta_{\mathcal{O}}+\ell-1)}{\Gamma(d-\Delta_{\mathcal{O}}+\ell)}\left[\frac{\ell}{2}\left(\Delta_{\mathcal{O}}-\frac{d}{2}\right)\left(\Delta_\phi-\Delta_v\right)\alpha_{\mathcal{O}}+\frac{1}{4}\left(\Delta_\phi-\Delta_v+\Delta_{\mathcal{O}}+\ell-1\right)\right. \\
&\qquad\qquad\left.\times\left(\left(\Delta_{\mathcal{O}}-1\right)\left(d-\Delta_{\mathcal{O}}+\ell-1\right)-\left(d-\Delta_{\mathcal{O}}-1\right)\left(\Delta_\phi-\Delta_v\right)\right)\beta_{\mathcal{O}}\right]. \qquad (2.23)
\end{aligned}$$

Then the mixing matrix

$$\alpha_{\phi v\widetilde{\mathcal{O}}} = M_\alpha{}^\alpha \alpha_{\phi v\mathcal{O}} + M_\alpha{}^\beta \beta_{\phi v\mathcal{O}}, \qquad \beta_{\phi v\widetilde{\mathcal{O}}} = M_\beta{}^\alpha \alpha_{\phi v\mathcal{O}} + M_\beta{}^\beta \beta_{\phi v\mathcal{O}}, \qquad (2.24)$$

can be read off from (2.22) and (2.23).

**Case $\langle\phi v\widetilde{\mathcal{A}}\rangle$:**

Finally, we turn to the mixed symmetry operator $\mathcal{A}_{b_1b_2c_1\cdots c_k}(x_3)$. Similar techniques to those employed above, results in

$$\begin{aligned}
\gamma_{\widetilde{\mathcal{A}}} = \pi^{d/2}&\frac{\Gamma(\Delta_{\mathcal{A}}+k)\Gamma(\Delta_{\mathcal{A}}-\frac{d}{2})\Gamma(\frac{1}{2}\left(d+\Delta_\phi-\Delta_v-\Delta_{\mathcal{A}}+k+1\right))}{\Gamma(\Delta_{\mathcal{A}})\Gamma(d-\Delta_{\mathcal{A}}+k+1)\Gamma(\frac{1}{2}\left(\Delta_\phi-\Delta_v+\Delta_{\mathcal{A}}+k+1\right))} \\
&\times\frac{\Gamma(\frac{1}{2}\left(d-\Delta_\phi+\Delta_v-\Delta_{\mathcal{A}}+k+1\right))}{\Gamma(\frac{1}{2}\left(-\Delta_\phi+\Delta_v+\Delta_{\mathcal{A}}+k+1\right))}\left(\Delta_{\mathcal{A}}-2\right)\gamma_{\mathcal{A}}. \qquad (2.25)
\end{aligned}$$

## 2.4 The mixed-symmetric shadow integral

We now turn our attention to the integral (2.6) for the case $O = \mathcal{A}$,

$$W_{\mathcal{A}} = M_{34\mathcal{A}}\mathcal{N}_{\mathcal{A}} \times \int d^d x_0 \, \langle \phi_1(x_1) v_{2\,a}(x_2) \mathcal{A}_{b_1\cdots b_\ell}(x_0)\rangle \cdot \left\langle \phi_3(x_3) v_{4\,b}(x_4) \widetilde{\mathcal{A}}(x_0)\right\rangle\Big|_{\mathcal{M}=e^{2\pi i \Delta_{\mathcal{A}}}} . \tag{2.26}$$

From the three-point function (2.13), we see that the integrand includes the following contraction of tensor structures

$$\left(\gamma_{12\mathcal{A}} m^{(20)}_{ac_1} k^{(012)}_{c_2} k^{(012)}_{d_1} \cdots k^{(012)}_{d_k}\right) \times \widetilde{\Pi}^{(k)\,c_1c_2d_1\cdots d_k}_{e_1e_2f_1\cdots f_k} \left(\gamma_{34\widetilde{\mathcal{A}}} m^{(40)}{}_b{}^{e_1} k^{(034)\,e_2} k^{(034)\,f_1} \cdots k^{(034)\,f_k}\right). \tag{2.27}$$

Now using (2.125), (2.128), and (2.129), as well as other identities from appendices A and B, allows us to rewrite this contraction as

$$\begin{aligned}
&\left(\gamma_{12\mathcal{A}} m^{(20)}_{ac_1} k^{(012)}_{c_2} k^{(012)}_{d_1} \cdots k^{(012)}_{d_k}\right) \times \widetilde{\Pi}^{(k)\,c_1c_2d_1\cdots d_k}_{e_1e_2f_1\cdots f_k} \left(\gamma_{34\widetilde{\mathcal{A}}} m^{(40)}{}_b{}^{e_1} k^{(034)\,e_2} k^{(034)\,f_1} \cdots k^{(034)\,f_k}\right) \\
&= \frac{1}{2}\gamma_{12\mathcal{A}}\gamma_{34\widetilde{\mathcal{A}}} \sqrt{\frac{x_{02}^2 x_{04}^2 x_{12}^2 x_{34}^2}{x_{01}^2 x_{03}^2}} \left\{ \frac{k+2}{k+1} p_{d,k+1}(t) \frac{\partial^2 t}{\partial x_2^a \partial x_4^b} \right. \\
&\quad - \frac{\partial^2}{\partial x_2^a \partial x_4^b} \left[ \frac{1}{(k+1)(k+2)} p_{d,k+2}(t) \right. \\
&\quad \left.\left. + \frac{k+2}{(d+2k)(d+2k-2)(d+k-2)} p_{d,k}(t) \right] \right\},
\end{aligned} \tag{2.28}$$

where $p_{d,\ell}(t)$ is proportional to the Gegenbauer polynomials (see (2.98)), and $t$ is given by

$$t = k^{(012)} \cdot k^{(034)} = \frac{1}{2}\left(x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2 x_{12}^2 x_{34}^2\right)^{-1/2} \times \left(-x_{01}^2 x_{03}^2 x_{24}^2 + x_{01}^2 x_{04}^2 x_{23}^2 + x_{02}^2 x_{03}^2 x_{14}^2 - x_{02}^2 x_{04}^2 x_{13}^2\right). \tag{2.29}$$

In appendix C we give more details and motivation for how we arrive at this expression.

As a comparison, using (2.8) and (2.10), the integrals (2.6) for the symmetric traceless exchange $O = \mathcal{O}$ in $\langle\phi\phi\phi\phi\rangle$, $\langle\phi v\phi\phi\rangle$, $\langle\phi\phi\phi v\rangle$ and $\langle\phi v\phi v\rangle$, are of the form

$$W^{rs}_{\mathcal{O}} = \sum_t (M_{34\mathcal{O}})^{st} \mathcal{N}_{\mathcal{O}} \left(S^r_{a\,00} \circ_{\ell,\Delta_{\mathcal{O}}} S^t_{b\,00}\right), \tag{2.30}$$

where we defined

$$\left(S^r_{a\,PQ} \circ_{\ell,\Delta_{\mathcal{O}}} S^t_{b\,RS}\right) \equiv \int d^d x_0 S^r_{a\,c_1\cdots c_\ell}(x_1,x_2,x_0;\Delta_1+P,\Delta_2+Q,\Delta_{\mathcal{O}}) \times S^{t\;\;c_1\cdots c_\ell}_{\;b}(x_3,x_4,x_0;\Delta_3+R,\Delta_4+S,d-\Delta_{\mathcal{O}}), \quad (2.31)$$

for $r,t \in \{\alpha,\beta,\lambda\}$. Given the form of the structures $S$, these contractions are given in terms of the polynomials $p_{d,\ell}(t)$ and its derivatives $\partial_t p_{d,\ell}(t)$, as shown in (2.96) and (2.103).

To see that (2.28) is indeed a combination of the traceless symmetric contractions $\left(S^r_{a\,PQ} \circ_{\ell,\Delta_{\mathcal{O}}} S^t_{b\,RS}\right)$, notice that, the terms in (2.28) corresponding to derivatives of $t$ with respect to $x_i$, produce the tensor structures from three-point functions of operators with spin (this is not surprising since we can increase the spin of three-point functions by the action of derivatives as in (2.11), (2.12)). More precisely,

$$\frac{\partial t}{\partial x_2^a} = \left(\frac{\partial}{\partial x_2^a} k^{(012)\,b}\right) k_b^{(034)} = -\sqrt{\frac{x_{01}^2}{x_{02}^2 x_{12}^2}}\left[\left(m^{20}\cdot k^{(034)}\right)_a + t k_a^{(201)}\right], \quad (2.32)$$

$$\begin{aligned}\frac{\partial^2 t}{\partial x_2^a \partial x_4^b} &= \sqrt{\frac{x_{01}^2 x_{03}^2}{x_{02}^2 x_{04}^2 x_{12}^2 x_{34}^2}}\left(m_{ab}^{(24)} - 2\sqrt{\frac{v}{u}} k_a^{(214)} k_b^{(432)}\right) \\ &\quad - \frac{1}{2}\frac{x_{01}^2 x_{03}^2 x_{24}^2 + x_{01}^2 x_{04}^2 x_{23}^2 + x_{02}^2 x_{03}^2 x_{14}^2 + x_{02}^2 x_{04}^2 x_{13}^2}{x_{02}^2 x_{04}^2 x_{12}^2 x_{34}^2} k_a^{(201)} k_b^{(403)} \\ &\quad + \sqrt{\frac{x_{03}^2 x_{14}^2}{x_{04}^2 x_{12}^2 x_{24}^2 x_{34}^2}} k_a^{(201)} k_b^{(412)} - \sqrt{\frac{x_{03}^2 x_{23}^2}{x_{04}^2 x_{24}^2}}\frac{x_{01}^2 x_{24}^2 + x_{02}^2 x_{14}^2}{x_{02}^2 x_{12}^2 x_{34}^2} k_a^{(201)} k_b^{(432)} \\ &\quad - \sqrt{\frac{x_{01}^2 x_{14}^2}{x_{02}^2 x_{24}^2}}\frac{x_{03}^2 x_{24}^2 + x_{04}^2 x_{23}^2}{x_{04}^2 x_{12}^2 x_{34}^2} k_a^{(214)} k_b^{(403)} + \sqrt{\frac{x_{01}^2 x_{23}^2}{x_{02}^2 x_{12}^2 x_{24}^2 x_{34}^2}} k_a^{(234)} k_b^{(403)}.\end{aligned} \quad (2.33)$$

Combining these expressions with (2.103), we obtain also

$$\begin{aligned}&\frac{\partial^2}{\partial x_2^a \partial x_4^b} p_{d,\ell}(t) \\ &= \ell^2 \sqrt{\frac{x_{01}^2 x_{03}^2}{x_{02}^2 x_{04}^2 x_{12}^2 x_{34}^2}}\left(m_{ac_1}^{(20)} + k_a^{(201)} k_{c_1}^{(012)}\right)\left(m^{(40)\;\;d_1}_{\;\;\;b} + k_b^{(403)} k^{(034)\,d_1}\right) \\ &\qquad \times k_{c_2}^{(012)} \cdots k_{c_\ell}^{(012)} \Pi^{(\ell)\;c_1\cdots c_\ell}_{d_1\cdots d_\ell} k^{(034)\,d_2} \cdots k^{(034)\,d_\ell}.\end{aligned} \quad (2.34)$$

Putting these results together leads to an expression for the mixed-symmetric partial wave, in terms of integrals from symmetric partial waves (2.30):

$$
\begin{aligned}
W_{\mathcal{A}} = \mathcal{N}_{\mathcal{A}} M_{34\mathcal{A}} \Bigg\{ & \frac{1}{2}\frac{k+2}{k+1}\Bigg[\left(m_{ab}^{(24)} - 2\sqrt{\frac{v}{u}} k_a^{(214)} k_b^{(432)}\right)\left(S_{00}^{\lambda} \circ_{k+1} S_{00}^{\lambda}\right) \\
& -\frac{1}{2}\frac{1}{\sqrt{x_{12}^2 x_{34}^2}}\left(x_{24}^2\left(S^{\alpha}_{a\,-\frac{1}{2}\,\frac{1}{2}} \circ_{k+1} S^{\alpha}_{b\,-\frac{1}{2}\,\frac{1}{2}}\right) + x_{23}^2\left(S^{\alpha}_{a\,-\frac{1}{2}\,\frac{1}{2}} \circ_{k+1} S^{\alpha}_{b\,\frac{1}{2}\,-\frac{1}{2}}\right)\right. \\
& \left. + x_{14}^2\left(S^{\alpha}_{a\,\frac{1}{2}\,-\frac{1}{2}} \circ_{k+1} S^{\alpha}_{b\,-\frac{1}{2}\,\frac{1}{2}}\right) + x_{13}^2\left(S^{\alpha}_{a\,\frac{1}{2}\,-\frac{1}{2}} \circ_{k+1} S^{\alpha}_{b\,\frac{1}{2}\,-\frac{1}{2}}\right)\right) \\
& -\sqrt{\frac{x_{14}^2}{x_{24}^2}} k_b^{(412)}\left(S^{\alpha}_{a\,\frac{1}{2}\,-\frac{1}{2}} \circ_{k+1} S^{\lambda}_{00}\right) + \sqrt{\frac{x_{23}^2 x_{24}^2}{x_{12}^2 x_{34}^2}} k_b^{(432)}\left(S^{\alpha}_{a\,-\frac{1}{2}\,\frac{1}{2}} \circ_{k+1} S^{\lambda}_{00}\right) \\
& + x_{14}^2\sqrt{\frac{x_{23}^2}{x_{12}^2 x_{24}^2 x_{34}^2}} k_b^{(432)}\left(S^{\alpha}_{a\,\frac{1}{2}\,-\frac{1}{2}} \circ_{k+1} S^{\lambda}_{00}\right) + \sqrt{\frac{x_{14}^2 x_{24}^2}{x_{12}^2 x_{34}^2}} k_a^{(214)}\left(S^{\lambda}_{00} \circ_{k+1} S^{\alpha}_{b\,-\frac{1}{2}\,\frac{1}{2}}\right) \\
& + x_{23}^2\sqrt{\frac{x_{14}^2}{x_{12}^2 x_{24}^2 x_{34}^2}} k_a^{(214)}\left(S^{\lambda}_{00} \circ_{k+1} S^{\alpha}_{b\,\frac{1}{2}\,-\frac{1}{2}}\right) - \sqrt{\frac{x_{23}^2}{x_{24}^2}} k_a^{(234)}\left(S^{\lambda}_{00} \circ_{k+1} S^{\alpha}_{b\,\frac{1}{2}\,-\frac{1}{2}}\right)\Bigg] \\
& -\frac{1}{2}\frac{k+2}{k+1}\Bigg[\left(S^{\alpha}_{a\,00} \circ_{k+2} S^{\alpha}_{b\,00}\right) - \left(S^{\alpha}_{a\,00} \circ_{k+2} S^{\beta}_{b\,00}\right) - \left(S^{\beta}_{a\,00} \circ_{k+2} S^{\alpha}_{b\,00}\right) \\
& + \left(S^{\beta}_{a\,00} \circ_{k+2} S^{\beta}_{b\,00}\right)\Bigg] - \frac{1}{2}\frac{k^2\,(k+2)}{(d+2k)\,(d+2k-2)\,(d+k-2)} \\
& \times\left[\left(S^{\alpha}_{a\,00} \circ_{k} S^{\alpha}_{b\,00}\right) - \left(S^{\alpha}_{a\,00} \circ_{k} S^{\beta}_{b\,00}\right) - \left(S^{\beta}_{a\,00} \circ_{k} S^{\alpha}_{b\,00}\right) + \left(S^{\beta}_{a\,00} \circ_{k} S^{\beta}_{b\,00}\right)\right]\Bigg\}\,.
\end{aligned}
\tag{2.35}
$$

Note that all the contractions $(S \circ S)$ have the same dependence on $\Delta_{\mathcal{A}}$, and thus we omitted it to save space.

To generalize this technique to other mixed-symmetric representations, one would need to write the corresponding tensor structure contractions (like (2.28)), as total derivatives of the symmetric contraction $p_{d,k}$ and undifferentiated polynomials times derivatives of $t$—then the relation of these expressions to those from symmetric exchanges would follow in analogy to our case. We believe that it will be possible to do this in more general situations, but this has not been definitively established. However, to support this conjecture, we present the contraction for $[k+1,1,1]$ in appendix C, which has an analogous form.

## 2.5 The closed form conformal blocks for $\langle \phi v \phi v \rangle$

Given the expression for the mixed-symmetric partial wave in (2.35), computing the mixed-symmetric conformal blocks involves inverting the definition for the symmetric contractions (2.30), and writing $W_{\mathcal{O}}$ in terms of conformal blocks via

(1.186). We describe such calculation in this section. For completeness, we also give explicit expressions for the spinning symmetric blocks in terms of scalar blocks.

**Conformal block definitions**

Using (1.185) and (1.186), we write the relevant conformal blocks expansions, where the tensor structures are constructed with the algorithm of subsection 1.3.2. These are

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \mathcal{K}_4^{\Delta_i}\sum_{\mathcal{O}}\lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}\, g(u,v;\Delta_i;\ell,\Delta_{\mathcal{O}}), \tag{2.36}$$

$$\begin{aligned}\langle\phi_1(x_1)v_a(x_2)\phi_3(x_3)\phi_4(x_4)\rangle &\\ = \mathcal{K}_4^{\Delta_i}\sum_{\mathcal{O}}\lambda_{34\mathcal{O}}\Big[&\Big(\alpha_{12\mathcal{O}}g_1^{\alpha\lambda}(u,v)+\beta_{12\mathcal{O}}g_1^{\beta\lambda}(u,v)\Big)\,k_a^{(214)}\\ &+\Big(\alpha_{12\mathcal{O}}g_2^{\alpha\lambda}(u,v)+\beta_{12\mathcal{O}}g_2^{\beta\lambda}(u,v)\Big)\,k_a^{(234)}\Big],\end{aligned} \tag{2.37}$$

$$\begin{aligned}\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)v_a(x_4)\rangle &\\ = \mathcal{K}_4^{\Delta_i}\sum_{\mathcal{O}}\lambda_{12\mathcal{O}}\Big[&\Big(\alpha_{34\mathcal{O}}g_1^{\lambda\alpha}(u,v)+\beta_{34\mathcal{O}}g_1^{\lambda\beta}(u,v)\Big)\,k_a^{(412)}\\ &+\Big(\alpha_{34\mathcal{O}}g_2^{\lambda\alpha}(u,v)+\beta_{34\mathcal{O}}g_2^{\lambda\beta}(u,v)\Big)\,k_a^{(432)}\Big].\end{aligned} \tag{2.38}$$

For $\langle\phi_1(x_1)v_{2\,a}(x_2)\phi_3(x_3)v_{4\,b}(x_4)\rangle$, the conformal block expansion is given by (2.5), with

$$\begin{aligned}q_0 = \sum_{\mathcal{O}}\Big(&\alpha_{12\mathcal{O}}\alpha_{34\mathcal{O}}g_0^{\alpha\alpha}+\alpha_{12\mathcal{O}}\beta_{34\mathcal{O}}g_0^{\alpha\beta}+\beta_{12\mathcal{O}}\alpha_{34\mathcal{O}}g_0^{\beta\alpha}+\beta_{12\mathcal{O}}\beta_{34\mathcal{O}}g_0^{\beta\beta}\Big)\\ &+\sum_{\mathcal{A}}\gamma_{12\mathcal{A}}\gamma_{34\mathcal{A}}g_0^{\gamma\gamma},\end{aligned} \tag{2.39}$$

$$\begin{aligned}q_{ij} = \sum_{\mathcal{O}}\Big(&\alpha_{12\mathcal{O}}\alpha_{34\mathcal{O}}g_{ij}^{\alpha\alpha}+\alpha_{12\mathcal{O}}\beta_{34\mathcal{O}}g_{ij}^{\alpha\beta}+\beta_{12\mathcal{O}}\alpha_{34\mathcal{O}}g_{ij}^{\beta\alpha}+\beta_{12\mathcal{O}}\beta_{34\mathcal{O}}g_{ij}^{\beta\beta}\Big)\\ &+\sum_{\mathcal{A}}\gamma_{12\mathcal{A}}\gamma_{34\mathcal{A}}g_{ij}^{\gamma\gamma}.\end{aligned} \tag{2.40}$$

### 2.5.1 Conformal blocks for $\langle\phi\phi\phi\phi\rangle$

First we start with the case of four (not necessarily identical) scalars. The conformal block expansion (2.36) is related to partial waves by (1.185), which in turn we write as (2.6). This implies that

$$\mathcal{K}_4^{\Delta_i}\lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}g(u,v;\Delta_i;\ell,\Delta_\mathcal{O}) = \mathcal{N}_\mathcal{O}\lambda_{12\mathcal{O}}\lambda_{34\widetilde{\mathcal{O}}}\left(S_{00}^\lambda \circ_{\ell,\Delta_\mathcal{O}} S_{00}^\lambda\right), \tag{2.41}$$

where we used the definition (2.31). From the results of Appendix B and the explicit form of $S^\lambda$ in (2.9), we see that relevant contraction of tensor structures $k$ is given by

$$k^{(012)}_{a_1}\cdots k^{(012)}_{a_\ell}\Pi^{(\ell)\,a_1\cdots a_\ell}_{b_1\cdots b_\ell}k^{(034)\,b_1}\cdots k^{(034)\,b_\ell} = p_{d,\ell}(t), \tag{2.42}$$

with $t$ defined in (2.29).

Let us now define integrals

$$I^{(\ell)}_{\alpha,\beta,\gamma,\delta} = \int\frac{d^dx_0\,p_{d,\ell}(t)}{\left(x_{01}^2\right)^\alpha\left(x_{02}^2\right)^\beta\left(x_{03}^2\right)^\gamma\left(x_{04}^2\right)^\delta}. \tag{2.43}$$

For $\ell = 0$ this integral is evaluated in (2.153). With this definition and the expressions (2.15) and (2.21), we write the conformal block $g$ as

$$\begin{aligned}
g(u,v;\Delta_i;\ell,\Delta_\mathcal{O}) &= \pi^{-d/2}\\
&\times\frac{\Gamma(\frac{1}{2}(d+\Delta_3-\Delta_4-\Delta_\mathcal{O}+\ell))\Gamma(\frac{1}{2}(d-\Delta_3+\Delta_4-\Delta_\mathcal{O}+\ell))\Gamma(\Delta_\mathcal{O}+\ell)}{\Gamma(\frac{1}{2}(\Delta_3-\Delta_4+\Delta_\mathcal{O}+\ell))\Gamma(\frac{1}{2}(-\Delta_3+\Delta_4+\Delta_\mathcal{O}+\ell))\Gamma(d-\Delta_\mathcal{O}+\ell-1)}\\
&\times\frac{\Gamma(d-\Delta_\mathcal{O}-1)}{\Gamma(\frac{d}{2}-\Delta_\mathcal{O})}\left(x_{12}^2\right)^{\frac{1}{2}\Delta_\mathcal{O}}\left(x_{13}^2\right)^{\frac{1}{2}(\Delta_3-\Delta_4)}\left(x_{14}^2\right)^{\frac{1}{2}(\Delta_1-\Delta_2-\Delta_3+\Delta_4)}\left(x_{24}^2\right)^{\frac{1}{2}(-\Delta_1+\Delta_2)}\\
&\times\left(x_{34}^2\right)^{\frac{1}{2}(d-\Delta_\mathcal{O})}I^{(\ell)}_{\frac{1}{2}(\Delta_1-\Delta_2+\Delta_\mathcal{O}),\frac{1}{2}(-\Delta_1+\Delta_2+\Delta_\mathcal{O}),\frac{1}{2}(d+\Delta_3-\Delta_4-\Delta_\mathcal{O}),\frac{1}{2}(d-\Delta_3+\Delta_4-\Delta_\mathcal{O})}.
\end{aligned} \tag{2.44}$$

Note that the prefactor $(x_{12}^2)^{\Delta_\mathcal{O}/2}$ already has the desired behavior under the monodromy $\mathcal{M}: x_{12}^2 \to e^{4\pi i}x_{12}^2$, so we will want to pick out the terms from the integral which are invariant under $\mathcal{M}$.

If we expand the polynomial $p_{d,\ell}(t)$ using (2.97) from Appendix B then the integral is simply a sum functions $f_{\alpha,\beta,\gamma,\delta}$ defined in Appendix D (for even dimensions $d$, it can be written in terms of hypergeometric functions, see (2.163)). For example, in the case $\ell = 0$, then $p_{d,0}(t) = 1$, and we have

$$\begin{aligned}
g(u,v;\Delta_i;0,\Delta_\mathcal{O}) &= \frac{\Gamma(\Delta_\mathcal{O})}{\Gamma(\frac{d}{2}-\Delta_\mathcal{O})}\\
&\times\frac{\Gamma(\frac{1}{2}(d-\Delta_3+\Delta_4-\Delta_\mathcal{O}))u^{\frac{1}{2}\Delta_\mathcal{O}}v^{\frac{1}{2}(-\Delta_3+\Delta_4-\Delta_\mathcal{O})}}{\Gamma(\frac{1}{2}(\Delta_1-\Delta_2+\Delta_\mathcal{O}))\Gamma(\frac{1}{2}(-\Delta_1+\Delta_2+\Delta_\mathcal{O}))\Gamma(\frac{1}{2}(-\Delta_3+\Delta_4+\Delta_\mathcal{O}))}\\
&\times\widehat{f}_{\frac{1}{2}(\Delta_1-\Delta_2+\Delta_\mathcal{O}),\frac{1}{2}(-\Delta_1+\Delta_2+\Delta_\mathcal{O}),\frac{1}{2}(d+\Delta_3-\Delta_4-\Delta_\mathcal{O}),\frac{1}{2}(d-\Delta_3+\Delta_4-\Delta_\mathcal{O})}(uv^{-1},v^{-1}),
\end{aligned} \tag{2.45}$$

where $\widehat{f}$ is defined in (2.156). Again, the $u^{\Delta_{\mathcal{O}}/2}$ factor behaves correctly under the monodromy $\mathcal{M}$, and thus the conformal block $g(u,v)$ is given by the monodromy invariant piece of $\widehat{f}$, which we call $f$ (see (2.157)).

The formula above shows explicity that $g(u,v;\Delta_i;\ell,\Delta_{\mathcal{O}})$ depends on the differences $\Delta_1-\Delta_2$ and $\Delta_3-\Delta_4$ only. Thus we adopt a condensed notation that will be useful below:

$$g_{\ell;P,Q}(u,v)=g(u,v;\Delta_1+P,\Delta_2,\Delta_3+Q,\Delta_4;\ell,\Delta_{\mathcal{O}}). \tag{2.46}$$

Here $P$ and $Q$, allows us to write conformal blocks where the difference in conformal dimensions is shifted by integer amounts. In this notation the dependence on the $\Delta_i$ and $\Delta_{\mathcal{O}}$ is implicit.

For $\ell>0$ one can exploit the recursion relations (2.101) to expand the numerator of the integrand in (2.43). As shown in [64], this results in

$$\begin{aligned} g_{\ell;0,0}(u,v)=&\ \frac{\Delta_{\mathcal{O}}+\ell-1}{d-\Delta_{\mathcal{O}}+\ell-2}\\ &\times\left[\frac{1}{2}\frac{d+\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell-2}{\Delta_3-\Delta_4+\Delta_{\mathcal{O}}+\ell-2}u^{-1/2}\left(g_{\ell-1;1,-1}(u,v)-g_{\ell-1;-1,-1}(u,v)\right)\right.\\ &\quad+\frac{1}{2}\frac{d-\Delta_3+\Delta_4-\Delta_{\mathcal{O}}+\ell-2}{-\Delta_3+\Delta_4+\Delta_{\mathcal{O}}+\ell-2}u^{-1/2}\left(vg_{\ell-1;-1,1}(u,v)-g_{\ell-1;1,1}(u,v)\right)\\ &\quad-\frac{(\Delta_{\mathcal{O}}+\ell-2)\,(d+\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell-2)\,(d-\Delta_3+\Delta_4-\Delta_{\mathcal{O}}+\ell-2)}{(d-\Delta_{\mathcal{O}}+\ell-3)\,(\Delta_3-\Delta_4+\Delta_{\mathcal{O}}+\ell-2)\,(-\Delta_3+\Delta_4+\Delta_{\mathcal{O}}+\ell-2)}\\ &\quad\left.\times\frac{(\ell-1)\,(d+\ell-4)}{(d+2\ell-4)\,(d+2\ell-6)}g_{\ell-2;0,0}(u,v)\right]. \end{aligned} \tag{2.47}$$

In $d=2$ the recursion can be solved explicitly in terms of hypergeometric functions, and in higher even dimensions solutions can also be constructed [68, 69]. At any rate, here we will assume that these conformal blocks are known, and we will express the new conformal blocks in terms of these.

### 2.5.2 Conformal blocks for $\langle\phi v\phi\phi\rangle$

Now for the case of three scalars, and a vector in position two, the conformal blocks are (c.f. (2.37), (1.185), and (2.6))

$$\begin{aligned} &\mathcal{K}_4^{\Delta_i}\lambda_{34\mathcal{O}}\left[\left(\alpha_{12\mathcal{O}}g_1^{\alpha\lambda}+\beta_{12\mathcal{O}}g_1^{\beta\lambda}\right)k_a^{(214)}+\left(\alpha_{12\mathcal{O}}g_2^{\alpha\lambda}+\beta_{12\mathcal{O}}g_2^{\beta\lambda}\right)k_a^{(234)}\right]\\ &\qquad=\mathcal{N}_{\mathcal{O}}\lambda_{34\widetilde{\mathcal{O}}}\left(\alpha_{12\mathcal{O}}\left(S^{\alpha}_{a\,00}\circ_{\ell,\Delta_{\mathcal{O}}}S^{\lambda}_{00}\right)+\beta_{12\mathcal{O}}\left(S^{\beta}_{a\,00}\circ_{\ell,\Delta_{\mathcal{O}}}S^{\lambda}_{00}\right)\right). \end{aligned} \tag{2.48}$$

Then, expressing $S^\alpha$ and $S^\beta$ in terms of $S^\lambda$ as in (2.11) and (2.12), and pulling the differential operators outside of the integral, allows us to express $g_i^{r\lambda}$ in terms of differential operators acting on $g$. This leads to

$$g_1^{\alpha\lambda} = \frac{1}{2\,(1-\Delta_{\mathcal{O}})} \left[(1 - \Delta_1 + \Delta_v + (1-v)\,(\Delta_3 - \Delta_4) + 2v\,(1-v)\,\partial_v - 2uv\partial_u) \right. \\ \times g(u,v;\Delta_1 - 1, \Delta_v, \Delta_3, \Delta_4; \ell, \Delta_{\mathcal{O}}) \\ \left. + \,(1 + \Delta_1 - \Delta_v - 2u\partial_u)\, g(u,v;\Delta_1+1,\Delta_v,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}})\right], \quad (2.49)$$

$$g_2^{\alpha\lambda} = \frac{\sqrt{uv}}{2\,(1-\Delta_{\mathcal{O}})} \left[(\Delta_3 - \Delta_4 + 2u\partial_u + 2v\partial_v)\, g(u,v;\Delta_1-1,\Delta_v,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}}) \right. \\ \left. -2\partial_v g(u,v;\Delta_1+1,\Delta_v,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}})\right], \quad (2.50)$$

Similarly,

$$g_1^{\beta\lambda} = \frac{\Delta_1 - \Delta_v - \Delta_{\mathcal{O}} + \ell + 1}{\ell} g_1^{\alpha\lambda}(u,v;\Delta_1,\Delta_v,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}}) \\ - \frac{1}{\ell}\,(1 + \Delta_1 - \Delta_v - 2u\partial_u)\, g(u,v;\Delta_1+1,\Delta_v,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}}), \quad (2.51)$$

$$g_2^{\beta\lambda} = \frac{\Delta_1 - \Delta_v - \Delta_{\mathcal{O}} + \ell + 1}{\ell} g_2^{\alpha\lambda}(u,v;\Delta_1,\Delta_v,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}}) \\ + \frac{2\sqrt{uv}}{\ell}\partial_v g(u,v;\Delta_1+1,\Delta_v,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}}). \quad (2.52)$$

Note that as with the scalar blocks, the expressions only depend on the difference $\Delta_1 - \Delta_v$ and $\Delta_3 - \Delta_4$, not on the weights individually. The other crucial property of these expressions is that the operators which act on $g$ on the right hand side involve only integer powers of $\sqrt{u}$, so they are all invariant under the monodromy $\mathcal{M}$. Therefore the results above have the correct boundary conditions.

### 2.5.3 Conformal blocks for $\langle \phi\phi\phi v \rangle$

When the vector is in the fourth position, the calculation is very similar to the previous case. The conformal blocks are computed with

$$\mathcal{K}_4^{\Delta_i} \lambda_{12\mathcal{O}} \left[\left(\alpha_{34\mathcal{O}} g_1^{\lambda\alpha} + \beta_{34\mathcal{O}} g_1^{\lambda\beta}\right) k_a^{(412)} + \left(\alpha_{34\mathcal{O}} g_2^{\lambda\alpha} + \beta_{34\mathcal{O}} g_2^{\lambda\beta}\right) k_a^{(432)}\right] \\ = \mathcal{N}_{\mathcal{O}} \lambda_{12\mathcal{O}} \left(\alpha_{34\widetilde{\mathcal{O}}} \left(S_{00}^\lambda \circ_{\ell,\Delta_{\mathcal{O}}} S_{a\,00}^\alpha\right) + \beta_{34\widetilde{\mathcal{O}}} \left(S_{00}^\lambda \circ_{\ell,\Delta_{\mathcal{O}}} S_{a\,00}^\beta\right)\right). \quad (2.53)$$

Note that $\alpha_{34\widetilde{\mathcal{O}}}$ and $\beta_{34\widetilde{\mathcal{O}}}$ must be expanded using (2.22) and (2.23). The results are

$$\begin{aligned} g_1^{\lambda\alpha} = & \frac{\sqrt{u}}{2\left(\Delta_{\mathcal{O}}-1\right)}\left[\left(1-\Delta_3+\Delta_v-2u\partial_u-2v\partial_v\right)\right. \\ & \times g(u,v;\Delta_1,\Delta_2,\Delta_3-1,\Delta_v;\ell,\Delta_{\mathcal{O}}) \\ & \left.+\left(1-\Delta_1+\Delta_2+\Delta_3-\Delta_v+2v\partial_v\right)g(u,v;\Delta_1,\Delta_2,\Delta_3+1,\Delta_v;\ell,\Delta_{\mathcal{O}})\right], \end{aligned} \tag{2.54}$$

$$\begin{aligned} g_2^{\lambda\alpha} = & \frac{\sqrt{v}}{2\left(1-\Delta_{\mathcal{O}}\right)}\left[\left(1-\Delta_3+\Delta_v-2u\partial_u+2\left(1-v\right)\partial_v\right)\right. \\ & \times g(u,v;\Delta_1,\Delta_2,\Delta_3-1,\Delta_v;\ell,\Delta_{\mathcal{O}}) \\ & \left.+\left(1+\Delta_3-\Delta_v-2u\partial_u\right)g(u,v;\Delta_1,\Delta_2,\Delta_3+1,\Delta_v;\ell,\Delta_{\mathcal{O}})\right], \end{aligned} \tag{2.55}$$

$$\begin{aligned} g_1^{\lambda\beta} = & \frac{\Delta_3-\Delta_v-\Delta_{\mathcal{O}}+\ell+1}{\ell}g_1^{\lambda\alpha}(u,v;\Delta_1,\Delta_2,\Delta_3,\Delta_v;\ell,\Delta_{\mathcal{O}}) \\ & +\frac{\sqrt{u}}{\ell}\left(1-\Delta_1+\Delta_2+\Delta_3-\Delta_v+2v\partial_v\right)g(u,v;\Delta_1,\Delta_2,\Delta_3+1,\Delta_v;\ell,\Delta_{\mathcal{O}}), \end{aligned} \tag{2.56}$$

$$\begin{aligned} g_2^{\lambda\beta} = & \frac{\Delta_3-\Delta_v-\Delta_{\mathcal{O}}+\ell+1}{\ell}g_2^{\lambda\alpha}(u,v;\Delta_1,\Delta_2,\Delta_3,\Delta_v;\ell,\Delta_{\mathcal{O}}) \\ & -\frac{\sqrt{v}}{\ell}\left(1+\Delta_3-\Delta_v-2u\partial_u\right)g(u,v;\Delta_1,\Delta_2,\Delta_3+1,\Delta_v;\ell,\Delta_{\mathcal{O}}), \end{aligned} \tag{2.57}$$

which again, only depend on the differences $\Delta_1-\Delta_2$ and $\Delta_3-\Delta_v$, and they have the correct boundary conditions.

### 2.5.4 Symmetric conformal blocks for $\langle \phi v \phi v \rangle$

For two scalars at positions 1 and 3, and two vectors at positions 2 and 4, we follow a completely analogous procedure as above which leads to

$$g_0^{\alpha\alpha} = \frac{1}{2(1-\Delta_{\mathcal{O}})}\left[\frac{1}{\sqrt{v}} g^{\alpha\lambda}_{2;\ell;0,-1} - \sqrt{u} g^{\alpha\lambda}_{1;\ell;0,1} - \sqrt{v} g^{\alpha\lambda}_{2;\ell;0,1}\right], \tag{2.58}$$

$$\begin{aligned} g_{11}^{\alpha\alpha} = &\frac{\sqrt{u}}{2(1-\Delta_{\mathcal{O}})}\left[-(1-\Delta_3+\Delta_4-2u\partial_u-2v\partial_v)\, g^{\alpha\lambda}_{1;\ell;0,-1}\right.\\ &\left.+(\Delta_1-\Delta_2-\Delta_3+\Delta_4-2v\partial_v)\, g^{\alpha\lambda}_{1;\ell;0,1}\right], \end{aligned} \tag{2.59}$$

$$\begin{aligned} g_{12}^{\alpha\alpha} = &\frac{\sqrt{v}}{2(1-\Delta_{\mathcal{O}})}\left[(1-\Delta_3+\Delta_4-2u\partial_u+2(1-v)\partial_v)\, g^{\alpha\lambda}_{1;\ell;0,-1}\right.\\ &\left.+(1+\Delta_3-\Delta_4-2u\partial_u)\, g^{\alpha\lambda}_{1;\ell;0,1}\right], \end{aligned} \tag{2.60}$$

$$\begin{aligned} g_{21}^{\alpha\alpha} = &\frac{\sqrt{u}}{2(1-\Delta_{\mathcal{O}})}\left[-(1-\Delta_3+\Delta_4-2u\partial_u-2v\partial_v)\, g^{\alpha\lambda}_{2;\ell;0,-1}\right.\\ &\left.-(1-\Delta_1+\Delta_2+\Delta_3-\Delta_4+2v\partial_v)\, g^{\alpha\lambda}_{2;\ell;0,1}\right], \end{aligned} \tag{2.61}$$

$$\begin{aligned} g_{22}^{\alpha\alpha} = &\frac{\sqrt{v}}{2(1-\Delta_{\mathcal{O}})}\left[\left(1-\frac{1}{v}-\Delta_3+\Delta_4-2u\partial_u+2(1-v)\partial_v\right) g^{\alpha\lambda}_{2;\ell;0,-1}\right.\\ &\left.+(2+\Delta_3-\Delta_4-2u\partial_u)\, g^{\alpha\lambda}_{2;\ell;0,1}\right], \end{aligned} \tag{2.62}$$

where in the spirit of (2.46), we used the following condensed representation

$$g^{rs}_{i;\ell;P,Q} = g^{rs}_i(u,v;\Delta_1+P,\Delta_2,\Delta_3+Q,\Delta_4;\ell,\Delta_{\mathcal{O}}), \quad r,s\in\{\lambda,\alpha,\beta\}. \tag{2.63}$$

The other components $\alpha\beta$, $\beta\alpha$, and $\beta\beta$ are given in appendix F.

Actually, the combination which occurs in the four-point function

$$\alpha_{12\mathcal{O}}\alpha_{34\mathcal{O}} g_p^{\alpha\alpha} + \alpha_{12\mathcal{O}}\beta_{34\mathcal{O}} g_p^{\alpha\beta} + \beta_{12\mathcal{O}}\alpha_{34\mathcal{O}} g_p^{\beta\alpha} + \beta_{12\mathcal{O}}\beta_{34\mathcal{O}} g_p^{\beta\beta}, \tag{2.64}$$

has a remarkably simple form if written in terms of scalar blocks. This is

$$A_1A_2\mathcal{D}_p^{--} g_{\ell;-1,-1} + A_1B_2\mathcal{D}_p^{-+} g_{\ell;-1,1} + B_1A_2\mathcal{D}_p^{+-} g_{\ell;1,-1} + B_1B_2\mathcal{D}_p^{++} g_{\ell;1,1}, \tag{2.65}$$

where

$$A_1 = \frac{1}{2(\Delta_{\mathcal{O}}-1)}\left(\alpha_{12\mathcal{O}} + (\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1)\frac{\beta_{12\mathcal{O}}}{\ell}\right), \tag{2.66}$$

$$A_2 = \frac{1}{2(\Delta_{\mathcal{O}}-1)}\left(\alpha_{34\mathcal{O}} + (\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1)\frac{\beta_{34\mathcal{O}}}{\ell}\right), \tag{2.67}$$

$$B_1 = \frac{1}{2(\Delta_{\mathcal{O}}-1)}\left(\alpha_{12\mathcal{O}} + (\Delta_1-\Delta_2+\Delta_{\mathcal{O}}+\ell-1)\frac{\beta_{12\mathcal{O}}}{\ell}\right), \tag{2.68}$$

$$B_2 = \frac{1}{2(\Delta_{\mathcal{O}}-1)}\left(\alpha_{34\mathcal{O}} + (\Delta_3-\Delta_4+\Delta_{\mathcal{O}}+\ell-1)\frac{\beta_{34\mathcal{O}}}{\ell}\right), \tag{2.69}$$

and $\mathcal{D}_p^{\pm\pm}$ are differential operators given in a table in appendix H.

### 2.5.5 Mixed-symmetric conformal blocks for $\langle \phi v \phi v \rangle$

Finally, we arrive at the conformal blocks of two vectors and two scalars, associated to the exchange of the mixed-symmetric operator $\mathcal{A}$. Our starting point is expression (2.35). To evaluate the contractions $(S \circ S)$, we first combine (1.186) with the conformal block expansions at the beginning of this section to obtain the corresponding expressions in terms of conformal blocks. Plugging these into (2.30) we obtain

$$\left(S^{\lambda}_{00} \circ_{\ell,\Delta_{\mathcal{A}}} S^{\lambda}_{00}\right) = \frac{\mathcal{K}_4^{\Delta_i}}{\mathcal{N}_{\mathcal{O}}\left(\lambda_{34\widetilde{\mathcal{O}}}/\lambda_{34\mathcal{O}}\right)} g_{\ell;0,0}\Big|_{\Delta_{\mathcal{O}}\to\Delta_{\mathcal{A}}} \tag{2.70}$$

where $\mathcal{N}_{\mathcal{O}}$, $\lambda_{34\widetilde{\mathcal{O}}}$, and $g_{\ell;0,0}$ are given by (2.15), (2.21), and (2.46) respectively. Similarly,

$$\left(S^{\alpha}_{a\,PQ} \circ_{\ell,\Delta_{\mathcal{A}}} S^{\lambda}_{00}\right) = \frac{\mathcal{K}_4^{\Delta_1+P,\Delta_2+Q,\Delta_3,\Delta_4}}{\mathcal{N}_{\mathcal{O}}\left(\lambda_{34\widetilde{\mathcal{O}}}/\lambda_{34\mathcal{O}}\right)\Big|_{\Delta_1\to\Delta_1+P,\Delta_2\to\Delta_2+Q}} \times \left[g^{\alpha\lambda}_{1;\ell;P-Q,0} k_a^{(214)} + g^{\alpha\lambda}_{2;\ell;P-Q,0} k_a^{(234)}\right]\Bigg|_{\Delta_{\mathcal{O}}\to\Delta_{\mathcal{A}}}, \tag{2.71}$$

$$\left(S^{\lambda}_{00} \circ_{\ell,\Delta_{\mathcal{A}}} S^{\alpha}_{a\,RS}\right) = \frac{\mathcal{K}_4^{\Delta_1,\Delta_2,\Delta_3+R,\Delta_4+S}}{\mathcal{N}_{\mathcal{O}}} \times \Bigg[\left(\left(M^{-1}\right)_{\alpha}^{\ \alpha}\Big|_{\Delta_3\to\Delta_3+R,\Delta_4\to\Delta_4+S}\right)\left(g^{\lambda\alpha}_{1;\ell;0,R-S} k_a^{(412)} + g^{\lambda\alpha}_{2;\ell;0,R-S} k_a^{(432)}\right) + \left(\left(M^{-1}\right)_{\beta}^{\ \alpha}\Big|_{\Delta_3\to\Delta_3+R,\Delta_4\to\Delta_4+S}\right)\left(g^{\lambda\beta}_{1;\ell;0,R-S} k_a^{(412)} + g^{\lambda\beta}_{2;\ell;0,R-S} k_a^{(432)}\right)\Bigg]\Bigg|_{\Delta_{\mathcal{O}}\to\Delta_{\mathcal{A}}}, \tag{2.72}$$

$$\left(S^{r}_{a\,PQ} \circ_{\ell,\Delta_{\mathcal{A}}} S^{s}_{b\,RS}\right) = \frac{\mathcal{K}_4^{\Delta_1+P,\Delta_2+Q,\Delta_3+R,\Delta_4+S}}{\mathcal{N}_{\mathcal{O}}} \times \sum_{p,l}\left(\left(M^{-1}\right)_{l}^{\ s}\Big|_{\Delta_1\to\Delta_1+P,\Delta_2\to\Delta_2+Q,\Delta_3\to\Delta_3+R,\Delta_4\to\Delta_4+S}\right) g^{rl}_{p;\ell;P-Q,R-S} t^{p}_{ab}\Big|_{\Delta_{\mathcal{O}}\to\Delta_{\mathcal{A}}}, \tag{2.73}$$

where the sums are over $l \in \{\alpha, \beta\}$, $p \in \{0, 11, 12, 21, 22\}$, the tensor structures are

$$t^{0}_{ab} = m^{(24)}_{ab}, \quad t^{11}_{ab} = k_a^{(214)} k_b^{(412)}, \quad t^{12}_{ab} = k_a^{(214)} k_b^{(432)}, \; t^{21}_{ab} = k_a^{(234)} k_b^{(412)}, \; t^{22}_{ab} = k_a^{(234)} k_b^{(432)}, \tag{2.74}$$

and $M$ is given in (2.24). Plugging in these results into (2.35), and collecting all the different tensor structures finally leads to

$$
\begin{aligned}
&g_{0}^{\gamma\gamma}(u,v;\Delta_1,\Delta_2,\Delta_3,\Delta_4;k,\Delta_{\mathcal{A}})\\
&\quad=\frac{1}{2}\frac{k+2}{k+1}\Bigg[C_1 g_{k+1;0,0}-\frac{1}{2\sqrt{u}}\Big(C_2\left(g^{\alpha\alpha}_{0;k+1;-1,-1}+g^{\alpha\alpha}_{0;k+1;1,-1}\right)\\
&\quad+C_3\left(g^{\alpha\beta}_{0;k+1;-1,-1}+g^{\alpha\beta}_{0;k+1;1,-1}\right)+C_4\left(vg^{\alpha\alpha}_{0;k+1;-1,1}+g^{\alpha\alpha}_{0;k+1;1,1}\right)\\
&\quad+C_5\left(vg^{\alpha\beta}_{0;k+1;-1,1}+g^{\alpha\beta}_{0;k+1;1,1}\right)\Big)\Bigg]\\
&\quad-\frac{1}{2}\frac{k+2}{k+1}\left[C_6\left(g^{\alpha\alpha}_{0;k+2;0,0}-g^{\beta\alpha}_{0;k+2;0,0}\right)+C_7\left(g^{\beta\beta}_{0;k+2;0,0}-g^{\alpha\beta}_{0;k+2;0,0}\right)\right]\\
&-\frac{1}{2}\frac{k^2(k+2)}{(d+2k)(d+2k-2)(d+k-2)}\left[C_8\left(g^{\alpha\alpha}_{0;k;0,0}-g^{\beta\alpha}_{0;k;0,0}\right)+C_9\left(g^{\beta\beta}_{0;k;0,0}-g^{\alpha\beta}_{0;k;0,0}\right)\right],
\end{aligned}
\tag{2.75}
$$

$$
\begin{aligned}
&g_{11}^{\gamma\gamma}(u,v;\Delta_1,\Delta_2,\Delta_3,\Delta_4;k,\Delta_{\mathcal{A}})=\frac{1}{2}\frac{k+2}{k+1}\Bigg[-\frac{1}{2\sqrt{u}}\Big(C_2\left(g^{\alpha\alpha}_{11;k+1;-1,-1}+g^{\alpha\alpha}_{11;k+1;1,-1}\right)\\
&\quad+C_3\left(g^{\alpha\beta}_{11;k+1;-1,-1}+g^{\alpha\beta}_{11;k+1;1,-1}\right)\\
&\quad+C_4\left(vg^{\alpha\alpha}_{11;k+1;-1,1}+g^{\alpha\alpha}_{11;k+1;1,1}\right)+C_5\left(vg^{\alpha\beta}_{11;k+1;-1,1}+g^{\alpha\beta}_{11;k+1;1,1}\right)\Big)\\
&\quad-C_1 g^{\alpha\lambda}_{1;k+1;1,0}+\frac{C_2}{\sqrt{u}}g^{\lambda\alpha}_{1;k+1;0,-1}+\frac{C_3}{\sqrt{u}}g^{\lambda\beta}_{1;k+1;0,-1}+\frac{C_4 v}{\sqrt{u}}g^{\lambda\alpha}_{1;k+1;0,1}+\frac{C_5 v}{\sqrt{u}}g^{\lambda\beta}_{1;k+1;0,1}\Bigg]\\
&\quad-\frac{1}{2}\frac{k+2}{k+1}\left[C_6\left(g^{\alpha\alpha}_{11;k+2;0,0}-g^{\beta\alpha}_{11;k+2;0,0}\right)+C_7\left(g^{\beta\beta}_{11;k+2;0,0}-g^{\alpha\beta}_{11;k+2;0,0}\right)\right]\\
&-\frac{1}{2}\frac{k^2(k+2)}{(d+2k)(d+2k-2)(d+k-2)}\left[C_8\left(g^{\alpha\alpha}_{11;k;0,0}-g^{\beta\alpha}_{11;k;0,0}\right)+C_9\left(g^{\beta\beta}_{11;k;0,0}-g^{\alpha\beta}_{11;k;0,0}\right)\right],
\end{aligned}
\tag{2.76}
$$

$$
\begin{aligned}
&g_{12}^{\gamma\gamma}(u,v;\Delta_1,\Delta_2,\Delta_3,\Delta_4;k,\Delta_{\mathcal{A}})\\
&\quad=\frac{1}{2}\frac{k+2}{k+1}\Bigg[-2\sqrt{\frac{v}{u}}C_1 g_{k+1;0,0}-\frac{1}{2\sqrt{u}}\Big(C_2\left(g^{\alpha\alpha}_{12;k+1;-1,-1}+g^{\alpha\alpha}_{12;k+1;1,-1}\right)\\
&\quad+C_3\left(g^{\alpha\beta}_{12;k+1;-1,-1}+g^{\alpha\beta}_{12;k+1;1,-1}\right)+C_4\left(vg^{\alpha\alpha}_{12;k+1;-1,1}+g^{\alpha\alpha}_{12;k+1;1,1}\right)\\
&\quad+C_5\left(vg^{\alpha\beta}_{12;k+1;-1,1}+g^{\alpha\beta}_{12;k+1;1,1}\right)\Big)+C_1\sqrt{\frac{v}{u}}\left(g^{\alpha\lambda}_{1;k+1;-1,0}+g^{\alpha\lambda}_{1;k+1;1,0}\right)\\
&\quad+\frac{C_2}{\sqrt{u}}g^{\lambda\alpha}_{2;k+1;0,-1}+\frac{C_3}{\sqrt{u}}g^{\lambda\beta}_{2;k+1;0,-1}+\frac{C_4 v}{\sqrt{u}}g^{\lambda\alpha}_{2;k+1;0,1}+\frac{C_5 v}{\sqrt{u}}g^{\lambda\beta}_{2;k+1;0,1}\Bigg]\\
&\quad-\frac{1}{2}\frac{k+2}{k+1}\left[C_6\left(g^{\alpha\alpha}_{12;k+2;0,0}-g^{\beta\alpha}_{12;k+2;0,0}\right)+C_7\left(g^{\beta\beta}_{12;k+2;0,0}-g^{\alpha\beta}_{12;k+2;0,0}\right)\right]\\
&-\frac{1}{2}\frac{k^2(k+2)}{(d+2k)(d+2k-2)(d+k-2)}\left[C_8\left(g^{\alpha\alpha}_{12;k;0,0}-g^{\beta\alpha}_{12;k;0,0}\right)+C_9\left(g^{\beta\beta}_{12;k;0,0}-g^{\alpha\beta}_{12;k;0,0}\right)\right],
\end{aligned}
\tag{2.77}
$$

$$
\begin{aligned}
g^{\gamma\gamma}_{21}(u,v;\Delta_1,\Delta_2,\Delta_3,\Delta_4;k,\Delta_{\mathcal{A}}) &= \frac{1}{2}\frac{k+2}{k+1}\Bigg[-\frac{1}{2\sqrt{u}}\left(C_2\left(g^{\alpha\alpha}_{21;k+1;-1,-1}+g^{\alpha\alpha}_{21;k+1;1,-1}\right)\right. \\
&+C_3\left(g^{\alpha\beta}_{21;k+1;-1,-1}+g^{\alpha\beta}_{21;k+1;1,-1}\right)+C_4\left(vg^{\alpha\alpha}_{21;k+1;-1,1}+g^{\alpha\alpha}_{21;k+1;1,1}\right) \\
&\left.+C_5\left(vg^{\alpha\beta}_{21;k+1;-1,1}+g^{\alpha\beta}_{21;k+1;1,1}\right)\right)-C_1 g^{\alpha\lambda}_{2;k+1;1,0}-C_4\sqrt{v}g^{\lambda\alpha}_{1;k+1;0,1}-C_5\sqrt{v}g^{\lambda\beta}_{1;k+1;0,1}\Bigg] \\
&-\frac{1}{2}\frac{k+2}{k+1}\left[C_6\left(g^{\alpha\alpha}_{21;k+2;0,0}-g^{\beta\alpha}_{21;k+2;0,0}\right)+C_7\left(g^{\beta\beta}_{21;k+2;0,0}-g^{\alpha\beta}_{21;k+2;0,0}\right)\right] \\
&-\frac{1}{2}\frac{k^2(k+2)}{(d+2k)(d+2k-2)(d+k-2)}\left[C_8\left(g^{\alpha\alpha}_{21;k;0,0}-g^{\beta\alpha}_{21;k;0,0}\right)+C_9\left(g^{\beta\beta}_{21;k;0,0}-g^{\alpha\beta}_{21;k;0,0}\right)\right],
\end{aligned}
\tag{2.78}
$$

$$
\begin{aligned}
g^{\gamma\gamma}_{22}(u,v;\Delta_1,\Delta_2,\Delta_3,\Delta_4;k,\Delta_{\mathcal{A}}) &= \frac{1}{2}\frac{k+2}{k+1}\Bigg[-\frac{1}{2\sqrt{u}}\left(C_2\left(g^{\alpha\alpha}_{22;k+1;-1,-1}+g^{\alpha\alpha}_{22;k+1;1,-1}\right)\right. \\
&+C_3\left(g^{\alpha\beta}_{22;k+1;-1,-1}+g^{\alpha\beta}_{22;k+1;1,-1}\right)+C_4\left(vg^{\alpha\alpha}_{22;k+1;-1,1}+g^{\alpha\alpha}_{22;k+1;1,1}\right) \\
&\left.+C_5\left(vg^{\alpha\beta}_{22;k+1;-1,1}+g^{\alpha\beta}_{22;k+1;1,1}\right)\right) \\
&+C_1\sqrt{\frac{v}{u}}\left(g^{\alpha\lambda}_{2;k+1;-1,0}+g^{\alpha\lambda}_{2;k+1;1,0}\right)-C_4\sqrt{v}g^{\lambda\alpha}_{2;k+1;0,1}-C_5\sqrt{v}g^{\lambda\beta}_{2;k+1;0,1}\Bigg] \\
&-\frac{1}{2}\frac{k+2}{k+1}\left[C_6\left(g^{\alpha\alpha}_{22;k+2;0,0}-g^{\beta\alpha}_{22;k+2;0,0}\right)+C_7\left(g^{\beta\beta}_{22;k+2;0,0}-g^{\alpha\beta}_{22;k+2;0,0}\right)\right] \\
&-\frac{1}{2}\frac{k^2(k+2)}{(d+2k)(d+2k-2)(d+k-2)}\left[C_8\left(g^{\alpha\alpha}_{22;k;0,0}-g^{\beta\alpha}_{22;k;0,0}\right)+C_9\left(g^{\beta\beta}_{22;k;0,0}-g^{\alpha\beta}_{22;k;0,0}\right)\right].
\end{aligned}
\tag{2.79}
$$

The constants appearing above are written in appendix G.

