A Study on Neutrino Masses and Mixings from Certain Flavor Symmetries

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Subhankar Roy

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Declaration of Authorship

I, Subhankar Roy, declare that this thesis titled, "A Study on Neutrino Masses and Mixings from Certain Flavor Symmetries", done under the supervision of Prof. N. Nimai Singh and the work presented in it are my own. I confirm that

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- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have discussed the published work of others, this is always clearly ascribed.
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This is to certify that Mr. Subhankar Roy has worked under my supervision for the thesis entitled "A Study on Neutrino Masses and Mixings from certain Flavor Symmetries", which is being submitted to the Gauhati University for the degree of Doctor of Philosophy in the faculty of science. The research work reported in this thesis is original and candidate's own work. He has fulfilled all the requirements under the Ph.D regulations of Gauhati University. To the best of my knowledge, the thesis or part thereof has not been submitted to any other university for any degree or diploma.

Prof N. Nimai Singh
(Ph.D Supervisor)

Date:

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Abbreviations

BL	**Bi-Large**
BM	**Bimaximal**
BTM	**Bi-Trimaximal**
CKM	**Cabibbo-Kobayashi-Maskawa**
CUORE	**Cryogenic Underground Observatory for Rare Events**
Gallex	**Gallium Experiment**
GR	**Golden Ratio**
GREDA	**GERmanium Detector Array**
GUT	**Grand Unified Theory**
HM	**Heidelberg-Moskow**
IGEX	**International Germanium EXperiment**
IH	**Inverted Hierarchy**
KamLand	**Kamioka Liquid Scintillator Antineutrino Detector**
KARMEN	**KArlsruhe Rutherford Medium Energy Neutrino**
KATRIN	**KArlsruhe TRItium Neutrino**
LBNE	**Long-Baseline Neutrino Experiment**
LSND	**Liquid Scintillator Neutrino Detector**
MACRO	**Monopole, Astrophysics and Cosmic Ray Observatory**
MINOS	**Main Injector Neutrino Oscillation Search**
MSW	**Golden Ratio**
NEMO	**Neutrino Ettore Majorana Observatory**
NH	**Normal Hierarchy**
NOva	**NuMI Off-Axis ν_e Appearence**
NuMI	**Neutrinos at the Main Injector**

PDG	**Particle Data Group**
PMNS	**Pontecorvo-Maki-Nakagawa-Sakata**
QDIH	**Quasi-Degenerate Inverted Hierarchy**
QDN	**Quasi-Degenerate**
QDNH	**Quasi-Degenerate Normal Hierarchy**
RENO	**Reactor Experiment for Neutrino Oscillations**
SAGE	**Soviet-American-Gallium-Experiment**
SK	**Super-Kamiokande**
SM	**Standard Model**
SNO	**Sudbury Neutrino Observatory**
SUSY	**Super Symmetry**
TBC	**Tri-Bimaximal Cabibbo**
TBM	**Tri-Bimaximal**
T2K	**Tokai to Kamioka**

Preface

The study of neutrinos is an indispensable limb of particle physics which conceives several unsolved questions yet to be addressed: both experimentally and theoretically. In simple words, the essence of neutrino physics phenomenology lies in how to explain the nine physical parameters related to neutrino masses, mixing angles and CP violation. The explanation involves : "Why the parameters appear as they do ?" and "How these parameters can be visualized in certain frame-works ?" In addition, it embraces the prediction of those parameters also which are not yet observed or measured precisely in experiments. All these interesting aspects persuade one to look beyond the standard model of particle physics. A first principle to answer to the former question is still concealed. Our endeavor centers round the possibilities to look into the second prospect. Mostly our work involves the data from the neutrino oscillation experiments. In our analysis, either the two Majorana phases remain covered or in some special occasion we switch to the CP conserving scenarios. We know that the neutrino oscillation experiments keep the ordering of neutrino masses indeterminate. In other words, three different orderings "Normal", "Inverted" and "Quasi-degenerate" are equally possible. The "Neutrino mass matrix" is a very important tool for the theorists, because it gives the information of both masses and mixing angles (and the CP phases). A proper parametrization of the same is requisite because it is essential not only in the oscillation phenomenon and neutrino-less double beta decay scenarios but also in cosmology. In a different approach, we can start from a proper parametrization of lepton mixing matrix also which embraces the information of mixing angles (and the CP phases) only.

The first part of the thesis is devoted to a proper parametrization of the neutrino mass matrix and the latter part is attributed to the estimation of lepton mixing matrix only.

Whether the neutrino mass matrix is dominated by "Randomness" or certain flavor symmetry is there to dictate the same is still an open question. But in the thesis we are guided by the second possibility. In fact the flavour symmetries can not answer to all the unsettled issues and the related frameworks are subjected to correction. For example, A_4 and S_4 discrete flavor symmetries are associated with 2-3 symmetry of neutrino mass matrix, which in turn predicts a vanishing reactor angle. But the present oscillation data rules out this possibility. How to bring about similar needful modifications to a certain framework and the necessity to look beyond the existing models are discussed in this thesis.

Our strategy is model-independent and perspective is "bottom-up". The Thesis is organized as follows.

- The first chapter discusses the general aspects of neutrino physics, like Dirac and Majorana nature of neutrino, flavor oscillation, matter effects, different theoretical aspects etc.
- The second chapter highlights the importance of proper parametrization of Quasi-degenerate neutrino mass matrix based on $\mu-\tau$ symmetric framework, with minimum number of parameters in the basis where charged lepton mass matrix is diagonal (or non-diagonal).
- The third chapter discusses the restraints or the sum rules that a neutrino mass matrix may encounter, if at least one of the mixing angles is connected to the corresponding mass ratio (a scenario similar to the case of quarks).
- The fourth chapter tries to modify the TBC and hence the TBM mixing by considering the contribution both from neutrino and charged lepton sectors.
- The fifth chapter highlights the possibilities to construct a framework where Cabibbo angle from quark sector can seed the parametrization both in charged lepton and neutrino sector in the light of BL mixing.
- In the sixth chapter, the concluding remarks are made.

Chapter 1

Introduction: Brief overview of Neutrinos

1.1 Introduction

The Standard Model (SM) of electro-weak and strong interactions is an profuse theory that describes how the elementary particles behave and the way they interact. The recent discovery of Higgs particle has declared the triumph of Standard model. But strong reasons are there to think beyond the same, not only from particle physics point of view but from cosmology as well. The Standard Model is unable to explain the inflation, dark matter and dark energy. On the other hand the inter conversion among the neutrinos from the three generations of flavors strongly supports for a finite nonzero neutrino mass and establishes conclusively that the neutrinos mix in a similar fashion as the quarks do (though the mixing is more intense for neutrinos than that for quarks). This is the first evidence that provides enough motivation to trace the footprints of new physics.

The SM cuddles three families of fermions. Each family of fermions contains two distinct types of quarks: "up-quark" with electric charge $Q = 2/3$ and "down-quark" with, $Q = -1/3$, and the two leptons: charged leptons with $Q = -1$ and neutrinos $Q = 0$. We have three replicas of such a family. But the particles from different families, sharing the same quantum number posses different mass. For example, the masses of the up quarks are $m_u \sim 2.3^{+0.7}_{-0.5}\, MeV$, $m_c \sim 1.275 \pm 0.025\, GeV$ and $m_t \sim 173.21 \pm 0.51 \pm 0.71\, GeV$ (from direct measurement)[2]. The down quarks and the charged leptons have got similar masses: $m_d \sim 4.8^{+0.5}_{-0.3}\,MeV$, $m_s \sim 95 \pm 5\,MeV$, $m_b \sim 4.18 \pm 0.03\,GeV$, $m_e \sim 0.511\,MeV$, $m_\mu \sim 105.658\,MeV$ and $m_\tau \sim 1776.82 \pm 0.16\,MeV$ [2]. The origin of the fermion masses are due to the Yukawa interaction which couples the left-handed and right-handed chiral components of specific fermion field. But in SM, in contrast to the charged leptons, the right handed chiral components of neutrino field are missing and hence they are deprived of masses.

In 1930, the American Physicist Wolfgang Pauli first introduced the idea of neutrinos. The motivation behind this idea was to compensate the apparent violation of energy in the nuclear *beta*- decay experiments. In *beta*-decay, a neutron from an unstable nucleus converts itself to a proton with the emission of an electron. The electron is expected to carry discrete energy like other radioactive processes like α-decay or γ emission. But in contrary to that the electron shows a continuous spectrum. So this problem can be forsaken if there is some particle (neutrino) which takes away some portion of the energy. According to Pauli, neutrino must be mass-less and must carry a spin of 1/2 (so that angular momentum is conserved) and electrically neutral and weakly interacting. As they are electrically neutral and a have a feeble interaction with matter, Pauli expected that hardly this particle could be detected. But in 1956, Clyde Cowan and Fred Reines detected the antineutrino emitted from a nuclear reactor, through inverse β decay processes, at Savannah River in South Carolina, USA.

The first signature that the neutrinos posses finite mass came from an experiment conducted by an American scientist Raymond Davis Jr. during the detection of solar neutrinos. It was found that the flux of the solar neutrinos was only one-third of what is predicted by the theories. This observation bewildered the physicists. Pontecorvo in 1957, first introduced the idea of "Neutrino oscillation", a phenomenon where the particular flavor of neutrino can convert itself to another. Independently, this idea was developed by Maki, Nakagawa and Sakata in 1962. Based on this idea of neutrino oscillation, some Russian researchers Mikheyev and Smirnov suggested that the solar neutrinos which are none other than the electron neutrinos (ν_e) are changing into some other forms like "ν_τ" or "ν_μ" which are unfounded in Davis's experiment. The theory they presented involves resonant enhancement of the oscillation due to matter effect within the sun (MSW effect).

In 1998, the Japanese experiment Super-Kamiokande first witnessed the existence of neutrino oscillation. When the cosmic rays strike the upper surface ofthe atmosphere, the so called muon neutrinos (are also called atmospheric neutrinos) are generated. Since the neutrinos pass the earth easily, the Super-Kamiokande was able to detect the neutrinos from above and below. The experiment revealed that the number of neutrinos coming from above are as expected, but this is not true for the neutrinos appearing from below. The number is just half of the expected one. This is because half of the muon neutrinos are converted into tau neutrinos while traversing through earth.

It was the Sudbury Neutrino Observatory (SNO) in Canada which finally confirmed the flavor conversion of the "solar neutrinos". The SNO data finally removed the doubts related with Solar model. The electron neutrinos are produced at the standard rate in Sun, but during transit, they oscillate into muon and tau neutrinos, and only one third of the original flux of the electron neutrinos can be detected on earth.

1.2 Theoretical motivation behind the neutrino mass

The "mass-term" is a a part of the Lagrangian which associates the left-handed and right-handed counterparts of a field [3]. We encounter two possibilities : "Dirac mass term" and "Majorana mass term" respectively,

$$\textbf{Dirac Mass term} \qquad m(\bar{\psi_R}\,\psi_L + h.c), \tag{1.1}$$

$$\textbf{Majorana Mass term} \qquad M(\bar{\psi_L}^c\,\psi_L + h.c). \tag{1.2}$$

The Dirac term accounts for the mass generation in quarks and the charged leptons while, the latter is pertinent in the context of neutral leptons, i.e, the neutrinos. The Dirac mass term is in harmony with the lepton number conservation. Since the Majorana condition makes a particle and an antiparticle indistinguishable [4, 5], the associated mass term violates the lepton number conservation by two units. In the above expression, ψ^c or equivalent terms represent the charge-conjugation state of the field ψ. The name "Majorana field" was originally attributed to a completely neutral fermionic field which can be constructed from a Dirac field, say χ by impelling the Majorana condition,

$$\chi^c = \pm\,\chi. \tag{1.3}$$

1.2.1 The massless neutrinos

Let us consider the Dirac-spinor [3],

$$\psi = \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \tag{1.4}$$

from which two component Weyl spinors ψ_L and ψ_R are generated as illustrated in the following,

$$\psi_L = \frac{1}{2}(1-\gamma_5)\,\psi = \begin{bmatrix} 0 \\ \eta \end{bmatrix}, \quad \psi_R = \frac{1}{2}(1+\gamma_5)\,\psi = \begin{bmatrix} \xi \\ 0 \end{bmatrix}, \tag{1.5}$$

where all the γ s are in accordance with the Weyl representation. On similar footing the charge conjugated field, $\psi^c = i\gamma^2\psi^*$ is set up. Now, out of these two Weyl spinors, two fields,

$$\chi = \psi_L + \psi_L^c, \tag{1.6}$$

$$\omega = \psi_R - \psi_R^c, \tag{1.7}$$

are constructed which respects the Majorana condition in Eq. (1.3). The kinetic term corresponding to the Weyl spinors is shown below,

$$\mathcal{L} = i\,\bar{\psi}_R\gamma^\mu\partial_\mu\psi_R + i\,\bar{\psi}_L\gamma^\mu\partial_\mu\psi_L, \tag{1.8}$$

now transforms into,

$$\mathcal{L} = \frac{i}{2}(\bar{\chi}\gamma^\mu\partial_\mu\chi + \bar{\omega}\gamma^\mu\partial_\mu\omega). \tag{1.9}$$

1.2.2 The Majorana neutrinos

If there exists only one Weyl field, say χ, then it can be made massive only through the Majorana mass term [3–5]. We present the Lagrangian along with the mass term,

$$\mathcal{L} = \frac{i}{2}\bar{\chi}\gamma^{\mu}\partial_{\mu}\chi - \frac{1}{2}m\bar{\chi}\chi, \tag{1.10}$$

where the mass m is real. A generalized choice of χ,

$$\chi = e^{i\alpha}\psi_L + e^{i\beta}\psi_L^c, \tag{1.11}$$

which satisfies the Majorana condition in the extended sense,

$$\chi^c = e^{-i(\alpha+\beta)}\chi, \tag{1.12}$$

leads to a Lagrangian,

$$\mathcal{L} = i\,\bar{\psi_L}\gamma^{\mu}\partial_{\mu}\psi_L - \frac{M}{2}\bar{\psi_L^c}\psi_L + h.c, \tag{1.13}$$

where, the mass, $M = m\,e^{i(\alpha-\beta)}$, is complex and by redefining ψ_L the phase term can be quenched. The second term violates the lepton number conservation.

1.2.3 The Dirac neutrinos

If there are two Weyl fields, there may exist a mass term that conserves the lepton number. If, $\psi_L^{(1)}$ and $\psi_L^{(2)}$ are two Weyl fields, the mass term can be constructed as shown in the following,

$$\mathcal{L} = -\frac{m_{ij}}{2}\overline{\psi_L^{(i)c}}\psi_L^j + h.c, \tag{1.14}$$

which is the generalization of the mass term in Eq. (1.13). Provided, the diagonal mass terms $m_{ii} = 0$, the lepton number, $L_i - L_j$, can be made conserved. On defining two fields, ψ_L and ψ_R as $\psi_L = \psi_L^{(i)}$ and $\psi_R = [\psi_L^{(j)}]^c$, we obtain the conventional Dirac mass term,

$$\mathcal{L} = -m\overline{\psi}\psi = -m(\bar{\psi_L}\psi_R + \bar{\psi_R}\psi_L). \tag{1.15}$$

On assigning $\psi_L^{(i)}$ and $\psi_L^{(j)c}$ different flavors, say, $\psi_L^{(1)} = \nu_{eL}$ and $\psi_L^{(2)} = \nu_{\mu L}$, we obtain a Dirac mass term with off-diagonal elements [3].

1.2.4 The massive neutrinos in the standard model

The above discussion, leads to the inference that following any of the mechanisms, the massive neutrinos can be embraced in the Electro-weak theory [6, 7]. If there exists ν_R field in addition to ν_L, then through Yukawa coupling, a Dirac mass term,

$$\mathcal{L}_{mass} = f_\nu \overline{\nu_R} \phi l_L, \tag{1.16}$$

can be instituted. And, if there exists no right handed counterpart ν_R, then the Majorana mass term as shown in the following,

$$\mathcal{L} = \frac{1}{2} G \, \overline{l_L^c} \, l_L \frac{\phi\phi}{M}, \tag{1.17}$$

is the only mean to incorporate neutrino mass in the standard framework. The Majorana neutrino mass will appear as shown below,

$$m_\nu = G \frac{\langle \phi_0^2 \rangle}{M}. \tag{1.18}$$

But, this mass term (see Eq. (1.17)) is necessarily non-renormalisable and is assessed as an effective interaction. The term, M is called the effective mass. This interaction transpires from more fundamental interactions at a higher energy scale and accounts for the smallness of m_ν. In Eq. (1.18) if, $M \to \propto$, then, $m_\nu \to 0$.

1.2.5 See-saw mechanism

The problem of understanding the smallness of neutrino masses plays the central role in the study of neutrino physics. The simplest possibility to understand the effective interaction as shown in Eq. (1.17) is perhaps the "See-Saw" mechanism [8–12]. The GUT scenarios initiate some framework where both Dirac and Majorana neutrinos coexist. Let us present the simple Lagrangian in support of this context,

$$\mathcal{L} = f \overline{\nu_R} \, \nu_L \langle \phi^0 \rangle + \frac{M}{2} \overline{\nu_R^c} \, \nu_R + h.c. \tag{1.19}$$

With the heavy field ($M \gg \langle \phi \rangle$) being integrated out, the Feynman diagram

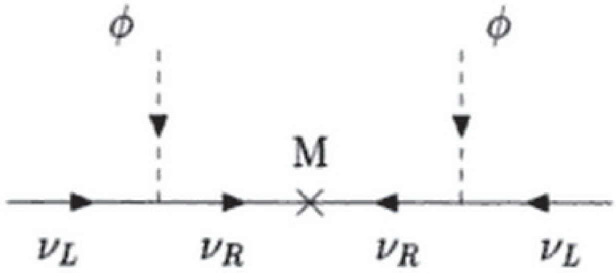

FIGURE 1.1: The Feynman diagram for See-saw mechanism responsible for neutrino mass

shown in Fig. (1.1), generates,

$$\mathcal{L}_{eff} = \frac{f^2}{2M}\phi^0\phi^0\,\overline{\nu_L^c}\,\nu_L, \tag{1.20}$$

The same can be obtained by diagonalizing the mass matrix as shown below,

$$\begin{array}{cc} \begin{array}{cc} \nu_L & \nu_R \end{array} & \\ \begin{pmatrix} 0 & m \\ m & M \end{pmatrix} & \begin{array}{c} \nu_L \\ \nu_R \end{array} \end{array}, \tag{1.21}$$

where, we see a mixing between the two left handed and the right handed sectors and with the Dirac mass, $m = f\langle\phi\rangle$, we obtain,

$$m_{\nu_L} \simeq \frac{m^2}{M}. \tag{1.22}$$

The Dirac mass responsible for the mixing between ν_L and ν_R is probably of the order of the mass of other charged particles. The smallness of the mass of left handed neutrino, m_{ν_L}, is owing to the large mass-scale, M. This mechanism plays a very significant role in Grand Unified Theories, where it is possible to relate the Dirac mass, m and the charge $-2/3$ quarks.

1.3 Neutrino Mixing

Here we present a comparative study between the Dirac and Majorana neutrinos. If the neutrinos are of "Dirac-type", then the corresponding mass term assumes the form [3],

$$\begin{aligned}\mathcal{L}_{mass} &= m\overline{\nu_R}\,\nu_L + h.c, \\ &= \overline{\nu_R^{\alpha}} m_{\alpha\beta}\nu_L^{\beta} + h.c.\end{aligned} \tag{1.23}$$

The mass matrix ($m_{\alpha\beta}$) is diagonalized using two unitary matrices U and V and the $\mathcal{L}_{mass}$ is presented as shown below,

$$\begin{aligned}\mathcal{L}_{mass} &= \overline{\nu_R}^i\, V^{\dagger}_{i\alpha}\, m_{\alpha\beta} U_{\beta j}\, \nu_L^j + h.c, \\ &= \overline{\nu_R^i} m_{ii}\nu_L^i + h.c,\end{aligned} \tag{1.24}$$

where ν_α (or, equivalent terms) represent the flavor eigenstates of the weak interaction and ν^i represents the mass eigenstates, which are related to each other in the following way,

$$\nu_L^{\alpha} = U_{\alpha i}\nu_L^i, \tag{1.25}$$

$$\nu_R^{\alpha} = V_{\alpha i}\nu_R^i. \tag{1.26}$$

And, we have,

$$V^{\dagger}\, m\, U = m^{diag}. \tag{1.27}$$

where, U and V are the two unitary matrices. Similarly, we diagonalize the Majorana mass term,

$$\begin{aligned}\mathcal{L}_{mass} &= -\frac{1}{2}\nu_L^{\alpha\, T}\, C^{-1}\, m_{\alpha\beta}\, \nu_L^{\beta} + h.c, \\ &= -\frac{1}{2}\nu_L^{i\,T}\, C^{-1}\, U_{i\alpha}\, m_{\alpha\beta}\, U_{\beta j}\, \nu_L^j + h.c, \\ &= -\frac{1}{2}\nu_L^{i\,T}\, C^{-1}\, m_{ii}\nu_L^i + h.c.\end{aligned} \tag{1.28}$$

In contrast to the previous case, we experience a single unitary transformation for Majorana scenario as shown below,

$$\nu_L^{\alpha} = U_{\alpha i}\nu_L^i, \tag{1.29}$$

and we see, that the unitary matrix U, appears twice in the mass term. We have,

$$U^{\dagger}\, m\, U = m^{diag}. \tag{1.30}$$

This is to be emphasized that in contrast to the Dirac neutrino mass matrix, the Majorana mass matrix is symmetric in general. The flavor states ν_α's are the eigenstates of weak interaction and appear in the laboratory experiments, while the mass eigenstate, ν^i are prominent in the oscillation experiments.

1.3.1 The Weak interaction Lagrangian

In weak interaction, only the left handed fields appear and the corresponding Lagrangian is shown as in the following,

$$\mathcal{L}_{int} = \frac{g}{\sqrt{2}} W^+_\mu \begin{bmatrix} \overline{\nu}^e_L & \overline{\nu}^\mu_L & \overline{\nu}^\tau_L \end{bmatrix} \gamma^\mu \begin{bmatrix} e_L \\ \mu_L \\ \tau_L \end{bmatrix} . \tag{1.31}$$

Now following Eq. (1.25) (or, Eq. (1.29)), $\mathcal{L}_{int}$, is expressed in terms of the neutrino mass eigenstates, ν^i_L,

$$\mathcal{L}_{int} = \frac{g}{\sqrt{2}} W^+_\mu \begin{bmatrix} \overline{\nu}^1_L & \overline{\nu}^2_L & \overline{\nu}^3_L \end{bmatrix} \gamma^\mu U^\dagger \begin{bmatrix} e_L \\ \mu_L \\ \tau_L \end{bmatrix} , \tag{1.32}$$

where U is termed as lepton mixing matrix or U_{PMNS} (Pontecorvo-Maki-Nakagawa-Sakata) [13, 14]. In discussion so far, we have assumed the charged lepton mass matrix to be diagonal.

1.3.2 Parametrization of U

The lepton mixing matrix U is an unitary matrix. We consider the general case [15] when U is an $n \times n$ matrix, and any unitary matrix can be represented as $U = e^{i\,H}$, where H is a $n \times n$ Hermatian matrix which has got n^2 independent real parameters. The number of angles required to parametrize H is the same to that used to parametrize a $n \times n$ orthogonal matrix, say O. Again, we can present, $O = e^A$. The orthogonality conditions, $O^T O = 1$ compels, A to be antisymmetric. Hence, the matrix, A has got $n\,(n-1)/2$ real non diagonal elements. This indicates that U has got,

$$N_\theta = \frac{n\,(n-1)}{2}, \tag{1.33}$$

number of angles. Hence, the number of phases required to characterize, U is,

$$N_\phi = n^2 - N_\theta = \frac{n\,(n+1)}{2}. \tag{1.34}$$

Considering the scenario, where only three generations of neutrinos (i.e., $n=3$)are allowed, we require three angles ($N_\theta = 3$) and six phases ($N_\phi = 6$) to parametrize the lepton mixing matrix. We present U [16],

$$U = \Psi_1\, R_{23}\, R_{13}\, \Psi_2\, R_{12}\, \Psi_3, \tag{1.35}$$

where, R_{ij} s are the orthogonal matrices, and Ψ_is are the diagonal matrices containing the phases, where,

$$R_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{bmatrix},\ R_{13} = \begin{bmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{bmatrix},\ R_{12} = \begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.36}$$

$$\Psi_1 = \begin{bmatrix} e^{i\,\phi_1} & 0 & 0 \\ 0 & e^{i\,\phi_2} & 0 \\ 0 & 0 & e^{i\,\phi_3} \end{bmatrix},\ \Psi_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\,\phi_4} & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \Psi_3 = \begin{bmatrix} e^{i\,\phi_5} & 0 & 0 \\ 0 & e^{i\,\phi_6} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{1.37}$$

where, $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$. The θ_{ij} s are the three independent Euler's rotation angles. If we deal with the Dirac neutrinos, we have the $\mathcal{L}_{int}$ (See Eq.(1.32)) appear as in the following,

$$\begin{aligned} \mathcal{L}_{int} &= \frac{g}{\sqrt{2}} W_\mu^+ \begin{bmatrix} \overline{\nu}_L^1 & \overline{\nu}_L^2 & \overline{\nu}_L^3 \end{bmatrix} \gamma^\mu\, \Psi_3^\dagger\, R_{12}^T\, \Psi_2^\dagger\, R_{13}^T\, R_{23}^T\, \Psi_1^\dagger \begin{bmatrix} e_L \\ \mu_L \\ \tau_L \end{bmatrix}, \\ &= \frac{g}{\sqrt{2}} W_\mu^+ \begin{bmatrix} \overline{\nu}_L^1 & \overline{\nu}_L^2 & \overline{\nu}_L^3 \end{bmatrix} \gamma^\mu \begin{bmatrix} e^{-i\,\phi_5} & 0 & 0 \\ 0 & e^{-i\,\phi_6} & 0 \\ 0 & 0 & 1 \end{bmatrix} R_{12}^T\, \Psi_2^\dagger\, R_{13}^T\, R_{23}^T\, \Psi_1^\dagger \begin{bmatrix} e_L \\ \mu_L \\ \tau_L \end{bmatrix} \end{aligned} \tag{1.38}$$

By redefining, $\nu_L^1 \leftrightarrow \nu_L^1 e^{i\,\phi_5}$ and $\nu_L^2 \leftrightarrow \nu_L^2 e^{i\,\phi_6}$, the diagonal phase matrix Ψ_3 can be omitted from U.

$$\begin{aligned}
\mathcal{L}_{int} &= \frac{g}{\sqrt{2}} W_\mu^+ \begin{bmatrix} \overline{\nu}_L^1 & \overline{\nu}_L^2 & \overline{\nu}_L^3 \end{bmatrix} \gamma^\mu\, R_{12}^T\, \Psi_2^\dagger\, R_{13}^T\, R_{23}^T\, \Psi_1^\dagger \begin{bmatrix} e_L \\ \mu_L \\ \tau_L \end{bmatrix}, \\
&= \frac{g}{\sqrt{2}} W_\mu^+ \begin{bmatrix} \overline{\nu}_L^1 & \overline{\nu}_L^2 & \overline{\nu}_L^3 \end{bmatrix} \gamma^\mu\, R_{12}^T\, \Psi_2^\dagger\, R_{13}^T\, R_{23}^T \begin{bmatrix} e^{-i\,\phi_1} & 0 & 0 \\ 0 & e^{-i\,\phi_2} & 0 \\ 0 & 0 & e^{-i\,\phi_3} \end{bmatrix} \begin{bmatrix} e_L \\ \mu_L \\ \tau_L \end{bmatrix}
\end{aligned} \tag{1.39}$$

Again, in the similar fashion, Ψ_1 is eliminated by redefining, $e_L \leftrightarrow e_L e^{i\,\phi_1}$, $\mu_L \leftrightarrow \mu_L e^{i\,\phi_2}$ and $\tau_L \leftrightarrow \tau_L e^{i\,\phi_3}$. Hence,

$$\mathcal{L}_{int} = \frac{g}{\sqrt{2}} W_\mu^+ \begin{bmatrix} \overline{\nu}_L^1 & \overline{\nu}_L^2 & \overline{\nu}_L^3 \end{bmatrix} \gamma^\mu\, R_{12}^T\, \Psi_2^\dagger\, R_{13}^T\, R_{23}^T \begin{bmatrix} e_L \\ \mu_L \\ \tau_L \end{bmatrix}. \tag{1.40}$$

From the above exercise, we understand that for Dirac type neutrinos, we can express the lepton mixing matrix in terms of three angles and one phase.

$$\textbf{Dirac neutrino :} \qquad U = U\,(\theta_{12},\, \theta_{13},\, \theta_{23},\, \phi_4). \tag{1.41}$$

But the situation differs if the neutrinos are of Majorana type. The charged lepton fields can easily absorb Ψ_1. And let us say, neutrino mass eigenstates, ν_L^i, absorbs Ψ_3, as before. But, in the Majorana mass term, ν_L^i appears twice. The mass term appears closely to the form, $\nu_L^{iT}\, U^T\, m\, U\, \nu_L^i$. Hence, although we remove the phases, ϕ_5 and ϕ_6 from U, it will appear in the mass matrix. The another alternative is to keep Ψ_3, as it is in U which leads to,

$$\mathcal{L}_{int} = \frac{g}{\sqrt{2}} W_\mu^+ \begin{bmatrix} \overline{\nu}_L^1 & \overline{\nu}_L^2 & \overline{\nu}_L^3 \end{bmatrix} \gamma^\mu \begin{bmatrix} e^{-i\,\phi_5} & 0 & 0 \\ 0 & e^{-i\,\phi_6} & 0 \\ 0 & 0 & 1 \end{bmatrix} R_{12}^T\, \Psi_2^\dagger\, R_{13}^T\, R_{23}^T \begin{bmatrix} e_L \\ \mu_L \\ \tau_L \end{bmatrix}, \tag{1.42}$$

and,

$$\textbf{Majorana neutrino :} \qquad U = U\,(\theta_{12}, \theta_{13}, \theta_{23}, \phi_4, \phi_5, \phi_6) \tag{1.43}$$

We have now nine (or seven) parameters and estimation of which at low energies will determine whether the neutrinos are of Majorana or Dirac type. Hence a suitable parametrization of U is the part and parcel of neutrino physics. This will help to deal with the experimental data and interpreting the underlying physics.