### 2.5.6 A note on using these results

One important detail is that, with our definition of the shadow integral (2.6), in the limit $u \to 0$, followed by $v \to 1$, the scalar blocks (2.41) are $c_\ell u^{\frac{1}{2}(\Delta_{\mathcal{O}}-\ell)}(1-v)^\ell$ with $c_\ell = (-1/2)^\ell$. Some authors use different normalizations, e.g. $c_\ell = 1$. However, since the expressions for $g^{\gamma\gamma}$ involve sums of scalar conformal blocks at different spins $\ell$, then using a normalization differentthan ours will produce incorrect results.

## 2.6 Discussion

In this chapter we studied the conformal block decomposition of the four-point function of two scalars and two vectors, $\langle\phi v\phi v\rangle$, in general spacetime dimension $d$.

We computed the conformal blocks associated to the exchange of symmetric traceless operators $\mathcal{O}$, by applying differential operators to scalar blocks as described in subsection 1.5.2. Furthermore we found that the conformal blocks of the mixed-symmetric operator $\mathcal{A}$ can also be written in this way, with the difference that they have shifted spins $k$,$k+1$,$k+2$.

We have verified numerically that our results satisfy the correct behavior under exchange symmetry mentioned at the end of subsection 1.7.2. Furthermore, we have checked that in $d = 3$, the mixed-symmetric conformal blocks become the parity-odd conformal blocks [31, 45, 91] as expected from (2.120). It would be interesting to understand our results in the context of holography, and particularly to compare with the bulk geometric quantities from [120].

A natural next step is to apply our results in the conformal bootstrap program to seek bounds on the CFT data of theories with vectors in their spectrum. This has partially been achieved analytically, using light-cone and causality techniques of sections 1.7.3 and 1.8. These results, based on [2], are presented in chapter 4. However, numerical results would also be interesting. In particular when the vectors are conserved, it could constrain theories with continuous global symmetries and their spectra of charged scalars.

So far it is not known if the methods presented in this chapter can be generalized to other mixed-symmetric representations. From the form of the contraction (2.141), it is plausible that this can be done for $(k+1, 1, 1)$ too. However, this has not been checked explicitly yet. It would be interesting to understand the connection between these contraction formulas and the formalism of subsection 1.6.4, where the mixed-symmetric blocks are computed via the application of differential operators acting on a pair of coordinates across different OPEs.

# Appendix A: Building blocks and identities

In this chapter, the physical space is flat $\mathbb{R}^d$ with Euclidean signature. Indices $a$, $b$, etc., are raised and lowered with the Kronecker delta $\delta_{ab}$. From the fact that $x_{ij} + x_{jk} = x_{ik}$, we can show the physical space version of (1.184)

$$k_a^{(ik\ell)} = -\sqrt{\frac{x_{i\ell}^2 x_{jk}^2}{x_{ij}^2 x_{k\ell}^2}} k_a^{(ijk)} + \sqrt{\frac{x_{ik}^2 x_{j\ell}^2}{x_{ij}^2 x_{k\ell}^2}} k_a^{(ij\ell)}. \tag{2.80}$$

Using the basic identity that

$$x_{ij} \cdot x_{k\ell} = \frac{1}{2}\left(-x_{ik}^2 + x_{i\ell}^2 + x_{jk}^2 - x_{j\ell}^2\right), \tag{2.81}$$

we can prove identities

$$k^{(ijk)} \cdot k^{(i\ell m)} = \frac{1}{2}\left(x_{ij}^2 x_{ik}^2 x_{i\ell}^2 x_{im}^2 x_{jk}^2 x_{\ell m}^2\right)^{-1/2} \left(-x_{ij}^2 x_{i\ell}^2 x_{km}^2 + x_{ij}^2 x_{im}^2 x_{k\ell}^2 + x_{ik}^2 x_{i\ell}^2 x_{jm}^2 - x_{ik}^2 x_{im}^2 x_{j\ell}^2\right), \tag{2.82}$$

$$m_{ab}^{(ij)} k^{(jk\ell)\,b} = -\sqrt{\frac{x_{ik}^2 x_{j\ell}^2}{x_{ij}^2 x_{k\ell}^2}} k_a^{(ijk)} + \sqrt{\frac{x_{i\ell}^2 x_{jk}^2}{x_{ij}^2 x_{k\ell}^2}} k_a^{(ij\ell)}, \tag{2.83}$$

and

$$\delta^{cd} m_{ac}^{(ik)} m_{db}^{(kj)} = m_{ab}^{(ij)} - 2k_a^{(ijk)} k_b^{(jik)}. \tag{2.84}$$

As special cases of these fomulae, we have

$$\left(k^{(ijk)}\right)^2 = 1, \qquad m_{ab}^{(ij)} k^{(jik)\,b} = k_a^{(ijk)}, \qquad \delta^{cd} m_{ac}^{(ij)} m_{db}^{(ji)} = \delta_{ab}. \tag{2.85}$$

One more useful identity is

$$\frac{\partial}{\partial x_k^a} k_b^{(ijk)} = -\sqrt{\frac{x_{ij}^2}{x_{ik}^2 x_{jk}^2}} \left(m_{ab}^{(ki)} + k_a^{(kij)} k_b^{(ijk)}\right). \tag{2.86}$$

# Appendix B: Lorentz representation projectors

We will be grouping tensor operators by their representations under SO($d$). There is a large body of work on irreducible representations of SO($d$) (for instance see the nice discussion in [29] and references therein), but we really don't need the full power of this theory for the current work.

Consider a tensor with $n$ indices. It must transform as a sub-representation of the tensor product $\mathbf{d}^{\otimes n}$ of $n$ copies of the $d$-dimensional vector representation. To distinguish the different irreducible representations $I$ which appear in the decomposition of $\mathbf{d}^{\otimes n}$, we can use projectors, $\Pi^{I\,a_1\cdots a_n}_{b_1\cdots b_n}$. Being projectors, these must satisfy

$$\Pi^{I\,a_1\cdots a_n}_{c_1\cdots c_n} \Pi^{J\,c_1\cdots c_n}_{b_1\cdots b_n} = \delta^{IJ} \Pi^{I\,a_1\cdots a_n}_{b_1\cdots b_n}. \tag{2.87}$$

The projectors are built exclusively with Kronecker deltas $\delta^{a_i}_{b_j}$, $\delta^{a_i a_j}$, or $\delta_{b_i b_j}$.

Below, we will need the projectors for the totally symmetric traceless representation of spin $\ell$ (i.e. with $\ell$ indices), and also for a mixed symmetry representation with $k+2$ indices which we will describe below.

**Totally symmetric:** Consider first the projector onto totally symmetric traceless representations, $\Pi^{(\ell)\,a_1\cdots a_\ell}_{b_1\cdots b_\ell}$. By the symmetries of the problem, it must have the form

$$\Pi^{(\ell)\,a_1\cdots a_\ell}_{b_1\cdots b_\ell} = A_0\delta^{(a_1}_{(b_1}\cdots\delta^{a_\ell)}_{b_\ell)} + \sum_{i=1}^{\lfloor \ell/2\rfloor} A_i \delta^{(a_1a_2}\cdots\delta^{a_{2i-1}a_{2i}}\delta_{(b_1b_2}\cdots\delta_{b_{2i-1}b_{2i}}\delta^{a_{2i+1}}_{b_{2i+1}}\cdots\delta^{a_\ell)}_{b_\ell)}, \tag{2.88}$$

where the $A_i$ are constants. For $\ell\geq 2$, taking the trace with $\delta^{b_{\ell-1}b_\ell}$ we get

$$\begin{aligned} &A_0\delta^{(a_1}_{(b_1}\cdots\delta^{a_{\ell-2}}_{b_{\ell-2})}\delta^{a_{\ell-1}a_\ell)} \\ &+\sum_{i=1}^{\lfloor \ell/2\rfloor} A_i\left\{\frac{(\ell-2i)\,(\ell-1-2i)}{\ell\,(\ell-1)}\delta^{(a_1a_2}\cdots\delta^{a_{2i+1}a_{2i+2}}\delta_{(b_1b_2}\cdots\delta_{b_{2i-1}b_{2i}}\delta^{a_{2i+3}}_{b_{2i+1}}\cdots\delta^{a_\ell)}_{b_{\ell-2})}\right. \\ &\left.+\frac{2i\,(d+2\ell-2i-2)}{\ell\,(\ell-1)}\delta^{(a_1a_2}\cdots\delta^{a_{2i-1}a_{2i}}\delta_{(b_1b_2}\cdots\delta_{b_{2i-3}b_{2i-2}}\delta^{a_{2i+1}}_{b_{2i-1}}\cdots\delta^{a_\ell)}_{b_{\ell-2})}\right\}. \end{aligned} \tag{2.89}$$

Thus tracelessness requires

$$A_i = -\frac{(\ell+2-2i)\,(\ell+1-2i)}{2i\,(d+2\ell-2-2i)}A_{i-1}, \qquad \text{for } 1\leq i\leq\lfloor \ell/2\rfloor, \tag{2.90}$$

or

$$A_i = (-1)^i\,\frac{\ell!\Gamma(\frac{d}{2}+\ell-i-1)}{2^{2i}\,(\ell-2i)!i!\Gamma(\frac{d}{2}+\ell-1)}A_0. \tag{2.91}$$

Finally, we can fix $A_0$ by the condition that $\Pi^2=\Pi$, i.e.

$$\Pi^{(\ell)\,a_1\cdots a_\ell}_{c_1\cdots c_\ell}\Pi^{(\ell)\,c_1\cdots c_\ell}_{b_1\cdots b_\ell} = \Pi^{(\ell)\,a_1\cdots a_\ell}_{b_1\cdots b_\ell}. \tag{2.92}$$

In fact we only need to check the leading terms, not the subleading traceless terms, because the latter can't contribute to the former when we square. Then since

$$\delta^{(a_1}_{(c_1}\cdots\delta^{a_\ell)}_{c_\ell)}\delta^{(c_1}_{(b_1}\cdots\delta^{c_\ell)}_{b_\ell)} = \delta^{(a_1}_{(b_1}\cdots\delta^{a_\ell)}_{b_\ell)}, \tag{2.93}$$

we require $A_0^2=A_0$, and hence we should take $A_0=1$, and we can write

$$\begin{aligned} \Pi^{(\ell)\,a_1\cdots a_\ell}_{b_1\cdots b_\ell} = \sum_{i=0}^{\lfloor \ell/2\rfloor}\left(-\frac{1}{4}\right)^i &\frac{\ell!\,\Gamma(\frac{d}{2}+\ell-i-1)}{i!\,(\ell-2i)!\,\Gamma(\frac{d}{2}+\ell-1)} \\ &\times\delta^{(a_1a_2}\cdots\delta^{a_{2i-1}a_{2i}}\delta_{(b_1b_2}\cdots\delta_{b_{2i-1}b_{2i}}\delta^{a_{2i+1}}_{b_{2i+1}}\cdots\delta^{a_\ell)}_{b_\ell)}. \end{aligned} \tag{2.94}$$

These projectors obey certain recursion relations. With the explicit expressions for coefficients above, one can show that

$$\Pi^{(\ell)\,a_1\cdots a_\ell}_{b_1\cdots b_\ell} = \delta^{(a_1}_{(b_1}\Pi^{(\ell-1)\,a_2\cdots a_\ell)}_{b_2\cdots b_\ell)} - \frac{(d+\ell-4)\,(\ell-1)}{(d+2\ell-6)\,(d+2\ell-4)}\delta^{(a_1 a_2}\delta_{(b_1 b_2}\Pi^{(\ell-2)\,a_3\cdots a_\ell)}_{b_3\cdots b_\ell)}. \tag{2.95}$$

Now we can define polynomials $p_{d,\ell}(t)$ by

$$X_{a_1}\cdots X_{a_\ell}\Pi^{(\ell)\,a_1\cdots a_\ell}_{b_1\cdots b_\ell}Y^{b_1}\cdots Y^{b_\ell} = \left(X^2Y^2\right)^{\ell/2}p_{d,\ell}(t), \qquad t = \frac{X\cdot Y}{\sqrt{X^2Y^2}}. \tag{2.96}$$

Explicitly, using (2.94), we have

$$p_{d,\ell}(t) = \sum_{i=0}^{\lfloor \ell/2\rfloor}\left(-\frac{1}{4}\right)^i\frac{\ell!\Gamma(\frac{d}{2}+\ell-i-1)}{i!(\ell-2i)!\Gamma(\frac{d}{2}+\ell-1)}t^{\ell-2i}. \tag{2.97}$$

These are related to the more familiar Gegenbauer polynomials by

$$p_{d,\ell}(t) = \frac{\ell!\Gamma(\frac{d}{2}-1)}{2^\ell\Gamma(\frac{d}{2}+\ell-1)}C_\ell^{(\frac{d}{2}-1)}(t). \tag{2.98}$$

They obey a simple differential identity,

$$p'_{d,\ell}(t) = \ell p_{d+2,\ell-1}(t), \tag{2.99}$$

and also

$$p_{d+2,\ell}(t) = p_{d,\ell}(t) + \frac{\ell\,(\ell-1)}{(d+2\ell-2)\,(d+2\ell-4)}p_{d+2,\ell-2}(t). \tag{2.100}$$

We can also prove a recursion relation for fixed $d$ from (2.95),

$$p_{d,\ell}(t) = tp_{d,\ell-1}(t) - \frac{(d+\ell-4)\,(\ell-1)}{(d+2\ell-4)\,(d+2\ell-6)}p_{d,\ell-2}(t). \tag{2.101}$$

The first few of these polynomials are

$$p_0 = 1, \qquad p_1 = t, \qquad p_2 = t^2 - \frac{1}{d}, \qquad p_3 = t^3 - \frac{3}{d+2}t,$$

$$p_4 = t^4 - \frac{6}{d+4}t^2 + \frac{3}{(d+2)(d+4)}, \quad p_5 = t^5 - \frac{10}{d+6}t^3 + \frac{15}{(d+4)(d+6)}t. \tag{2.102}$$

We will also need the result of the following partial contractions of $\Pi^{(\ell)}$,

$$\begin{aligned}
&X_{c_1}\cdots X_{c_{\ell-1}}\Pi^{(\ell)\,ac_1\cdots c_{\ell-1}}_{bd_1\cdots d_{\ell-1}}Y^{d_1}\cdots Y^{d_{\ell-1}} = \frac{1}{\ell^2}\frac{\partial}{\partial X_a}\frac{\partial}{\partial Y^b}\left[\left(X^2Y^2\right)^{\ell/2}p_\ell(t)\right]\\
&= \frac{1}{\ell^2}\left(X^2Y^2\right)^{\frac{\ell-1}{2}}\left[\delta^a_b\partial_t + \left(\frac{X^aX_b}{X^2}+\frac{Y^aY_b}{Y^2}\right)\left((\ell-1)\,\partial_t - t\partial_t^2\right)\right.\\
&\quad\left.+\frac{X^aY_b}{\sqrt{X^2Y^2}}\left(\ell^2-(2\ell-1)\,t\partial_t+t^2\partial_t^2\right)+\frac{Y^aX_b}{\sqrt{X^2Y^2}}\partial_t^2\right]p_\ell(t)\\
&= \frac{1}{\ell}\left(X^2Y^2\right)^{\frac{\ell-1}{2}}\left[\delta^a_b p_{d+2,\ell-1}(t)+(\ell-1)\left(\frac{X^aX_b}{X^2}+\frac{Y^aY_b}{Y^2}\right)\left(p_{d+2,\ell-1}(t)-tp_{d+4,\ell-2}(t)\right)\right.\\
&\quad+\frac{X^aY_b}{\sqrt{X^2Y^2}}\left(\ell p_{d,\ell}(t)-(2\ell-1)\,tp_{d+2,\ell-1}(t)+(\ell-1)\,t^2p_{d+4,\ell-2}(t)\right)\\
&\quad\left.+(\ell-1)\,\frac{Y^aX_b}{\sqrt{X^2Y^2}}p_{d+4,\ell-2}(t)\right].
\end{aligned} \tag{2.103}$$

**Mixed symmetry:** Now we would like to find projectors onto the mixed symmetry representations that we need for the scalar-vector bootstrap. Recall that these tensors are antisymmetric in their first two indices, totally symmetric in their remaining $k$ indices, they vanish when antisymmetrized over any three indices (this condition is trivial unless the three are the first two indices plus one more), and are completely traceless. We will write the corresponding projectors $\widetilde{\Pi}^{(k)\,a_1a_2b_1\cdots b_k}_{c_1c_2d_1\cdots d_k}$ with tildes to distinguish from the totally symmetric case considered above.

For $k=0$, the only index structure compatible with antisymmetry is

$$\widetilde{\Pi}^{(0)\,a_1a_2}_{c_1c_2} = A_0\,(\delta^{a_1}_{c_1}\delta^{a_2}_{c_2}-\delta^{a_1}_{c_2}\delta^{a_2}_{c_1})\,. \tag{2.104}$$

Imposing $\widetilde{\Pi}^2=\widetilde{\Pi}$ then implies $A_0=1/2$.

For $k=1$, there are three terms compatible with the antisymmetry in the $a_i$ and the $c_i$,

$$\begin{aligned}
\widetilde{\Pi}^{(1)\,a_1a_2b}_{c_1c_2d} &= A_0\,(\delta^{a_1}_{c_1}\delta^{a_2}_{c_2}-\delta^{a_1}_{c_2}\delta^{a_2}_{c_1})\,\delta^b_d + B_0\left(\delta^{a_1}_{c_1}\delta^{a_2}_{d}\delta^b_{c_2}-\delta^{a_1}_{c_2}\delta^{a_2}_{d}\delta^b_{c_1}-\delta^{a_1}_{d}\delta^{a_2}_{c_1}\delta^b_{c_2}+\delta^{a_1}_{d}\delta^{a_2}_{c_2}\delta^b_{c_1}\right)\\
&\quad+C_1\left(\delta^{a_1b}\delta_{c_1d}\delta^{a_2}_{c_2}-\delta^{a_1b}\delta_{c_2d}\delta^{a_2}_{c_1}-\delta^{a_2b}\delta_{c_1d}\delta^{a_1}_{c_2}+\delta^{a_2b}\delta_{c_2d}\delta^{a_1}_{c_1}\right).
\end{aligned} \tag{2.105}$$

Demanding that this vanish on antisymetrizing $[a_1a_2b]$ leads to the constraint

$$2A_0-4B_0=0, \tag{2.106}$$

while demanding that it vanishes when we trace with $\delta^{c_2d}$ gives

$$A_0+B_0+(d-1)\,C_1=0. \tag{2.107}$$

Finally, demanding that $\widetilde{\Pi}^2=\widetilde{\Pi}$ requires

$$2A_0^2+4B_0^2=A_0,\quad 4A_0B_0-2B_0^2=B_0,\quad\text{and}\quad 4A_0C_1+4B_0C_1+2\,(d-1)\,C_1^2=C_1. \tag{2.108}$$

The unique non-vanishing solution to these constraints is that

$$A_0 = \frac{1}{3}, \qquad B_0 = \frac{1}{6}, \qquad C_1 = -\frac{1}{2\,(d-1)}, \tag{2.109}$$

so

$$\begin{aligned}\widetilde{\Pi}^{(1)\,a_1a_2b}_{c_1c_2d} &= \frac{1}{3}\left(\delta^{a_1}_{c_1}\delta^{a_2}_{c_2} - \delta^{a_1}_{c_2}\delta^{a_2}_{c_1}\right)\delta^b_d + \frac{1}{6}\left(\delta^{a_1}_{c_1}\delta^{a_2}_{d}\delta^{b}_{c_2} - \delta^{a_1}_{c_2}\delta^{a_2}_{d}\delta^{b}_{c_1} - \delta^{a_1}_{d}\delta^{a_2}_{c_1}\delta^{b}_{c_2} + \delta^{a_1}_{d}\delta^{a_2}_{c_2}\delta^{b}_{c_1}\right)\\ &\quad - \frac{1}{2\,(d-1)}\left(\delta^{a_1b}\delta_{c_1d}\delta^{a_2}_{c_2} - \delta^{a_1b}\delta_{c_2d}\delta^{a_2}_{c_1} - \delta^{a_2b}\delta_{c_1d}\delta^{a_1}_{c_2} + \delta^{a_2b}\delta_{c_2d}\delta^{a_1}_{c_1}\right). \end{aligned}\tag{2.110}$$

For $k > 1$, the following structure is the most general consistent with antisymmetry of the $a_i$ and $c_i$, symmetry of the $b_i$ and $d_i$, and symmetry between upper and lower indices,

$$\begin{aligned}&\widetilde{\Pi}^{(k)\,a_1a_2b_1\cdots b_k}_{c_1c_2d_1\cdots d_k}\\ &= \sum_{i=0}^{\lfloor k/2\rfloor} A_i\left(\delta^{a_1}_{c_1}\delta^{a_2}_{c_2} - \delta^{a_1}_{c_2}\delta^{a_2}_{c_1}\right)\delta^{(b_1b_2}\cdots\delta^{b_{2i-1}b_{2i}}\delta_{(d_1d_2}\cdots\delta_{d_{2i-1}d_{2i}}\delta^{b_{2i+1}}_{d_{2i+1}}\cdots\delta^{b_k)}_{d_k)}\\ &\quad + \sum_{i=0}^{\lfloor (k-1)/2\rfloor} B_i\left(\delta^{a_1}_{c_1}\delta^{a_2}_{(d_1}\delta^{(b_1}_{|c_2|} - \delta^{a_1}_{c_2}\delta^{a_2}_{(d_1}\delta^{(b_1}_{|c_1|} - \delta^{a_1}_{(d_1}\delta^{a_2}_{|c_1}\delta^{(b_1}_{c_2|} + \delta^{a_1}_{(d_1}\delta^{a_2}_{|c_2}\delta^{(b_1}_{c_1|}\right)\\ &\qquad \times\delta^{b_2b_3}\cdots\delta^{b_{2i}b_{2i+1}}\delta_{d_2d_3}\cdots\delta_{d_{2i}d_{2i+1}}\delta^{b_{2i+2}}_{d_{2i+2}}\cdots\delta^{b_k)}_{d_k)}\\ &\quad + \sum_{i=1}^{\lfloor (k+1)/2\rfloor} C_i\left(\delta^{a_1(b_1}\delta_{c_1(d_1}\delta^{|a_2|}_{|c_2|} - \delta^{a_1(b_1}\delta_{c_2(d_1}\delta^{|a_2|}_{|c_1|} - \delta^{a_2(b_1}\delta_{c_1(d_1}\delta^{|a_1|}_{|c_2|} + \delta^{a_2(b_1}\delta_{c_2(d_1}\delta^{|a_1|}_{|c_1|}\right)\\ &\qquad \times\delta^{b_2b_3}\cdots\delta^{b_{2i-2}b_{2i-1}}\delta_{d_2d_3}\cdots\delta_{d_{2i-2}d_{2i-1}}\delta^{b_{2i}}_{d_{2i}}\cdots\delta^{b_k)}_{d_k)}\\ &\quad + \sum_{i=1}^{\lfloor k/2\rfloor} D_i\Big(\delta^{a_1(b_1}\delta_{c_1(d_1}\delta^{|a_2|}_{d_2}\delta^{b_2}_{|c_2|} - \delta^{a_1(b_1}\delta_{c_2(d_1}\delta^{|a_2|}_{d_2}\delta^{b_2}_{|c_1|} - \delta^{a_2(b_1}\delta_{c_1(d_1}\delta^{|a_1|}_{d_2}\delta^{b_2}_{|c_2|}\\ &\qquad + \delta^{a_2(b_1}\delta_{c_2(d_1}\delta^{|a_1|}_{d_2}\delta^{b_2}_{|c_1|}\Big)\delta^{b_3b_4}\cdots\delta^{b_{2i-1}b_{2i}}\delta_{d_3d_4}\cdots\delta_{d_{2i-1}d_{2i}}\delta^{b_{2i+1}}_{d_{2i+1}}\cdots\delta^{b_k)}_{d_k)}\\ &\quad + \sum_{i=1}^{\lfloor k/2\rfloor} E_i\Big(\delta^{a_1(b_1}\delta_{(d_1d_2}\delta^{|a_2|}_{|c_1}\delta^{b_2}_{c_2|} - \delta^{a_1(b_1}\delta_{(d_1d_2}\delta^{|a_2|}_{|c_2}\delta^{b_2}_{c_1|} - \delta^{a_2(b_1}\delta_{(d_1d_2}\delta^{|a_1|}_{|c_1}\delta^{b_2}_{c_2|}\\ &\qquad + \delta^{a_2(b_1}\delta_{(d_1d_2}\delta^{|a_1|}_{|c_2}\delta^{b_2}_{c_1|} + \delta^{(b_1b_2}\delta_{c_1(d_1}\delta^{|a_1}_{|c_2|}\delta^{a_2|}_{d_2} - \delta^{(b_1b_2}\delta_{c_2(d_1}\delta^{|a_1}_{|c_1|}\delta^{a_2|}_{d_2} - \delta^{(b_1b_2}\delta_{c_1(d_1}\delta^{|a_1}_{d_2}\delta^{a_2|}_{|c_2|}\\ &\qquad + \delta^{(b_1b_2}\delta_{c_2(d_1}\delta^{|a_1}_{d_2}\delta^{a_2|}_{|c_1|}\Big)\delta^{b_3b_4}\cdots\delta^{b_{2i-1}b_{2i}}\delta_{d_3d_4}\cdots\delta_{d_{2i-1}d_{2i}}\delta^{b_{2i+1}}_{d_{2i+1}}\cdots\delta^{b_k)}_{d_k)}. \end{aligned}\tag{2.111}$$

Demanding this vanish when we antisymmetrize over $[a_1a_2b_1]$, when we trace with $\delta^{d_{k-1}d_k}$, and when we trace with $\delta^{c_2d_k}$ fixes everything up to one constant $A_0$ which can then be fixed by the condition that $\widetilde{\Pi}^2 = \widetilde{\Pi}$. The result is that

$$B_i = \frac{k-2i}{2} A_i, \tag{2.112}$$

$$C_i = -\frac{k-2i+2}{d+k-2}\left[\frac{i\,(d+k-1)}{d+2k-2i} + \frac{1}{2}\right] A_{i-1}, \tag{2.113}$$

$$D_i = \frac{i\,(d+2k)}{d+k-2} A_i, \tag{2.114}$$

$$E_i = -iA_i, \tag{2.115}$$

while the $A_i$ are given by

$$A_0 = \frac{1}{k+2}, \tag{2.116}$$

and the recursion

$$A_i = -\frac{(k-2i+1)\,(k-2i+2)}{2i\,(d+2k-2i)} A_{i-1}, \tag{2.117}$$

solved by

$$A_i = \left(-\frac{1}{4}\right)^i \frac{k!\,\Gamma(\frac{d}{2}+k-i)}{(k+2)\,i!\,(k-2i)!\,\Gamma(\frac{d}{2}+k)}. \tag{2.118}$$

For $d \leq 4$, the story so far is not quite complete.

In $d = 2$, these mixed symmetry tensors labeled by $k$ are equivalent to spin-$k$ symmetric traceless tensors, with the map

$$\mathcal{A}_{a_1 a_2 b_1 \cdots b_k} = \epsilon_{a_1 a_2} \mathcal{O}_{b_1 \cdots b_k}, \qquad \mathcal{O}_{a_1 \cdots a_k} = \frac{1}{2} \epsilon^{b_1 b_2} \mathcal{A}_{b_1 b_2 a_1 \cdots a_k}. \tag{2.119}$$

In $d = 3$ similarly, there is an isomorphism between mixed symmetry labeled by $k$ and traceless symmetric of spin $k+1$, via

$$\mathcal{A}_{a_1 a_2 b_1 \cdots b_k} = \epsilon_{a_1 a_2}{}^{c} \mathcal{O}_{b_1 \cdots b_k c} + \frac{k}{2}\left(\epsilon_{a_1 (b_1}{}^{c} \mathcal{O}_{|a_2| b_2 \cdots b_k) c} - \epsilon_{a_2 (b_1}{}^{c} \mathcal{O}_{|a_1| b_2 \cdots b_k) c}\right), \tag{2.120}$$

and

$$\mathcal{O}_{a_1 \cdots a_{k+1}} = \frac{1}{k+2} \epsilon_{(a_1}{}^{c_1 c_2} \mathcal{A}_{|c_1 c_2| a_2 \cdots a_{k+1})}. \tag{2.121}$$

Finally, in $d = 4$ we don't have to worry about any isomorphisms of this sort, but we instead need to recognize that our mixed symmetry representations are in fact reducible. To split the two pieces apart, we can define

$$\Pi^{(\pm)\,a_1 a_2}_{b_1 b_2} = \frac{1}{4}\left(\delta^{a_1}_{b_1}\delta^{a_2}_{b_2} - \delta^{a_1}_{b_2}\delta^{a_2}_{b_1} \pm \epsilon^{a_1 a_2}{}_{b_1 b_2}\right), \tag{2.122}$$

and then define

$$\widetilde{\Pi}^{(k\pm)\,a_1 a_2 b_1 \cdots b_k}_{c_1 c_2 d_1 \cdots d_k} = \Pi^{(\pm)\,a_1 a_2}_{e_1 e_2} \Pi^{(\pm)\,f_1 f_2}_{c_1 c_2} \widetilde{\Pi}^{(k)\,e_1 e_2 b_1 \cdots b_k}_{f_1 f_2 d_1 \cdots d_k}. \tag{2.123}$$

As in the symmetric case, we will need to consider the result of contracting these projectors with vectors $X$ and $Y$, so we consider the expression

$$X_{c_1} \cdots X_{c_{k+1}} \widetilde{\Pi}^{(k)\,a c_1 \cdots c_{k+1}}_{b d_1 \cdots d_{k+1}} Y^{d_1} \cdots Y^{d_{k+1}}. \tag{2.124}$$

The free indices $a$ and $b$ can only be carried by a Kronecker delta $\delta^a_b$ or by the vectors $X^a$ and $Y^a$. Moreover, the expression must be symmetric under simultaneous interchange of $X$ with $Y$ and $a$ with $b$, and it must be identically zero when we contract with $X_a$ or with $Y^b$. These conditions imply that it must have the form

$$X_{c_1}\cdots X_{c_{k+1}}\widetilde{\Pi}^{(k)\,ac_1\cdots c_{k+1}}_{bd_1\cdots d_{k+1}}Y^{d_1}\cdots Y^{d_{k+1}} = \left(X^2Y^2\right)^{\frac{k+1}{2}}\left[\left(-\delta^a_b+\frac{X^aX_b}{X^2}+\frac{Y^aY_b}{Y^2}-\frac{X^aY_b}{\sqrt{X^2Y^2}}t\right)f_{k-1}(t)+\left(-\delta^a_b t+\frac{Y^aX_b}{\sqrt{X^2Y^2}}\right)g_k(t)\right], \quad (2.125)$$

for some polynomials $f_{k-1}(t)$ and $g_k(t)$ of degree $k-1$ and $k$ respectively, with $t = X\cdot Y/\sqrt{X^2Y^2}$ as before. These polynomials can be determined by explicit contraction of (2.111) and use of the solutions for coefficients determined above. The result is

$$f_{k-1}(t) = \frac{1}{2}\sum_{i=0}^{\lfloor\frac{k-1}{2}\rfloor}\left(-\frac{1}{4}\right)^i\frac{k!\Gamma(\frac{d}{2}+k-i)}{i!\,(k-2i-1)!\,(d+k-2)\,\Gamma(\frac{d}{2}+k)}t^{k-2i-1}, \quad (2.126)$$

and

$$g_k(t) = -\frac{1}{2}\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}\left(-\frac{1}{4}\right)^i\frac{k!\,(d+k-2i-2)\,\Gamma(\frac{d}{2}+k-i)}{i!\,(k-2i)!\,(d+k-2)\,\Gamma(\frac{d}{2}+k)}t^{k-2i}. \quad (2.127)$$

Actually, these can be recast in terms of the polynomials $p_{d,\ell}(t)$ which we defined in the symmetric case (and which are related to the usual Gegenbauer polynomials),

$$\begin{aligned} f_{k-1}(t) &= \frac{1}{2\,(k+1)\,(d+k-2)}p''_{d,k+1}(t) = \frac{k}{2\,(d+k-2)}p_{d+4,k-1}(t), \quad (2.128)\\ g_k(t) &= -\frac{1}{2\,(k+1)\,(d+k-2)}\left((d-2)\,p'_{d,k+1}(t)+tp''_{d,k+1}(t)\right)\\ &= -\frac{1}{2\,(d+k-2)}\left((d-2)\,p_{d+2,k}(t)+ktp_{d+4,k-1}(t)\right), \quad (2.129)\end{aligned}$$

where a prime denotes the derivative with respect to the argument $t$.

# Appendix C: Mixed symmetric contractions

In general one expects that the contraction of the projector $\Pi^{[\lambda]}$ associated to some Young symmetry $\lambda$ is given by

$$X_{f_1}\cdots X_{f_k}\Pi^{[\lambda]e_1\cdots e_n f_1\cdots f_k}_{g_1\cdots g_n h_1\cdots h_k}Y^{h_1}\cdots Y^{h_k} = \sum_i T_i(X,Y)^{e_1\cdots e_n}_{g_1\cdots g_n}\mathcal{P}_i(t), \quad (2.130)$$

where $T_i$ are tensor structures made out of combinations of $X$, $Y$, and the Kronecker delta, and $\mathcal{P}$ polynomials on $t\equiv\frac{X\cdot Y}{\sqrt{X^2Y^2}}$. In the previous section we showed this, explicitly, for $[k+1,1]$. More generally, from the work of [121–123] (see also [37]), one can understand this expression as the result of a particular differential operator (say in $X$) acting on the symmetric contraction of $\lambda_1$ indices

$$X_{f_1}\cdots X_{f_k}\Pi^{[\lambda]e_1\cdots e_n f_1\cdots f_k}_{g_1\cdots g_n h_1\cdots h_k}Y^{h_1}\cdots Y^{h_k}=\mathcal{D}^{[\lambda]e_1\cdots e_n}_{g_1\cdots g_n}(X)H_{\lambda_1}(X\cdot Y)^{\lambda_1}, \tag{2.131}$$

where

$$H_{\lambda_1}(X\cdot Y)^{\lambda_1}=X_{f_1}\cdots X_{f_{\lambda_1}}\Pi^{[\lambda_1]f_1\cdots f_{\lambda_1}}_{h_1\cdots h_{\lambda_1}}Y^{h_1}\cdots Y^{h_{\lambda_1}}=(X^2Y^2)^{\lambda_1/2}p_{d,\lambda_1}(t), \tag{2.132}$$

and $\lambda_1$ is the length of the top row of the Young pattern $[\lambda]$ (in our case this is $k+1$). In the context of conformal blocks, the extra indices $e_j$ are contracted with $m^{(10)}$, $m^{(20)}$, and the indices $g_j$ with $m^{(30)}$, $m^{(40)}$. Furthermore, $X\equiv k^{(012)}$, $Y\equiv k^{(034)}$ with $X^2=Y^2=1$. Thus a generic contraction with $T_i$

$$m^{10}\cdots m^{10}m^{20}\cdots m^{20}\cdot T_i(k^{(012)},k^{(034)})\cdot m^{(30)}\cdots m^{(30)}m^{(40)}\cdots m^{(40)}, \tag{2.133}$$

can include combinations of 1- and 2-index elements

$$m^{(i0)}_{ab}k^{(012)\,b},\quad m^{(i0)}_{ab}k^{(034)\,b},\quad m^{(i0)}_{ab}m^{(0j)\,b}{}_c, \tag{2.134}$$

with $i,j=1,2,3,4$. The result presented in this chapter suggests that one can write the contractions (2.133) as derivatives of $t$ only. For example, for (2.125) we have

$$\begin{aligned} m^{(20)}_{ae}T_1{}^e{}_c m^{(04)\,c}{}_b &= m^{(20)}_{ae}(t\delta^e_c-k^{(012)}_c k^{(034)\,e})m^{(04)\,c}{}_b \\ &= \sqrt{\frac{x_{02}^2x_{04}^2x_{12}^2x_{34}^2}{x_{01}^2x_{03}^2}}\left(t\frac{\partial^2 t}{\partial x_2^a\partial x_4^b}-\frac{\partial t}{\partial x_2^a}\frac{\partial t}{\partial x_4^b}\right). \end{aligned} \tag{2.135}$$

$$\begin{aligned} &m^{(20)}_{ae}T_2{}^e{}_c m^{(04)\,c}{}_b \\ &= m^{(20)}_{ae}\left((t^2-1)\delta^e_c+k^{(012)}_c k^{(012)\,e}+k^{(034)}_c k^{(034)\,e}-t(k^{(012)}_c k^{(034)\,e}+k^{(034)}_c k^{(012)\,e})\right)m^{(04)\,c}{}_b \\ &= \sqrt{\frac{x_{02}^2x_{04}^2x_{12}^2x_{34}^2}{x_{01}^2x_{03}^2}}\left((t^2-1)\frac{\partial^2 t}{\partial x_2^a\partial x_4^b}-t\frac{\partial t}{\partial x_2^a}\frac{\partial t}{\partial x_4^b}\right), \end{aligned} \tag{2.136}$$

where we have extracted $T_1$ and $T_2$ by rewriting (2.125) as

$$-T_1{}^a{}_b\,(tf_{k-1}+g_k)+T_2{}^a{}_b f_{k-1}(t), \tag{2.137}$$

and we picked the particular combinations because $f_{k-1}$ and $tf_{k-1}+g_k$ are just constant multiples of $p_{d+4,k-1}$ and $p_{d+2,k}$ respectively. These results rely on the fact that

$$\frac{\partial k^{(0ij)}_c}{\partial x^a_j}=-\sqrt{\frac{x_{0i}^2}{x_{0j}^2x_{ij}^2}}\left(m^{(0j)}_{ac}+k^{(j0i)}_a k^{(0ij)}_c\right), \tag{2.138}$$

and the key observation is that the particular combinations of $\delta$, $k^{(012)}$, $k^{(034)}$, that appear in $T_i$, are such that the terms $k^{(j0i)}_a$ cancel out, leaving only the terms $m^{(0j)}$ that we want. This leads to

$$m_{ac}^{(20)}k_{d_1}^{(012)}\cdots k_{d_{k+1}}^{(012)}\widetilde{\Pi}^{(k)\,cd_1\cdots d_{k+1}}_{ef_1\cdots f_{k+1}}m^{(04)\,e}{}_{b}k^{(034)\,f_1}\cdots k^{(034)\,f_{k+1}}$$
$$=\sqrt{\frac{x_{02}^2x_{04}^2x_{12}^2x_{34}^2}{x_{01}^2x_{03}^2}}\frac{1}{2(d+k-2)}\left(k\left((t^2-1)\frac{\partial^2 t}{\partial x_2^a\partial x_4^b}-t\frac{\partial t}{\partial x_2^a}\frac{\partial t}{\partial x_4^b}\right)p_{d+4,k-1}\right.$$
$$\left.+(d-2)\left(t\frac{\partial^2 t}{\partial x_2^a\partial x_4^b}-\frac{\partial t}{\partial x_2^a}\frac{\partial t}{\partial x_4^b}\right)p_{d+2,k}\right). \quad (2.139)$$

Equation (2.28) then follows from the chain rule and simple Gegenbauer identities listed in appendix B.

As an extra result, we present the contraction under the projector $\Pi^{[k+1,1,1]}$ associated to the Young pattern $[k+1,1,1]$. Using techniques from [121–123] one obtains

$$\mathcal{P}^{[k+1,1,1]e_1e_2}_{c_1c_2}\equiv X_{f_1}\cdots X_{f_{k+1}}\Pi^{[k+1,1,1]e_1e_2f_1\cdots f_{k+1}}_{c_1c_2d_1\cdots d_{k+1}}Y^{d_1}\cdots Y^{d_{k+1}}$$
$$\propto\frac{(k+1)(X^2Y^2)^{\frac{k+1}{2}}}{(k+3)(d+k-3)}\delta^{[d}_{c_1}\delta^{f]}_{c_2}\delta^{e_1}_{[g}\delta^{e_2}_{h]}\left((d-3)\delta^g_d\left[t\delta^h_f-2\frac{X_fY^h}{\sqrt{X^2Y^2}}\right]p_{d+2,k}(t)\right.$$
$$\left.+2k\left[\delta^g_d\left(\frac{(t^2-1)\delta^h_f}{2}+\frac{X_fX^h}{X^2}-\frac{t\left(X_fY^h+Y_fX^h\right)}{\sqrt{X^2Y^2}}+\frac{Y_fY^h}{Y^2}\right)-\frac{X_dY_fX^gY^h}{X^2Y^2}\right]p_{d+4,k-1}\right). \quad (2.140)$$

Thus from the previous discussion one finds that

$$m^{(10)}_{ae_1}m^{(20)}_{be_2}\mathcal{P}^{[k+1,1,1]e_1e_2}_{c_1c_2}m_c^{(30)\,c_1}m_d^{(40)\,c_2}$$
$$=\frac{x_{02}^2x_{04}^2x_{12}^2x_{34}^2}{x_{01}^2x_{03}^2}m^{(12)}_{ab'}m^{(34)}_{cd'}\left\{\frac{\partial^2 t}{\partial x_2^b\partial x_4^{[d|}}\frac{\partial^2 t}{\partial x_2^{b'}\partial x_4^{|d']}}p_{d,k+1}-2\frac{\partial^2 t}{\partial x_2^{[b|}\partial x_4^{[d|}}\frac{\partial^2}{\partial x_2^{|b']}\partial x_4^{|d']}}\right.$$
$$\left.\times\left[\frac{1}{(k+2)(k+3)}p_{d,k+2}+\frac{(k+1)}{(d+2k)(d+2k-2)(d+k-3)}p_{d,k}\right]\right\}, \quad (2.141)$$

where in the second term, the square bracket notation is indicating that $b$ ($d$) is antisymmetrized with $b'$ ($d'$).

This is one of the two new contractions that appear in the conformal blocks for the $[k+1,1,1]$ exchange in $\langle VVVV\rangle$. Those conformal blocks are left for future work.

# Appendix D: Integrals

Much of the material in this appendix follows [28].

The basic building block for our integrals is

$$\int d^Dx_0\left(\sum_i a_ix_{0i}^2\right)^{-d}=\frac{2^{1-d}\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})}\left(\sum_{i<j}a_ia_jx_{ij}^2\right)^{-d/2}, \quad (2.142)$$

along with the Feynman-Schwinger trick which uses the identity

$$\frac{1}{\prod_{i=1}^{n} X_i^{c_i}} = \frac{\Gamma(\sum_{i=1}^{n} c_i)}{\prod_{j=1}^{n} \Gamma(c_j)} \left( \prod_{k=2}^{n} \int_0^\infty d\mu_k \, \mu^{c_k - 1} \right) \frac{1}{\left( X_1 + \sum_{\ell=2}^{n} \mu_\ell X_\ell \right)^{\sum_{m=1}^{n} c_m}}. \quad (2.143)$$

**Three-point integrals:** Suppose $\alpha + \beta + \gamma = d$. Then the integral

$$I_{\alpha,\beta,\gamma}(x_1, x_2, x_3) = \int \frac{d^D x_0}{\left(x_{01}^2\right)^\alpha \left(x_{02}^2\right)^\beta \left(x_{03}^2\right)^\gamma}, \quad (2.144)$$

will be a conformal scalar of weight $\alpha$, $\beta$, and $\gamma$ under conformal transformations of $x_1$, $x_2$, and $x_3$ respectively. To evaluate the integral, we first use (2.143) and then (2.142) to write

$$\begin{aligned} I_{\alpha,\beta,\gamma}(x_1, x_2, x_3) &= \frac{\Gamma(d)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int d^D x_0 \int_0^\infty ds\, s^{\beta - 1} \int_0^\infty dt\, t^{\gamma - 1} \frac{1}{\left(x_{01}^2 + s x_{02}^2 + t x_{03}^2\right)^d} \\ &= \frac{2^{1-d} \pi^{\frac{d+1}{2}} \Gamma(d)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\frac{d+1}{2})} \int_0^\infty ds\, s^{\beta-1} \int_0^\infty dt\, t^{\gamma-1} \left(s x_{12}^2 + t x_{13}^2 + st x_{23}^2\right)^{-d/2}. \end{aligned} \quad (2.145)$$

To perform the remaining integrals, we recall one of the representations of the beta function

$$\int_0^\infty du \frac{u^{x-1}}{(1+u)^{x+y}} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \quad (2.146)$$

Then

$$\begin{aligned} &I_{\alpha,\beta,\gamma}(x_1, x_2, x_3) \\ &= \frac{2^{1-d} \pi^{\frac{d+1}{2}} \Gamma(d)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\frac{d+1}{2})} \int_0^\infty ds\, s^{\beta-1} \left(s x_{12}^2\right)^{-d/2} \int_0^\infty dt\, t^{\gamma-1} \left(1 + t\left(\frac{x_{13}^2 + s x_{23}^2}{s x_{12}^2}\right)\right)^{-d/2} \\ &= \frac{2^{1-d} \pi^{\frac{d+1}{2}} \Gamma(d)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\frac{d+1}{2})} \\ &\quad \times \left(x_{12}^2\right)^{\gamma - \frac{d}{2}} \int_0^\infty ds\, s^{\beta+\gamma-\frac{d}{2}-1} \left(x_{13}^2 + s x_{23}^2\right)^{-\gamma} \int_0^\infty du\, u^{\gamma-1} (1+u)^{-d/2} \\ &= \frac{2^{1-d} \pi^{\frac{d+1}{2}} \Gamma(d)\Gamma(\frac{d}{2} - \gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\frac{d+1}{2})\Gamma(\frac{d}{2})} \left(x_{12}^2\right)^{\gamma-\frac{d}{2}} \left(x_{13}^2\right)^{-\gamma} \int_0^\infty ds\, s^{\beta+\gamma-\frac{d}{2}-1} \left(1 + s\frac{x_{23}^2}{x_{13}^2}\right)^{-\gamma} \\ &= \frac{\pi^{d/2}\Gamma(\frac{d}{2} - \gamma)}{\Gamma(\alpha)\Gamma(\beta)} \left(x_{12}^2\right)^{\gamma-\frac{d}{2}} \left(x_{13}^2\right)^{\beta-\frac{d}{2}} \left(x_{23}^2\right)^{\frac{d}{2}-\beta-\gamma} \int_0^\infty dv\, v^{\beta+\gamma-\frac{d}{2}-1} (1+v)^{-\gamma} \\ &= \frac{\pi^{d/2}\Gamma(\frac{d}{2} - \gamma)\Gamma(\beta + \gamma - \frac{d}{2})\Gamma(\frac{d}{2} - \beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \left(x_{12}^2\right)^{\gamma-\frac{d}{2}} \left(x_{13}^2\right)^{\beta-\frac{d}{2}} \left(x_{23}^2\right)^{\frac{d}{2}-\beta-\gamma} \\ &= \pi^{d/2} \frac{\Gamma(\frac{d}{2} - \alpha)\Gamma(\frac{d}{2} - \beta)\Gamma(\frac{d}{2} - \gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \left(x_{12}^2\right)^{\gamma-\frac{d}{2}} \left(x_{13}^2\right)^{\beta-\frac{d}{2}} \left(x_{23}^2\right)^{\alpha-\frac{d}{2}}, \end{aligned} \quad (2.147)$$

where we have also made use of the duplication formula for the gamma function, which in this case tells us

$$\Gamma(\frac{d}{2})\Gamma(\frac{d+1}{2}) = 2^{1-d}\sqrt{\pi}\Gamma(d). \quad (2.148)$$

Similarly, we will need to evaluate

$$I_{\alpha\beta\gamma;a_1\cdots a_n}(x_1,x_2,x_3) = \Pi^{(n)\,b_1\cdots b_n}_{a_1\cdots a_n} \int \frac{d^D x_0}{\left(x_{01}^2\right)^{\alpha}\left(x_{02}^2\right)^{\beta}\left(x_{03}^2\right)^{\gamma}} k^{(302)}_{b_1}\cdots k^{(302)}_{b_n}, \tag{2.149}$$

which for $\alpha+\beta+\gamma = d$ will be a conformal scalar of weight $\alpha$ ($\beta$) under conformal transformations of $x_1$ ($x_2$), and a traceless symmetric tensor of conformal weight $\gamma$ under transformations of $x_3$. We compute by doing a binomial expansion of the $k^{(302)}$'s,

$$\begin{aligned}
I_{\alpha\beta\gamma;a_1\cdots a_n}(x_1,x_2,x_3) &= \Pi^{(n)\,b_1\cdots b_n}_{a_1\cdots a_n} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} (-1)^k \left(x_{23}^2\right)^{\frac{n-2k}{2}} \left(x_{23}\right)_{b_1}\cdots\left(x_{23}\right)_{b_k} \\
&\times \int \frac{d^D x_0}{\left(x_{01}^2\right)^{\alpha}\left(x_{02}^2\right)^{\beta+\frac{n}{2}}\left(x_{03}^2\right)^{\gamma+\frac{n-2k}{2}}} \left(x_{03}\right)_{b_{k+1}}\cdots\left(x_{03}\right)_{b_n} \\
&= \Pi^{(n)\,b_1\cdots b_n}_{a_1\cdots a_n} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} (-1)^k \left(x_{23}^2\right)^{\frac{n-2k}{2}} \left(x_{23}\right)_{b_1}\cdots\left(x_{23}\right)_{b_k} \\
&\times \frac{\Gamma(\gamma-\frac{n}{2})}{2^{n-k}\Gamma(\gamma+\frac{n}{2}-k)} \frac{\partial}{\partial x_3^{b_{k+1}}}\cdots\frac{\partial}{\partial x_3^{b_n}} I_{\alpha,\beta+\frac{n}{2},\gamma-\frac{n}{2}}(x_1,x_2,x_3) \\
&= \pi^{d/2}\Pi^{(n)\,b_1\cdots b_n}_{a_1\cdots a_n} \sum_{k=0}^{n}\sum_{m=0}^{n-k} \frac{n!}{k!m!\,(n-k-m)!} (-1)^k \\
&\times \frac{\Gamma(\frac{d}{2}+n-m-k-\alpha)\Gamma(\frac{d}{2}-\frac{n}{2}+m-\beta)\Gamma(\frac{d}{2}+\frac{n}{2}-\gamma)}{\Gamma(\alpha)\Gamma(\beta+\frac{n}{2})\Gamma(\gamma+\frac{n}{2}-k)} \\
&\times \left(x_{12}^2\right)^{\gamma-\frac{n}{2}-\frac{d}{2}} \left(x_{13}^2\right)^{\beta+\frac{n}{2}-m-\frac{d}{2}} \left(x_{23}^2\right)^{\alpha-\frac{n}{2}+m-\frac{d}{2}} \left(x_{13}\right)_{b_1}\cdots\left(x_{13}\right)_{b_m}\left(x_{23}\right)_{b_{m+1}}\cdots\left(x_{23}\right)_{b_n} \\
&= \pi^{d/2}\Pi^{(n)\,b_1\cdots b_n}_{a_1\cdots a_n} \sum_{m=0}^{n} \frac{n!}{m!\,(n-m)!} (-1)^{n-m} \frac{\Gamma(\frac{d}{2}-\alpha)\Gamma(\frac{d}{2}+\frac{n}{2}-\beta)\Gamma(\frac{d}{2}+\frac{n}{2}-\gamma)}{\Gamma(\alpha)\Gamma(\beta+\frac{n}{2})\Gamma(\gamma+\frac{n}{2})} \\
&\times \left(x_{12}^2\right)^{\gamma-\frac{n}{2}-\frac{d}{2}} \left(x_{13}^2\right)^{\beta+\frac{n}{2}-m-\frac{d}{2}} \left(x_{23}^2\right)^{\alpha-\frac{n}{2}+m-\frac{d}{2}} \left(x_{13}\right)_{b_1}\cdots\left(x_{13}\right)_{b_m}\left(x_{23}\right)_{b_{m+1}}\cdots\left(x_{23}\right)_{b_n} \\
&= \pi^{d/2}\Pi^{(n)\,b_1\cdots b_n}_{a_1\cdots a_n} \frac{\Gamma(\frac{d}{2}-\alpha)\Gamma(\frac{d}{2}+\frac{n}{2}-\beta)\Gamma(\frac{d}{2}+\frac{n}{2}-\gamma)}{\Gamma(\alpha)\Gamma(\beta+\frac{n}{2})\Gamma(\gamma+\frac{n}{2})} \\
&\times \left(x_{12}^2\right)^{\gamma-\frac{d}{2}} \left(x_{13}^2\right)^{\beta-\frac{d}{2}} \left(x_{23}^2\right)^{\alpha-\frac{d}{2}} k^{(312)}_{b_1}\cdots k^{(312)}_{b_n},
\end{aligned} \tag{2.150}$$

where we used the identity

$$\sum_{k=0}^{N} \frac{N!}{k!\,(N-k)!} (-1)^k \frac{\Gamma(x-k)}{\Gamma(y-k)} = (-1)^N \frac{\Gamma(x-N)\Gamma(y-x+N)}{\Gamma(y)\Gamma(y-x)}, \tag{2.151}$$

with $N = n-m$, $x = \frac{d}{2}+n-m-\alpha$, and $y = \gamma+\frac{n}{2}$.

We will also need one more result along these lines,

$$\begin{aligned}
I_{\alpha,\beta,\gamma;a;b_1\cdots b_n} &= \Pi^{(n)\,c_1\cdots c_n}_{b_1\cdots b_n}\int \frac{d^D x_0}{\left(x_{01}^2\right)^{\alpha}\left(x_{02}^2\right)^{\beta}\left(x_{03}^2\right)^{\gamma}} k_a^{(203)} k_{c_1}^{(302)}\cdots k_{c_n}^{(302)} \\
= \Pi^{(n)\,c_1\cdots c_n}_{b_1\cdots b_n} &\left\{ \frac{\sqrt{x_{23}^2}}{2\beta+n-1}\frac{\partial}{\partial x_2^a} I_{\alpha,\beta-\frac12,\gamma+\frac12;c_1\cdots c_n} + \frac{2\beta-1}{2\beta+n-1}\frac{(x_{23})_a}{\sqrt{x_{23}^2}} I_{\alpha,\beta-\frac12,\gamma+\frac12;c_1\cdots c_n} \right. \\
&\left. + \frac{n m_{ac_1}^{(23)}}{2\beta+n-1} I_{\alpha,\beta,\gamma;c_2\cdots c_n} \right\} \\
= \pi^{d/2}\Pi^{(n)\,c_1\cdots c_n}_{b_1\cdots b_n} &\frac{\Gamma(\frac d2-\alpha)\Gamma(\frac d2+\frac{n-1}{2}-\beta)\Gamma(\frac d2+\frac{n-1}{2}-\gamma)}{\Gamma(\alpha)\Gamma(\beta+\frac{n+1}{2})\Gamma(\gamma+\frac{n+1}{2})}\left(x_{12}^2\right)^{\gamma-\frac d2}\left(x_{13}^2\right)^{\beta-\frac d2} \\
\times \left(x_{23}^2\right)^{\alpha-\frac d2} &\left[\left(\frac d2+\frac{n-1}{2}-\beta\right)\left(\frac d2+\frac{n-1}{2}-\gamma\right)k_a^{(213)}k_{c_1}^{(213)} + \frac n2\left(\frac d2-\alpha\right)m_{ac_1}^{(23)}\right] \\
&\times k_{c_2}^{(312)}\cdots k_{c_n}^{(312)}. \qquad (2.152)
\end{aligned}$$

In this case we made use of (2.86).

**Four-point integrals:** As with the previous section, we start with integrals of the form

$$I_{\alpha,\beta,\gamma,\delta}(x_1,x_2,x_3,x_4) = \int \frac{d^D x_0}{\left(x_{01}^2\right)^{\alpha}\left(x_{02}^2\right)^{\beta}\left(x_{03}^2\right)^{\gamma}\left(x_{04}^2\right)^{\delta}}, \qquad (2.153)$$

where $\alpha+\beta+\gamma+\delta=d$. Using (2.142) and (2.143) we can show

$$\begin{aligned}
I_{\alpha,\beta,\gamma,\delta} = \frac{2^{1-d}\pi^{\frac{d+1}{2}}\Gamma(d)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\delta)\Gamma(\frac{d+1}{2})}&\int_0^\infty ds\, s^{\beta-1}\int_0^\infty dt\, t^{\gamma-1}\int_0^\infty dq\, q^{\delta-1} \\
&\times\left(sx_{12}^2+tx_{13}^2+qx_{14}^2+stx_{23}^2+sqx_{24}^2+tqx_{34}^2\right)^{-d/2}. \qquad (2.154)
\end{aligned}$$

After a change of variables we can do one of the three integrals, giving us a result

$$\begin{aligned}
I_{\alpha,\beta,\gamma,\delta} = \pi^{d/2}\frac{\Gamma(\frac d2-\delta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}&\left(x_{14}^2\right)^{-\alpha}\left(x_{23}^2\right)^{\delta-\frac d2}\left(x_{24}^2\right)^{\frac d2-\beta-\delta}\left(x_{34}^2\right)^{\frac d2-\gamma-\delta} \\
&\times \widehat{f}_{\alpha,\beta,\gamma,\delta}(uv^{-1},v^{-1}), \qquad (2.155)
\end{aligned}$$

where we have defined

$$\widehat{f}_{\alpha,\beta,\gamma,\delta}(z_1,z_2) = \int_0^\infty ds\, s^{\beta-1}\int_0^\infty dt\, t^{\gamma-1}\left(sz_1+tz_2+st\right)^{\frac{\delta-\alpha-\beta-\gamma}{2}}(1+s+t)^{-\delta}, \qquad (2.156)$$

and $u$ and $v$ are the usual invariant cross-ratios defined in (1.119).

As explained in Section 2.5.1, the monodromy projection requires us to keep only the terms in $\widehat{f}_{\alpha,\beta,\gamma,\delta}(z_1,z_2)$ which are invariant under $z_1\to e^{4\pi i}z_1$. In [28] it is shown how to do this very elegantly using contour deformation arguments, with the result that the invariant pieces are given precisely by

$$f_{\alpha,\beta,\gamma,\delta}(z_1,z_2) = \widehat{f}_{\alpha,\beta,\gamma,\delta}(z_1,z_2)\Big|_{\text{monodromy}-\text{invariant}}$$
$$= \frac{\sin(\pi\delta)}{\sin(\frac{\pi}{2}(\gamma+\delta-\alpha-\beta))}\int_0^\infty ds\, s^{\beta-1}\int_{s+1}^\infty dt\, t^{\gamma-1}\,(st+tz_2-sz_1)^{\frac{\delta-\alpha-\beta-\gamma}{2}}\,(t-s-1)^{-\delta}\,. \tag{2.157}$$

The function $f_{\alpha,\beta,\gamma,\delta}(z_1,z_2)$ obeys several easily verified identities (also $\widehat{f}$ obeys the same identities),

$$\frac{\partial}{\partial z_1}f_{\alpha,\beta,\gamma,\delta}(z_1,z_2) = \frac{\delta-\alpha-\beta-\gamma}{2}f_{\alpha+1,\beta+1,\gamma,\delta}(z_1,z_2), \tag{2.158}$$
$$\frac{\partial}{\partial z_2}f_{\alpha,\beta,\gamma,\delta}(z_1,z_2) = \frac{\delta-\alpha-\beta-\gamma}{2}f_{\alpha+1,\beta,\gamma+1,\delta}(z_1,z_2), \tag{2.159}$$

as well as

$$f_{\alpha,\beta,\gamma,\delta}(z_1,z_2) = f_{\alpha+1,\beta,\gamma,\delta+1}(z_1,z_2)+f_{\alpha,\beta+1,\gamma,\delta+1}(z_1,z_2)+f_{\alpha,\beta,\gamma+1,\delta+1}(z_1,z_2), \tag{2.160}$$

and

$$f_{\alpha,\beta,\gamma,\delta}(z_1,z_2) = f_{\alpha,\beta+1,\gamma+1,\delta}(z_1,z_2) + z_1 f_{\alpha+1,\beta+1,\gamma,\delta}(z_1,z_2) + z_2 f_{\alpha+1,\beta,\gamma+1,\delta}(z_1,z_2). \tag{2.161}$$

When $\alpha+\beta+\gamma+\delta$ is an even integer, which we will call $2h$ (so $h=d/2$ in the four-point integral above, and this would be valid in even dimensions), then $f_{\alpha,\beta,\gamma,\delta}$ can actually be evaluated explicitly in terms of hypergeometric functions. First we change from $z_1$ and $z_2$ to a complex variable $x$ related by

$$z_1 = \frac{x\bar{x}}{(1-x)(1-\bar{x})}, \qquad z_2 = \frac{1}{(1-x)(1-\bar{x})}, \tag{2.162}$$

and then it can be shown [28, 64] that

$$f_{\alpha,\beta,\gamma,\delta}(z_1,z_2) = \frac{\Gamma(\alpha)\Gamma(1-h+\beta)\Gamma(1-\delta)\Gamma(h-\gamma)\Gamma(\gamma+\delta-h)}{\Gamma(\delta)\Gamma(h-\delta)\Gamma(1+h-\gamma-\delta)}\left((1-x)(1-\bar{x})\right)^{h-\delta}$$
$$\times\left(\frac{1}{x-\bar{x}}\left(x\partial_x-\bar{x}\partial_{\bar{x}}\right)\right)^{h-1}\left[{}_2F_1(1-h+\beta,1-\delta,1+h-\gamma-\delta;x)\right.$$
$$\left.\times\,{}_2F_1(1-h+\beta,1-\delta,1+h-\gamma-\delta;\bar{x})\right]. \tag{2.163}$$

# Appendix E: Mixing matrices and normalization factors

For the case of two scalars and a symmetric traceless tensor, inserting (2.17) into (2.8) leads to

$$\left\langle\phi_1(x_1)\phi_2(x_2)\widetilde{\mathcal{O}}_{a_1\cdots a_\ell}(x_3)\right\rangle = \Pi^{(\ell)\,b_1\cdots b_\ell}_{a_1\cdots a_\ell}\int\frac{d^Dx_0}{(x_{03}^2)^{d-\Delta_{\mathcal{O}}}}m^{(03)}{}_{b_1}{}^{c_1}\cdots m^{(03)}{}_{b_\ell}{}^{c_\ell}$$
$$\times\left(\lambda_{\mathcal{O}}\left(x_{01}^2\right)^{\frac{1}{2}(-\Delta_1+\Delta_2-\Delta_{\mathcal{O}})}\left(x_{02}^2\right)^{\frac{1}{2}(\Delta_1-\Delta_2-\Delta_{\mathcal{O}})}\left(x_{12}^2\right)^{\frac{1}{2}(-\Delta_1-\Delta_2+\Delta_{\mathcal{O}})}\right.$$
$$\left.\times\Pi^{(\ell)\,d_1\cdots d_\ell}_{c_1\cdots c_\ell}k^{(012)}_{d_1}\cdots k^{(012)}_{d_\ell}\right). \tag{2.164}$$

Since (as reviewed in Appendix A) $m^{(03)}{}_a{}^c m^{(03)}_{bc} = \delta_{ab}$, and since $\Pi^{(\ell)}$ removes traces, it follows that

$$\Pi^{(\ell)\,b_1\cdots b_\ell}_{a_1\cdots a_\ell} m^{(03)}{}_{b_1}{}^{c_1}\cdots m^{(03)}{}_{b_\ell}{}^{c_\ell}\Pi^{(\ell)\,d_1\cdots d_\ell}_{c_1\cdots c_\ell} k^{(012)}_{d_1}\cdots k^{(012)}_{d_\ell} = \Pi^{(\ell)\,b_1\cdots b_\ell}_{a_1\cdots a_\ell} y_{b_1}\cdots y_{b_\ell}, \quad (2.165)$$

where

$$y_a = m^{(03)}_{ab} k^{(012)\,b} = \left(\sqrt{\frac{x_{01}^2x_{23}^2}{x_{03}^2x_{12}^2}} - \frac{x_{02}^2x_{13}^2}{\sqrt{x_{01}^2x_{03}^2x_{12}^2x_{23}^2}}\right) k_a^{(302)} + \sqrt{\frac{x_{02}^2x_{13}^2}{x_{01}^2x_{23}^2}} k_a^{(312)}. \quad (2.166)$$

Expanding in a trinomial expansion, we then obtain

$$\begin{aligned}
&\left\langle \phi_1(x_1)\phi_2(x_2)\widetilde{\mathcal{O}}_{a_1\cdots a_\ell}(x_3)\right\rangle \\
&= \lambda_{\mathcal{O}}\Pi^{(\ell)\,b_1\cdots b_\ell}_{a_1\cdots a_\ell}\sum_{k=0}^{\ell}\sum_{m=0}^{\ell-k}\frac{\ell!}{k!m!\,(\ell-k-m)!}(-1)^m \\
&\quad\times\left(x_{12}^2\right)^{\frac{1}{2}(-\Delta_1-\Delta_2+\Delta_{\mathcal{O}}-k-m)}\left(x_{13}^2\right)^{\frac{1}{2}(\ell-k+m)}\left(x_{23}^2\right)^{k-\frac{\ell}{2}} k^{(312)}_{b_1}\cdots k^{(312)}_{b_{\ell-k-m}} \\
&\quad\times I_{\frac{1}{2}(\Delta_1-\Delta_2+\Delta_{\mathcal{O}}+\ell-2k),\frac{1}{2}(-\Delta_1+\Delta_2+\Delta_{\mathcal{O}}-\ell+k-m),d-\Delta_{\mathcal{O}}+\frac{k+m}{2};b_{\ell-k-m+1}\cdots b_\ell}(x_1,x_2,x_3) \\
&= \pi^{d/2}\lambda_{\mathcal{O}}\Pi^{(\ell)\,b_1\cdots b_\ell}_{a_1\cdots a_\ell}k^{(312)}_{b_1}\cdots k^{(312)}_{b_\ell}\sum_{k=0}^{\ell}\sum_{m=0}^{\ell-k}\frac{\ell!}{k!m!\,(\ell-k-m)!}(-1)^m \\
&\quad\times\frac{\Gamma(\frac{1}{2}(d-\Delta_1+\Delta_2-\Delta_{\mathcal{O}}-\ell)+k)\Gamma(\frac{1}{2}(d+\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell)+m)\Gamma(\Delta_{\mathcal{O}}-\frac{d}{2})}{\Gamma(\frac{1}{2}(\Delta_1-\Delta_2+\Delta_{\mathcal{O}}+\ell)-k)\Gamma(\frac{1}{2}(-\Delta_1+\Delta_2+\Delta_{\mathcal{O}}-\ell)+k)\Gamma(d-\Delta_{\mathcal{O}}+k+m)} \\
&\quad\times\left(x_{12}^2\right)^{\frac{1}{2}(d-\Delta_1-\Delta_2-\Delta_{\mathcal{O}})}\left(x_{13}^2\right)^{\frac{1}{2}(-\Delta_1+\Delta_2+\Delta_{\mathcal{O}}-d)}\left(x_{23}^2\right)^{\frac{1}{2}(\Delta_1-\Delta_2+\Delta_{\mathcal{O}}-d)} \\
&= \pi^{d/2}\Pi^{(\ell)\,b_1\cdots b_\ell}_{a_1\cdots a_\ell}k^{(312)}_{b_1}\cdots k^{(312)}_{b_\ell}\left(x_{12}^2\right)^{\frac{1}{2}\left(-\Delta_1-\Delta_2+\Delta_{\widetilde{\mathcal{O}}}\right)}\left(x_{13}^2\right)^{\frac{1}{2}\left(-\Delta_1+\Delta_2-\Delta_{\widetilde{\mathcal{O}}}\right)} \\
&\quad\times\left(x_{23}^2\right)^{\frac{1}{2}\left(\Delta_1-\Delta_2-\Delta_{\widetilde{\mathcal{O}}}\right)}\frac{\Gamma(\Delta_{\mathcal{O}}-\frac{d}{2})\Gamma(\Delta_{\mathcal{O}}+\ell-1)}{\Gamma(\Delta_{\mathcal{O}}-1)\Gamma(d-\Delta_{\mathcal{O}}+\ell)} \\
&\quad\times\frac{\Gamma(\frac{1}{2}(d+\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell))\Gamma(\frac{1}{2}(d-\Delta_1+\Delta_2-\Delta_{\mathcal{O}}+\ell))}{\Gamma(\frac{1}{2}(\Delta_1-\Delta_2+\Delta_{\mathcal{O}}+\ell))\Gamma(\frac{1}{2}(-\Delta_1+\Delta_2+\Delta_{\mathcal{O}}+\ell))}\lambda_{\mathcal{O}},
\end{aligned} \quad (2.167)$$

where we use the notation and results for integrals defined in Appendix D, and we evaluated the sums, first over $m$ and then over $k$, using the identities

$$\sum_{k=0}^{N}\frac{N!}{k!\,(N-k)!}(-1)^k\frac{\Gamma(x+k)}{\Gamma(y+k)} = \frac{\Gamma(x)\Gamma(y-x+N)}{\Gamma(y+N)\Gamma(y-x)}, \quad (2.168)$$

which is equivalent to (2.151), and

$$\sum_{k=0}^{N}\frac{N!}{k!\,(N-k)!}\frac{1}{\Gamma(x+k)\Gamma(y-k)} = \frac{\Gamma(x+y+N-1)}{\Gamma(x+N)\Gamma(y)\Gamma(x+y-1)}. \quad (2.169)$$

Thus comparing with (2.20) one can read off (2.21).

Now for a scalar, a vector, and a traceless symmetric tensor we have

$$
\begin{aligned}
&\left\langle \phi(x_1) v_a(x_2) \widetilde{\mathcal{O}}_{b_1\cdots b_\ell}(x_3) \right\rangle = \Pi^{(\ell)\,c_1\cdots c_\ell}_{b_1\cdots b_\ell} \\
&\quad\times \int \frac{d^D x_0}{\left(x_{03}^2\right)^{d-\Delta_\mathcal{O}}} m^{(03)}{}_{c_1}{}^{d_1} \cdots m^{(03)}{}_{c_\ell}{}^{d_\ell} \left(x_{01}^2\right)^{\frac{1}{2}\left(-\Delta_\phi+\Delta_v-\Delta_\mathcal{O}\right)} \left(x_{02}^2\right)^{\frac{1}{2}\left(\Delta_\phi-\Delta_v-\Delta_\mathcal{O}\right)} \\
&\quad\times \left(x_{12}^2\right)^{\frac{1}{2}\left(-\Delta_\phi-\Delta_v+\Delta_\mathcal{O}\right)} \Pi^{(\ell)\,e_1\cdots e_\ell}_{d_1\cdots d_\ell} \left[-\alpha_\mathcal{O} k_a^{(201)} k_{e_1}^{(012)} + \beta_\mathcal{O} m_{ae_1}^{(20)}\right] k_{e_2}^{(012)} \cdots k_{e_\ell}^{(012)} \\
&\qquad\qquad\qquad = \Pi^{(\ell)\,c_1\cdots c_\ell}_{b_1\cdots b_\ell} \left(x_{12}^2\right)^{\frac{1}{2}\left(-\Delta_\phi-\Delta_v+\Delta_\mathcal{O}\right)} \\
&\qquad\quad\times \int d^D x_0 \left(x_{01}^2\right)^{\frac{1}{2}\left(-\Delta_\phi+\Delta_v-\Delta_\mathcal{O}\right)} \left(x_{02}^2\right)^{\frac{1}{2}\left(\Delta_\phi-\Delta_v-\Delta_\mathcal{O}\right)} \left(x_{03}^2\right)^{\Delta_\mathcal{O}-d} \\
&\quad\times \left[-\alpha_\mathcal{O}\left(\sqrt{\frac{x_{03}^2 x_{12}^2}{x_{01}^2 x_{23}^2}} k_a^{(203)} - \sqrt{\frac{x_{02}^2 x_{13}^2}{x_{01}^2 x_{23}^2}} k_a^{(213)}\right) y_{c_1} + \beta_\mathcal{O}\left(m_{ac_1}^{(23)} - 2k_a^{(203)} k_{c_1}^{(302)}\right)\right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times y_{c_2} \cdots y_{c_\ell}, \qquad (2.170)
\end{aligned}
$$

using identities from Appendix A.

We then proceed as before, performing trinomial expansions on the $y_a$'s, perform the integrals using the results of Appendix D, and the identities (2.168) and (2.169). This results in (2.22) and (2.23).

Related to these integration techniques is the determination of the normalization factor $\mathcal{N}_\mathcal{O}$ that appears in the shadow projector $P_\mathcal{O}$. As discussed in the main text, this is fixed by requiring

$$\begin{aligned}
&\langle\varphi_1(x_1)\varphi_2(x_2)\mathcal{O}_{a_1\cdots a_\ell}(x_3)\rangle = \langle\mathcal{O}_{a_1\cdots a_\ell}(x_3)P_{\mathcal{O}}\varphi_1(x_1)\varphi_2(x_2)\rangle \\
&= \mathcal{N}_{\mathcal{O}}\int d^D x_0\,\langle\mathcal{O}_{a_1\cdots a_\ell}(x_3)\mathcal{O}_{b_1\cdots b_\ell}(x_0)\rangle\left\langle\widetilde{\mathcal{O}}^{b_1\cdots b_\ell}(x_0)\varphi_1(x_1)\varphi_2(x_2)\right\rangle \\
&= \mathcal{N}_{\mathcal{O}}\lambda_{12\widetilde{\mathcal{O}}}\Pi^{(\ell)\,c_1\cdots c_\ell}_{b_1\cdots b_\ell}\Pi^{(\ell)\,d_1\cdots d_\ell}_{a_1\cdots a_\ell}\int d^D x_0\left(x_{12}^2\right)^{\frac{1}{2}(d-\Delta_1-\Delta_2-\Delta_{\mathcal{O}})}\left(x_{01}^2\right)^{\frac{1}{2}(-\Delta_1+\Delta_2+\Delta_{\mathcal{O}}-d)} \\
&\quad\times\left(x_{02}^2\right)^{\frac{1}{2}(\Delta_1-\Delta_2+\Delta_{\mathcal{O}}-d)}k^{(012)}_{c_1}\cdots k^{(012)}_{c_\ell}\left(x_{03}^2\right)^{-\Delta_{\mathcal{O}}}m^{(03)}{}_{d_1}{}^{b_1}\cdots m^{(03)}{}_{d_\ell}{}^{b_\ell} \\
&= \mathcal{N}_{\mathcal{O}}\lambda_{12\widetilde{\mathcal{O}}}\Pi^{(\ell)\,b_1\cdots b_\ell}_{a_1\cdots a_\ell}\left(x_{12}^2\right)^{\frac{1}{2}(d-\Delta_1-\Delta_2-\Delta_{\mathcal{O}})}\int d^D x_0\left(x_{01}^2\right)^{\frac{1}{2}(-\Delta_1+\Delta_2+\Delta_{\mathcal{O}}-d)} \\
&\quad\times\left(x_{02}^2\right)^{\frac{1}{2}(\Delta_1-\Delta_2+\Delta_{\mathcal{O}}-d)}\left(x_{03}^2\right)^{-\Delta_{\mathcal{O}}}y_{b_1}\cdots y_{b_\ell} \\
&= \mathcal{N}_{\mathcal{O}}\lambda_{12\widetilde{\mathcal{O}}}\Pi^{(\ell)\,b_1\cdots b_\ell}_{a_1\cdots a_\ell}\left(x_{12}^2\right)^{\frac{1}{2}(-\Delta_1-\Delta_2+\Delta_{\mathcal{O}})}\left(x_{13}^2\right)^{\frac{1}{2}(-\Delta_1+\Delta_2-\Delta_{\mathcal{O}})}\left(x_{23}^2\right)^{\frac{1}{2}(\Delta_1-\Delta_2-\Delta_{\mathcal{O}})} \\
&\quad\times k^{(312)}_{b_1}\cdots k^{(312)}_{b_\ell}\frac{\Gamma(\frac{d}{2}-\Delta_{\mathcal{O}})}{\Gamma(d-\Delta_{\mathcal{O}}-1)}\frac{\Gamma(d-\Delta_{\mathcal{O}}+\ell-1)}{\Gamma(\Delta_{\mathcal{O}}+\ell)} \\
&\quad\times\frac{\Gamma(\frac{1}{2}\left(\Delta_1-\Delta_2+\Delta_{\mathcal{O}}+\ell\right))}{\Gamma(\frac{1}{2}\left(d+\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell\right))}\frac{\Gamma(\frac{1}{2}\left(-\Delta_1+\Delta_2+\Delta_{\mathcal{O}}+\ell\right))}{\Gamma(\frac{1}{2}\left(d-\Delta_1+\Delta_2-\Delta_{\mathcal{O}}+\ell\right))} \\
&= \langle\varphi_1(x_1)\varphi_2(x_2)\mathcal{O}_{a_1\cdots a_\ell}(x_3)\rangle\,\mathcal{N}_{\mathcal{O}}\pi^d \\
&\quad\times\frac{\Gamma(\Delta_{\mathcal{O}}-\frac{d}{2})\Gamma(\frac{d}{2}-\Delta_{\mathcal{O}})}{\left(\Delta_{\mathcal{O}}+\ell-1\right)\left(d-\Delta_{\mathcal{O}}+\ell-1\right)\Gamma(\Delta_{\mathcal{O}}-1)\Gamma(d-\Delta_{\mathcal{O}}-1)},
\end{aligned} \tag{2.171}$$

where we read off (2.15).

# Appendix F: $\alpha\beta$, $\beta\alpha$, and $\beta\beta$ components of the $\langle SVSV\rangle$ blocks

Here we write the additional conformal block components that appear for $\ell > 0$. These are expressed in condensed notation where the blocks on the LHS, as well as the $\alpha\alpha$ blocks on the right-hand-sides have unshifted arguments, $g_p^{rs}(u,v;\Delta_1,\Delta_2,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}})$. The others follow the conventions in the main text

$$\begin{aligned}
g_0^{\alpha\beta} &= \frac{\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1}{\ell}g_0^{\alpha\alpha}+\frac{1}{\ell}\left[\sqrt{u}g^{\alpha\lambda}_{1;\ell;0,1}+\sqrt{v}g^{\alpha\lambda}_{2;\ell;0,1}\right], \\
g_{11}^{\alpha\beta} &= \frac{\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1}{\ell}g_{11}^{\alpha\alpha}-\frac{1}{\ell}\sqrt{u}\left(\Delta_1-\Delta_2-\Delta_3+\Delta_4-2v\partial_v\right)g^{\alpha\lambda}_{1;\ell;0,1}, \\
g_{12}^{\alpha\beta} &= \frac{\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1}{\ell}g_{12}^{\alpha\alpha}-\frac{1}{\ell}\sqrt{v}\left(\Delta_3-\Delta_4+1-2u\partial_u\right)g^{\alpha\lambda}_{1;\ell;0,1}, \\
g_{21}^{\alpha\beta} &= \frac{\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1}{\ell}g_{21}^{\alpha\alpha}-\frac{1}{\ell}\sqrt{u}\left(\Delta_1-\Delta_2-\Delta_3+\Delta_4-1-2v\partial_v\right)g^{\alpha\lambda}_{2;\ell;0,1}, \\
g_{22}^{\alpha\beta} &= \frac{\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1}{\ell}g_{22}^{\alpha\alpha}-\frac{1}{\ell}\sqrt{v}\left(\Delta_3-\Delta_4+2-2u\partial_u\right)g^{\alpha\lambda}_{2;\ell;0,1},
\end{aligned} \tag{2.172}$$

$$\begin{aligned}
g_0^{\beta\alpha} &= \frac{\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1}{\ell}g_0^{\alpha\alpha}+\frac{1}{\ell}\left[g_{1;\ell;1,0}^{\lambda\alpha}+\sqrt{\frac{u}{v}}g_{2;\ell;1,0}^{\lambda\alpha}\right],\\
g_{11}^{\beta\alpha} &= \frac{\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1}{\ell}g_{11}^{\alpha\alpha}-\frac{1}{\ell}\left(\Delta_1-\Delta_2+2-2u\partial_u\right)g_{1;\ell;1,0}^{\lambda\alpha},\\
g_{12}^{\beta\alpha} &= \frac{\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1}{\ell}g_{12}^{\alpha\alpha}-\frac{1}{\ell}\left(\Delta_1-\Delta_2+1-2u\partial_u\right)g_{2;\ell;1,0}^{\lambda\alpha},\\
g_{21}^{\beta\alpha} &= \frac{\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1}{\ell}g_{21}^{\alpha\alpha}+\frac{1}{\ell}\sqrt{\frac{u}{v}}2v\partial_v g_{1;\ell;1,0}^{\lambda\alpha},\\
g_{22}^{\beta\alpha} &= \frac{\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1}{\ell}g_{22}^{\alpha\alpha}-\frac{1}{\ell}\sqrt{\frac{u}{v}}\left(1-2v\partial_v\right)g_{2;\ell;1,0}^{\lambda\alpha},
\end{aligned} \tag{2.173}$$

$$\begin{aligned}
g_0^{\beta\beta} &= \frac{\left(\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1\right)\left(\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1\right)}{\ell^2}g_0^{\alpha\alpha}\\
&+\frac{\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1}{\ell^2}\left[\sqrt{u}g_{1;\ell;0,1}^{\alpha\lambda}+\sqrt{v}g_{2;\ell;0,1}^{\alpha\lambda}\right]\\
&+\frac{\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1}{\ell^2}\left[g_{1;\ell;1,0}^{\lambda\alpha}+\sqrt{\frac{u}{v}}g_{2;\ell;1,0}^{\lambda\alpha}\right]\\
&-\frac{1}{\ell^2}\sqrt{u}\left(\Delta_1-\Delta_2+1-2u\partial_u-2v\partial_v\right)g_{\ell;1,1},\\
g_{11}^{\beta\beta} &= \frac{\left(\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1\right)\left(\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1\right)}{\ell^2}g_{11}^{\alpha\alpha}\\
&-\frac{\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1}{\ell^2}\sqrt{u}\left(\Delta_1-\Delta_2-\Delta_3+\Delta_4-2v\partial_v\right)g_{1;\ell;0,1}^{\alpha\lambda}\\
&-\frac{\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1}{\ell^2}\left(\Delta_1-\Delta_2+2-2u\partial_u\right)g_{1;\ell;1,0}^{\lambda\alpha}\\
&+\frac{1}{\ell^2}\sqrt{u}\left(\Delta_1-\Delta_2+1-2u\partial_u\right)\left(\Delta_1-\Delta_2-\Delta_3+\Delta_4-2v\partial_v\right)g_{\ell;1,1},\\
g_{12}^{\beta\beta} &= \frac{\left(\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1\right)\left(\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1\right)}{\ell^2}g_{12}^{\alpha\alpha}\\
&-\frac{\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1}{\ell^2}\sqrt{v}\left(\Delta_3-\Delta_4+1-2u\partial_u\right)g_{1;\ell;0,1}^{\alpha\lambda}\\
&-\frac{\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1}{\ell^2}\left(\Delta_1-\Delta_2+1-2u\partial_u\right)g_{2;\ell;1,0}^{\lambda\alpha}\\
&+\frac{1}{\ell^2}\sqrt{v}\left(\Delta_1-\Delta_2+1-2u\partial_u\right)\left(\Delta_3-\Delta_4+1-2u\partial_u\right)g_{\ell;1,1},
\end{aligned} \tag{2.174}$$

$$
\begin{aligned}
g_{21}^{\beta\beta} \quad = \quad & \frac{(\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1)(\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1)}{\ell^2} g_{21}^{\alpha\alpha} \\
& -\frac{\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1}{\ell^2}\sqrt{u}\left(\Delta_1-\Delta_2-\Delta_3+\Delta_4-1-2v\partial_v\right) g_{2;\ell;0,1}^{\alpha\lambda} \\
& +\frac{\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1}{\ell^2}\sqrt{\frac{u}{v}}2v\partial_v g_{1;\ell;1,0}^{\lambda\alpha} \\
& -\frac{1}{\ell^2}\frac{u}{\sqrt{v}}2v\partial_v\left(\Delta_1-\Delta_2-\Delta_3+\Delta_4-2v\partial_v\right) g_{\ell;1,1}, \\
g_{22}^{\beta\beta} \quad = \quad & \frac{(\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1)(\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1)}{\ell^2} g_{22}^{\alpha\alpha} \\
& -\frac{\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1}{\ell^2}\sqrt{v}\left(\Delta_3-\Delta_4+2-2u\partial_u\right) g_{2;\ell;0,1}^{\alpha\lambda} \\
& -\frac{\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1}{\ell^2}\sqrt{\frac{u}{v}}\left(1-2v\partial_v\right) g_{2;\ell;1,0}^{\lambda\alpha} \\
& -\frac{1}{\ell^2}\sqrt{u}2v\partial_v\left(\Delta_3-\Delta_4+1-2u\partial_u\right) g_{\ell;1,1},
\end{aligned}
$$

# Appendix G: Mixed symmetric constants

The constants appearing in the mixed-symmetric conformal blocks are defined by

$$
C_1 = \frac{\mathcal{N}_{\mathcal{A}}\gamma_{34\widetilde{\mathcal{A}}}/\gamma_{34\mathcal{A}}}{\left(\mathcal{N}_{\mathcal{O}}\lambda_{34\widetilde{\mathcal{O}}}/\lambda_{34\mathcal{O}}\right)_{k+1\,00}} = \frac{d-\Delta_{\mathcal{A}}-1}{d-\Delta_{\mathcal{A}}-2}, \tag{2.175}
$$

$$
\begin{aligned}
C_2 &= \mathcal{N}_{\mathcal{A}}\gamma_{34\widetilde{\mathcal{A}}}/\gamma_{34\mathcal{A}}\left(\mathcal{N}_{\mathcal{O}}^{-1}\left(M^{-1}\right)_{\alpha}{}^{\alpha}\right)_{k+1\,-\frac{1}{2}\,\frac{1}{2}} \\
&= \frac{(\Delta_{\mathcal{A}}-1)(d-\Delta_{\mathcal{A}}+k)-(d-\Delta_{\mathcal{A}}-1)(\Delta_3-\Delta_4-1)}{(d-\Delta_{\mathcal{A}}-2)(d-\Delta_3+\Delta_4-\Delta_{\mathcal{A}}+k+1)},
\end{aligned} \tag{2.176}
$$

$$
\begin{aligned}
C_3 &= \mathcal{N}_{\mathcal{A}}\gamma_{34\widetilde{\mathcal{A}}}/\gamma_{34\mathcal{A}}\left(\mathcal{N}_{\mathcal{O}}^{-1}\left(M^{-1}\right)_{\beta}{}^{\alpha}\right)_{k+1\,-\frac{1}{2}\,\frac{1}{2}} \\
&= \frac{(k+1)(d-2\Delta_{\mathcal{A}})(\Delta_3-\Delta_4-1)}{(d-\Delta_{\mathcal{A}}-2)(d-\Delta_3+\Delta_4-\Delta_{\mathcal{A}}+k+1)(\Delta_3-\Delta_4+\Delta_{\mathcal{A}}+k-1)}
\end{aligned} \tag{2.177}
$$

$$
\begin{aligned}
C_4 &= \mathcal{N}_{\mathcal{A}}\gamma_{34\widetilde{\mathcal{A}}}/\gamma_{34\mathcal{A}}\left(\mathcal{N}_{\mathcal{O}}^{-1}\left(M^{-1}\right)_{\alpha}{}^{\alpha}\right)_{k+1\,\frac{1}{2}\,-\frac{1}{2}} \\
&= \frac{(\Delta_3-\Delta_4+\Delta_{\mathcal{A}}+k+1)\left((\Delta_{\mathcal{A}}-1)(d-\Delta_{\mathcal{A}}+k)-(d-\Delta_{\mathcal{A}}-1)(\Delta_3-\Delta_4+1)\right)}{(d-\Delta_{\mathcal{A}}-2)(d+\Delta_3-\Delta_4-\Delta_{\mathcal{A}}+k+1)(-\Delta_3+\Delta_4+\Delta_{\mathcal{A}}+k-1)},
\end{aligned} \tag{2.178}
$$

$$
\begin{aligned}
C_5 &= \mathcal{N}_{\mathcal{A}}\gamma_{34\widetilde{\mathcal{A}}}/\gamma_{34\mathcal{A}}\left(\mathcal{N}_{\mathcal{O}}^{-1}\left(M^{-1}\right)_{\beta}{}^{\alpha}\right)_{k+1\,\frac{1}{2}\,-\frac{1}{2}} \\
&= \frac{(k+1)(d-2\Delta_{\mathcal{A}})(\Delta_3-\Delta_4+1)}{(d-\Delta_{\mathcal{A}}-2)(d+\Delta_3-\Delta_4-\Delta_{\mathcal{A}}+k+1)(-\Delta_3+\Delta_4+\Delta_{\mathcal{A}}+k-1)},
\end{aligned} \tag{2.179}
$$

$$
\begin{aligned}
C_6 = \mathcal{N}_{\mathcal{A}} \gamma_{34\widetilde{\mathcal{A}}}/\gamma_{34\mathcal{A}} & \left(\mathcal{N}_{\mathcal{O}}^{-1}\left(\left(M^{-1}\right)_{\alpha}{}^{\alpha}-\left(M^{-1}\right)_{\alpha}{}^{\beta}\right)\right)_{k+2\,00} \\
=& \frac{(d-\Delta_{\mathcal{A}}+k)\,(\Delta_3-\Delta_4+\Delta_{\mathcal{A}}+k+1)}{(d-\Delta_{\mathcal{A}}-2)\,(\Delta_{\mathcal{A}}+k+1)} \\
& \times \frac{(d-\Delta_{\mathcal{A}}-1)\,(d-\Delta_{\mathcal{A}}+k+1)-(\Delta_{\mathcal{A}}-1)\,(\Delta_3-\Delta_4)}{(d+\Delta_3-\Delta_4-\Delta_{\mathcal{A}}+k+1)\,(d-\Delta_3+\Delta_4-\Delta_{\mathcal{A}}+k+1)},
\end{aligned} \tag{2.180}
$$

$$
\begin{aligned}
C_7 = \mathcal{N}_{\mathcal{A}} \gamma_{34\widetilde{\mathcal{A}}}/\gamma_{34\mathcal{A}} & \left(\mathcal{N}_{\mathcal{O}}^{-1}\left(\left(M^{-1}\right)_{\beta}{}^{\beta}-\left(M^{-1}\right)_{\beta}{}^{\alpha}\right)\right)_{k+2\,00} = \frac{d-\Delta_{\mathcal{A}}+k}{d-\Delta_{\mathcal{A}}-2} \\
\times \frac{(d-\Delta_{\mathcal{A}}-1)\,(\Delta_{\mathcal{A}}+k+1)\,(d-\Delta_{\mathcal{A}}+k+1)-(\Delta_{\mathcal{A}}-1)\,(\Delta_3-\Delta_4)^2}{(\Delta_{\mathcal{A}}+k+1)\,(d+\Delta_3-\Delta_4-\Delta_{\mathcal{A}}+k+1)\,(d-\Delta_3+\Delta_4-\Delta_{\mathcal{A}}+k+1)},
\end{aligned} \tag{2.181}
$$

$$
\begin{aligned}
C_8 = \mathcal{N}_{\mathcal{A}} \gamma_{34\widetilde{\mathcal{A}}}/\gamma_{34\mathcal{A}} & \left(\mathcal{N}_{\mathcal{O}}^{-1}\left(\left(M^{-1}\right)_{\alpha}{}^{\alpha}-\left(M^{-1}\right)_{\alpha}{}^{\beta}\right)\right)_{k\,00} \\
= & \frac{(\Delta_{\mathcal{A}}+k)\,((d-\Delta_{\mathcal{A}}-1)\,(d-\Delta_{\mathcal{A}}+k-1)-(\Delta_{\mathcal{A}}-1)\,(\Delta_3-\Delta_4))}{(d-\Delta_{\mathcal{A}}-2)\,(d-\Delta_{\mathcal{A}}+k-1)\,(-\Delta_3+\Delta_4+\Delta_{\mathcal{A}}+k-1)},
\end{aligned} \tag{2.182}
$$

$$
\begin{aligned}
C_9 = \mathcal{N}_{\mathcal{A}} \gamma_{34\widetilde{\mathcal{A}}}/\gamma_{34\mathcal{A}} & \left(\mathcal{N}_{\mathcal{O}}^{-1}\left(\left(M^{-1}\right)_{\beta}{}^{\beta}-\left(M^{-1}\right)_{\beta}{}^{\alpha}\right)\right)_{k\,00} \\
= \frac{(\Delta_{\mathcal{A}}+k)\left((d-\Delta_{\mathcal{A}}-1)\,(\Delta_{\mathcal{A}}+k-1)\,(d-\Delta_{\mathcal{A}}+k-1)-(\Delta_{\mathcal{A}}-1)\,(\Delta_3-\Delta_4)^2\right)}{(d-\Delta_{\mathcal{A}}-2)\,(d-\Delta_{\mathcal{A}}+k-1)\,(\Delta_3-\Delta_4+\Delta_{\mathcal{A}}+k-1)\,(-\Delta_3+\Delta_4+\Delta_{\mathcal{A}}+k-1)}&.
\end{aligned} \tag{2.183}
$$

In computing these constants we have used notation where a subscript on a quantity in parentheses, $(f)_{k'\,P\,Q}$ means that we should evaluate $f$ (which is given in terms of three-point function data) for external fields of dimensions $\Delta_3+P$ and $\Delta_4+Q$, and an exchange operator of spin $\ell=k'$ and dimension $\Delta_{\mathcal{O}}=\Delta_{\mathcal{A}}$.

# Appendix H: Operators appearing in symmetric exchange blocks

Define

$$
\begin{gathered}
\delta_1=\Delta_3-\Delta_4+2u\partial_u+2v\partial_v, \qquad \delta_2=\Delta_1-\Delta_2-\Delta_3+\Delta_4-2v\partial_v, \\
\delta_3=2v\partial_v, \qquad \delta_4=\Delta_1-\Delta_2-2u\partial_u-2v\partial_v.
\end{gathered} \tag{2.184}
$$

Then the operators which appear in the expression (2.65) are

$$
\mathcal{D}_0^{--}=\sqrt{u}\,(\delta_1-1)\,, \quad \mathcal{D}_0^{-+}=\sqrt{u}\,(\delta_2-2)\,, \quad \mathcal{D}_0^{+-}=-\frac{\sqrt{u}}{v}\delta_3, \quad \mathcal{D}_0^{++}=-\sqrt{u}\,(\delta_4+1)\,, \tag{2.185}
$$

$$
\begin{gathered}
\mathcal{D}_{11}^{--}=-\sqrt{u}\,(\delta_2+v\,(\delta_1+1))\,(\delta_1-1)\,, \quad \mathcal{D}_{11}^{-+}=-\sqrt{u}\,(\delta_2+v\,(\delta_1+1))\,(\delta_2-2)\,, \\
\mathcal{D}_{11}^{+-}=\sqrt{u}\,(\delta_1-1)\,(\delta_3+\delta_4+1)\,, \quad \mathcal{D}_{11}^{++}=\sqrt{u}\delta_2\,(\delta_3+\delta_4+1)\,,
\end{gathered} \tag{2.186}
$$

$$\mathcal{D}_{12}^{--} = -\frac{1}{\sqrt{v}}\left(\delta_3 - v\left(\delta_1 - 1\right)\right)\left(\delta_2 + v\left(\delta_1 - 1\right)\right),$$
$$\mathcal{D}_{12}^{-+} = \sqrt{v}\left(\delta_2 - 2 + v\left(\delta_1 + 1\right)\right)\left(\delta_2 - \delta_4 - 1\right),$$
$$\mathcal{D}_{12}^{+-} = \frac{1}{\sqrt{v}}\left(\delta_3 - v\left(\delta_1 - 1\right)\right)\left(\delta_3 + \delta_4 + 1\right), \quad \mathcal{D}_{12}^{++} = -\sqrt{v}\left(\delta_2 - \delta_4 - 1\right)\left(\delta_3 + \delta_4 + 1\right), \tag{2.187}$$

$$\mathcal{D}_{21}^{--} = u\sqrt{v}\left(\delta_1 - 1\right)\left(\delta_1 + 1\right), \quad \mathcal{D}_{21}^{-+} = u\sqrt{v}\left(\delta_1 + 1\right)\left(\delta_2 - 2\right),$$
$$\mathcal{D}_{21}^{+-} = -\frac{u}{\sqrt{v}}\left(\delta_1 - 1\right)\delta_3, \quad \mathcal{D}_{21}^{++} = -\frac{u}{\sqrt{v}}\delta_2\delta_3, \tag{2.188}$$

$$\mathcal{D}_{22}^{--} = \sqrt{u}\left(\delta_3 - v\left(\delta_1 + 1\right)\right)\left(\delta_1 - 1\right), \quad \mathcal{D}_{22}^{-+} = -\sqrt{u}v\left(\delta_1 + 1\right)\left(\delta_2 - \delta_4 - 1\right),$$
$$\mathcal{D}_{22}^{+-} = -\frac{\sqrt{u}}{v}\left(\delta_3 - 2 - v\left(\delta_1 - 1\right)\right)\delta_3, \quad \mathcal{D}_{22}^{++} = \sqrt{u}\left(\delta_2 - \delta_4 - 1\right)\delta_3. \tag{2.189}$$

# 3 Seeds for spinning partial waves

On seed spinning conformal partial waves for four-point functions of generic traceless-symmetric tensors in arbitrary dimensions

The main obstacle for studying correlators of spinning operators, in the context of the bootstrap program, is that the spinning partial waves for the exchange of operators in mixed-symmetric $SO(d)$ representations are not readily available in arbitrary dimensions $d$. In this chapter, based on [4], we give a closed form expression for any spinning seed conformal partial wave that can appear in the four-point function of traceless-symmetric operators in arbitrary dimensions $d$.

## 3.1 Introduction

As discussed previously in section 1.7, conformal partial waves play a crucial role in the bootstrap program. In chapter 2 we presented the computation of spinning (seed) conformal blocks for the exchange of an operator in the $(\ell, 1)$ representation of $SO(d)$. However, for generic spinning four-point functions, other representations can also be exchanged (see footnote 16). In this chapter we extend the results of chapter 2 by giving an expression for the seed partial wave associated to an exchanged operator in the representation $(\ell_1, \ell_2, \ell_3)$ for $0 < \ell_3 \leq \ell_2 \leq \ell_1$. This is the most general operator that can be exchanged in the four-point function of traceless-symmetric operators. To understand why, we can use expression (1.134) for a three-point function between the representations $(l_1)$, $(l_2)$, and $(\ell_1, \ell_2, \ell_3)$. In order to construct a singlet from the tensor product, we need to restrict the three-row diagram down to two rows by filling the $\ell_3$ boxes in the bottom with $V$. Finally we contract the boxes of $(l_1)$ and $(l_2)$ with the remaining two rows $\ell_1$ and $\ell_2$ respectively (after restriction, if required). However, if the mixed-symmetric operator has four rows or more, then this procedure is null since one of the traceless-symmetric representations will end up antisymmetrized.

In contrast with chapter 2, where the shadow projector technique was employed, here we use the weight-shifting operator formalism discussed in subsection 1.6.4. Our main result is a couple of recursion relations for the seed partial wave of $(\ell_1, \ell_2, \ell_3)$, that lowers the number of boxes in the second and third rows, one unit at a time. Hence the number of recursive steps to go from spinning seed partial wave in the $(\ell_1, \ell_2, \ell_3)$ representation to the usual scalar (seed) partial wave of [68,69] is $\ell_2 + \ell_3$. By combining the result presented in this chapter with (1.217) one can write all spinning partial waves involved in the four-point function of traceless-symmetric operators.

This chapter is organized as follows. In section 3.2 we extend the discussion of subsection 1.6.4 in order to give a simplified version of (1.224) for the case of seed partial waves, which results in an expression that relates seed partial waves in different representations. Then we specialize to the case where the exchanged representation is $(\ell_1, \ell_2, \ell_3)$, and pose the required intermediate calculations needed to write its recursion. In the following two sections we carry out the intermediate steps; in section 3.3 we compute the vector weight-shifting differential operators for three-row representations, and in section 3.4 we calculate the required coefficients relating 2- and 3-point functions between different representations. In section 3.5 we write the final recursion relations and we conclude in section 3.6.

## 3.2 Seed conformal partial waves

In subsection 1.6.4 we reviewed an algorithm based on [46] for computing the spinning partial wave of an exchanged operator in some representation $[\Delta, \rho]$, in terms of derivatives acting on the spinning partial wave for an operator in $[\Delta', \rho']$. The generic equation is (1.224). In particular, if the representations of the external operators $\Phi_i$ are chosen such that all the three-point functions appearing in (1.224) have only one independent tensor structure (which we call seed-like), then it provides a relation between seed partial waves associated to different exchanged representations. However, as shown in [46] there is a more efficient algorithm for computing seed partial waves, which we review first.

Let us consider an exchanged operator $\mathcal{O} = [\Delta, \rho]$ and four external operators $\Phi_i$ where $\varphi_2 \equiv \Phi_2 = [\Delta_2, (0)]$ and $\varphi_4 \equiv \Phi_4 = [\Delta_4, (0)]$ are scalars, while $\Phi_1 =$

$[\Delta_1, \rho_1]$ and $\Phi_3 = [\Delta_3, \rho_3]$ are in particular representations such that the three-point functions $\langle \Phi_1 \varphi_2 \mathcal{O} \rangle$ and $\langle \Phi_3 \varphi_4 \mathcal{O} \rangle$ have a unique tensor structure. This can be achieved, for example, by setting $\rho_1$ and $\rho_3$ to be $\rho$ with the first row removed.

Now focusing on one three-point function, say $\langle \Phi_1 \varphi_2 \mathcal{O} \rangle$, we can apply (1.179) for the case where both correlators have a unique tensor structure. Explicitly,

$$\mathcal{D}_1^{(n)} \cdot \mathcal{D}_0^{(m)} \langle \Phi_1' \varphi_2 O \rangle = C_{mn} \langle \Phi_1 \varphi_2 \mathcal{O} \rangle, \tag{3.1}$$

where $\mathcal{D}_1^{(n)} : \Phi_1' \to \Phi_1$ and $\mathcal{D}_0^{(m)} : O \to \mathcal{O}$, and the choice of $m, n$ is not unique; the only condition is that $\langle \Phi_1' \varphi_2 O \rangle$ remains seed-like and that the Young diagram of $O$ is contained in that of $\mathcal{O}$. Next, using (1.221) and (1.176) leads to

$$\begin{aligned} \langle \Phi_1 \varphi_2 \mathcal{O} \rangle \bowtie \langle \mathcal{O} \Phi_3 \varphi_4 \rangle = C_{mn}^{-1} \sum_{\Phi_3' \in \Omega \otimes \Phi_3, a, b, c} & \begin{Bmatrix} \Phi_3 & \varphi_4 & \Phi_3' \\ O & \Omega & \mathcal{O} \end{Bmatrix}_{ab}^{\bullet c} \begin{Bmatrix} O & \mathbf{1} & \mathcal{O} \\ \mathcal{O} & \Omega & O \end{Bmatrix}_{\bullet c}^{\bullet m} \\ & \times \mathcal{D}_1^{(n)} \cdot \mathcal{D}_3^{(b)} \langle \Phi_1' \varphi_2 O \rangle \bowtie \langle O \Phi_3' \varphi_4 \rangle^{(a)}, \end{aligned} \tag{3.2}$$

where now $\mathcal{D}_3^{(b)} : \Phi_3' \to \Phi_3$. However, notice that $\langle O \Phi_3' \varphi_4 \rangle^{(a)}$ is not seed-like. Thus we need to invert (1.179) to write this three-point function in terms of seed-like correlators. In other words, to reproduce the tensor structures $(a)$ we must find a linear combination of contracted weight–shifting operators: $\sum_{e, \Phi_3''} \mathcal{W}_{(e)}^{(a)} \mathscr{D}_{\Phi_3'' \varphi_4'}^{(e) \Phi_3' \varphi_4} \langle O \Phi_3'' \varphi_4' \rangle$, such that the $SO(d)$ representations of $\Phi_3''$ correspond to the Young diagram of $O$ with the first row removed, and that $\varphi_4'$ remains a scalar. Using this we obtain an expression that relates two seed partial waves from different representations:

$$\begin{aligned} \langle \Phi_1 \varphi_2 \mathcal{O} \rangle \bowtie \langle \mathcal{O} \Phi_3 \varphi_4 \rangle = C_{mn}^{-1} \sum_{\Phi_3' \in \Omega \otimes \Phi_3, \Phi_3'', a, b, c, e} & \begin{Bmatrix} \Phi_3 & \varphi_4 & \Phi_3' \\ O & \Omega & \mathcal{O} \end{Bmatrix}_{ab}^{\bullet c} \begin{Bmatrix} O & \mathbf{1} & \mathcal{O} \\ \mathcal{O} & \Omega & O \end{Bmatrix}_{\bullet c}^{\bullet m} \\ & \times \mathcal{W}_{(e)}^{(a)} \mathcal{D}_1^{(n)} \cdot \mathcal{D}_3^{(b)} \mathscr{D}_{\Phi_3'' \varphi_4'}^{(e) \Phi_3' \varphi_4} \langle \Phi_1' \varphi_2 O \rangle \bowtie \langle O \Phi_3'' \varphi_4' \rangle. \end{aligned} \tag{3.3}$$

The advantage of this expression over (1.224) is that we have chosen a fixed $\Phi_1'$ such that one of the three-point functions is already seed-like, thereby avoiding the application of an extra operator $\mathscr{D}$.

### 3.2.1 Connecting all seeds recursively

The objective of this chapter is then to use (3.3) in order to give a recursive expression for the spinning seed partial wave associated to the exchange of an operator in the representation $(\ell_1, \ell_2, \ell_3)$, $0 < \ell_3 \leq \ell_2 \leq \ell_1$. Given that (3.3) allows us to write the seed partial wave for $\mathcal{O}$ in terms of partials waves for $O$, abbreviated $\mathcal{O} \to O$, then wee need to construct two classes of recursions:

$$\begin{aligned} \text{I}: &\quad (\ell_1,\ell_2,\ell_3) \to (\ell_1,\ell_2,\ell_3-1)\,, \\ \text{II}: &\quad (\ell_1,\ell_2) \to (\ell_1,\ell_2-1)\,. \end{aligned}$$

One can then compose several of these to go down to the symmetric traceless representation: $(\ell_1,\ell_2,\ell_3) \to \cdots \to (\ell_1,\ell_2) \to \cdots \to (\ell_1)$, which is the well-known scalar partial wave of [68, 69].

In what follows, we describe the necessary calculations in order to achieve this.

### Recursion I

Let us first consider the case $\mathcal{O} = [\Delta,(\ell_1,\ell_2,\ell_3)] \to O = [\Delta,(\ell_1,\ell_2,\ell_3-1)]$, which implies $\Omega = (1)$. To have a seed partial wave, we choose $\Phi_1 = [\Delta_1,(\ell_2,\ell_3)]$, $\Phi_3 = [\Delta_3,(\ell_2,\ell_3)]$. From the discussion above, in order to preserve the seed-like structure of the first three-point function on the right hand side of (3.3), we need an operator such that $\mathcal{D}_1^{(n)} : \Phi_1' = [\Delta_1,(\ell_2,\ell_3-1)] \to \Phi_1$. The other derivative in (3.3) sends $\mathcal{D}_3^{(b)} : \Phi_3' \to \Phi_3$, where the explicit values for $\Phi_3'$ are, by (1.167),

$$\begin{aligned} \Phi_3' \in (1) \otimes [\Delta_3,(\ell_2,\ell_3)] &= [\Delta_3-1,(\ell_2,\ell_3)] \oplus [\Delta_3,(\ell_2+1,\ell_3)] \oplus [\Delta_3,(\ell_2,\ell_3+1)] \\ &\oplus [\Delta_3,(\ell_2,\ell_3,1)] \oplus [\Delta_3,(\ell_2-1,\ell_3)] \oplus [\Delta_3,(\ell_2,\ell_3-1)] \oplus [\Delta_3+1,(\ell_2,\ell_3)]. \quad (3.4) \end{aligned}$$

Here we are taking the most generic case where the $\ell_i$ are such that all the Young diagrams are valid. However, notice that when inserting these representations into the three-point function $\langle \Phi_3' \phi_3 O \rangle$, not all three-point functions are non-zero. In particular $\langle [\Delta_3,(\ell_2-1,\ell_3)][\Delta_4,(0)][\Delta,(\ell_1,\ell_2,\ell_3-1)] \rangle = 0$. Finally, the differential operators $\mathscr{D}^{(e)\Phi_3'\varphi_4}_{\Phi_3''\varphi_4'}$ must be such that

$$\mathscr{D}^{\Phi_3'\varphi_4}_{\Phi_3''\varphi_4'} : [\Delta_3'',(\ell_2,\ell_3-1)] \times [\Delta_4',(0)] \to (1) \otimes [\Delta_3,(\ell_2,\ell_3)] \ni \Phi_3' \times [\Delta_4,(0)].$$

### Recursion II

Now for $\mathcal{O} = [\Delta,(\ell_1,\ell_2)] \to O = [\Delta,(\ell_1,\ell_2-1)]$, we have a very similar situation except the terms are simpler. The initial representations are $\Phi_1 = [\Delta_1,(\ell_2)]$, $\Phi_3 = [\Delta_3,(\ell_2)]$, and the choice of $\Phi_1'$ is given by $\mathcal{D}_1^{(n)} : \Phi_1' = [\Delta_1,(\ell_2-1)] \to \Phi_1$. The representations $\Phi_3'$ that appear in the sum and in $\mathcal{D}_3^{(b)} : \Phi_3' \to \Phi_3$ are

$$\begin{aligned} \Phi_3' \in (1) \otimes [\Delta_3,(\ell_2)] = [\Delta_3-1,(\ell_2)] &\oplus [\Delta_3,(\ell_2+1)] \\ &\oplus [\Delta_3,(\ell_2,1)] \oplus [\Delta_3,(\ell_2-1)] \oplus [\Delta_3+1,(\ell_2)], \quad (3.5) \end{aligned}$$

where now all the three-point functions with $\Phi_3'$ are non-zero. The differential basis for $\langle O\Phi_3'\varphi_4\rangle$ is, in this case,

$$\mathscr{D}^{\Phi_3'\varphi_4}_{\Phi_3''\varphi_4'} : [\Delta_3'', (\ell_2-1)] \times [\Delta_4', (0)] \to (1) \otimes [\Delta_3, (\ell_2)] \ni \Phi_3' \times [\Delta_4, (0)].$$

Currently, there are no known generic results for the $6j$ symbols nor for the weight-shifting operators that are required to achieve our objective in general dimensions. Therefore before setting up the recursion relations, we need to:

- compute the 8 vector weight-shifting differential operators for mixed-symmetric representations of three rows [1]
- compute the two- and three-point $6j$ symbols appearing on the right hand side of (3.3), as well as the overall coefficient $C_{mn}$
- find a differential basis to write the second three-point function of (3.3) so that the right hand side is written in terms of seed partial waves only
- put everything together and write both recursions I and II.