1.3.2.1 Standard parametrization scheme

The Particle Data Group (PDG) adopted a parametrization of U which is called the Standard parametrization [2]. According to this parametrization, U is described as the product of three consecutive rotation matrices multiplied with a diagonal matrix containing phases.

$$U = R_{23}\,(\theta_{23}\,:\,0)\,U_{13}(\theta_{13}\,:\,\delta)\,R_{12}\,(\theta_{12}\,:\,0)\,P; \tag{1.44}$$

where,

$$U_{13} = \begin{bmatrix} c_{13} & 0 & s_{13}\,e^{-i\,\delta} \\ 0 & 1 & 0 \\ -s_{13}\,e^{i\,\delta} & 0 & c_{13} \end{bmatrix}, \quad P = \begin{bmatrix} e^{i\,\alpha} & 0 & 0 \\ 0 & e^{i\,\beta} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{1.45}$$

where δ is called the Dirac-type phase and α and β are called the Majorana phases. For Dirac neutrinos, $\alpha,\,\beta = 0$. The lepton mixing matrix, U, is presented as in the following,

$$U = \begin{bmatrix} c_{12}c_{13} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - c_{23}s_{12}s_{13}e^{i\delta} & c_{13}c_{23} \end{bmatrix} \begin{bmatrix} e^{i\,\alpha} & 0 & 0 \\ 0 & e^{i\,\beta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.46}$$

The angles, θ_{12}, θ_{13} and θ_{23} are called the three mixing angles.

1.3.2.2 Symmetric parametrization scheme

Besides the standard parametrization there is also another parametrization scheme called Symmetric [17] which describes U in the following way,

$$U = U_{23}\,(\theta_{23}\,:\,\omega_{23})\,U_{13}(\theta_{13}\,:\,\omega_{13})\,R_{12}\,(\theta_{12}\,:\,\omega_{12}); \tag{1.47}$$

where,

$$U_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23}\, e^{-i\,\omega_{23}} \\ 0 & -s_{23}\, e^{i\,\omega_{23}} & c_{23} \end{bmatrix}, \tag{1.48}$$

$$U_{13} = \begin{bmatrix} c_{13} & 0 & s_{13}\, e^{-i\,\omega_{13}} \\ 0 & 1 & 0 \\ -s_{13}\, e^{i\,\omega_{13}} & 0 & c_{13} \end{bmatrix}, \tag{1.49}$$

$$U_{12} = \begin{bmatrix} c_{12} & s_{12}\, e^{-i\,\omega_{12}} & 0 \\ -s_{12}\, e^{i\,\omega_{12}} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.50}$$

All the phases appearing in U are physical. The "Symmetric" parametrization differs from the "Standard" one, in expressing the Dirac and Majorana CP phases. The equivalence of the two parametrization schemes says,

$$\delta = \omega_{13} - \omega_{12} - \omega_{23}, \tag{1.51}$$

$$\alpha = \omega_{12} + \omega_{23}, \tag{1.52}$$

$$\beta = \omega_{23}. \tag{1.53}$$

Both the parametrization schemes are equally relevant, but the Standard parametrization is the most preferred one. Here we want to emphasize on an important point that we are working in a basis where charged lepton mass matrix is assumed to be diagonal (in general termed "Flavor-basis"). In this basis, only neutrino mixing contributes towards the final lepton mixing matrix, U. For further discussion, we refer to Eq. (1.29), for which the three generations of neutrinos can be represented in the following fashion,

$$\underbrace{\begin{bmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{bmatrix}}_{|\nu_{\alpha\mathbf{L}}\rangle} = \underbrace{\begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{bmatrix}}_{\mathbf{U}} \underbrace{\begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix}}_{|\nu_{\mathbf{i}}\rangle} \tag{1.54, 1.55}$$

The above expressions indicate that the left handed flavor eigenstates are just the linear superposition of neutrino mass eigenstates. For example,

$$|\nu_{eL}\rangle = U_{e1}|\nu_1\rangle + U_{e2}|\nu_2\rangle + U_{e3}|\nu_3\rangle. \tag{1.56}$$

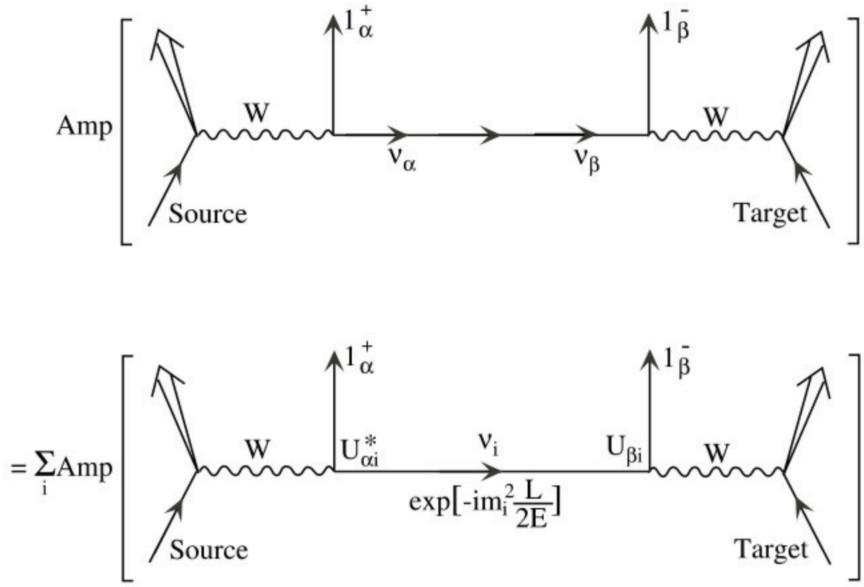

FIGURE 1.2: The flavor changing process involving $\nu_\alpha \to \nu_\beta$ from source to detector.

Certain observable parameters like, θ_{13}, θ_{12}, θ_{23} and δ can be derived out from the above expression of U in the following way,

$$\begin{aligned}
\sin^2\theta_{13} &= |U_{e3}|^2, && (1.57)\\
\sin^2\theta_{13} &= \frac{|U_{e2}|^2}{1-|U_{e3}|^2}, && (1.58)\\
\sin^2\theta_{23} &= \frac{|U_{\mu 3}|^2}{1-|U_{e3}|^2}, && (1.59)\\
\delta &= -Arg\,[U_{e3}], && (1.60)
\end{aligned}$$

which are in accordance with the Standard parametrization.

Next we shall discuss several possibilities to measure the unknown quantities like the three mass parameters, three mixing angles and the three phases. The neutrino oscillation experiments deal with the mass parameters, mixing angles and the Dirac CP phase, whereas, the experiments like neutrino-less double beta decay experiments deal with the mass parameters and the Majorana phases.

1.4 Neutrino Oscillation

Let us consider a weak interaction process where a neutrino (ν_α) and a charged lepton (l_α) with certain flavor α [3, 18–34] are engendered. The neutrino, transits through a finite macroscopic distance L from the source (where it is produced) and interacts with the target, and finally begets a charged lepton (l_β) with a different flavor β. So, during the advance, ν_α has evolved to ν_β. This phenomenon which involves the change in neutrino flavor is called, $\nu_\alpha \to \nu_\beta$, oscillation (See Fig.(1.2)).

1.4.1 Oscillation Probability

In the intermediate state, the neutrino may attain any of the mass eigenstates ν_i. In other words, the neutrinos are produced and detected as flavor eigenstates and they transit as a coherent superposition of neutrino mass eigenstates [18, 19, 35, 36]. Hence, the probability amplitude $\mathcal{A}\,(\nu_\alpha \to \nu_\beta)$ is the product of the following three individual amplitudes as shown below,

- $\mathbf{U^*_{\alpha_i}}$ at the source. This contribution is the outcome of the fact that ν_α is a linear superposition of ν_i s,
- $\mathbf{U_{\beta_i}}$ appears at the target due to the similar reason discussed above,
- $\mathbf{e^{-i\,m_i^2 \frac{L}{2E}}}$, due to the travel by a distance L with energy E.

After proper transformation, we obtain the expression of probability, $\mathcal{P}\,(\nu_\alpha \to \nu_\beta)$ for neutrino-neutrino oscillation processes as shown in the following [18, 19],

$$\begin{aligned} \mathcal{P}(\nu_\alpha \to \nu_\beta) &= |\mathcal{A}(\nu_\alpha \to \nu_\beta)|^2 \\ &= \delta_{\alpha\beta} - 4\sum_{i>j} Re(U^*_{\alpha\,i}\,U_{\beta\,i}\,U_{\alpha\,j}\,U^*_{\beta\,j})\,\sin^2\left[\Delta\,m^2_{ij}\,\frac{L}{4E}\right] \\ &\quad +2\sum_{i>j} Im(U^*_{\alpha\,i}\,U_{\beta\,i}\,U_{\alpha\,j}\,U^*_{\beta\,j})\,\sin^2\left[\Delta\,m^2_{ij}\,\frac{L}{2E}\right], \end{aligned} \tag{1.61}$$

where, $\Delta\,m^2_{ij} = m_i^2 - m_j^2$. The above expression for oscillation probability holds good for any number of neutrino mass eigenstates. We emphasize on the following important features.

- Neutrino oscillation depends on the mass-squared difference parameter, $\Delta\,m^2_{ij}$. If the mass eigenvalues are zero or strictly degenerate, $\mathcal{P}(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta}$, indicating no oscillation. In other words, if there is oscillation the state ν_i has to be massive.

- $\mathcal{P}$ oscillates as function of L/E.
- On assuming CPT invariance, we have,

$$\mathcal{P}(\overline{\nu_\alpha} \to \overline{\nu_\beta}) = \mathcal{P}(\nu_\beta \to \nu_\alpha); \tag{1.62}$$

Again, following Eq. (1.61), we can produce, $\mathcal{P}(\nu_\beta \to \nu_\alpha \,:\, U) = \mathcal{P}(\nu_\alpha \to \nu_\beta \,:\, U^*)$, leading to the following relation [18],

$$\mathcal{P}(\overline{\nu_\alpha} \to \overline{\nu_\beta} \,:\, U) = \mathcal{P}(\nu_\alpha \to \nu_\beta \,:\, U^*); \tag{1.63}$$

This says that if U is replaced by U^*, the probability of occurrence of, $\overline{\nu_\alpha} \to \overline{\nu_\beta}$ and that for $\nu_\alpha \to \nu_\beta$ are equal. But certainly they will diverge if U is not real [18]. This signifies a violation of CP symmetry in the neutrino oscillation processes.

$$\mathcal{P}(\nu_\alpha \to \nu_\beta) - \mathcal{P}(\overline{\nu_\alpha} \to \overline{\nu_\beta}) = 4 \sum_{i>j} Im(U^*_{\alpha\, i}\, U_{\beta\, i}\, U_{\alpha\, j}\, U^*_{\beta\, j}) \sin^2 \left[\Delta\, m^2_{ij} \frac{L}{2E}\right]. \tag{1.64}$$

The discussion so far concerns the propagation of neutrinos through vacuum only. In the next section we shall discuss briefly the influence of matter on neutrino oscillation.

1.4.2 Matter Effect

The properties of neutrinos, concerning the mixing angles and the effective masses are affected if it travels through certain medium [37, 38]. The difference between the coherent forward scattering amplitudes (on nucleons and electrons)for different neutrinos is interpreted in terms of "n",a quantity equivalent to the refractive index in Optics. A Sterile neutrino which does not interact has $n = 1$. In matter all formulae carrying the neutrino momentum "p" is modified by "$n\,p$". This introduces an additional phase factor in the neutrino wave function. The neutrino mass matrix is also modified. For example, the 1-1 element m_{ee}, is modified to [18],

$$m^2_{ee} \to m^2_{ee} - 2p^2\,(n_e - n_\mu) = m^2_{ee} + 2\sqrt{2}\, G_F\, N_e\, p, \tag{1.65}$$

where, G_F is the Fermi Constant and N_e is the electron density. Hence, after diagonalization, the mass matrix will lead to certain mass eigenstates which are different from those obtained in vacuum.

Let us look into the probability expression for two neutrino scenario (say, $\nu_e \to \nu_\mu$) in vacuum,

$$\mathcal{P}(\nu_e \to \nu_\mu) = \sin^2 2\theta \, \sin^2 \left[\Delta\, m^2 \, \frac{L}{4E} \right]. \tag{1.66}$$

If matter effect is taken into consideration, the mixing angle, θ and $\Delta\, m^2$ is modified as shown in the following,

$$\sin^2 2\theta \;\; \to \;\; \frac{\sin^2 2\theta}{\sin^2 2\theta + \left(\cos 2\theta - \frac{2\sqrt{2}\, G_F\, N_e\, E}{\Delta\, m^2}\right)^2}, \tag{1.67}$$

$$\Delta\, m^2 \;\; \to \;\; \Delta\, m^2 \sqrt{\sin^2 2\theta + \left(\cos 2\theta - \frac{2\sqrt{2}\, G_F\, N_e\, E}{\Delta\, m^2}\right)^2}. \tag{1.68}$$

This is to be noted that since ν_μ and ν_τ have same neutral current interaction with ordinary matter, the $\nu_\mu - \nu_\tau$ oscillations are invariant under Matter effect. Because of the difference in signs of the scattering amplitudes for neutrinos and antineutrinos, the concerned matter effects are also different. This in turn can conceal the CP violation effects.

1.4.2.1 MSW Effect

From the above discussion, it is clear that the effective neutrino mass (similarly, the mixing angles also) changes as a function of the neutrino energy. For certain matter density, two neutrino mass eigenstates may turn degenerate also in this modified scenario. In the present scenario, the oscillation parameters may undergo a resonance condition, producing maximal oscillation even if the vacuum oscillation amplitude is small. The neutrinos when travel through the interior of Sun faces continuously varying matter density. In this course, a certain resonant point is achieved when a complete flavor conversion takes place; i.e, all ν_e become ν_μ and vice-versa. This is known as "Mikheyev-Smirnov-Wolfenstein" (MSW) effect [39–42].

1.4.3 Neutrino Oscillation-Detection

Following the oscillation probability expression in Eq. (1.61), we see that the neutrino detection is possible through two different ways [18, 43]:

- Appearance channel- detection of ν_β in the beam of ν_α, provided $\alpha \neq \beta$,
- Disappearance channel-disappearance of ν_α into other flavors.

This is to be emphasized that the neutrino flavor is defined by the charged lepton produced in the charged current interaction. Hence for an appearance measurement, it implies that the neutrino energy is sufficient enough to produce the final state charged lepton. Below, we shall put forward some essential equations related to the detection of neutrino oscillation in several neutrino experiments.

Accelerator neutrino experiments: ($E \sim 1\,GeV$, $L \sim 1 - 1000\,km$)

We have the following probability equations used in the accelerator experiments [2],

$$\begin{aligned}
\mathcal{P}(\nu_\mu \to \nu_\tau) &= \sin^2 2\theta_{23}\, \cos^4\theta_{13} \sin^2\left[\Delta\, m^2_{32}\, \frac{L}{4E}\right], &(1.69)\\
\mathcal{P}(\nu_\mu \to \nu_e) &= \sin^2 2\theta_{13}\, \sin^2\theta_{23} \sin^2\left[\Delta\, m^2_{32}\, \frac{L}{4E}\right], &(1.70)\\
\mathcal{P}(\nu_e \to \nu_\mu) &= \sin^2 2\theta_{13}\, \sin^2\theta_{23} \sin^2\left[\Delta\, m^2_{32}\, \frac{L}{4E}\right], &(1.71)\\
\mathcal{P}(\nu_e \to \nu_\tau) &= \sin^2 2\theta_{13}\, \cos^2\theta_{23} \sin^2\left[\Delta\, m^2_{32}\, \frac{L}{4E}\right], &(1.72)
\end{aligned}$$

where, the angle θ_{13} and CP violation effect are neglected. But the present and future long baseline experiments are sensitive to nonzero θ_{13} and δ. Including δ and low mass scale, the expression for $\mathcal{P}(\nu_\mu \to \nu_e)$ is modified to,

$$\begin{aligned}
\mathcal{P}(\nu_\mu \to \nu_e) &= \sin^2\theta_{23}\, \sin^2 2\theta_{13} \sin^2\left[\Delta\, m^2_{32}\, \frac{L}{4E}\right] + \cos^2\theta_{23}\, \sin^2 2\theta_{13} \sin^2\left[\Delta\, m^2_{21}\, \frac{L}{4E}\right]\\
&\quad + \cos\theta_{13} \sin 2\theta_{12} \sin 2\theta_{13} \sin 2\theta_{23} \sin\left[\Delta m^2_{32} \frac{L}{4E}\right] \sin\left[\Delta m^2_{21} \frac{L}{4E}\right] \times\\
&\quad \left(\cos\delta \cos\left[\Delta m^2_{32} \frac{L}{4E}\right] \pm \sin\delta \sin\left[\Delta m^2_{32} \frac{L}{4E}\right]\right), \qquad (1.73)
\end{aligned}$$

where the negative sign appears for neutrinos and for antineutrinos it is positive.
Experiments involved: KARMEN, LSND, LBNE, MINOS, MINOS+, NOvA, NuMI, T2K etc.

Reactor neutrino experiments: ($E \sim 1\, MeV$, $L \sim 1 - 100\, km$)

The oscillation probability is expressed as [2],

$$\begin{aligned}
\mathcal{P}(\overline{\nu_e} \to \overline{\nu_e}) &= 1 - \cos^4\theta_{13}\, \sin^2 2\theta_{12} \sin^2\left[\Delta\, m^2_{21}\, \frac{L}{4E}\right] \\
&\quad - \cos^2\theta_{12}\, \sin^2 2\theta_{13} \sin^2\left[\Delta\, m^2_{31}\, \frac{L}{4E}\right] \\
&\quad - \sin^2\theta_{12}\, \sin^2 2\theta_{13} \sin^2\left[\Delta\, m^2_{32}\, \frac{L}{4E}\right]. \qquad (1.74)
\end{aligned}$$

The $\overline{\nu_e}$ have the energy close to $4\, MeV$. For short distances, $L < 5\, km$, the above expression is modified to,

$$\mathcal{P}(\overline{\nu_e} \to \overline{\nu_e}) = 1 - \sin^2 2\theta_{13} \sin^2\left[\Delta\, m^2_{32}\, \frac{L}{4E}\right]. \qquad (1.75)$$

Experiments involved: CHOOZ, DAYA BAY, Double Chooz, KamLAND, RENO etc.

Solar ($E \sim 1\, MeV$, $L \sim 1.5 \times 10^8\, km$) and Atmospheric neutrino ($E \sim 1\, GeV$, $L \sim 10^4\, km$) experiments

The solar neutrino experiments are concerned with ν_e disappearance channel and are sensitive to $\Delta\, m^2_{21}$ and θ_{12}. Solar neutrinos are highly influenced by matter effect. The *pp* (^7Be) have mean energy of $0.2\, MeV$ ($0.9\, MeV$) and these neutrinos are little influenced by matter . But the ^8B neutrinos having mean energy of the order of $10\, MeV$ are subjected to the matter effect. These are produced and exit Sun as ν_2 mass eigenstate and do not undergo vacuum oscillation [43]. Therefore on reaching earth, the solar neutrinos are "effectively incoherent". Hence the survival probability is given as in the following,

$$\langle \mathcal{P}(\nu_e \to \nu_e) \rangle = f_1 \cos^2\theta_{sol} + f_2 \sin^2\theta_{sol}, \qquad (1.76)$$

where f_1 and f_2 are the fractions of neutrinos that are ν_1 and ν_2 respectively, satisfying the relation, $f_1 + f_2 = 1$. Due to MSW effect $f_2 \approx 0.9$, for ^8B neutrinos. Atmospheric neutrino experiments deal with ν_μ disappearance channel and are sensitive to θ_{23} and $\Delta\, m^2_{32}$. In vacuum, the survival probability is given as [43],

$$\begin{aligned} \mathcal{P}(\nu_\mu \to \nu_\mu) &= 1 - 4|U_{\mu 3}|^2|U_{\mu 1}|^2 \sin^2\left[\Delta m^2_{31}\frac{L}{4E}\right] - 4|U_{\mu 3}|^2|U_{\mu 2}|^2 \sin^2\left[\Delta m^2_{32}\frac{L}{4E}\right] \\ &\quad -4|U_{\mu 2}|^2|U_{\mu 1}|^2 \sin^2\left[\Delta m^2_{21}\frac{L}{4E}\right] \end{aligned} \tag{1.77}$$

For experiments at the atmospheric L/E (around $500\, km/GeV$), we have

$$\mathcal{P}(\nu_\mu \to \nu_\mu) \approx 1 - 4|U_{\mu 3}|^2(1 - |U_{\mu 3}|^2) \sin^2\left[\Delta m^2_{\mu\mu}\frac{L}{4E}\right], \tag{1.78}$$

where, $\Delta m^2_{\mu\mu}$ is the effective atmospheric $\Delta\, m^2$ for ν_μ disappearance channel,

$$\Delta\, m^2_{\mu\mu} \equiv \frac{|U_{\mu 1}|^2|\Delta m^2_{31}| + |U_{\mu 2}|^2|\Delta m^2_{32}|}{|U_{\mu 1}|^2 + |U_{\mu 2}|^2}. \tag{1.79}$$

Experiments involved: Homestake, SAGE, Gallex, SK, SNO, MACRO
In the next section we shall summarize the results of the neutrino oscillation experiments.

1.4.4 Present status of the oscillation parameters

The oscillation experiments comprehend six observational parameters: the three mixing angles, θ_{12}, θ_{23}, θ_{13} , Dirac CP violating phase, δ and the mass square differences, Δm^2_{21} and $|\Delta m^2_{31}|$. But the experiments are unable to ascertain the three individual masses m_1, m_2 and m_3. It is found that the mass eigenstate ν_2 is more massive than ν_1. Again, the ignorance of the exact sign of Δm^2_{31}, leads to two possibilities: either, $m_3 > m_1$ (**Normal**), or $m_3 < m_1$ (**Inverted**). Besides, there are two possibilities for θ_{23}: either $\theta_{23} > \pi/4$ or, $\theta_{23} < \pi/4$. The central

Parameter	central value	1σ range	2σ range	3σ range
$\Delta\, m^2_{21}$ $\left[10^{-5}\, eV^2\right]$	7.60	7.42 – 7.79	7.26 – 7.99	7.11 – 8.18
$\lvert\Delta\, m^2_{31}\rvert$ $\left[10^{-3}\, eV^2\right]$ (NH)	2.48	2.41 – 2.53	2.35 – 2.59	2.30 – 2.65
$\lvert\Delta\, m^2_{31}\rvert$ $\left[10^{-3}\, eV^2\right]$ (IH)	2.38	2.32 – 2.43	2.26 – 2.48	2.20 – 2.54
$\sin^2\theta_{12}$	0.323	0.307 – 0.339	0.292 – 0.357	0.278 – 0.375
$\sin^2\theta_{23}$ (NH)	0.567 (0.467)	.439 – 0.599	0.413 – 0.623	0.392 – 0.643
$\sin^2\theta_{23}$ (IH)	0.573	.530 – 0.598	0.432 – 0.621	0.403 – 0.640
$\sin^2\theta_{13}$ (NH)	0.0234	.0214 – 0.0254	0.0195 – 0.0274	0.0177 – 0.0294
$\sin^2\theta_{23}$ (IH)	0.0240	.0221 – 0.0259	0.0202 – 0.0278	0.0183 – 0.0297
δ/π (NH)	1.34	0.96 – 1.98	0 – 2	0 – 2
δ/π (NH)	1.48	1.16 – 1.82	0 – 0.14 & 0.81 – 2.0	0 – 2

Table 1.1: The observational parameters of neutrino oscillation experiments for Normal (NH) and Inverted (IH) ordering of neutrino masses [1].

values of the observational parameters are depicted below [1].

$$\Delta m^2_{21} = 7.6 \times 10^{-5}\, eV^2, \tag{1.80}$$

$$|\Delta m^2_{31}| = 2.48 \times 10^{-3}\, eV^2 \quad \text{(NH)}, \tag{1.81}$$

$$= 2.38 \times 10^{-3}\, eV^2 \quad \text{(IH)}, \tag{1.82}$$

$$\theta_{12} = 34.63^0, \tag{1.83}$$

$$\theta_{13} = 8.79^0 \qquad (NH), \tag{1.84}$$

$$= 8.91^0 \qquad (IH), \tag{1.85}$$

$$\theta_{23} = 48.85^0\,(43.10^0) \qquad (NH), \tag{1.86}$$

$$= 49.20^0 \qquad (IH), \tag{1.87}$$

$$\delta = 1.34\,\pi \qquad (NH), \tag{1.88}$$

$$= 1.48\,\pi \qquad (IH), \tag{1.89}$$

and a summary encompassing the 1σ, 2σ and 3σ range of the observational parameters are presented in Table. (1.1). The limitations of the oscillation experiments lies in incapability of predicting the absolute scale of neutrino masses and the Majorana CP violating phases.

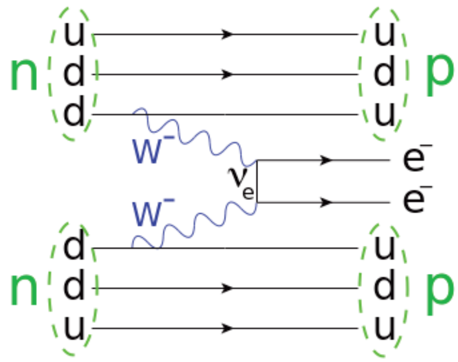

FIGURE 1.3: The neutrino-less double β decay process

1.5 Towards the absolute neutrino mass scale

Several strides to realize the absolute scale of neutrino masses are discussed very briefly in this section. They are as in the following [2, 44],

- **The observation of the end-point of ^3H β decay spectrum**

  We define a quantity called "effective electron neutrino mass",

$$m_\beta = \sqrt{m_1^2|U_{e1}|^2 + m_2^2|U_{e2}|^2 + m_3^2|U_{e3}|^2}. \tag{1.90}$$

  According to the Troitzk experiment [45], $m_\beta < 2.05\,eV$. Similar results are obtained in the Mainz experiment [46] which says, $m_\beta < 2.05\,eV$. The future prospect of KATRIN experiment [47] is to achieve a sensitivity of $m_\beta \sim 0.20\,eV$.

- **The neutrino-less double β decay ($0\nu2\beta$):** A successful observation of $0\nu2\beta$ decay process will give credence to the Majorana nature of neutrinos. The Feynman diagram of the same is shown in Fig. (1.3)

  This process involves the prediction of following quantity termed "effective mass of $0\nu2\beta$ decay ",

$$\begin{aligned} m_\beta &= |m_1 {U_{e1}}^2 + m_2 {U_{e2}}^2 + m_3 {U_{e3}}^2|, \\ &= |m_1\, c_{12}^2 c_{13}^2 e^{2i\alpha} + m_2\, c_{13}^2 s_{12}^2 e^{2i\beta} + m_3\, s_{13}^2 e^{-2i\delta}|\ \textbf{(Standard)}, \quad (1.91) \\ &= |m_1\, c_{12}^2 c_{13}^2 + m_2\, c_{13}^2 s_{12}^2 e^{2i\omega_{12}} + m_3\, s_{13}^2 e^{2i\omega_{13}}|\ \textbf{(Symmetric)} \quad (1.92) \end{aligned}$$

  which directly entails the two Majorana CP violating phases. There was a claim of a signal $m_{\beta\beta} = 0.11 - 0.56\,eV$ at 95% confidence level [48] but was criticized [49, 50]. A more refined result is awaited from the experiments such as GREDA [51].

- **Large Scale structures in the early universe :** The contribution of the relic neutrinos towards the density of universe is connected with a quantity $\sum m_i = |m_1| + |m_2| + |m_3|$. In 2013, Planck Collaboration put forward their results which on considering the contribution from Baryon Acoustic Oscillation (BAO)comes out to be [52],

$$\sum m_i < 0.23\, eV. \tag{1.93}$$

1.6 Majorana CP violating phases, α and β

In Eq. (1.64), we have already discerned the possibilities to experience the signature of CP violation in scenarios, involving the $\nu_\alpha \to \nu_\beta$ or $\overline{\nu_\alpha} \to \overline{\nu_\beta}$ oscillations. In a much more elegant way, we define a term called Jarlskog invariant parameter J_{CP} [53, 54],

$$\begin{aligned} J_{CP} &= Im[U_{\alpha i}U_{\beta j}U^*_{\alpha j}U^*_{\beta i}] \\ &= c_{12}s_{12}c^2_{13}s_{13}c_{23}s_{23}\sin\delta. \end{aligned} \tag{1.94}$$

The CP violating antisymmetry between $\mathcal{P}(\nu_\alpha \to \nu_\beta)$ and $\mathcal{P}(\overline{\nu_\alpha} \to \overline{\nu_\beta})$ can highlight only the Dirac CP violating Phase, δ. As it was pointed out that although the $0\nu 2\beta$ decay processes can substantiate the Majorana CP phases, yet cannot envisage the phases. The future long baseline experiments are trying to probe δ with higher precision. A more challenging task will be to predict α and β. A systematic analysis encompassing the $\nu_\alpha \to \overline{\nu_\beta}$ and $\overline{\nu_\alpha} \to \nu_\beta$ oscillation processes can shed light in this regard. Concerning the CP antisymmetry that may arise between the oscillation probabilities, $\mathcal{P}(\nu_\alpha \to \overline{\nu_\beta})$ and $\mathcal{P}(\overline{\nu_\alpha} \to \nu_\beta)$, certain "Jarsklog like parameters" [55], $\mathcal{V}^{ij}_{\alpha\beta}$ are defined as shown in the following,

$$\mathcal{V}^{ij}_{\alpha\beta} = Im[U_{\alpha i}U_{\beta i}U^*_{\alpha j}U^*_{\beta j}]. \tag{1.95}$$

These parameters satisfy the following relations,

$$\mathcal{V}^{ij}_{\alpha\beta} = \mathcal{V}^{ij}_{\beta\alpha} = -\mathcal{V}^{ji}_{\alpha\beta} = -\mathcal{V}^{ji}_{\beta\alpha} \tag{1.96}$$

We present a few of these parameters like ,

$$\mathcal{V}^{12}_{ee} = c^2_{12} s^2_{12} c^4_{13} \sin 2(\alpha - \beta), \quad \mathcal{V}^{13}_{ee} = c^2_{12} c^2_{13} s^2_{13} \sin 2(\delta + \alpha) \tag{1.97}$$

$$\mathcal{V}^{23}_{ee} = s^2_{12} c^2_{13} s^2_{13} \sin 2(\delta + \beta), ... \tag{1.98}$$

...etc. The details of similar parameters are illustrated in Ref [55]. In the limit, $\delta = 0$, we have $J = 0$. But $\mathcal{V}^{ij}_{\alpha\beta}$ s are in general non-vanishing. In this limit, there is no signature of CP violation in $\nu - \nu$ or $\overline{\nu} - \overline{\nu}$ oscillation, a strong CP violation may transpire in $\nu - \overline{\nu}$ oscillation processes. Also, in the limit $\theta_{13} = 0$, the parameters which involve U_{e3}, like, J, $\mathcal{V}^{13}_{ee}$, $\mathcal{V}^{23}_{ee}$, $\mathcal{V}^{13}_{e\mu}$, $\mathcal{V}^{13}_{e\tau}$..etc. are extinguished. Only if the Majorana nature of neutrinos is confirmed in $0\nu 2\beta$ decay experiments, these CP phases like α and β can be planned to be figured out in distant future experiments involving neutrino-antineutrino oscillations.