We present these results in the remaining of the chapter.

## 3.3 Mixed-symmetric weight shifting operators

We proceed now to computing all the vector weight-shifting differential operators acting on general $SO(d)$ representations of at most three rows: $[\Delta, \rho = (\ell_1, \ell_2, \ell_3)]$. The defining properties of a weight-shifting operator that maps $\mathcal{D}_A : [\Delta, \rho] \to [\Delta + i, \lambda]$, are given by (1.170). In summary, the weights of the operator must be such that it changes the homogeneity of $P$ and $Z^j$ by $i$ and $(\rho)_i - (\lambda)_i$, respectively. Furthermore, it must respect the mixed-symmetric properties of the $SO(d)$ representation, it must be transverse, and interior to the null-cone. The latter condition is summarized in (1.172).

The first step to construct the operators is then to define a basis of monomials with which one can modify any weight of the conformal representation by one unit.

[1] From (3.4) it might seem like we only need 7 operators. However, $\mathscr{D}$ needs to map two-row diagrams to one of the representations in $\Phi_3'$ with three rows. Therefore we need an extra one that increases the number of boxes in the third row.

Inspired by equation (2.44) in [46], a convenient basis is given by [2]

$$M(T)^{AB} = \left(c\delta^{AB} + T^A \partial_T^B\right), \tag{3.6}$$

where $c$ is a constant, and $T$ symbolizes any of the embedding space vectors $P$, $Z^j$. By construction, the action of $M(T)$ does not alter any of the conformal weights. Therefore contracting $M$ with $P$ or $\partial_P$ decreases or increases the conformal dimension $\Delta$ respectively. On the other hand, the contraction of $M$ with $Z^j$ or $\partial_{Z^j}$ increases or decreases the number of boxes in the $j$-th row $\ell_j$ respectively. Constructing the weight-shifting operators for each case is clear now; depending on how the operator is supposed to alter the conformal weights, we start with any of $P, Z^j, \partial_P, \partial_{Z^j}$ and contract it with many different $M$ until the other conditions are satisfied. Note that although there are several ways of contracting the indices between different $M$, not all of them are useful. For example if we contract two $M$ depending on the same vector $T$ like $M(T)^{AB}M(T)_B{}^C$, produces a term proportional to $T \cdot \partial_T$. However since all the fields are homogeneous, this will become a numerical weight when it hits the field. Similarly, one can find contractions that produce terms that vanish due to transversality and mixed symmetry. By making a sensible ansatz and demanding that the properties mentioned before are satisfied, then fixes all the coefficients. The results are presented below.

**Case** $[\Delta, (\ell_1, \ell_2, \ell_3)] \to [\Delta + 1, (\ell_1, \ell_2, \ell_3)]$**:**

$$\begin{aligned}\mathcal{D}_M^{+(000)} &= \frac{1}{a_{22}a_{30}(-1 - a_{30} + a_{33})}\left(a_{00}\delta_M^A + P_M\partial_P^A\right)\left(a_{01}\delta_A^B + Z_A^1\partial_{Z^1}^B\right)\\ &\quad\times\left(a_{02}\delta_B^C + Z_B^2\partial_{Z^2}^C\right)\left(a_{03}\delta_C^D + Z_C^3\partial_{Z^3}^D\right)\left(\left(a_{31} - a_{01}\right)\delta_D^E + Z^{3E}\partial_{Z^3 D}\right)\\ &\quad\times\left(\left(a_{21} - a_{01}\right)\delta_E^F + Z^{2F}\partial_{Z^2 E}\right)\left(\left(a_{13} - a_{03}\right)\delta_F^G + Z^{1G}\partial_{Z^1 F}\right)\partial_{PG}.\end{aligned} \tag{3.7}$$

**Case** $[\Delta, (\ell_1, \ell_2\ell_3)] \to [\Delta - 1, (\ell_1 + 1, \ell_2, \ell_3)]$**:**

$$\mathcal{D}_M^{-(000)} = X_M. \tag{3.8}$$

**Case** $[\Delta, (\ell_1, \ell_2, \ell_3)] \to [\Delta, (\ell_1 + 1, \ell_2, \ell_3)]$**:**

$$\mathcal{D}_M^{0(+00)} = \left(\left(a_{00} - a_{10}\right)\delta_M^A + P_M\partial_P^A\right)Z_A^1. \tag{3.9}$$

[2] During the final preparation of this chapter, a paper appeared [124] which computed the weight-shifting operators for diagrams of two rows in a similar way. Setting $\ell_3 = 0$ in our results reproduces theirs, up to normalizations.

**Case** $[\Delta,(\ell_1,\ell_2,\ell_3)]\to[\Delta,(\ell_1-1,\ell_2,\ell_3)]$**:**

$$\mathcal{D}_M^{0(-00)} = \frac{1}{a_{13}a_{22}(a_{30}-a_{10})}\left(a_{10}\delta_M^A + P_M\partial_P^A\right)\left(a_{11}\delta_A^B + Z_A^1\partial_{Z^1}^B\right)\left(a_{12}\delta_B^C + Z_B^2\partial_{Z^2}^C\right) \times\left(a_{13}\delta_C^D + Z_C^3\partial_{Z^3}^D\right)\left(\left(a_{30}-a_{10}\right)\delta_D^E + Z^{3E}\partial_{Z^3D}\right) \times\left(\left(a_{20}-a_{10}\right)\delta_E^F + Z^{2F}\partial_{Z^2E}\right)\partial_{Z^1F}. \quad (3.10)$$

**Case** $[\Delta,(\ell_1,\ell_2,\ell_3)]\to[\Delta,(\ell_1,\ell_2+1,\ell_3)]$**:**

$$\mathcal{D}_M^{0(0+0)} = \frac{1}{a_{11}-a_{12}}\left(\left(a_{00}-a_{20}\right)\delta_M^A + P_M\partial_P^A\right)\left(\left(a_{11}-a_{12}\right)\delta_A^B + Z_A^1\partial_{Z^1}^B\right)Z_B^2. \quad (3.11)$$

**Case** $[\Delta,(\ell_1,\ell_2,\ell_3)]\to[\Delta,(\ell_1,\ell_2-1,\ell_3)]$**:**

$$\mathcal{D}_M^{0(0-0)} = \frac{1}{a_{22}a_{23}}\left(a_{02}\delta_M^A + P_M\partial_P^A\right)\left(a_{12}\delta_A^B + Z_A^1\partial_{Z^1}^B\right)\left(a_{22}\delta_B^C + Z_B^2\partial_{Z^2}^C\right) \times\left(a_{32}\delta_C^D + Z_C^3\partial_{Z^3}^D\right)\left(\left(a_{30}-a_{20}\right)\delta_D^E + Z^{3E}\partial_{Z^3D}\right)\partial_{Z^2E}. \quad (3.12)$$

**Case** $[\Delta,(\ell_1,\ell_2,\ell_3)]\to[\Delta,(\ell_1,\ell_2,\ell_3+1)]$**:**

$$\mathcal{D}_M^{0(00+)} = \frac{1}{a_{22}-a_{23}}\left(\left(a_{00}-a_{03}\right)\delta_M^A + P_M\partial_P^A\right)\left(\left(a_{11}-a_{13}\right)\delta_A^B + Z_A^1\partial_{Z^1}^B\right) \times\left(\left(a_{22}-a_{23}\right)\delta_B^C + Z_B^2\partial_{Z^2}^C\right)Z_C^3. \quad (3.13)$$

**Case** $[\Delta,(\ell_1,\ell_2,\ell_3)]\to[\Delta,(\ell_1,\ell_2,\ell_3-1)]$**:**

$$\mathcal{D}_M^{0(00-)} = \frac{1}{a_{23}a_{33}}\left(a_{03}\delta_M^A + P_M\partial_P^A\right)\left(a_{13}\delta_A^B + Z_A^1\partial_{Z^1}^B\right)\left(a_{23}\delta_B^C + Z_B^2\partial_{Z^2}^C\right) \times\left(a_{33}\delta_C^D + Z_C^3\partial_{Z^3}^D\right)\partial_{Z^3D}. \quad (3.14)$$

Here we defined the constants $a_{ij}$ as

$$a_{ij} = 1+i+j+\delta_{ij}-(\ell_i+\ell_j+d), \quad \ell_0\equiv-\Delta, \quad (3.15)$$

and we defined some overall normalization in some cases to make the $6j$ symbols more compact. However, one can easily reabsorb these constants. Note that each of these operators depend on the representation they act on. However, one can produce representation-independent expressions by removing the overall normalizations and substituting the coefficients $-\Delta$ and $\ell_i$ by $P\cdot\partial_P$ and $Z^i\cdot\partial_{Z^i}$ respectively.

## 3.4 Computation of coefficients

Having the explicit expressions for the weight-shifting operators, we can start computing the $6j$ symbols required for writing (3.3). All of this follow from straightforward but tedious application of (1.179) and its reduction to two-point functions (1.218). All the required expressions for two- and three-point functions are explicitly written in section 1.4.

### 3.4.1 Two-point $6j$ symbols

The $6j$ symbols from two-point functions in (3.3) is computed via (1.218). It is easy to check that

$$\partial_{Z_1^i} \cdot \partial_{Z_1^j} \langle [\Delta,\rho](P_1, Z_1^k)[\Delta,\rho](P_2, Z_2^l)\rangle = O(X_2^2, Z_2^m \cdot X_2, Z_2^m \cdot Z_2^n), \tag{3.16}$$

$$\partial_{Z_1^i} \cdot \partial_{P_1} \langle [\Delta,\rho](P_1, Z_1^k)[\Delta,\rho](P_2, Z_2^l)\rangle = O(X_2^2, Z_2^m \cdot X_2, Z_2^m \cdot Z_2^n), \tag{3.17}$$

hence we can ignore all the terms containing these derivatives in the weight-shifting differential operators which reduces the complexity of the computation to a minimum. Performing the computation results in

$$\begin{Bmatrix} [\Delta,(\ell_1,\ell_2-1)] & \mathbf{1} & [\Delta,(\ell_1,\ell_2,)] \\ [\Delta,(\ell_1,\ell_2)] & (1) & [\Delta,(\ell_1,\ell_2-1)] \end{Bmatrix}^{\bullet 0(0+)}_{\bullet 0(0-)} = -\frac{(\Delta+\ell_2-2)}{\ell_2(\ell_2+1)(d-3-\Delta+\ell_2)(d-4+\ell_1+\ell_2)}, \tag{3.18}$$

for the recursion II, and

$$\begin{Bmatrix} [\Delta,(\ell_1,\ell_2,\ell_3-1)] & \mathbf{1} & [\Delta,(\ell_1,\ell_2,\ell_3)] \\ [\Delta,(\ell_1,\ell_2,\ell_3)] & (1) & [\Delta,(\ell_1,\ell_2,\ell_3-1)] \end{Bmatrix}^{\bullet 0(00+)}_{\bullet 0(00-)} = \frac{(\Delta+\ell_3-3)(\ell_3-\ell_1-2)}{\ell_3(d-4-\Delta+\ell_3)(d-5+\ell_1+\ell_3)}, \tag{3.19}$$

for the recursion I.

### 3.4.2 Three-point $6j$ symbols

To compute the three-point function $6j$ symbols, we use (1.179), where the 'bubble' coefficients are defined in (1.178). The advantage of this expression compared to (1.176) is that it does not involve a sum over different three-point functions on the right hand side, and moreover it only involves conformally invariant quantities.[3] The bubble coefficients can be read off from (1.178) by applying it to the corresponding two-point function, i.e.

$$\mathcal{D}_i^{(a)} \cdot \mathcal{D}_i^{(b)} \langle \Phi_i'(P_i)\Phi_i'(P_j)\rangle = \begin{pmatrix} \Phi_i' \\ \Phi_i \ (1) \end{pmatrix}^{ab} \langle \Phi_i'(P_i)\Phi_i'(P_j)\rangle, \tag{3.20}$$

where $\mathcal{D}_i^{(b)} : \Phi_i' \to \Phi_i$ and $\mathcal{D}_i^{(a)} : \Phi_i \to \Phi_i'$.

Now we present a few of the required $6j$ symbols for recursion I in (3.3). Some of the expressions are quite lengthy so we have compiled the complete list into a `Mathematica` file which can be found at [125].

$$\begin{Bmatrix} [\Delta_3,(\ell_2)] & [\Delta_4,(0)] & [-1+\Delta_3,(\ell_2)] \\ [\Delta,(\ell_1,-1+\ell_2)] & (1) & [\Delta,(\ell_1,\ell_2)] \end{Bmatrix}^{\bullet\ 0(0-0)}_{1\ +(000)}$$
$$= \frac{(-6+d)(\Delta-\Delta_3+\Delta_4+\ell_1)\ell_2(1+\ell_2)(-5+d+\ell_2)(-4+d+2\ell_2)}{2(d-2\Delta_3)(-2+d-\Delta_3)(-3+\Delta_3)(-2+\Delta_3)(-1+d-\Delta_3+\ell_2)(-1+\Delta_3+\ell_2)(-6+d+2\ell_2)}, \tag{3.21}$$

$$\begin{Bmatrix} [\Delta_3,(\ell_2)] & [\Delta_4,(0)] & [-1+\Delta_3,(\ell_2)] \\ [\Delta,(\ell_1,-1+\ell_2)] & (1) & [\Delta,(\ell_1,\ell_2)] \end{Bmatrix}^{\bullet\ 0(0-0)}_{2\ +(000)}$$
$$= \frac{(-6+d)\ell_2(1+\ell_2)(-5+d+\ell_2)(-3+d-\Delta+\Delta_3-\Delta_4+\ell_2)(-4+d+2\ell_2)}{2(d-2\Delta_3)(-2+d-\Delta_3)(-3+\Delta_3)(-2+\Delta_3)(-1+d-\Delta_3+\ell_2)(-1+\Delta_3+\ell_2)(-6+d+2\ell_2)}, \tag{3.22}$$

$$\begin{Bmatrix} [\Delta_3,(\ell_2)] & [\Delta_4,(0)] & [\Delta_3,(-1+\ell_2)] \\ [\Delta,(\ell_1,-1+\ell_2)] & (1) & [\Delta,(\ell_1,\ell_2,)] \end{Bmatrix}^{\bullet\ 0(0-0)}_{1\ 0(+00)}$$
$$= \frac{\ell_2(1+\ell_2)(-5+d+\ell_2)(-2+d-\Delta+\ell_2)(-2+d-\Delta_3+\ell_2)(-3+d+\ell_1+\ell_2)(-4+d+2\ell_2)^2}{(-3+d+\ell_2)(-1+d-\Delta_3+\ell_2)(-1+\Delta_3+\ell_2)(-6+d+2\ell_2)(-2+d+2\ell_2)}, \tag{3.23}$$

$$\begin{Bmatrix} [\Delta_3,(\ell_2)] & [\Delta_4,(0)] & [\Delta_3,(\ell_2,1)] \\ [\Delta,(\ell_1,-1+\ell_2)] & (1) & [\Delta,(\ell_1,\ell_2)] \end{Bmatrix}^{\bullet\ 0(0-0)}_{1\ 0(0-0)}$$
$$= -\frac{(-\Delta+\Delta_3-\Delta_4-\ell_1)(-\Delta+\Delta_3+\Delta_4+\ell_1)(-1+\ell_2)\ell_2(-5+d+\ell_2)(-4+d+2\ell_2)}{4(-2+\Delta_3)(2-d+\Delta_3)(-3+d+\ell_2)(-6+d+2\ell_2)}, \tag{3.24}$$

$$\begin{Bmatrix} [\Delta_3,(\ell_2)] & [\Delta_4,(0)] & [\Delta_3,(1+\ell_2)] \\ [\Delta,(\ell_1,-1+\ell_2)] & (1) & [\Delta,(\ell_1,\ell_2)] \end{Bmatrix}^{\bullet\ 0(0-0)}_{1\ 0(-00)}$$
$$= -\frac{(-6+d)(\Delta-\Delta_3-\Delta_4-\ell_1)(\Delta-\Delta_3+\Delta_4+\ell_1)\ell_2(-5+d+\ell_2)(-4+d+2\ell_2)}{2(2+\ell_2)(-3+d+\ell_2)(-1+d-\Delta_3+\ell_2)(-1+\Delta_3+\ell_2)(-6+d+2\ell_2)(-2+d+2\ell_2)}. \tag{3.25}$$

[3] The one disadvantage of this method is that, for some cases, contracting two weight-shifting operators and applying it to a three-point function is computationally intensive. For those cases, one can use (1.176) and solve for the missing $6j$ symbols.

### 3.4.3 Normalization coefficient

Now we want to compute the coefficients $C_{mn}$ in (3.3), given by

$$\begin{aligned}&C_{0(0+)0(+0)}\langle[\Delta_1,(\ell_2)](P_1,Z_1^k)[\Delta_2,(0)](P_2)[\Delta,(\ell_1,\ell_2)](P_0,Z_0^l)\rangle\\&=\mathcal{D}_1^{0(+00)}\cdot\mathcal{D}_0^{0(0+0)}\langle[\Delta_1,(\ell_2-1)](P_1,Z_1^k)[\Delta_2,(0)](P_2)[\Delta,(\ell_1,\ell_2-1)](P_0,Z_0^l)\rangle,\end{aligned} \tag{3.26}$$

for the recursion II, and

$$\begin{aligned}&C_{0(00+)0(0+)}\langle[\Delta_1,(\ell_2,\ell_3)](P_1,Z_1^k)[\Delta_2,(0)](P_2)[\Delta,(\ell_1,\ell_2,\ell_3)](P_0,Z_0^l)\rangle\\&=\mathcal{D}_1^{0(0+)}\cdot\mathcal{D}_0^{0(00+)}\langle[\Delta_1,(\ell_2,\ell_3-1)](P_1,Z_1^k)[\Delta_2,(0)](P_2)[\Delta,(\ell_1,\ell_2,\ell_3-1)](P_0,Z_0^l)\rangle,\end{aligned} \tag{3.27}$$

for the recursion I. This results in

$$C_{0(0+)0(+0)}=(\Delta+\ell_2-2)(\Delta_1+\ell_2-2), \tag{3.28}$$

$$C_{0(00+)0(0+)}=\frac{(\ell_3-\ell_1-2)(\ell_3-\ell_2-2)}{(\ell_3-\ell_2-1)}(\Delta+\ell_3-3)(\Delta_1+\ell_3-3). \tag{3.29}$$

### 3.4.4 Differential basis

The last ingredient for the recursion (3.3) is to find a differential basis

$$\langle\Phi_3'\varphi_4O\rangle^{(a)}=\sum_{e,\Phi_3''}\mathcal{W}_{(e)}^{(a)}\mathscr{D}_{\Phi_3''\varphi_4'}^{(e)\Phi_3'\varphi_4}\langle\Phi_3''\varphi_4'O\rangle, \tag{3.30}$$

where the operators $\mathscr{D}$ are built out of contractions of weight-shifting differential operators:

$$\mathcal{D}_{ij}^{a;b}\equiv(\mathcal{D}_i^a\cdot\mathcal{D}_j^b). \tag{3.31}$$

In particular, for the recursion I these must implement the transformations

$$[\Delta_4',(0)]\times[\Delta_3'',(\ell_2,\ell_3-1)]\to[\Delta_4,(0)]\times\begin{cases}[\Delta_3-1,(\ell_2,\ell_3)]\\ [\Delta_3,(\ell_2+1,\ell_3)]\\ [\Delta_3,(\ell_2,\ell_3+1)]\\ [\Delta_3,(\ell_2,\ell_3,1)]\\ [\Delta_3,(\ell_2-1,\ell_3)]\\ [\Delta_3+1,(\ell_2,\ell_3)]\end{cases}, \tag{3.32}$$

whereas for the recursion II we need

$$[\Delta_4', (0)] \times [\Delta_3'', (\ell_2 - 1)] \to [\Delta_4, (0)] \times \begin{cases} [\Delta_3 - 1, (\ell_2)] \\ [\Delta_3, (\ell_2 + 1)] \\ [\Delta_3, (\ell_2, 1)] \\ [\Delta_3 + 1, (\ell_2)] \end{cases}. \tag{3.33}$$

Note that $\Delta_3''$ and $\Delta_4'$ are generally given by $\Delta_3 + i$ and $\Delta_4 + j$, respectively, for arbitrary $i, j \in \mathbb{Z}$. In principle, adding enough weight-shifting operator contractions, we can have a differential basis with $i = j = 0$ at the expense of increasing the order of derivatives. In here we will keep the order to a minimum, thus for the cases where (3.32) and (3.33) require changing the spin weights by two units, we use the following basis

$$\begin{aligned} &\mathcal{D}_{43}^{+(00);(a)}\mathcal{D}_{43}^{+(00);(b)}, \quad \mathcal{D}_{43}^{+(00);(a)}\mathcal{D}_{43}^{-(00);(b)} \\ &\mathcal{D}_{43}^{-(00);(a)}\mathcal{D}_{43}^{+(00);(b)}, \quad \mathcal{D}_{43}^{-(00);(a)}\mathcal{D}_{43}^{-(00);(b)}. \end{aligned} \tag{3.34}$$

On the other hand, for the cases in (3.32) and (3.33) where the spin weights change by one unit, we use

$$\mathcal{D}_{43}^{+(00);(b)}, \quad \mathcal{D}_{43}^{-(00);(b)}. \tag{3.35}$$

Therefore the differential operators that implement (3.32) and (3.33) must be either

$$\begin{aligned} \langle \Phi_3' \varphi_4 O \rangle^{(a)} = {} & \mathcal{W}^{(b)(c);(a)}_{\Phi_3'\varphi_4 O;1} \mathcal{D}_{43}^{+(00);(b)} \mathcal{D}_{43}^{+(00);(c)} \langle \Phi_3'^{-b-c} \varphi_4^{-2(0)} O \rangle \\ & + \mathcal{W}^{(b)(c);(a)}_{\Phi_3'\varphi_4 O;2} \mathcal{D}_{43}^{+(00);(b)} \mathcal{D}_{43}^{-(00);(c)} \langle \Phi_3'^{-b-c} \varphi_4 O \rangle \\ & + \mathcal{W}^{(b)(c);(a)}_{\Phi_3'\varphi_4 O;3} \mathcal{D}_{43}^{-(00);(b)} \mathcal{D}_{43}^{+(00);(c)} \langle \Phi_3'^{-b-c} \varphi_4 O \rangle \\ & + \mathcal{W}^{(b)(c);(a)}_{\Phi_3'\varphi_4 O;4} \mathcal{D}_{43}^{-(00);(b)} \mathcal{D}_{43}^{-(00);(c)} \langle \Phi_3'^{-b-c} \varphi_4^{2(0)} O \rangle, \end{aligned} \tag{3.36}$$

or

$$\begin{aligned} \langle \Phi_3' \varphi_4 O \rangle^{(a)} = {} & \mathcal{W}^{(c);(a)}_{\Phi_3'\varphi_4 O;1} \mathcal{D}_{43}^{+(00);(c)} \langle \Phi_3'^{-c} \varphi_4^{-1(0)} O \rangle \\ & + \mathcal{W}^{(c);(a)}_{\Phi_3'\varphi_4 O;2} \mathcal{D}_{43}^{-(00);(c)} \langle \Phi_3'^{-c} \varphi_4^{1(0)} O \rangle, \end{aligned} \tag{3.37}$$

where the short hand notation of an operator $\phi$ with a superscript $a$ means that a numerical factor has been added to the conformal weights of the representation of $\phi = [\Delta, (\ell)]$. For example if $a = 0(+1)$ then the representation of $\phi^{-a}$ is $[\Delta, (\ell-1)]$, etc. All the coefficients $\mathcal{W}$ for both recursions are given in [125]. Here we give one example to elucidate the form of these expressions:

$$\langle[\Delta_3-1,(\ell_2)][\Delta_4,(0)][\Delta,(\ell_1,\ell_2-1)]\rangle^{(2)} = \frac{1}{2(d-2\Delta_4)(\ell_1-\ell_2+1)(\Delta+\ell_2-2)(\Delta_3+\ell_2-3)}$$
$$\times\Bigg(\frac{(d-6)(\Delta_4-\Delta_3-\Delta+\ell_1-2\ell_2+4)}{(\Delta_4-d+2)(\Delta_4-d+1)(\Delta_4-3)(\Delta_4-2)}\mathcal{D}_{43}^{+(00);0(+0)}\Sigma^{0,-1}$$
$$+(\Delta-\Delta_3+\Delta_4+\ell_1)(\Delta_3+\Delta_4+\ell_1-\Delta-2)(\Delta_3+\Delta_4+2\ell_2-\ell_1-4)\mathcal{D}_{43}^{-(00);0(+0)}\Sigma^{0,1}\Bigg)$$
$$\times\langle[\Delta_3-1,(\ell_2-1)][\Delta_4,(0)][\Delta,(\ell_1,\ell_2-1)]\rangle, \quad (3.38)$$

where $\Sigma^{a,b}$ increases the dimension of $\Delta_3$ by $a$ and the dimension of $\Delta_4$ by $b$.

## 3.5 Seed partial wave recursion relations

Putting everything together results in the following recursion relations

### Recursion I

$$W_{(\ell_1,\ell_2,\ell_3)} = \left(\bar{d}_1\Sigma^{0,-2}+\bar{d}_2\Sigma^{-1,-1}+\bar{d}_3\Sigma^{-1,1}+\bar{d}_4+\bar{d}_5\Sigma^{1,-1}+\bar{d}_6\Sigma^{1,1}+\bar{d}_7\Sigma^{0,2}\right)W_{(\ell_1,\ell_2,\ell_3-1)}, \quad (3.39)$$

where $\Sigma$ is defined the same way as above and the differential operators $\bar{d}_i$ are given by

$$\bar{d}_1 = \bar{v}_{11}\mathcal{D}_{13}^{0(0+);-(00)}\mathcal{D}_{43}^{+(00);0(+0)}\mathcal{D}_{43}^{+(00);0(0+)}+\bar{v}_{12}\mathcal{D}_{13}^{0(0+);0(0-)}\mathcal{D}_{43}^{+(00);0(0+)}\mathcal{D}_{43}^{+(00);0(0+)}$$
$$+\bar{v}_{13}\mathcal{D}_{13}^{0(0+);0(00-)}\mathcal{D}_{43}^{+(00);0(00+)}\mathcal{D}_{43}^{+(00);0(0+)}, \quad (3.40)$$

$$\bar{d}_2 = \bar{v}_2\mathcal{D}_{13}^{0(0+);+(00)}\mathcal{D}_{43}^{+(00);0(0+)}, \quad \bar{d}_3 = \bar{v}_3\mathcal{D}_{13}^{0(0+);+(00)}\mathcal{D}_{43}^{-(00);0(0+)},$$
$$\bar{d}_5 = \bar{v}_5\mathcal{D}_{13}^{0(0+);-(00)}\mathcal{D}_{43}^{+(00);0(0+)}, \quad \bar{d}_6 = \bar{v}_6\mathcal{D}_{13}^{0(0+);-(00)}\mathcal{D}_{43}^{-(00);0(0+)}, \quad (3.41)$$

$$\bar{d}_4 = \bar{v}_{41}\mathcal{D}_{13}^{0(0+);0(0+)}+\bar{v}_{42}\mathcal{D}_{13}^{0(0+);0(-0)}\mathcal{D}_{43}^{-(00);0(+0)}\mathcal{D}_{43}^{+(00);0(0+)}$$
$$+\bar{v}_{43}\mathcal{D}_{13}^{0(0+);0(-0)}\mathcal{D}_{43}^{+(00);0(+0)}\mathcal{D}_{43}^{-(00);0(0+)}+\bar{v}_{44}\mathcal{D}_{13}^{0(0+);0(0-)}\mathcal{D}_{43}^{-(00);0(0+)}\mathcal{D}_{43}^{+(00);0(0+)}$$
$$+\bar{v}_{45}\mathcal{D}_{13}^{0(0+);0(0-)}\mathcal{D}_{43}^{+(00);0(0+)}\mathcal{D}_{43}^{-(00);0(0+)},+\bar{v}_{46}\mathcal{D}_{13}^{0(0+);0(00-)}\mathcal{D}_{43}^{-(00);0(00+)}\mathcal{D}_{43}^{+(00);0(0+)}$$
$$+\bar{v}_{47}\mathcal{D}_{13}^{0(0+);0(00-)}\mathcal{D}_{43}^{+(00);0(00+)}\mathcal{D}_{43}^{-(00);0(0+)}, \quad (3.42)$$

$$\bar{d}_7 = \bar{v}_{71}\mathcal{D}_{13}^{0(0+);0(-0)}\mathcal{D}_{43}^{-(00);0(+0)}\mathcal{D}_{43}^{-(00);0(0+)}+\bar{v}_{72}\mathcal{D}_{13}^{0(0+);0(0-)}\mathcal{D}_{43}^{-(00);0(0+)}\mathcal{D}_{43}^{-(00);0(0+)}$$
$$+\bar{v}_{73}\mathcal{D}_{13}^{0(0+);0(00-)}\mathcal{D}_{43}^{-(00);0(00+)}\mathcal{D}_{43}^{-(00);0(0+)}. \quad (3.43)$$

**Recursion II**

$$W_{(\ell_1,\ell_2,0)} = \left(d_1\Sigma^{0,-2} + d_2\Sigma^{-1,-1} + d_3\Sigma^{-1,1} + d_4 + d_5\Sigma^{1,-1} + d_6\Sigma^{1,1} + d_7\Sigma^{0,2}\right) W_{(\ell_1,\ell_2-1,0)}, \tag{3.44}$$

where the differential operators $d_i$ are

$$d_1 = v_{11}\mathcal{D}_{13}^{0(+0);0(-0)}\mathcal{D}_{43}^{+(00);0(+0)}\mathcal{D}_{43}^{+(00);0(+0)} + v_{12}\mathcal{D}_{13}^{0(+0);0(0-)}\mathcal{D}_{43}^{+(00);0(+0)}\mathcal{D}_{43}^{+(00);0(0+)}, \tag{3.45}$$

$$d_2 = v_2\mathcal{D}_{13}^{0(+0);+(00)}\mathcal{D}_{43}^{+(00);0(+0)}, \quad d_3 = v_3\mathcal{D}_{13}^{0(+0);+(00)}\mathcal{D}_{43}^{-(00);0(+0)}, \quad d_5 = v_5\mathcal{D}_{13}^{0(+0);-(00)}\mathcal{D}_{43}^{+(00);0(+0)}, \quad d_6 = v_6\mathcal{D}_{13}^{0(+0);-(00)}\mathcal{D}_{43}^{-(00);0(+0)}, \tag{3.46}$$

$$d_4 = v_{41}\mathcal{D}_{13}^{0(+0);0(+0)} + v_{42}\mathcal{D}_{13}^{0(+0);0(-0)}\mathcal{D}_{43}^{-(00);0(+0)}\mathcal{D}_{43}^{+(00);0(+0)} + v_{43}\mathcal{D}_{13}^{0(+0);0(-0)}\mathcal{D}_{43}^{+(00);0(+0)}\mathcal{D}_{43}^{-(00);0(+0)} + v_{44}\mathcal{D}_{13}^{0(+0);0(0-)}\mathcal{D}_{43}^{-(00);0(+0)}\mathcal{D}_{43}^{+(00);0(0+)} + v_{45}\mathcal{D}_{13}^{0(+0);0(0-)}\mathcal{D}_{43}^{+(00);0(+0)}\mathcal{D}_{43}^{-(00);0(0+)}, \tag{3.47}$$

$$d_7 = v_{71}\mathcal{D}_{13}^{0(+0);0(-0)}\mathcal{D}_{43}^{-(00);0(+0)}\mathcal{D}_{43}^{-(00);0(+0)} + v_{72}\mathcal{D}_{13}^{0(+0);0(0-)}\mathcal{D}_{43}^{-(00);0(+0)}\mathcal{D}_{43}^{-(00);0(0+)}. \tag{3.48}$$

For both cases, the coefficients $\bar{v}_{ij}$ and $v_{ij}$ are numerical factors depending on the conformal weights of both the external and exchanged operators. The explicit form of these coefficients is given in [125].

## 3.6 Discussion

The main results in this chapter are the recursion relations (3.39) and (3.44). These allow us to write any spinning seed partial wave $W_{[\Delta,(\ell_1,\ell_2,\ell_3)]}$ in terms of the scalar partial waves $W_{[\Delta,(\ell_1)]}$ of [68, 69]. Note that in its current form, $W_{[\Delta,(\ell_1,\ell_2,\ell_3)]}$ appears in the four-point function

$$\langle[\Delta_1,(\ell_2,\ell_3)][\Delta_2,(0)][\Delta_3,(\ell_2,\ell_3)][\Delta_4,(0)]\rangle, \tag{3.49}$$

which can be related to the partial wave of the same representation in

$$\langle[\Delta_1,(l_1)][\Delta_2,(l_2)][\Delta_3,(l_3)][\Delta_4,(l_4)]\rangle, \tag{3.50}$$

for arbitrary $l_i$ via (1.217). We do not present the explicit expressions for doing that here, since in general, the operators in a representation $[\Delta, (\ell_1, \ell_2, \ell_3)]$ can appear in more than one four-point function of traceless-symmetric operators. For example, $[\Delta, (\ell, 2)]$ appears in both $\langle T_1 \phi_2 T_3 \phi_4 \rangle$ and $\langle V_1 V_2 V_3 V_4 \rangle$, where $\phi$, $V$, and $T$ represent a scalar, a vector and a spin-2 tensor respectively [37]. Nonetheless, it would be interesting to set up these equations for particular interesting cases such as for four conserved spin-2 tensors.

In the process of deriving (3.39) and (3.44) we also computed useful quantities in the context of the weight-shifting operator formalism, first coined in [46] (also reviewed in section 1.5). These include the computation of many coefficients that relate two- and three-point functions in different representations, given partially in section 3.4. The full set of coefficients can be found in [125] in a more usable form. Another result is the computation of explicit weight-shifting operators associated to the vector representation for operators of up to three rows, given in section 3.3.

From the factorized form of these weight-shifting operators, it seems like a generalization to arbitrary number of rows should be feasible. It could be useful to first understand how each of the terms in parenthesis behaves under (1.170) and then figure out how to construct the coefficients in general. The basis of coefficients used in section 3.3 seems to shed some light onto the possible generalization; namely, one of the indices in $a_{ij}$ corresponds to the polatization of each term in parenthesis ($P$ for $i = 0$), while the other index is the label of the polarization whose weight is being altered by the differential operator.[4] Another interesting generalization is to consider the weight-shifting operators for finite-dimensional representations other than the vector. As seen in subsection 3.4.4, we used products of vector weight-shifting operators in order to change the conformal weights of a single field by more than one unit. However, this generically increases the order of derivatives. By constructing higher representation weight-shifting operators, one could achieve the same result with less derivatives. A known example of this is the differential operator $D_{1\,ij}$, defined in (1.180), which is of order one in derivatives but deceases the dimension by one and increases the spin by one, for traceless-symmetric fields.

It would be interesting to use (3.39) and (3.44) to compute all conformal blocks of $\langle TTTT \rangle$ in arbitrary dimensions $d$. This can then be used in both numerical and analytical approaches to the bootstrap program for finding universal bounds on the CFT data of conformal field theories. The firs step in this direction has already been taken in $d = 3$ by [126] , where bounds on the stress-tensor OPE coefficients and the central charge where found numerically, and when assuming

[4]There are some exceptions to that rule for the terms given by a difference of coefficients. Nevertheless, they also show an interesting pattern.

gaps in the spectrum of lower spin operators, it reduces the space of consistent theories to a finite region in CFT data space.

The bootstrap equations (1.227) are usually set up in terms of conformal blocks (1.186). This implies passing the derivatives of (3.39) and (3.44) through the kinetic factor $\mathcal{K}_4$ as well as the four-point tensor structures $Q$, to obtain an expression involving derivatives with respect to the cross-ratios $U, V$ from (1.119). However, in its present form, this is not easy to do given the large order in derivatives. As mentioned before, this could be simplified by using higher representation mixed-symmetric operators. An alternative is to use the conformal symmetry to fix the coordinates $P_i$ to a two-dimensional region, known as the conformal frame [38], and map the derivatives to this space. A similar technique was applied successfully in [126], and it could reduce the computational load when computing derivatives. The final result in terms of $U$ and $V$ then applies to any configuration of points due to conformal symmetry.

# 4 Analytic bounds from spinning correlators

On analytic constraints from causality and light-cone bootstrap for correlators with spin

Even though conformal symmetry places a large amount of constrains on quantum field theories, carving out the space of consistent CFT data is no easy task. In order to get a handle on this problem, many tools have been developed both analytical and numerical. In this chapter, based on [2], we employ two analytical techniques to extract non-trivial universal information for conformal field theories in arbitrary dimensions.

## 4.1 Introduction

Among the many bootstrap approaches, here we will focus on analytical techniques. In particular we explore two techniques. One of them involves studying the bootstrap in the lightcone limit, presented in subsection 1.7.3. This reveals a direct relation between couplings of low-twist operators and the asymptotic behavior of CFT spectra at large spin [88–92,96–99,102,127–129]. The other technique, presented in section 1.8 (based on [94]) uses the fact that CFT unitarity/reflection positivity implies both causality and sum rules, leading to constraints on the signs of products of OPE coefficients. These constraints are closely related to the bound on chaos [130,131] as discussed in the context of CFT correlators in [132–135].

As a result, we obtain that the 'conformal collider bounds', first derived in [136], must hold in any unitary, parity-preserving conformal field theory with a unique stress energy tensor. In addition, by relating these bounds to the anomalous dimensions of high spin double-trace operators we find that the dual gravitational theory must be attractive.

For context let us quickly review the conformal collider bounds as proposed in [136]. We first produce a localized perturbation via the insertion of an operator $O$ near the origin and then we measure the integrated energy flux in a fixed spatial direction $\vec{n}$ at infinity, using a 'calorimeter' $\mathcal{E}(\vec{n})$. In a CFT this corresponds to

$$\langle \mathcal{E}(\vec{n}) \rangle \equiv \frac{\langle O \mathcal{E}(\vec{n}) O \rangle}{\langle OO \rangle}, \tag{4.1}$$

where the energy flux is given by

$$\mathcal{E}(\vec{n}) = \lim_{r\to\infty} r^2 \int_{-\infty}^{\infty} n_i T^{0i}(t, r\vec{n}) dt, \tag{4.2}$$

with $T$ the stress-tensor. Then the assumption of [136] is that these 'calorimeters' should measure positive energies and thus $\langle \mathcal{E}(\vec{n}) \rangle \geq 0$. In CFT language this means that the possible values of the OPE coefficients in the three-point function $\langle OTO \rangle$ must be constrained. This is a very strong statement since the bounds are theory-independent. For example, for $O = T$ in $d = 4$, the OPE coefficients of $\langle TTT \rangle$ are related to the conformal trace anomaly charges $a$ and $c$ on a general background [137]. Thus positivity of the integrated energy flux implies the conformal collider bounds

$$\frac{31}{18} \geq \frac{a}{c} \geq \frac{1}{3}, \tag{4.3}$$

which are universal for any 4d CFT with a stress-tensor.

Note that in [136], assuming the positivity of $\langle \mathcal{E}(\vec{n}) \rangle$ leads to the conformal collider bounds. However, here they will be a consequence of CFT first principles, such as unitarity, associativity of the operator algebra, and causality.

This chapter is organized as follows. In section 4.2 we present the four-point functions of interest as well as their conformal block expansion characteristics for different OPE channels. In section 4.3 we use the light-cone bootstrap to find how the spectrum of large-spin double-twist operators must be related to the minimal twist operators, in order for the CFT to be consistent. In particular we find their twist, anomalous dimensions, and OPE coefficients in terms of the twist and OPE coefficients of the stress-tensor. In section 4.4 we use causality arguments for spinning correlators in order to obtain constraints on particular combinations of OPEs involving the stress-tensor. Finally in section 4.5 we combine the results of the previous two sections to conclude that the anomalous dimensions of large-spin double-twist operators must have a fixed sign. We conclude in section 4.6. Details and extensions of our computations are presented in appendices A, B, and C.

## 4.2 Spinning conformal block expansions

The objects of study in this chapter are the spinning correlators $\langle J\phi\phi J\rangle$, $\langle V\phi\phi V\rangle$, and $\langle T\phi\phi T\rangle$, where the symbols $\phi$, $J$, $V$, and $T$ represent a scalar $\phi = [\Delta_\phi, (0)]$, a conserved vector $J = [\Delta_J = d-1, (1)]$, a non-conserved vector $V = [\Delta_V, (1)]$, and the stress-tensor $T = [\Delta_T = d, (2)]$ respectively. Recall from section 1.6 that for generic symmetric traceless spinning fields $O_i^{(\ell_i)}(x_i, \epsilon_i) = O^{\mu_1\cdots\mu_\ell}\epsilon_{\mu_1}\ldots\epsilon_{\mu_\ell} \in [\Delta_i, (\ell_i)]$, the four-point has the following s-channel conformal block expansion

$$\langle O_1^{\ell_1}(x_1,\epsilon_1)O_2^{\ell_2}(x_2,\epsilon_2)O_3^{\ell_3}(x_3,\epsilon_3)O_4^{\ell_4}(x_4,\epsilon_4)\rangle = \mathcal{K}_4^{\Delta_i}\sum_{\mathcal{O},a,b,p}\lambda^a_{12\mathcal{O}}\lambda^b_{34\mathcal{O}}g^{\Delta_{12},\Delta_{34}}_{\mathcal{O},a,b,p}(z,\bar z)Q^p(x_i,\epsilon_i), \quad (4.4)$$

where the polarizations $\epsilon$ satisfy the properties defined in subsection 1.2.2, and the cross-ratios $z, \bar z$ are given in (1.230). Additionally, $\mathcal{O}$ runs over the operators in $O_1\times O_2$, $a(b)$ runs over the three-point function structures of $\langle O_1O_2\mathcal{O}\rangle(\langle O_3O_4\mathcal{O}\rangle)$, and $p$ runs over the four-point function tensor structures $Q^p$. Analogous expansions can be written for the t- and u-channels. In particular, for $\langle J\phi\phi J\rangle$, we define the correlator [1]

$$G_J^{\mu\nu}(z,\bar z) \equiv \langle J^\mu(0)\phi(z,\bar z)\phi(1)J^\nu(\infty)\rangle, \quad (4.5)$$

where the configuration of points is as in equations (1.250) and (1.251), and consider the following channel decompositions

$$\text{s-channel:} \quad G_J^{\mu\nu}(z,\bar z) = (z\bar z)^{-\frac{1}{2}(\Delta_\phi-\Delta_J)}\sum_{\mathcal{O}}\lambda_{J\phi\mathcal{O}}\lambda_{\phi J\mathcal{O}}g_{\mathcal{O}}^{\Delta_{J\phi},\Delta_{\phi J},\mu\nu}(z,\bar z) = G^{\mu\nu}_{J,STT}(z,\bar z) + G^{\mu\nu}_{J,A}(z,\bar z), \quad (4.6)$$

$$\text{t-channel:} \quad G_J^{\mu\nu}(z,\bar z) = [(1-z)(1-\bar z)]^{-\Delta_\phi}\sum_{\mathcal{O},b}\lambda^b_{JJ\mathcal{O}}\lambda_{\phi\phi\mathcal{O}}g^{0,0,\mu\nu}_{\mathcal{O},b}(1-z,1-\bar z), \quad (4.7)$$

where we have absorbed the tensor structures $Q^p$ into the conformal blocks $g$, since in our configuration they will also become functions of $z$ and $\bar z$ (which is not true for generic configurations). In addition, as seen in chapter 2, the $J\times\phi$ OPE contains in general two families of operators. Thus we split the full contribution of each family to the four-point function into $G^{\mu\nu}_{J,STT}(z,\bar z)$ and $G^{\mu\nu}_{J,A}(z,\bar z)$, where $STT \equiv \mathcal{O}_{[\ell]} = [\Delta_{\mathcal{O}_{[\ell]}}, (\ell)]$ is traceless and symmetric, and $A \equiv \mathcal{O}_{[\ell,1]} = [\Delta_{\mathcal{O}_{[\ell,1]}}, (\ell,1)]$ has a pair of antisymmetrized indices and the other $(\ell-1)$ indices symmetrized.

[1] Here we define $O(\infty)$ as $\lim_{x\to\infty} x^{2\Delta_O}O(x)$.

In this case $J$ is a conserved current, so $\Delta_J = d-1$ (c.f. subsection 1.4.3). In the $J \times J$ OPE, operators with spin can appear with two independent parity-preserving tensor structures, while scalars have a unique tensor structure. Thus the index $b$ accounts for that. Note that the u-channel is similar to the s-channel with $z \to 1/z, \bar{z} \to 1/\bar{z}$, but we will not need it explicitly.

Similarly, for the case of a non-conserved current $V$, $\langle V\phi\phi V\rangle$, we have

$$G_V^{\mu\nu}(z,\bar{z}) \equiv \langle V^\mu(0)\phi(z,\bar{z})\phi(1)V^\nu(\infty)\rangle, \tag{4.8}$$

$$\text{s-channel:} \qquad G_V^{\mu\nu}(z,\bar{z}) = (z\bar{z})^{-\frac{1}{2}(\Delta_\phi-\Delta_V)} \sum_{\mathcal{O},a,b} \lambda^a_{V\phi\mathcal{O}}\lambda^b_{\phi V\mathcal{O}} g_{\mathcal{O}}^{\Delta_{V\phi},\Delta_{\phi V},\mu\nu}(z,\bar{z})$$
$$= G^{\mu\nu}_{V,STT}(z,\bar{z}) + G^{\mu\nu}_{V,A}(z,\bar{z}), \tag{4.9}$$

$$\text{t-channel:} \qquad G_V^{\mu\nu}(z,\bar{z}) = [(1-z)(1-\bar{z})]^{-\Delta_\phi} \sum_{\mathcal{O},c} \lambda^c_{VV\mathcal{O}}\lambda_{\phi\phi\mathcal{O}} g_{\mathcal{O},b}^{0,0,\mu\nu}(1-z,1-\bar{z}), \tag{4.10}$$

where $\Delta_V$ is not fixed. As seen in chapter 2 the $\phi \times V$ OPE has two independent tensor structures for general spinning $\mathcal{O}$, so the indices $a, b$ run over $\{1,2\}$. On the other hand, $c$ runs over tensor structures in the $V \times V$ OPE. For example for $\mathcal{O} = T$, $c \in \{1,2,3\}$.

Finally, for the stress-tensor case $\langle T\phi\phi T\rangle$,

$$G_T^{\mu\nu\rho\sigma}(z,\bar{z}) \equiv \langle T^{\mu\nu}(0)\phi(z,\bar{z})\phi(1)T^{\rho\sigma}(\infty)\rangle, \tag{4.11}$$

$$\text{s-channel:} \qquad G_T^{\mu\nu\rho\sigma}(z,\bar{z}) = (z\bar{z})^{-\frac{1}{2}(\Delta_\phi-\Delta_T)} \sum_{\mathcal{O}} \lambda_{T\phi\mathcal{O}}\lambda_{\phi T\mathcal{O}} g_{\mathcal{O}}^{\Delta_{T\phi},\Delta_{\phi T},\mu\nu\rho\sigma}(z,\bar{z})$$
$$= G^{\mu\nu\rho\sigma}_{T,STT}(z,\bar{z}) + G^{\mu\nu\rho\sigma}_{T,A}(z,\bar{z}) + G^{\mu\nu\rho\sigma}_{T,B}(z,\bar{z}), \tag{4.12}$$

$$\text{t-channel:} \qquad G_T^{\mu\nu\rho\sigma}(z,\bar{z})$$
$$= [(1-z)(1-\bar{z})]^{-\Delta_\phi} \sum_{\mathcal{O},b} \lambda^b_{TT\mathcal{O}}\lambda_{\phi\phi\mathcal{O}} g_{\mathcal{O},b}^{0,0,\mu\nu\rho\sigma}(1-z,1-\bar{z}), \tag{4.13}$$

where conservation of $T$ implies $\Delta_T = d$ (c.f. subsection 1.4.3). In general $d \geq 4$, there are three types of operators in the $T \times \phi$ OPE, so we split the four-point function into $G^{\mu\nu\rho\sigma}_{T,STT}(z,\bar{z})$, $G^{\mu\nu\rho\sigma}_{T,A}(z,\bar{z})$, and $G^{\mu\nu\rho\sigma}_{T,B}(z,\bar{z})$, where $STT$ and $A$ are defined above, and $B \equiv \mathcal{O}_{[\ell,2]} = [\Delta_{\mathcal{O}_{[\ell,2]}}, (\ell,2)]$ has two pairs of indices antisymmetrized and the other $(\ell-2)$ indices symmetrized. The index $b$ in (4.13) sums over the the different tensor structures for operators in the $T \times T$ OPE. For general operators $\mathcal{O}$ with spin, $b$ runs over three different parity-preserving structures in $d \geq 4$, whereas a scalar $\mathcal{O}$ has a unique tensor structure.

## 4.3 The light-cone bootstrap

### 4.3.1 Overview

First we look at the light-cone bootstrap technique presented in subsection 1.7.3. The main idea is to study the bootstrap equation (1.227) for the s- and t-channels, in the light-cone limit [2]

$$\bar{z} \to 1, \quad z \text{ finite.} \tag{4.14}$$

The $t$-channel expansion is then organized as a sum over twists $\tau = \Delta - \ell$ and spins $\ell$, where the leading order contribution is given by the identity operator (zero twist). Using the particular t-channel expansions (4.7), (4.10), (4.13) we see that this corresponds to a power law singularity $(1-\bar{z})^{-\Delta_\phi}$. On the other hand, the s-channel expansions (4.6), (4.9), (4.12) contain at most a $\ln(1-\bar{z})$ singularity, as seen in subsection 1.7.3. Therefore to reproduce this singularity, the s-channel must contain infinite large-spin operators as established in [88].

In what follows, we study the implications of this technique for the particular spinning correlators mentioned in the previous section. The algorithm for extracting information regarding the large-spin operators is already described in subsection 1.7.3 for scalar four-point functions. In our case, the four-point functions carry the spin of the external operators, thus we apply the same algorithm for the different components. The relevant spinning conformal blocks for each case are computed in appendix A, in terms of the well-known scalar conformal blocks.

### 4.3.2 Bounds from $\langle J\phi\phi J\rangle$

We will work with two polarizations of the 4-point function (following [91]), $G_J^{++}$ and $G_J^{tt}$, where $+$ is in the direction of $z$ (i.e. $\epsilon = (i, 1, 0, 0, \dots)$), and $t$ is transverse to the $z$ plane (e.g. $\epsilon = (0, 0, 1, 0, \dots)$, etc.). Recall that the s-channel expansion contains contributions from both $G_{J,STT}$ and $G_{J,A}$, but as shown in (4.117) and (4.108), $G_{J,A}^{++}$ is subleading with respect to $G_{J,STT}^{++}$ in the light-cone limit (at finite $z$), while both $G_{J,STT}^{tt}$ and $G_{J,A}^{tt}$ contribute equally to $G_J^{tt}$. In other words, when matching singularities in $(1-\bar{z})$, the following 'triangular structure' holds:

$$\begin{aligned} G_J^{++} &= G_{J,STT}^{++} \\ G_J^{tt} &= G_{J,STT}^{tt} + G_{J,A}^{tt}. \end{aligned} \tag{4.15}$$

[2] Note that in 1.7.3 we took the small $z$ limit, so the roles of the s- and t-channels are reversed here.

This means that when solving the light-cone bootstrap constraints, we first fix the CFT data for $STT$ from the $++$ polarization, and then plug that result into the $tt$ polarization to fix the CFT data for $A$.

**Leading order matching**

To reproduce the singularity from the identity at leading order in $(1-\bar{z})$, the s-channel expansion (4.6) must contain two families of large spin double-twist operators: $[J\phi]^{[\ell]}_{n,\ell} \in STT$, with the schematic form $J^{\mu_1}\partial^{(\mu_2}\dots\partial^{\mu_\ell)}\partial^{2n}\phi$, and $[J\phi]^{[\ell,1]}_{n,\ell} \in A$, with the schematic form $J^{[\mu_1}\partial^{(\mu_2]}\partial^{\mu_3}\dots\partial^{\mu_\ell)}\partial^{2n}\phi$. In the $\ell\to\infty$ limit, the anomalous dimensions of these operators vanish as a power law in $\ell$ and their twists must be of the form

$$\tau_{[J\phi]^{[\ell]}_{n,\ell}}\Big|_{\ell\to\infty} = d-2+\Delta_\phi+2n, \tag{4.16}$$

$$\tau_{[J\phi]^{[\ell,1]}_{n,\ell}}\Big|_{\ell\to\infty} = d-1+\Delta_\phi+2n, \tag{4.17}$$

where $n\geq 0$ is a finite integer. Furthermore, the large $\ell$ asymptotics of the OPE coefficients for double twist operators are fixed to (see Appendix B for a formula to do the sum over $n$)

$$(\lambda_{J\phi[J\phi]^{[\ell]}_{n,\ell}})^2 = \frac{(1-\frac{d}{2}+\Delta_\phi)_n(\frac{d}{2}-1)_n}{4^n n!(\Delta_\phi+n-1)_n}(\lambda_{J\phi[J\phi]^{[\ell]}_{0,\ell}})^2, \tag{4.18}$$

$$(\lambda_{J\phi[J\phi]^{[\ell,1]}_{n,\ell}})^2 = \frac{(1-\frac{d}{2}+\Delta_\phi)_n(\frac{d}{2})_n}{4^n n!(\Delta_\phi+n)_n}(\lambda_{J\phi[J\phi]^{[\ell,1]}_{0,\ell}})^2, \tag{4.19}$$

where

$$(\lambda_{J\phi[J\phi]^{[\ell]}_{0,\ell}})^2 = \frac{C_J\sqrt{\pi}2^{-\Delta_\phi-d+5}}{2^\ell\Gamma(\Delta_\phi)\Gamma(d)}\ell^{\frac{1}{2}(2\Delta_\phi+2d-7)}, \tag{4.20}$$

$$(\lambda_{J\phi[J\phi]^{[\ell,1]}_{0,\ell}})^2 = \frac{C_J\sqrt{\pi}(d/2-1)2^{-\Delta_\phi-d+4}}{2^\ell\Gamma(\Delta_\phi)\Gamma(d)}\ell^{\frac{1}{2}(2\Delta_\phi+2d-5)}. \tag{4.21}$$

**Next-to-leading order matching**

The next-to-leading order contribution in the t-channel comes from the stress tensor conformal block, which will contain a $\log(z)$ term.[3] Given that the s-channel blocks are proportional to $z^{\frac{\tau}{2}}$, it is clear that adding an anomalous dimension $\tau\to\tau+\gamma$ and expanding for small $\gamma$ reproduces the $\log(z)$. This implies that at

[3] This block is computed via a different operator (4.152) acting on the scalar block, thus the log term appears in analogy with (1.236)

next order in $O(1/\ell)$, the twists of the large-spin double-twist operators must be given by

$$\tau_{[J\phi]^{[\ell]}_{n,\ell}} = d - 2 + \Delta_\phi + 2n + \frac{\gamma_{[J\phi]^{[\ell]}_n}}{\ell^{d-2}} + \dots, \tag{4.22}$$

$$\tau_{[J\phi]^{[\ell,1]}_{n,\ell}} = d - 1 + \Delta_\phi + 2n + \frac{\gamma_{[J\phi]^{[\ell,1]}_n}}{\ell^{d-2}} + \dots. \tag{4.23}$$

The factor $\ell^{d-2}$ is chosen such that after the large spin sum in the s-channel, we get the correct $(1-\bar{z})$ factor multiplying the $\log(z)$ term in the t-channel stress tensor block.