1.7 The Theory underlying?

We see that the neutrino oscillation experiments have confirmed the neutrinos as massive. This in turn triggers the possibilities to look beyond Standard model of particle physics. Not only the issue of masses but also the elucidation of other parameters like the mixing angles and the CP violating phases are the part and parcel of this puzzle. We abridge the several questions that the theorists have to nourish as in the following,

- Why neutrino masses are so small?
- What is exact ordering of neutrino masses ?
- Why the lepton mixing differs a lot from that of quarks?
- $\theta_{13} \sim \theta_c$, where θ_c is the Cabibbo angle, What does it imply?
- $\theta_{23} > \frac{\pi}{4}$ or, $\theta_{23} < \frac{\pi}{4}$?
- Neutrinos are Dirac or Majorana particles?

- Do sterile neutrinos exist?

But till now there is no thoroughgoing framework that can vindicate all the issues simultaneously. We are still in a journey and the goal is far away. The theorists in general adopt two strategies to attack the problem : “Bottom-up” and “Top-down”. The first standpoint is somewhat an expedition from a smaller picture to a wider landscape, while the second viewpoint is exactly opposite to the former.

Bottom-up strategies start with the observation. For example, under this approach one may try to construct the neutrino mass matrix in the basis charged lepton mass matrix is diagonal. The information of masses and mixing are collected from the oscillation data. Taking into account the Renormalization group equations, one tries to establish the neutrino mass matrix at the new scale physics. Also, one can try to relate the neutrino mass matrix with possible flavor symmetry groups and if needed can try to include the mechanism of symmetry violation also [56, 57].

On the contrary, for Top-down approaches, the underlying motivation is independent of neutrino phenomenology. This includes -Grand unified theory, TeV scale theory, Extra-dimension or string theories etc, which are used to predict the parameters and the textures of neutrino mass matrix etc. But both of the standpoints “Top-down” and “Bottom-up” must coincide at particular occasion. It seems impossible to unveil the hidden physics entirely based on observation. At some point of time one has to embrace some apriori reasoning motivated by GUT or super-symmetry etc. Similarly, working solely from top-down perspective, may hide some important elements, where Bottom-up strategies may assist. Hence both of the strategies are important. We put forward briefly some important pathways in relation with Top-down and Bottom-up approaches.

1.7.1 Top-Down approach

- **Anarchy approach**
  This approach assumes that the large leptonic mixing results from the lack of structure [58]. The allowed mass matrix is filled with random numbers. In fact, the earlier result, $\theta_{13} = 0$, was against this viewpoint. But present observation that θ_{13} is not at all vanishing, but indeed large ($\theta_{13} \sim 9^0$) clearly enhances the present stand-point. Since the anarchy approach predicts only

the probabilities and not specific numerals, it will be too hard to substantiate this stand through experiments.

- **Family symmetries**
  With family symmetry G_{family}, the Standard model symmetry is extended to,

$$G = SU(3)_C \times SU(2)_L \times U(1)_Y \times G_{family}, \tag{1.99}$$

  provided the Lagrangian remain invariant under the following transformation,

$$l_L \to X_L l_L, \quad l_R \to X_R l_R, \quad \nu_R \to X_R \nu_R, \tag{1.100}$$

  where, l_L, l_R and ν_R are the left-handed lepton doublet, right-handed charged lepton singlets and the right handed Majorana neutrinos respectively, and X_L, X_R and X_ν are certain representations belonging to the group G_{family}. This approach tries to explain the mixing properties and the mass hierarchies related to quarks and leptons. In this regard, some discrete flavor symmetry groups like A_4, S_4 etc. [59–71] are very popular.

- **Grand Unified Theories**
  The SU(5) GUT group is the simplest of all [72]. The original model does not hold the right handed neutrinos and the original model is extended to encompass the same. However, The SO(10) [73]group automatically accommodate the right-handed neutrinos and the other fifteen fermions of the standard model. The right-handed neutrinos get mass via breaking of B-L subgroup of SO(10) symmetry. The left-right symmetric SO(10) is a very attractive tool and predicts small neutrino masses via see-saw mechanism.

- **Extra Dimension and String Theories**
  The motivation of small mass and large mixing can be understood from the string theory point of view also. In one approach, the smallness of Dirac type neutrinos are explained in terms of the small overlap of neutrino wave function with four dimensional brane [74, 75]. Again from a different perspective, the "instanton effect" [76, 77] in string theory may account for the small neutrino masses and large lepton mixing.

1.7.2 Bottom-Up approach

- **Bi-Maximal mixing**

  This mixing pattern posits [78],

$$\theta_{12}^{PMNS} = 45^0, \quad \theta_{23}^{PMNS} = 45^0, \quad \theta_{13}^{PMNS} = 0^0. \tag{1.101}$$

  In the light of present experimental results, certainly the θ_{12} and θ_{23} are to be corrected.

- **Tri-Bimaximal mixing**

  According to this mixing pattern [79],

$$\theta_{12}^{PMNS} = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right), \quad \theta_{23}^{PMNS} = 45^0, \quad \theta_{13}^{PMNS} = 0^0. \tag{1.102}$$

  The two mixing angles, θ_{13}^{PMNS} and θ_{23}^{PMNS} are congruous with the 1σ range of experimental data. This mixing pattern can be considered as a plausible leading order structure.

- **Golden-Ratio**

  For, Golden ratio mixing [80–82], the θ_{12} is presented as,

$$\tan\theta_{12} = \frac{1}{\phi}, \quad \text{where,} \quad \phi = \frac{1+\sqrt{5}}{2}, \tag{1.103}$$

  gives, $\theta_{12} = 31.7^0$, There is another version of Golden ratio scheme which says, $\cos\theta_{12} = \phi/2$, i.e., $\theta_{12} = 36^0$ [83, 84].

- **Tri-Bimaximal Cabibbo mixing**

  This mixing pattern [85] which assumes,

$$\theta_{12}^{PMNS} = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right), \quad \theta_{23}^{PMNS} = 45^0, \quad \theta_{13}^{PMNS} = \frac{\theta_c}{\sqrt{2}}, \tag{1.104}$$

  can be considered as a good starting point for model-builders.

- **Bi-Trimaximal mixing**

  According to this mixing pattern [86],

$$\sin\theta_{12}^{PMNS} = \sin\theta_{23}^{PMNS} = \sqrt{\frac{8-2\sqrt{2}}{13}} \approx 0.591 \quad \sin\theta_{13}^{PMNS} = 0.211 \tag{1.105}$$

- **Bi-Large mixing**

  In this case [87], the reactor angle is taken to be the seed for solar and atmospheric angles.

$$\sin\theta_{13}^{PMNS} = \epsilon, \quad \sin\theta_{23}^{PMNS} = a\,\lambda, \quad \sin\theta_{12}^{PMNS} = s\,\lambda, \tag{1.106}$$

  where, $a \simeq s$. In particular, for, $\epsilon = \lambda$, where, $\lambda = \sin\theta_c \sim 0.22$, we have $a = s \simeq 3$.

- **QLC relations**

  The quark-lepton complementarity (QLC) establishes a tie-in between the quark and lepton mixing angles [88–90],

$$\theta_{12}^{PMNS} + \theta_c = 45^0, \tag{1.107}$$

  It is found that by multiplying Bi-maximal mixing matrix times the CKM mixing matrix, the above relation can be obtained. In the light of present measurement, we have a new QLC relation [91],

$$\theta_{13}^{PMNS} = \frac{\theta_c}{\sqrt{2}}, \tag{1.108}$$

  which propounds the signature of an underlying GUT.

1.8 Scope of the Thesis

From the discussion above, we understand that a vivid idea of neutrino mass ordering is still lacking and a specific choice of the same is model-dependent. In other words, all the three possible orderings, involving Normal, Inverted, and Degenerate spectrum are relevant from phenomenological point of view. If the ordering is strictly "Normal" or "Inverted", then on setting the lowest mass eigenvalue to zero, in either of the two cases, one can make the parametrization simpler. But, if the spectrum is degenerate, the parametrization is comparatively challenging. Besides, the two Majorana phase parameters are still in the dark. For simplicity, we shall concentrate only to the Majorana CP conserving scenarios. Besides, the octant where θ_{23} must lie is uncertain. In the thesis, we shall try to address the problems related with large reactor angle, the uncertainty associated with the octant of atmospheric angle and a successful prediction of Dirac-type CP violating phase. Also the issues such as the relevance of μ-τ symmetry, Tri-bimaximal mixing, and the requirement (and the possibilities) to go beyond the same will be discussed.

The "Neutrino mass matrix" conceives the information of both: masses, mixing angles and the (three) CP phases. On the, other hand, the "Lepton mixing matrix" carries the information of mixing angles and (one or three) CP phases. In the first half of the thesis, a proper parametrization of the neutrino mass matrix is emphasized on and in the latter half, different ways to parametrize the PMNS matrix is discussed. The study we put forward is kept model-independent and a bottom-up perspective is followed throughout. The thesis is organized as in the following.

- In the second chapter, the importance of the Quasi-degenerate neutrino mass models along with a proper parametrization of the related neutrino mass matrices in the μ-τ symmetric environment is discussed. The six different cases associated with Quasi-degenerate (QDN) neutrino models are studied. The related mass matrices, m_{LL}^{ν}s are parametrized with two free parameters (α, η), standard Wolfenstein parameter (λ) and input mass scale is chosen, $m_0 \sim 0.08\, eV$. The μ-τ symmetry cannot engender the nonzero θ_{13}. In this regard, the charged lepton correction is taken into consideration. Four independent building block matrices, I_0, I_1, I_2 and I_3 are highlighted in order to parametrize the neutrino mass matrix with μ-τ-symmetric texture. This building blocks treat all ranges of solar angle equally. But certain condition involving the suppression of the number free parameters restricts the BM mixing but allows both TBM and TBM deviated scenarios with ease. In this work, the solar mixing angle is controlled from neutrino sector and the other two angles, the reactor and atmospheric ones are controlled from the charged lepton sector. In the framework of oscillation experiments, cosmological observation and future experiments involving the phenomena like β-decay and $0\nu\beta\beta$ decay, the six different QDN models are appraised and hardly, any reason to discard any one of the QDN mass models is unfounded. The QDNH-*TypeA* model shows strong preference for $\sin^2\theta_{12} = 0.32$. The present work leaves a scope to extend the search of most favorable QDN mass model from observed baryon asymmetry of the Universe.

- In the third chapter, we try to look into the texture of neutrino mass matrix beyond the μ-τ symmetry. In the quark sector, the mixing angles are related to the quark mass ratios. In this chapter, it is tried to see whether a similar tie-up between the mixing angles and the neutrino mass ratio is possible or not. The oscillations experiments at present have confirmed the large value of the reactor angle. So we cannot decline the possibility that $\sin\theta_{13}$ and $\sqrt{m_1/m_3}$ are correlated, $\epsilon \sim \sin\theta_{13} \sim \sqrt{m_1/m_3}$. At the same time, we have the other possibility: $\eta \sim \sin\theta_{23} \sim \sqrt{m_2/m_3}$. But the prospect, $\sin\theta_{12} \sim \sqrt{m_1/m_2}$, is not realized in the light of present data from oscillation experiments. Again, the remaining two possibilities are not realizable simultaneously. So, unlike quarks, the apprehension of the correlation between the mixing angles and mass ratios is partial in the neutrino sector. This ansatz cannot answer to the hierarchy problem of neutrinos, but constrains the parametrization of neutrino masses and mixing in several ways, and hints for a predictive framework. Two different parametrization schemes: ϵ-based and η-based, are developed. When the above ansatz and the related constraints are induced in the general neutrino mass matrix in flavor basis, five different hierarchy dependent textures are encountered. The number of independent parameters in the mass matrices are found to be $\leq$ 4. For simplicity, all the parameters in the neutrino mass matrix are kept real.

- In the fourth chapter, certain possibilities to amend the Tri-bimaximal mixing are invoked. The TBM mixing is partially successful in predicting the mixing angles, θ_{12} and θ_{23} which are found to be consistent within 1σ error limit. But the observed rector angle deviates a lot from the TBM prediction. It is seen, $\theta_{13} \approx \theta_{13}^{TBM} + \mathcal{O}(\theta_c)$, $\theta_{12} \approx \theta_{12}^{TBM} - \mathcal{O}(\theta_{23}^{CKM})$ and $\theta_{23} \approx \theta_{23}^{TBM} \pm \mathcal{O}(\theta_{23}^{CKM})$. Next to TBM mixing scheme, it is the Tribimaximal-Cabibbo mixing, which predicts a non-zero reactor angle, $\theta_{13} \sim \mathcal{O}(\theta_c)$. In order to bring about the correction to TBM mixing, we consider the contribution both from the neutrino and charged lepton sector. The contribution from neutrino sector is motivated by an unique feature of μ-τ symmetry by which one can control θ_{12} with ease. In another approach, instead of considering the correction from neutrino sector, we use a CKM-like charged lepton diagonalizing matrix. In the former approach, the information of the Dirac-type CP violating phase, δ_{CP} is suppressed. In the latter approach, a correlation can be seen between $\sin^2\theta_{12}$ and δ_{CP}. This leads to a precise prediction of θ_{12} and δ_{CP} is found close to $1.5\,\pi$. The TBM deviated scenarios concerning both $\theta_{23} \geq 45^0$ and $\theta_{23} \leq 45^0$, are discussed.

- In the fifth chapter, the importance of Bi-Large neutrino mixing in the light of recent oscillation data is discussed. Bi-large neutrino mixing is motivated in the F-Theory GUT and has geometrical origin. The Bi-large neutrino mixing says, $\sin\theta_{13} \simeq \theta_c$ and $\sin\theta_{12} = \sin\theta_{23} = \psi\lambda$, where, ψ is an unknown parameter. But in the present scenario, the solar and atmospheric angles are not at all equal. We try to modify the same by considering the charged lepton correction. The GUT models relate the charged lepton and the down-type quark mass matrices. According to $SO(10)$ GUT, we can choose, $M_e \sim M_d$, whereas, the correlation, $M_e \sim M_d^T$ is motivated in SU(5) GUT. Depending upon this we can choose, either, $U_l \approx V_{CKM}$, or $U_l \approx V_{CKM}^\dagger$ respectively. The neutrino mixing matrix, U_{BL} is parametrized in terms of ψ, λ and phase δ. For, U_l, both the "exact-CKM" and "CKM-like" textures are taken into consideration. The latter differs from the former concerning the inclusion of certain extra phases. If, U_l holds exact CKM texture, there are only two unknown parameters: ψ and δ. In terms of observed θ_{12} and θ_{13}, the quantities ψ and δ are parametrized. The other observables like, θ_{23} and δ_{CP} are treated as the prediction of certain frame-works. If U_l follows exact CKM texture, the contribution towards the observed δ_{CP} appears solely from the neutrino sector. On the other hand, both charged lepton and neutrino sectors contribute towards δ_{CP}, if U_l is CKM-like. Even if the neutrino sector does not accord to δ_{CP}, the charged lepton sector can lead the latter. The analysis embraces both kinds of possibilities, $\theta_{23} \leq 45^0$ and $\theta_{23} \geq 45^0$. Concerning the generalized CKM-like correction, the analysis addresses both standard and symmetric parametrization.

- In the sixth chapter, the motivation, results and shortcomings related with the works presented in Chapters 2-5, are summarized. The different prospects to extend the investigation in several areas are also discussed.

Chapter 2

Quasi-degenerate Neutrino mass models and their significance: A model independent investigation

The prediction of possible ordering of neutrino masses relies mostly on the model selected. Alienating the $\mu-\tau$ interchange symmetry from discrete flavour symmetry based models, turns the neutrino mass matrix less predictive. But this inspires one to seek the answer from other phenomenological frameworks. We need a proper parametrization of the neutrino mass matrices concerning individual hierarchies. In the present work, we attempt to study the six different cases of Quasi-degenerate (QDN) neutrino models. The related mass matrices, m_{LL}^{ν} are parametrized with two free parameters (α, η) and standard Wolfenstein parameter (λ). The input mass scale m_0 is selected around $\sim 0.08\,eV$. We begin with a $\mu-\tau$ symmetric neutrino mass matrix tailed by a correction from charged lepton sector. The parametrization accentuates the existence of four independent texture zero building block matrices which are common to all the QDN models under $\mu-\tau$ symmetric framework. These remain invariant irrespective of any choice of solar angle. In our parametrization, the neutrino sector controls the solar angle, whereas the reactor and atmospheric angles are dictated by the charged lepton sector. In the framework of oscillation experiments, cosmological observation and future experiments involving β-decay and $0\nu\beta\beta$ experiments, all QDN models are tested and a reason to rule out anyone out of the six models is unfounded. A strong preference for $\sin^2\theta_{12} = 0.32$ is observed for QDNH-TypeA model.

2.1 Introduction

One of the most challenging riddles of neutrino physics is to trace out the exact ordering of the absolute neutrino masses. The Quasi degenerate hierarchy [92–105] among all the three possibilities, refers to the scenario when the three mass eigenvalues are of similar order, $m_1 \sim m_2 \sim m_3$. As the solar mass squared difference (Δm^2_{21}) is positive and the the sign of atmospheric mass squared difference (Δm^2_{31}) is unspecified, we encounter two divisions of QDN patterns: they are,

- " Quasi-degenerate Normal Hierarchy (QDNH) type" : $m_1 \lesssim m_2 \lesssim m_3$,
- "Quasi-degenerate Inverted Hierarchy type" (QDIH): $m_3 \lesssim m_1 \lesssim m_2$.

Besides, the remaining possibilities are strict "Normal hierarchy" (NH): $m_1 << m_2 << m_3$, $m_1 \sim 0$ and "Inverted hierarchy" (IH): $m_3 << m_1 << m_2$, $m_3 \sim 0$. The two Majorana phases (α, β) are admitted to the diagonalized neutrino mass matrix m^d_{LL}, where, $m^d_{LL} = diag(m_1, m_2\, e^{i\alpha}, m_3\, e^{i\beta})$ [106]. On adopting the CP conserving cases, three subclasses corresponding to each model is generated. The CP parity patterns of the sub classes are :

- Type IA: $m^d_{LL} = diag\,(+m_1, -m_2, +m_3)$,
- Type IB: $m^d_{LL} = diag\,(+m_1, +m_2, +m_3)$ and,
- Type IC: $m^d_{LL} = diag\,(+m_1, +m_2, -m_3)$.

The QDN model were very often forsaken [107, 108] in view of the neutrino-less double β decay experiments and cosmological data. The range of absolute neutrino mass scale, m_0 was chosen as, $0.1\,eV - 0.4\,eV$ [109] in earlier QDN models. But, the Cosmological data in concern with the sum of the three absolute neutrino masses, $\Sigma|m_i| \leq 0.28\,eV$ [110], strongly abandons any possibility of quasi-degenerate neutrinos to exist with absolute mass scale more than $0.1\,eV$. The Σm_i corresponding to strict NH and IH scenarios are approximately $0.06\,eV$ and $0.1\,eV$ respectively. Hence the validity of both the models are beyond dispute. In the context of cosmological observation on $\Sigma\, m_i$ and the future experiments, we shall try to look into the possibilities related to the reanimation of the QDN models, with comparatively lower mass scale, $m_0 \lesssim 0.1\,eV$.

Regarding the three unknown absolute masses, only two relations involving $m^2_{i=1,2,3}$ are known so far. In NH and IH models, this problem can be easily overcome as the lowest mass (either m_1 or m_2) is set to zero. In case of QDN model, we consider the largest mass m_0 as a input. Besides, there are also three mixing angles: reactor (θ_{13}), solar (θ_{12}) and atmospheric (θ_{23}). A general neutrino mass matrix m^{ν}_{LL} carries the information of all these six quantities. For a phenomenological analysis of the QDN model, a suitable parametrization of m^{ν}_{LL} is an essential part. We shall try to design the general neutrino mass matrix, m^{ν}_{LL} with minimum numbers of free parameters. As a first approximation, m^{ν}_{LL} is assumed to follow $\mu-\tau$ symmetry [60, 79, 111–114]. This symmetry keeps θ_{12} arbitrary and hence can handle both Tri-Bimaximal(TBM) mixing and deviation from it as well [103–105, 115, 116]. This characteristic feature of $\mu-\tau$ symmetry bears immense phenomenological importance. The expected deviations to $\theta_{13}=0$ and $\theta_{23}=\pi/4$, are controlled from charged lepton sector[16, 117–126].

We hope, this investigation on QDN mass models will serve as a platform for our future study of Baryogenesis and leptogenesis [103–105, 127]. This investigation will require the knowledge of the texture of left handed neutrino mass matrices, m^{ν}_{LL}.

2.2 Need for parametrization of a general $\mu-\tau$ symmetric mass matrix.

The present neutrino oscillation data reports the lepton mixing angles, $\theta_{23}\sim 40^0$ and $\theta_{13}\sim 9^0$ [128–134] which are undoubtedly deviated from what TBM mixing and BM mixing says: $\theta_{23}=45^0$ and $\theta_{13}=0^0$. A neutrino mass matrix which satisfies these properties (of BM/TBM mixing)[78, 79, 111, 135–139], in the basis where charged lepton mass matrix, m^{l}_{LL} is diagonal, $m^{l}_{LL}=diag(m_e,m_\mu,m_\tau)$, exhibits a $\mu-\tau$ interchange symmetry. With a permutation matrix, T which conducts a flavor interchange $\mu\leftrightarrow\tau$, we express the texture of a $\mu-\tau$ symmetric mass matrix as in the following [140, 141],

$$T\, M_{\mu\tau}\, T = M_{\mu\tau}, \Longrightarrow M_{\mu\tau} = \begin{bmatrix} x & y & y \\ y & z & w \\ y & w & z \end{bmatrix}, \tag{2.1}$$

where,

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{2.2}$$

We experience another form $M_{\mu\tau}$ [142], different from that in Eq.(2.1),

$$M_{\mu\tau} = \begin{bmatrix} x & y & -y \\ y & z & w \\ -y & w & z \end{bmatrix}. \tag{2.3}$$

The matrix element is invariant under the flavor interchange of $\mu \leftrightarrow -\tau$. The permutation matrix responsible for this symmetry is T'.

$$T'\, M_{\mu\tau}\, T' = M_{\mu\tau}, \Longrightarrow M_{\mu\tau}, \tag{2.4}$$

where,

$$T' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}. \tag{2.5}$$

The presence of a $-ve$ sign before y in the 1-3 the element of $M_{\mu\tau}$ (see Eq.(2.3)) ensures the positivity of the mixing angles.

Except the maximal atmospheric and vanishing reactor angle, $\mu-\tau$ symmetry has no further prediction. Different discrete symmetry groups are very often combined with $\mu-\tau$ interchange symmetry in order to obtain a predictive neutrino mass matrix [143]. For example, in the original Altarelli-Feruglio model [65, 144], the neutrino mass matrix takes the form (see Eq.(2.1)),

$$M_{\mu\tau} = \begin{bmatrix} a + \frac{2}{3}b & -\frac{b}{3} & -\frac{b}{3} \\ -\frac{b}{3} & \frac{2}{3}b & a - \frac{b}{3} \\ -\frac{b}{3} & a - \frac{b}{3} & \frac{2}{3}b \end{bmatrix}, \tag{2.6}$$

the mass eigenvalues are $m_1 = a + b$, $m_2 = a$ and $m_3 = b - a$ which gives, $\Delta m^2_{sol} = (-b^2 - 2ab)$ and $\Delta m^2_{atm} = -4ab$. Since it is known that $\Delta m^2_{sol} > 0$, which implies $ab < 0$. Hence, a and b must have opposite signs which in turn says, $\Delta m^2_{atm} > 0$. This model advocates for normal hierarchy of the absolute neutrino masses. In addition, it supports for TBM mixing: $\theta_{12} = \sin^{-1}(1/\sqrt{3})$. Similarly, a neutrino mass matrix of following kind (see Eq.(2.1)),

$$M_{\mu\tau} = \begin{bmatrix} 0 & a & a \\ a & b & c \\ a & c & b \end{bmatrix}, \tag{2.7}$$

can be related with an interesting mixing scheme called Golden ratio [145] and dictates the mass pattern to be of normal hierarchy type.

One of the most interesting property of the $\mu - \tau$ (interchange) symmetry which is very often neglected is the arbitrariness of the solar angle θ_{12}. With a proper choice of the parameters, x, y, w and z, θ_{12} is controlled with the following relation [60], intrinsic to $M_{\mu\tau}$ in Eq.(2.1),

$$\tan 2\theta_{12} = \frac{2\sqrt{2}y}{x - w - z}. \tag{2.8}$$

For, $M_{\mu\tau}$ in Eq.(2.3) the expression for $\tan 2\theta_{12}$ is,

$$\tan 2\theta_{12} = \frac{2\sqrt{2}y}{x + w - z} \tag{2.9}$$

It seems that $\mu{-}\tau$ symmetry is more natural and BM and TBM mixing schemes are certain special cases of this symmetry. In fact ,the recent result $\sin^2\theta_{12} \sim 0.32$ [132] deviated a little from TBM prediction ($\sin^2\theta_{12} = 0.33$), can be accommodated within the $\mu - \tau$ symmetry regime [103–105, 115, 116]. Neglecting the small deviations (though significant) for θ_{23} and θ_{13}, we first approximate, $m^\nu_{LL} = M_{\mu\tau}$ and the charged lepton diagonalizing matrix, $U_{eL} = I$.

This is an undeniable fact that the predictions of a suitable order of absolute neutrino masses are not unique and differs with the choice of models. Keeping aside all the models, which associate $M_{\mu\tau}$ with different discrete flavour symmetries, here we concentrate on a general parametrization of $M_{\mu\tau}$. The idea behind this decoupling is not to overlook the necessity of different symmetry groups, but to look into the subtle aspects of $\mu-\tau$ symmetry, starting from a phenomenological point of view. Here we emphasize on the facts that $\mu-\tau$ interchange symmetry is not partial to any hierarchy of absolute neutrino masses and has a good control over the solar angle. In the Refs. [146, 147], we put forward another possible way to parametrize the neutrino mass matrix based on μ-τ symmetry.

In the present article, concerning the parametrization of the $\mu-\tau$ symmetric mass matrices for different hierarchical cases, we shall stick to the second convention (see Eq.(2.3)).

2.3 Invariant building blocks of $\mu-\tau$ symmetric mass matrix

We want to draw attention on the general texture of $M_{\mu\tau}$s satisfying BM [78, 135, 136] and TBM [79, 111, 137–139] mixing schemes.

$$M_{\mu\tau}^{BM} = \begin{bmatrix} x & y & y \\ y & z & x-z \\ y & x-z & z \end{bmatrix} , \tag{2.10}$$

$$M_{\mu\tau}^{TBM} = \begin{bmatrix} x & y & y \\ y & z & x+y-z \\ y & x+y-z & z \end{bmatrix} . \tag{2.11}$$

It is to be noted that the above two forms of $M_{\mu\tau}$'s are in accordance with the first convention (see Eq.(2.1)). In terms of the three parameters x, y and z, the respective mass matrices can be decomposed with certain building block matrices I_x, $I_y^{BM,TBM}$ and I_z in the following way.

$$\begin{aligned} M_{\mu\tau}^{BM} &= xI_x + yI_y^{BM} + zI_z, & (2.12) \\ M_{\mu\tau}^{TBM} &= xI_x + yI_y^{TBM} + zI_z. & (2.13) \end{aligned}$$

Where,

$$I_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \; I_z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \tag{2.14}$$

$$I_y^{BM} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; I_y^{TBM} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \tag{2.15}$$

There is a distinct change in the texture of I_y, as the mixing pattern transits from BM to TBM. I_y^{BM} and I_y^{TBM} have the diagonalizing matrices, $U_{BM} = R_{23}(\theta_{23} = -\pi/4).R_{13}(\theta_{13} = 0).R_{12}(\theta_{12} = -\pi/4)$ and $U_{TBM} = R_{23}(\theta_{23} = -\pi/4).R_{13}(\theta_{13} = 0).R_{12}(\theta_{12} = \sin^{-1}(1/\sqrt{3}))$ respectively and thus carry the signatures of respective models. For, $I_{x,z}$ the diagonalizing matrices are, $U_{x,y} = R_{23}(\theta_{23} = -\pi/4)$.

We insist on the possibility of finding out certain building blocks of $M_{\mu\tau}$ that will remain invariant at the face of any mixing schemes (BM or TBM) or simply independent of any θ_{12} in general. With this idea, four such independent texture-zero matrices, $I_{i=0,1,2,3}$ are posited (see Table.(2.1)). On considering the fact that a general $M_{\mu\tau}$ is capable of holding four free parameters at the most (if α and β are specified), we parametrize $M_{\mu\tau}$ for QDNH-*Type IA* case in the following way.

$$M_{\mu\tau} = I_0 - (\beta - \frac{\alpha}{2})I_1 + 2\alpha(\eta^2 - \frac{1}{4})I_2 + \alpha\eta(1-2\eta^2)^{1/2}I_3. \tag{2.16}$$

$$= \begin{pmatrix} \alpha - \beta - 2\alpha\eta^2 & -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \alpha\eta(1-2\eta^2)^{\frac{1}{2}} \\ -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{1}{2} - \frac{\beta}{2} + \alpha\eta^2 & \frac{1}{2} + \frac{\beta}{2} - \alpha\eta^2 \\ \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{1}{2} + \frac{\beta}{2} - \alpha\eta^2 & \frac{1}{2} - \frac{\beta}{2} + \alpha\eta^2 \end{pmatrix} \tag{2.17}$$

Here, α, β and η are three free parameters and the mass matrix is normalized with input parameter m_0. The parameters, α and β are related with absolute masses of three neutrinos. The quantity, m_0 signifies the largest neutrino mass. It can be seen that whatever may be the changes in mixing schemes, the basic building blocks are not affected. The free parameter η dictates the solar angle. $\eta = 1/2$, $1/\sqrt{6}$, correspond to BM and TBM mixing respectively. In contrast to

I_i		I_i^{diag}	U_i
I_0	$\frac{1}{2}\begin{bmatrix}0 & 0 & 0\\0 & 1 & 1\\0 & 1 & 1\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix}$	$\begin{bmatrix}1 & 0 & 0\\0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$
I_1	$\frac{1}{2}\begin{bmatrix}2 & 0 & 0\\0 & 1 & -1\\0 & -1 & 1\end{bmatrix}$	$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}1 & 0 & 0\\0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$
I_2	$\frac{1}{2}\begin{bmatrix}-2 & 0 & 0\\0 & 1 & -1\\0 & -1 & 1\end{bmatrix}$	$\begin{bmatrix}-1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}1 & 0 & 0\\0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$
I_3	$\begin{bmatrix}0 & -1 & 1\\-1 & 0 & 0\\1 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}-\sqrt{2} & 0 & 0\\0 & \sqrt{2} & 1\\0 & 1 & 0\end{bmatrix}$	$\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\-\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}}\\\frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}}\end{bmatrix}$

TABLE 2.1: The texture of the invariant building blocks $I_{i=0,1,2,3}$, the diagonalized blocks $I^{diag}_{i=0,1,2,3}$ and the corresponding diagonalizing matrices (U_i).

$M_{\mu\tau}$s in Eqs. (2.12)-(2.13), the corresponding mass matrices are,

$$M_{\mu\tau}^{BM} = I_0 - (\beta - \frac{\alpha}{2})I_1 + 0I_2 + \frac{1}{2\sqrt{2}}\alpha I_3, \tag{2.18}$$

$$M_{\mu\tau}^{TBM} = I_0 - (\beta - \frac{\alpha}{2})I_1 - \frac{1}{6}\alpha I_2 + \frac{1}{3}\alpha I_3. \tag{2.19}$$

Here we want to add that with $\eta = 2/5$, $\sin^2\theta_{12} = 0.32$ (best-fit)[132] can be obtained. It can be seen that, $I_0 + I_1 = I$, the identity matrix. Also, from Table.(2.1), this is interesting to note that the diagonalizing matrix of I_3 is none other than U_{BM}.

There are certain significant features of this parametrization. With same building block matrices, we can extend the parametrization of $M_{\mu\tau}$s for other five QDN and even for the NH and IH cases also. For example, similar to Eq. (2.16), a rearrangement of the free parameters (α, β, η), and I_is we parametrize $M_{\mu\tau}$ for QDIH-*Type IA* case in the following way.

$$M_{\mu\tau} = \beta I_0 - \left(1 - \frac{\alpha}{2}\right) I_1 + 2\alpha\left(\eta^2 - \frac{1}{4}\right) I_2 + \alpha\eta\left(1 - 2\eta^2\right)^{1/2} I_3. \tag{2.20}$$

Upon considering $\beta = \alpha$, in Eq.(2.16), we get $M_{\mu\tau}$ satisfying strict NH-$Type\, IA$ condition.

$$M_{\mu\tau} = I_0 - \frac{\alpha}{2} I_1 + 2\alpha \left(\eta^2 - \frac{1}{4}\right) I_2 + \alpha\eta \left(1 - 2\eta^2\right)^{1/2} I_3. \tag{2.21}$$

Similarly, with $\beta = 0$, in Eq.(2.20), we obtain a $M_{\mu\tau}$ that represents IH-$Type\, IA$ case.

$$M_{\mu\tau} = 0 I_0 - \left(1 - \frac{\alpha}{2}\right) I_1 + 2\alpha \left(\eta^2 - \frac{1}{4}\right) I_2 + \alpha\eta \left(1 - 2\eta^2\right)^{1/2} I_3. \tag{2.22}$$

Similarly, we can formulate the same for other cases also. The details are shown in Table.(2.2) and Table.(2.3)

In this present approach, the mass parameters and the mixing angle parameters are decoupled. A single expression of $\tan 2\theta_{12}$ for all the eleven cases is,

$$\tan 2\theta_{12} = \frac{2\sqrt{2}\eta(1 - 2\eta^2)^{1/2}}{1 - 4\eta^2}, \tag{2.23}$$

$$\text{or,} \quad \sin^2\theta_{12} = 2\eta^2. \tag{2.24}$$

.

2.4 The input parameter m_0 for QDN model

In either of the two QDN cases, m_0 represents the largest absolute neutrino mass. For QDNH cases, we use the following relations to work out the neutrino masses m_i.

$$m_1 = m_0 \sqrt{1 - \frac{\Delta m^2_{atm}}{m_0^2}}, \tag{2.25}$$

$$m_2 = m_0 \sqrt{1 + \frac{\Delta m^2_{sol}}{m_0^2} - \frac{\Delta m^2_{atm}}{m_0^2}}, \tag{2.26}$$

$$m_3 = m_0. \tag{2.27}$$

QDN-NH,IH	$M_{\mu\tau}(\alpha,\beta,\eta)/m_0$	m_i/m_0
QDNH-IA :	$\begin{bmatrix} \alpha-\beta-2\alpha\eta^2 & -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \alpha\eta(1-2\eta^2)^{\frac{1}{2}} \\ -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{1}{2}-\frac{\beta}{2}+\alpha\eta^2 & \frac{1}{2}+\frac{\beta}{2}-\alpha\eta^2 \\ \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{1}{2}+\frac{\beta}{2}-\alpha\eta^2 & \frac{1}{2}-\frac{\beta}{2}+\alpha\eta^2 \end{bmatrix}$ $= I_0-(\beta-\frac{\alpha}{2})I_1+2\alpha(\eta^2-\frac{1}{4})I_2+\alpha\eta(1-2\eta^2)^{1/2}I_3.$	$\alpha-\beta$ $-\beta$ 1
QDNH-IB :	$\begin{bmatrix} \beta+2\alpha\eta^2-\alpha & \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} \\ \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{1}{2}+\frac{\beta}{2}-\alpha\eta^2 & \frac{1}{2}-\frac{\beta}{2}+\alpha\eta^2 \\ -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{1}{2}-\frac{\beta}{2}+\alpha\eta^2 & \frac{1}{2}+\frac{\beta}{2}-\alpha\eta^2 \end{bmatrix}$ $= I_0+(\beta-\frac{\alpha}{2})I_1-2\alpha(\eta^2-\frac{1}{4})I_2-\alpha\eta(1-2\eta^2)^{1/2}I_3$	$\beta-\alpha$ β 1
QDNH-IC :	$\begin{bmatrix} \beta+2\alpha\eta^2-\alpha & \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} \\ \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{\beta}{2}-\alpha\eta^2-\frac{1}{2} & \alpha\eta^2-\frac{1}{2}-\frac{\beta}{2} \\ -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \alpha\eta^2-\frac{1}{2}-\frac{\beta}{2} & \frac{\beta}{2}-\alpha\eta^2-\frac{1}{2} \end{bmatrix}$ $= -I_0+(\beta-\frac{\alpha}{2})I_1-2\alpha(\eta^2-\frac{1}{4})I_2-\alpha\eta(1-2\eta^2)^{1/2}I_3$	$\beta-\alpha$ β -1
QDIH-IA :	$\begin{bmatrix} \alpha-2\alpha\eta^2-1 & -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \alpha\eta(1-2\eta^2)^{\frac{1}{2}} \\ -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{\beta}{2}+\alpha\eta^2-\frac{1}{2} & \frac{1}{2}+\frac{\beta}{2}-\alpha\eta^2 \\ \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{1}{2}+\frac{\beta}{2}-\alpha\eta^2 & \frac{\beta}{2}+\alpha\eta^2-\frac{1}{2} \end{bmatrix}$ $= \beta I_0-(1-\frac{\alpha}{2})I_1+2\alpha(\eta^2-\frac{1}{4})I_2+\alpha\eta(1-2\eta^2)^{1/2}I_3.$	$\alpha-1$ -1 β
QDIH-IB :	$\begin{bmatrix} 1-\alpha+2\alpha\eta^2 & \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} \\ \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{1}{2}+\frac{\beta}{2}-\alpha\eta^2 & \frac{\beta}{2}+\alpha\eta^2-\frac{1}{2} \\ -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{\beta}{2}+\alpha\eta^2-\frac{1}{2} & \frac{1}{2}+\frac{\beta}{2}-\alpha\eta^2 \end{bmatrix}$ $= \beta I_0+(1-\frac{\alpha}{2})I_1-2\alpha(\eta^2-\frac{1}{4})I_2-\alpha\eta(1-2\eta^2)^{1/2}I_3.$	$1-\alpha$ 1 β
QDIH-IC :	$\begin{bmatrix} 1-\alpha+2\alpha\eta^2 & \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} \\ \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{1}{2}-\frac{\beta}{2}-\alpha\eta^2 & \alpha\eta^2-\frac{\beta}{2}-\frac{1}{2} \\ -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \alpha\eta^2-\frac{\beta}{2}-\frac{1}{2} & \frac{1}{2}-\frac{\beta}{2}-\alpha\eta^2 \end{bmatrix}$ $= -\beta I_0+(1-\frac{\alpha}{2})I_1-2\alpha(\eta^2-\frac{1}{4})I_2-\alpha\eta(1-2\eta^2)^{1/2}I_3.$	$1-\alpha$ 1 $-\beta$

TABLE 2.2: The parametrization of $M_{\mu\tau}$ for six different QDN cases with three free parameters (α,β,η) with four basic building blocks $I_{i=0,1,2,3}$. m_0 is the input parameter.

NH,IH	$M_{\mu\tau}(\alpha,\eta)/m_0$	m_i/m_0
NH-IA :	$\begin{bmatrix} -2\alpha\eta^2 & -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \alpha\eta(1-2\eta^2)^{\frac{1}{2}} \\ -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{1}{2}-\frac{\alpha}{2}+\alpha\eta^2 & \frac{1}{2}+\frac{\alpha}{2}-\alpha\eta^2 \\ \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{1}{2}+\frac{\alpha}{2}-\alpha\eta^2 & \frac{1}{2}-\frac{\alpha}{2}+\alpha\eta^2 \end{bmatrix}$ $= I_0 - \frac{\alpha}{2}I_1 + 2\alpha(\eta^2-\frac{1}{4})I_2 + \alpha\eta(1-2\eta^2)^{1/2}I_3.$	0 $-\alpha$ 1
NH-IB :	$\begin{bmatrix} 2\alpha\eta^2 & \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} \\ \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{1}{2}+\frac{\alpha}{2}-\alpha\eta^2 & \frac{1}{2}-\frac{\alpha}{2}+\alpha\eta^2 \\ -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{1}{2}-\frac{\alpha}{2}+\alpha\eta^2 & \frac{1}{2}+\frac{\alpha}{2}-\alpha\eta^2 \end{bmatrix}$ $= I_0 + \frac{\alpha}{2}I_1 - 2\alpha(\eta^2-\frac{1}{4})I_2 - \alpha\eta(1-2\eta^2)^{1/2}I_3$	0 α 1
NH-IC :	$\begin{bmatrix} 2\alpha\eta^2 & \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} \\ \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{\alpha}{2}-\alpha\eta^2-\frac{1}{2} & \alpha\eta^2-\frac{\alpha}{2}-\frac{1}{2} \\ -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \alpha\eta^2-\frac{\alpha}{2}-\frac{1}{2} & \frac{\alpha}{2}-\alpha\eta^2-\frac{1}{2} \end{bmatrix}$ $= -I_0 - \frac{\alpha}{2}I_1 - 2\alpha(\eta^2-\frac{1}{4})I_2 - \alpha\eta(1-2\eta^2)^{1/2}I_3.$	0 α -1
IH-IA :	$\begin{bmatrix} \alpha-2\alpha\eta^2-1 & -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \alpha\eta(1-2\eta^2)^{\frac{1}{2}} \\ -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \alpha\eta^2-\frac{1}{2} & \frac{1}{2}-\alpha\eta^2 \\ -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{1}{2}-\alpha\eta^2 & \alpha\eta^2-\frac{1}{2} \end{bmatrix}$ $= 0I_0 - (1-\frac{\alpha}{2})I_1 + 2\alpha(\eta^2-\frac{1}{4})I_2 + \alpha\eta(1-2\eta^2)^{1/2}I_3.$	$\alpha-1$ -1 0
IH-IB :	$\begin{bmatrix} 1-\alpha+2\alpha\eta^2 & \alpha\eta(1-\eta^2)^{\frac{1}{2}} & -\alpha\eta(1-\eta^2)^{\frac{1}{2}} \\ \alpha\eta(1-\eta^2)^{\frac{1}{2}} & \frac{1}{2}-\alpha\eta^2 & \alpha\eta^2-\frac{1}{2} \\ -\alpha\eta(1-\eta^2)^{\frac{1}{2}} & \alpha\eta^2-\frac{1}{2} & \frac{1}{2}-\alpha\eta^2 \end{bmatrix}$ $= 0I_0 - (1-\frac{\alpha}{2})I_1 + 2\alpha(\eta^2-\frac{1}{4})I_2 + \alpha\eta(1-2\eta^2)^{1/2}I_3.$	$1-\alpha$ 1 0

TABLE 2.3: The extension of the parametrization to NH and IH models. It can be seen that only two free parameters α and η are required to parametrize the mass matrices.

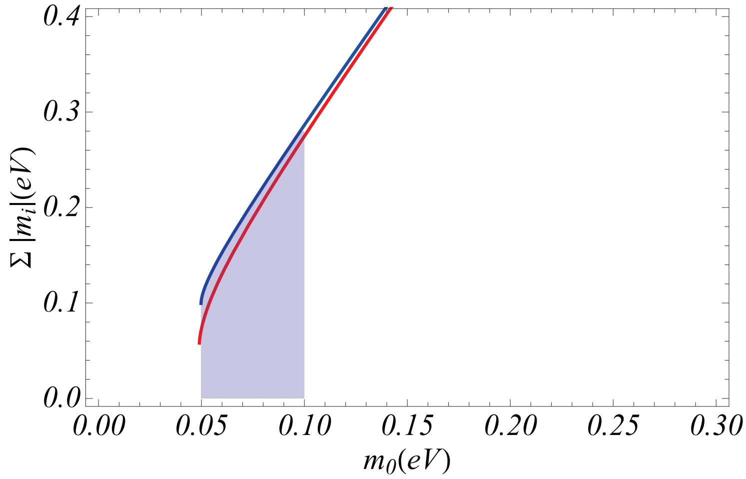

FIGURE 2.1: Σm_i vs the input parameter m_0. Corresponding to the cosmological upper bound $\Sigma m_i \lesssim 0.28\, eV$ and beyond $m_0 > 0.05\, eV$, Σm_i is imaginary, we get a range of m_0 as $[\,0.05,\, 0.1\,]\, eV$. The Red stands for QDNH case while Blue signifies the case of QDIH

For, QDIH cases, we use,

$$m_1 = m_0\sqrt{1 - \frac{\Delta m^2_{sol}}{m_0^2}}, \tag{2.28}$$

$$m_2 = m_0, \tag{2.29}$$

$$m_3 = m_0\sqrt{1 - \frac{\Delta m^2_{sol}}{m_0^2} - \frac{\Delta m^2_{atm}}{m_0^2}}. \tag{2.30}$$

Also, we have,

$$\Sigma m_i = |m_1| + |m_2| + |m_3|. \tag{2.31}$$

The present cosmological upper bound on Σm_i is $0.28\, eV$ [110] and the best-fit values of the mass squared differences are approximately: $\Delta m^2_{21} \sim 7.6 \times 10^{-5}\, eV^2$, $\Delta m^2_{31} \sim 2.4 \times 10^{-3}\, eV^2$ [132–134]. From a graphical analysis of $\Sigma\,|m_i|$ vs. m_0 reveals that the absolute mass scale m_0 must lie approximately within $0.05\, eV - 0.1\, eV$ (Fig.(2.1)). The upper limit of m_0 is the direct outcome of the cosmological upper bound [110]. The lower limit arises because, when $m_0 \lesssim 0.05\, eV$, m_1, m_2 for QDNH case and m_3 for QDIH case become imaginary. By studying the variation of m_i and corresponding slopes (dm_i/dm_0) with respect to m_0 (Fig.(2.2)), we expect that the level of degeneracy is better for $m_0 > 0.07\, eV$ and approximate the range of m_0 from $0.07 - 0.1\, eV$. For all numerical studies we adhere to $m_0 \sim 0.08\, eV$.

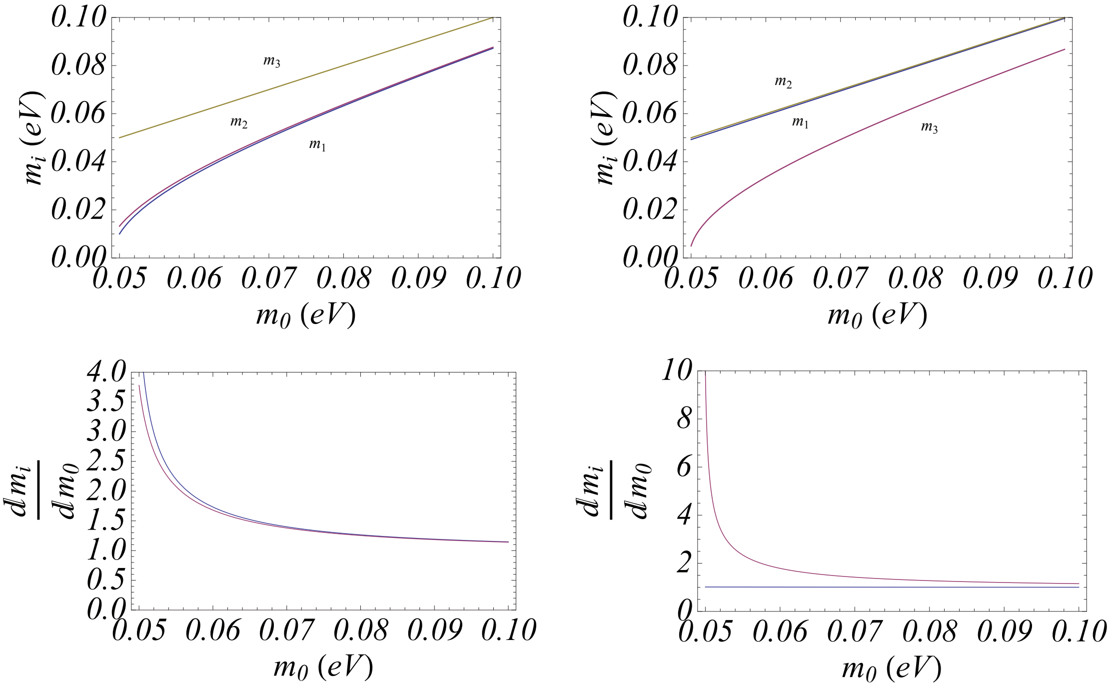

FIGURE 2.2: Study of m_i vs m_0 (top-left: QDNH case, top-right: QDIH case) and dm_i/dm_0 vs m_0 (bottom-left: QDNH case, bottom-right: QDIH case).

2.5 Endeavor to suppress the number of free parameters in QDN models

This is clear that only two free parameters α and η are required to parametrize $M_{\mu\tau}$ for NH and IH models (see Table.(2.3)); whereas QDN model requires three (α, β, η) (see Table.(2.2)). The rejection of one parameter for NH and IH cases is natural. But we shall try to see whether under certain logical ground we can suppress the number of free parameters for QDN model or not.

We consider the example of QDNH-Type IA case. With $m_0 \sim 0.08\,eV$, we study the ratio $\alpha : \beta$ and $\beta : \eta$ for the 2σ and 3σ ranges of the three parameters based on the Global data analysis [133]. The idea behind this approach is to detect whether there exists a simple linear correlation between the parameters or not.

Fig (2.3)reveals such a quest is not absurd at all and we can assume, $\alpha \approxeq 2\beta$ and $\beta \approxeq 2\eta$. But the first ansatz leads to $\Delta m^2_{21} = 0$ and turns out insignificant. We stick to the second ansatz. An immediate outcome is that the parameter η which is responsible only for the mixing angle θ_{12} in the earlier parametrization of $M_{\mu\tau}(\alpha, \beta, \eta)$ is now capable of driving the mass parameters also. In other words,

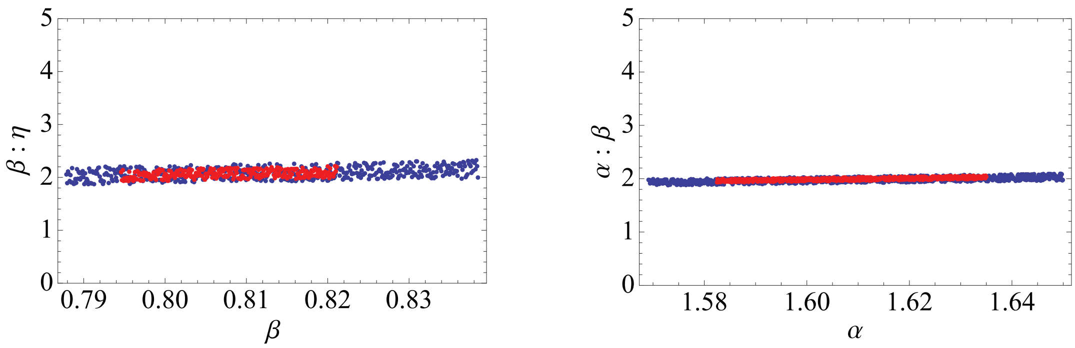

FIGURE 2.3: In order to check the validity of the assumptions: there may lie a linear correlation between the parameters (α, β, η) for QDNH-IA case, we check graphically $\alpha : \beta$ and $\beta : \eta$. The analysis hints for $\beta = 2\eta$ and $\alpha = 2\beta$. The colour red and blue stand for the 2σ and the 3σ range respectively.

the arbitrariness of θ_{12} is now reduced a little. In contrast to Eq.(2.16), for QDNH-*TypeA* case, with normalized m_0, we have,

$$M_{\mu\tau}(\alpha,\eta) = I_0 - (2\eta - \frac{\alpha}{2})I_1 + 2\alpha(\eta^2 - \frac{1}{4})I_2 + \alpha\eta(1-2\eta^2)^{1/2}I_3, \tag{2.32}$$

$$= \begin{bmatrix} \alpha - 2\eta - 2\alpha\eta^2 & -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \alpha\eta(1-2\eta^2)^{\frac{1}{2}} \\ -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{1}{2} - \eta + \alpha\eta^2 & \frac{1}{2} + \eta - \alpha\eta^2 \\ \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{1}{2} + \eta - \alpha\eta^2 & \frac{1}{2} - \eta + \alpha\eta^2 \end{bmatrix}. \tag{2.33}$$

The ansatz $\beta = 2\eta$ is applicable to other remaining QDN cases also (see Table.(2.4)). Needless to say, that the suppression of parameters does not affect $\tan 2\theta_{12}$ in Eq.(2.23).

2.6 TBM, deviation from TBM and BM mixing

We experience that $M_{\mu\tau}$ parametrized with (α, η) (see Table.(2.4)) gives certain correlation between absolute masses and θ_{12} ,

$$\sin\theta_{12} = \frac{1}{\sqrt{2}}\frac{m_2}{m_3} \text{ (QDNH case)}, \tag{2.34}$$

$$\sin\theta_{12} = \frac{1}{\sqrt{2}}\frac{m_3}{m_2} \text{ (QDIH case)}. \tag{2.35}$$

QDN-NH,IH	$M_{\mu\tau}(\alpha,\eta)/m_0$	m_i/m_0
QDNH-IA :	$\begin{bmatrix} \alpha-2\eta-2\alpha\eta^2 & -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \alpha\eta(1-2\eta^2)^{\frac{1}{2}} \\ -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{1}{2}-\eta+\alpha\eta^2 & \frac{1}{2}+\eta-\alpha\eta^2 \\ \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{1}{2}+\eta-\alpha\eta^2 & \frac{1}{2}-\eta+\alpha\eta^2 \end{bmatrix}$ $= I_0-(2\eta-\frac{\alpha}{2})I_1+2\alpha(\eta^2-\frac{1}{4})I_2+\alpha\eta(1-2\eta^2)^{1/2}I_3.$	$\alpha-2\eta$, -2η, 1
QDNH-IB :	$\begin{bmatrix} 2\eta+2\alpha\eta^2-\alpha & \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} \\ \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{1}{2}+\eta-\alpha\eta^2 & \frac{1}{2}-\eta+\alpha\eta^2 \\ -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{1}{2}-\eta+\alpha\eta^2 & \frac{1}{2}+\eta-\alpha\eta^2 \end{bmatrix}$ $= I_0+(2\eta-\frac{\alpha}{2})I_1-2\alpha(\eta^2-\frac{1}{4})I_2-\alpha\eta(1-2\eta^2)^{1/2}I_3$	$2\eta-\alpha$, 2η, 1
QDNH-IC :	$\begin{bmatrix} 2\eta+2\alpha\eta^2-\alpha & \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} \\ \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \eta-\alpha\eta^2-\frac{1}{2} & \alpha\eta^2-\frac{1}{2}-\eta \\ -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \alpha\eta^2-\frac{1}{2}-\eta & \eta-\alpha\eta^2-\frac{1}{2} \end{bmatrix}$ $= -I_0+(2\eta-\frac{\alpha}{2})I_1-2\alpha(\eta^2-\frac{1}{4})I_2-\alpha\eta(1-2\eta^2)^{1/2}I_3$	$2\eta-\alpha$, 2η, -1
QDIH-IA :	$\begin{bmatrix} \alpha-2\alpha\eta^2-1 & -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \alpha\eta(1-2\eta^2)^{\frac{1}{2}} \\ -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \eta+\alpha\eta^2-\frac{1}{2} & \frac{1}{2}+\eta-\alpha\eta^2 \\ \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{1}{2}+\eta-\alpha\eta^2 & \eta+\alpha\eta^2-\frac{1}{2} \end{bmatrix}$ $= 2\eta I_0-(1-\frac{\alpha}{2})I_1+2\alpha(\eta^2-\frac{1}{4})I_2+\alpha\eta(1-2\eta^2)^{1/2}I_3.$	$\alpha-1$, -1, 2η
QDIH-IB :	$\begin{bmatrix} 1-\alpha+2\alpha\eta^2 & \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} \\ \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{1}{2}+\eta-\alpha\eta^2 & \eta+\alpha\eta^2-\frac{1}{2} \\ -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \eta+\alpha\eta^2-\frac{1}{2} & \frac{1}{2}+\eta-\alpha\eta^2 \end{bmatrix}$ $= 2\eta I_0+(1-\frac{\alpha}{2})I_1-2\alpha(\eta^2-\frac{1}{4})I_2-\alpha\eta(1-2\eta^2)^{1/2}I_3.$	$1-\alpha$, 1, 2η
QDIH-IC :	$\begin{bmatrix} 1-\alpha+2\alpha\eta^2 & \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} \\ \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{1}{2}-\eta-\alpha\eta^2 & \alpha\eta^2-\eta-\frac{1}{2} \\ -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \alpha\eta^2-\eta-\frac{1}{2} & \frac{1}{2}-\eta-\alpha\eta^2 \end{bmatrix}$ $= -2\eta I_0+(1-\frac{\alpha}{2})I_1-2\alpha(\eta^2-\frac{1}{4})I_2-\alpha\eta(1-2\eta^2)^{1/2}I_3.$	$1-\alpha$, 1, -2η

TABLE 2.4: The parametrization of $M_{\mu\tau}$ for six different QDN cases with two free parameters (α,η) with four basic building blocks $I_{i=0,1,2,3}$. m_0 is the input parameter.

Considering QDNH case as an example, we find,

$$\Delta m^2_{21} = \alpha(2\sqrt{2}\sin\theta_{12} - \alpha)\, m_0^2, \tag{2.36}$$

$$\Delta m^2_{31} = (1 - \alpha + \sqrt{2}\sin\theta_{12})(1 + \alpha - \sqrt{2}\sin\theta_{12})\, m_0^2. \tag{2.37}$$

For all the QDNH cases, we fix the input, $m_0 = 0.082\, eV$. TBM condition implies $\theta_{12} = \sin^{-1}(1/\sqrt{3})$. A choice of the free parameter, $\alpha = 1.626$ (QDNH-IA), We obtain $\Delta m^2_{21} \sim 7.6 \times 10^{-5}\, eV^2$ and $\Delta m^2_{31} \sim 2.32 \times 10^{-3} eV^2$ [132–134].

If we expect a little deviation from TBM mixing, say $\sin^2\theta_{12} = 0.32$ then along with a choice of $\alpha = 1.5929$, we obtain $\Delta m^2_{21} \sim 7.6 \times 10^{-5}\, eV^2$ and $\Delta m^2_{31} \sim 2.49 \times 10^{-3} eV^2$[132]. Similar treatment holds good for the remaining cases also. The parametrization of the mass matrix with two free parameters (α, η) is compatible with both TBM mixing and with deviation from TBM as well, and agrees to the global data [132–134].

But the BM mixing ($\sin\theta_{12} = 1/\sqrt{2}$) is somehow disfavoured by all the six QDN mass models $M_{\mu\tau}$ (α, η) (see Table.(2.4)). The BM mixing will lead to, $m_2 = m_3$, which implies $\Delta m^2_{21} = \Delta m^2_{31}$. (See Eqs. (2.34)-(2.35) Needless to mention that this is problem never arises if we adopt the general parametrization with three free parameters (α, β, η) (see Table.(2.2)).