At this point the $(1-\bar{z})$ dependence is compatible on both sides. What is left is to perform the sum over $n$ and demand that the terms $z^k \log(z)$ match for all $k$. For this to work, the $n$-dependent coefficients $\gamma_n$ in (4.22) and (4.23) must be

$$\begin{aligned}\gamma_{[J\phi]^{[\ell]}_n} = & \frac{(-1)^n n! \Gamma(\Delta_\phi - \frac{d}{2} + 1)\Gamma(\Delta_\phi + n - \frac{d}{2} + 1)}{(\frac{d}{2} - 1)_n \Gamma(\frac{d}{2} + 1)^2} \gamma_{[J\phi]^{[\ell]}_0} \\ & \times \sum_{i=0}^{n} \frac{(-1)^i (i+1)^2_{\frac{d}{2}} (\Delta_\phi + n - 1)_i}{(n-i)! \Gamma(\Delta_\phi - \frac{d}{2} + 1 + i)^2} {}_3F_2 \left( \begin{matrix} -i, -i, \Delta_\phi - d \\ -\frac{d}{2} - i, -\frac{d}{2} - i \end{matrix} ; 1 \right),\end{aligned} \tag{4.24}$$

$$\begin{aligned}\gamma_{[J\phi]^{[\ell,1]}_n} = & \frac{(-1)^n n! \Gamma(\Delta_\phi - \frac{d}{2} + 1)\Gamma(\Delta_\phi + n - \frac{d}{2} + 1)}{(\frac{d}{2})_n \Gamma(\frac{d}{2} + 1)^2} \gamma_{[J\phi]^{[\ell,1]}_0} \\ & \times \sum_{i=0}^{n} \frac{(-1)^i (i+1)^2_{\frac{d}{2}} (\Delta_\phi + n)_i}{(n-i)! \Gamma(\Delta_\phi - \frac{d}{2} + 1 + i)^2} {}_3F_2 \left( \begin{matrix} -i, -i, \Delta_\phi - d \\ -\frac{d}{2} - i, -\frac{d}{2} - i \end{matrix} ; 1 \right),\end{aligned} \tag{4.25}$$

where

$$\gamma_{[J\phi]^{[\ell]}_0} = \lambda_{\phi\phi T} \frac{(d-2)\Gamma(d+1)\Gamma(d+2)\Gamma(\Delta_\phi) \left[ d\Gamma\left(\frac{d}{2}\right) C_J - 4\pi^{\frac{d}{2}} \lambda_{JJT} \right]}{16\pi^{\frac{d}{2}} \sqrt{C_T} C_J \Gamma\left(\frac{d}{2} + 1\right)^3 \Gamma\left(-\frac{d}{2} + \Delta_\phi + 1\right)}, \tag{4.26}$$

$$\gamma_{[J\phi]^{[\ell,1]}_0} = \lambda_{\phi\phi T} \frac{\Gamma(d+1)\Gamma(d+2)\Gamma(\Delta_\phi) \left[ 2(d-1)\pi^{\frac{d}{2}} \lambda_{JJT} - (d-2)\Gamma\left(\frac{d}{2} + 1\right) C_J \right]}{4\pi^{\frac{d}{2}} \sqrt{C_T} C_J (d-2) \Gamma\left(\frac{d}{2} + 1\right)^3 \Gamma\left(-\frac{d}{2} + \Delta_\phi + 1\right)}. \tag{4.27}$$

Details of this calculation are given in Appendix B. Our conventions for the OPE coefficients and the two-point function normalizations $C_J$ and $C_T$ are given in Appendix C.

### 4.3.3 Bounds from $\langle V\phi\phi V\rangle$

We now generalize the above discussion to external non-conserved operators (4.8), for unitary $V$, i.e. $\Delta_V > d-1$. For symmetric traceless $\mathcal{O}$, the three-point function $\langle V\phi\mathcal{O}\rangle$, can be computed as

$$(b_1 D_{1\,12}\Sigma^{1,0} + b_2 D_{2\,12}\Sigma^{0,1})\frac{V_3^\ell}{P_{12}^{\frac{1}{2}(\Delta_V+\Delta_\phi-\Delta_\mathcal{O}-\ell)}P_{13}^{\frac{1}{2}(\Delta_V+\Delta_\mathcal{O}+\ell-\Delta_\phi)}P_{23}^{\frac{1}{2}(\Delta_\phi+\Delta_\mathcal{O}+\ell-\Delta_V)}}, \tag{4.28}$$

where $D_{l\,ij}$ are in (1.180) and $\Sigma^{i,j}$ increases the dimension of the operator at 1(2) by $i(j)$. Using these, one can compute the conformal blocks as in Appendix A. For mixed-symmetric $A = [\Delta_A, (\ell, 1)]$, the three-point function $\langle V\phi A\rangle$ has one tensor structure, as in the conserved case (c.f. (1.145)). In this case, the conformal blocks are the same as the ones for conserved $V$.

#### Leading order matching

Matching the identity block in the t-channel leads again to the existence of two classes of large spin double-twist operators $[V\phi]^{[\ell]}_{n,\ell}$ and $[V\phi]^{[\ell,1]}_{n,\ell}$, with twists

$$\tau_{[V\phi]^{[\ell]}_{n,\ell}}\Big|_{\ell\to\infty} = \tau_V + \Delta_\phi + 2n, \tag{4.29}$$

$$\tau_{[V\phi]^{[\ell,1]}_{n,\ell}}\Big|_{\ell\to\infty} = \tau_V + 1 + \Delta_\phi + 2n. \tag{4.30}$$

In addition, the effect of the $b_1$ term must be subleading at small $(1-\bar{z})$. Therefore the relevant OPE coefficient is $b_2 \equiv \lambda_{V\phi[V\phi]^{[\ell]}_{n,\ell}}$. For this case we also find the analogous triangular structure for polarizations (4.15). For $n = 0$, we find that the $z$ dependence is matched whenever the OPE coefficients are

$$(\lambda_{V\phi[V\phi]^{[\ell]}_{0,\ell}})^2 = \frac{\sqrt{\pi}2^{-\Delta_V-\Delta_\phi+4}}{\Gamma(\Delta_V+1)\Gamma(\Delta_\phi)}\ell^{\frac{1}{2}(2\Delta_V+2\Delta_\phi-5)}, \tag{4.31}$$

$$(\lambda_{V\phi[V\phi]^{[\ell,1]}_{0,\ell}})^2 = \frac{\sqrt{\pi}(\Delta_V-1)2^{-\Delta_V-\Delta_\phi+2}}{\Gamma(\Delta_V+1)\Gamma(\Delta_\phi)}\ell^{\frac{1}{2}(2\Delta_V+2\Delta_\phi-3)}, \tag{4.32}$$

where we normalized the two-point function $\langle VV\rangle$ as

$$\langle V^\mu(x)V^\nu(0)\rangle = \frac{I^{\mu\nu}(x)}{x_{12}^{2\Delta_V}}. \tag{4.33}$$

#### Next-to-leading order matching

For the next-to-leading order matching, the t-channel conformal block is computed by differential operator that changes $\langle\phi\phi T\rangle \to \langle VVT\rangle$. That is

$$\langle VVT\rangle = \Big(e_1 D_{1\,12}D_{1\,21} + e_2 D_{2\,12}D_{2\,21} + e_3 H_{12}\Big)\Sigma^{1,1}\frac{V_3^2}{P_{12}^{\Delta_V-d/2-1}P_{13}^{d/2+1}P_{23}^{d/2+1}} \tag{4.34}$$

The relation between the basis $e_i$ and the basis used in [138] is

$$e_1 = a_3, \qquad e_2 = -2a_2 - a_3, \qquad e_3 = a_1 - 2(d/2-1)a_2 + a_3(1-d), \tag{4.35}$$

where the Ward identity additionally imposes the condition

$$a_1 = -(\Delta_V - d + 1)(a_2 + a_3). \tag{4.36}$$

This implies that at next order in $O(1/\ell)$, the large spin double-twist operators $[V\phi]$ must have twists

$$\tau_{[V\phi]^{[\ell]}_{n,\ell}}\big|_{\ell\to\infty} = \tau_V + \Delta_\phi + 2n + \frac{\gamma_{[V\phi]^{[\ell]}_n}}{\ell^{d-2}} + \ldots, \tag{4.37}$$

$$\tau_{[V\phi]^{[\ell,1]}_{n,\ell}}\big|_{\ell\to\infty} = \tau_V + 1 + \Delta_\phi + 2n + \frac{\gamma_{[V\phi]^{[\ell,1]}_n}}{\ell^{d-2}} + \ldots, \tag{4.38}$$

where

$$\gamma_{[V\phi]^{[\ell]}_0} = \lambda_{\phi\phi T}\frac{\Gamma(d+2)\Gamma(\Delta_\phi)\Gamma(\Delta_V+1)}{4\sqrt{C_T}\Gamma\left(\frac{d}{2}+1\right)^2\Gamma\left(-\frac{d}{2}+\Delta_\phi+1\right)\Gamma\left(-\frac{d}{2}+\Delta_V+2\right)} \times\left[a_2\left(d^2-6d+4\Delta_V+4\right)+4a_3(-d+\Delta_V+1)\right], \tag{4.39}$$

$$\gamma_{[V\phi]^{[\ell,1]}_0} = \lambda_{\phi\phi T}\frac{\Gamma(d+2)\Gamma(\Delta_\phi)\Gamma(\Delta_V+1)}{2\sqrt{C_T}(\Delta_V-1)\Gamma\left(\frac{d}{2}+1\right)^2\Gamma\left(-\frac{d}{2}+\Delta_\phi+1\right)\Gamma\left(-\frac{d}{2}+\Delta_V+2\right)} \times\left[2a_2\left(-d\Delta_V+d+\Delta_V^2-1\right)+a_3\left(-d\Delta_V+2d+2\Delta_V^2-2\right)\right]. \tag{4.40}$$

### 4.3.4 Bounds from $\langle T\phi\phi T\rangle$

Now we repeat the same calculation for $\langle T\phi\phi T\rangle$. In contrast with the vector cases, now we four external indices. However, similar to the previous cases, we show in appendix A that at the s-channel expansion has the following triangular structure in the light-cone limit

$$\begin{aligned} G_T^{++++} &= G_{T,STT}^{++++}, \\ G_T^{+3+3} &= G_{T,STT}^{+3+3} + G_{T,A}^{+3+3}, \\ G_T^{34} &= G_{T,STT}^{34} + G_{T,A}^{34} + G_{T,B}^{34}, \end{aligned} \tag{4.41}$$

where $G_T^{34} \equiv \frac{1}{2}\langle (T^{33}-T^{44})\phi\phi(T^{33}-T^{44})\rangle$ so that we can ignore all trace terms in the four-point function tensor structures, thus simplifying our analysis.

Again, using this structure we can obtain the light-cone bootstrap constraints for each type of exchange operator separately.

### Leading order matching

At leading order in $(1-\bar{z})$, matching the identity implies the existence of large spin operators with the following twist values:

$$\tau_{[T\phi]^{[\ell]}_{n,\ell}}\Big|_{\ell\to\infty} = d-2+\Delta_\phi+2n, \tag{4.42}$$

$$\tau_{[T\phi]^{[\ell,1]}_{n,\ell}}\Big|_{\ell\to\infty} = d-1+\Delta_\phi+2n, \tag{4.43}$$

$$\tau_{[T\phi]^{[\ell,2]}_{n,\ell}}\Big|_{\ell\to\infty} = d+\Delta_\phi+2n. \tag{4.44}$$

Next we perform the sum over $n$ which leads to large $\ell$ asymptotics of the double twist OPE coefficients:

$$(\lambda_{T\phi[T\phi]^{[\ell]}_{n,\ell}})^2 = \frac{(1-\frac{d}{2}+\Delta_\phi)_n(\frac{d}{2}-1)_n}{4^n n!(\Delta_\phi+n-1)_n}(\lambda_{T\phi[T\phi]^{[\ell]}_{0,\ell}})^2, \tag{4.45}$$

$$(\lambda_{T\phi[T\phi]^{[\ell,1]}_{n,\ell}})^2 = \frac{(1-\frac{d}{2}+\Delta_\phi)_n(\frac{d}{2})_n}{4^n n!(\Delta_\phi+n)_n}(\lambda_{T\phi[T\phi]^{[\ell,1]}_{0,\ell}})^2, \tag{4.46}$$

$$(\lambda_{T\phi[T\phi]^{[\ell,2]}_{n,\ell}})^2 = \frac{(1-\frac{d}{2}+\Delta_\phi)_n(\frac{d}{2}+1)_n}{4^n n!(\Delta_\phi+n+1)_n}(\lambda_{T\phi[T\phi]^{[\ell,2]}_{0,\ell}})^2, \tag{4.47}$$

where

$$(\lambda_{T\phi[T\phi]^{[\ell]}_{0,\ell}})^2 = C_T\frac{\sqrt{\pi}2^{-\Delta_\phi-d+6}}{\Gamma(\Delta_\phi)\Gamma(d+2)}2^{-\ell}\ell^{\frac{1}{2}(-7+2\Delta_\phi+2d)}, \tag{4.48}$$

$$(\lambda_{T\phi[T\phi]^{[\ell,1]}_{0,\ell}})^2 = C_T\frac{\sqrt{\pi}(d-1)2^{-\Delta_\phi-d+5}}{\Gamma(\Delta_\phi)\Gamma(d+2)}2^{-\ell}\ell^{\frac{1}{2}(2\Delta_\phi+2d-5)}, \tag{4.49}$$

$$(\lambda_{T\phi[T\phi]^{[\ell,2]}_{0,\ell}})^2 = C_T\frac{\sqrt{\pi}d(d-1)2^{-\Delta_\phi-d+2}}{\Gamma(\Delta_\phi)\Gamma(d+2)}2^{-\ell}\ell^{\frac{1}{2}(2\Delta_\phi+2d-3)}. \tag{4.50}$$

Details on the summation over $n$ are given in appendix B.

### Next-to-leading order matching

Matching the $\log(z)$ terms in the t-channel stress-tensor contribution, we get that the large spin anomalous dimensions are corrected to:

$$\tau_{[T\phi]^{[\ell]}_{n,\ell}} = d-2+\Delta_\phi+2n+\frac{\gamma_{[T\phi]^{[\ell]}_n}}{\ell^{d-2}}+\dots, \tag{4.51}$$

$$\tau_{[T\phi]^{[\ell,1]}_{n,\ell}} = d-1+\Delta_\phi+2n+\frac{\gamma_{[T\phi]^{[\ell,1]}_n}}{\ell^{d-2}}+\dots, \tag{4.52}$$

$$\tau_{[T\phi]^{[\ell,2]}_{n,\ell}} = d+\Delta_\phi+2n+\frac{\gamma_{[T\phi]^{[\ell,2]}_n}}{\ell^{d-2}}+\dots. \tag{4.53}$$

Finally, performing the sum over $n$ and matching the $z$ dependence in the s- and t-channels results in the following anomalous dimensions coefficients:

$$\gamma_{[T\phi]_n^{[\ell]}} = \frac{(-1)^n n!\Gamma(\Delta_\phi - \frac{d}{2}+1)\Gamma(\Delta_\phi + n - \frac{d}{2}+1)}{(\frac{d}{2}-1)_n \Gamma(\frac{d}{2}+1)^2}\gamma_{[T\phi]_0^{[\ell]}} \times \sum_{i=0}^{n} \frac{(-1)^i (i+1)^2_{\frac{d}{2}} (\Delta_\phi + n - 1)_i}{(n-i)!\Gamma(\Delta_\phi - \frac{d}{2}+1+i)^2} {}_3F_2\left(\begin{matrix}-i,-i,\Delta_\phi - d \\ -\frac{d}{2}-i, -\frac{d}{2}-i\end{matrix};1\right), \quad (4.54)$$

$$\gamma_{[T\phi]_n^{[\ell,1]}} = \frac{(-1)^n n!\Gamma(\Delta_\phi - \frac{d}{2}+1)\Gamma(\Delta_\phi + n - \frac{d}{2}+1)}{(\frac{d}{2})_n \Gamma(\frac{d}{2}+1)^2}\gamma_{[T\phi]_0^{[\ell,1]}} \times \sum_{i=0}^{n} \frac{(-1)^i (i+1)^2_{\frac{d}{2}} (\Delta_\phi + n)_i}{(n-i)!\Gamma(\Delta_\phi - \frac{d}{2}+1+i)^2} {}_3F_2\left(\begin{matrix}-i,-i,\Delta_\phi - d \\ -\frac{d}{2}-i, -\frac{d}{2}-i\end{matrix};1\right), \quad (4.55)$$

$$\gamma_{[T\phi]_n^{[\ell,2]}} = \frac{(-1)^n n!\Gamma(\Delta_\phi - \frac{d}{2}+1)\Gamma(\Delta_\phi + n - \frac{d}{2}+1)}{(\frac{d}{2}+1)_n \Gamma(\frac{d}{2}+1)^2}\gamma_{[T\phi]_0^{[\ell,2]}} \times \sum_{i=0}^{n} \frac{(-1)^i (i+1)^2_{\frac{d}{2}} (\Delta_\phi + n + 1)_i}{(n-i)!\Gamma(\Delta_\phi - \frac{d}{2}+1+i)^2} {}_3F_2\left(\begin{matrix}-i,-i,\Delta_\phi - d \\ -\frac{d}{2}-i, -\frac{d}{2}-i\end{matrix};1\right), \quad (4.56)$$

where

$$\gamma_{[T\phi]_{0,\ell}^{[\ell]}} = \lambda_{\phi\phi T} \frac{2^{2d-5}(d-2)\pi^{-\frac{d}{2}-1}\Gamma\left(\frac{d-1}{2}\right)^2\Gamma(\Delta_\phi)}{\sqrt{C_T}(d-1)\Gamma\left(-\frac{d}{2}+\Delta_\phi+1\right)} \times \left[(d+1)((d-3)t_2+d-1)+((d-1)d-4)t_4\right], \quad (4.57)$$

$$\gamma_{[T\phi]_{0,\ell}^{[\ell,1]}} = \lambda_{\phi\phi T} \frac{2^{2d-5}\pi^{-\frac{d}{2}-1}\Gamma\left(\frac{d-1}{2}\right)^2\Gamma(\Delta_\phi)\left[(d+1)(d(t_2+2)-3t_2-2)-4t_4\right]}{\sqrt{C_T}\Gamma\left(-\frac{d}{2}+\Delta_\phi+1\right)}, \quad (4.58)$$

$$\gamma_{[T\phi]_{0,\ell}^{[\ell,2]}} = \lambda_{\phi\phi T} \frac{2^{2d-3}\pi^{-\frac{d}{2}-1}\Gamma\left(\frac{d-1}{2}\right)^2\Gamma(\Delta_\phi)\left[(d+1)(d-t_2-1)-2t_4\right]}{\sqrt{C_T}\Gamma\left(-\frac{d}{2}+\Delta_\phi+1\right)}. \quad (4.59)$$

## 4.4 Causality

### 4.4.1 Overview

Now we move on to studying the constraints from causality, presented in section 1.8, for $\langle J\phi\phi J\rangle$, $\langle V\phi\phi V\rangle$, and $\langle T\phi\phi T\rangle$. Recall that in Lorentzian signature the four-point function is multi-valued and has a complex structure of branch points and branch cuts given by the light-cones of operators within the correlator. Causality is trivially satisfied in the first sheet of Lorentzian correlators $G(z,\bar{z})$ (as in, e.g. (4.5)) that have been analyticity continued from Euclidean ones, following the recipie of section 1.8. This can be tested by checking that $G(z,\bar{z})$ is analytic away from the light-cones. However, constraints on the CFT data appear when requiring that causality holds on the second sheet $G(ze^{-2\pi i},\bar{z})$ also, which is obtained by taking the operator at $z$ around the branch point at $z=0$. In particular, we want to check that $G(ze^{-2\pi i},\bar{z})$ is analytic in a small region $(z,\bar{z})\sim(1,1)$ near the light-cone of $\phi(1)$. Following the discussion in subsection 1.8.2, the region of interest is

$$D=\{\,(\sigma\in\mathbb{C},\eta\in\mathbb{R})\mid \operatorname{Im}\sigma\geq 0,\,|\sigma|\leq R\in\mathbb{R},\,0<\eta\ll R\ll 1\,\},\tag{4.60}$$

where $\sigma$ and $\eta$ are defined by

$$z=1+\sigma,\tag{4.61}$$

$$\bar{z}=1+\eta\sigma.\tag{4.62}$$

With these coordinates, the objects of study are the normalized spinning four-point functions

$$\begin{aligned}G^{\mu\nu}_{J,\eta}(\sigma)&\equiv(\eta\sigma^2)^{\Delta_\phi}G^{\mu\nu}_{J}(1+\sigma,1+\eta\sigma),\\ \widehat{G}^{\mu\nu}_{J,\eta}(\sigma)&\equiv(\eta\sigma^2)^{\Delta_\phi}G^{\mu\nu}_{J}((1+\sigma)e^{-2\pi i},1+\eta\sigma),\end{aligned}\tag{4.63}$$

$$\begin{aligned}G^{\mu\nu}_{V,\eta}(\sigma)&\equiv(\eta\sigma^2)^{\Delta_\phi}G^{\mu\nu}_{V}(1+\sigma,1+\eta\sigma),\\ \widehat{G}^{\mu\nu}_{V,\eta}(\sigma)&\equiv(\eta\sigma^2)^{\Delta_\phi}G^{\mu\nu}_{V}((1+\sigma)e^{-2\pi i},1+\eta\sigma),\end{aligned}\tag{4.64}$$

and

$$\begin{aligned}G^{\mu\nu,\rho\sigma}_{T,\eta}(\sigma)&\equiv(\eta\sigma^2)^{\Delta_\phi}G^{\mu\nu,\rho\sigma}_{T}(1+\sigma,1+\eta\sigma),\\ \widehat{G}^{\mu\nu,\rho\sigma}_{T,\eta}(\sigma)&\equiv(\eta\sigma^2)^{\Delta_\phi}G^{\mu\nu,\rho\sigma}_{T}((1+\sigma)e^{-2\pi i},1+\eta\sigma),\end{aligned}\tag{4.65}$$

for $\langle J\phi\phi J\rangle$, $\langle V\phi\phi V\rangle$, and $\langle T\phi\phi T\rangle$ respectively.

As we saw in subsection 1.8.2, in order to check for analyticity on the second sheet, one uses reflection positivity of the Euclidean correlator to show that the s-channel expansion is a polynomial in $z,\bar{z}$ with positive coefficients. In the case of scalar correlators this implied that $\widehat{G}_\eta$ is analytic in $D$, and furthermore that $\operatorname{Re}(G_\eta(\sigma)-\widehat{G}_\eta(\sigma))\geq 0$ for $\sigma\in[-R,R]$.

### Reflection positivity for spinning correlators

For the case of spinning correlators, we consider the following states in the Hilbert space of radial quantization

$$|f,\epsilon\rangle = \int_0^1 dr_1 \int_0^{2\pi} d\theta_1 r_1^{\Delta_J+\Delta_\phi} f(r_1,\theta_1)\phi(r_1 e^{i\theta_1}, r_1 e^{-i\theta_1}) J^\mu \epsilon_\mu(0)\,|0\rangle\,, \tag{4.66}$$

$$\langle f,\epsilon^*| = \langle 0|\,\epsilon^*_\nu I^\nu_\rho(\infty\hat{x}^1) J^\rho(\infty\hat{x}^1) \int_0^1 dr_2 \int_0^{2\pi} d\theta_2 r_2^{\Delta_J-\Delta_\phi} f^*(r_2,\theta_2)\phi\left(\frac{1}{r_2}e^{i\theta_2}, \frac{1}{r_2}e^{-i\theta_2}\right), \tag{4.67}$$

where

$$I^{\mu\nu}(x) = \eta^{\mu\nu} - 2\frac{x^\mu x^\nu}{x^2}, \tag{4.68}$$

and $f(\infty\hat{x}^1) \equiv \lim_{r\to\infty} f(r\hat{x}^1)$ with $\hat{x}^1$, the unit vector pointing at the $x^1 = \frac{1}{2}(x^+ + x^-)$ direction. In contrast with the scalar case, the inversion tensor $I^\nu_\rho$ can get a sign for different polarizations. In particular $I^-_+(\infty\hat{x}^1) = -1$, implies that for $\epsilon_\mu$ pointing in the + direction, reflection positivity $\langle f,\epsilon^*|\, f,\epsilon\rangle \geq 0$ leads to a power series in $z,\bar{z}$ with negative semidefinite coefficients for the s-channel expansion of $G_J^{++}(z,\bar{z})$.

As in the scalar case, these positivity conditions lead to the analyticity of $\widehat{G}^{++}_{J,\eta}(\sigma)$ and $\widehat{G}^{tt}_{J,\eta}(\sigma)$ in $D$, and furthermore (notice the negative sign in $G_J^{++}$)

$$\mathrm{Re}(-G^{++}_{J,\eta}(\sigma) + \widehat{G}^{++}_{J,\eta}(\sigma)) \geq 0, \tag{4.69}$$

$$\mathrm{Re}(G^{tt}_{J,\eta}(\sigma) - \widehat{G}^{tt}_{J,\eta}(\sigma)) \geq 0, \quad \sigma\in[-R,R]. \tag{4.70}$$

For the case of $G_V$, a completely analogous expression holds. For $G_T$, only $G_T^{+t+t}$ changes sign:

$$\mathrm{Re}(G^{++++}_{T,\eta}(\sigma) - \widehat{G}^{++++}_{T,\eta}(\sigma)) \geq 0, \tag{4.71}$$

$$\mathrm{Re}(-G^{+3+3}_{T,\eta}(\sigma) + \widehat{G}^{+3+3}_{T,\eta}(\sigma)) \geq 0, \tag{4.72}$$

$$\mathrm{Re}(G^{34}_{T,\eta}(\sigma) - \widehat{G}^{34}_{T,\eta}(\sigma)) \geq 0, \quad \sigma\in[-R,R], \tag{4.73}$$

From the refined positivity condition $\langle f,\epsilon^*|P_{\mathcal{O}}|f,\epsilon\rangle \geq 0$, we see that analyticity and boundedness still holds for each of the exchanged representations (i.e. $STT$, $A$, $B$) that contribute to the s-channel correlator.

### Correlators on the second sheet

Note from the previous section that, in the light-cone, ($\bar{z}\to 1$, $z$ finite), the t-channel has a $\log(z)$ dependence which develops a term proportional to $2\pi i$ when going around $z = 0$. Therefore, given that the spinning correlators in the t-channel are obtained by applying differential operators on the scalar correlator, the spinning expressions for $\widehat{G}$ can be computed from (1.261) for $\mathcal{O}_m = T$, the stress-tensor. This leads, generically, to [4]

$$\widehat{G}_\eta(\sigma) = G_\eta(\sigma) - i\beta\frac{\eta^{\frac{d}{2}-1}}{\sigma} + \dots . \qquad (4.74)$$

Then following the integration procedure in subsection 1.8.2 implies

$$\beta = \frac{1}{\pi}\lim_{R\to 0}\lim_{\eta\to 0}\eta^{1-\frac{d}{2}}\int_{-R}^{R} d\sigma\,\mathrm{Re}(G_\eta(\sigma) - \widehat{G}_\eta(\sigma)), \qquad (4.75)$$

where the sign of the integrand implies a constraint on the sign of $\beta$.

### Constraints from different contributions to the s-channel

Recall that in our polarizations, the s-channel has a triangular structure in the light-cone: (4.15), (4.41). Moreover, using light-cone bootstrap techniques, we have shown in Appendix A that, order-by-order in $(1-\bar{z})$, different polarizations of the same exchanged operator are related to each other. More precisely, for $G_J$, we have that the s-channel expansion must be

$$\begin{aligned} G^{++}_{J,STT} &= (1-\bar{z})^{-\Delta_\phi} g^{++}_{J,STT}(z) + (1-\bar{z})^{-\Delta_\phi+\frac{d}{2}-1} h^{++}_{J,STT}(z) + \cdots , \\ G^{tt}_{J,STT} &= (1-\bar{z})^{-\Delta_\phi} g^{tt}_{J,STT}(z) + (1-\bar{z})^{-\Delta_\phi+\frac{d}{2}-1} h^{tt}_{J,STT}(z) + \cdots , \end{aligned} \qquad (4.76)$$

with $g^{tt}_{J,STT}(z) = \frac{g^{++}_{J,STT}(z)}{2(1-d)}$ and $h^{tt}_{J,STT}(z) = -\frac{h^{++}_{J,STT}(z)}{d}$. Using these results in the second line of (4.15), implies that the contribution from the operator $A$ is

$$\begin{aligned} G^{tt}_{J,A}(z,\bar{z}) &= G^{tt}_{J}(z,\bar{z}) - G^{tt}_{J,STT}(z,\bar{z}) \\ &= G^{tt}_{J}(z,\bar{z}) - (1-\bar{z})^{-\Delta_\phi} g^{tt}_{J,STT}(z) - (1-\bar{z})^{-\Delta_\phi+\frac{d}{2}-1} h^{tt}_{J,STT}(z) - \cdots \\ &= G^{tt}_{J}(z,\bar{z}) - (1-\bar{z})^{-\Delta_\phi}\frac{g^{++}_{J,STT}(z)}{2(1-d)} + (1-\bar{z})^{-\Delta_\phi+\frac{d}{2}-1}\frac{h^{++}_{J,STT}(z)}{d} - \cdots . \end{aligned} \qquad (4.77)$$

But from the first line of (4.15), we see that the functions $g^{++}_{J,STT}$ and $h^{++}_{J,STT}$ can be computed from the t-channel expansion of $G^{++}_J$. Indeed the term $g^{++}_{J,STT}(z)$ is the coefficient of $(1-\bar{z})^{-\Delta_\phi}$ in the identity exchange, while $h^{++}_{J,STT}(z)$ is the coefficient of $(1-\bar{z})^{-\Delta_\phi+\frac{d}{2}-1}$ in the stress-tensor exchange. Therefore (4.77) provides a formula for computing the t-channel expansion of $G^{tt}_J$, coming only from the exchange of the operators $A$ in the s-channel.

[4]For the cases with the sign reversed, e.g. $G^{++}_J$, we have $-\widehat{G}_\eta(\sigma) = -G_\eta(\sigma) - i\beta\frac{\eta^{\frac{d}{2}-1}}{\sigma} + \dots$ and $\beta = \frac{1}{\pi}\lim_{R\to 0}\lim_{\eta\to 0}\eta^{1-\frac{d}{2}}\int_{-R}^{R} d\sigma\,\mathrm{Re}(\widehat{G}_\eta(\sigma) - G_\eta(\sigma))$.

With (4.77) we define the normalized t-channel expansion and its continuation to the second sheet:

$$\begin{aligned} G^{tt}_{J,\eta,(A)}(\sigma) &= (\eta\sigma^2)^{\Delta_\phi} G^{tt}_{J,A}(1+\sigma, 1+\eta\sigma), \\ \widehat{G}^{tt}_{J,\eta,(A)}(\sigma) &= (\eta\sigma^2)^{\Delta_\phi} G^{tt}_{J,A}((1+\sigma)e^{-2\pi i}, 1+\eta\sigma). \end{aligned} \tag{4.78}$$

From the positivity arguments discussed above, these functions satisfy the same relations as (4.70). Note that in the ++ polarization, the only contribution is from $STT$. Thus $G^{++}_{J,\eta,(STT)}(\sigma)$ is simply $G^{++}_{J,\eta}(\sigma)$. For $G_V$ we have a completely analogous story with $g^{tt}_{V,STT}(z) = -\frac{g^{++}_{V,STT}(z)}{2\Delta_V}$ and $h^{tt}_{V,STT}(z) = -\frac{h^{++}_{V,STT}(z)}{2(\Delta_V+1-d/2)}$. While for $G_T$ the relations between different polarizations are summarized in table 4.1. Thus the contribution of each s-channel exchange to the t-channel expansion is computed analogously, and we define the following normalized correlators (for $V$ we have the same definitions (4.78) with $J \to V$)

$$\begin{aligned} G^{+3+3}_{T,\eta,(A)}(\sigma) &= (\eta\sigma^2)^{\Delta_\phi} G^{+3+3}_{T,A}(1+\sigma, 1+\eta\sigma), \\ \widehat{G}^{+3+3}_{T,\eta,(A)}(\sigma) &= (\eta\sigma^2)^{\Delta_\phi} G^{+3+3}_{T,A}((1+\sigma)e^{-2\pi i}, 1+\eta\sigma), \end{aligned} \tag{4.79}$$

$$\begin{aligned} G^{34}_{T,\eta,(B)}(\sigma) &= (\eta\sigma^2)^{\Delta_\phi} G^{34}_{T,B}(1+\sigma, 1+\eta\sigma), \\ \widehat{G}^{34}_{T,\eta,(B)}(\sigma) &= (\eta\sigma^2)^{\Delta_\phi} G^{34}_{T,B}((1+\sigma)e^{-2\pi i}, 1+\eta\sigma). \end{aligned} \tag{4.80}$$

In the rest of the section we will apply (4.75) and the positivity conditions presented above to these normalized correlators,[5] in order to find bounds on the OPE coefficients of $\langle JJT\rangle$, $\langle VVT\rangle$, and $\langle TTT\rangle$.

### 4.4.2 Bounds from $\langle J\phi\phi J\rangle$

$\langle J^+\phi\phi J^+\rangle$

Using the t-channel spinning conformal blocks for stress tensor exchange, the correlation function on the second sheet $\widehat{G}^{++}_\eta(\sigma)$ at next-to-leading order in $\eta$ is:

$$\begin{aligned} -\widehat{G}^{++}_{J,\eta}(\sigma) = 2C_J - i\lambda_{\phi\phi T} & \frac{d(d-2)\Gamma(d+1)}{\pi^{\frac{d}{2}}\sqrt{C_T}\Gamma\left(\frac{d}{2}+1\right)^3} \frac{\eta^{\frac{d}{2}-1}}{\sigma} \\ & \times \left[2^{d+2}\pi^{\frac{d+1}{2}}\Gamma\left(\frac{d+3}{2}\right)\lambda_{JJT} - \pi\Gamma(d+2)C_J\right] + \ldots, \end{aligned} \tag{4.81}$$

$$\lambda_{\phi\phi T} = -\frac{d\Delta_\phi}{d-1}\frac{1}{\sqrt{C_T}}, \tag{4.82}$$

[5]One could alternatively use the full t-channel expansion $G^{tt}_J$, and similar for $G_T$, but the bounds will be weaker as shown in [95].

where the form of $\lambda_{\phi\phi T}$ comes from the Ward identities [137].

Note that for this polarization, the positivity condition reverses it sign (4.69), thus using the integral in footnote 4, we have

$$\pi\Gamma(d+2)C_J - 2^{d+2}\pi^{\frac{d+1}{2}}\Gamma\left(\frac{d+3}{2}\right)\lambda_{JJT}$$
$$= N \lim_{R\to 0}\lim_{\eta\to 0}\eta^{1-\frac{d}{2}}\int_{-R}^{R} d\sigma\, \mathrm{Re}(-G^{++}_{J,\eta}(\sigma)+\widehat{G}^{++}_{J,\eta}(\sigma)) \geq 0, \quad (4.83)$$

where $N$ is a positive constant. This implies

$$\lambda_{JJT} \leq \frac{\Gamma\left(\frac{d}{2}+1\right)}{2\pi^{\frac{d}{2}}}C_J, \quad (4.84)$$

which is one of the conformal collider bounds, saturated by a theory of free fermions.

$\langle J^t \phi\phi J^t\rangle$

With the arguments of the previous subsection, we compute the normalized contribution from the operator $A$ to the t-channel expansion via (4.77). Then the continuation to the second sheet, defined in (4.78), results in

$$\widehat{G}^{tt}_{J,\eta,(A)}(\sigma) = \frac{d-2}{d-1}C_J + i\lambda_{\phi\phi T}\frac{2\Gamma(d+2)}{\pi^{\frac{d}{2}}\sqrt{C_T}\Gamma\left(\frac{d}{2}+1\right)^3}\frac{\eta^{\frac{d}{2}-1}}{\sigma}$$
$$\times\left[2^{d+1}\pi^{\frac{d+1}{2}}(d-1)\Gamma\left(\frac{d+1}{2}\right)\lambda_{JJT} - \pi(d-2)\Gamma(d+1)C_J\right] + \ldots. \quad (4.85)$$

Hence by the integral (4.75) and the positivity condition (4.70) (recall that from the refined positivity condition, the same relation applies to $G_{J,A}$)

$$\lambda_{JJT} \geq \frac{(d-2)\Gamma(\frac{d}{2}+1)}{2(d-1)\pi^{\frac{d}{2}}}C_J. \quad (4.86)$$

This inequality is the other conformal collider bound on $\langle JJT\rangle$, saturated by a theory of free bosons.

Note that the supersymmetric conformal collider bounds follow from the general bounds derived above [136]. If the current is not the R symmetry current, then it is contained in a multiplet with a scalar. In this case supersymmetry fixes $\lambda_{JJT}$ in terms of $C_J$ via [136, 137]

$$\lambda_{JJT} = \frac{d(d-2)\Gamma\left(\frac{d}{2}+1\right)}{2(d-1)^2\pi^{\frac{d}{2}}}C_J, \quad (4.87)$$

which satisfies the conformal collider bounds. In $d = 4$ with $\mathcal{N} = 1$ supersymmetry and assuming $J$ is the superconformal $\mathrm{U}(1)_R$ current, then we have[6]

$$\lambda_{JJT} = \frac{2(a+3c)}{9c\pi^2} C_J \quad \Rightarrow \quad \frac{3}{2} \geq \frac{a}{c} \geq 0, \tag{4.88}$$

where $a$ is the Euler anomaly and $c$ is proportional to the central charge $C_T$ (see appendix C for the precise relation). However, this is not the strongest lower bound for $\frac{a}{c}$. A stronger bound, $\frac{3}{2} \geq \frac{a}{c} \geq \frac{1}{2}$, comes from the stress tensor bounds of subsection 4.4.4. For $\mathcal{N} = 2$ and with $J$ the superconformal $\mathrm{SU}(2)_R$ current, we have instead

$$\lambda_{JJT} = \frac{4(a+c)}{9c\pi^2} C_J \quad \Rightarrow \quad \frac{5}{4} \geq \frac{a}{c} \geq \frac{1}{2}, \tag{4.89}$$

which are the strongest bounds for $d = 4$ $\mathcal{N} = 2$ theories. Finally for $\mathcal{N} = 4$ we have $a = c$, so the bounds are always satisfied.

### 4.4.3 Bounds from $\langle V\phi\phi V\rangle$

Now we repeat the exact same procedure for the non-conserved vector $V$. We compute the normalized correlator in the second sheet via (4.64) for the $++$ polarization, and use the definitions (4.78) and (4.77), with $J \to V$ and the corresponding polarization ratios, to compute the contribution from $A$ to the t-channel. We find

$$\begin{aligned} -\widehat{G}^{++}_{V,\eta}(\sigma) &= 2 + i\lambda_{\phi\phi T} \frac{2^{2d+1}\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{d+3}{2}\right)}{\sqrt{C_T}\Gamma\left(\frac{d}{2}+1\right)^2} \frac{\eta^{\frac{d}{2}-1}}{\sigma} \\ &\quad \times \left[a_2\left(d^2 - 6d + 4\Delta_V + 4\right) + 4a_3(-d + \Delta_V + 1)\right] + \ldots \end{aligned} \tag{4.90}$$

$$\begin{aligned} \widehat{G}^{tt}_{V,\eta,(A)}(\sigma) &= \frac{\Delta_V - 1}{\Delta_V} - i\lambda_{\phi\phi T} \frac{4^{d+1}\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{d+3}{2}\right)}{\sqrt{C_T}\Gamma\left(\frac{d}{2}+1\right)^2} \frac{\eta^{\frac{d}{2}-1}}{\sigma} \\ &\times \left[\frac{a_3\left(d(\Delta_V - 2) - 2\Delta_V^2 + 2\right) + 2(\Delta_V - 1)a_2(d - \Delta_V - 1)}{(2(\Delta_V + 1) - d)}\right] + \ldots . \end{aligned} \tag{4.91}$$

Using the integral (4.75) (and the one in footnote 4 for the $++$ polarization) and the analogous positivity conditions to (4.69), (4.70), gives the bounds

$$a_2 \leq 0 \quad \& \quad a_3 \geq \frac{a_2\left(d^2 - 6d + 4\Delta_V + 4\right)}{4(d - \Delta_V - 1)}, \tag{4.92}$$

or

$$a_2 > 0 \quad \& \quad a_3 \geq \frac{2a_2\left(-d\Delta_V + d + \Delta_V^2 - 1\right)}{d(\Delta_V - 2) - 2\Delta_V^2 + 2}, \tag{4.93}$$

[6] These relations are straightforwardly derived using the covariant formalism of [139, 140] or in superembedding space [141–149].

where we assumed $\Delta_V > d-1$ satisfies the unitarity bound as a non-conserved vector.

In [138] it was observed that the bounds derived via deep inelastic scattering (DIS) are weaker than the bounds derived from the positivity of the energy one-point function $\langle \mathcal{E}(\vec{n}) \rangle$. Our results coincide with the results of the DIS argument.

### 4.4.4 Bounds from $\langle T\phi\phi T \rangle$

Now for the stress-tensor case, we will parametrize $\langle TTT \rangle$ in general dimensions by $C_T$, $t_2$, and $t_4$, where $C_T$ is the central charge which appears in the 2-point function of the stress tensor. The relation between $t_2$, $t_4$ and the basis used in [137] is given in appendix C. The procedure for obtaining the OPE coefficient bounds is completely analogous to the previous two cases.

$\langle T^{++}\phi\phi T^{++} \rangle$

The t-channel conformal block of the stress energy tensor can be used to compute the following normalized correlation function on the second sheet, defined in (4.65):

$$\widehat{G}^{++++}_{T,\eta}(\sigma) = 4C_T \; + i\lambda_{\phi\phi T} \frac{\sqrt{C_T} 2^d \pi^{\frac{1}{2}-\frac{d}{2}} (d-2)(d+4)\Gamma\left(\frac{d+3}{2}\right)\Gamma(d+3)}{\left(d^2-1\right)^3 \Gamma\left(\frac{d}{2}+1\right)^2} \frac{\eta^{\frac{d}{2}-1}}{\sigma}$$
$$\times \left[(d+1)((d-3)t_2+d-1)+((d-1)d-4)t_4\right]+\dots. \tag{4.94}$$

Applying the integral (4.75), and reflection positivity (4.71) implies the bound

$$\left(1-\frac{1}{d-1}t_2-\frac{2}{d^2-1}t_4\right)+\frac{d-2}{d-1}(t_2+t_4)\geq 0. \tag{4.95}$$

$\langle T^{+t}\phi\phi T^{+t} \rangle$

Now we isolate the $G^{+3+3}_{T,A}$ contribution by the analogue of (4.77), where $G^{+3+3}_{T}$ is computed with the t-channel conformal block, and we use (4.94) to extract the factors $g^{++++}_{T,STT}$ and $h^{++++}_{T,STT}$. Using the definitions (4.79) we obtain:

$$-\widehat{G}^{+3,+3}_{T,\eta,(A)}(\sigma) = \frac{d-1}{1+d}C_T + i\lambda_{\phi\phi T} \frac{\sqrt{C_T} 2^{d-3} \pi^{\frac{1}{2}-\frac{d}{2}} \Gamma\left(\frac{d-1}{2}\right)\Gamma(d+3)}{(d+1)^2 \Gamma\left(\frac{d}{2}+1\right)^2} \frac{\eta^{\frac{d}{2}-1}}{\sigma}$$
$$\times \left[(d+1)(d(t_2+2)-3t_2-2)-4t_4\right]+\dots. \tag{4.96}$$

Then using the footnote 4 and (4.72) implies the bound

$$\left(1-\frac{1}{d-1}t_2-\frac{2}{d^2-1}t_4\right)+\frac{1}{2}t_2\geq 0. \tag{4.97}$$

$\langle T^{tt} \phi\phi T^{tt} \rangle$

Finally to get the contribution of the $B$ operators, due to the triangular structure (4.41), we need to subtract the contributions from $STT$ and $A$. This is done by a straightforward generalization of (4.77) to include the polarization ratios for $A$ given in table 4.1. Thus $G_T^{34}$ is computed with the t-channel conformal block of the stress tensor, and we relate $G^{34}_{T,STT}$ to $G^{++++}_{T,STT}$, $G^{34}_{T,A}$ to $G^{+3+3}_{T,A}$, and compute the coefficients with (4.94), and (4.96). The result is

$$\widehat{G}^{34}_{T,B}(\sigma) = \frac{(d-1)^2}{d(d+1)} C_T + i\lambda_{\phi\phi T} \frac{\sqrt{C_T} 2^{2d+1} \pi^{-\frac{d}{2}} \Gamma\left(\frac{d+1}{2}\right)^2}{(d+1)\Gamma\left(\frac{d}{2}+1\right)} \frac{\eta^{\frac{d}{2}-1}}{\sigma} \times [(d+1)(d-t_2-1)-2t_4] + \dots . \tag{4.98}$$

Then using (4.75) and (4.73) implies the bound

$$\left(1 - \frac{1}{d-1} t_2 - \frac{2}{d^2-1} t_4\right) \geq 0. \tag{4.99}$$

Each of the bounds (4.95), (4.97), and (4.99) corresponds to a conformal collider bound in general dimensions as can be seen from [150,151]. Furthermore, in $d = 3$, there is a degeneracy in tensor structures (see Appendix C) which implies $t_2 = 0$, and the third bound becomes equivalent to the second. In figure 4.1 we illustrate the bounds in $t_2$, $t_4$ in a few different dimensions.

In $d = 4$ these bounds are given by

$$\frac{31}{18} \geq \frac{a}{c} \geq \frac{1}{3}, \tag{4.100}$$

in the basis $a$ and $c$, as described in Appendix C. These are also strengthened in the presence of supersymmetry. For $\mathcal{N} = 1$ supersymmetry, we have the relations

$$t_2 = 6\left(1 - \frac{a}{c}\right), \quad t_4 = 0 \quad \Rightarrow \quad \frac{3}{2} \geq \frac{a}{c} \geq \frac{1}{2}, \tag{4.101}$$

which is a stronger version of the lower bound that we obtained from subsection 4.4.2 for $U(1)_R$ currents. For $\mathcal{N} = 2$, the lower bound is identical to the one in (4.89) but the upper bound is weaker [152], so there are no new constraints from $\langle TTT \rangle$. Finally for $\mathcal{N} = 4$, where $a = c$, the bounds are trivially satisfied.

## 4.5 Mixing light-cone bootstrap with causality

Comparing the anomalous dimensions (4.26) and (4.27) with the inequalities (4.84) and (4.86), and the negativity of $\lambda_{\phi\phi T}$ (4.82) (for unitarity theories), proves that in

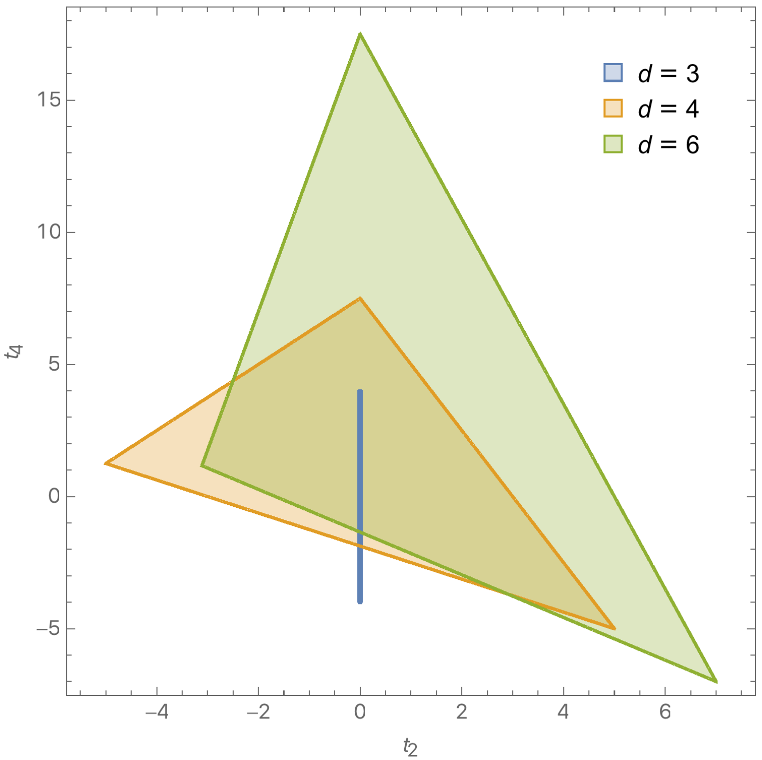


Figure 4.1: Conformal collider bounds in several dimensions $d$. Filled regions correspond to the allowed values of the parameters $t_2$, $t_4$.

the large $\ell$ limit, the symmetric traceless double-twist states $[J\phi]_{0,\ell}^{[\ell]}$ and the mixed symmetry double-twist states $[J\phi]_{0,\ell}^{[\ell,1]}$ have negative anomalous dimensions arising from the exchange of the stress tensor. In fact, in the regime when $\ell \gg n \geq 0$, the formulas (4.24) and (4.25) for $\gamma_{[J\phi]_{n,\ell}^{[\ell]}}$ and $\gamma_{[J\phi]_{n,\ell}^{[\ell,1]}}$ respectively, imply that they are all negative semidefinite because of the conformal collider bounds.

In a quantum gravitational theory in AdS dual to a CFT, the double-twist states correspond to two-particle bound states and the anomalous dimensions from the $T$ exchange correspond to the gravitational binding energy between the particles [88]. Hence the negativity of the anomalous dimensions implies that AdS gravity is attractive at super-horizon distances. From the point of view of large $N$ CFTs, it is understood that causality of bulk gravity is related to the collider bounds [150–156] and furthermore the anomalous dimensions of double-twist states formed from scalars are related to Shapiro time delay (related to causality) in the bulk [157–160]. Hence bulk causality, collider bounds and attractiveness of gravity at long distances have been shown to be intimately connected for large $N$ theories. However, unlike the literature cited above, we have found that the negativity of the anomalous dimensions is a consequence of unitarity and crossing symmetry of the CFT alone, and it does not rely on any large $N$ limit. We have therefore provided a field theoretical proof that the dual gravitational theory in AdS must be attractive between a scalar particle and a gauge boson separated at super-horizon distances. Furthermore, because of (4.95), (4.97), (4.99), and $\lambda_{\phi\phi T} < 0$, the anomalous dimensions (4.54)–(4.59) are all negative too for $n > 0$ in the $\ell \gg n$ limit. Hence gravity is also attractive between a scalar particle and a graviton separated by super-horizon distances.

## 4.6 Discussion

In this chapter, we have successfully applied two analytical methods for constraining the space of consistent conformal field theories (CFT). In the first part we used the light-cone bootstrap and found that the strong connection between low-twist operators and large-spin double-trace operators, first found in [88, 89], still holds for the spinning correlators $\langle J\phi\phi J\rangle$, $\langle V\phi\phi V\rangle$, and $\langle T\phi\phi T\rangle$. Techniques of this type have crucial relevance to quantum gravitational theories in AdS. For example, AdS observables such as binding energies [88, 89, 91, 92] and Eikonal phases [157–159] are directly related to the anomalous dimensions of classes of CFT operators with large dimensions and large spin. The fact that this can be achieved shows the power of the bootstrap methods. In its strongest form, unitarity and crossing symmetry may contain all necessary information to classify the whole landscape of conformal field theories without any extra input. If it is possible to define all CFTs through the bootstrap method, it would be a significant step forward in the understanding of strongly coupled quantum field theory as they are connected to CFTs by renormalization group flows.

In the second part of the chapter we have used constraints from causality of the Lorentzian CFT to prove that the "conformal collider bounds", originally proposed in [136], hold for any unitary parity-preserving conformal field theory (CFT) in arbitrary dimensions $d \geq 3$, where the lowest non-trivial twist operator is a unique stress tensor. While there was a large amount of evidence suggesting the result was indeed correct, the material presented here, based on [2], is a proof purely based on unitarity, conformal symmetry, and quantum field theory axioms.

Interestingly, the combination of both techniques implies that the gravitational interaction is attractive between two particles separated by super-horizon distances in AdS, as a direct consequence of the unitarity of the underlying quantum theory. More generally, once we understand CFTs, we can use them as starting points to answer important questions in quantum gravity. An especially exciting question is the quantum origin of universal features of gravitational interactions, such as causality, (non-)locality and attractiveness [88–90, 132, 161–163]. The bounds proved in this chapter are directly related to properties of 3-particle vertices in the bulk, including at least one graviton [136].

# Appendix A: Spinning Conformal Blocks at Large Spin

In this appendix we derive the relevant s-channel spinning conformal blocks in the light-cone limit. We find that they can be written as (derivatives of) a single scalar conformal block. This simplification occurs because the sum over spins in the s-channel is performed via

$$\int_0^\infty d\ell\, \ell^\alpha K_\nu(2\ell\sqrt{1-\bar{z}}) = \frac{(1-\bar{z})^{-(\alpha+1)/2}}{4}\Gamma\left(\frac{1+\alpha-\nu}{2}\right)\Gamma\left(\frac{1+\alpha+\nu}{2}\right), \tag{4.102}$$

so that terms with higher powers in $1/\ell$ result in subleading terms in $(1-\bar{z})$, for $(1-\bar{z}) \ll 1$ and $\ell \gg 1$ with $(1-\bar{z})\ell^2 \lesssim 1$. Therefore the idea is to count the relative powers of $1/\ell$, where $(1-\bar{z})$ has a weight of $O(1/\ell^2)$.

Our strategy will be to write the conformal blocks as differential operators acting on a basic set of 'seed' blocks, following the general approach developed in [1, 29, 30, 34, 45, 54, 55]. For the seed blocks we expect, motivated by the results in [1], that they can be written as $g_{\text{seed}} \sim g_{\text{scalar}} + (\dots)$, where the term $(\dots)$ includes $STT$ conformal blocks of higher-spin correlators (see for example Eq. (4.87) in that paper). Moreover one can check, using the results of [45], that the spinning blocks in $(\dots)$ are sub-leading in $1/\ell$ with respect to $g_{\text{scalar}}$.[7] For blocks that can be derived from seed blocks, the simplifications can be inferred by looking at the differential operators of [45] when acting on seeds.

The projection of these results into different polarizations gives an explicit check of the the triangular structure (4.41) for finite $z$. Furthermore, contributions from each irreducible representation are related by a $z$-independent factor, at each order in $(1-\bar{z})$.

**Seed Blocks:** First we look at the seed conformal blocks for the $[\ell, 1]$ and $[\ell, 2]$ representations that appear in $\langle J\phi\phi J\rangle$ and $\langle T\phi\phi T\rangle$ respectively.

Notice that the simplification for $[\ell, 1]$ can be easily obtained by taking the $\bar{z} \to 1$, $(1-\bar{z})\ell^2 \lesssim 1$ limit in the expressions for $g_A$ given in (4.87)–(4.91) of [1]. Nonetheless we include this calculation given that the logic is the same as for the $[\ell, 2]$ blocks, where the explicit expressions are not known yet.

[7] The differential operators are constructed in such a way that the spin is increased while maintaining the dimensions of the original three-point function. Thus there are no relative powers of $(1-\bar{z})$ coming from the difference in external dimensions (see (4.114)). The sub-leading powers of $\ell$ then come from the matrix transforming the differential basis to the standard one.