2.7 Charged lepton correction

We derive the diagonalizing matrix for both $M_{\mu\tau}(\alpha, \beta, \eta)$ (see Table. (2.2)) and $M_{\mu\tau}(\alpha, \eta)$ (see Table.(2.4)) in the exact form as shown below,

$$U_{\nu L} = \begin{pmatrix} (1-2\eta^2)^{1/2} & \sqrt{2}\eta & 0 \\ -\eta & \frac{1}{\sqrt{2}}(1-2\eta^2)^{1/2} & \frac{1}{\sqrt{2}} \\ \eta & -\frac{1}{\sqrt{2}}(1-2\eta^2)^{1/2} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{2.38}$$

Indeed, θ_{13} is zero and θ_{23} is $\pi/4$. We have to include some extra ingredient in order to deviate θ_{13} and θ_{23} from what $M_{\mu\tau}$ says.

The mixing matrix in the lepton sector, U_{PMNS}, appears in the electro-weak coupling to the W bosons and is expressed in terms of lepton mass eigenstates. We have,

$$\mathcal{L} = -\bar{e_L} M_e e_R - \frac{1}{2}\bar{\nu_L} m_{LL}^{\nu} \nu_L^c + H.c, \tag{2.39}$$

A transformation from flavour to mass basis: $U_{eL}^{\dagger} M_e U_{eR} = diag(m_e, m_\mu, m_\tau)$ and $U_{\nu L}^{\dagger} m_{LL}^{\nu} U_{\nu R} = diag(m_1, m_2, m_3)$ gives [16, 117–122],

$$U_{PMNS} = U_{eL}^{\dagger} U_{\nu L}. \tag{2.40}$$

As stated earlier, it was assumed $U_{eL} = I$ and hence $U_{PMNS} = U_{\nu L}(\eta)$. Probably a suitable texture of U_{eL} other than I, satisfying the unitary condition, may give rise to the desired deviation in the mixing angles. The mixing angle, θ_{12} is controlled efficiently with $\mu - \tau$ symmetry. We want to preserve this important property even though contribution from charged lepton sector is considered.

2.7.1 The charged lepton mixing matrix

In the absence of any CP phases, the charged lepton mixing matrix takes the form of a general 3×3 orthogonal matrix. In order to parametrize a 3×3 orthogonal matrix we require three rotational matrices of the following form.

$$R_{12}(\theta) = \begin{bmatrix} c_\theta & s_\theta & 0 \\ -s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.41}$$

$$R_{23}(\sigma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\sigma & s_\sigma \\ 0 & -s_\sigma & c_\sigma \end{bmatrix}, \tag{2.42}$$

$$R_{23}(\sigma) = \begin{bmatrix} c_\tau & 0 & s_\tau \\ 0 & 1 & 0 \\ -s_\tau & 0 & c_\tau \end{bmatrix}, \tag{2.43}$$

where, $s_\omega = \sin\omega$ and $c_\omega = \cos\omega$. We experience nine independent choices of combining these independent rotational matrices in order to generate the general orthogonal matrix [148]. Out of all these choices, we prefer $R = R_{12}(\theta) R_{31}(\tau) R_{23}(\sigma)$ in the charged lepton sector, which is different from the standard parametrization scheme. Again keeping in mind the fact that $R_{ij}^{-1}(\omega)$ plays an equivalent role as

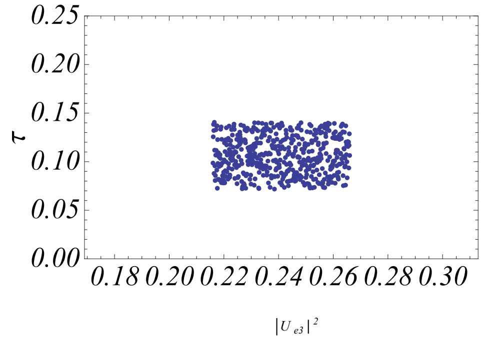

FIGURE 2.4: Graphical analysis to fix the parameter, τ against the 1σ range of $\sin^2\theta_{13} = |U_{e3}|^2$.

$R_{ij}(\omega)$ [148] in the construction of the general orthogonal matrix, we parametrize the charged lepton mixing matrix, U_{PMNS} (see Eq.(2.40)),

$$\tilde{U}_{eL} = \tilde{R}_{12}^{-1}(\theta)\tilde{R}_{31}^{-1}(\sigma)\tilde{R}_{23}(\tau), \tag{2.44}$$

and along with the small angle approximation: $s_\omega = \omega$ and $c_\omega = 1 - \omega^2/2$, we finally construct the PMNS matrix, of which the three important elements are,

$$U_{e2} \approx s_{12}^\nu + \tfrac{c_{12}^\nu}{\sqrt{2}}(\theta - \sigma) - \tfrac{s_{12}^\nu}{2}(\theta^2 + \sigma^2), \tag{2.45}$$

$$U_{e3} \approx \tfrac{1}{\sqrt{2}}(\theta + \sigma), \tag{2.46}$$

$$U_{\mu 3} \approx \tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}\tau - \tfrac{1}{2\sqrt{2}}(\theta^2 + \sigma^2), \tag{2.47}$$

where $s_{12}^\nu = \sqrt{2}\eta$ and $c_{12}^\nu = \sqrt{1 - 2\eta^2}$. The choice of the σ, θ and τ are arbitrary. So that $\sin\theta_{12}$ as obtained from $M_{\mu\tau}$ is not disturbed, the middle term in the expression of U_{e2} must vanish, $\theta - \sigma = 0$. We choose, θ, $\sigma = \lambda/2$, ($\lambda = 0.2253 \pm 0.0007$, standard Wolfenstein parameter[149]) and get $\sin\theta_{13} = |U_{e3}| = \lambda/\sqrt{2}$ [91]. Once, θ and σ are fixed, the choice of τ is guided by the requirement of necessary deviation of θ_{23} from the maximal condition. We see, in Fig (2.4). with respect to 1σ range of $|U_{e3}|^2$, τ centers around $\tau \sim 0.1 \sim \lambda/2$. Finally we model

$$\tilde{U}_{eL} = \tilde{R}_{12}^{-1}(\lambda/2)\tilde{R}_{31}^{-1}(\lambda/2)\tilde{R}_{23}(\lambda/2). \tag{2.48}$$

so that,

$$\tilde{U}^{\dagger}_{eL} \approx \begin{bmatrix} 1-\frac{\lambda^2}{4} & \frac{\lambda}{2} & \frac{\lambda}{2} \\ -\frac{\lambda}{2}+\frac{\lambda^2}{4} & 1-\frac{\lambda^2}{4} & -\frac{\lambda}{2} \\ -\frac{\lambda}{2}-\frac{\lambda^2}{4} & \frac{\lambda}{2}-\frac{\lambda^2}{4} & 1-\frac{\lambda^2}{4} \end{bmatrix} + \mathcal{O}(\lambda^3). \tag{2.49}$$

2.7.2 Breaking the $\mu - \tau$ interchange symmetry

Once, the charged lepton contributions are taken into consideration the $\mu - \tau$ symmetry will be perturbed. Finally, we obtain, the corrected neutrino mass matrix, $m^{\nu}_{LL}(\alpha, \eta, \lambda) = \tilde{U}^{\dagger}_{eL}.M_{\mu\tau}\tilde{U}_{eL}$. The invariant building blocks $I_{i=0,1,2,3}$ (see Table.(2.1)) of $M_{\mu\tau}$ will now change to,

$$\begin{aligned} I^{\lambda}_{i=0,..,3} &= \tilde{U}^{\dagger}_{eL}.I_{i=0,..,3}.\tilde{U}_{eL} \\ &= I_{i=0,..,3} + \Delta I^{\lambda}_{i=0,..,3} + \mathcal{O}(\lambda^3). \end{aligned} \tag{2.50}$$

The matrices ΔI^{λ}_i s are listed in Table.(2.5). We consider the case of $M_{\mu\tau}$ with two free parameters (α, η) for QDNH-$TypeA$ case (see Eq.(2.32)), as example.

$$\begin{aligned} m^{\nu}_{LL}(\alpha,\eta,\lambda) &= \tilde{U}^{\dagger}_{eL}.M_{\mu\tau}(\alpha,\eta)\tilde{U}_{eL} \\ &= I^{\lambda}_0 - (2\eta - \frac{\alpha}{2})I^{\lambda}_1 + 2\alpha(\eta^2 - \frac{1}{4})I^{\lambda}_2 + \alpha\eta(1-2\eta^2)^{1/2}I^{\lambda}_3, \\ &= \begin{bmatrix} \alpha - 2\eta - 2\alpha\eta^2 & -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \alpha\eta(1-2\eta^2)^{\frac{1}{2}} \\ -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{1}{2}-\eta+\alpha\eta^2 & \frac{1}{2}+\eta-\alpha\eta^2 \\ \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{1}{2}+\eta-\alpha\eta^2 & \frac{1}{2}-\eta+\alpha\eta^2 \end{bmatrix} \\ &\quad + \frac{\lambda}{2}\left(1 - 2\eta - \frac{\alpha}{2}\right) \begin{bmatrix} \lambda & 1-\frac{1}{2}\lambda & 1+\frac{1}{2}\lambda \\ 1-\frac{1}{2}\lambda & -1-\frac{1}{4}\lambda & -\lambda \\ 1+\frac{1}{2}\lambda & -\lambda & 1-\frac{3}{4}\lambda \end{bmatrix} \\ &\quad + \alpha\lambda\left(\eta^2 - \frac{1}{4}\right) \begin{bmatrix} \lambda & -\frac{\lambda}{2} & 1+\frac{1}{2}\lambda \\ -\frac{\lambda}{2} & 1-\frac{3}{4}\lambda & 0 \\ 1+\frac{1}{2}\lambda & 0 & -1-\frac{1}{4}\lambda \end{bmatrix} \\ &\quad + \frac{\alpha\eta\lambda}{2}(1-2\eta^2)^{1/2} \begin{bmatrix} 0 & -1+\lambda & -1-\frac{1}{2}\lambda \\ -1+\lambda & 2 & 2\lambda \\ -1-\frac{1}{2}\lambda & 2\lambda & -2 \end{bmatrix} + \mathcal{O}(\lambda^3) \end{aligned} \tag{2.51}$$

ΔI_i^λ

$$\Delta I_0^\lambda \approx \frac{1}{2}\lambda\begin{bmatrix} \lambda & 1-\frac{1}{2}\lambda & 1+\frac{1}{2}\lambda \\ 1-\frac{1}{2}\lambda & -1-\frac{1}{4}\lambda & -\lambda \\ 1+\frac{1}{2}\lambda & -\lambda & 1-\frac{3}{4}\lambda \end{bmatrix}$$

$$\Delta I_1^\lambda = -\Delta I_0^\lambda$$

$$\Delta I_2^\lambda \approx \frac{1}{2}\lambda\begin{bmatrix} \lambda & -\frac{1}{2}\lambda & 1+\frac{1}{2}\lambda \\ -\frac{1}{2}\lambda & 1-\frac{3}{4}\lambda & 0 \\ -\frac{1}{2}\lambda & 0 & -1-\frac{1}{4}\lambda \end{bmatrix}$$

$$\Delta I_3^\lambda \approx \frac{1}{2}\lambda\begin{bmatrix} 0 & -1+\lambda & -1-\frac{1}{2}\lambda \\ -1+\lambda & 2 & 2\lambda \\ -1-\frac{1}{2}\lambda & 2\lambda & -2 \end{bmatrix}$$

$m_{LL}^\nu(\alpha,\eta,\lambda)$

QDNH-IA : $(I_0+\Delta I_0^\lambda)-(2\eta-\frac{\alpha}{2})(I_1-\Delta I_0^\lambda)+2\alpha(\eta^2-\frac{1}{4})(I_2+\Delta I_2^\lambda)+\alpha\eta(1-2\eta^2)^{1/2}(I_3+\Delta I_3^\lambda)$

QDNH-IB : $(I_0+\Delta I_0^\lambda)+(2\eta-\frac{\alpha}{2})(I_1-\Delta I_0^\lambda)-2\alpha(\eta^2-\frac{1}{4})(I_2+\Delta I_2^\lambda)-\alpha\eta(1-2\eta^2)^{1/2}(I_3+\Delta I_3^\lambda)$

QDNH-IC : $-(I_0+\Delta I_0^\lambda)+(2\eta-\frac{\alpha}{2})(I_1-\Delta I_0^\lambda)-2\alpha(\eta^2-\frac{1}{4})(I_2\Delta+I_2^\lambda)-\alpha\eta(1-2\eta^2)^{1/2}(I_3+\Delta I_3^\lambda)$

QDIH-IA : $2\eta(I_0+\Delta I_0^\lambda)-(1-\frac{\alpha}{2})(I_1-\Delta I_0^\lambda)+2\alpha(\eta^2-\frac{1}{4})(I_2+\Delta I_2^\lambda)+\alpha\eta(1-2\eta^2)^{1/2}(I_3+\Delta I_3^\lambda)$

QDIH-IB : $2\eta(I_0+\Delta I_0^\lambda)+(1-\frac{\alpha}{2})(I_1-\Delta I_0^\lambda)-2\alpha(\eta^2-\frac{1}{4})(I_2+\Delta I_2^\lambda)-\alpha\eta(1-2\eta^2)^{1/2}(I_3+\Delta I_3^\lambda)$.

QDIH-IC : $-2\eta(I_0+\Delta I_0^\lambda)+(1-\frac{\alpha}{2})(I_1-\Delta I_0^\lambda)-2\alpha(\eta^2-\frac{1}{4})(I_2+\Delta I_2^\lambda)-\alpha\eta(1-2\eta^2)^{1/2}(I_3+\Delta I_3^\lambda)$.

$$U_{PMNS} \approx \begin{bmatrix} c_{12}^\nu(1-\frac{1}{4}\lambda^2) & s_{12}^\nu(1-\frac{1}{4}\lambda^2) & \frac{\lambda}{\sqrt{2}} \\ -\frac{s_{12}^\nu}{\sqrt{2}}-\frac{\lambda}{2}(1-\frac{\lambda}{2})(\frac{s_{12}^\nu}{\sqrt{2}}+c_{12}^\nu) & \frac{c_{12}^\nu}{\sqrt{2}}+\frac{\lambda}{2}(1-\frac{\lambda}{2})(\frac{c_{12}^\nu}{\sqrt{2}}-s_{12}^\nu) & \frac{1}{\sqrt{2}}-\frac{\lambda}{2\sqrt{2}} \\ \frac{1}{\sqrt{2}}(1-\frac{\lambda}{2})s_{12}^\nu-\frac{\lambda}{2}(1+\frac{\lambda}{2})c_{12}^\nu & -\frac{1}{\sqrt{2}}(1-\frac{\lambda}{2})c_{12}^\nu-\frac{\lambda}{2}(1+\frac{\lambda}{2})s_{12}^\nu & \frac{1}{\sqrt{2}}+\frac{\lambda}{2\sqrt{2}} \end{bmatrix},$$

$$s_{12}^\nu = \sqrt{2}\eta,\ c_{12}^\nu = (1-2\eta^2)^{1/2}$$

TABLE 2.5: The perturbation to the respective building block matrices, I_is are estimated in terms of ΔI_is. The corresponding textures of the corrected mass matrices $m_{LL}^\nu(\alpha,\eta,\lambda)$ are also described. The lepton mixing matrix which is now modified from $U_{\nu L}$ to $U_{eL}^\dagger.U_{\nu L}$ is also presented.

The details of the texture for other QDN cases are described in Table.(2.5). The texture of the PMNS matrix, $U_{PMNS} = U_{eL}^{\dagger}.U_{\nu L}$, is presented in Table.(2.5). We obtain,

$$\sin^2\theta_{12} = 2\eta^2 + \mathcal{O}(\lambda^3), \tag{2.52}$$
$$\sin^2\theta_{13} = \frac{1}{2}\lambda^2 + \mathcal{O}(\lambda^3), \tag{2.53}$$
$$\sin^2\theta_{23} = \frac{1}{2} - \frac{1}{2}\lambda + \frac{1}{8}\lambda^2 + \mathcal{O}(\lambda^3). \tag{2.54}$$

2.8 Numerical calculation

We assign certain ranges to the free parameter α and η respectively. Based on the 1σ range of the physical observable quantities available from Global data analysis [133], we assign $\alpha = 1.5939 - 1.6239$ (QDNH-IA), $0.0080 - 0.0220$ (QDNH-IB,IC), $1.9945 - 1.9948$ (QDIH-IA), $0.0052 - 0.0055$ (QDIH-IB,IC) , and $\eta = 0.3814 - 0.4031$. The input parameter $m_0 \sim 0.08\,eV$. and $\lambda = 0.2253$. We have now four parameters, out of which α and η are free and the number of unknowns present is six.

2.8.1 Observable parameters in oscillation experiments and cosmological observation

We apply the six QDN neutrino mass matrices $m_{LL}^{\nu}(\alpha, \eta, \lambda)$ to study their relevance in the oscillation experiments. It is found that under a suitable choice of the free parameters (α, η), all the six QDN models are equally capable of describing both TBM and TBM-deviated scenarios (see Table.(2.6)) and are indistinguishable. QDNH model says, $|m_1|, |m_2| \sim 0.06\,eV$, $|m_3| \sim 0.08\,eV$, while $|m_2|, |m_3| \sim 0.08\,eV$, $m_1 \sim 0.06\,eV$ for QDIH case. For both the cases, $\Delta m_{21}^2 \sim 7.6 \times 10^{-5}\,eV^2$ and $\Delta m_{31}^2 \sim 2.4 \times 10^{-3}\,eV^2$. The mixing angle parameters are $\sin^2\theta_{13} \simeq 0.025$, $\sin^2\theta_{12} \simeq 0.32$ and $\sin^2\theta_{23} \simeq 0.39$. Also $\Sigma|m_i| \simeq 0.21\,eV$ (QDNH case) and $\Sigma|m_i| \simeq 0.23\,eV$ (QDIH case).

We study a quantity $\sqrt{\Delta m_{21}^2}/\sqrt{\Delta m_{31}^2}$, which according to the global data analysis lies near to 0.2. The correlation plots in the plane $\sqrt{\Delta m_{21}^2}/\sqrt{\Delta m_{31}^2}$ and $\sin^2\theta_{12}$

QD	NH-IA	NH-IB	NH-IC	IH-IA	IH-IB	IH-IC
α (TBM)	1.626	0.0068	0.0068	1.9946	0.0054	0.0054
α	1.5929	0.0071	0.0071	1.9946	0.0054	0.0054
η (TBM)	0.4083	0.4083	0.4083	0.4083	0.4083	0.4083
η	0.40	0.40	0.40	0.3987	0.3987	0.3987
$m_0\ eV$	0.082	0.082	0.082	0.084	0.084	0.084
$m_1\ eV$(TBM)	0.06638	0.06639	0.06639	0.08355	0.08355	0.08355
$m_2\ eV$(TBM)	- 0.06695	0.06695	0.06695	-0.084	0.084	0.084
$m_3\ eV$(TBM)	0.082	0.082	-0.082	0.06859	0.06859	-0.06859
$m_1\ eV$	0.06502	0.06502	0.06502	0.08355	0.08355	0.08355
$m_2\ eV$	-0.0656	0.0656	0.0656	-0.084	0.084	0.084
$m_3\ eV$	0.082	0.082	-0.082	0.0672	0.0672	-0.0672
$\Delta m^2_{21}(10^{-5}eV^2)$ (TBM)	7.645	7.435	7.435	7.60	7.60	7.60
$\Delta m^2_{31}(10^{-3}eV^2)$ (TBM)	2.318	2.316	2.316	-2.352	-2.28	-2.28
$\Delta m^2_{21}(10^{-5}eV^2)$	7.605	7.605	7.605	7.60	7.60	7.60
$\Delta m^2_{21}(10^{-3}eV^2)$	2.497	2.497	2.497	-2.464	-2.464	-2.464
$\Sigma\, m_i\ eV$ (TBM)	0.2153	0.2154	0.2154	0.23613	0.23613	0.23613
$\Sigma\, m_i\ eV$	0.21262	0.21262	0.21262	0.23475	0.23475	0.23475
$\sin^2\theta_{12}$	0.319	0.319	0.319	0.3195	0.3195	0.3195
$\sin^2\theta_{13}$	0.0252	0.0252	0.0252	0.0252	0.0252	0.0252
$\sin^2\theta_{23}$	0.3943	0.3943	0.3943	0.3943	0.3943	0.3943
$m_{\nu_e}\ eV$	0.06582	0.06582	0.06582	0.0835	0.0835	0.0835
$m_{ee}\ eV$	0.02452	0.06590	0.06174	0.03063	0.083625	0.08021

Table 2.6: The study of the six cases of Quasi degenerate neutrino mass model for both TBM mixing and deviation from TBM mixing. The analysis is done with the parameters (α, η, λ) and input m_0. m_0 is fixed at $0.082\,eV$ (QDNH) and $0.084\,eV$ (QDIH) respectively. The free parameter α is related with absolute masses. The free parameter η controls both masses and the solar angle. $\lambda = 0.2253$, the Wolfenstein parameter is related with deviation of reactor angle from zero and that for atmospheric from maximal condition.

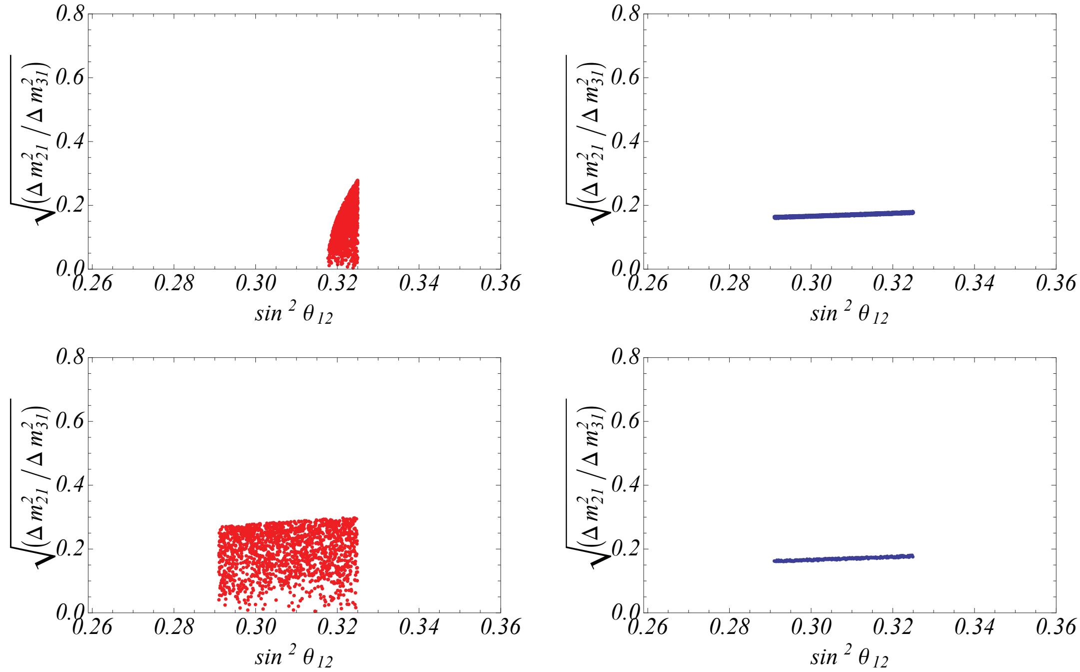

FIGURE 2.5: The correlation plots in the plane of $\sqrt{\Delta m^2_{21}/\Delta m^2_{31}}$ and $\sin^2\theta_{sol}$ for different cases of QDNH-IA (top-left), QDNH-IB,IC (bottom-left), QDIH-IA (top-right) and QDIH-IB,IC (bottom-right). The bounds on $\sqrt{\Delta m^2_{21}/\Delta m^2_{31}}$ are found to be sharp for QDIH cases. The experimental value of this quantity must lie close to 0.2. For QDNH-IA case, we obtain a bound on $\sin^2\theta_{sol}$ around a value of 0.32.

for all QDN models are shown in Fig (2.4). We see for QDNH-Type IA case, there exists a sharp bound on $\sin^2\theta_{12}$ around 0.32 which is the experimental best-fit of $\sin^2\theta_{12}$ according to Global data analysis [132].

2.8.2 Absolute electron neutrino mass (m_{ν_e}) and Effective Majorana neutrino mass (m_{ee})

Besides the oscillation experiments and the cosmological bound on $\Sigma|m_i|$, There are other two important quantities : effective electron neutrino mass, m_{ν_e} appearing in β-decay and effective Majorana mass m_{ee}, appearing in neutrino-less double β-decay experiment and are useful for the study of nature of the neutrino masses.

$$m_{\nu_e} = (\Sigma m_i^2 |U_{ei}|^2)^{1/2}, \tag{2.55}$$

$$m_{ee} = |m_1|U_{e1}|^2 + m_2|U_{e2}|^2 + m_3|U_{e3}|^2|. \tag{2.56}$$

The results of Mainz [46] and Toitsk[150] Tritium β-decay experiments,we obtain, $m_{\nu_e} < 2.2\,eV$. The upcoming KATRIN experiment [151], expects the sensitivity

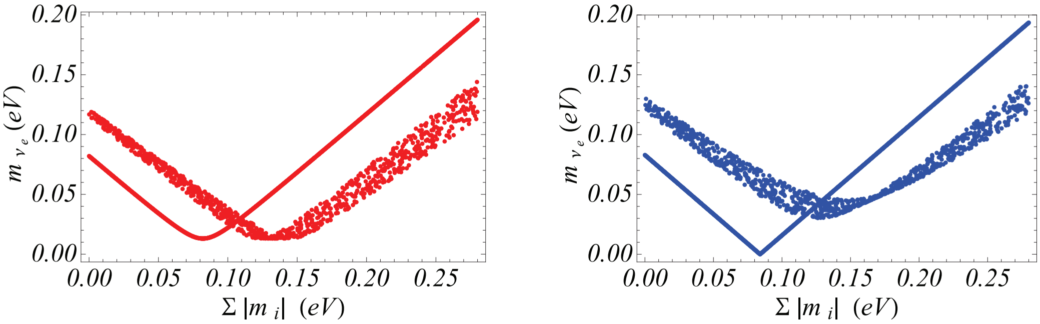

FIGURE 2.6: A study of the correlation in the plane of m_{ν_e} and Σm_i. Left: QDNH case, Right: QDIH case.

upto $m_{\nu_e} \sim 0.3\,eV$. In the present work, the QDNH and QDIH models predict, $m_{\nu_e} \sim 0.07\,eV$ and $m_{\nu_e} \sim 0.08\,eV$ respectively.

The HM group [152–154] and IGEX [50, 155, 156] groups reported the upper limit of m_{ee} to $0.3 - 1.3\,eV$. The CUORICINO[157] experiment gives an improved upper bound on m_{ee}, $m_{ee} < 0.23 - 0.85\,eV$. This is still considered somewhat controversial[50, 158], and requires independent confirmation. The experiments such as CUORE [159, 160], GERDA[161], NEMO[162–164] and Majorana[165, 166] will attempt to improve the sensitivity of the measurement down to about $m_{ee} \simeq (0.05 - 0.09)\,eV$. Hence in that respect the QDN models are of immense importance. In our present work, QDNH and QDIH-both Type IB and Type IC models predict $m_{ee} \simeq 0.06\,eV$ and $m_{ee} \simeq 0.08\,eV$ respectively. The predictions given by Type IA cases of both QDNH and QDIH models are interesting in the sense that they leave a scope for the future experiments to go down upto a sensitivity of $m_{ee} \simeq 0.02\,eV$ and $m_{ee} \simeq 0.03\,eV$ respectively. The correlation plots are studied in the plane m_{ν_e} (and m_{ee}) and $\Sigma|m_i|$ (Fig (2.6) and Fig (2.7)).

2.9 Discussion: How to discriminate different QDN models?

We have tried to bring all the eleven cases involving six QDN, three NH and two IH cases under same roof of parametrization by introducing four common independent building block matrices, $I_{i=0,1,2,3}$. The idea of fragmentation is guided by the quest of some mechanism to save the internal texture of $M_{\mu\tau}$ against the changing solar

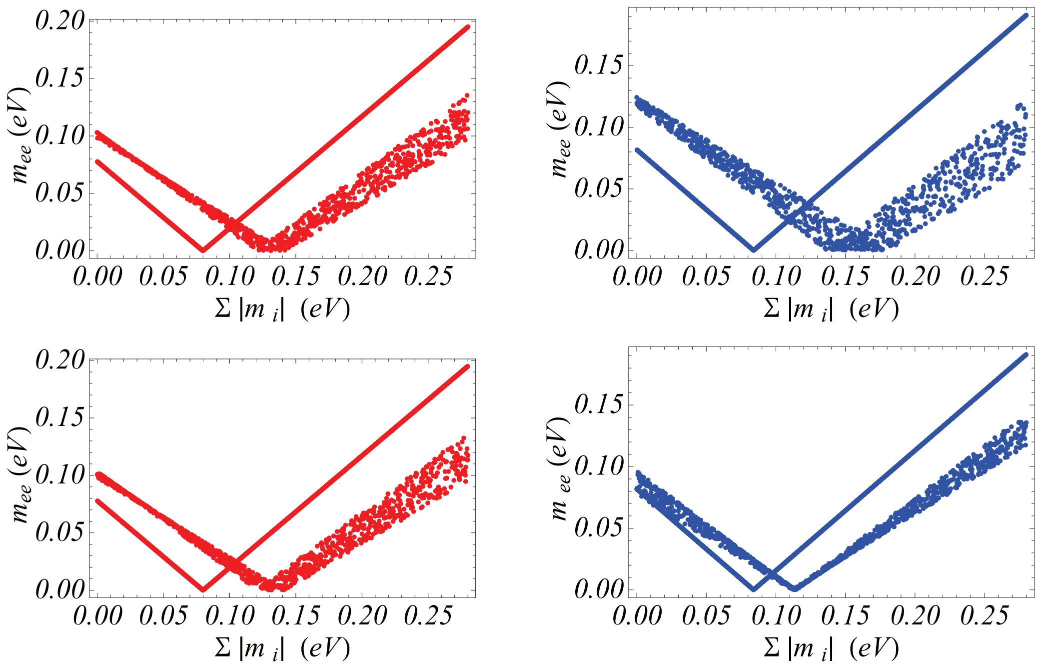

FIGURE 2.7: A study of the correlation in the plane of m_{ee} and Σm_i. Top-left: QDNH-Type IA case; top-right: QDNH-Type IB, IC cases; bottom-left: QDIH-Type IA case; bottom-right: QDIH-Type IB, IC cases.

angle. The I_is when incorporated with the free parameters in a proper way, lead to an important feature of $M_{\mu\tau}$, $\sin\theta_{12} = \sqrt{2}\eta$. θ_{12} is expressible in terms single parameter only, unlike the general $M_{\mu\tau} = M_{\mu\tau}(x, y, z, w)$ where θ_{12} requires the knowledge of four free parameters (x, y, z, w) (see Eq.(2.8)). This is also interesting to note that one of the building blocks, I_3 has got the same eigenstates as predicted by BM mixing (see Table.(2.1)). The existence of this invariant textures within the mass matrix seems to be relevant and we hope that a fruitful investigation is subjected to the study of underlying discrete flavour symmetry groups.

Charged lepton correction is considered as a significant tool in order to break the $\mu - \tau$ symmetry [16, 117–122]. The models where θ_{13}^{ν} is very small, contributions to θ_{13} is implemented mostly from charged lepton sector. Also, this tool is very important for those models where $\theta_{23} = \pi/4$ and it may provide consistency with the LMA MSW solutions [44]. In GUT scenarios also, one finds in addition to the breaking of $\mu - \tau$ symmetry in the neutrino sector, charged lepton corrections are unavoidable [167, 168]. Regarding the parametrization of U_{eL}, we have followed a parametrization scheme different from that of standard one. This step is motivated by the fact that a particular choice of parametrization does not affect the final observables, but a suitable choice can make the mathematics easier. The parametrization of U_{eL} respects the GUT motivated, new QLC relation, $\theta_{13} \sim \theta_c/\sqrt{2}$ [91]. In our parametrization U_{eL} does not affect the prediction of θ_{12} from neutrino sector.

Our work started with the following motivations. They are,

(1) Whether QDN neutrino mass models are equally possible like that of NH and IH models?

(2) How to discriminate the QDN models?

In the background of the oscillation experiments, we have tried to answer to the first question by testing the efficiency of each $m^{\nu}_{LL}(\alpha, \eta, \lambda)$ in predicting the values of five observational parameters and comparing those with Global data. In that context we find the existence of QDN neutrinos of both NH and IH pattern is undisputed. Above all, all the six QDN models sound equally possible (see Table.(2.6)). Hence only the oscillation experiments are not sufficient enough to answer to the second question. But here we want to mention that QDNH-Type A model shows a strong preference for $\sin^2\theta_{12} = 0.32$, which is the best-fit result according to Global analysis done by Forero *et.al* [132], evident from the correlation plot in Fig(2.5).

We have tried to find out the answer of the second question in the framework of β-decay and $0\nu\beta\beta$-decay experiments. But all the six QDN models predict the quantities m_{ν_e} and m_{ee}, below the upper bounds of the past experiments and interestingly they are much closer to the sensitivities expected to be achieved in the future experiments. In our analysis QDNH-$Type-A$ model leaves a scope for future experiments to go down upto $m_{ee} \sim 0.02\, eV$.

In section (2.4), it has been shown that QDN nature of neutrinos permits the mass scale, $0.05 \leq m_0 \leq 0.1\, eV$. But, concerning a fair degree of degeneracy, the range is modified to, $0.07 \leq m_0 \leq 0.1\, eV$. The ansatz regarding the correlation, $\beta : \eta \simeq 2$ plays an important role in the transition from $M_{\mu\tau}(\alpha, \beta, \eta) \rightarrow M_{\mu\tau}(\alpha, \eta)$. This ansatz holds good for the mass scale $m_0 \sim 0.07 - 0.09\, eV$, over the 3σ range of β and η. If there are three free parameters (α, β, η) present in m^{ν}_{LL}, degrees of arbitrariness is also quite higher. Although this ansatz restricts the arbitrariness of θ_{12} to some extent, yet only two free parameters α and η (with $\lambda = 0.2253$ and input m_0 fixed at $0.08\, eV$) are sufficient to predict five observational parameters (related with oscillation experiments), in close agreement with that of experimental 1σ range of data. The parametrization respects both TBM and small TBM-deviated cases. In this context the anstaz $\beta = 2\eta$, appears to be relevant and natural .

We hope that perhaps the cosmological upper bound on Σm_i have some relevance with the discrimination of the six models. So long we adhere to $\Sigma m_i < 0.28\,eV$[110], both QDNH and QDIH models are safe which predict, $\Sigma m_i = 0.212$ and $\Sigma m_i = 0.235$ respectively for input mass scale $m_0 \sim 0.08\,eV$. But the recent analysis supports for a tighter upper bound, $\Sigma m_i < 0.23\,eV$ [52]. If so, the QDIH model seems to be insecure in our analysis. If we believe the ansatz, $\beta : \eta = 2$ to be natural, then with lowering the mass scale from $0.08\,eV$, and controlling α and η, we can achieve $\Sigma m_i < 0.23\,eV$ for QDIH case, and also this will favor the TBM deviated condition. But at the same time, it will give rise to a serious problem that QDIH model with $m_0 < 0.08\,eV$ will completely discard $\theta = \sin^{-1}(1/\sqrt{3})$ (TBM), because corresponding to that angle, Δm^2_{31} will be outside the 3σ range. But the solar angle $\theta = \sin^{-1}(1/\sqrt{3})$, is still relevant within 1σ[132]or 2σ[133] range. So on this basis can we discard QDIH model? But we hope it will be too hurry to come to any conclusion. There is a possibility that by assuming $\beta : \eta = c$, where $c \neq 2$ but $c \sim 2$ (which is allowed indeed), and then lowering of m_0, may solve this problem and can make $QDIH$ models safe.

The discussion so far tells that on phenomenological ground, there is no dispute on the existence of quasi-degenerate neutrinos with $m_i < 0.1eV$, in nature. But the question whether it is of NH type or IH type, is still not clear. In this regard, we expect that possible answer may emerge from the observed Baryon asymmetry of the universe ($\eta_B = 6.5^{+0.4}_{-0.5} \times 10^{-10}$) [103–105, 127]. The calculation of η_B requires the texture of heavy right handed Majorana neutrino mass matrix, M_{RR}. With a suitable choice of Dirac neutrino mass matrix, m_{LR} allowed by $SO(10)$ GUT, we can transit from m^{ν}_{LL} (parametrized so far) to M_{RR} by employing the inversion of Type I see-saw formula, $M_{RR} = -m^T_{LR} m^{\nu^{-1}}_{LL} m_{LR}$. We hope that significant physical insight can be fetched from this approach and it would be possible to figure out the most favorable QDN models out of the six. Unlike, in Refs. [103, 104], the parametrization of mass matrices involves only two free parameters (α, η) and no other input constant terms which are also different for TBM and TBM deviated scenarios. The prediction of θ_{12} involves (η, ϵ, c, d), whereas in our parametrization it depends on η only. With minimum number of parameters, we have achieved a better control over mass matrices. In contrast to Refs. [103, 104], with our parametrization of m^{ν}_{LL}, certain analytical structure of $U_{\nu L}$ is also possible. We hope, this parametrization will be useful for other phenomenological studies also.

Chapter 3

The mixing angle as a function of neutrino mass ratio

In the quark sector, we experience a correlation between the mixing angles and the mass ratios. A partial realization of the similar tie-up in the neutrino sector helps to constrain the parametrization of masses and mixing, and hints for a predictive framework. We derive five hierarchy dependent textures of neutrino mass matrix with minimum number of parameters ($\leq$ 4), following a model-independent strategy.

3.1 Introduction

The neutrino mass matrix plays the central role in the study of neutrino physics, as it contains the information of both the masses and mixing. In that sense, it is more fundamental than the PMNS matrix. It is always desirable to derive a texture of the mass matrix which leads to significant prediction. If the neutrino mass matrix, $\mathcal{M}_\nu$ follows μ-τ symmetry [112, 142, 169–175] , we obtain two constraints on the matrix elements; they are: $(\mathcal{M}_\nu)_{12} = (\mathcal{M}_\nu)_{13}$ and $(\mathcal{M}_\nu)_{22} = (\mathcal{M}_\nu)_{33}$. These two constraints generate : $\theta_{13} = 0$ and $\theta_{23} = 45^0$. But the μ-τ symmetric texture does not tell anything about the neutrino mass hierarchy and solar angle. The texture becomes predictive only when it is associated with certain flavor symmetries [59–61, 63–66, 69, 176]. On the contrary, the present experimental data strongly rule out any possibility of a vanishing reactor angle [128–130] and the central value of θ_{23} is more than 45^0 (and is close to 49^0) [1]. These deviations undoubtedly questions the credibility of μ-τ symmetry.

Visualization of a more realistic neutrino mixing pattern and mass matrix, demands perturbation to the μ-τ symmetry [177–179]. In the present article we emphasize more on the possibility to perceive an exact texture of M_ν which is *model independent*, with minimum number of efficient elements, than following perturbation techniques. Our approach is bottom-up and inspired by the phenomenology of quark sector.

3.2 The angle and the mass ratio

The Cabibbo angle (θ_c) [149, 180] is a parameter which plays a significant role in describing the quark masses and mixing. It is anticipated that this angle might be a function of the ratio of down and strange quark masses [181],

$$\sin\theta_c \simeq \sqrt{\frac{m_d}{m_s}}. \tag{3.1}$$

It is an esteemed endeavor of particle physicists to unify the quark and lepton sectors, or to realize similar kind of happenings in both the sectors otherwise. Based on this, is it possible to extend a similar idea in the form of an *ansatz* in the neutrino sector also, as in the following,

$$\sqrt{\frac{m_i}{m_j}} = \sin\theta_{ij}. \quad (i,j=1,2,3)\,? \tag{3.2}$$

Undoubtedly there are several hurdles which will arise both from the theoretical and phenomenological perspectives. The reason lies in the difference between the mixing mechanism in both the sectors. The CKM matrix is very close to unit matrix and the spectra of "up" and "down" quarks are strongly hierarchical. But, for neutrinos we are ignorant of the exact hierarchy of the masses. Unlike the quarks, the mixing is quite large in lepton sector and the PMNS matrix is far from being an unit matrix.

So long the reactor angle was predicted to be vanishing, such development in Eq.(3.2) seems obsolete. But, in the light of present data, when, $\theta_{13} \sim \theta_c$, one

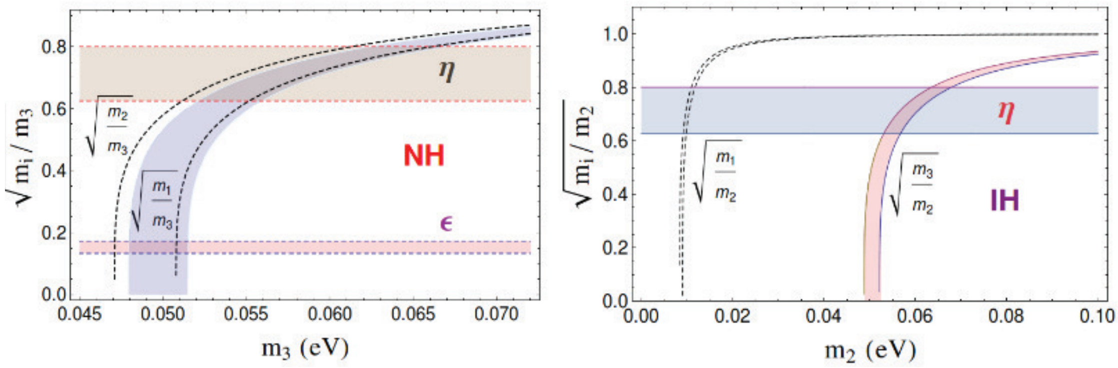

FIGURE 3.1: The evolution of the neutrino mass ratios with respect to the absolute mass scale for both normal (**left**)and inverted ordering (**right**) of the neutrino masses (Corresponding to 3σ range of Δm^2_{21} and Δm^2_{31}) are illustrated.

cannot deny the possible existence of the following relation,

$$\sqrt{\frac{m_1}{m_3}} \simeq \sin\theta_{13} = \epsilon, \quad \text{(say)}, \tag{3.3}$$

in the non-degenerate spectrum of neutrino masses, obeying normal ordering (see Fig.(3.1)). We emphasize that the realization of the ansatz in Eq. (3.2) is not "full", but "*partial*". Because, it seems impossible to realize all the three possibilities (See in Eq. (3.2)) *simultaneously*. For example, if we realize Eq. (3.3), then another possibility,

$$\sqrt{\frac{m_2}{m_3}} \simeq \sin\theta_{23} = \eta; \tag{3.4}$$

is ruled out and vice-versa (See Fig. (3.1)). Again, a similar realization in 1-2 sector is forbidden because of the smallness of the solar mass squared difference, which insists the mass ratio, $m_1 : m_2$ to be constant, and this ratio tends towards unity.

Because of similar reasons, for inverted ordering of the neutrino masses, only the 2-3 realization,

$$\sqrt{\frac{m_3}{m_2}} \simeq \sin\theta_{23} = \eta; \tag{3.5}$$

is possible.

Now even though there are several clampdowns, a partial realization of the ansatz (Eq. (3.2)) clutches certain positive aspects like it puts some constraints on the parametrization of neutrino masses and mixing. Let us discuss some implications of the ansatz in Eq. (3.2).

- It categorizes the parametrization in both ways: "ϵ-based" or "η-based".
- The "ϵ-based" parametrization only encompasses the Normal ordering of the masses with non-degenerate spectrum (**NH-ND**), with absolute mass scale, $0.047\,eV \leq m_3 \leq 0.05\,eV$. This parametrization rules out any possibility of vanishing m_1 (Strict-NH case). Since the reactor angle is not zero as depicted in Eq. (3.3).
- "η-based" parametrization encompasses both the Normal ordering and inverted ordering of neutrino masses with degenerate spectrum (**NH-QD** and **IH-QD**) only, with absolute mass scale, $0.05\,eV \leq m_3(\,m_2\,) \leq 0.067\,eV$.
- The present ansatz (Eq. (3.2)), rules out the possibility of non-degenerate inverted spectrum of neutrino masses.
- In the degenerate limit, the ansatz sets the upper limit on the sum of the neutrino masses, $\Sigma\, m_i \leq 0.17\,eV$. This prediction is relevant in the light of present Cosmological observation.

3.3 The Parametrization

We can see that although the ansatz, (Eq. (3.2)) cannot solve the hierarchy issue, yet it can put some constraints on the mass spectrum. This are subjected to the sensitivities of the future experiments. The ϵ and η based mass spectrum can be represented in the following way,

$$M_\nu^d(\epsilon, s) = \begin{bmatrix} s\epsilon^2 & 0 & 0 \\ 0 & s\epsilon & 0 \\ 0 & 0 & 1 \end{bmatrix} m_3, \qquad \textbf{(NH-ND)} \tag{3.6}$$

$$M_\nu^d(\eta, c) = \begin{bmatrix} c\eta^2 & 0 & 0 \\ 0 & \eta^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} m_3, \qquad \textbf{(NH-QD)} \tag{3.7}$$

$$M_\nu^d(\eta, c) = \begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \eta^2 \end{bmatrix} m_2, \qquad \textbf{(IH-QD)} \tag{3.8}$$

where, s and c are $\mathcal{O}(1)$ coefficients: $\epsilon < s < \epsilon^{-1}$, $\eta < c < \eta^{-1}$. The positivity of $\Delta\, m_{21}^2$ enforces, $c\, <\, 1$.

We are working in a basis, where the charged lepton mass matrix is diagonal. We represent the PMNS matrix, $U(\epsilon)$ which is "ϵ" motivated as in the following,

$$U(\epsilon, d, f) \approx \begin{bmatrix} 1 - \frac{1}{2} f^2\epsilon^2 - \frac{\epsilon^2}{2} & f\epsilon & \epsilon \\ -f\epsilon - d\epsilon^2 & 1 - \frac{1}{2} f^2\epsilon^2 - \frac{d^2\epsilon^2}{2} & d\epsilon \\ -\epsilon + cd\epsilon^2 & -d\epsilon - f\epsilon^2 & 1 - \frac{1}{2} d^2\epsilon^2 - \frac{\epsilon^2}{2} \end{bmatrix}, \qquad \textbf{(PMNS-I)} \tag{3.9}$$

Where, d and f are $\mathcal{O}(1)$ coefficients. We put forward another possible form of PMNS matrix, which is motivated by η-based parametrization, $U(\eta)$,

$$U(\eta, b, c) \approx \begin{bmatrix} \sqrt{\frac{2}{3}} c' & \frac{c}{\sqrt{3}} & b\gamma \\ -\frac{c\kappa}{\sqrt{3}} - \sqrt{\frac{2}{3}} b\gamma\eta c' & \sqrt{\frac{2}{3}}\kappa c' - \frac{bc\gamma\eta}{\sqrt{3}} & \eta \\ \frac{c\eta}{\sqrt{3}} - \sqrt{\frac{2}{3}} b\gamma\kappa c' & -\frac{bc\gamma\kappa}{\sqrt{3}} - \sqrt{\frac{2}{3}}\eta c' & \kappa \end{bmatrix} \qquad \textbf{(PMNS-II)} \tag{3.10}$$

where, $\gamma(\eta) = \eta^8$, $\kappa(\eta) = \cos\sin^{-1}(\eta)$ and $c' = (3 - c^2)^{\frac{1}{2}}/2$. Let us summarize some important features of the above two non-familiar parametrization of PMNS matrix.

- This is to be highlighted that in either of the two possibilities **PMNS-I** or **PMNS II**, a vanishing reactor angle is not possible. If this is so, the mass eigenvalues will also disappear. Hence, the present parametrization cannot hold the Tri-Bimaximal (TB) and Bi-maximal (BM) framework in exact form, which assumes predominantly the reactor angle to be zero.
- The **PMNS-II** allows only the possibilities, $\theta_{23} > 45^0$ and $\theta_{12} \lesssim \sin^{-1}(1/\sqrt{3})$.
- The free parameter, $\epsilon \sim \mathcal{O}(\lambda)$ and $\eta \sim 4\,\lambda$, where λ is the Wolfenstein parameter.

$\mathcal{A} \approx -f^2 s\epsilon^4 + f^2 s\epsilon^3 - s\epsilon^4 + s\epsilon^2 + \epsilon^2$
$\mathcal{B} \approx -\frac{1}{2}d^2 f s\epsilon^4 - ds\epsilon^4 - \frac{d\epsilon^4}{2} + d\epsilon^2 - \frac{1}{2}f^3 s\epsilon^4 - \frac{1}{2}fs\epsilon^4 - fs\epsilon^3 + fs\epsilon^2$
$\mathcal{C} \approx -\frac{d^2\epsilon^3}{2} + dfs\epsilon^4 - dfs\epsilon^3 - f^2 s\epsilon^4 - s\epsilon^3 - \frac{\epsilon^3}{2} + \epsilon$
$\mathcal{D} \approx -d^2 s\epsilon^3 - d^2\epsilon^4 + d^2\epsilon^2 - 2dfs\epsilon^4 + f^2 s\epsilon^4 - f^2 s\epsilon^3 + s\epsilon$
$\mathcal{E} \approx \frac{1}{2}d^3 s\epsilon^4 - \frac{d^3\epsilon^3}{2} + df^2 s\epsilon^4 - ds\epsilon^2 - d\epsilon^3 + d\epsilon + fs\epsilon^4 - fs\epsilon^3$
$\mathcal{F} \approx d^2 s\epsilon^3 + d^2\epsilon^4 - d^2\epsilon^2 + 2dfs\epsilon^4 + s\epsilon^4 + \frac{\epsilon^4}{4} - \epsilon^2 + 1$

TABLE 3.1: The elements of the general neutrino mass matrix (Normal Hierarchy-non-degenerate (**NH-ND**) case)

3.4 The Texture of the Neutrino mass matrix

Next, we try to understand how the ansatz in Eq. (3.2) will help to understand the texture of neutrino mass matrix? The ansatz reduces the number of free parameters, and the number of working parameters are less than that of physical ones. Hence we expect that the mass matrix is little predictive. To construct the same, we concentrate on the finding out some *exact* sum rules to relate different matrix elements.

Here, we construct the neutrino mass matrix, $\mathcal{M}_\nu = \mathcal{U}.{\mathcal{M}_\nu}^d.U^T$. For a lucid flow of the present discussion we keep aside the numerical description of the internal parameters like ϵ, η etc. The details of the same can be found in the Table.(3.4). We have the general texture of left-handed Majorana neutrino mass matrix, as shown below.

$$M_\nu \sim \begin{bmatrix} \mathcal{A} & \mathcal{B} & \mathcal{C} \\ \mathcal{B} & \mathcal{D} & \mathcal{E} \\ \mathcal{C} & \mathcal{E} & \mathcal{F} \end{bmatrix}. \tag{3.11}$$

If, the parametrization is ϵ-based, we have $\mathcal{M}_\nu = \mathcal{M}_\nu(\epsilon, d, f, s)$ and $\mathcal{M}_\nu = \mathcal{M}_\nu(\eta, b, c)$, if it is η-based. We start with the **NH-ND** case, parametrization of which is ϵ based. The matrix elements are tabulated in Table. (3.1).

Choosing the working parameters, (ϵ, d, f, s) properly, we derive the following exact relations connecting the matrix elements.

$$\text{Sum rule 1:} \quad 2(\mathcal{A}+\mathcal{C})-(\mathcal{E}-\mathcal{B})=0, \tag{3.12}$$

$$\text{Sum rule 2:} \quad \mathcal{D}-(\mathcal{A}+\mathcal{F})=0, \tag{3.13}$$

$$\text{Sum rule 3:} \quad 2\mathcal{A}-(\mathcal{B}+\mathcal{C})=0, \tag{3.14}$$

$$\text{Sum rule 4:} \quad 2\mathcal{A}-(\mathcal{D}-\mathcal{E})=0, \tag{3.15}$$

These sum-rules promote a framework with,

$$\theta_{13} \approx 8.73^0, \quad \theta_{23} \approx 48.96^0, \quad \theta_{12} \approx 32.51^0. \tag{3.16}$$

Except the solar angle which is consistent with 2σ range (by sacrificing one of the sum rules, the solar angle can be made precise.), the rest lies within 1σ range. The above sum rules lead to the following texture,

$$M_\nu = \begin{bmatrix} \mathcal{A} & 2\,\mathcal{A}-\mathcal{C} & \mathcal{C} \\ 2\,\mathcal{A}-\mathcal{C} & 6\,\mathcal{A}+\mathcal{C} & 4\,\mathcal{A}+\mathcal{C} \\ \mathcal{C} & 4\,\mathcal{A}+\mathcal{C} & 5\,\mathcal{A}+\mathcal{C} \end{bmatrix} m_3, \qquad (\textbf{Texture-I}) \tag{3.17}$$

where, $\mathcal{A} > \mathcal{C}$. It is interesting to note that the **Texture-I** is nothing but combination of a pattern akin to the modified Fritzsch-like texture of quark mass matrices [182, 183],

$$\begin{bmatrix} 0 & 2\,\mathcal{A} & 0 \\ 2\,\mathcal{A} & 6\,\mathcal{A} & 4\,\mathcal{A} \\ 0 & 4\,\mathcal{A} & 5\,\mathcal{A} \end{bmatrix}, \tag{3.18}$$

and, a μ-τ symmetric texture as shown below,

$$\begin{bmatrix} \mathcal{A} & -\mathcal{C} & \mathcal{C} \\ -\mathcal{C} & \mathcal{C} & \mathcal{C} \\ \mathcal{C} & \mathcal{C} & \mathcal{C} \end{bmatrix}. \tag{3.19}$$

Next, we turn towards another possibility, i.e., the **NHQD** scenario which is motivated by η-based parametrization. The matrix elements of the concerned neutrino mass matrix are illustrated in Table.(3.2).

$\mathcal{A} \approx b^2\gamma^2 + \frac{c^2\eta^2}{3} + \frac{2}{3}c\eta^2 c'^2$
$\mathcal{B} \approx -\frac{1}{3}\eta\{b\gamma\left(c^2\eta^2 - 3\right) + 2bc\gamma\eta^2 c'^2 + \sqrt{2}(c-1)c\eta\kappa c'\}$
$\mathcal{C} \approx \frac{1}{3}\{b\gamma\kappa\left(3 - c^2\eta^2\right) - 2bc\gamma\eta^2\kappa c'^2 + \sqrt{2}(c-1)c\eta^3 c'\}$
$\mathcal{D} \approx \frac{1}{9}\eta^2\{c\left(\sqrt{6}b\gamma\eta c' + \sqrt{3}c\kappa\right)^2 + 3\left(bc\gamma\eta - \sqrt{2}\kappa c'\right)^2 + 9\}$
$\mathcal{E} \approx -\frac{1}{3}\eta\{-2\eta^2\kappa\left(c'\right)^2\left(b^2c\gamma^2 - 1\right) + \kappa\left(-b^2\gamma^2c^2\eta^2 + c^3\eta^2 - 3\right) + \sqrt{2}b(c-1)c\gamma\eta c'\left(\eta^2 - \kappa^2\right)\}$
$\mathcal{F} \approx \frac{1}{9}\eta^2\left(\sqrt{3}bc\gamma\kappa + \sqrt{6}\eta c'\right)^2 + \frac{1}{9}c\eta^2\left(\sqrt{3}c\eta - \sqrt{6}b\gamma\kappa c'\right)^2 + \kappa^2$

TABLE 3.2: The elements of the general neutrino mass matrix (**NH-QD** case)

As before, we encounter the following exact sum rules,

$$\text{Sum rule 1:} \qquad \mathcal{D} - \mathcal{F} - \mathcal{B} = 0, \tag{3.20}$$

$$\text{Sum rule 2:} \qquad 4\,\mathcal{E} - \mathcal{D} = 0, \tag{3.21}$$

$$\text{Sum rule 3:} \qquad 3\,\mathcal{C} - 2(\mathcal{D} - \mathcal{F}) = 0, \tag{3.22}$$

which prescribe the following texture of the neutrino mass matrix guided by three parameters,

$$M_\nu = \begin{bmatrix} \mathcal{A} & \mathcal{B} & \frac{2}{3}\mathcal{B} \\ \mathcal{B} & 4\mathcal{E} & \mathcal{E} \\ \frac{2}{3}\mathcal{B} & \mathcal{E} & 4\mathcal{E} - \mathcal{B} \end{bmatrix} m_3, \qquad (\textbf{Texture-II}) \tag{3.23}$$

with, $4\,\mathcal{E} > \mathcal{A} > \mathcal{E} > \mathcal{B}$. The above neutrino mass matrix is consistent with the prediction,

$$\theta_{12} \approx 34.08^0, \quad \theta_{23} \approx 49.66^0, \quad \theta_{13} \approx 10^0. \tag{3.24}$$

We see that the above framework predicts a reactor angle, lying slightly above the experimental observation. Here also, we experience a μ-τ symmetric texture with a perturbation matrix,

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} & \mathcal{B} \\ \mathcal{B} & 4\mathcal{E} & \mathcal{E} \\ \mathcal{B} & \mathcal{E} & 4\mathcal{E} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{1}{3}\mathcal{B} \\ 0 & 0 & 0 \\ \frac{1}{3}\mathcal{B} & 0 & -\mathcal{B} \end{bmatrix}. \tag{3.25}$$

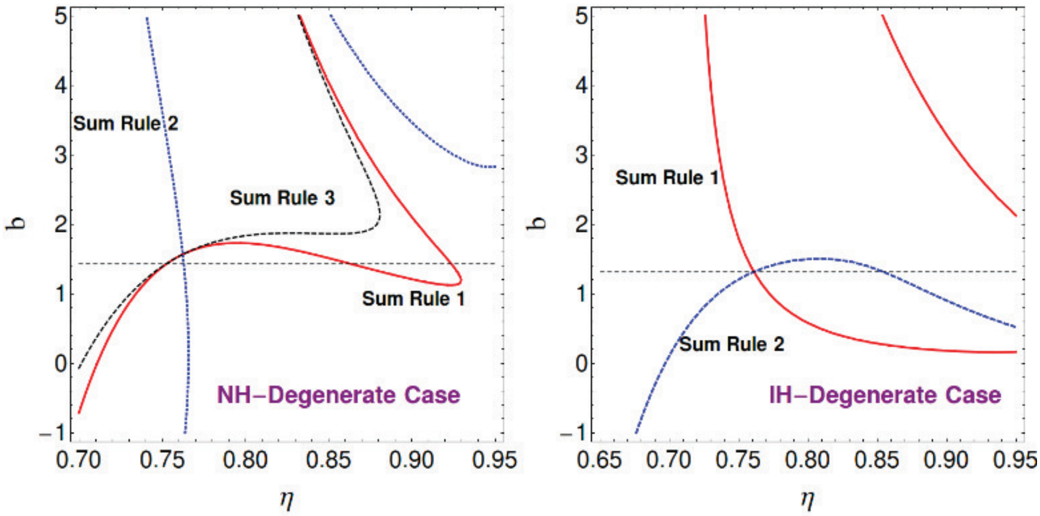

FIGURE 3.2: The illustration of the Sum rules, in the plane b-η for the mass matrices corresponding to **Texture-II** (**left**) and **Texture-V** (**right**).

A precise value $\theta_{13} \approx 9.3^0$, consistent within 1σ bound is obtainable at the cost of sacrificing the Sum rule 2 in Eq. (3.21) (See Fig.(3.2)). The other angles remain untouched. And, the corresponding texture of the neutrino mass matrix appears as in the following,

$$M_\nu = \begin{bmatrix} \mathcal{A} & \mathcal{B} & \frac{2}{3}\mathcal{B} \\ \mathcal{B} & \mathcal{D} & \mathcal{E} \\ \frac{2}{3}\mathcal{B} & \mathcal{E} & \mathcal{D}-\mathcal{B} \end{bmatrix} m_3, \qquad (\textbf{Texture-III}) \qquad (3.26)$$

with $\mathcal{D} > \mathcal{A} > \mathcal{E} > \mathcal{B}$. The discussion related to the hidden μ-τ symmetric texture is similar to that for **Texture-II**. Similarly for the case of Inverted hierarchy (**IH-QD**), the matrix elements are shown in Table.(3.3). The realization of the following sum rules:

$$\text{Sum rule 1:} \qquad \mathcal{B} - \mathcal{C} = 0, \qquad (3.27)$$

$$\text{Sum rule 2:} \qquad \mathcal{A} + \mathcal{E} - \mathcal{D} + \mathcal{B} = 0, \qquad (3.28)$$

promote the neutrino mass matrix, M_ν, to assume the following texture,

$$M_\nu = \begin{bmatrix} \mathcal{A} & \mathcal{B} & \mathcal{B} \\ \mathcal{B} & \mathcal{D} & \mathcal{D}-\mathcal{A}-\mathcal{B} \\ B & \mathcal{D}-\mathcal{A}-\mathcal{B} & \mathcal{F} \end{bmatrix} m_2, \qquad (\textbf{Texture-IV}). \qquad (3.29)$$

with $\mathcal{A} > \mathcal{F} > \mathcal{D} > |\mathcal{B}|$.