**$[\ell, 1]$ Seed**

The integral representation of the $[\ell, 1]$ conformal block in $\langle J\phi\phi J\rangle$ is given by

$$\begin{aligned} & g_A^{\Delta_{J\phi},\Delta_{\phi J},\mu\nu}(z,\bar z) \\ & = \frac{\mathcal{N}_A(\lambda_{\phi J\widetilde{A}}/\lambda_{\phi JA})}{X_{\Delta_i}} \int d^d x_0 \, \langle J_\mu(x_1)\phi(x_2)A(x_0)\rangle \Pi^{[\ell,1]} \langle \widetilde{A}(x_0)\phi(x_3)J_\nu(x_4)\rangle, \end{aligned} \tag{4.103}$$

where the tensor contraction is $\Pi^{[\ell,1]} = m^{(10)}_{\mu\rho}\mathcal{P}^{[\ell,1]\,\rho}_{\sigma} m^{(40)\,\sigma}_{\nu}$, with

$$\mathcal{P}^{[\ell,1]\,\rho}_{\sigma}(k^{(012)}, k^{(034)}) \equiv k^{(012)}_{\rho_1}\cdots k^{(012)}_{\rho_\ell}\Pi^{[\ell,1]\,\rho\rho_1\cdots\rho_\ell}_{\sigma\sigma_1\cdots\sigma_\ell} k^{(034)\,\sigma_1}\cdots k^{(034)\,\sigma_\ell}. \tag{4.104}$$

Here $k$ and $m$ are given in (A.3) and (A.4) of [1] respectively, $\tilde{A}$ is a shadow operator, and the integral has an implicit monodromy projection (as discussed in [28]). Using the results of [123] we can write this tensor as

$$\mathcal{P}^{[\ell,1]\,\rho}_{\sigma}(X,Y) = \frac{1}{\ell+1}\left(\ell\delta^\rho_\sigma + \frac{X^2\partial_\sigma\partial^\rho - (\ell-1)X^\rho\partial_\sigma}{d+\ell-3} - X_\sigma\partial^\rho\right)\mathcal{P}^{[\ell]}(X,Y), \tag{4.105}$$

where $\mathcal{P}^{[\ell]}$ is the traceless-symmetric contraction of $\ell$ indices,

$$\mathcal{P}^{[\ell]}(X,Y) = X_{a_1}\cdots X_{a_\ell}\Pi^{[\ell]\,a_1\cdots a_\ell}_{b_1\cdots b_\ell} Y^{b_1}\cdots Y^{b_\ell}. \tag{4.106}$$

Notice that derivatives acting on $\mathcal{P}^{[\ell]}$ are structures that appear in $STT$ spinning blocks, and thus sub-leading with respect to scalar blocks. Therefore keeping only the first term in (4.105) leads to a single scalar block times a tensor structure,

$$\begin{aligned} & g_A^{\Delta_{J\phi},\Delta_{\phi J},\mu\nu}(z,\bar z) \\ & = \frac{\mathcal{N}_A(\lambda_{\phi J\widetilde{A}}/\lambda_{\phi JA})}{\mathcal{N}_{\mathcal{O}}(\lambda_{\phi J\widetilde{\mathcal{O}}}/\lambda_{\phi J\mathcal{O}})} g^{\Delta_{J\phi},\Delta_{\phi J}}_{(\Delta_A,\ell)}(z,\bar z)\left(m^{(14)}_{\mu\nu} + 2(z\bar z)^{-\frac{1}{2}}k^{(124)}_\mu k^{(413)}_\nu\right) + O(1/\ell), \end{aligned} \tag{4.107}$$

where the prefactor is given in (G.1) of [1]. Notice that this prefactor can always be set to one by changing the normalization of $A$. Evaluating the relevant polarizations leads to

$$\begin{aligned} g_A^{\Delta_{J\phi},\Delta_{\phi J},++}(z,\bar z) &= O((1-\bar z)^1), \\ g_A^{\Delta_{J\phi},\Delta_{\phi J},tt}(z,\bar z) &= g^{\Delta_{J\phi},\Delta_{\phi J}}_{\Delta_A,\ell}(z,\bar z) + O(1/\ell), \end{aligned} \tag{4.108}$$

where the tensor structure of the last term is of order $O((1-\bar z)^0)$.

**$[\ell, 2]$ Seed**

For $\langle T\phi\phi T\rangle$ we have

$$\begin{aligned} & g_B^{\Delta_{T\phi},\Delta_{\phi T},\mu\nu\rho\sigma}(z,\bar z) \\ & = \frac{\mathcal{N}_B(\lambda_{\phi T\widetilde{B}}/\lambda_{\phi TB})}{X_{\Delta_i}} \int d^d x_0 \, \langle T_{\mu\nu}(x_1)\phi(x_2)B(x_0)\rangle \Pi^{[\ell,2]} \langle \widetilde{B}(x_0)\phi(x_3)T_{\rho\sigma}(x_4)\rangle, \end{aligned} \tag{4.109}$$

where $B \in [\ell,2]$. Here the contraction is $\Pi^{[\ell,2]} = m^{(10)}_{\mu\alpha_1} m^{(10)}_{\nu\alpha_2} \mathcal{P}^{[\ell,2]\,\alpha_1\alpha_2}_{\beta_1\beta_2} m^{(40)\,\beta_1}_{\rho} m^{(40)\,\beta_2}_{\sigma}$, with [123]

$$\mathcal{P}^{[\ell,2]\,\alpha_1\alpha_2}_{\beta_1\beta_2}(X,Y) = \left(\frac{\ell-1}{\ell+1}\delta^{\alpha_1}_{(\beta_1}\delta^{\alpha_2}_{\beta_2)} + \text{derivatives}\right)\mathcal{P}^{[\ell]}(X,Y). \tag{4.110}$$

By the same arguments as in the previous case

$$\begin{aligned} &g_B^{\Delta_{T\phi},\Delta_{\phi T},\mu\nu\rho\sigma}(z,\bar{z}) \\ &= \frac{\mathcal{N}_B(\lambda_{\phi T\widetilde{B}}/\lambda_{\phi TB})}{\mathcal{N}_{\mathcal{O}}(\lambda_{\phi T\widetilde{\mathcal{O}}}/\lambda_{\phi T\mathcal{O}})} g^{\Delta_{T\phi},\Delta_{\phi T}}_{(\Delta_B,\ell)}(z,\bar{z})\Pi^{[2]\,\mu\nu;\alpha\beta}\Pi^{[2]\,\rho\sigma;\gamma\delta}\left(m^{(14)}_{\alpha\gamma} + 2(z\bar{z})^{-\frac{1}{2}}k^{(124)}_{\alpha}k^{(413)}_{\gamma}\right) \\ &\qquad\qquad \times\left(m^{(14)}_{\beta\delta} + 2(z\bar{z})^{-\frac{1}{2}}k^{(124)}_{\beta}k^{(413)}_{\delta}\right) + O(1/\ell), \end{aligned} \tag{4.111}$$

Notice that we can set the prefactor to one by a suitable normalization of $B$. Evaluating the relevant polarizations at lowest order in $O(1-\bar{z})$ gives

$$\begin{aligned} g_B^{\Delta_{T\phi},\Delta_{\phi T},++++}(z,\bar{z}) &= O((1-\bar{z})^2), \\ g_B^{\Delta_{T\phi},\Delta_{\phi T},+3+3}(z,\bar{z}) &= O((1-\bar{z})^1), \\ g_B^{\Delta_{T\phi},\Delta_{\phi T},34}(z,\bar{z}) &= g^{\Delta_{T\phi},\Delta_{\phi T}}_{\Delta_B,\ell}(z,\bar{z}) + O(1/\ell), \end{aligned} \tag{4.112}$$

where the last term's tensor structure is of order $O((1-\bar{z})^0)$.

**Derived Blocks:** Now we turn to the conformal blocks that can be obtained from seeds, by acting with the differential operators $D_{ij}$ of [45].[8] The $STT$ exchange in both $\langle J\phi\phi J\rangle$ and $\langle T\phi\phi T\rangle$ can be computed from the lightcone approximation to the scalar block [88]

$$g^{\Delta_{12},\Delta_{34}}_{\tau,\ell}(u,v) \overset{\substack{\ell\gg 1,\ (1-\bar{z})\ell^2\lesssim 1 \\ v\ll u<1}}{=} f_1^{\Delta_{12},\Delta_{34}}(\ell,1-\bar{z})f_2^{\Delta_{12},\Delta_{34}}(\tau,u)(1+O(1/\sqrt{\ell},\sqrt{1-\bar{z}})), \tag{4.113}$$

where $v\approx(1-\bar{z})(1-u)$, $u\approx z$, and

$$f_1^{\Delta_{12},\Delta_{34}}(\ell,x) = \left(-\frac{1}{2}\right)^{\ell}\pi^{-\frac{1}{2}}2^{2\ell}\ell^{\frac{1}{2}}x^{\frac{\Delta_{12}-\Delta_{34}}{4}}K_{\frac{\Delta_{34}-\Delta_{12}}{2}}(2\ell\sqrt{x}), \tag{4.114}$$

$$f_2^{\Delta_{12},\Delta_{34}}(\tau,u) = \frac{2^{\tau}u^{\frac{\tau}{2}}}{(1-u)^{\frac{d}{2}-1}}{}_2F_1\left(\frac{\tau-d+2-\Delta_{12}}{2},\frac{\tau-d+2+\Delta_{34}}{2},\tau-d+2;u\right), \tag{4.115}$$

with $K$ a modified Bessel function of the second kind. This limit holds for even $d\geq 2$ as long as the sum over $\ell$ only receives contributions in the region where the product $\ell^2(1-\bar{z})$ is kept fixed. For the $[\ell,1]$ exchange in $\langle T\phi\phi T\rangle$ the procedure is completely analogous given that its seed is also a scalar conformal block, as shown in (4.108). In both cases one can analyze the differential operators and drop derivatives as well as powers of $\ell$ and $(1-\bar{z})$ that produce subleading terms. The results are summarized below.

[8] It may also be interesting to derive these blocks more directly by expressing the OPE in embedding space [65].

**STT**

For $\langle J\phi\phi J\rangle$, the differential operator is

$$\left(a_1^L D_{11}\Sigma_L^{1,0} + D_{12}\Sigma_L^{0,1}\right)\left(a_1^R D_{44}\Sigma_R^{0,1} + D_{43}\Sigma_R^{1,0}\right). \tag{4.116}$$

The $a_1^{L,R}$ terms can be found by imposing conservation, but their effect is subleading in the lightcone limit. The action of the differential operators on partial waves leads to

$$\begin{aligned} g_{\Delta_{\mathcal{O}},\ell}^{\Delta_{J\phi},\Delta_{\phi J},++}(u,v) &= 2[v\partial_v - \Delta_{J\phi}]g_{\Delta_{\mathcal{O}},\ell}^{\Delta_{J\phi},\Delta_{\phi J},tt}(u,v), \\ g_{\Delta_{\mathcal{O}},\ell}^{\Delta_{J\phi},\Delta_{\phi J},tt}(u,v) &= \frac{1}{2}\sqrt{u}[\Delta_{J\phi} - 1 - v\partial_v]g_{\Delta_{\mathcal{O}},\ell}^{\Delta_{J\phi}-1,\Delta_{\phi J}+1}(u,v)(1+O(1/\ell)). \end{aligned} \tag{4.117}$$

For $\langle T\phi\phi T\rangle$, the differential operator is

$$\begin{aligned} &\left(b_1^L (D_{11})^2\Sigma_L^{2,0} + b_2^L D_{12}D_{11}\Sigma_L^{1,1} + (D_{12})^2\Sigma_L^{0,2}\right) \\ &\qquad\times\left(b_1^R (D_{44})^2\Sigma_R^{0,2} + b_2^R D_{43}D_{44}\Sigma_R^{1,1} + (D_{43})^2\Sigma_R^{2,0}\right). \end{aligned} \tag{4.118}$$

Similar to the previous case, the contribution of the $b_{1,2}^{L,R}$ terms are fixed by conservation and subleading in $1/\ell$. Counting powers in the differential operator gives

$$\begin{aligned} g_{\Delta_{\mathcal{O}},\ell}^{\Delta_{T\phi},\Delta_{\phi T},++++}(u,v) &= 2[v\partial_v - (\Delta_{T\phi}+1)]g_{\Delta_{\mathcal{O}},\ell}^{\Delta_{T\phi},\Delta_{\phi T},+3,+3}(u,v), \\ g_{\Delta_{\mathcal{O}},\ell}^{\Delta_{T\phi},\Delta_{\phi T},+3+3}(u,v) &= [v\partial_v - \Delta_{T\phi}]g_{\Delta_{\mathcal{O}},\ell}^{\Delta_{T\phi},\Delta_{\phi T},34}(u,v), \\ g_{\Delta_{\mathcal{O}},\ell}^{\Delta_{T\phi},\Delta_{\phi T},34}(u,v) &= \frac{u}{2}[(\Delta_{T\phi}-2)(\Delta_{T\phi}-1) \\ &\quad + v(4-2\Delta_{T\phi}+v\partial_v)\partial_v]g_{\Delta_{\mathcal{O}},\ell}^{\Delta_{T\phi}-2,\Delta_{\phi T}+2}(u,v)(1+O(1/\ell)). \end{aligned} \tag{4.119}$$

**$[\ell, 1]$ case**

In this case the seed 3-point function is given by (here we are using the formalism of [29])

$$\langle J(P_1;Z_1)\phi(P_2)A(X_3;Z_3,\Theta_3)\rangle = \frac{V_3^{(\Theta_3)}H_{13}^{(Z_1,\Theta_3)}(V_3^{(Z)})^{\ell-1}}{P_{12}^{\frac{1}{2}(\Delta_\phi+\Delta_J-\Delta_A-\ell)}P_{13}^{\frac{1}{2}(\Delta_{J\phi}+\Delta_A+\ell+2)}P_{23}^{\frac{1}{2}(\Delta_A-\Delta_{J\phi}+\ell)}}. \tag{4.120}$$

To construct $\langle T(P_1;Z_1)\phi(P_2)A(X_3;Z_3,\Theta_3)\rangle$ we act with a linear combination of $D_{11}\Sigma^{1,0}$ and $D_{12}\Sigma^{0,1}$ and impose conservation. The spinning blocks for this exchange are then given by acting on partial waves $W_A$ with the differential operator

$$\left(\lambda_1^L D_{11}\Sigma_L^{1,0} + D_{12}\Sigma_L^{0,1}\right)\left(\lambda_1^R D_{44}\Sigma_R^{1,0} + D_{43}\Sigma_R^{0,1}\right), \tag{4.121}$$

where

$$\lambda_1^L = \lambda_1^R = \left(-\frac{(\Delta_\phi - \Delta_A + \ell - 1)(-\Delta_\phi + \Delta_A + d + \ell - 1)}{(\Delta_\phi - \Delta_A)(\Delta_\phi + \Delta_A - d) - (\ell-1)(d+\ell-1)}\right). \tag{4.122}$$

This leads to

$$\begin{aligned} g_A^{\Delta_{T\phi},\Delta_{\phi T},++++}(u,v) &= 0, \\ g_A^{\Delta_{T\phi},\Delta_{\phi T},+3+3}(u,v) &= \frac{1}{2}[v\partial_v - \Delta_{T\phi}]g_A^{\Delta_{T\phi},\Delta_{\phi T},34}(u,v), \\ g_A^{\Delta_{T\phi},\Delta_{\phi T},34}(u,v) &= -\frac{1}{2}\sqrt{u}[1-\Delta_{T\phi}+v\partial_v]g_{\Delta_A,\ell}^{\Delta_{T\phi}-1,\Delta_{\phi T}+1}(u,v)(1+O(1/\ell)), \end{aligned} \tag{4.123}$$

where we used the approximation given in (4.108).

**Polarization Ratios:** Now we check that the different polarizations of the 4-point function $G(z,\bar{z})$ are related to each other by a $z$-independent factor. To see this we perform the sum over spins in the s-channel, via (4.102). For $\langle J\phi\phi J\rangle$ this results in

$$\begin{aligned} G_{J,STT}^{++} \propto &-\frac{\Gamma(d)\Gamma(\Delta_\phi)}{2^4}\sum_n (\lambda_{J\phi[J\phi]_n^{[\ell]}})^2 F_n(u) \\ &-\frac{\Gamma(\frac{d}{2}+1)\Gamma(\Delta_\phi-\frac{d}{2}+1)}{2^5}(1-\bar{z})^{\frac{d}{2}-1}\sum_n (\lambda_{J\phi[J\phi]_n^{[\ell]}})^2\gamma_{[J\phi]_n^{[\ell]}}F_n(u)\ln(u), \end{aligned} \tag{4.124}$$

$$\begin{aligned} G_{J,STT}^{tt} \propto &\frac{\Gamma(d-1)\Gamma(\Delta_\phi)}{2^5}\sum_n (\lambda_{J\phi[J\phi]_n^{[\ell]}})^2 F_n(u) \\ &+\frac{\Gamma(\frac{d}{2})\Gamma(\Delta_\phi-\frac{d}{2}+1)}{2^6}(1-\bar{z})^{\frac{d}{2}-1}\sum_n (\lambda_{J\phi[J\phi]_n^{[\ell]}})^2\gamma_{[J\phi]_n^{[\ell]}}F_n(u)\ln(u), \end{aligned} \tag{4.125}$$

where we defined

$$\begin{aligned} F_n(u) &\equiv \frac{2^{\Delta_\phi+d+2n}u^n}{\sqrt{\pi}(1-u)^{1-\frac{d}{2}}}\,{}_2F_1\left(\frac{d}{2}+n-1,\frac{d}{2}+n-1;\Delta_\phi+2n;u\right), \\ (\lambda_{J\phi[J\phi]_n^{[\ell]}})^2 &\equiv 2^\ell \ell^{-\frac{1}{2}(2\Delta_\phi+2d-7)}(\lambda_{J\phi[J\phi]_{n,\ell}^{[\ell]}})^2, \quad \gamma_{[J\phi]_n^{[\ell]}} \equiv \ell^{d-2}\gamma_{[J\phi]_{n,\ell}^{[\ell]}}, \end{aligned}$$

and used (4.22). The proportionality coefficient is the kinematical term in front of the 4-point function. The ratios $G_{J,STT}^{tt}/G_{J,STT}^{++}$ are then $\frac{1}{2(1-d)}$ at order $O((1-\bar{z})^0)$ and $-\frac{1}{d}$ at order $O((1-\bar{z})^{\frac{d}{2}-1})$. Similarly, for $\langle T\phi\phi T\rangle$ we have

$$\begin{aligned} G_{T,STT}^{++++} \propto &\frac{\Gamma(d+2)\Gamma(\Delta_\phi)}{2^4}\sum_n (\lambda_{T\phi[T\phi]_n^{[\ell]}})^2 F_n(u) \\ &+\frac{\Gamma(\frac{d}{2}+3)\Gamma(\Delta_\phi-\frac{d}{2}+1)}{2^5}(1-\bar{z})^{\frac{d}{2}-1}\sum_n (\lambda_{T\phi[T\phi]_n^{[\ell]}})^2\gamma_{[T\phi]_n^{[\ell]}}F_n(u)\ln(u), \end{aligned} \tag{4.126}$$

$$\begin{aligned} G_{T,STT}^{+3+3} \propto &-\frac{\Gamma(d+1)\Gamma(\Delta_\phi)}{2^5}\sum_n (\lambda_{T\phi[T\phi]_n^{[\ell]}})^2 F_n(u) \\ &-\frac{\Gamma(\frac{d}{2}+2)\Gamma(\Delta_\phi-\frac{d}{2}+1)}{2^6}(1-\bar{z})^{\frac{d}{2}-1}\sum_n (\lambda_{T\phi[T\phi]_n^{[\ell]}})^2\gamma_{[T\phi]_n^{[\ell]}}F_n(u)\ln(u), \end{aligned} \tag{4.127}$$

$$G^{34}_{T,STT} \propto \frac{\Gamma(d)\Gamma(\Delta_\phi)}{2^5}\sum_n (\lambda_{T\phi[T\phi]^{[\ell]}_n})^2 F_n(u) + \frac{\Gamma(\frac{d}{2}+1)\Gamma(\Delta_\phi-\frac{d}{2}+1)}{2^6}(1-\bar{z})^{\frac{d}{2}-1}\sum_n (\lambda_{T\phi[T\phi]^{[\ell]}_n})^2\gamma_{[T\phi]^{[\ell]}_n} F_n(u)\ln(u), \quad (4.128)$$

$$G^{+3+3}_{T,A} \propto -\frac{\Gamma(d+1)\Gamma(\Delta_\phi)}{2^5}\sum_n (\lambda_{T\phi[T\phi]^{[\ell,1]}_n})^2 \tilde{F}_n(u) - \frac{\Gamma(\frac{d}{2}+2)\Gamma(\Delta_\phi-\frac{d}{2}+1)}{2^6}(1-\bar{z})^{\frac{d}{2}-1}\sum_n (\lambda_{T\phi[T\phi]^{[\ell,1]}_n})^2\gamma_{[T\phi]^{[\ell,1]}_n} \tilde{F}_n(u)\ln(u), \quad (4.129)$$

$$G^{34}_{T,A} \propto \frac{\Gamma(d)\Gamma(\Delta_\phi)}{2^4}\sum_n (\lambda_{T\phi[T\phi]^{[\ell,1]}_n})^2 \tilde{F}_n(u) + \frac{\Gamma(\frac{d}{2}+1)\Gamma(\Delta_\phi-\frac{d}{2}+1)}{2^5}(1-\bar{z})^{\frac{d}{2}-1}\sum_n (\lambda_{T\phi[T\phi]^{[\ell,1]}_n})^2\gamma_{[T\phi]^{[\ell,1]}_n} \tilde{F}_n(u)\ln(u), \quad (4.130)$$

where the twist for $A$ is given by (4.23) and

$$\tilde{F}_n(u) \equiv \frac{2^{\Delta_\phi+d+2n}u^n}{\sqrt{\pi}(1-u)^{-\frac{d}{2}}}\, {}_2F_1\left(\frac{d}{2}+n, \frac{d}{2}+n; \Delta_\phi+2n+1; u\right),$$

$$(\lambda_{T\phi[T\phi]^{[\ell]}_n})^2 \equiv 2^\ell \ell^{-\frac{1}{2}(2\Delta_\phi+2d-7)}(\lambda_{T\phi[T\phi]^{[\ell]}_{n,\ell}})^2, \quad \gamma_{[T\phi]^{[\ell]}_n} \equiv \ell^{d-2}\gamma_{[T\phi]^{[\ell]}_{n,\ell}},$$

$$(\lambda_{T\phi[T\phi]^{[\ell,1]}_n})^2 \equiv 2^{\ell-1} \ell^{-\frac{1}{2}(2\Delta_\phi+2d-5)}(\lambda_{T\phi[T\phi]^{[\ell,1]}_{n,\ell}})^2, \quad \gamma_{[T\phi]^{[\ell,1]}_n} \equiv \ell^{d-2}\gamma_{[T\phi]^{[\ell,1]}_{n,\ell}}.$$

For this case the ratios are summarized in table 4.1.

Table 4.1: Ratios for the different polarizations of $\langle T\phi\phi T\rangle$ in the lightcone limit.

| | $O((1-\bar{z})^0)$ | $O((1-\bar{z})^{\frac{d}{2}-1})$ |
|---|---|---|
| $G^{+3+3}_{T,STT}/G^{++++}_{T,STT}$ | $-\frac{1}{2(d+1)}$ | $-\frac{1}{d+4}$ |
| $G^{34}_{T,STT}/G^{++++}_{T,STT}$ | $\frac{1}{2d(d+1)}$ | $\frac{2}{(d+2)(d+4)}$ |
| $G^{34}_{T,A}/G^{+3+3}_{T,A}$ | $-\frac{2}{d}$ | $-\frac{4}{d+2}$ |

# Appendix B: Anomalous Dimensions for Non-Zero $n$

In this appendix we generalize our results for anomalous dimensions to $n > 0$ in the regime $\ell \gg n$.

**$\langle J\phi\phi J\rangle$:** In order to match the identity at all orders in $z$, we use the summation formula

$$(1-x)^b = \sum_{n\geq 0} \frac{x^n (b)_n (c)_n}{n!(b+c+n-1)_n}\, {}_2F_1(b+n, b+n; b+c+2n; x) \quad (4.131)$$

in the s-channel expansion. This fixes the OPE coefficients, which we write in terms of the $n=0$ result:

$$(\lambda_{J\phi[J\phi]^{[\ell]}_{n,\ell}})^2 = \frac{(1-\frac{d}{2}+\Delta_\phi)_n(\frac{d}{2}-1)_n}{4^n n!(\Delta_\phi+n-1)_n}(\lambda_{J\phi[J\phi]^{[\ell]}_{0,\ell}})^2, \tag{4.132}$$

$$(\lambda_{J\phi[J\phi]^{[\ell,1]}_{n,\ell}})^2 = \frac{(1-\frac{d}{2}+\Delta_\phi)_n(\frac{d}{2})_n}{4^n n!(\Delta_\phi+n)_n}(\lambda_{J\phi[J\phi]^{[\ell,1]}_{0,\ell}})^2. \tag{4.133}$$

Now we split the anomalous dimensions as

$$\gamma_{[J\phi]^{[\ell]}_{n,\ell}} = \tilde{\gamma}_{[J\phi]^{[\ell]}_n}\gamma_{[J\phi]^{[\ell]}_{0,\ell}}, \quad \gamma_{[J\phi]^{[\ell,1]}_{n,\ell}} = \tilde{\gamma}_{[J\phi]^{[\ell,1]}_n}\gamma_{[J\phi]^{[\ell,1]}_{0,\ell}}, \tag{4.134}$$

and match the stress-tensor at all orders in $z$. This leads to the following equations

$$\begin{aligned}\frac{\Gamma(\frac{d}{2}-1)\Gamma(\Delta_\phi-\frac{d}{2}+1)(\frac{d}{2}+1)_j^2}{(j!)^2\Gamma(\Delta_\phi-\frac{d}{2}+1+j)^2}\,{}_3F_2\left(\begin{matrix}-j,-j,\Delta_\phi-d\\-\frac{d}{2}-j,-\frac{d}{2}-j\end{matrix};1\right)&\\ =\sum_{n=0}^{j}\frac{(\Delta_\phi+2n-1)\Gamma(\frac{d}{2}+n-1)\Gamma(\Delta_\phi+n-1)}{n!(j-n)\Gamma(\Delta_\phi+n-\frac{d}{2}+1)\Gamma(\Delta_\phi+n+j)}\tilde{\gamma}_{[J\phi]^{[\ell]}_n},&\end{aligned} \tag{4.135}$$

$$\begin{aligned}\frac{\Gamma(\frac{d}{2})\Gamma(\Delta_\phi-\frac{d}{2}+1)(\frac{d}{2}+1)_j^2}{(j!)^2\Gamma(\Delta_\phi-\frac{d}{2}+1+j)^2}\,{}_3F_2\left(\begin{matrix}-j,-j,\Delta_\phi-d\\-\frac{d}{2}-j,-\frac{d}{2}-j\end{matrix};1\right)&\\ =\sum_{n=0}^{j}\frac{(\Delta_\phi+2n)\Gamma(\frac{d}{2}+n)\Gamma(\Delta_\phi+n)}{n!(j-n)\Gamma(\Delta_\phi+n-\frac{d}{2}+1)\Gamma(\Delta_\phi+n+j+1)}\tilde{\gamma}_{[J\phi]^{[\ell,1]}_n},&\end{aligned} \tag{4.136}$$

where $j$ represents the power of $z$ in the Taylor expansion. Using the techniques of [96,99], we write $\tilde{\gamma}$ in terms of terminating hypergeometric functions:

$$\begin{aligned}\tilde{\gamma}_{[J\phi]^{[\ell]}_n} =& \frac{(-1)^n n!\Gamma(\Delta_\phi-\frac{d}{2}+1)\Gamma(\Delta_\phi+n-\frac{d}{2}+1)}{(\frac{d}{2}-1)_n\Gamma(\frac{d}{2}+1)^2}\\ &\times\sum_{i=0}^{n}\frac{(-1)^i(i+1)^2_{\frac{d}{2}}(\Delta_\phi+n-1)_i}{(n-i)!\Gamma(\Delta_\phi-\frac{d}{2}+1+i)^2}\,{}_3F_2\left(\begin{matrix}-i,-i,\Delta_\phi-d\\-\frac{d}{2}-i,-\frac{d}{2}-i\end{matrix};1\right),\end{aligned} \tag{4.137}$$

$$\begin{aligned}\tilde{\gamma}_{[J\phi]^{[\ell,1]}_n} =& \frac{(-1)^n n!\Gamma(\Delta_\phi-\frac{d}{2}+1)\Gamma(\Delta_\phi+n-\frac{d}{2}+1)}{(\frac{d}{2})_n\Gamma(\frac{d}{2}+1)^2}\\ &\times\sum_{i=0}^{n}\frac{(-1)^i(i+1)^2_{\frac{d}{2}}(\Delta_\phi+n)_i}{(n-i)!\Gamma(\Delta_\phi-\frac{d}{2}+1+i)^2}\,{}_3F_2\left(\begin{matrix}-i,-i,\Delta_\phi-d\\-\frac{d}{2}-i,-\frac{d}{2}-i\end{matrix};1\right).\end{aligned} \tag{4.138}$$

One can check that this solves (4.135) and (4.136) order by order in $n$, for arbitrarily high values.

**$\langle T\phi\phi T\rangle$:** Following the same steps as in the previous case, we find the OPE coefficients

$$(\lambda_{T\phi[T\phi]^{[\ell]}_{n,\ell}})^2 = \frac{(1-\frac{d}{2}+\Delta_\phi)_n(\frac{d}{2}-1)_n}{4^n n!(\Delta_\phi+n-1)_n}(\lambda_{T\phi[T\phi]^{[\ell]}_{0,\ell}})^2, \tag{4.139}$$

$$(\lambda_{T\phi[T\phi]^{[\ell,1]}_{n,\ell}})^2 = \frac{(1-\frac{d}{2}+\Delta_\phi)_n(\frac{d}{2})_n}{4^n n!(\Delta_\phi+n)_n}(\lambda_{T\phi[T\phi]^{[\ell,1]}_{0,\ell}})^2, \tag{4.140}$$

$$(\lambda_{T\phi[T\phi]^{[\ell,2]}_{n,\ell}})^2 = \frac{(1-\frac{d}{2}+\Delta_\phi)_n(\frac{d}{2}+1)_n}{4^n n!(\Delta_\phi+n+1)_n}(\lambda_{T\phi[T\phi]^{[\ell,2]}_{0,\ell}})^2. \tag{4.141}$$

Notice that for $[\ell]$ and $[\ell,1]$, the n-dependence is the same as in $\langle J\phi\phi J\rangle$. Finally, we define anomalous dimensions for $n\geq 0$ as

$$\gamma_{[T\phi]^{[\ell]}_{n,\ell}} = \tilde{\gamma}_{[T\phi]^{[\ell]}_n}\gamma_{[T\phi]^{[\ell]}_{0,\ell}}, \quad \gamma_{[T\phi]^{[\ell,1]}_{n,\ell}} = \tilde{\gamma}_{[T\phi]^{[\ell,1]}_n}\gamma_{[T\phi]^{[\ell,1]}_{0,\ell}}, \quad \gamma_{[T\phi]^{[\ell,2]}_{n,\ell}} = \tilde{\gamma}_{[T\phi]^{[\ell,2]}_n}\gamma_{[T\phi]^{[\ell,2]}_{0,\ell}}. \tag{4.142}$$

For $STT$ and $A$ we find the same equations as in $\langle J\phi\phi J\rangle$. Therefore $\tilde{\gamma}_{[T\phi]^{[\ell]}_n} = \tilde{\gamma}_{[J\phi]^{[\ell]}_n}$ and $\tilde{\gamma}_{[T\phi]^{[\ell,1]}_n} = \tilde{\gamma}_{[J\phi]^{[\ell,1]}_n}$. On the other hand, for $B$ we have

$$\begin{aligned}&\frac{\Gamma(\frac{d}{2}+1)\Gamma(\Delta_\phi-\frac{d}{2}+1)(\frac{d}{2}+1)_j^2}{(j!)^2\Gamma(\Delta_\phi-\frac{d}{2}+1+j)^2}\,{}_3F_2\left(\begin{matrix}-j,-j,\Delta_\phi-d\\ -\frac{d}{2}-j,-\frac{d}{2}-j\end{matrix};1\right)\\ &\qquad=\sum_{n=0}^{j}\frac{(\Delta_\phi+2n+1)\Gamma(\frac{d}{2}+n+1)\Gamma(\Delta_\phi+n+1)}{n!(j-n)\Gamma(\Delta_\phi+n-\frac{d}{2}+1)\Gamma(\Delta_\phi+n+j+2)}\tilde{\gamma}_{[T\phi]^{[\ell,2]}_n}.\end{aligned} \tag{4.143}$$

The solution is

$$\begin{aligned}\tilde{\gamma}_{[T\phi]^{[\ell,2]}_n} =& \frac{(-1)^n n!\Gamma(\Delta_\phi-\frac{d}{2}+1)\Gamma(\Delta_\phi+n-\frac{d}{2}+1)}{(\frac{d}{2}+1)_n\Gamma(\frac{d}{2}+1)^2}\\ &\times\sum_{i=0}^{n}\frac{(-1)^i(i+1)^2_{\frac{d}{2}}(\Delta_\phi+n+1)_i}{(n-i)!\Gamma(\Delta_\phi-\frac{d}{2}+1+i)^2}\,{}_3F_2\left(\begin{matrix}-i,-i,\Delta_\phi-d\\ -\frac{d}{2}-i,-\frac{d}{2}-i\end{matrix};1\right).\end{aligned} \tag{4.144}$$

**Examples:** Now using the identities in the appendices of [96, 99] we can rewrite the terminating hypergeometric and perform the sum over $i$ for specific even dimensions. In $d=4$ we have

$$\begin{aligned}\tilde{\gamma}_{[T\phi]^{[\ell]}_n} &= 1+\frac{3n(\Delta_\phi+n-1)(\Delta_\phi+n(\Delta_\phi+n-1))}{\Delta_\phi(\Delta_\phi-1)},\\ \tilde{\gamma}_{[T\phi]^{[\ell,1]}_n} &= \frac{(n+1)(\Delta_\phi+n-1)(\Delta_\phi+n(\Delta_\phi+n))}{\Delta_\phi(\Delta_\phi-1)},\\ \tilde{\gamma}_{[T\phi]^{[\ell,2]}_n} &= \frac{(n+1)(n+2)(\Delta_\phi+n-1)(\Delta_\phi+n)}{2\Delta_\phi(\Delta_\phi-1)},\end{aligned} \tag{4.145}$$

whereas in $d=6$

$$\begin{aligned}
\tilde{\gamma}_{[T\phi]_n^{[\ell]}} &= \frac{(n+1)(\Delta_\phi+n-2)(5n^2(n-1)^2+2\Delta_\phi(5n^3-5n-3)+\Delta_\phi^2(5n(n+2)+6))}{6\Delta_\phi(\Delta_\phi-1)(\Delta_\phi-2)},\\
\tilde{\gamma}_{[T\phi]_n^{[\ell,1]}} &= \frac{(n+1)(n+2)(\Delta_\phi+n-2)(\Delta_\phi+n-1)(3\Delta_\phi+2n(\Delta_\phi+n))}{6\Delta_\phi(\Delta_\phi-1)(\Delta_\phi-2)},\\
\tilde{\gamma}_{[T\phi]_n^{[\ell,2]}} &= \frac{(n+1)(n+2)(n+3)(\Delta_\phi+n-2)(\Delta_\phi+n-1)(\Delta_\phi+n)}{6\Delta_\phi(\Delta_\phi-1)(\Delta_\phi-2)}.
\end{aligned} \tag{4.146}$$

It is easy to check that for $\Delta_\phi \geq \frac{d}{2}-1$ these expressions are positive for all $n$.

# Appendix C: Correlation Functions of Conserved Operators

In this appendix we will provide more details on the 3-point functions $\langle JJJ\rangle$ and $\langle TTT\rangle$.

**Tensor Structures:** We follow the notation and techniques of [31, 45]. See also [91] for more details on the differential representation on the 3-point functions.

Here we define the basic buildings blocks as

$$\begin{aligned}
H_{ij} &= -2[(Z_i\cdot Z_j)(P_i\cdot P_j-(Z_i\cdot P_j)(Z_j\cdot P_i))],\\
V_{i,jk} &= \frac{(Z_i\cdot P_j)(P_i\cdot P_k)-(Z_i\cdot P_k)(P_i\cdot P_j)}{P_j\cdot P_k}.
\end{aligned} \tag{4.147}$$

We will use the shorthand $V_1=V_{1,23}$, $V_2=V_{2,31}$, and $V_3=V_{3,1,2}$.

**$\langle JJT\rangle$:** We will normalize the operators as follows:

$$\langle J(P_1,Z_1)J(P_2,Z_2)\rangle = C_J\frac{H_{12}}{P_{12}^d},\quad \langle T(P_1,Z_1)T(P_2,Z_2)\rangle = C_T\frac{H_{12}^2}{P_{12}^{d+2}}. \tag{4.148}$$

The general form of the $\langle JJT\rangle$ 3-point function, after imposing symmetry under $1\leftrightarrow 2$, is given by

$$\langle J(P_1;Z_1)J(P_2;Z_2)T(P_3;Z_3)\rangle = \frac{\alpha V_1V_2V_3^2+\beta(H_{13}V_2+H_{23}V_1)V_3+\gamma H_{12}V_3^2+\eta H_{13}H_{23}}{(P_{12})^{\frac{d}{2}-1}(P_{13})^{\frac{d}{2}+1}(P_{23})^{\frac{d}{2}+1}}.$$

Imposing conservation implies

$$\begin{aligned}
-\alpha-d\beta+(2+d)\gamma &= 0,\\
-2\beta+2\gamma+(2-d)\eta &= 0.
\end{aligned} \tag{4.149}$$

The relation between our basis and that used in [137], see Eqs. (3.11-3.14) is given by[9]

$$\begin{aligned} &\eta = 2\tilde{e}, \quad \beta = -2\tilde{c}, \\ &\gamma = \tilde{a} - \frac{\tilde{b}}{d} - \frac{4\tilde{c}}{d}, \quad \alpha = 2\tilde{a} + \tilde{b}\big(1 - \frac{2}{d}\big) - \frac{8\tilde{c}}{d}, \end{aligned} \tag{4.150}$$

They also found that the Ward identity for the stress energy tensor implies

$$2S_d(\tilde{c} + \tilde{e}) = dC_J. \tag{4.151}$$

Where $S_d$ is the volume of a $(d-1)$-dimensional sphere, $S_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$. So $\langle JJT\rangle$ is fixed up to one OPE coefficient, $\tilde{c}$, and $C_J$. We labeled the parameter $\tilde{c}$ as $\lambda_{JJT}$ in the main text, following the conventions of [91]. In the rest of this appendix we will also adopt this convention. To construct the conformal block corresponding to $T^{\mu\nu}$ exchange in the s-channel of $\langle JJ\phi\phi\rangle$ we apply the following differential operator on the scalar partial wave,

$$\begin{aligned} &D_{L,T} \\ &= \left[\left(2\lambda_{JJT} - \frac{C_J d(d-2)}{(d-1)S_d}\right) D_{11}D_{22} + \left(2\lambda_{JJT} + \frac{C_J d^2}{S_d(1-d)}\right) D_{12}D_{21} - 2\lambda_{JJT} H_{12}\right] \Sigma_L^{1,1}. \end{aligned} \tag{4.152}$$

The conformal block for $T^{\mu\nu}$ exchange in the t-channel of $\langle J\phi\phi J\rangle$ is then found by letting $2 \leftrightarrow 4$ everywhere in the resulting expression.

Finally, in [136] the parameter $a_2$ was introduced, distinct from the $a_2$ OPE coefficient used in $\langle VVT\rangle$, which gives the energy distribution for a state created by a conserved current:

$$\langle \mathcal{E}(n)\rangle_{\epsilon\cdot j} = \frac{1}{S_d}\left(1 + a_2\big(\cos^2(\theta) - \frac{1}{d-1}\big)\right) \tag{4.153}$$

where $\theta$ is the angle between the spatial polarization $\epsilon^i$ and the point on $S^{d-1}$ labelled by $n^i$. Requiring that the energy one point function be positive yields the bounds

$$-\frac{d-1}{d-2} \le a_2 \le d-1. \tag{4.154}$$

The upper bound is saturated in a theory of free bosons and the lower bound is saturated in a theory of free fermions. The relation between $\lambda_{JJT}$ and $a_2$ is given by

$$\lambda_{JJT} = -\frac{C_J(d-2)d\pi^{-\frac{d}{2}}\left(a_2 - d^2 + d\right)\Gamma\left(\frac{d}{2}\right)}{4(d-1)^3}. \tag{4.155}$$

[9] We add tildes to the variables to avoid confusion between these variables, the conformal anomalies $a$ and $c$, and the $\langle TTT\rangle$ OPE coefficients.

**$\langle TTT \rangle$:** In this section we will review the connection between the parametrization of $\langle TTT \rangle$ in terms of the variables $\hat{c}$, $\hat{e}$, and $C_T$ as defined in [137], the $t_2$, $t_4$, $C_T$ parametrization used in studies of the energy one point function [151], and the free field theory results.

We start by defining the following basis of parity-even tensor structures for $\langle TTT \rangle$,

$$Q_1 = V_1^2 V_2^2 V_3^2, \tag{4.156}$$
$$Q_2 = H_{23} V_1^2 V_2 V_3 + H_{13} V_1 V_2^2 V_3, \tag{4.157}$$
$$Q_3 = H_{12} V_1 V_2 V_3^2, \tag{4.158}$$
$$Q_4 = H_{12} H_{13} V_2 V_3 + H_{12} H_{23} V_1 V_3, \tag{4.159}$$
$$Q_5 = H_{13} H_{23} V_1 V_2, \tag{4.160}$$
$$Q_6 = H_{12}^2 V_3^2, \tag{4.161}$$
$$Q_7 = H_{13}^2 V_2^2 + H_{23}^2 V_1^2, \tag{4.162}$$
$$Q_8 = H_{12} H_{13} H_{23}. \tag{4.163}$$

In [137] they parametrized the correlation function in general dimensions in terms of 8 variables: $\hat{a}$, $\hat{b}$, $\hat{b}'$, $\hat{c}$, $\hat{c}'$, $\hat{e}$, $\hat{e}'$, and $\hat{f}$. Labeling the coefficients of $Q_i$ by $x_i$, the relation between the bases is given by

$$x_1 = 8(\hat{c} + \hat{e}) + \hat{f}, \quad x_2 = -4(4\hat{b}' + \hat{e}'), \quad x_3 = 4(2\hat{c} + \hat{e}), \tag{4.164}$$
$$x_4 = -8\hat{b}', \quad x_5 = 8\hat{b} + 16\hat{a}, \quad x_6 = 2\hat{c}, \tag{4.165}$$
$$x_7 = 2\hat{c}', \quad x_8 = 8\hat{a}. \tag{4.166}$$

Conservation of the stress-energy tensor implies

$$x_1 = 2x_2 + \frac{1}{4}(d^2 + 2d - 8)x_4 - \frac{1}{2}d(2+d)x_7, \qquad x_8 = \frac{x_2 - (\frac{d}{2} + 1)x_4 + 2dx_7}{\frac{d^2}{2} - 2}, \tag{4.167}$$
$$x_2 = x_3, \qquad x_4 = x_5, \qquad x_6 = x_7, \tag{4.168}$$

which is consistent with the conservation constraints of [137]. Finally, they found that solving the Ward identity yields

$$4S_d \frac{(d-2)(d+3)\hat{a} - 2\hat{b} - (d+1)\hat{c}}{d(d+2)} = C_T. \tag{4.169}$$

In $d > 3$ dimensions we can parametrize the parity-even structures in $\langle TTT \rangle$ by $\hat{c}$, $\hat{e}$, and $C_T$, while in $d = 3$ the $H_{12}H_{13}H_{23}$ structure is not linearly independent and $\langle TTT \rangle$ is fixed up to two parameters, $2\hat{a} - \hat{c}$ and $C_T$.

The relation between this basis and the $t_2$ and $t_4$ basis is given by

$$\hat{c} = -\frac{C_T \pi^{-\frac{d}{2}} \Gamma\left(\frac{d}{2} + 2\right)}{2(d-1)^3(d+1)^2(d+2)} \times \left[\left(d\left(-3d^2 + d + 2\right) + 4\right) t_4 + (d+1)\left(2d^4 - d^3(t_2 + 4) + d^2 + d + 3t_2\right)\right], \tag{4.170}$$

$$\hat{e} = \frac{C_T \pi^{-\frac{d}{2}} \Gamma\left(\frac{d}{2}+2\right)}{4(d-1)^3(d+1)^2}$$
$$\times \left[(d+1)\left((d-3)\left(d^2-3\right)t_2 + 2(d-2)d^2+2\right) + (2(d-5)d^2+4d+12)t_4\right]. \quad (4.171)$$

Finally, as noted in [151], in even dimensions we can parametrize $\langle TTT \rangle$ by its expressions in free field theories of conformally coupled scalars, fermions, and $(\frac{d}{2}-1)$-forms:

$$\langle TTT \rangle = n_s \langle TTT \rangle_s + n_f \langle TTT \rangle_f + n_t \langle TTT \rangle_t, \quad (4.172)$$

where $n_s$, $n_f$, and $n_t$ give the effective number of real scalars, Dirac fermions, and $(\frac{d}{2}-1)$ forms, although there may not necessarily be any connection to the actual field content. The conformal collider constraints can then be written as [150, 151],

$$\left(1 - \frac{1}{d-1}t_2 - \frac{2}{d^2-1}t_4\right) + \frac{d-2}{d-1}(t_2+t_4) \propto n_s \geq 0, \quad (4.173)$$

$$\left(1 - \frac{1}{d-1}t_2 - \frac{2}{d^2-1}t_4\right) + \frac{1}{2}t_2 \propto n_f \geq 0, \quad (4.174)$$

$$\left(1 - \frac{1}{d-1}t_2 - \frac{2}{d^2-1}t_4\right) \propto n_t \geq 0. \quad (4.175)$$

The constraints (4.173), (4.174), and (4.175) are equivalent to the constraints derived by considering $\langle T^{++}\phi\phi T^{++}\rangle$, $\langle T^{+3}\phi\phi T^{+3}\rangle$, and $\langle (T^{33}-T^{44})\phi\phi(T^{33}-T^{44})\rangle$, respectively. In three dimensions $t_2 = 0$ and the second and third constraints are redundant. Finally in four dimensions we have [136, 137, 164]

$$\frac{a}{c} = \frac{2n_s + 124n_t + 22n_f}{6n_s + 72n_t + 36n_f}, \quad (4.176)$$

where $n_t$ now counts the number of real free vectors, $a$ is the Euler anomaly, and $c$ is related to central charge $C_T$ as $c = \frac{\pi^4}{40}C_T$. The bounds from equations (4.173) and (4.175) then imply

$$\frac{31}{18} \geq \frac{a}{c} \geq \frac{1}{3}. \quad (4.177)$$

# 5 Gravitational interactions à la CFT and vice versa

On the relationship between spinning conformal partial waves and geodesic Witten diagrams

The main tool that allows conformal field theories to be classified by the bootstrap program is the decomposition of correlators into conformal partial waves. In particular, as seen in chapter 4, when these correspond to spinning correlators one can extract universal information concerning the nature of the dual gravitational theory through the holographic principle. Based on [3], in this chapter we present the necessary tools to understand how spinning conformal partial waves are represented in AdS.

## 5.1 Introduction

Throughout this thesis the recurring theme has been the exploitation of the conformal symmetry as an efficient organizational principle for the observables in the theory. In particular, the conformal block decomposition of four-point correlation functions is such a principle: it is natural to cast the four point function into portions that are purely determined by symmetries—conformal partial waves—and the theory dependent CFT data. The aim of this chapter is to apply the efficiency of the conformal block decomposition to holography: can we organize observables in AdS gravity as we do in a CFT? This question has been at the heart of holography since its conception [165–167], with perhaps the most influential result the prescription to evaluate CFT correlation functions via Witten diagrams [167]. But only until very recently the concept of conformal partial wave was addressed directly in holography: the authors in [120] proposed that the counterpart of a $\mathrm{CFT}_d$ conformal partial wave is a *geodesic Witten diagram* in $\mathrm{AdS}_{d+1}$. This is basically a conventional Witten diagram with a different integration region. Namely the contact terms of the fields are projected over geodesics rather than integrated over the entire AdS volume.

The goal of this chapter is twofold: to give a method to evaluate a spinning conformal partial waves using holography, and to show how Witten diagrams decompose in terms of these. The strategy is to employ the techniques of subsection 1.6.4 along the lines of the AdS proposal in [120]. In particular, we will show how to decode the tensor structures appearing in three point functions and conformal partial waves in terms of bulk differential operators acting on geodesic diagrams.

This chapter is organized as follows. Section 5.2 is a review of the embedding space formalism to describe both $\mathrm{CFT}_d$ and $\mathrm{AdS}_{d+1}$ quantities in a common language. The main result is in section 5.3 where we device a differential basis on AdS that generates three-point tensor structures from bulk objects. This shows how one can obtain any spinning conformal partial wave via an appropriate geodesic Witten diagram with perfect agreement with the CFT. In section 5.4 we discuss certain features of this method by focusing mostly on low spin examples. We first discuss the relation among gravitational interactions and OPE structures using geodesic diagrams, and contrast it with the reconstruction done using Witten diagrams. Even though there are non-trivial cancellations in the geodesic diagrams (which do not occur with volume integrals), in section 5.5 we show how to decompose four point exchange Witten diagrams in terms of geodesic diagrams. We conclude in section 5.6. Extended calculations that complement the results of the main text are written in appendices A and B.

## 5.2 Embedding space formalism

The simplest way of computing CFT objects is by working on the embedding space formalism where the index structure of tensor quantities is encoded as homogeneous polynomials. This has been discussed at length in section 1.2, but here we include an executive summary to give a better contrast to the embedding formalism of AdS.

A conformal field $\phi^{\mu_1,\ldots,\mu_\ell}(x)$, $x \in \mathbb{R}^d$ of dimension $\Delta$ in the symmetric-traceless representation $(\ell)$, is encoded as an embedded polynomial $\Phi(P, Z)$, $P \in$

$\mathbb{R}^{d+1,1}$, where

$$\Phi(\lambda P, \alpha Z + \beta P) = \lambda^{-\Delta}\alpha^{\ell}\Phi(P,Z), \quad P^2 = Z\cdot P = Z^2 = 0, \quad \forall \lambda, \alpha, \beta. \tag{5.1}$$

The original field $\phi(x,z)$ is recovered by replacing $P^A = \left(x^\alpha, \frac{1-x^2}{2}, \frac{1+x^2}{2}\right)$ and $Z^A = (z^\alpha, -z\cdot x, z\cdot x)$. Furthermore, to recover the tensor indices we apply the differential operator in (2.1) to obtain

$$\phi^{\mu_1,\dots,\mu_\ell}(x) = \frac{1}{\ell!(d/2-1)_\ell} D^{\mu_1}\cdots D^{\mu_\ell}\phi(x,z) \tag{5.2}$$

This operator is also convenient for other purposes. For example, we can do full contractions via the polynomial directly: given two encoded tensors in $\mathbb{R}^d$, their index contraction is

$$f_{a_1\cdots a_n} g^{a_1\cdots a_n} = \frac{1}{n!(d/2-1)_n} f(x,D) g(x,z)\ . \tag{5.3}$$

In the $(d+2)$-dimensional variables we have

$$f_{a_1\cdots a_n} g^{a_1\cdots a_n} = \frac{1}{n!(d/2-1)_n} F(P,D) G(P,Z)\ , \tag{5.4}$$

where

$$D_A = \left(\frac{d}{2} - 1 + Z\cdot\frac{\partial}{\partial Z}\right)\frac{\partial}{\partial Z^A} - \frac{1}{2} Z_A \frac{\partial^2}{\partial Z\cdot\partial Z}\ . \tag{5.5}$$

All other definitions regarding embedding formalism, correlation functions, and partial waves on the CFT side can be found in sections 1.2, 1.3, 1.4, 1.5, and 1.6.

To describe both CFT and AdS quantities in a more homogeneous manner, it is then useful to formulate the AdS version of the embedding space formalism, which we present in the rest of this section.

### 5.2.1 Embedding formalism for AdS

In this exposition of the embedding formalism for AdS we will follow [168, 169].[1] Euclidean AdS$_{d+1}$ in Poincare coordinates is given by the following metric

$$ds^2_{\text{AdS}} = \frac{1}{r^2}\left(dr^2 + dx^a dx_a\right)\ , \tag{5.6}$$

where we take the AdS radius to be one and $a = 1,\dots,d$. The isometries of this metric are given by the $SO(d+1,1)$ group. Therefore following an analysis along the lines of section 1.2, implies mapping the AdS$_{d+1}$ coordinates $y^\mu = (r, x^a)$,

[1] See also [170–173] for recent work using this formalism in the context of higher spin gravity.

$\mu = 1, \ldots, d+1$, to $\widetilde{\mathcal{M}}^{d+1,1} \subset \mathbb{R}^{d+1,1}$ coordinates $Y^A$, $A = 1, \ldots, d, d+1, d+2$, in such a way that the action of $SO(d+1,1)$ on $\widetilde{\mathcal{M}}^{d+1,1}$ corresponds to isometries in $\text{AdS}_{d+1}$. In order for this construction to be consistent we have to demand that $SO(d+1,1)$ is closed on $\widetilde{\mathcal{M}}^{d+1,1}$. As in the case for conformal coordinates, this can be done by imposing a constraint on the square of the length, i.e. $Y^2 = \text{const.}$ Moreover, this reduces the dimensionality of the space to $d+1$, which is what we want for $\text{AdS}_{d+1}$. This implies that

$$Y^2 = -1, \quad Y^{d+2} > 0, \tag{5.7}$$

with the map

$$Y^a = \frac{x^a}{r}, \quad Y^{d+1} = \frac{1 - r^2 - x^2}{2r}, \quad Y^{d+2} = \frac{1 + r^2 + x^2}{2r}. \tag{5.8}$$

Pulling the flat $\mathbb{R}^{d+1,1}$ metric onto $\widetilde{\mathcal{M}}^{d+1,1}$ is therefore

$$\begin{aligned} dY^A dY_A = dY^\mu dY_\mu + (dY^{d+1})^2 \\ - (dY^{d+2})^2 \Big|_{Y^a = \frac{x^a}{r}, Y^{d+1} = \frac{1-r^2-x^2}{2r}, Y^{d+2} = \frac{1+r^2+x^2}{2r}} = ds^2_{\text{AdS}}, \end{aligned} \tag{5.9}$$

which is $SO(d+1,1)$ invariant and equal to the $\text{AdS}_{d+1}$ metric. The AdS boundary points are obtained by sending $r \to 0$, which approaches the projective null-cone (1.74). In other words $P^A = \lim_{r\to 0} rY^A$.

Similar to section 1.2, tensors on $\widetilde{\mathcal{M}}^{d+1,1}$ are projected back to $\text{AdS}_{d+1}$ space by

$$t_{\mu_1 \cdots \mu_n} = \frac{\partial Y^{A_1}}{\partial y^{\mu_1}} \cdots \frac{\partial Y^{A_n}}{\partial y^{\mu_n}} \mathcal{T}_{A_1 \cdots A_n}(Y) \ , \tag{5.10}$$

where the projectors are

$$\frac{\partial Y^A}{\partial r} = -\frac{1}{r} Y^A + \bar{P}^A, \quad \frac{\partial Y^A}{\partial x^b} = \frac{1}{r} \left( \delta^a_b, -x_b, x_b \right), \tag{5.11}$$

and $\bar{P}$ is defined in (1.94). With these, it is easily shown that $\frac{\partial Y^A}{\partial y^\mu} Y_A = 0$, which implies that a tensor of the type $\mathcal{T}_{A_1 \cdots A_n}(Y) = Y_{(A_1} \mathcal{T}_{A_2 \cdots A_n)}(Y)$ is unphysical, i.e. it has a vanishing projection to $\text{AdS}_{d+1}$. Another consequence of this redundancy is that the induced AdS metric can be written as

$$G_{AB} = \eta_{AB} + Y_A Y_B \ , \tag{5.12}$$

which plays a role as a projector.