$\mathcal{A} \approx b^2\gamma^2\eta^2 + \frac{c^2}{3} + \frac{2}{3}cc'^2$
$\mathcal{B} \approx -\frac{1}{3}bc^2\gamma\eta - \frac{2}{3}bc\gamma\eta c'^2 + b\gamma\eta^3 + \frac{1}{3}\sqrt{2}c\kappa c' - \frac{1}{3}\sqrt{2}c^2\kappa c'$
$\mathcal{C} \approx -\frac{1}{3}b\gamma c^2\kappa - \frac{2}{3}b\gamma c\kappa c'^2 + b\gamma\eta^2\kappa - \frac{1}{3}\sqrt{2}c\eta c' + \frac{1}{3}\sqrt{2}c^2\eta c'$
$\mathcal{D} \approx \frac{1}{3}b^2c^2\gamma^2\eta^2 + \frac{2}{3}b^2c\gamma^2\eta^2c'^2 - \frac{2}{3}\sqrt{2}bc\gamma\eta\kappa c' + \frac{1}{3}\sqrt{2}2bc^2\gamma\eta\kappa c' + \frac{c^3\kappa^2}{3} + \frac{2}{3}\kappa^2c'^2 + \eta^4$
$\mathcal{E} \approx \frac{1}{3}b^2\gamma^2c^2\eta\kappa + \frac{2}{3}b^2\gamma^2c\eta\kappa c'^2 + \frac{1}{3}\sqrt{2}b\gamma c\eta^2c' - \frac{1}{3}\sqrt{2}b\gamma c\kappa^2c' - \frac{1}{3}\sqrt{2}b\gamma c^2\eta^2c' + \frac{1}{3}\sqrt{2}b\gamma c^2\kappa^2c'$ $-\frac{1}{3}c^3\eta\kappa - \frac{2}{3}\eta\kappa c'^2 + \eta^3\kappa$
$\mathcal{F} \approx \frac{1}{3}b^2\gamma^2c^2\kappa^2 + \frac{2}{3}b^2\gamma^2c\kappa^2c'^2 + \frac{1}{3}\sqrt{2}2b\gamma c\eta\kappa c' - \frac{2}{3}\sqrt{2}b\gamma c^2\eta\kappa c' + \frac{c^3\eta^2}{3} + \frac{2}{3}\eta^2c'^2 + \eta^2\kappa^2$

TABLE 3.3: The elements of the general neutrino mass matrix (**IH-QD** case)

The above sum rules, restricts the reactor angle at,

$$\theta_{13} \approx 7.1^0, \tag{3.30}$$

Which is little lower than what we observe experimentally.The other predictions are,

$$\theta_{12} \approx 34.84^0, \quad \theta_{23} \approx 50.76^0. \tag{3.31}$$

which are consistent within 1σ. By changing the Sum rules a little, (See Fig.(3.2)) as in the following,

$$\text{Sum rule 1:} \qquad \mathcal{B} - \mathcal{C} = 0, \tag{3.32}$$

$$\text{Sum rule 2:} \qquad \mathcal{A} - \mathcal{B} - \mathcal{F} + \mathcal{E} = 0. \tag{3.33}$$

We can set the reactor angle within the 1σ bound ($\theta_{13} \approx 8.5^0$). The corresponding neutrino mass matrix appears as in the following,

$$M_\nu = \begin{bmatrix} \mathcal{A} & \mathcal{B} & \mathcal{B} \\ \mathcal{B} & \mathcal{D} & \mathcal{F} - \mathcal{A} + \mathcal{B} \\ B & \mathcal{F} - \mathcal{A} + \mathcal{B} & \mathcal{F} \end{bmatrix} m_2, \qquad (\textbf{Texture-V}) \tag{3.34}$$

Where, $\mathcal{A} > \mathcal{F} > \mathcal{D} > |\mathcal{B}|$.

	Parameters	Prediction	Texture
NH-ND	$\epsilon = 0.151859,$ $f = 3.53882,$ $d = 4.96783,$ $s = 1.21652\ ,$ $m_3 = 0.0483055\, eV.$	$m_1 = 0.00135518\, eV,$ $m_2 = 0.00892393\, eV,$ $\Delta\, m_{21}^2 = 7.78 \times 10^{-5}\, eV^2,$ $\Delta\, m_{31}^2 = 2.33 \times 10^{-3}\, eV^2,$ $\sin^2\theta_{12} = 0.289,$ $\sin^2\theta_{23} = 0.569,$ $\sin^2\theta_{13} = 0.0233,$ $\Sigma\, m_i = 0.0585846\, eV.$	$\begin{bmatrix} \mathcal{A} & 2\,\mathcal{A}-\mathcal{C} & \mathcal{C} \\ 2\,\mathcal{A}-\mathcal{C} & 6\,\mathcal{A}+\mathcal{C} & 4\ \mathcal{A}+\mathcal{C} \\ \mathcal{C} & 4\,\mathcal{A}+\ \mathcal{C} & 5\,\mathcal{A}+\mathcal{C} \end{bmatrix}$ $\mathcal{A} = 0.0946855,\ \ \mathcal{C} = 0.038355.$
NH-QD	$\eta = 0.762386\ ,$ $b = 1.56568,$ $c = 0.96663,$ $m_3 = 0.0588\, eV.$	$m_1 = 0.0330417\, eV,$ $m_2 = 0.0341791\, eV,$ $\Delta\, m_{21}^2 = 7.65 \times 10^{-5}\, eV^2,$ $\Delta\, m_{31}^2 = 2.37 \times 10^{-3}\, eV^2,$ $\sin^2\theta_{12} = 0.315,$ $\sin^2\theta_{23} = 0.581,$ $\sin^2\theta_{13} = 0.0319,$ $\Sigma\, m_i = 0.126036\, eV.$	$\begin{bmatrix} \mathcal{A} & \mathcal{B} & \frac{2}{3}\mathcal{B} \\ \mathcal{B} & 4\mathcal{E} & \mathcal{E} \\ \frac{2}{3}\mathcal{B} & \mathcal{E} & 4\mathcal{E}-\mathcal{B} \end{bmatrix}$ $\mathcal{A} = 0.581819,\ \ \mathcal{B} = 0.0636482,$ $\mathcal{E} = 0.203162.$
	$\eta = 0.756875,$ $b = 1.4945,$ $c = 0.96663,$ $m_3 = 0.0588\, eV.$	$m_1 = 0.0325638\, eV,$ $m_2 = 0.0336859\, eV,$ $\Delta\, m_{21}^2 = 7.43 \times 10^{-5}\, eV^2,$ $\Delta\, m_{31}^2 = 2.40 \times 10^{-3}\, eV^2,$ $\sin^2\theta_{12} = 0.314,$ $\sin^2\theta_{23} = 0.573,$ $\sin^2\theta_{13} = 0.0259,$ $\Sigma\, m_i = 0.12506\, eV.$	$\begin{bmatrix} \mathcal{A} & \mathcal{B} & \frac{2}{3}\mathcal{B} \\ \mathcal{B} & \mathcal{D} & \mathcal{E} \\ \frac{2}{3}\mathcal{B} & \mathcal{E} & \mathcal{D}-\mathcal{B} \end{bmatrix}$ $\mathcal{A} = 0.571196,\ \ \mathcal{B} = 0.058653,$ $\mathcal{D} = 0.80716,\ \ \mathcal{E} = 0.208885.$
IH-QD	$\eta = 0.774459,$ $b = 0.950921,$ $c = 0.989598,$ $m_2 = 0.0612\, eV.$	$m_1 = 0.0605614\, eV,$ $m_3 = 0.0367071\, eV,$ $\Delta\, m_{21}^2 = 7.75 \times 10^{-5}\, eV^2,$ $\Delta\, m_{31}^2 = 2.32 \times 10^{-3}\, eV^2,$ $\sin^2\theta_{12} = 0.3264,$ $\sin^2\theta_{23} = 0.599,$ $\sin^2\theta_{13} = 0.0151,$ $\Sigma\, m_i = 0.158464\, eV.$	$\begin{bmatrix} \mathcal{A} & \mathcal{B} & \mathcal{B} \\ \mathcal{B} & \mathcal{D} & \mathcal{D}-\mathcal{A}-\mathcal{B} \\ B & \mathcal{D}-\mathcal{A}-\mathcal{B} & \mathcal{F} \end{bmatrix}$ $\mathcal{A} = 0.987095,\ \mathcal{B} = -0.0341297$ $\mathcal{D} = 0.761603,\ \mathcal{F} = 0.840778.$
	$\eta = 0.761187,$ $b = 1.31518,$ $c = 0.989553,$ $m_3 = 0.06047\, eV.$	$m_1 = 0.0598426\, eV,$ $m_3 = 0.0350405\, eV,$ $\Delta\, m_{21}^2 = 7.57 \times 10^{-5}\, eV^2,$ $\Delta\, m_{31}^2 = 2.35 \times 10^{-3}\, eV^2,$ $\sin^2\theta_{12} = 0.3264,$ $\sin^2\theta_{23} = 0.579,$ $\sin^2\theta_{13} = 0.0219,$ $\Sigma\, m_i = 0.155355\, eV.$	$\begin{bmatrix} \mathcal{A} & \mathcal{B} & \mathcal{B} \\ \mathcal{B} & \mathcal{D} & \mathcal{F}-\mathcal{A}+\mathcal{B} \\ B & \mathcal{F}-\mathcal{A}+\mathcal{B} & \mathcal{F} \end{bmatrix}$ $\mathcal{A} = 0.983997,\ \mathcal{B} = -0.0430051$ $\mathcal{D} = 0.759459,\ \mathcal{F} = 0.825692,$

TABLE 3.4: The summary of parametrization and the neutrino mass matrix texture

3.5 Discussion

In the above discussion, for all the five textures, the parameters are chosen in such a way that the two observational mass parameters can lie always within the 1σ boundary of experimental data. Also, it is seen that the number of independent matrix elements is 2, 3 or 4. Needless to say, that textures we have displayed are hierarchy dependent and exact. Also, one interesting fact that we see is the existence of a μ-τ symmetric or partially broken μ-τ symmetric textures is unavoidable in the above patterns. But this does not allow us to take the initial choice of M_ν to be μ-τ symmetric. Since, both ϵ-based or η-based parametrization are reluctant to a assume a vanishing θ_{13} (which is one of the important traces of the μ-τ symmetry), certainly, we have to look into some other possibilities.
Trying to understand the underlying flavor symmetry groups or to unveil the first principle working behind the above textures will be a fascinating exercise for the model builders. We hope that Refs. [184–187], can provide some important clue in this line. In order to keep the discussion simple, all the parameters are treated as real. However, we have stressed on different phenomenological facets when it is possible to express at least one of the mixing angles in terms of the mass ratio.

Chapter 4

Realistic lepton mixing matrices deviated from Tri-Bimaximal condition

A model independent endeavor is attempted to amend the existing Tri-bimaximal (TBM) mixing scheme in terms of the Wolfestein parameters, λ and A, which finally leads seven different possible TBM-deviated scenarios. The present investigation embraces the possibilities both $\theta_{23} \leq 45^0$ and $\theta_{23} \geq 45^0$ in addition to a non-zero θ_{13} and Dirac-type CP violating phase. We emphasize on the modification to Tri-Bimaximal Cabibbo (TBC) mixing and the utility of μ-τ symmetry in shifting θ_{12} a little from $\sin^{-1}(1/\sqrt{3})$.

4.1 Introduction

The Tribimaximal mixing (TBM) is a popular neutrino mixing framework, important both from phenomenological as well as from theoretical point of view [59–61, 63–69, 79, 176]. It predicts distinctly the solar angle, $\theta_{12} = \sin^{-1}(1/\sqrt{3}) \approx 35.26^0$, consistent within the 1σ error [1]. The present best-fit is slightly lower than the predicted one, $\theta_{12} \approx 34.63^0$. But, one of the predictions of TBM model, i.e., reactor angle, $\theta_{13} = 0$, at the same time, is strictly ruled out by the present conspicuous observation : $\theta_{13} \approx 9^0 \sim \mathcal{O}(\theta_c)$ [128–130]. This interesting observation on the other hand enhances the possibilities to bridge the quark and lepton sectors, both from top down and bottom up perspectives. But at the same time, the TBM mixing becomes apparently irresolute. Clasping the flavor of TBM mixing, a new idea of mixing called Tri-bimaximal Cabibbo mixing (TBC) [85] is put forward, which differs from the former in anticipating the reactor angle. According to the latter, θ_{13} is rather $\theta_c/\sqrt{2}$ [91], than zero. Both the mixing schemes agree to maximal atmospheric mixing, $\theta_{23} = 45^0$ (consistent within 1σ error). In the present scenario, next to the reactor one, the prediction of atmospheric angle θ_{23} is very interesting. Upto last year, there was stronger indication for $\theta_{23} < 45^0$. But present results [1] indicate, $\theta_{23} \approx 49^0$. But still the possibility of $\theta_{23} \approx 43^0$ is not excluded.

We require:

$$\theta_{13} \approx \theta_{13}^{TBM} + \mathcal{O}(\theta_c). \tag{4.1}$$

This relation indicates the approximate equivalence between 1-2 and 1-3 mixing in quark and neutrino sectors respectively. But we want to highlight the following possible relations,

$$\theta_{12} \approx \theta_{12}^{TBM} - \mathcal{O}(\tilde{\theta}), \tag{4.2}$$

$$\theta_{23} \approx \theta_{23}^{TBM} \pm \mathcal{O}(\tilde{\theta}). \tag{4.3}$$

where $\tilde{\theta} = \theta_{23}^{CKM} \approx A\lambda^2$.

Whether these observations are just accidental or they carry the signature of some underlying theory is subjected to investigation. But the appearance of the two CKM parameters [149] as a source of correction to the TBM model is interesting. If we believe in the flavor basis where, $U_{PMNS} = U_{TB}$, then it appears that the TB mixing is least credible. But a proper choice of the symmetry basis, $U_{PMNS} = U_l^\dagger.U_{TB}$, (where, U_l is the charged lepton diagonalizing matrix) leaves enough scope to generate the corrections to original TB platform. In fact, on choosing a CKM-like U_l, $U_l \approx R_{12}^l(\theta_{12}^l = \theta_c)$, one can induce: $\theta_{13} = \theta_c/\sqrt{2}$. Again, another choice of CKM-like U_l, $U_l = R_{23}^l(\theta_{23}^l = \tilde{\theta})$, leads to,

$$\theta_{23} = \frac{\pi}{4} - A\lambda^2. \tag{4.4}$$

We posit another possibility, $U_l = R_{23}^{l^{-1}}(\theta_{23}^l = \tilde{\theta})$ which leads to an important result,

$$\theta_{23} = \frac{\pi}{4} + A\lambda^2. \tag{4.5}$$

At the same time, we highlight a rare but efficient possibility [188] that the corrections may even arise from neutrino sector itself, $U_{PMNS} = U_l^\dagger.U_{TB}.W$, where W is the correction matrix in the neutrino sector[189].

4.2 Controlling θ_{12} from neutrino sector

Next question is how to generate the observed deviation (though small) in solar angle? We follow the strategy as shown below. If neutrinos mix in the TBM way, then the concerned neutrino mass matrix follows a particular symmetry called μ-τ symmetry [59–61, 63–69, 79, 142, 176]. An idiosyncratic characteristic of this symmetry is that it keeps θ_{12} free, $\theta_{13} = 0$ and $\theta_{23} = 45^0$. In fact, TBM mixing is a special case associated with μ-τ symmetry. We want to emphasize on this salient feature of μ-τ symmetry that allows a small deviation of θ_{12} from TBM prediction, leaving the other predictions unhindered. This unique trait most often illuminates several strategies from model independent perspectives[103, 104, 115, 116, 127, 190, 191].

In Ref.[191], following a model independent scheme, it is shown that a general μ-τ symmetric matrix can be expressed in terms of four *invariant building block* matrices [1] : $M_{\mu\tau} = a\, I_0 + b\, I_1 + c\, I_2 + d\, I_3$, where,

$$I_0 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad I_1 = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \tag{4.6}$$

$$I_1 = \frac{1}{2}\begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{4.7}$$

[1]These building blocks are termed *invariant*, in the sense, whenever we consider other mixing patterns consistent with μ-τ symmetry (for both normal as well as inverted ordering), only the coefficients a, b, c and d are to be changed. For example, a choice, $a = 1$, $b = \beta - \alpha/2$, $c = 0$ and $d = -\alpha/(2\sqrt{2})$ leads to a μ-τ symmetric mass matrix consistent with Bi-maximal mixing.

On choosing $a = 1$, $b = \beta - \alpha/2$, $c = \alpha/6$ and $d = -\alpha/3$, we obtain the neutrino mass matrix for normal ordering of the masses, $m_1 : m_2 : m_3 = \beta - \alpha : \alpha : 1$,: $M_{TBM} = I_0 + (\beta - \alpha/2)I_1 + \alpha/6 I_2 - \alpha/3 I_3$, which can be reduced further to the strict normal ordering case on choosing $\beta = \alpha$,

$$M_{TBM} = I_0 + \frac{\alpha}{2}I_1 + \frac{\alpha}{6}I_2 - \frac{\alpha}{3}I_3, \tag{4.8}$$

A choice of $\alpha = A\lambda$ (Where, $A = 0.790$, $\lambda = 0.2252$) and absolute mass scale of $0.0495\,eV$ gives, $\Delta m^2_{sol} \approx 7.75 \times 10^{-5}\,eV^2$ and $\Delta m^2_{atm} \approx 2.45 \times 10^{-3}\,eV^2$. Needless to say that the diagonalizing matrix of M_{TBM} in eq. (4.8), upto the Majorana phases is,

$$U_{TB} = \begin{bmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}. \tag{4.9}$$

We note that a μ-τ preserving perturbation of the following kind,

$$M_s = \frac{\epsilon\alpha}{3}\left(I_2 + \frac{1}{4}I_3\right), \tag{4.10}$$

to M_{TBM}, in eq. (4.8), induces a least change of $\mathcal{O}(\epsilon^2)$ to the mass eigenvalues [192], whereas affects the mass eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$, by an amount $\mathcal{O}(\epsilon)$. The state $|\nu_3\rangle$ is kept intact. In view of the mass eigenvalues, the new perturbation parameter ϵ is expected to be small. The diagonalizing matrix of the new μ-τ symmetric mass matrix, $M_{TBM} + M_s$ is,

$$U_\nu \approx \begin{bmatrix} \sqrt{\frac{2}{3}} + \frac{\epsilon}{2\sqrt{6}} & \frac{1}{\sqrt{3}} - \frac{\epsilon}{2\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} + \frac{\epsilon}{2\sqrt{6}} & \frac{1}{\sqrt{3}} + \frac{\epsilon}{4\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} - \frac{\epsilon}{2\sqrt{6}} & -\frac{1}{\sqrt{3}} - \frac{\epsilon}{4\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \tag{4.11}$$

resembling, $s^2_{12} \approx 1/3(1 - \epsilon) + \mathcal{O}(\epsilon^2)$. On tuning $\epsilon = A\lambda^2$, we see that $s^2_{12} \approx 0.32$, which is in close agreement with the observation. Needless to mention that, θ_{13} and θ_{23} remain same as before.

An attempt to deviate θ_{13} and θ_{23} will certainly break down this symmetry but the prediction of the latter can be sustained [167, 168]. We shall try to look into some possible textures of charged lepton diagonalizing matrix.

We have the TBC mixing matrix, upto the Majorana phases, as in the following,

$$U_{\text{TBC}} \approx \begin{bmatrix} \sqrt{\frac{2}{3}} - \frac{\lambda^2}{2\sqrt{6}} & \frac{1}{\sqrt{3}} - \frac{\lambda^2}{4\sqrt{3}} & \frac{e^{-i\delta}\lambda}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} - \frac{e^{i\delta}\lambda}{\sqrt{6}} & \frac{1}{\sqrt{3}} - \frac{e^{i\delta}\lambda}{2\sqrt{3}} & \frac{1}{\sqrt{2}} - \frac{\lambda^2}{4\sqrt{2}} \\ \frac{1}{\sqrt{6}} - \frac{e^{i\delta}\lambda}{\sqrt{6}} & -\frac{1}{\sqrt{3}} - \frac{e^{i\delta}\lambda}{2\sqrt{3}} & \frac{1}{\sqrt{2}} - \frac{\lambda^2}{4\sqrt{2}} \end{bmatrix}. \tag{4.12}$$

. For convenience, we assume that $U_{TBC} = U_{l_1}^{\dagger}.U_{TB}$, so that we can identify the contribution from the charged lepton sector responsible for uplifting θ_{13}, We obtain, $U_{l_1} = U_{13}^{-1}(\theta_c/2 : \delta).U_{12}^{-1}(\theta_c/2 : \delta).R_{23}(\theta_c^2/8)$, where

$$U_{l_1} \approx \begin{bmatrix} 1 - \frac{\lambda^2}{4} & -\frac{1}{2}e^{-i\delta}\lambda & -\frac{1}{2}e^{-i\delta}\lambda \\ \frac{1}{2}e^{i\delta}\lambda & 1 - \frac{\lambda^2}{8} & -\frac{\lambda^2}{8} \\ \frac{1}{2}e^{i\delta}\lambda & -\frac{\lambda^2}{8} & 1 - \frac{\lambda^2}{8} \end{bmatrix}. \tag{4.13}$$

We have seen that the U_{l_1} contributes only towards a nonzero θ_{13}, and, does not touch θ_{12} and θ_{23} at all. But the consistency with the Global data demands modification of U_{l_1}. Based on the earlier discussion, to include the possibility of $\theta_{23} < \pi/4$, a CKM like correction can be associated with U_{l_1}. We redefine U_{l_1} (see Eq. (4.13)) as $U_{l_2} = U_{l_1}.R_{23}(\tilde{\theta})$, where $\tilde{\theta} = A\,\lambda^2$ [126, 193, 194],

$$U_{l_2} \approx \begin{bmatrix} 1 - \frac{\lambda^2}{4} & -\frac{1}{2}e^{-i\delta}\lambda & -\frac{1}{2}e^{-i\delta}\lambda \\ \frac{1}{2}e^{i\delta}\lambda & 1 - \frac{\lambda^2}{8} & A\lambda^2 - \frac{\lambda^2}{8} \\ \frac{1}{2}e^{i\delta}\lambda & -A\lambda^2 - \frac{\lambda^2}{8} & 1 - \frac{\lambda^2}{8} \end{bmatrix}, \tag{4.14}$$

which is capable of generating,

$$s_{23}^2 \approx \frac{1}{2} - A\lambda^2. \tag{4.15}$$

In another prescription, we define U_l as, $U_{l_3} = U_{l_1}.R_{23}^{-1}(\tilde{\theta})$, where,

$$U_{l_3} \approx \begin{bmatrix} 1 - \frac{\lambda^2}{4} & -\frac{1}{2}e^{-i\delta}\lambda & -\frac{1}{2}e^{-i\delta}\lambda \\ \frac{1}{2}e^{i\delta}\lambda & 1 - \frac{\lambda^2}{8} & -A\lambda^2 - \frac{\lambda^2}{8} \\ \frac{1}{2}e^{i\delta}\lambda & A\lambda^2 - \frac{\lambda^2}{8} & 1 - \frac{\lambda^2}{8} \end{bmatrix}, \tag{4.16}$$

gives rise to,

$$s_{23}^2 \approx \frac{1}{2} + A\lambda^2. \tag{4.17}$$

This is to be emphasized that in eq. (4.11), only θ_{12} is deviated from TBM prediction, whereas, the U_l's mentioned in eqs. (4.13)-(4.16)can deviate θ_{13} and θ_{23}, leaving the rest undisturbed. We hope that the neutrino mass matrix follows μ-τ symmetry, consistent with eq. (4.11) and the symmetry is finally broken down with any of the U_l's, mentioned in eqs. (4.13)-(4.16). Hence we obtain,

- **Case 1:** $(U_{lep})_1 = U_{l_1}^\dagger . U_\nu$.

$$(U_{lep})_1 \approx \begin{bmatrix} \sqrt{\frac{2}{3}} - \frac{\lambda^2}{2\sqrt{6}} + \frac{A\lambda^2}{2\sqrt{6}} & \frac{1}{\sqrt{3}} - \frac{\lambda^2}{4\sqrt{3}} - \frac{A\lambda^2}{2\sqrt{3}} & \frac{e^{i\delta}\lambda}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} - \frac{e^{-i\delta}\lambda}{\sqrt{6}} + \frac{A\lambda^2}{2\sqrt{6}} & \frac{1}{\sqrt{3}} - \frac{e^{-i\delta}\lambda}{2\sqrt{3}} + \frac{A\lambda^2}{4\sqrt{3}} & \frac{1}{\sqrt{2}} - \frac{\lambda^2}{4\sqrt{2}} \\ \frac{1}{\sqrt{6}} - \frac{e^{-i\delta}\lambda}{\sqrt{6}} - \frac{A\lambda^2}{2\sqrt{6}} & -\frac{1}{\sqrt{3}} - \frac{e^{-i\delta}\lambda}{2\sqrt{3}} - \frac{A\lambda^2}{4\sqrt{3}} & \frac{1}{\sqrt{2}} - \frac{\lambda^2}{4\sqrt{2}} \end{bmatrix} .P, \quad (4.18)$$

consistent with,

$$s_{13}^2 \approx \frac{\lambda^2}{2}, \quad s_{12}^2 \approx \frac{1}{3}(1 - A\lambda^2), \quad s_{23}^2 = \frac{1}{2}. \quad (4.19)$$

- **Case 2:** $(U_{lep})_2 = U_{l_2}^\dagger . U_\nu$.

$$(U_{lep})_2 \approx \begin{bmatrix} \sqrt{\frac{2}{3}} - \frac{\lambda^2}{2\sqrt{6}} + \frac{A\lambda^2}{2\sqrt{6}} & \frac{1}{\sqrt{3}} - \frac{\lambda^2}{4\sqrt{3}} - \frac{A\lambda^2}{2\sqrt{3}} & \frac{e^{i\delta}\lambda}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} - \frac{e^{-i\delta}\lambda}{\sqrt{6}} - \frac{A\lambda^2}{2\sqrt{6}} & \frac{1}{\sqrt{3}} - \frac{e^{-i\delta}\lambda}{2\sqrt{3}} + \frac{5A\lambda^2}{4\sqrt{3}} & \frac{1}{\sqrt{2}} - \frac{\lambda^2}{4\sqrt{2}} - \frac{A\lambda^2}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} - \frac{e^{-i\delta}\lambda}{\sqrt{6}} - \frac{\sqrt{3}A\lambda^2}{2\sqrt{2}} & -\frac{1}{\sqrt{3}} - \frac{e^{-i\delta}\lambda}{2\sqrt{3}} + \frac{\sqrt{3}A\lambda^2}{4} & \frac{1}{\sqrt{2}} - \frac{\lambda^2}{4\sqrt{2}} + \frac{A\lambda^2}{\sqrt{2}} \end{bmatrix} .P, \quad (4.20)$$

consistent with,

$$s_{13}^2 \approx \frac{\lambda^2}{2}, \quad s_{12}^2 \approx \frac{1}{3}(1 - A\lambda^2), \quad s_{23}^2 = \frac{1}{2} - A\lambda^2. \quad (4.21)$$

- **Case 3:** $(U_{lep})_3 = U_{l_3}^\dagger . U_\nu$.

$$(U_{lep})_3 \approx \begin{bmatrix} \sqrt{\frac{2}{3}} - \frac{\lambda^2}{2\sqrt{6}} + \frac{A\lambda^2}{2\sqrt{6}} & \frac{1}{\sqrt{3}} - \frac{\lambda^2}{4\sqrt{3}} - \frac{A\lambda^2}{2\sqrt{3}} & \frac{e^{i\delta}\lambda}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} - \frac{e^{-i\delta}\lambda}{\sqrt{6}} + \frac{\sqrt{3}A\lambda^2}{2\sqrt{2}} & \frac{1}{\sqrt{3}} - \frac{e^{-i\delta}\lambda}{2\sqrt{3}} - \frac{\sqrt{3}A\lambda^2}{4} & \frac{1}{\sqrt{2}} - \frac{\lambda^2}{4\sqrt{2}} + \frac{A\lambda^2}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} - \frac{e^{-i\delta}\lambda}{\sqrt{6}} + \frac{A\lambda^2}{2\sqrt{6}} & -\frac{1}{\sqrt{3}} - \frac{e^{-i\delta}\lambda}{2\sqrt{3}} - \frac{5A\lambda^2}{4\sqrt{3}} & \frac{1}{\sqrt{2}} - \frac{\lambda^2}{4\sqrt{2}} - \frac{A\lambda^2}{\sqrt{2}} \end{bmatrix} .P, \quad (4.22)$$

consistent with,

$$s_{13}^2 \approx \frac{\lambda^2}{2}, \quad s_{12}^2 \approx \frac{1}{3}(1 - A\lambda^2), \quad s_{23}^2 = \frac{1}{2} + A\lambda^2. \quad (4.23)$$

In the above expressions: $P = diag\,\{e^{i\alpha}, e^{i\beta}, 1\}$, where α and β are the Majorana phases. This is evident that the above three cases are the three reformed versions of TBC mixing.

4.3 CKM-like charged lepton correction

We can think of certain possibilities, where the charged lepton sector itself is capable of administering all the three mixing angles.