The next step is to encode symmetric traceless tensors by contracting their indices with a polarization vector $W$:

$$\mathcal{T}(Y, W) \equiv W^{A_1} \cdots W^{A_n} \mathcal{T}_{A_1 \cdots A_n}(Y) \ . \tag{5.13}$$

Then these correspond to encoded $\mathrm{AdS}_{d+1}$ tensors $t(y,w)$ (with $w$ a polarization in AdS), provided that

1. $W^2=0$—it encodes the tracelessness condition.
2. $W\cdot Y=0$—given that $Y^A$ projects to zero.
3. $\mathcal{T}(Y,W+\alpha Y)=\mathcal{T}(Y,W)$—it makes the tensor transverse to the surface $Y^2=-1$.
4. $(Y\cdot\partial_Y+W\cdot\partial_W+\mu)\mathcal{T}(Y,W)=0$ for some given value of $\mu$.[2]

Note that the transversality condition can be implemented via the induced metric $G$:

$$(G\mathcal{T})_{A_1\ldots A_n}(Y)\equiv G_{A_1}^{B_1}\cdots G_{A_n}^{B_n}\mathcal{T}_{B_1\ldots B_n}(Y),\quad Y^{A_i}(G\mathcal{T})_{A_1\ldots A_i\ldots A_n}(Y)=0. \tag{5.14}$$

As on the CFT side, to recover the tensor components we use a projector analogous to (5.5). Given [3]

$$\begin{aligned}K_A=&\frac{d-1}{2}\left(\frac{\partial}{\partial W^A}+Y_AY\cdot\frac{\partial}{\partial W}\right)+W\cdot\frac{\partial}{\partial W}\frac{\partial}{\partial W^A}\\&+Y_A\left(W\cdot\frac{\partial}{\partial W}\right)\left(Y\cdot\frac{\partial}{\partial W}\right)-\frac{1}{2}W_A\left(\frac{\partial^2}{\partial W\cdot\partial W}+Y\cdot\frac{\partial}{\partial W}Y\cdot\frac{\partial}{\partial W}\right),\end{aligned} \tag{5.15}$$

we obtain transverse, symmetric and traceless tensor via

$$\mathcal{T}_{A_1\cdots A_n}(Y)=\frac{1}{n!\left(\frac{d-1}{2}\right)_n}K_{A_1}\cdots K_{A_n}\mathcal{T}(Y,W)\ . \tag{5.16}$$

A covariant derivative in AdS is defined in the ambient space $\widetilde{\mathcal{M}}^{d+1,1}$ as

$$\nabla_A=\frac{\partial}{\partial Y^A}+Y_A\left(Y\cdot\frac{\partial}{\partial Y}\right)+W_A\left(Y\cdot\frac{\partial}{\partial W}\right)\ . \tag{5.17}$$

When acting on an transverse tensor we have

$$\nabla_B\mathcal{T}_{A_1\cdots A_n}(Y)=G_B^{B_1}G_{A_1}^{C_1}\cdots G_{A_n}^{C_n}\frac{\partial}{\partial Y^{B_1}}\mathcal{T}_{C_1\cdots C_n}(Y)\ , \tag{5.18}$$

where $G_{AB}$ is the induced AdS metric (5.12). In polynomial notation, the divergence of a tensor can be written as

$$\nabla\cdot(K\mathcal{T}(Y,W))\ , \tag{5.19}$$

[2]For a bulk massive spin-$J$ field in $\mathrm{AdS}_{d+1}$, we have $\mu=\Delta+J$ with $M^2=\Delta(\Delta-d)-J$.
[3]The form of this projector is chosen so that the transversality condition holds.

which projects to $\nabla^{\mu} t_{\mu\mu_2\ldots\mu_n}$ in $\mathrm{AdS}_{d+1}$. Similarly, index contractions are enconded by

$$t_{\mu_1\ldots\mu_n}\nabla^{\mu_1}\cdots\nabla^{\mu_n}\phi = \frac{1}{n!\left(\frac{d-1}{2}\right)_n}\mathcal{T}(Y,K)(W\cdot\nabla)^n\Phi(Y)\ ,$$
$$t_{\mu_1\ldots\mu_n}f^{\mu_1\ldots\mu_n} = \frac{1}{n!\left(\frac{d-1}{2}\right)_n}\mathcal{T}(Y,K)\mathcal{F}(Y,W)\ , \tag{5.20}$$

where $t$ and $f$ are symmetric and traceless tensors. Note that for transverse polynomials, the projector $K$ commutes with covariant derivative:

$$\nabla\cdot K = K\cdot\nabla\ . \tag{5.21}$$

It is useful to notice that for encoded polynomials in embedding space (5.13), where the tensor is already symmetric, traceless and transverse, the projector $K$ acquires a simpler form $K = \left(\frac{d-1}{2}+n-1\right)\partial_W$. Since this will be the case in all our calculations, we will simply use $\partial_W$ to contract indices.

### $\mathbf{AdS}_{d+1}$ propagators

Here we follow [169] and review some results of [174]; propagators in the AdS coordinates can be found in e.g. [175, 176] among many other references. We are interested in describing the propagator of a spin-$J$ field. In AdS coordinates, this field is a symmetric tensor that, in addition, satisfies the Fierz conditions

$$\nabla^2 h_{\mu_1\ldots\mu_J} = M^2 h_{\mu_1\ldots\mu_J}\ ,\quad \nabla^{\mu_1}h_{\mu_1\ldots\mu_J}=0\ ,\quad h^{\mu}{}_{\mu\mu_3\ldots\mu_J}=0\ . \tag{5.22}$$

These equations fully determine the AdS propagators, which we now write in the embedding formalism. The bulk–to–boundary propagator of a symmetric traceless field of rank $J$ is

$$G_{b\partial}^{\Delta|J}(Y_j,P_i;W_j,Z_i) = \mathcal{C}_{\Delta,J}\frac{\mathcal{H}_{ij}(Z_i,W_j)^J}{\Psi_{ij}^{\Delta}}\ , \tag{5.23}$$

where $\mathcal{C}_{\Delta,J}$ is a normalization (which we will ignore), and we defined

$$\Psi_{ij}\equiv -2P_i\cdot Y_j\ ,\quad \mathcal{H}_{ij}(Z_i,W_j)\equiv Z_i\cdot W_j + 2\frac{(W_j\cdot P_i)(Z_i\cdot Y_j)}{\Psi_{ij}}\ . \tag{5.24}$$

The mass squared is related to the conformal weight $\Delta$ of the dual operator as $M^2=\Delta(\Delta-d)-J$. This is the analogue of the CFT two point function (1.141). It will be also useful to rewrite the bulk–to–boundary propagator as [174]

$$G_{b\partial}^{\Delta|J}(Y,P;W,Z) = \frac{1}{(\Delta)_J}\left(\mathscr{D}_P(W,Z)\right)^J G_{b\partial}^{\Delta|0}(Y,P)\ , \tag{5.25}$$

where [4]

$$\mathscr{D}_P(W,Z) = (Z\cdot W)\left(Z\cdot\frac{\partial}{\partial Z} - P\cdot\frac{\partial}{\partial P}\right) + (P\cdot W)\left(Z\cdot\frac{\partial}{\partial P}\right) \ . \tag{5.26}$$

And it will also be convenient to cast the $n$-th derivative of $G_{b\partial}^{\Delta|J}$ in terms of scalar propagators:

$$\begin{aligned}(W'\cdot\partial_Y)^n G_{b\partial}^{\Delta|J}(Y,P;W,Z)&\\ = 2^n\Gamma(\Delta+n)\sum_{i=0}^{J}\sum_{k=0}^{i}\binom{J}{i}\binom{i}{k}&\frac{(n-k+1)_k}{\Gamma(\Delta+i)}(W\cdot P)^i(W\cdot Z)^{J-i}\\ &\times (W'\cdot Z)^k(W'\cdot P)^{n-k}(Z\cdot\partial_P)^{i-k}G_{b\partial}^{\Delta+n|0}(Y,P) \ . \end{aligned}\tag{5.27}$$

The bulk–to–bulk propagator of a spin-$J$ fields can be written as[5]

$$G_{bb}^{\Delta|J}(Y_i,Y_j;W_i,W_j) = \sum_{k=0}^{J}(W_i\cdot W_j)^{J-k}(W_i\cdot Y_j W_j\cdot Y_i)^k g_k(u) \ , \tag{5.28}$$

where $u = -1 + Y_{ij}/2$ and $Y_{ij} \equiv -2Y_i\cdot Y_j$. The functions $g_k$ can be written in terms of hypergeometric functions via

$$g_k(u) = \sum_{i=k}^{J}(-1)^{i+k}\left(\frac{i!}{j!}\right)^2\frac{h_i^{(k)}(u)}{(i-k)!} \ , \tag{5.29}$$

where $h_i$ is given by a recursion:

$$\begin{aligned} h_k &= c_k\Big((d-2k+2J-1)\left[(d+J-2)h_{k-1}+(1+u)h'_{k-1}\right]+(2-k+J)h_{k-2}\Big) \ ,\\ h_0 &= \frac{\Gamma(\Delta)}{2\pi^h\Gamma(\Delta+1-h)}(2u)^{-\Delta}{}_2F_1\left(\Delta,\Delta-h+\frac{1}{2},2\Delta-2h+1,-\frac{2}{u}\right),\\ c_k &= -\frac{1+J-k}{k(d+2J-k-2)(\Delta+J-k-1)(d-\Delta+J-k-1)} \ , \end{aligned}\tag{5.30}$$

## 5.3 Spinning geodesic Witten diagrams

The general objective of this section is to use CFT techniques in order to increase the spin of the external legs of Witten diagrams. Let us consider the three-point

[4]This differential operator $\mathscr{D}$ should not be confused with similar named CFT operators in (1.215).

[5]Note that (5.28) is not a homogeneous function of $Y$. In solving for the bulk-to-bulk operator the constrain $Y^2 = -1$ is used, which breaks the homogeneity property of the polynomials in embedding space.

Witten diagram for a moment

$$\int dy\, G_{b\partial}^{\Delta_1|0}(y,x_1)G_{b\partial}^{\Delta_2|0}(y,x_2)G_{b\partial}^{\Delta_3|0}(y,x_3)\,, \tag{5.31}$$

where the integral is over the AdS volume. It is well known [175] that it reproduces the three-point function of scalar operators with dimensions $\Delta_1$, $\Delta_2$, $\Delta_3$. Therefore exploiting a relation of the type (1.179) allows us to increase the spin of two legs in a three-point Witten diagram by applying differential operators. This was done in [174], and more recently, the action of single weight-shifting operators (discussed in section 1.5) on Witten diagrams was characterized in [124]. Moreover, using the split representation of the bulk–to–bulk propagator [169], these techiniques carry over to spinning four-point Witten diagrams at tree-level [177]. Applications to two-point spinning loop diagrams are also possible [178].

However, here we take an alternate approach. It turns out that the integral over the AdS volume in the three-point Witten diagram (5.31) can be reproduced by a one dimensional integral over a geodesic $\gamma_{ij}$ that connects a pair of endpoints $(x_i, x_j)$ on the boundary [179], called three-point *geodesic Witten diagram*:

$$\int_{\gamma_{ij}} d\lambda\, G_{b\partial}^{\Delta_1|0}(y(\lambda),x_1)G_{b\partial}^{\Delta_2|0}(y(\lambda),x_2)G_{b\partial}^{\Delta_3|0}(y(\lambda),x_3)\;, \tag{5.32}$$

where the equivalence is regardless the choice of endpoints; different choices just giving different numerical pre-factors.[6] Moreover, it was discovered in [120] that the restriction of the double-volume integral in the four-point Witten diagram to a double-line integral over the geodesics $\gamma_{ij}, \gamma_{kl}$ (which connect the boundary points $(x_i, x_j), (x_k, x_l)$) reproduces the conformal partial wave expansion $W_{\Delta|0}$ in the $(ij)(kl)$ channel (as defined in (1.185)):

$$\begin{aligned} W_{\Delta|0}(x_i) = \mathcal{W}_{\Delta|0}(x_i) \equiv \int_{\gamma_{12}} d\lambda \int_{\gamma_{34}} d\lambda' G_{b\partial}^{\Delta_1|0}(y(\lambda),x_1)G_{b\partial}^{\Delta_2|0}(y(\lambda),x_2) \\ \times G_{bb}^{\Delta|0}(y(\lambda),y'(\lambda'))G_{b\partial}^{\Delta_3|0}(y'(\lambda'),x_3)G_{b\partial}^{\Delta_4|0}(y'(\lambda'),x_4)\;, \end{aligned} \tag{5.33}$$

where $\lambda$ is an affine parameter for $\gamma_{12}$ and $\lambda'$ for $\gamma_{34}$. This is the simplest version of a four-point geodesic Witten diagram: the expression involves bulk–to–boundary and bulk–to–bulk propagators in AdS projected along geodesics connecting the endpoints, as depicted in Fig. 5.1. There was also evidence that this map worked correctly for more general partial waves [120, 179].

Our interest here is to explore cases where the legs of the three- and four-point geodesic Witten diagrams have non-trivial spin. In this section we develop the basis of AdS$_{d+1}$ differential operators that implement the CFT relation

[6]The results in [180, 181] as well suggested that (5.32) reproduces correlation functions of three scalar primaries.

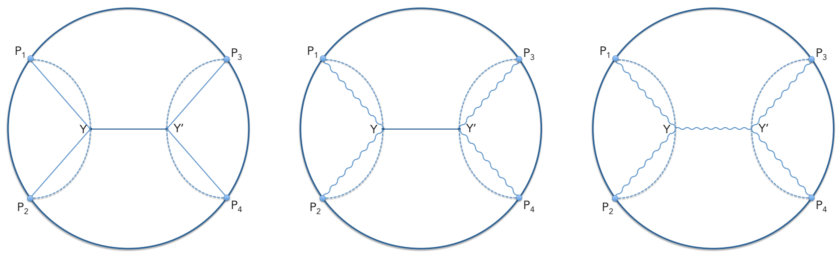


Figure 5.1: Examples of geodesic Witten diagrams in $\mathrm{AdS}_{d+1}$. The doted line indicates that we are projecting the propagators over a geodesic that connects the endpoints. Straight lines correspond to scalar fields, while wavy lines are symmetric traceless tensors of spin $J$. The first diagram corresponds to the scalar block in (5.33). The middle diagram (with scalar propagator in the exchange) will be the focus of section 5.3.1 and the last diagram (with a spin-$J$ field exchanged) is the focus of section 5.3.2.

(1.215) for symmetric-traceless representations, when acting on geodesic integrals (these are not necessarily three-point Witten diagrams, see Fig. 5.2). Therefore by (1.217) these give a prescription on how to obtain spinning partial waves $W_{\Delta|l}^{l_1,l_2,l_3,l_4}(x_1,x_2,x_3,x_4)$ from the geodesic Witten diagram (5.33). We stress that we will not use local cubic interactions to capture the conformal partial wave in this section. We postpone to section 5.4 the interpretation of this construction in terms of cubic interactions in the bulk.

### 5.3.1 Bulk differential basis: scalar exchanges

Recall from section 1.5.2 that for symmetric-traceless three-point functions, a convenient basis of differential operators are

$$D_{1\,ij}\ , \qquad D_{2\,ij}\ , \qquad \text{and} \qquad H_{ij}\ , \tag{5.34}$$

defined in (1.180). The operator $D_{1\,ij}$ increases the spin at position $i$ by one and decreases the dimension by one at position $i$; $D_{2\,ij}$ increases the spin at position $i$ by one and decreases the dimension by one at position $j$. $H_{ij}$ increases the spin by one at both $i$ and $j$ and leaves the conformal dimensions unchanged. These operators we will map to differential operators acting on bulk coordinates, except for $H_{ij}$, whose action will remain unchanged. $H_{ij}$ does induce a cubic interaction and we will discuss its effect in section 5.4.

The action of a single operator in (5.34) on an s-channel conformal partial wave $W_{\Delta|l}(P_i)$ will affect either the pair $(P_1, P_2)$ or $(P_3, P_4)$, but not all points simultaneously. So let us consider the components in the integral (5.33) that only depend on $\gamma_{12}$—which connects $(P_1, P_2)$:

$$\int_{\gamma_{12}} d\lambda\, G_{b\partial}^{\Delta_1|0}(Y_\lambda, P_1) G_{b\partial}^{\Delta_2|0}(Y_\lambda, P_2) G_{bb}^{\Delta|0}(Y_\lambda, Y') \ , \tag{5.35}$$

where we casted the propagators in embedding space, and used the notation $Y_\lambda$ to denote the embedded bulk point evaluated at the geodesic.[7] Fig. 5.2 depicts diagramatically the content in (5.35), and we note that $Y'$ is not necessarily projected over $\gamma_{34}$. Here $G_{b\partial}^{\Delta_1|0}(Y, P_1) \equiv G_{b\partial}^{\Delta_1|0}(Y, P_1; 0, 0)$ given in (5.23); in general we will omit dependence on variables that are not crucial for the equation in hand.

In Poincare coordinates, a geodesic connecting $x_i$ with $x_j$ is

$$\gamma_{ij}: \quad y^\mu(\lambda) = (r(\lambda), x^a(\lambda)) = \left( \frac{(x_{ij}^2)^{\frac{1}{2}}}{2\cosh(\lambda)}, \frac{x_i^a + x_j^a}{2} + \frac{(x_{ij})^a}{2} \tanh(\lambda) \right) \ ,$$
$$x_{ij} \equiv x_i - x_j \ . \tag{5.36}$$

The corresponding expression in embedding space is

$$\gamma_{ij}: \qquad Y_\lambda^A \equiv \frac{e^{-\lambda} P_i^A + e^{\lambda} P_j^A}{\sqrt{P_{ij}}} \ , \quad P_{ij} = -2 P_i \cdot P_j \ , \tag{5.37}$$

where we used (5.8) and (1.78). Replacing these expressions into (5.35) gives

$$\frac{1}{(P_{12})^{(\Delta_1+\Delta_2)/2}} \int_{-\infty}^{\infty} d\lambda\, e^{-\Delta_{12}\lambda}\, G_{bb}^{\Delta|0}(Y_\lambda, Y') \ , \qquad \Delta_{12} = \Delta_1 - \Delta_2 \ , \tag{5.38}$$

where we used (5.23). To increase the spin at $P_1$ and/or $P_2$ we would act on (5.38) with a combination of the differential operators in (5.34). By inspection of the integral in (5.38), $D_{i\,jk}$ has only a non-trivial action over the bulk–to–bulk propagators. In other words

$$D_{k\,ij} G_{b\partial}^{\Delta_n|0}(Y_\lambda, P_n) = O(P^2, Z \cdot P) \ , \qquad n = 1, 2 \ . \tag{5.39}$$

Hence, the task ahead is to build a bulk differential operator that acts on the third leg of the diagram: $G_{bb}^{\Delta|0}(Y_\lambda, Y')$, and reproduces $D_{k,ij}$.

Let us consider a general function $G(Y_\lambda \cdot Y')$ (not necessarily a bulk–to–bulk propagator) with no explicit dependence on $P_i$ (only through the geodesics in $Y_\lambda$),

[7] Throughout this chapter, we will use the symbols $Y^A$ and $W^A$ to denote embedded AdS points, and their auxiliary polarization vectors, respectively. Whereas $P^A$ and $Z^A$ denote their respective CFT counterparts.

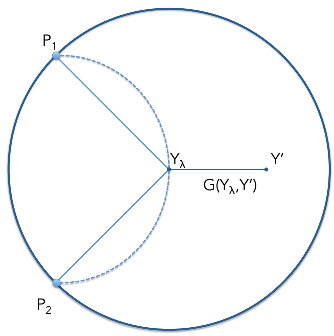


Figure 5.2: A precursor diagram where two legs are in the boundary and one in the bulk. This type of object appears at intermediate steps when evaluating conformal blocks.

and furthermore with no $W$ dependence. Then we want to construct differential operators $\mathcal{D}$ (not to be confused with the weight-shifting operators of section 1.5) such that

$$D_{k\,ij} G\left(Y_\lambda \cdot Y'\right) = \mathcal{D}_{k\,ij} G\left(Y_\lambda \cdot Y'\right) \ , \tag{5.40}$$

where $\mathcal{D}_{k\,ij}$ has derivatives with respect to $Y'$ only. Moreover, $\mathcal{D}$ has to satisfy the same properties of $D$. Namely, it has to be: transverse with respect to $P_i$, of order one in $Z_i$, of order zero in $Y'$, and with the correct transformation properties under $P \to \lambda P$. Hence the most generic form of $\mathcal{D}$ is $Y'^A Z_i^B S_{ABC} \partial^C_{Y'}$, where $S$ is independent of $Y'$ and $Z_i$. Assuming $S$ is a function of the flat metric $\eta$ and $P$, then transversality implies that $S$ has two classes of solutions compatible with (5.40):

$$\begin{aligned} S_{ABC} &= \eta_{AB}(P_i)_C - (P_i)_A \eta_{BC}, \\ S_{ABC} &= \left(\eta_{AB} + 2\frac{(P_i)_A (P_j)_B}{P_{ij}}\right)(P_j)_C - (P_j)_A \left(\eta_{BC} + 2\frac{(P_j)_B (P_i)_C}{P_{ij}}\right). \end{aligned} \tag{5.41}$$

Therefore we can write the dual operators $\mathcal{D}$ as

$$\begin{aligned} \mathcal{D}_{1\,ij} &= Z_i \cdot Y'\, P_i \cdot \partial_{Y'} + \frac{1}{2} \Psi_{iY'}\, Z_i \cdot \partial_{Y'} \ , \\ \mathcal{D}_{2\,ij} &= H_{ij}(Z_i, Y') P_j \cdot \partial_{Y'} + \frac{1}{2} \Psi_{jY'} H_{ij}(Z_i, \partial_{Y'}) \ , \end{aligned} \tag{5.42}$$

where $\Psi_{ij}$ is given in (5.24) and we have defined

$$H_{ij}(M, N) \equiv H_{ij}^{(M,N)}, \tag{5.43}$$

with $H_{ij}^{(M,N)}$ as in (1.125). As their name indicate, these operators have the same properties as their counterparts $D$. More precisely, $\mathcal{D}_{1\,ij}$ is increasing the spin by one and decreasing the dimension by one at position $i$, while $\mathcal{D}_{2\,ij}$ increases the spin at position $i$ by one and decreases the dimension by one at position $j$.

To verify that $\mathcal{D}$ has exactly the same effect as $D$, it is instructive to go through some identities. One can show the following relation by direct calculation (notice that the first operator in the commutator of the left hand side is $D$ and not $\mathcal{D}$)

$$[D_{k\,ij}, \mathcal{D}_{k'\,i'j'}]f(Y') = [\mathcal{D}_{k\,ij}, \mathcal{D}_{k'\,i'j'}]f(Y') \; . \tag{5.44}$$

Let us call $D_1$, $D_2$ two generic operators of the form $D_{k\,ij}$, then

$$\begin{aligned} D_1 D_2 (Y_\lambda \cdot Y') &= (D_1 Y_\lambda) \cdot (\mathcal{D}_2 Y') + Y_\lambda \cdot (D_1 \mathcal{D}_2 Y') \\ &= Y_\lambda \cdot (\mathcal{D}_2 \mathcal{D}_1 Y') + Y_\lambda \cdot ([D_1, \mathcal{D}_2] Y') \\ &= Y_\lambda \cdot (\mathcal{D}_1 \mathcal{D}_2 Y') = \mathcal{D}_1 \mathcal{D}_2 (Y_\lambda \cdot Y') \end{aligned} \tag{5.45}$$

where in the first line we used (5.40) for $D_2$ and the product rule for $D_1$, in the second line we used (5.40) for $D_1$ in the first term and the fact that $D_1 Y' = 0$ in the second, and finally in the third line we used used (5.44). Then for the product of an arbitrary number of operators,

$$\begin{aligned} D_1 D_2 \cdots D_n Y_\lambda \cdot Y' &= Y_\lambda \cdot (\mathcal{D}_2 \cdots \mathcal{D}_n \mathcal{D}_1 Y') + Y_\lambda \cdot (D_1 \mathcal{D}_2 \cdots \mathcal{D}_n Y') \\ &= Y_\lambda \cdot (\mathcal{D}_1 \mathcal{D}_2 \cdots \mathcal{D}_n Y') = \mathcal{D}_1 \mathcal{D}_2 \cdots \mathcal{D}_n Y_\lambda \cdot Y' \end{aligned} \tag{5.46}$$

where in the first line we used the induction hypothesis for $n-1$ operators and in the second line we pushed $D_1$ through and used (5.44) to put everything in terms of $\mathcal{D}$. The conclusion is that repeated application of boundary derivatives on geodesic integrals can be replaced by bulk derivatives in reverse order:

$$\begin{aligned} &H_{12}^{n_{12}} (\mathcal{D}_{2,12}^{n_1} \mathcal{D}_{2,21}^{n_2} \mathcal{D}_{1,12}^{m_1} \mathcal{D}_{1,21}^{m_2} - D_{2,12}^{n_1} D_{2,21}^{n_2} D_{1,12}^{m_1} D_{1,21}^{m_2}) \\ &\qquad \times \int_{\gamma_{12}} d\lambda \, G_{b\partial}^{\Delta_1|0}(Y_\lambda, P_1) G_{b\partial}^{\Delta_2|0}(Y_\lambda, P_2) G_{bb}^{\Delta|0}(Y_\lambda, Y') = 0 \; . \end{aligned} \tag{5.47}$$

We just found that the dual of $D$ are derivatives with respect to $Y'$. However, given that the generic form of these differential operators is $\mathcal{D}(Y') = Y'^A Z_i^B S_{ABC} \partial_{Y'}^C$, and $S$ is antisymmetric under $A \leftrightarrow C$ as seen in (5.41), then we have

$$\begin{aligned} Y'^A Z_i^B S_{ABC} \partial_{Y'}^C Y_\lambda \cdot Y' &= -Y_\lambda^A Z_i^B S_{ABC} \partial_{Y_\lambda}^C Y_\lambda \cdot Y' \\ &\Rightarrow \quad \mathcal{D}_{k\,ij}(Y') Y_\lambda \cdot Y' = -\mathcal{D}_{k\,ij}(Y_\lambda) Y_\lambda \cdot Y' \; . \end{aligned} \tag{5.48}$$

Using (5.48) it is easy to show that for more derivatives,

$$\mathcal{D}_{k_1\,i_1 j_1}(Y') \cdots \mathcal{D}_{k_n\,i_n j_n}(Y') Y_\lambda \cdot Y' = (-1)^n \mathcal{D}_{k_n\,i_n j_n}(Y_\lambda) \cdots \mathcal{D}_{k_1\,i_1 j_1}(Y_\lambda) Y_\lambda \cdot Y' \; . \tag{5.49}$$

This of course also holds when the derivatives act on $G(Y_\lambda \cdot Y')$. It is interesting to note that the action of a single $\mathcal{D}(Y_\lambda)$ on bulk–to–boundary operators is trivial, i.e.

$$\mathcal{D}_{k\,ij}(Y_\lambda) G_{b\partial}^{\Delta_{1,2}|0}(Y_\lambda, P_{1,2}) = O(P^2, Z\cdot P) \ , \tag{5.50}$$

which is consistent to how $D_{k\,ij}$ acts on bulk–to–boundary propagators at the geodesic (5.39). However,

$$\mathcal{D}_{k'\,i'j'}(Y_\lambda)\cdots \mathcal{D}_{k\,ij}(Y_\lambda) G_{b\partial}^{\Delta_{1,2}|0}(Y_\lambda, P_{1,2}) \neq 0 \ , \tag{5.51}$$

because (5.50) relies on properties of the geodesic $\gamma_{12}$, and in (5.51) the operation of taking derivatives with respect to $Y$ does not commute with projecting on $\gamma_{12}$.[8] Hence, as we generate tensorial structures using $\mathcal{D}(Y_\lambda)$, it only acts on $G_{bb}$, i.e.

$$\begin{aligned}(-1)^N \int_{\gamma_{12}} d\lambda\, G_{b\partial}^{\Delta_1|0}(Y_\lambda, P_1) G_{b\partial}^{\Delta_2|0}(Y_\lambda, P_2) \mathcal{D}_{1,21}^{m_2}\mathcal{D}_{1,12}^{m_1}\mathcal{D}_{2,21}^{n_2}\mathcal{D}_{2,12}^{n_1} G_{bb}^{\Delta_3|0}(Y_\lambda, Y') = \\ D_{2,12}^{n_1} D_{2,21}^{n_2} D_{1,12}^{m_1} D_{1,21}^{m_2} \int_{\gamma_{12}} d\lambda\, G_{b\partial}^{\Delta_1|0}(Y_\lambda, P_1) G_{b\partial}^{\Delta_2|0}(Y_\lambda, P_2) G_{bb}^{\Delta_3|0}(Y_\lambda, Y') \ ,\end{aligned} \tag{5.52}$$

where $N \equiv m_1 + m_2 + n_1 + n_2$.

From here we see how to cast conformal partial waves where the exchanged field is a scalar field (dual to a scalar primary $\mathcal{O}$ of conformal dimension $\Delta$): the version of (1.217) in gravitational language is

$$W_{\Delta|0}^{l_1,l_2,l_3,l_4}(P_i; Z_i) = \mathcal{W}_{\Delta|0}[\mathcal{D}_{\rm left}(Y_\lambda), \mathcal{D}_{\rm right}(Y'_{\lambda'})] \ , \tag{5.53}$$

where we define

$$\begin{aligned}\mathcal{W}_{\Delta|0}[\mathcal{D}_{\rm left}(Y_\lambda), \mathcal{D}_{\rm right}(Y'_{\lambda'})] \equiv \int_{\gamma_{12}}\int_{\gamma_{34}} G_{b\partial}^{\Delta_1|0}(Y_\lambda, P_1) G_{b\partial}^{\Delta_2|0}(Y_\lambda, P_2) \\ \times \left[\mathcal{D}_{\rm left}(Y_\lambda)\mathcal{D}_{\rm right}(Y'_{\lambda'}) G_{bb}^{\Delta|0}(Y_\lambda, Y'_{\lambda'})\right] G_{b\partial}^{\Delta_3|0}(P_3, Y'_{\lambda'}) G_{b\partial}^{\Delta_4|0}(P_4, Y'_{\lambda'}) \ .\end{aligned} \tag{5.54}$$

To close this subsection, we record another convenient way to re-write (5.42):

$$\begin{aligned}\mathcal{D}_{1\,ij}(Y_\lambda) &= \frac{\Psi_{i\lambda}}{2}\mathcal{H}_{i\lambda}(Z_i, \partial_{Y_\lambda}) \ , \\ \mathcal{D}_{2\,ij}(Y_\lambda) &= \frac{\Psi_{j\lambda}}{2}\left[\mathcal{H}_{i\lambda}(Z_i, \partial_{Y_\lambda}) + 2\mathcal{V}_{\partial\, i,j\lambda}(Z_i)\mathcal{V}_{b\,\lambda,ij}(\partial_{Y_\lambda})\right] \ ,\end{aligned} \tag{5.55}$$

[8]For $\mathcal{D}_{1\,21}$ and $\mathcal{D}_{2\,12}$, (5.50) is true without projecting on $\gamma_{12}$. Furthermore, (5.51) is true only if the $\mathcal{D}$'s do not commute. However, we will use (5.52) to treat all the $\mathcal{D}$'s in the same footing.

where $\mathcal{H}_{ij}$ is given in (5.24), and we defined

$$\mathcal{V}_{\partial\, i,jm}(Z_i) = \frac{\Psi_{im} Z_i \cdot P_j - P_{ij} Z_i \cdot Y_m}{\sqrt{\Psi_{im}\Psi_{jm}P_{ij}}} \ , \tag{5.56}$$

$$\mathcal{V}_{b\, m,ij}(W_m) = \frac{\Psi_{jm} W_m \cdot P_i - \Psi_{im} W_m \cdot P_j}{\sqrt{\Psi_{im}\Psi_{jm}P_{ij}}} \ , \tag{5.57}$$

which can be viewed as the generalizations of the CFT building block (1.124).

### 5.3.2 Bulk differential basis: spin exchanges

In the previous subsection we considered the geodesic integral (5.35) where the bulk–to–bulk propagator has no spin. We now generalize the discussion to include spin. The prescription given in [120] for spinning exchanged operators is that the bulk–to–bulk propagator for the spin $J$ field is contracted with the velocities of $Y_\lambda$ and $Y'_{\lambda'}$, i.e.

$$G_{bb}^{\Delta|J}(Y_\lambda, Y'_{\lambda'}) \equiv G_{bb}^{\Delta|J}\left(Y_\lambda, Y'_{\lambda'}; \frac{dY_\lambda}{d\lambda}, \frac{dY'_{\lambda'}}{d\lambda'}\right) \ . \tag{5.58}$$

This corresponds to the pullback of the propagator (5.28) along both geodesics in the diagram. Hence, a geodesic diagram that evaluates the conformal partial wave with a spin exchange is

$$\mathcal{W}_{\Delta|J}(P_1,P_2,P_3,P_4) = \int_{\gamma_{12}}\int_{\gamma_{34}} G_{b\partial}^{\Delta_1|0}(Y_\lambda,P_1)G_{b\partial}^{\Delta_2|0}(Y_\lambda,P_2)G_{bb}^{\Delta|J}(Y_\lambda,Y'_{\lambda'})G_{b\partial}^{\Delta_3|0}(Y'_{\lambda'},P_3)G_{b\partial}^{\Delta_4|0}(Y'_{\lambda'},P_4) \ . \tag{5.59}$$

In manipulating (5.58) to increase the spin of the external legs, we need to treat the contractions with $\frac{dY_\lambda}{d\lambda}$ with some care. First, it is important to note that $D_{k\,ij}$ commutes with $\frac{d}{d\lambda}$, and hence its action on $G_{bb}^{\Delta|J}(Y_\lambda, Y'_{\lambda'})$ in (5.59) is straightforward. However, we need to establish how $\mathcal{D}_{k\,ij}$ acts on (5.58), and this requires understanding how to cast $\frac{d}{d\lambda}$ as a covariant operation. It is easy to check by direct computation that this can be done in two ways:

$$\frac{d}{d\lambda} = -2P_{12}^{-1}\Psi_{2\lambda}P_1 \cdot \nabla_{Y_\lambda} = 2P_{12}^{-1}\Psi_{1\lambda}P_2 \cdot \nabla_{Y_\lambda} \ . \tag{5.60}$$

But the commutator of $\mathcal{D}_{k\,ij}$ with $\frac{d}{d\lambda}$ will depend on which equality we use. For example

$$D_{1\,12}\frac{dY_\lambda}{d\lambda} = -\mathcal{D}_{1\,12}(Y_\lambda)(-2P_{12}^{-1}\Psi_{2\lambda}P_1\cdot\nabla_{Y_\lambda})Y_\lambda\ , \tag{5.61}$$

$$D_{2\,21}\frac{dY_\lambda}{d\lambda} = -\mathcal{D}_{2\,21}(Y_\lambda)(-2P_{12}^{-1}\Psi_{2\lambda}P_1\cdot\nabla_{Y_\lambda})Y_\lambda\ , \tag{5.62}$$

which is the expected result by (5.46) and (5.49). Unfortunately, the two other $D$'s have the wrong sign relative to (5.46) and (5.49):

$$D_{1\,21}\frac{dY_\lambda}{d\lambda} = \mathcal{D}_{1\,21}(Y_\lambda)(-2P_{12}^{-1}\Psi_{2\lambda}P_1\cdot\nabla_{Y_\lambda})Y_\lambda\ , \tag{5.63}$$

$$D_{2\,12}\frac{dY_\lambda}{d\lambda} = \mathcal{D}_{2\,12}(Y_\lambda)(-2P_{12}^{-1}\Psi_{2\lambda}P_1\cdot\nabla_{Y_\lambda})Y_\lambda\ . \tag{5.64}$$

Using the other implementation of $\frac{d}{d\lambda}$ alternates the signs. In order to avoid this implementation problem, we formally define

$$\left[\mathcal{D}_{k\,ij}(Y_\lambda), \frac{d}{d\lambda}\right] \equiv 0\ . \tag{5.65}$$

This implies that as we encounter quantities that contain explicit derivatives of $\lambda$ we will manipulate them by first acting with $\mathcal{D}_{k\,ij}(Y_\lambda)$ and then taking the derivative with respect to $\lambda$. For instance,

$$\begin{aligned} D_{k\,ij}\frac{dY_\lambda}{d\lambda}\cdot\frac{dY'_{\lambda'}}{d\lambda'} &= \frac{d}{d\lambda}\frac{d}{d\lambda'}D_{k\,ij}Y_\lambda\cdot Y'_{\lambda'} \\ &= -\frac{d}{d\lambda}\frac{d}{d\lambda'}\mathcal{D}_{k\,ij}(Y_\lambda)Y_\lambda\cdot Y'_{\lambda'}\ . \end{aligned} \tag{5.66}$$

Given this implementation of the differential operators, the partial wave in gravitational language (5.53) generalizes to spinning exchanges by using (5.58) and (5.65). This shows that for each partial wave $W_{\Delta|J}^{l_1,l_2,l_3,l_4}(P_i;Z_i)$ in the boundary CFT there is a counterpart geodesic integral in AdS $\mathcal{W}_{\Delta|J}^{l_1,l_2,l_3,l_4}(P_i;Z_i)$ that reproduces the same quantity.

## 5.4 Gravitational interactions via geodesic diagrams

We have given in the previous section a systematic procedure to build the appropriate tensor structures $V_{i,jk}$ and $H_{ij}$ appearing in conformal partial waves by using directly bulk differential operators $\mathcal{D}_{i\,jk}(Y_\lambda)$. Using this method, we would like to identify the gravitational interactions that the operators $\mathcal{D}_{i\,jk}(Y_\lambda)$ are capturing.

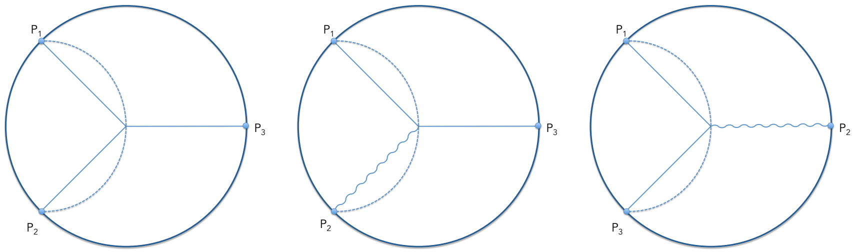


Figure 5.3: Examples of geodesic Witten diagrams in $\text{AdS}_{d+1}$ that capture three point functions. Straight lines correspond to scalar propagators, while wavy lines denote symmetric traceless spin-$J$ fields; $P_i$ is the boundary position in embedding formalism. The dotted line denotes the geodesic over which we integrate. Note that the second and third diagram only differ by the choice of geodesic.

The identification of tensor structures with gravitational interactions has been successfully carried out in [173]: all possible cubic vertices in $\text{AdS}_{d+1}$ where mapped to the tensor structures of a $\text{CFT}_d$ via Witten diagrams for three point functions. Here we would like to revisit this identification using instead as a building block diagrams in AdS that are projected over geodesic integrals rather than volume integrals; and as we will show below, the geodesic diagrams do suffer from some non-trivial cancellations for certain derivative interactions.

For the discussion in this section it is sufficient to consider the three-point geodesic Witten diagram of (5.32). The type of diagrams we will be considering in this section are depicted in Fig. 5.3, where the dotted line represents which geodesic we will integrate over. We will first attempt to rebuild interactions using these geodesic integrals, and at the end of this section we will contrast with the results in [173].

## 5.4.1 From the bulk differential basis to cubic interactions

In this subsection we will go through some explicit computations of three point functions using the method developed in section 5.3.1. Our goal is not to check that our bulk results match with the CFT values (which they do); our goal is to illustrate how these operators $\mathcal{D}_{i\,jk}(Y_\lambda)$, and hence $(V_{i,jk}, H_{ij})$, map up to local AdS interactions.

Our seed to all further computation is the three point function of three scalar primaries $\mathcal{K}_3^{\Delta_1,\Delta_2,\Delta_3}(P_i)$ defined in (1.118). In terms of geodesic integrals, this is given by

$$\mathcal{K}_3^{\Delta_1,\Delta_2,\Delta_3}(P_i) = c_{\Delta_1,\Delta_2,\Delta_3} \int_{\gamma_{12}} d\lambda\, G_{b\partial}^{\Delta_1|0}(Y_\lambda, P_1) G_{b\partial}^{\Delta_2|0}(Y_\lambda, P_2) G_{b\partial}^{\Delta_3|0}(Y_\lambda, P_3) \ , \tag{5.67}$$

where

$$c_{\Delta_1\Delta_2\Delta_3} = \frac{2\Gamma(\Delta_3)}{\Gamma\left(\frac{-\Delta_1+\Delta_2+\Delta_3}{2}\right)\Gamma\left(\frac{\Delta_1-\Delta_2+\Delta_3}{2}\right)} \ . \tag{5.68}$$

Here we are ignoring the normalization of $G_{b\partial}$ in (5.23) and the gamma functions in $c_{\Delta_1\Delta_2\Delta_3}$ result from the integration over the geodesic $\gamma_{12}$. $G_{\Delta_1,\Delta_2,\Delta_3|0,0,0} = \mathcal{K}_3^{\Delta_1,\Delta_2,\Delta_3}$ is the CFT$_d$ three point function casted as a geodesic integral in AdS$_{d+1}$.

**Example: Vector-scalar-scalar**

To start, we consider the three point function of one vector and two scalar operators as built from scalar operators. Following the CFT discussion in sections 1.4.2 and 1.5.2, in this case there is only one tensor structure which can be written in two ways:

$$\begin{aligned} G_{\Delta_1,\Delta_2,\Delta_3|1,0,0} &= V_{1,23}\mathcal{K}_3^{\Delta_1,\Delta_2,\Delta_3} \\ &= \frac{2D_{1\,12}}{-1-\Delta_1+\Delta_2+\Delta_3}\mathcal{K}_3^{\Delta_1+1,\Delta_2,\Delta_3} \\ &= \frac{2D_{2\,12}}{-1+\Delta_1-\Delta_2+\Delta_3}\mathcal{K}_3^{\Delta_1,\Delta_2+1,\Delta_3} \ . \end{aligned} \tag{5.69}$$

We would like to extract which local bulk interaction can capture the left hand side of (5.69). Let us choose the first equality for concreteness. Using (5.52) and the bulk differential basis definitions (5.55), the bulk calculation is

$$\begin{aligned} &\frac{2c_{\Delta_1+1\Delta_2\Delta_3}}{1+\Delta_1-\Delta_2-\Delta_3} \int_{\gamma_{12}} d\lambda\, G_{b\partial}^{\Delta_1+1|0}(Y_\lambda, P_1) G_{b\partial}^{\Delta_2|0}(Y_\lambda, P_2) \mathcal{D}_{1,12}(Y_\lambda) G_{b\partial}^{\Delta_3|0}(Y_\lambda, P_3) \\ &= \frac{c_{\Delta_1+1\Delta_2\Delta_3}}{1+\Delta_1-\Delta_2-\Delta_3} \\ &\quad\times \int_{\gamma_{12}} d\lambda\, G_{b\partial}^{\Delta_1+1|0}(Y_\lambda, P_1) G_{b\partial}^{\Delta_2|0}(Y_\lambda, P_2) \Psi_{1\lambda}\mathcal{H}_{1\lambda}(Z_1, \partial_{Y_\lambda}) G_{b\partial}^{\Delta_3|0}(Y_\lambda, P_3) \\ &= \frac{c_{\Delta_1+1\Delta_2\Delta_3}}{1+\Delta_1-\Delta_2-\Delta_3} \\ &\quad\times \int_{\gamma_{12}} d\lambda\, G_{b\partial}^{\Delta_1|1}(Y_\lambda, P_1; \partial_W, Z_1) G_{b\partial}^{\Delta_2|0}(Y_\lambda, P_2)(W\cdot\partial_{Y_\lambda}) G_{b\partial}^{\Delta_3|0}(Y_\lambda, P_3) \ , \end{aligned} \tag{5.70}$$

where in the second equality we used the definition of the spinning bulk-to-boundary propagator (5.23). The contraction appearing inside the integral can be attributed to the following local AdS interaction

$$A_1^\mu \phi_2 \partial_\mu \phi_3 \ , \tag{5.71}$$

where $\phi_i$ is a bulk scalar of mass $M_i^2 = \Delta_i(\Delta_i - d)$ and the massive vector $A_{1\mu}$ has $M_1^2 = \Delta_1(\Delta_1 - d) - 1$. It is interesting to note that from this computation alone we could not infer that there is another potential interaction: $A_1^\mu \phi_3 \partial_\mu \phi_2$. This particular interaction is absent because $A_1^\mu \partial_\mu \phi_2$ vanishes when evaluated over the geodesic $\gamma_{12}$ due to (5.50). However, it would have been the natural interaction if we instead perform the integral over $\gamma_{13}$ in (5.70). Hence a natural identification of the tensor structure in (5.69) with gravitational interactions is

$$V_{1,23} : \quad A_1^\mu \phi_2 \partial_\mu \phi_3 \qquad \text{and} \qquad A_1^\mu \phi_3 \partial_\mu \phi_2 \ . \tag{5.72}$$

If we used gauge invariance we could constraint this combination to insist that $A_1$ couples to a conserved current (for us, however, the vector $A_1$ is massive). From the perspective of the usual Witten diagrams, which involve bulk integrals, these two interactions are indistinguishable up to normalizations, since they can be related after integrating by parts. In a geodesic diagram one has to take both into account; in our opinion, it is natural to expect that all pairings of endpoints $P_i$ have to reproduce the same tensor structure.

**Example: Vector-vector-scalar**

Moving on to the next level of complexity, we now consider the geodesic integral that would reproduce the three point function of two spin-1 fields and one scalar field. There are two tensor structures involved in this correlator, which can be written as

$$V_{1,23} V_{2,13} \mathcal{K}_3^{\Delta_1,\Delta_2,\Delta_3} = -\frac{4 D_{1\,12} D_{1\,21} \mathcal{K}_3^{\Delta_1+1,\Delta_2+1,\Delta_3}}{(\Delta_1 - \Delta_2)^2 - \Delta_3^2} + \frac{H_{12} \mathcal{K}_3^{\Delta_1,\Delta_2,\Delta_3}}{-\Delta_1 + \Delta_2 + \Delta_3} \ , \tag{5.73}$$

and

$$H_{12} \, \mathcal{K}_3^{\Delta_1,\Delta_2,\Delta_3} \ . \tag{5.74}$$

$G_{\Delta_1,\Delta_2,\Delta_3|1,1,0}$ is the linear superposition of (5.73) and (5.74).

As it was already hinted by our previous example, the identification of the interaction will depend on the geodesic we choose to integrate over. To start, let us consider casting $\mathcal{K}_3^{\Delta_1,\Delta_2,\Delta_3}$ exactly as in (5.67): the geodesic is $\gamma_{12}$ which connects at the positions with non-trivial spin (first diagram in Fig. 5.4). For this choice of geodesic, the second tensor structure is straightforward to cast as a bulk interaction integrated over the geodesic. From the definitions (1.125) and (5.24), one can show that

$$H_{12} = \mathcal{H}_{1\lambda}(Z_1, \partial_W) \mathcal{H}_{2\lambda}(Z_2, W) \ , \tag{5.75}$$

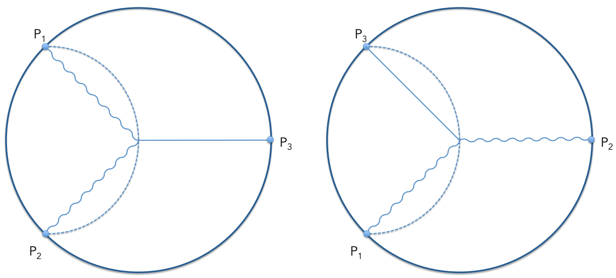


Figure 5.4: The diagrams here differ by the choice of geodesic. Depending on this choice, a given interaction will give rise to a different tensor structure.

where the right hand side is evaluated over the geodesic $\gamma_{12}$. Replacing this identity in (5.74), we find

$$H_{12}\,\mathcal{K}_3^{\Delta_1,\Delta_2,\Delta_3} = c_{\Delta_1,\Delta_2,\Delta_3}\int_{\gamma_{12}} G_{b\partial}^{\Delta_1|1}(Y_\lambda,P_1;\partial_W,Z_1)G_{b\partial}^{\Delta_2|1}(Y_\lambda,P_2;W,Z_2)G_{b\partial}^{\Delta_3|0}(Y_\lambda,P_3)\ . \tag{5.76}$$

This contact term is simply in physical space the interaction

$$H_{12}: \qquad A_{1\mu}A_2^\mu\phi_3\ . \tag{5.77}$$

This contraction will be generic every time our tensorial structure involves $H_{12}$. In general we will have the following relation

$$(H_{12})^n = (\mathcal{H}_{1\lambda}(Z_1,\partial_W)\mathcal{H}_{2\lambda}(Z_2,W))^n: \qquad h_{1\mu_1\ldots\mu_n}h_2^{\mu_1\ldots\mu_n}\phi_3\ , \tag{5.78}$$

where $(H_{12})^n$ generates one of the tensor structures for a tensor-tensor-scalar three point function, and the natural bulk interaction is the contraction of symmetric traceless tensors coupled minimally with a scalar.

For the other tensor structure, a bit more work is required. Let us first manipulate the first term in (5.73); using (5.49) we can write

$$\begin{aligned} D_{1\,12}D_{1\,21}G_{b\partial}^{\Delta_3|0}(Y_\lambda,P_3) &= \mathcal{D}_{1\,21}(Y_\lambda)\mathcal{D}_{1\,12}(Y_\lambda)G_{b\partial}^{\Delta_3|0}(Y_\lambda,P_3)\\ &= \frac{1}{8}\Psi_{1\lambda}\Psi_{2\lambda}\mathcal{H}_{1\lambda}(Z_1,\partial_W)\mathcal{H}_{2\lambda}(Z_2,\partial_W)(W\cdot\partial_{Y_\lambda})^2G_{b\partial}^{\Delta_3|0}(Y_\lambda,P_3)\\ &\qquad + \frac{1}{2}H_{12}\Psi_{2\lambda}P_1\cdot\partial_{Y_\lambda}G_{b\partial}^{\Delta_3|0}(Y_\lambda,P_3)\ . \end{aligned} \tag{5.79}$$

Applying this expression to (5.73) gives[9]

$$
\begin{aligned}
&-\frac{4D_{1\,12}D_{1\,21}}{(\Delta_1-\Delta_2)^2-\Delta_3^2}\mathcal{K}_3^{\Delta_1+1,\Delta_2+1,\Delta_3}\\
&\qquad=-\frac{4c_{\Delta_1+1,\Delta_2+1,\Delta_3}}{(\Delta_1-\Delta_2)^2-\Delta_3^2}\int_{\gamma_{12}}G_{b\partial}^{\Delta_1+1|0}G_{b\partial}^{\Delta_2+1|0}\mathcal{D}_{1,21}\mathcal{D}_{1,12}G_{b\partial}^{\Delta_3|0}\\
&\quad=-\frac{1}{2}\frac{c_{\Delta_1+1,\Delta_2+1,\Delta_3}}{(\Delta_1-\Delta_2)^2-\Delta_3^2}\int_{\gamma_{12}}G_{b\partial}^{\Delta_1|1}(\partial_W)G_{b\partial}^{\Delta_2|1}(\partial_W)(W\cdot\partial_{Y_\lambda})^2G_{b\partial}^{\Delta_3|0}\\
&\qquad\qquad\qquad-\frac{1}{-\Delta_1+\Delta_2+\Delta_3}H_{12}\mathcal{K}_3^{\Delta_1,\Delta_2,\Delta_3}\ .
\end{aligned}
\tag{5.80}
$$

Replacing (5.80) in (5.73) results in

$$
\begin{aligned}
&V_{1,23}V_{2,13}\mathcal{K}_3^{\Delta_1,\Delta_2,\Delta_3}\\
&\quad=-\frac{c_{\Delta_1+1\Delta_2+1\Delta_3}}{2((\Delta_1-\Delta_2)^2-\Delta_3^2)}\int_{\gamma_{12}}G_{b\partial}^{\Delta_1|1}(\partial_W)G_{b\partial}^{\Delta_2|1}(\partial_W)(W\cdot\partial_{Y_\lambda})^2G_{b\partial}^{\Delta_3|0}\ .
\end{aligned}
\tag{5.81}
$$

From here we see that another natural relation arises between the tensor structures and interactions:

$$
V_{1,23}V_{2,13}:\quad A_1^\mu A_2^\nu\partial_{(\mu}\partial_{\nu)}\phi_3\quad\sim\quad A_1^\mu A_2^\nu\left(\nabla_{(\mu}\nabla_{\nu)}+\Delta_3 g_{\mu\nu}\right)\phi_3\ . \tag{5.82}
$$

where the sign $\sim$ here means that the relation is schematic: to rewrite interactions with partial derivatives as covariant derivatives, we are using homogeneity properties of fields in the embedding formalism in (5.81). In what follows we will keep most of our expressions in terms of partial derivatives.

Now let us consider building $G_{\Delta_1,\Delta_2,\Delta_3|1,1,0}$ starting from a geodesic diagram where we integrate over $\gamma_{13}$ instead of $\gamma_{12}$ (second diagram in Fig. 5.4). The diagram with $\gamma_{12}$ already suggested as candidate interactions (5.77) and (5.82). If we integrate those interactions over $\gamma_{13}$ we find[10]

$$
\int_{\gamma_{13}}A_1^\mu A_2^\nu\partial_{(\mu}\partial_{\nu)}\phi_3=0\ , \tag{5.83}
$$

and $A_1^\mu A_{2\mu}\phi_3$ gives a linear combination of $V_{1,23}V_{2,13}$ and $H_{12}$. The identifications we made in (5.77) and (5.82) are obviously sensitive to the geodesic we select (there is a non-trivial kernel), and this is somewhat unsatisfactory. We can partially

[9]The fastest way to reproduce (5.80) from (5.79) is by using the explicit form of $G_{b\partial}^{\Delta_3|0}(Y_\lambda,P_3)$. An alternative route, which is more general, is to use (5.60): from here we can integrate by parts and rearrange the terms appropriately. This second route allows us to use (5.81) when at the third leg of the vertex we have bulk–to–bulk propagators rather than bulk–to–boundary.

[10]We are being schematic and brief in (5.83): it is implicit that we are using bulk–to–boundary propagators.

overcome this pathology by considering a wider set of interactions. By inspection we find that the tensor structure $V_{1,23}V_{2,13}$ is simultaneously captured by $\gamma_{13}$ and $\gamma_{12}$ by the interactions

$$V_{1,23}V_{2,13}: \qquad \alpha_1 A_1^\nu A_2^\mu \partial_\nu \partial_\mu \phi_3 \\ - \beta_1 \left( (\Delta_1+\Delta_2)\phi_3 \partial_\mu A_1^\nu \partial_\mu A_2^\nu - (1+\Delta_1\Delta_2)\phi_3 \partial_\nu A_1^\mu \partial_\mu A_2^\nu \right) . \quad (5.84)$$

The choice of geodesic affects the overall normalization, controlled by the choice of constants $\alpha_1$ and $\beta_1$. When projected over $\gamma_{12}$, the terms multiplying $\beta_1$ are proportional to $H_{12}$ and their coefficients are chosen such that they cancel each other. The interaction multiplying $\alpha_1$ is identically zero when integrated over $\gamma_{13}$. To capture $H_{12}$ along both $\gamma_{13}$ and $\gamma_{12}$ we just need

$$H_{12}: \qquad \phi_3 F_{1\mu\nu} F_2^{\mu\nu} . \quad (5.85)$$

Here it is important to note we are not using $A_1^\mu A_{2\mu}\phi_3$ as we did in (5.77), and we still find the correct result when using $\gamma_{12}$. This is because there are many ways we can cast $H_{12}$ as bulk quantities along $\gamma_{12}$: the relation (5.75) is not unique. For instance, one can check that

$$G_{b\partial}^{\Delta_1|1}(Y_\lambda, P_1; \partial_W, Z_1) G_{b\partial}^{\Delta_2|1}(Y_\lambda, P_2; W, Z_2) \\ = -\frac{1}{2(\Delta_1+\Delta_2)} (\partial_W \cdot \partial_{Y'})(\partial_{W'} \cdot \partial_Y) G_{b\partial}^{\Delta_1|1}(Y', P_1; W', Z_1) G_{b\partial}^{\Delta_2|1}(Y, P_2; W, Z_2)\big|_{Y=Y'=Y_\lambda} \\ = -\frac{1}{2(1+\Delta_1\Delta_2)} (\partial_Y \cdot \partial_{Y'}) G_{b\partial}^{\Delta_1|1}(Y', P_1; \partial_W, Z_1) G_{b\partial}^{\Delta_2|1}(Y, P_2; W, Z_2)\big|_{Y=Y'=Y_\lambda} \quad (5.86)$$

This type of relations are due to the projections over the geodesic, and they generate quite a bit of ambiguity as one tries to re-cast a given geodesic diagram as arising from a cubic interaction. Establishing relations such as (5.84) and (5.85) are not fundamental, and their ambiguity is not merely due to integrating by parts or using equations of motion. In appendix B we provide further examples on how to rewrite certain tensor structures as interactions, but we have not taken into account ambiguities such as those in (5.86). Generalizing (5.84) and (5.85) for higher spin fields is somewhat cumbersome (but not impossible). We comment in the discussion what are the computational obstructions we encounter to carry this out explicit.

### 5.4.2 Basis of cubic interactions via Witten diagrams

In the above we made use of our bulk differential basis to identify which interactions capture the suitable tensor structures that label the various correlation functions in the bulk. It is time now to compare with the results in [173].

The most general cubic vertex among the symmetric-traceless fields of spin $J_i$ and mass $M_i$ ($i = 1, 2, 3$) is a linear combination of interactions [168, 182–184]

$$V_3 = \sum_{n_i=0}^{J_i} g(n_i) I^{n_1,n_2,n_3}_{J_1,J_2,J_3}(Y_i)|_{Y_i=Y} \ , \tag{5.87}$$

where $g(n_i)$ are arbitrary coupling constants, and

$$\begin{aligned} I^{n_1,n_2,n_3}_{J_1,J_2,J_3}(Y_i) = \mathcal{Y}_1^{J_1-n_2-n_3}\mathcal{Y}_2^{J_2-n_3-n_1}\mathcal{Y}_3^{J_3-n_1-n_2} \\ \times \mathcal{H}_1^{n_1}\mathcal{H}_2^{n_2}\mathcal{H}_3^{n_3}\, \mathcal{T}_{J_1}(Y_1,W_1)\mathcal{T}_{J_2}(Y_2,W_2)\mathcal{T}_{J_3}(Y_3,W_3) \ . \end{aligned} \tag{5.88}$$

Here $\mathcal{T}_{J_i}(Y_i, W_i)$ are polynomials in the embedding formalism that contain the components of the symmetric traceless tensor field in AdS. This cubic interaction is built out of six basic contractions which are defined as[11]

$$\begin{aligned} \mathcal{Y}_1 = \partial_{W_1}\cdot\partial_{Y_2} \ , \quad \mathcal{Y}_2 = \partial_{W_2}\cdot\partial_{Y_3} \ , \quad \mathcal{Y}_3 = \partial_{W_3}\cdot\partial_{Y_1} \ , \\ \mathcal{H}_1 = \partial_{W_2}\cdot\partial_{W_3} \ , \quad \mathcal{H}_2 = \partial_{W_1}\cdot\partial_{W_3} \ , \quad \mathcal{H}_3 = \partial_{W_1}\cdot\partial_{W_2} \ . \end{aligned} \tag{5.89}$$

For more details on the construction of this vertex we refer to [168]. What is important to highlight here are the following two features. First, $V_3$ is the most general interaction modulo field re-parametrization and total derivatives. Second, the number of terms in (5.87) is exactly the same as the number of independent structures in a CFT three point function (1.156).

The precise map between these interactions and tensor structures is in appendix A of [173] (which is too lengthy to reproduce here). The first few terms give the following map:[12]

$$\begin{aligned} I^{0,0,0}_{1,0,0} &= A_1^\mu(\partial_\mu\phi_2)\phi_3 \underset{\text{bulk}}{\longrightarrow} V_{1,23} \\ I^{1,0,0}_{1,1,0} &= A_1^\mu A_{2\mu}\phi_3 \underset{\text{bulk}}{\longrightarrow} \left((\Delta_1-\Delta_2)^2-\Delta_3^2\right)V_{1,23}V_{2,13} \\ &\qquad - (-2\Delta_1\Delta_2+\Delta_1+\Delta_2-\Delta_3)H_{12} \\ I^{0,0,0}_{1,1,0} &= A_1^\mu(\partial_\mu A_2^\nu)\partial_\nu\phi_3 \underset{\text{bulk}}{\longrightarrow} (\Delta_1+\Delta_2-\Delta_3-2)V_{1,23}V_{2,13}+H_{12} \end{aligned} \tag{5.90}$$

In a nutshell this map is done by evaluating suitable Witten diagrams that capture three point functions and identify the resulting tensor structures. In appendix A

[11] As mentioned before all derivatives here are partial, but by using the homogeneity of $\mathcal{T}_{J_i}(Y_i, W_i)$ one can relate them to covariant derivatives.

[12] Here the notation $\underset{\text{bulk}}{\longrightarrow}$ means that the identification between the interaction and tensor structure is done via a bulk integral, i.e. a three-point Witten diagram. Similarly, $\underset{\gamma_{ij}}{\longrightarrow}$ denotes an analogous integral over a geodesic.

we derive specific examples to illustrate the mapping. Using this same basis of interactions and integrating them along $\gamma_{12}$ gives the following map

$$\begin{aligned} I^{0,0,0}_{1,0,0} &= A^\mu_1(\partial_\mu\phi_2)\phi_3 &\underset{\gamma_{12}}{\longrightarrow}\quad & 0 \\ I^{1,0,0}_{1,1,0} &= A^\mu_1 A_{2\mu}\phi_3 &\underset{\gamma_{12}}{\longrightarrow}\quad & H_{12} \\ I^{0,0,0}_{1,1,0} &= A^\mu_1(\partial_\mu A^\nu_2)\partial_\nu\phi_3 &\underset{\gamma_{12}}{\longrightarrow}\quad & H_{12} \end{aligned} \tag{5.91}$$

Clearly there is a tension between the tensor structures we assign to an interaction if we use a regular Witten diagram versus a geodesic diagram. The mismatch is due to the fact that certain derivatives contracted along $\gamma_{ij}$ are null. This reflects upon that a geodesic diagram is sensitive to the arrangement of derivatives which, for good reasons, are discarded in (5.87).

Nonetheless, some agreements do occur when considering interactions compatible with all geodesic integrations $\gamma_{ij}$. From (5.85) we have (up to overall normalizations)

$$\phi_3 F_{1\mu\nu}F^{\mu\nu}_2 \quad \underset{\gamma_{ij}}{\longrightarrow} \quad H_{12} \tag{5.92}$$

If we use these interactions on Witten diagrams, we obtain exactly the same map

$$\phi_3 F_{1\mu\nu}F^{\mu\nu}_2 \quad \underset{\text{bulk}}{\longrightarrow} \quad H_{12}\ . \tag{5.93}$$

The details of the computations leading to (5.93) are shown in appendix A. Moreover, we find that the interaction (5.84), which is $V_{1,23}V_{2,13}$ for the geodesic Witten diagram, gives the same tensor structure if we integrate over the bulk, as shown in (5.135). These relations indicate that it is possible to a have a compatible map among interactions in geodesic diagrams and Witten diagrams, even though there is disagreement at intermediate steps. However, from a bulk perspective the interaction selected in (5.93) is not in any special footing relative to those in (5.87).

## 5.5 Conformal block decomposition of spinning four-point Witten diagrams

For a fixed cubic interaction, there is generically a mismatch among tensor structures captured by Witten diagrams versus geodesic Witten diagrams. In this section we will analyse how this affects the decomposition of four-point Witten diagrams in terms of geodesic diagrams.

Our discussion is based in the four-point exchange diagram for four scalars fields done in [120], which we quickly review here. In Fig. 5.5 we represent the exchange: all fields involved are scalars, where the external legs have dimension $\Delta_i$ and the exchange field has dimension $\Delta$. The corresponding Witten diagram is

$$\begin{aligned}&\mathcal{A}^{\text{Exch}}_{0,0,0,0}(P_i)\\&=\int dY\int dY' G^{\Delta_1|0}_{b\partial}(Y,P_1)G^{\Delta_2|0}_{b\partial}(Y,P_2)G^{\Delta|0}_{bb}(Y,Y')G^{\Delta_3|0}_{b\partial}(Y',P_3)G^{\Delta_4|0}_{b\partial}(Y',P_4)\,.\end{aligned} \tag{5.94}$$

Here "$dY$" represents volume integrals in $\text{AdS}_{d+1}$. To write this expression as geodesic integrals, the crucial observation is that

$$G^{\Delta_1|0}_{b\partial}(Y,P_1)G^{\Delta_2|0}_{b\partial}(Y,P_2)=\sum_{m=0}^{\infty}a_m^{\Delta_1,\Delta_2}\varphi_m(\Delta_1,\Delta_2;Y)\;, \tag{5.95}$$

where

$$\varphi_m(\Delta_1,\Delta_2;Y)\equiv\int_{\gamma_{12}}G^{\Delta_1|0}_{b\partial}(Y_\lambda,P_1)G^{\Delta_2|0}_{b\partial}(Y_\lambda,P_2)G^{\Delta_m|0}_{bb}(Y_\lambda,Y)\;. \tag{5.96}$$

The field $\varphi_m(Y)$ is a normalizable solution of the Klein-Gordon equation with a source concentrated at $\gamma_{12}$ and mass $M^2=\Delta_m(\Delta_m-d)$. The equality in (5.95) holds whenever

$$a_m^{\Delta_1,\Delta_2}=\frac{(-1)^m}{m!}\frac{(\Delta_1)_m(\Delta_2)_m}{\beta_m(\Delta_1+\Delta_2+m-d/2)_m}\;,\quad \Delta_m=\Delta_1+\Delta_2+2m\;. \tag{5.97}$$

The constant $\beta_m$ soaks the choice of normalizations used in (5.96). Replacing (5.96) twice in (5.94) then gives

$$\begin{aligned}\mathcal{A}^{\text{Exch}}_{0,0,0,0}(P_i)&=\sum_{m,n}a_m^{\Delta_1,\Delta_2}a_n^{\Delta_3,\Delta_4}\\&\times\int_{\gamma_{12}}\int_{\gamma_{34}}G^{\Delta_1|0}_{b\partial}(Y_\lambda,P_1)G^{\Delta_2|0}_{b\partial}(Y_\lambda,P_2)G^{\Delta_3|0}_{b\partial}(Y'_{\lambda'},P_3)G^{\Delta_4|0}_{b\partial}(Y'_{\lambda'},P_4)\\&\qquad\times\int dY\int dY' G^{\Delta_m|0}_{bb}(Y_\lambda,Y)G^{\Delta|0}_{bb}(Y,Y')G^{\Delta_n|0}_{bb}(Y',Y'_{\lambda'})\;.\end{aligned} \tag{5.98}$$

The integrals in the last line can be simplified by using

$$G^{\Delta|0}_{bb}(Y,Y')=\langle Y|\frac{1}{\nabla^2-M^2}|Y'\rangle\;,\qquad\int dY|Y\rangle\langle Y|=1\;, \tag{5.99}$$

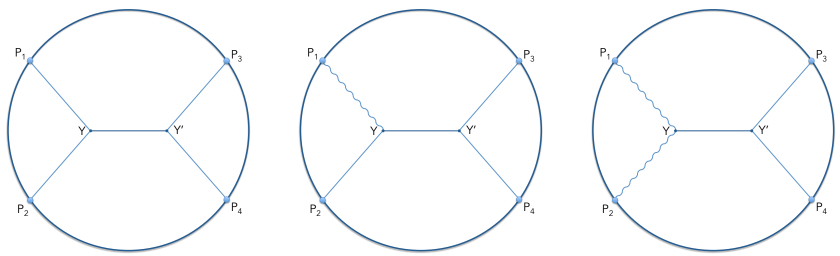


Figure 5.5: Four-point exchange Witten diagrams in $\mathrm{AdS}_{d+1}$, where the exchanged field is a scalar field of dimension $\Delta$. The first diagram corresponds to $\mathcal{A}^{\text{Exch}}_{0,0,0,0}$ in (5.94), the second diagram to $\mathcal{A}^{\text{Exch}}_{1,0,0,0}$ in (5.102), and the third diagram to $\mathcal{A}^{\text{Exch}}_{1,1,0,0}$ in (5.107).

which leads to

$$\int dY \int dY' G^{\Delta_m|0}_{bb}(Y_\lambda, Y) G^{\Delta|0}_{bb}(Y, Y') G^{\Delta_n|0}_{bb}(Y', Y'_{\lambda'}) =$$
$$\frac{G^{\Delta|0}_{bb}(Y_\lambda, Y'_{\lambda'})}{(M^2_\Delta - M^2_m)(M^2_\Delta - M^2_n)} + \frac{G^{\Delta_m|0}_{bb}(Y_\lambda, Y'_{\lambda'})}{(M^2_m - M^2_\Delta)(M^2_m - M^2_n)} + \frac{G^{\Delta_n|0}_{bb}(Y_\lambda, Y'_{\lambda'})}{(M^2_n - M^2_\Delta)(M^2_n - M^2_m)} . \tag{5.100}$$

And hence the four-point exchange diagram for scalars is

$$\mathcal{A}^{\text{Exch}}_{0,0,0,0}(P_i) = C_\Delta \mathcal{W}_{\Delta|0}(P_i) + \sum_m C_{\Delta_m} \mathcal{W}_{\Delta_m|0}(P_i) + \sum_n C_{\Delta_n} \mathcal{W}_{\Delta_n|0}(P_i) , \tag{5.101}$$

where we organized the expression in terms of the geodesic integral that defines $\mathcal{W}_{\Delta|0}$ in (5.33); the coefficients $C_\Delta$ basically follow from the contributions in (5.98) and (5.100).

### 5.5.1 Scalar exchange with one vector leg

Now let us generalize this decomposition to the case where the external legs have spin. The first non-trivial example is to just add a spin-1 field in one external leg and all other fields involved are scalar. The diagram is depicted in Fig. 5.5, and the integral expression is

$$\mathcal{A}^{\text{Exch}}_{1,0,0,0} = \int dY \int dY' G^{\Delta_1|1}_{b\partial}(Y, P_1, Z_1, \partial_W) \left( W \cdot \partial_Y G^{\Delta_2|0}_{b\partial}(Y, P_2) \right) G^{\Delta|0}_{bb}(Y, Y')$$
$$\times G^{\Delta_3|0}_{b\partial}(Y', P_3) G^{\Delta_4|0}_{b\partial}(Y', P_4) , \tag{5.102}$$

where we used one of the vertex interactions in (5.87). Using (5.25) and (5.27) we can rewrite this diagram in terms of the four-point scalar exchange (5.94) as

$$\mathcal{A}^{\text{Exch}}_{1,0,0,0}(\Delta_1,\Delta_2,\Delta_3,\Delta_4) = \frac{2\Delta_2}{\Delta_1}\, D_{2\,12}\mathcal{A}^{\text{Exch}}_{0,0,0,0}(\Delta_1,\Delta_2+1,\Delta_3,\Delta_4)\ , \tag{5.103}$$

and $D_{2\,12}$ is defined in (1.180). From here the path is clear: using the geodesic decomposition (5.101) and trading $D_{212}$ by $-\mathcal{D}_{2\,12}(Y_\lambda)$ we obtain

$$\mathcal{A}^{\text{Exch}}_{1,0,0,0} = \tilde{C}_\Delta \mathcal{W}^{1,0,0,0}_{\Delta|0} + \sum_m \tilde{C}_{\Delta_m}\mathcal{W}^{1,0,0,0}_{\Delta_m|0} + \sum_n \tilde{C}_{\Delta_n}\mathcal{W}^{1,0,0,0}_{\Delta_n|0}\ , \tag{5.104}$$

with suitable constants $\tilde{C}$ and

$$\begin{aligned}\mathcal{W}^{1,0,0,0}_{\Delta|0}&(\Delta_1,\Delta_2,\Delta_3,\Delta_4) = D_{2\,12}\mathcal{W}_{\Delta|0}(\Delta_1,\Delta_2+1,\Delta_3,\Delta_4)\\ &= -\frac{1}{2}\int_{\gamma_{12}}\int_{\gamma_{34}} G^{\Delta_1|1}_{b\partial}(Y_\lambda,P_1,Z_1,\partial_W)G^{\Delta_2|0}_{b\partial}(Y_\lambda,P_2)W\cdot\partial_{Y_\lambda}G^{\Delta|0}_{bb}(Y_\lambda,Y'_{\lambda'})\\ &\qquad\qquad\times G^{\Delta_3|0}_{b\partial}(Y'_{\lambda'},P_3)G^{\Delta_4|0}_{b\partial}(Y'_{\lambda'},P_4)\ ,\end{aligned} \tag{5.105}$$

where we used (5.70). It is interesting to note how the interaction gets slightly modified due to the cancellations that occur in the geodesic integrals: in (5.102) the derivative is acting on $G^{\Delta_2|0}_{b\partial}$, but the geodesic decomposition moves it to position of the exchanged field.

In this example it is also worth discussing the generalization of (5.95). Our decomposition of the bulk–to–boundary operators on position 1 and 2 reads

$$\begin{aligned}&G^{\Delta_1|1}_{b\partial}(Y,P_1,Z_1,\partial_W)W\cdot\partial_Y G^{\Delta_2|0}_{b\partial}(Y,P_2)\\ &= \frac{2\Delta_2}{\Delta_1}D_{2\,12}\left(G^{\Delta_1|0}_{b\partial}(Y,P_1)G^{\Delta_2+1|0}_{b\partial}(Y,P_2)\right)\\ &= \frac{2\Delta_2}{\Delta_1}\sum_{m=0}^{\infty} a_m^{\Delta_1,\Delta_2+1}\mathcal{D}_{2\,12}(Y)\varphi_m(\Delta_1,\Delta_2+1;Y)\\ &= -\frac{\Delta_2}{\Delta_1}\sum_{m=0}^{\infty} a_m^{\Delta_1,\Delta_2+1}\\ &\quad\times\int_{\gamma_{12}} G^{\Delta_1|1}_{b\partial}(Y_\lambda,P_1,Z_1,\partial_W)G^{\Delta_2|0}_{b\partial}(Y_\lambda,P_2)\,W\cdot\partial_{Y_\lambda}G^{\Delta_m|0}_{bb}(Y_\lambda,Y)\ .\end{aligned} \tag{5.106}$$

It is interesting to note the different interpretations one could give to the product $A_1^\mu\partial_\mu\phi_2$ (first line) in terms of resulting bulk fields. Very crudely, from the third line one would like to say that we just have a suitable differential operator acting on the field, while from the fourth line we would say that the product induces an interaction integrated along the geodesic. This type of decompositions of bulk fields would be interesting in the context of developing further a relation between an OPE expansion in the CFT to local bulk fields as done in [180, 181, 185].

### 5.5.2 Scalar exchange with two vector legs

It is instructive as well to discuss an example with two spin-1 fields as shown in the third diagram of Fig. 5.5. For sake of simplicity we will use the cubic interaction $A_{1\mu}A_2^{\mu}\phi$, which is part of the basis in (5.87). The four-point exchange is

$$\mathcal{A}^{\text{Exch}}_{1,1,0,0} = \int dY \int dY' G^{\Delta_1|1}_{b\partial}(Y, P_1, Z_1, \partial_W) G^{\Delta_2|1}_{b\partial}(Y, P_2, Z_2, W) G^{\Delta|0}_{bb}(Y, Y') \\ \times G^{\Delta_3|0}_{b\partial}(Y', P_3) G^{\Delta_4|0}_{b\partial}(Y', P_4) \ . \quad (5.107)$$

The new pieces are due to the presence of the spin-1 fields so we will focus on how to manipulate the propagators at position 1 and 2; the rest follows as in previous examples. Using (5.25) allows us to remove the tensorial pieces in (5.107) and recast it in terms of tensor structures. For this case in particular we have

$$\begin{aligned} G^{\Delta_1|1}_{b\partial}&(Y, P_1, Z_1, \partial_W) G^{\Delta_2|1}_{b\partial}(Y, P_2, Z_2, W) \\ &= \frac{1}{\Delta_1\Delta_2} \mathscr{D}_{P_1}(\partial_W, Z_1) \mathscr{D}_{P_2}(W, Z_2) G^{\Delta_1|0}_{b\partial}(Y, P_1) G^{\Delta_2|0}_{b\partial}(Y, P_2) \\ &= \frac{1}{\Delta_1\Delta_2} \mathscr{D}_{P_1}(\partial_W, Z_1) \mathscr{D}_{P_2}(W, Z_2) \sum_{m=0}^{\infty} a_m^{\Delta_1,\Delta_2} \varphi_m(\Delta_1, \Delta_2; Y) \ . \end{aligned} \quad (5.108)$$

From here we can relate the combination of $\mathscr{D}_P$'s acting on $\varphi_m$ to tensorial structures:

$$\begin{aligned} \mathscr{D}_{P_1}&(\partial_W, Z_1) \mathscr{D}_{P_2}(W, Z_2) \varphi_m(\Delta_1, \Delta_2; Y) = \\ &- 2 D_{1\,12} D_{1\,21} \int_{\gamma_{12}} G^{\Delta_1+1|0}_{b\partial}(Y_\lambda, P_1) G^{\Delta_2+1|0}_{b\partial}(Y_\lambda, P_2) G^{\Delta_m|0}_{bb}(Y_\lambda, Y) \\ &- \Delta_1(1-\Delta_2) H_{12} \int_{\gamma_{12}} G^{\Delta_1|0}_{b\partial}(Y_\lambda, P_1) G^{\Delta_2|0}_{b\partial}(Y_\lambda, P_2) G^{\Delta_m|0}_{bb}(Y_\lambda, Y) \ . \end{aligned} \quad (5.109)$$

This equality can be checked explicitly from the definitions of each term involved. From here we can trade $D_{i\,jk}$ for $\mathcal{D}_{i\,jk}$, and then further use (5.80) and (5.76) to write them as smeared interactions. Without taking into account any normalizations, what we find for the contraction of two gauge fields decomposed in terms of geodesic integrals is

$$\begin{aligned} G^{\Delta_1|1}_{b\partial}&(Y, P_1, Z_1, \partial_W) G^{\Delta_2|1}_{b\partial}(Y, P_2, Z_2, W) \sim \\ &\sum_m \int_{\gamma_{12}} G^{\Delta_1|1}_{b\partial}(Y_\lambda; \partial_W) G^{\Delta_2|1}_{b\partial}(Y_\lambda; \partial_W) (W \cdot \partial_{Y_\lambda})^2 G^{\Delta_m|0}_{bb}(Y_\lambda, Y) \\ &+ \sum_m \int_{\gamma_{12}} G^{\Delta_1|1}_{b\partial}(Y_\lambda; \partial_W) G^{\Delta_2|1}_{b\partial}(Y_\lambda; W) G^{\Delta_m|0}_{bb}(Y_\lambda, Y) \ , \end{aligned} \quad (5.110)$$

where we are suppressing as well most of the variables in the propagators. This example illustrates how more interactions are needed when we decompose a Witten diagram in terms of geodesic diagrams; or in other words, how the product expansion of the bulk fields requires different interactions than those used in the direct evaluation of a three point function. But more importantly, we should highlight that casting $G_{b\partial}^{\Delta_1|1}(Y,P_1,Z_1,\partial_W)G_{b\partial}^{\Delta_2|1}(Y,P_2,Z_2,W)$ as local interactions integrated along a geodesic is ambiguous. Consider as an example the last term in (5.110). We could have written it in multiple ways due to the degeneracies shown in (5.86): the product of two gauge fields could be casted as integrals of the interaction of $\phi A_\mu A^\mu$ or $\phi F_{\mu\nu}F^{\mu\nu}$ or similar contractions. And these interactions are not related by equations of motion nor field redefinitions. As we discussed in section 5.4.2, the identifications of gravitational interactions in a geodesic diagram is not unique and seems rather ad hoc. It would be interesting to understand if there is a more fundamental principle underlying products such as those in (5.110).

### 5.5.3 Scalar exchange with four spinning legs

In a nutshell, this is how we are decomposing a four-point scalar exchange Witten diagram in terms of geodesics diagrams:

1. Consider a cubic interaction $I^{n_1,n_2,n_3}_{J_1,J_2,0}$ of the form (5.88), where at position 1 and 2 we place bulk–to–boundary propagators and at position 3 we have a bulk–to–bulk propagator. From (5.25) and (5.27) we will be able to strip off the tensorial part of the interaction, i.e. schematically we will have
$$I^{n_1,n_2,n_3}_{J_1,J_2,0} = \mathscr{D}\cdots\mathscr{D}\, I^{0,0,0}_{0,0,0}\ . \tag{5.111}$$
Here "$\mathscr{D}\cdots\mathscr{D}$" symbolizes a chain of contractions of operators appearing in (5.25) and (5.27), and the precise contraction depends on the interaction. The important feature is that $\mathscr{D}\cdots\mathscr{D}$ involves only derivatives with respect to $Z_i$ or $P_i$ (and not $Y$) which allows us to take this portion outside of the volume integral in a Witten diagram. Here $I^{0,0,0}_{0,0,0}$ is a cubic interaction for three scalars with the appropriate propagators used, i.e.
$$I^{0,0,0}_{0,0,0} = G_{b\partial}^{\Delta_1|0}(Y,P_1)G_{b\partial}^{\Delta_2|0}(Y,P_2)G_{bb}^{\Delta|0}(Y,Y')\ . \tag{5.112}$$

2. The map among tensor structures and cubic interactions in [186] implies that we will always be able to write the combination of $\mathscr{D}$'s in terms of CFT operators:
$$\mathscr{D}\cdots\mathscr{D} I^{0,0,0}_{0,0,0} = D\cdots D\, I^{0,0,0}_{0,0,0}\ . \tag{5.113}$$
This tells us which are the tensor structures appearing in the Witten diagram.

3. Next we can rewrite $I^{0,0,0}_{0,0,0}$ as a sum over geodesic integrals via (5.95). This allows us to trade $D$ for our geodesic operators $\mathcal{D}(Y)$ as given in (5.47):

$$\mathscr{D}\cdots\mathscr{D}I^{0,0,0}_{0,0,0} = D\cdots D\, I^{0,0,0}_{0,0,0} = \mathcal{D}\cdots\mathcal{D}I^{0,0,0}_{0,0,0} \ . \tag{5.114}$$

4. And if desired, we can as well write the action of $\mathcal{D}$ on $I^{0,0,0}_{0,0,0}$ as an interaction via the map in (5.143). This gives a more local description of the OPE of the bulk fields in $I^{n_1,n_2,n_3}_{J_1,J_2,0}$ in terms of smeared interactions along the geodesic.

A four-point exchange Witten diagram, where the exchange particle is a scalar field, is build out of two vertices of the form $I^{n_1,n_2,n_3}_{J_1,J_2,0}$. So, keeping the loose schematic equalities, we can establish the following chain of equalities

$$\begin{aligned}\mathcal{A}^{\text{exch}}_{J_1,J_2,J_3,J_4} &\sim \mathscr{D}_{\text{left}}\mathscr{D}_{\text{right}}\mathcal{A}^{\text{exch}}_{0,0,0,0}\\ &\sim D_{\text{left}}D_{\text{right}}\,\mathcal{A}^{\text{exch}}_{0,0,0,0}\\ &\sim \sum_m \mathcal{W}_{\Delta_m|0}[\mathcal{D}_{\text{left}}(Y_\lambda),\mathcal{D}_{\text{right}}(Y'_{\lambda'})] \ .\end{aligned} \tag{5.115}$$

where $\mathscr{D}_{\text{left}}$ corresponds to product of differential operators that recast the vertex to the left in terms boundary operators acting on position $(P_1, P_2)$, and the analogously for $\mathscr{D}_{\text{right}}$ acting on $(P_3, P_4)$. And in the last line we used (5.101).

### 5.5.4 Spinning exchanges

In this last portion we will address examples where the exchanged field has spin, and illustrate how the four-point exchange diagram can be decomposed in terms of the geodesic integrals. First consider the following Witten diagram

$$\begin{aligned}\mathcal{A}^{\text{Exch|spin}}_{0,0,0,0} = \int dY\int dY' G^{\Delta_1|0}_{b\partial}(Y,P_1)\partial_W\cdot\Big(\partial_Y G^{\Delta_2|0}_{b\partial}(Y,P_2)\Big)\, G^{\Delta|1}_{bb}(Y,Y',W,\partial_{W'})\\ \times W'\cdot\Big(\partial_{Y'}G^{\Delta_3|0}_{b\partial}(Y',P_3)\Big)\, G^{\Delta_4|0}_{b\partial}(Y',P_4) \ .\end{aligned} \tag{5.116}$$

In this diagram we are using the interaction $\phi_1\partial_\mu\phi_2 A^\mu$ on both ends, and it is depicted in Fig. 5.6. The decomposition of (5.116) in terms of geodesic integrals was done in [120] and we will not repeat it here—it decomposes roughly into geodesic Witten diagrams with both scalar and vector exchanges. Next, let us consider a diagram where the field at position $P_2$ is a massive vector, i.e.

$$\begin{aligned}\mathcal{A}^{\text{Exch|spin}}_{0,1,0,0} = \int dY\int dY' G^{\Delta_1|0}_{b\partial}(Y,P_1)G^{\Delta_2|1}(Y,P_2;\partial_W,Z_2)G^{\Delta|1}_{bb}(Y,Y',W,\partial_{W'})\\ \times W'\cdot\partial_{Y'}\Big[G^{\Delta_3|0}_{b\partial}(Y',P_3)\Big]\, G^{\Delta_4|0}_{b\partial}(Y',P_4) \ .\end{aligned} \tag{5.117}$$

This would be the second diagram in Fig. 5.6, and we decided to use the interaction $\phi_1 A_2^\mu A_\mu$ for the cubic interaction on the left of the diagram. We can relate (5.117) to (5.116) by noticing the that the bulk–to–boundary operators satisfy the following series of identities

$$\begin{aligned} G_{b\partial}^{\Delta_1|0}(Y,P_1)G_{b\partial}^{\Delta_2|1}(Y,P_2;\partial_W,Z_2) &= \frac{1}{\Delta_2}\mathscr{D}_{P_2}(\partial_W,Z_2)G_{b\partial}^{\Delta_1|0}(Y,P_1)G_{b\partial}^{\Delta_2|0}(Y,P_2) \\ &= \frac{\Delta_2-1}{\Delta_2(\Delta_1-1)}D_{1\,21}\left[\frac{1}{P_{12}}G_{b\partial}^{\Delta_1-1|0}(Y,P_1)(\partial_W\cdot\partial_Y)G_{b\partial}^{\Delta_2|0}(Y,P_2)\right] \\ &\quad -\frac{1}{\Delta_2-1}D_{2\,21}\left[\frac{1}{P_{12}}G_{b\partial}^{\Delta_1|0}(Y,P_1)(\partial_W\cdot\partial_Y)G_{b\partial}^{\Delta_2-1|0}(Y,P_2)\right] \end{aligned} \quad (5.118)$$

Here we used (5.25), and then using the explicit polynomial dependence of $G_{b\partial}^{\Delta|0}(Y,P)$ to obtain the equality in the last line. It is interesting to note that we can now write

$$\begin{aligned} \mathcal{A}_{0,1,0,0}^{\text{Exch|spin}} &= \frac{\Delta_2-1}{\Delta_2(\Delta_1-1)}D_{1\,21}\left[\frac{1}{P_{12}}\mathcal{A}_{0,0,0,0}^{\text{Exch|spin}}(\Delta_1-1,\Delta_2,\Delta_3,\Delta_4)\right] \\ &\quad -\frac{1}{\Delta_2-1}D_{2\,21}\left[\frac{1}{P_{12}}\mathcal{A}_{0,0,0,0}^{\text{Exch|spin}}(\Delta_1,\Delta_2-1,\Delta_3,\Delta_4)\right]. \end{aligned} \quad (5.119)$$

And from here we can proceed by using the explicit decomposition of $\mathcal{A}_{0,0,0,0}^{\text{Exch|spin}}$ in terms of geodesic diagrams in [120] and then trading $D_{i\,jk}$ by $\mathcal{D}_{i\,jk}$ (just as we we did in the previous examples in this section).[13]

The manipulations shown here are very explicit for the interaction we have selected, but they are robust and not specific to the example. We expect that in general we will be able to carry out a decomposition such as the one in (5.118) and have generalizations of (5.119) without much difficulty.

## 5.6 Discussion

Our main result was to give a systematic method to evaluate conformal partial waves as geodesic integrals in AdS. From the CFT perspective, a spinning conformal partial wave is built from differential operators acting on the scalar conformal

[13]Note that the factor of $P_{12}$ can be reabsorbed into bulk–to–boundary propagators projected along geodesics, i.e

$$\frac{1}{P_{12}}G_{b\partial}^{\Delta_1|0}(Y_\lambda,P_1)G_{b\partial}^{\Delta_2|0}(Y_\lambda,P_2) = G_{b\partial}^{\Delta_1+1|0}(Y_\lambda,P_1)G_{b\partial}^{\Delta_2+1|0}(Y_\lambda,P_2)\ .$$

Hence, as we cast (5.119) as a sum over geodesic integrals, all terms will have a bulk interpretation.

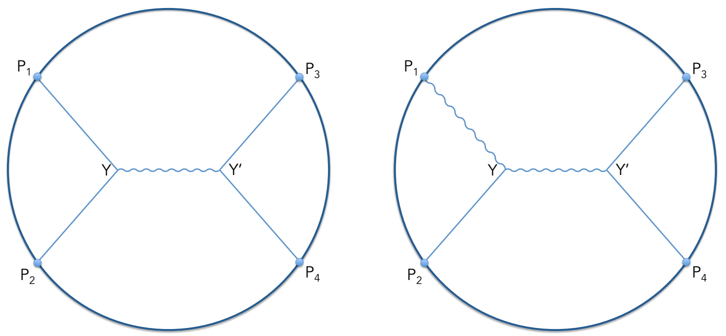


Figure 5.6: Four-point exchange Witten diagrams in $\mathrm{AdS}_{d+1}$, where the exchanged field is a symmetric tensor field of dimension $\Delta$ and spin $J$. In (5.116) and (5.117) we consider explicit examples where $J = 1$ for the external and exchanged field.

partial wave as seen in subsection 1.6.4; here we presented the analogue of these differential operators (for symmetric-traceless representations) in AdS and showed that they reproduce the same effect as in the CFT. More succinctly, we established that for arbitrary representations $[\Delta_i, (l_i)]$ and $[\Delta, (l)]$, we have

$$W^{l_1,l_2,l_3,l_4}_{\Delta|l}(P_i; Z_i) = D_{\text{left}} D_{\text{right}} W_{\Delta|l}(P_1, P_2, P_3, P_4) = \mathcal{W}_{\Delta|l}[\mathcal{D}_{\text{left}}(Y_\lambda), \mathcal{D}_{\text{right}}(Y'_{\lambda'})] \,, \tag{5.120}$$

where the last equality is a purely AdS object build out of geodesic integrals, while the left hand side are CFT quantities.

In section 5.4 we saw that the bulk differential operators (which generate three-point function tensor structures by construction) induce cubic vertex interactions when acting on three-point geodesic Witten diagrams. A feature of these geodesic Witten diagrams is that the mapping between CFT tensor structures and cubic interactions depends on which geodesic we integrate over—as in which pair of boundary points the geodesic joins. This is simple to understand since there exist interactions which vanish exactly when projected onto a particular choice of geodesic. Therefore by combining the induced interactions from all possible geodesics, we can engineer a geodesic-independent mapping between tensor structures and cubic interactions. Moreover we found that, for the particular cases we analyzed, the geodesic independent mapping is in agreement with what one would obtain by integrating over the whole volume (i.e. evaluating the Witten diagram), and we expect this to hold for generic cases. However, the cubic interactions from our geodesic-independent mapping are not the same as the basis of cubic vertices from [173]. This is because projecting onto a particular geodesic does not commute with partial integration in the full volume integral.

The immediate use of an object like $\mathcal{W}_{\Delta|l}$ is to evaluate correlation functions in holography. But relating the geodesic diagrams to regular Witten diagrams is a non-trivial task: interactions projected on geodesic integrals behave starkly different to interactions in volume integrals, as mentioned above. This mismatch between the two objects makes more delicate the decomposition of a Witten diagram in terms of geodesics. We carry out explicit examples in section 5.5, and discuss the general relation when the exchanged field is a scalar. The strategy we adopt for this decomposition is inspired by the identities used in [173]: one rewrites all tensorial properties of the interactions among bulk–to–boundary fields in terms of boundary operators acting on a scalar seed. This allows us to identify the CFT operators $D_{i\,jk}$, and use then our bulk operators $\mathcal{D}_{i\,jk}$ to write a final answer in terms of a sum of geodesic integrals. As a result, the set of cubic interactions needed to decompose a Witten diagram in terms of geodesic diagrams is larger than the basis in (5.87). Each individual geodesic integral is, however, much easier to evaluate.

We have not discussed contact Witten diagrams here, but actually they can be treated very similarly as we did in section 5.5. The scalar case was done in [120], so the task is to manipulate the vertex along the lines of the discussion in section 5.5.3: the analog of (5.111) for a quartic interaction would allow us to identify the suitable tensor structures. Note that in a quartic interaction all propagators involved are bulk–to–boundary and hence we can strip off its tensorial features.

In subsection 5.3.2 we gave a prescription on how to evaluate conformal partial waves via geodesic diagrams when the exchanged field has non-trivial spin. And the general strategy we have adopted in this work allowed us to relate the geodesic diagrams to Witten diagrams, as we discussed in subsection 5.5.4. From this method it is not straightforward to infer the gravitational interaction, as we did in section 5.4, with the main obstacle being the contractions of $dY^\mu/d\lambda$ appearing in the integrand. It might be interesting to improve our prescription, to make this connection more evident.

Another future direction that would be interesting to pursue is the addition of loops on the gravitational side. Very little is known about how to evaluate Witten diagrams beyond tree level, with the exception of the recent work in [187]. It would interesting to see how the geodesic diagram decomposition of a Witten diagram is affected by the presence of loops: since the geodesic diagrams are conformal partial waves, we would expect that loops only modify the OPE coeffcients in the decomposition and the relation between masses in AdS and conformal dimensions in the CFT.

# Appendix A: Tensor structures in Witten diagrams

In this appendix we will evaluate three point Witten diagrams explicitly to illustrate how the tensor structures appear in the final answer. We will focus on the following interactions:

$$A_1^\mu\, \partial_\mu \partial_\nu \phi_2\, A_3^\nu\ , \quad \partial_\mu A_1^\nu\, \phi_2\, \partial_\mu A_3^\nu\ , \quad \partial_\mu A_1^\nu\, \phi_2\, \partial_\nu A_3^\mu\ . \tag{5.121}$$

We will do this by using the techniques in [173, 174], where they write the $J$ spinning bulk to boundary propagator and its derivatives in terms of the scalar propagators. This allows us to express the three point function of our interest in terms of scalar three point functions. In our case, we will just need the following identities for the spin-1 case, which follow from (5.25) and (5.27):

$$\Delta\, G_{b\partial}^{\Delta|1}(Y,P;W,Z) = \mathscr{D}_P(W,Z)\, G_{b\partial}^{\Delta|0}(Y,P)\ , \tag{5.122}$$

$$(W'\cdot\partial_Y) G_{b\partial}^{\Delta|1}(Y,P;W,Z) = \mathscr{D}'_P(W',W,Z)\, G_{b\partial}^{\Delta+1|0}(Y,P)\ , \tag{5.123}$$

where $\mathscr{D}_P$ are differential operators defined as

$$\mathscr{D}_P(W,Z) = (Z\cdot W)\left(Z\cdot\frac{\partial}{\partial Z} - P\cdot\frac{\partial}{\partial P}\right) + (P\cdot W)\left(Z\cdot\frac{\partial}{\partial P}\right)\ , \tag{5.124}$$

$$\mathscr{D}'_P(W',W,Z) = 2\left((Z\cdot W')(P\cdot W) + \Delta(P\cdot W')(Z\cdot W) + (P\cdot W')(P\cdot W)\left(Z\cdot\frac{\partial}{\partial P}\right)\right)\ . \tag{5.125}$$

We start by evaluating a Witten diagram using the interaction $A_1^\mu\, \partial_\mu\partial_\nu\phi_2\, A_3^\nu$. We have

$$\int dY\, G_{b\partial}^{\Delta_1|1}(Y,P_1;\partial_{W_1},Z_1) G_{b\partial}^{\Delta_3|1}(Y,P_3;\partial_{W_3},Z_3)(W_1\cdot\partial_Y)(W_3\cdot\partial_Y)\, G_{b\partial}^{\Delta_2|0}(Y,P_2)\,. \tag{5.126}$$

Here $dY$ denotes an integral over the volume of AdS. Using (5.23) and (5.125) gives

$$\begin{aligned}
&\frac{4\Delta_2(\Delta_2+1)}{\Delta_1\Delta_3}\int dY\, \mathscr{D}_{P_1}(\partial_{W_1},Z_1) G_{b\partial}^{\Delta_1|0}(Y,P_1)\mathscr{D}_{P_3}(\partial_{W_3},Z_3) G_{b\partial}^{\Delta_3|0}(Y,P_3)\\
&\qquad\qquad \times (W_1\cdot P_2)(W_3\cdot P_2)\, G_{b\partial}^{\Delta_2+2|0}(Y,P_2)\\
&= \frac{4\Delta_2(\Delta_2+1)}{\Delta_1\Delta_3}\mathscr{D}_{P_1}(P_2,Z_1)\mathscr{D}_{P_3}(P_2,Z_3)\int dY\, G_{b\partial}^{\Delta_1|0}(Y,P_1) G_{b\partial}^{\Delta_3|0}(Y,P_3)\, G_{b\partial}^{\Delta_2+2|0}(Y,P_2)\\
&= \frac{4\Delta_2(\Delta_2+1)\mathsf{C}_{\Delta_1,\Delta_2+2,\Delta_3}}{\Delta_1\Delta_3}\mathscr{D}_{P_1}(P_2,Z_1)\mathscr{D}_{P_3}(P_2,Z_3)\mathcal{K}_3^{\Delta_1,\Delta_2+2,\Delta_3}\ ,
\end{aligned} \tag{5.127}$$

where

$$\mathsf{C}_{\Delta_1,\Delta_2,\Delta_3} = g\frac{\pi^h}{2}\Gamma\left(\frac{\Delta_1+\Delta_2+\Delta_3-2h}{2}\right)\frac{\Gamma\left(\frac{\Delta_1+\Delta_2-\Delta_3}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_3-\Delta_2}{2}\right)\Gamma\left(\frac{\Delta_2+\Delta_3-\Delta_1}{2}\right)}{\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3)}\,. \tag{5.128}$$

Notice that $\mathscr{D}_{P_1}(P_2,Z_1) = D_{2,12}$, and $\mathscr{D}_{P_3}(P_2,Z_3) = D_{2,32}$. Now, applying the differential operators to the scalar 3-point function we find that tensor structure corresponding to the previous diagram is the following linear combination:

$$A_1^\mu\,\partial_\mu\partial_\nu\phi_2\,A_3^\nu\ :\ \frac{\Delta_2(\Delta_2+1)(\Delta_1-\Delta_2+\Delta_3-2)\mathsf{C}_{\Delta_1,\Delta_2+2,\Delta_3}\left(H_{13}+(\Delta_1-\Delta_2+\Delta_3-2)\ V_{1,23}V_{3,21}\right)}{\Delta_1\Delta_3}\,. \tag{5.129}$$

For the interaction $\partial_\mu A_1^\nu\,\phi_2\,\partial_\mu A_3^\nu$ we have

$$\int dY\,(\partial_{W'}\cdot\partial_Y)\,G_{b\partial}^{\Delta_1|1}(Y,P_1;\partial_W,Z_1)(W'\cdot\partial_Y)G_{b\partial}^{\Delta_3|1}(Y,P_3;W,Z_3)\,G_{b\partial}^{\Delta_2|0}(Y,P_2)\,, \tag{5.130}$$

which using (5.123) is equivalent to

$$\begin{aligned}&\mathscr{D}'_{P_1}(\partial_{W'},\partial_W,Z_1)\mathscr{D}'_{P_3}(W',W,Z_3)\int dY\,G_{b\partial}^{\Delta_1+1|0}(Y,P_1)G_{b\partial}^{\Delta_3+1|0}(Y,P_3)\,G_{b\partial}^{\Delta_2|0}(Y,P_2)\\&=\mathsf{C}_{\Delta_1+1,\Delta_2,\Delta_3+1}\mathscr{D}'_{P_1}(\partial_{W'},\partial_W,Z_1)\mathscr{D}'_{P_3}(W',W,Z_3)\,\mathcal{K}_3^{\Delta_1+1,\Delta_2,\Delta_3+1}\,.\end{aligned} \tag{5.131}$$

Contracting the $W$'s in the differential operators gives

$$\begin{aligned}&\mathscr{D}'_{P_1}(\partial_{W'},\partial_W,Z_1)\mathscr{D}'_{P_3}(W',W,Z_3)=(\Delta_1+\Delta_3)(Z_1\cdot P_3)(Z_3\cdot P_1)+\Delta_1\Delta_3(Z_1\cdot Z_3)(P_1\cdot P_3)\\&+(P_1\cdot P_3)\left((Z_1\cdot\partial_{P_1})(Z_3\cdot\partial_{P_3})+(1+\Delta_3)(P_1\cdot Z_3)(Z_1\cdot\partial_{P_1})+(1+\Delta_1)(P_3\cdot Z_1)(Z_3\cdot\partial_{P_3})\right)\,,\end{aligned}$$

which leads to the following identification

$$\begin{aligned}&\partial_\mu A_1^\nu\,\phi_2\,\partial_\mu A_3^\nu\ :\ \mathsf{C}_{\Delta_1+1,\Delta_2,\Delta_3+1}\\&\quad\times\left((\Delta_1-\Delta_2+\Delta_3-2\Delta_1\Delta_3)H_{13}-(\Delta_1-\Delta_2-\Delta_3)(\Delta_1+\Delta_2-\Delta_3)\ V_{1,23}V_{3,21}\right)\,.\end{aligned} \tag{5.132}$$

The interaction $\partial_\mu A_1^\nu\,\phi_2\,\partial_\nu A_3^\mu$ is computed analogously as the previous, but with different $W$ contractions

$$\begin{aligned}&\int dY\,(\partial_{W'}\cdot\partial_Y)\,G_{b\partial}^{\Delta_1|1}(Y,P_1;\partial_W,Z_1)(W\cdot\partial_Y)G_{b\partial}^{\Delta_3|1}(Y,P_3;W',Z_3)\,G_{b\partial}^{\Delta_2|0}(Y,P_2)\\&=\mathsf{C}_{\Delta_1+1,\Delta_2,\Delta_3+1}\mathscr{D}'_{P_1}(\partial_{W'},\partial_W,Z_1)\mathscr{D}'_{P_3}(W,W',Z_3)\,\mathcal{K}_3^{\Delta_1+1,\Delta_2,\Delta_3+1}\,.\end{aligned} \tag{5.133}$$

This contraction of the differential operators gives

$$\begin{aligned}&\mathscr{D}'_{P_1}(\partial_{W'},\partial_W,Z_1)\mathscr{D}'_{P_3}(W',W,Z_3)=\Delta_1\Delta_3(Z_1\cdot P_3)(Z_3\cdot P_1)+(\Delta_1+\Delta_3)(Z_1\cdot Z_3)(P_1\cdot P_3)\\&+(P_1\cdot P_3)\left((Z_1\cdot\partial_{P_1})(Z_3\cdot\partial_{P_3})+(1+\Delta_3)(P_1\cdot Z_3)(Z_1\cdot\partial_{P_1})+(1+\Delta_1)(P_3\cdot Z_1)(Z_3\cdot\partial_{P_3})\right)\,,\end{aligned}$$

which applying it to the scalar three point function gives

$$\begin{aligned}&\partial_\mu A_1^\nu\,\phi_2\,\partial_\nu A_3^\mu\ :\ \mathsf{C}_{\Delta_1+1,\Delta_2,\Delta_3+1}\\&\quad(-(\Delta_1+\Delta_2+\Delta_3-2)H_{13}-(\Delta_1-\Delta_2-\Delta_3)(\Delta_1+\Delta_2-\Delta_3)\ V_{1,23}V_{3,21})\,.\end{aligned} \tag{5.134}$$

Based on these three interactions, we can make the following map

$$\begin{aligned} H_{13}: &\quad \partial_\mu A_1^\nu\,\phi_2\,\partial_\mu A_3^\nu - \partial_\mu A_1^\nu\,\phi_2\,\partial_\nu A_3^\mu\ , \\ V_{1,23}V_{3,21}: &\quad \alpha\, A_1^\mu\,\partial_\mu\partial_\nu\phi_2\,A_3^\nu \\ &\qquad - (\Delta_1+\Delta_3)\partial_\mu A_1^\nu\,\phi_2\,\partial_\mu A_3^\nu + (1+\Delta_1\Delta_3)\partial_\mu A_1^\nu\,\phi_2\,\partial_\nu A_3^\mu\ , \end{aligned} \tag{5.135}$$

where

$$\alpha = \frac{(\Delta_1-1)(\Delta_3-1)(\Delta_1-\Delta_2+\Delta_3)(2+\Delta_1-\Delta_2+\Delta_3)}{(\Delta_1+\Delta_2-\Delta_3)(\Delta_1-\Delta_2-\Delta_3)}\ . \tag{5.136}$$

Modulo normalizations, this identification is compatible with the identification using geodesic diagrams (5.84) and (5.85).

# Appendix B: Tensor-tensor-scalar structures via geodesic diagrams

Based on the two examples in sections 5.4.1 and 5.4.1, we can make a general identification between tensorial structures and a minimal set of gravitational interactions that will capture them for a fixed choice of the geodesic given by the first diagram in Fig. 5.4. We saw that the simplest way to identify $H_{12}$ in the bulk is by an interaction that contracts indices among symmetric tensors at position 1 and 2, and the $V$'s added derivatives on position 3 with suitable contractions on legs 1 and 2. Hence, it seems like each tensor structure $H_{12}^p V_{1,23}^q V_{2,13}^r \mathcal{K}_3^{\Delta_1,\Delta_2,\Delta_3}$ is reproduced by a geodesic integral of the form

$$\int_{\gamma_{12}} d\lambda \frac{\mathcal{H}_{1\lambda}(Z_1,\partial_W)^q \mathcal{H}_{1\lambda}(Z_1,\partial_{W'})^p}{\Psi_{1\lambda}^{\Delta_1}} \frac{\mathcal{H}_{2\lambda}(Z_2,\partial_W)^r \mathcal{H}_{2\lambda}(Z_2,W')^p}{\Psi_{2\lambda}^{\Delta_2}} (W\cdot\partial_{Y_\lambda})^{q+r}\Psi_{3\lambda}^{-\Delta_3}\ . \tag{5.137}$$

This is a claim we can prove. The proof requires the following identities which are easily obtained by induction:

$$\begin{aligned} (W\cdot\partial_{Y_\lambda})^n \Psi_{3\lambda}^{-\Delta_3} &= (-2)^n(-\Delta_3-n+1)_n (W\cdot P_3)^n \Psi_{3\lambda}^{-\Delta_3-n}\ , \\ (\mathcal{H}_{i\lambda}(Z_i,\partial_W))^n (W\cdot P_3)^l &= (l-n+1)_n (W\cdot P_3)^{l-n}\left(\sqrt{\frac{P_{i3}\Psi_{3\lambda}}{\Psi_{i\lambda}}}\mathcal{V}_{\partial\, i,3\lambda}(Z_i)\right)^n\ , \\ \mathcal{H}_{1\lambda}(Z_1,\partial_{W'})^p \mathcal{H}_{2\lambda}(Z_2,W')^p|_{\gamma_{12}} &= p! H_{12}^p\ . \end{aligned} \tag{5.138}$$

Applying these to the integral gives

$$2^{q+r}p!q!(-\Delta_3-q-r+1)_{q+r}(q+1)_r\left(\frac{P_{13}P_{23}}{P_{12}}\right)^{\frac{q+r}{2}} H_{12}^p V_{1,23}^q V_{2,13}^r \times \int_{\gamma_{12}} \Psi_{1\lambda}^{-\Delta_1}\Psi_{2\lambda}^{-\Delta_2}\Psi_{3\lambda}^{-\Delta_3-q-r}\ , \tag{5.139}$$

where we used

$$\sqrt{\frac{P_{i3}\Psi_{31}}{\Psi_{i1}}}\mathcal{V}_{\partial\, i,31}(Z_i) = -\sqrt{\frac{P_{13}P_{23}}{P_{12}}}\begin{cases} V_{1,23} & \text{if } i=1 \\ V_{2,13} & \text{if } i=2 \end{cases} \tag{5.140}$$

The remaining integral evaluates to

$$\int_{\gamma_{12}} \Psi_{1\lambda}^{-\Delta_1} \Psi_{2\lambda}^{-\Delta_2} \Psi_{3\lambda}^{-\Delta_3-q-r} = \frac{\mathcal{K}_3^{\Delta_1,\Delta_2,\Delta_3+q+r}}{c_{\Delta_1\Delta_2\Delta_3+q+r}} \ , \tag{5.141}$$

by (5.67). Therefore (5.137) results in

$$\frac{2^{q+r} p! q! (-\Delta_3 - q - r + 1)_{q+r} (q+1)_r}{c_{\Delta_1\Delta_2\Delta_3+q+r}} H_{12}^p V_{1,23}^q V_{2,13}^r \mathcal{K}_3^{\Delta_1,\Delta_2,\Delta_3} \ , \tag{5.142}$$

which completes the proof. Hence, from the analysis of the integrals over the geodesic $\gamma_{12}$ (which connects the fields with spin), we find the following identification

$$H_{12}^p V_{1,23}^q V_{2,13}^r : \quad h_{1\mu_1\cdots\mu_p}{}^{\alpha_1\cdots\alpha_q} h_2{}^{\mu_1\cdots\mu_p\beta_1\cdots\beta_r} \partial_{\alpha_1}\cdots\partial_{\alpha_q}\partial_{\beta_1}\cdots\partial_{\beta_r}\phi_3 \ . \tag{5.143}$$

As we have noticed in section 5.4.1 this identification is not unique. It is sensitive to the choice of geodesic, and moreover to redundancies that appear as derivatives are contracted along $\gamma_{12}$ (i.e. generalizations of (5.86)).

# 6 Conclusion

The idea of exploiting conformal symmetry to completely solve a CFT dates back to the early 1970s [7, 8]. However, actual results for the case of $d > 2$ dimensions only started appearing after the 2008 seminal paper [188]. The two main ingredients that made this study possible are the development of readily available expressions for conformal partial waves [64, 68, 69] as well recasting the conformal bootstrap equations (1.227) in a way that the CFT data can be solved via known algorithms (a linear programming problem in the case of [188]). Since then the bootstrap program literature has moved in two main directions [73, 74]:

- numerical: techniques have been developed for obtaining numerical bounds on CFT data for theories in 3, 4, as well as other dimensions, with and without global symmetries, as well as supersymmetries. Studies of this type include the determination of constraints arising from mixing more than one type of four-point function, as well as correlators with conserved currents.

- analytical: progress has been made for analytically solving the bootstrap equations in particular limits. This includes studying certain kinematic configurations such as the lightcone and Regge limits, as well as taking large dimension limits. Other progress involves considering slightly broken symmetries in the large $N$ and $\epsilon$-expansions. Furthermore, techniques exploiting Lorentzian causality and inversion formulas have also been developed.

One interesting consequence of the non-perturbative constraints from the bootstrap program is the possibility of gaining insight into features of quantum gravity via the AdS/CFT correspondence. For example, by studying CFTs in particular limits that correspond to local high energy scattering deep in the bulk [110, 189].

Another direction is to extend the holographic dictionary by better understanding the connections between the physics of bulk scattering and CFT correlators [3, 120, 124, 177, 178, 190–193]. One can then use bulk intuition to get insight into the CFT dynamics and inversely use CFT methods to discover nontrivial physics in the bulk.

Although the bootstrap literature has seen many developments in recent years, the work is far from done. The ultimate objective is to create a map of the space of non-trivial CFTs. This involves refining our current understanding of known CFTs as well as possibly discovering previously unknown theories. Given that conformal field theories appear in many areas of physics, one expects that improving the bootstrap technology could lead to breakthroughs in topics such as quantum gravity, condensed matter systems, and quantum field theory. To achieve all this we need to improve our current techniques for working with spinning operators, as well as inventing more efficient numerical and analytical tools for solving the bootstrap equations.

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