The SO(10) or SU(5) GUT models initiate certain logical choice of U_l. In SO(10) background, the charged lepton mass matrix is approximated to that of down type quark, $M_e \sim M_d$. Hence we can approximate, $U_l \approx V_{CKM}$. Again $SU(5)$ based models posit, $M_e \sim M_d^T$. Hence another possibility is, $U_l \approx V_{CKM}^\dagger$. In quark sector, the most dominant parameter is $\theta_{12}^{CKM} = \theta_c$, which is followed by $\theta_{23}^{CKM} = \tilde{\theta} = A\lambda^2$. So we can express U_l with 1-2 rotation only or with 2-3 rotation in addition to the same. As suggested in Ref. [148], we associate a complex phase parameter δ with 1-2 rotation, so that $U_{12} \to U_{12}(\theta_c, \delta)$. We have, the following four choices of U_l's,

$$U_{l_4} = \Omega.R_{12}^l(\theta_c).\Omega' \approx \begin{bmatrix} 1-\frac{\lambda^2}{2} & e^{-i\delta}\lambda & 0 \\ -e^{i\delta}\lambda & 1-\frac{\lambda^2}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4.24}$$

$$U_{l_5} = R_{23}^l(\tilde{\theta}).\Omega.R_{12}^l(\theta_c).\Omega' \approx \begin{bmatrix} 1-\frac{\lambda^2}{2} & e^{-i\delta}\lambda & 0 \\ -e^{i\delta}\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ 0 & -A\lambda^2 & 1 \end{bmatrix}, \tag{4.25}$$

$$U_{l_6} = \Omega.R_{12}^{l^T}(\theta_c).\Omega' \approx \begin{bmatrix} 1-\frac{\lambda^2}{2} & -e^{-i\delta}\lambda & 0 \\ e^{i\delta}\lambda & 1-\frac{\lambda^2}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4.26}$$

$$U_{l_7} = \Omega.R_{12}^{l^T}(\theta_c)\Omega'.R_{23}^{l^T}(\tilde{\theta}) \approx \begin{bmatrix} 1-\frac{\lambda^2}{2} & -e^{-i\delta}\lambda & 0 \\ e^{i\delta}\lambda & 1-\frac{\lambda^2}{2} & -A\lambda^2 \\ 0 & A\lambda^2 & 1 \end{bmatrix}. \tag{4.27}$$

where, we have $\Omega = diag\{e^{-i\delta/2}, e^{i\delta/2}, 1\}$ and $\Omega' = \Omega^\dagger$. With these four CKM-like U_l s, we develop the following landscapes with $U_\nu = U_{TB}$.

- **Case 4:** $(U_{lep})_4 = U_{l_4}^\dagger . U_{TB}$

$$(U_{lep})_4 \approx \begin{bmatrix} \sqrt{\frac{2}{3}} + \frac{e^{-i\delta}\lambda}{\sqrt{6}} - \frac{\lambda^2}{\sqrt{6}} & \frac{1}{\sqrt{3}} - \frac{e^{-i\delta}\lambda}{\sqrt{3}} - \frac{\lambda^2}{2\sqrt{3}} & -\frac{e^{-i\delta}\lambda}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} + \sqrt{\frac{2}{3}} e^{i\delta}\lambda + \frac{\lambda^2}{2\sqrt{6}} & \frac{1}{\sqrt{3}} + \frac{e^{i\delta}\lambda}{\sqrt{3}} - \frac{\lambda^2}{2\sqrt{3}} & \frac{1}{\sqrt{2}} - \frac{\lambda^2}{2\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix} .P \quad (4.28)$$

which results in,

$$s_{13}^2 = \frac{\lambda^2}{2}, \quad s_{12}^2 \approx \frac{1}{3} + \frac{\lambda^2}{6} - \frac{2}{3}\lambda\cos\delta, \quad s_{23}^2 \approx \frac{1}{2} - \frac{\lambda^2}{4}. \quad (4.29)$$

- **Case 5:** $(U_{lep})_5 = U_{l_5}^\dagger . U_{TB}$

$$(U_{lep})_5 \approx \begin{bmatrix} \sqrt{\frac{2}{3}} + \frac{e^{-i\delta}\lambda}{\sqrt{6}} - \frac{\lambda^2}{\sqrt{6}} & \frac{1}{\sqrt{3}} - \frac{e^{-i\delta}\lambda}{\sqrt{3}} - \frac{\lambda^2}{2\sqrt{3}} & -\frac{e^{-i\delta}\lambda}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} + \sqrt{\frac{2}{3}} e^{i\delta}\lambda + \frac{\lambda^2}{2\sqrt{6}} - \frac{A\lambda^2}{\sqrt{6}} & \frac{1}{\sqrt{3}} + \frac{e^{i\delta}\lambda}{\sqrt{3}} - \frac{\lambda^2}{2\sqrt{3}} + \frac{A\lambda^2}{\sqrt{3}} & \frac{1}{\sqrt{2}} - \frac{\lambda^2}{2\sqrt{2}} - \frac{A\lambda^2}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} - \frac{A\lambda^2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} + \frac{A\lambda^2}{\sqrt{3}} & \frac{1}{\sqrt{2}} + \frac{A\lambda^2}{\sqrt{2}} \end{bmatrix} .P, \quad (4.30)$$

giving rise to,

$$s_{13}^2 = \frac{\lambda^2}{2}, \quad s_{12}^2 \approx \frac{1}{3} + \frac{\lambda^2}{6} - \frac{2}{3}\lambda\cos\delta, \quad s_{23}^2 \approx \frac{1}{2} - \frac{\lambda^2}{4} - A\lambda^2. \quad (4.31)$$

- **Case 6:** $(U_{lep})_6 = U_{l_6}^\dagger . U_{TB}$

$$(U_{lep})_6 \approx \begin{bmatrix} \sqrt{\frac{2}{3}} - \frac{e^{-i\delta}\lambda}{\sqrt{6}} - \frac{\lambda^2}{\sqrt{6}} & \frac{1}{\sqrt{3}} + \frac{e^{-i\delta}\lambda}{\sqrt{3}} - \frac{\lambda^2}{2\sqrt{3}} & \frac{e^{-i\delta}\lambda}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} - \sqrt{\frac{2}{3}} e^{i\delta}\lambda + \frac{\lambda^2}{2\sqrt{6}} & \frac{1}{\sqrt{3}} - \frac{e^{i\delta}\lambda}{\sqrt{3}} - \frac{\lambda^2}{2\sqrt{3}} & \frac{1}{\sqrt{2}} - \frac{\lambda^2}{2\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix} .P \quad (4.32)$$

giving,

$$s_{13}^2 = \frac{\lambda^2}{2}, \quad s_{12}^2 \approx \frac{1}{3} + \frac{\lambda^2}{6} + \frac{2}{3}\lambda\cos\delta, \quad s_{23}^2 \approx \frac{1}{2} - \frac{\lambda^2}{4}. \quad (4.33)$$

Model	θ_{13}	θ_{12}	θ_{23}	δ_{CP}/π
1	9.2^0	35.25^0	45^0	–
2	9.2^0	34.45^0	42.70^0	–
3	9.2^0	34.45^0	47.30^0	–
4	9.2^0	34.63^0	44.26^0	1.54
5	9.2^0	34.63^0	41.97^0	1.54
6	9.2^0	34.63^0	44.26^0	1.46
7	9.2^0	34.63^0	46.57^0	1.46

TABLE 4.1: Summary of all the TBM deviated models.

- **Case 7:** $(U_{lep})_7 = U^{\dagger}_{l_7}.U_{TB}$

$$(U_{lep})_6 \approx \begin{bmatrix} \sqrt{\frac{2}{3}} - \frac{e^{-i\delta}\lambda}{\sqrt{6}} - \frac{\lambda^2}{\sqrt{6}} & \frac{1}{\sqrt{3}} + \frac{e^{-i\delta}\lambda}{\sqrt{3}} - \frac{\lambda^2}{2\sqrt{3}} & \frac{e^{-i\delta}\lambda}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} - \sqrt{\frac{2}{3}}e^{i\delta}\lambda + \frac{\lambda^2}{2\sqrt{6}} + \frac{A\lambda^2}{\sqrt{6}} & \frac{1}{\sqrt{3}} - \frac{e^{i\delta}\lambda}{\sqrt{3}} - \frac{\lambda^2}{2\sqrt{3}} - \frac{A\lambda^2}{\sqrt{3}} & \frac{1}{\sqrt{2}} - \frac{\lambda^2}{2\sqrt{2}} \\ \frac{1}{\sqrt{6}} + \frac{A\lambda^2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} - \frac{A\lambda^2}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}.P \tag{4.34}$$

giving,

$$s_{13}^2 = \frac{\lambda^2}{2}, \quad s_{12}^2 \approx \frac{1}{3} + \frac{\lambda^2}{6} + \frac{2}{3}\lambda\cos\delta, \quad s_{23}^2 \approx \frac{1}{2} - \frac{\lambda^2}{4} + A\lambda^2. \tag{4.35}$$

Here, we want to mention that the **Cases 4** and **5**, appear in literature in different ways. We summerize the results of all seven models in Table. (4.1). The interesting part of the **Cases 3** and **7**, is that they are very close to the present observation of $\theta_{23} \approx 49^0$. The other **Cases** like **2**, **4**, and **6** are consistent with the other possibility $\theta_{23} \approx 43^0$. In the **Cases 1-3**, δ_{CP} is a free parameter. The **Cases 6** and **7** predict the same to be $1.46\,\pi$, which is very close to that from experimental observation $\delta_{cp} \approx 1.48\,\pi$.

4.4 Discussion

Trying to relate the CKM and Lepton parameters is an esteemed goal of particle physics. Though lepton mixing is quite different from that of quarks, yet from unification point of view, we can expect the presence of some CKM parameters in the texture of PMNS matrix. The TBM mixing is unable to describe completely the mixing of leptons. But it provides a strong platform to discern several realistic models. In the first part of our discussion, we have tried to modify the TBC mixing by introducing CKM-like correction from charged lepton sector to make θ_{23} relevant in view of the experiments. Also we have discussed the possibilities to slim down θ_{12} in the μ-τ symmetric regime. Finally, we have tried to amend the original TBM model by strapping it with different choices of CKM-like charged lepton diagonalizing matrices. In the present note the approach is bottom-up and model independent. We hope that a more sophisticated appraisal in this line will help to unveil the underlying tie-in among quark, charged lepton and neutrino sectors.

Chapter 5

The Cabibbo angle as a universal seed for quark and lepton mixings

A model-independent ansatz to describe lepton and quark mixing in a unified way is suggested based upon the Cabibbo angle. In our framework neutrinos mix in a "Bi-Large" fashion, while the charged leptons mix as the "down-type" quarks do. In addition to the standard Wolfenstein parameters (λ, A) two other free parameters (ψ, δ) are needed to specify the physical lepton mixing matrix. Through this simple assumption one makes specific predictions for the atmospheric angle as well as leptonic CP violation in good agreement with current observations.

5.1 Introduction

A striking observation vindicated by recent experimental neutrino data is that the smallest of the lepton mixing angles is surprisingly large, similar to the largest of the quark mixing parameters, namely the Cabibbo angle (θ_c) [1, 132]. An interesting lepton mixing scheme called "Bi-Large" (BL) mixing has been proposed recently [87] and subsequently studied in Refs [195–197]. This mixing scheme assumes the atmospheric and the solar mixing angles to be equal and proportional to the reactor angle. In contrast to the Bi-maximal (BM) scenario [78, 136], within the BL scheme the atmospheric mixing angle does not need to be strictly "Maximal", but simply "Large" in general. In summary, BL mixing posits, $\sin\theta_{13} \simeq \lambda$, $\sin\theta_{12} = \sin\theta_{23} \sim \lambda$, where $\lambda = \sin\theta_c$.

Such BL mixing ansatz can be motivated in string theories. Indeed, in F-theory motivated Grand Unified Theory (GUT) models, a geometrical unification of charged lepton and neutrino sectors leads to a mild hierarchy in the neutrino mixing matrix in which θ_{12}^{ν} and θ_{23}^{ν} become large and comparable while $\theta_{13}^{\nu} \sim \theta_c \sim \sqrt{\alpha_{GUT}} \sim 0.2$ [198] [1]. Understanding the origin of the above relation from first principles is beyond the scope of this note. We stress however that this ansatz can be associated to specific flavor symmetries as suggested in Ref.[195] or Ref. [86] , rather than being a mere "numerical coincidence".

A successful framework for attacking the flavor problem constitutes an important quest in contemporary particle physics. A relevant question arises as to whether attempted solutions to the flavour problem may indicate foot-prints of unification or not. In the present note we look into some possible links between quark and lepton mixing parameters from a phenomenological "bottom-up perspective" [2]

In the quark sector the largest mixing is between the flavor states d and s, and is interpreted in terms of the Cabibbo angle [203] which is approximately 13^0. The matrix V_{CKM} is parametrized in terms of three independent angles and one complex CP phase [17, 204, 205]. A clever approximate presentation was proposed by Wolfenstein [206], and is by now standard, namely

$$V_{CKM} = \begin{bmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{bmatrix} \tag{5.1}$$

up to $\mathcal{O}(\lambda^4)$ where, λ, A, η and ρ are four independent Wolfenstein parameters, with $\lambda = \sin\theta_c \approx 0.22$.

In contrast, the mixing in the lepton sector is very different from quark mixing. While the solar and atmospheric angles are large: $\theta_{12} \approx 35^0$ and $\theta_{23} \approx 49^0$, the 1-3 mixing parameter in the lepton sector is the smallest and was believed to vanish according to the earlier results. However in last few years it has been established [128–130] that this mixing, now precisely measured, is almost as large as the d-s mixing in quark sector, $\theta_{13} \approx 9^0 \sim \mathcal{O}(\theta_c)$. This excludes the simplest proposed

[1] Neglecting the contribution from the charged lepton sector.

[2] An earlier alternative in the literature is "Quark-Lepton complementarity (QLC)"[89, 90, 199–202] .

schemes of neutrino mixing, which need to be revised in order to be consistent with observation [207].

5.2 Parametrization of Bi-Large neutrino mixing matrix

Up to Majorana phases the Bi-Large mixing factor may be parametrized as follows,

$$U_{BL} \approx \begin{bmatrix} c\left(1-\frac{\lambda^2}{2}\right) & s\left(1-\frac{\lambda^2}{2}\right) & \lambda e^{-i\delta} \\ -sc' - cs'\lambda e^{i\delta} & c'c - s's\lambda e^{i\delta} & s'\left(1-\frac{\lambda^2}{2}\right) \\ s's - c'c\lambda e^{i\delta} & -cs' - c'e^{i\delta}s\lambda & c'\left(1-\frac{\lambda^2}{2}\right) \end{bmatrix}, \tag{5.2}$$

with,

$$\tilde{J}_{CP} \approx -csc's'\lambda \sin\,\delta. \tag{5.3}$$

Where, s and s' represents the sines of solar and atmospheric angles and that c and c' represents the cosines of the same respectively. One sees that $\sin\theta_{12} = \sin\theta_{23} = \psi\lambda$, with $\sin\theta_{13} = \lambda$. With this parametrization it is evident that the Cabibbo angle is the *seed* for the mixing in both the quark and the lepton sector. In what follows we discuss the possible forms of the charged lepton contribution to the lepton mixing matrix.

5.3 Charged lepton diagonalizing matrix

In simplest SO(10) schemes the charged lepton mass matrix is approximated to that of down type quarks, $M_e \sim M_d$. Hence we have a choice [208],

$$\textbf{Type-I} \qquad U_l = V_{CKM}. \tag{5.4}$$

Similarly, within simplest SU(5) scheme one expects, $M_e \sim M_d^T$. Hence, we have another choice,

$$\textbf{Type-II} \qquad U_l = V_{CKM}^\dagger. \tag{5.5}$$

The physical lepton mixing matrix is simply

$$U_{lep} = U_l^{\dagger}.U_{BL}.I_{\phi}\,, \tag{5.6}$$

where U_{BL}, represents the Bi-Large neutrino mixing matrix and $I_{\phi} = diag(e^{i\alpha}, e^{i\beta}, 1)$, where α and β are the two additional CP violating phases associated to the Majorana nature of the neutrinos [209] [3]. One important point that is to be emphasized on is the position of positive and negative sign before the 1-2 and 2-3 elements of U_{ν}. If we are working in the basis where M_e is diagonal this sign ambiguity hardly matters. But on the contrary, when we stick to a particular texture of U_l, a particular sign convention may alter the prediction. So, let us look into the different possibilities related with the sign of θ_{12}^{ν} and θ_{23}^{ν}. In connection with the BL mixing for neutrinos, the following four possibilities are prominent.

$$\mathbf{A} \qquad s = -s' = \psi\lambda, \tag{5.7}$$
$$\mathbf{B} \qquad s = s' = \psi\lambda, \tag{5.8}$$
$$\mathbf{C} \qquad -s = s' = \psi\lambda, \tag{5.9}$$
$$\mathbf{D} \qquad -s = -s' = \psi\lambda. \tag{5.10}$$

Perhaps, it would have been more significant if we redefine the original BL ansatz, rather in the way, $|s| = |s'| = \psi\lambda$, than $s = s' = \psi\lambda$. If U_l is *exactly* V_{CKM} (or $V_{CKM}^{\dagger}$), we encounter different schemes like: **Type-IA**, **Type-IB**, and **Type-IIA** etc.

5.3.1 When U_l is exactly V_{CKM} (or, $V_{CKM}^{\dagger}$)

The **Type-I** (and **Type-II**) U_l on association with U_{BL} in Eq. (5.2) generates the following expressions for mixing angles.

$$s_{12}^2 \approx s^2 \mp 2c'cs\lambda + \left(c'^2c^2 + s'^2s^2 - s^2\right)\lambda^2, \tag{5.11}$$
$$s_{13}^2 \approx \left(1 + s'^2 \mp 2s'\cos\delta\right)\lambda^2, \tag{5.12}$$
$$s_{23}^2 \approx s'^2 + \lambda^2\left(s'^4 \mp 2Ac's' \mp 2s'^3\cos\delta - s'^2 \pm 2s'\cos\delta\right), \tag{5.13}$$

[3] As shown in [210] these phases are physical and affect lepton number violating processes such as neutrinoless double beta decay [211, 212].

upto $\mathcal{O}(\lambda^3)$. Since 1-3 angle in V_{CKM} (and hence in U_l) is very small, $\theta_{13}^q \sim \mathcal{O}(\lambda^3)$, it's contribution is very little towards the prediction of the above parameters. But the θ_{13}^q, masks the CP violating phase in CKM sector and in order to understand how CKM parameters may contribute towards δ_{CP} in lepton sector, we express J_{CP} upto $\mathcal{O}(\lambda^3)$.

$$\begin{aligned} J_{CP} \approx \tilde{J}_{CP} &\pm \lambda^2 \left(c^2 s' - c^2 s'^3 - s^2 c'^2 s'\right) \sin\delta \\ &\pm \lambda^3 cs(Ac'^2 - As'^2 - 2c's'^2 \cos\delta \pm 3c's') \sin\delta \\ &\pm Acc'^2 ss'\eta\lambda^3. \end{aligned} \tag{5.14}$$

So, here we see that there exists a constant background , $Acc'^2 ss'\eta\lambda^3$ in J_{CP} and exists even if the internal CP phase $\delta = 0$. But this contributes little towards the observable J_{CP}. Hence if U_l is exactly V_{CKM} the presence of internal CP phase, δ is undeniable. We express δ_{CP} as in the following,

$$\begin{aligned} \textbf{Type-I}: \delta_{CP} &\approx n\pi - \tan^{-1}\frac{\sin\delta}{\cos\delta - s'} \\ &+\frac{Ac'\lambda^2}{1 - 2s'\cos\delta + s'^2}(\eta\cos\delta - \rho\sin\delta + \sin\delta - \eta s'), \end{aligned} \tag{5.15}$$

$$\begin{aligned} \textbf{Type-II}: \delta_{CP} &\approx n\pi - \tan^{-1}\frac{\sin\delta}{\cos\delta + s'} \\ &-\frac{Ac'\lambda^2}{1 + 2s'\cos\delta + s'^2}(\eta\cos\delta - \rho\sin\delta - \eta s'), \end{aligned} \tag{5.16}$$

In the above expressions only two free parameters ψ and δ are there. How to choose ψ and δ? In fact, this task is not too complicated. One can choose ψ and δ in such a way, that any two of the three observable parameters, solar, reactor and atmospheric mixing angles are consistent with the neutrino oscillation data [1, 132], while the prediction for the remaining one will determine the tenability of the model.

First note that the determination of solar and reactor angles is rather stable irrespective of the neutrino mass spectrum. Hence it seems reasonable to use solar and reactor angles for the parametrization of the two unknowns. Hence we focus upon the predictions for θ_{23} and δ_{CP}, given their current indeterminacy from global neutrino oscillation data analysis [1].. Although consistent with maximal mixing, the possibility of θ_{23} lying within the first octant is certainly not excluded for normal ordering of neutrino masses. Moreover, probing for CP violation in the lepton sector is the next challenge for neutrino oscillation experiments. Hence

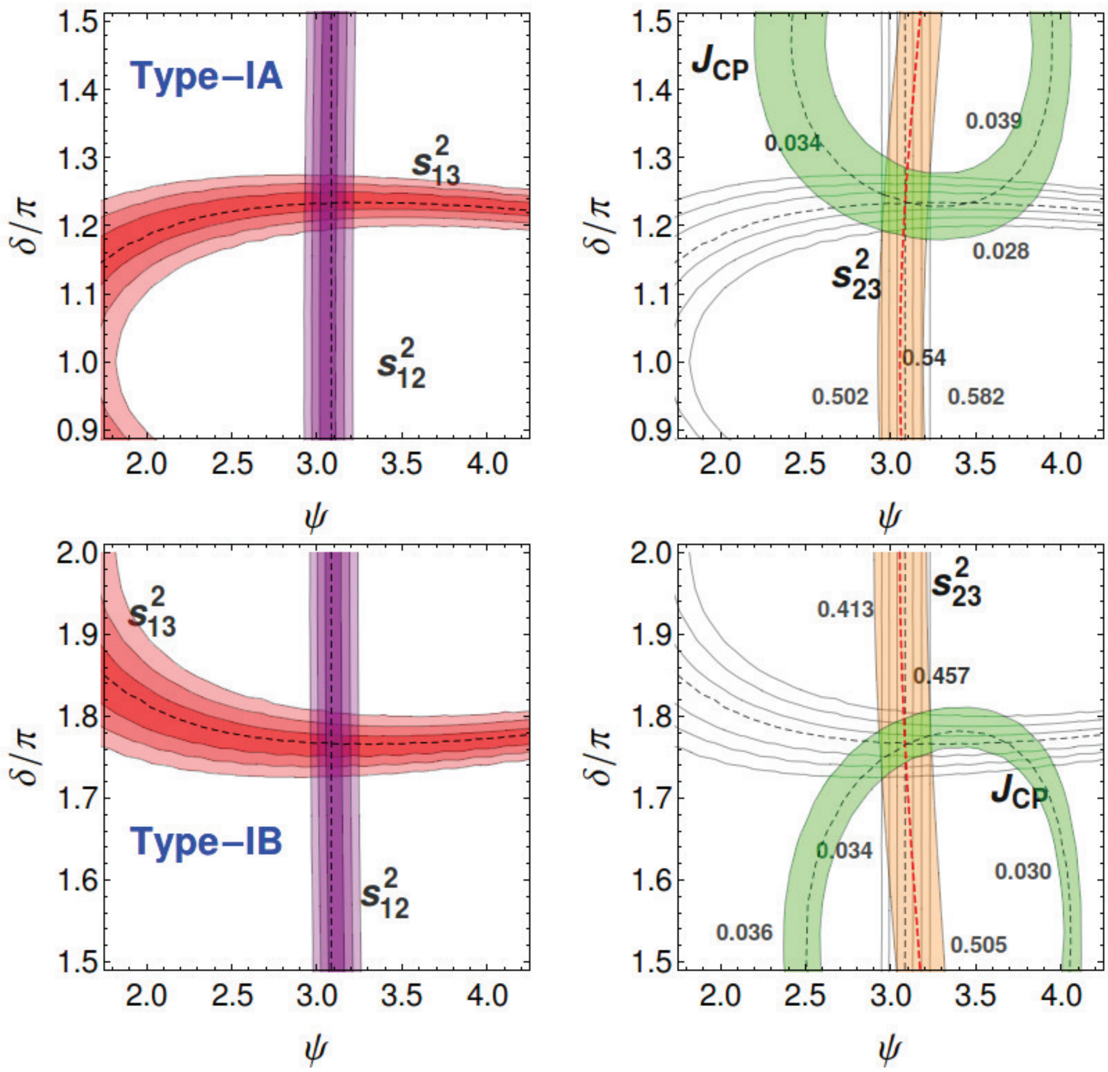

FIGURE 5.1: We show the overlap of the contour-plots corresponding to the central value,1σ, 2σ and 3σ ranges of s_{13}^2 and s_{23}^2 in $\psi - \delta$ plane for **Type-IA** (**Top-Left**) and **Type-IB** (**Bottom-Left**) schemes. The prediction for s_{23}^2 and J_{CP} in the overlapping region is studied for **Type-IA** (**Top-Right**) and **Type-IB** (**Bottom-Right**) respectively.

in addition to the prediction for the atmospheric angle, we use the prediction of our ansatz for δ_{CP} in order to scrutinize the viability of our ansatz, in any of the above forms. The results are summarized in Table 5.1. For definiteness we present a detailed discussion concerned only with the results for the **Type-IA** scheme (which obeys $s = -s' = \psi\lambda$), and similar results can be found for the other cases in the Table. (5.1). For numerical part, we fix the CKM parameters at their central values. In Fig. (5.1) we plot the free parameters δ and ψ. In the left panel we show the contour plot for s_{13} (horizontal band) and s_{12} (vertical band). The best fit value $s_{12}^2 \approx 0.323$ and $s_{13}^2 \approx 0.023$ [1] correspond to choosing $\psi \approx 3.08$

and $\delta \approx 1.23\,\pi$. The above choice leads to, $\theta_{23} \approx 47^0$. The CP phase, δ_{CP} is approximately $1.51\,\pi$, indicating a maximal CP violation. Similarly, for **Type-IB** scheme, which obeys, $s = s' = \psi\lambda$, the choice, $\psi \approx 3.08$ and $\delta \approx 1.765\pi$, gives rise to the best-fit value of θ_{13} and θ_{12}. But it predicts, $\theta_{23} \approx 42^0$ and $\delta_{CP} \approx 1.47\,\pi$. We see that the **Type-IA** scheme supports θ_{23} to lie within the second octant and that **Type-IB** scheme sets θ_{23} in the first.

Out of all the possible schemes, we single out only four possibilities: **Type-IA**, **Type-IIC**, **Type-IB** and **Type-IID**. The rest fails in predicting the atmospheric angle precisely within 3σ error. Interestingly, out of the remaining four, the **Type-IA** and **Type-IIC** describes almost similar landscape. And, the same observation holds good for the schemes **Type-IB** and **Type-IID**. The lepton mixing matrix corresponding to **Type-IA** scheme is presented as shown in the following,

$$U \approx \begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{bmatrix}, \tag{5.17}$$

where,

$$\begin{aligned}
U_{e1} &\approx c - c\lambda^2 - ce^{i\delta}s\lambda^2 + cs\lambda, && (5.18)\\
U_{e2} &\approx s - \lambda c^2 - e^{i\delta}s^2\lambda^2 - s\lambda^2, && (5.19)\\
U_{e3} &\approx e^{-i\delta}\lambda + s\lambda, && (5.20)\\
U_{\mu 1} &\approx As^2\lambda^2 + \frac{1}{2}cs\lambda^2 + c\lambda + ce^{i\delta}s\lambda - cs, && (5.21)\\
U_{\mu 2} &\approx s\lambda - \frac{1}{2}\lambda^2c^2 + c^2 - As\lambda^2c + e^{i\delta}s^2\lambda, && (5.22)\\
U_{\mu 3} &\approx e^{-i\delta}\lambda^2 - Ac\lambda^2 + s\lambda^2 - s, && (5.23)\\
U_{\tau 1} &\approx -e^{i\delta}\lambda c^2 - As\lambda^2c - s^2, && (5.24)\\
U_{\tau 2} &\approx Ac^2\lambda^2 - ce^{i\delta}s\lambda + cs, && (5.25)\\
U_{\tau 3} &\approx c - As\lambda^2 - \frac{c\lambda^2}{2}. && (5.26)
\end{aligned}$$

Type	ψ	δ/π	$\sin^2\theta_{23}$	δ_{CP}/π	$\lvert J_{CP}\rvert$
IA, **IIC**	$3.085^{+0.145}_{-0.137}$	$1.234^{+0.0426}_{-0.0417}$	$0.5398^{+0.0419}_{-0.0382}$	$1.509^{+0.0615}_{-0.0533}$	$0.0344^{+0.0052}_{-0.0061}$
				$1.520^{+0.0655}_{-0.0566}$	$0.0333^{+0.0053}_{-0.0061}$
IB, **IID**	$3.085^{+0.145}_{-0.137}$	$1.766^{+0.0417}_{-0.0426}$	$0.457^{+0.0481}_{-0.0445}$	$1.467^{+0.0303}_{-0.0312}$	$0.03408^{+0.0024}_{-0.0038}$
				$1.476^{+0.0305}_{-0.0314}$	$0.0332^{+0.0024}_{-0.0038}$

TABLE 5.1: Summary of the results corresponding to four BL schemes: **Type-IA, IB, IIC** and **IID** are shown. The **Type-I** and **Type-II** corresponds to the choices $U_l = V_{CKM}$ and $U_l = V_{CKM}^\dagger$ respectively. Also, **A**, **B**, **C** and **D** are associated with different sign convention discussed in Eqs. (5.7)-(5.10). Out of all possible combinations only **IA**, **IB**, **IIC** and **IID** survive and the rest are ruled out because those do not reproduce fruitful prediction of θ_{23}. The parameters ψ and δ are calibrated with respect to the central value $\pm$ 3σ range of s_{12} and s_{23} and are used to predict the observational parameters s_{23} and J_{CP} (or δ_{CP}).

5.3.2 When U_l is ***CKM-like***

Now let us turn towards the scenario, when U_l is not exactly V_{CKM} or $V_{CKM}^\dagger$, but *CKM-like*, which is a more generalized texture of U_l. Without disturbing the angular part we include two new complex phases ϕ_{12} and ϕ_{23} in the texture of V_{CKM}. A simplified texture of such a scenario is presented below.

$$\textbf{Type-III} \qquad U_l = \begin{bmatrix} 1-\frac{\lambda^2}{2} & \lambda e^{-i\phi_{12}} & A\lambda^3(\rho - i\eta) \\ -\lambda e^{i\phi_{12}} & 1-\frac{\lambda^2}{2} & A\lambda^2 e^{-i\phi_{23}} \\ A\lambda^3\left(e^{i\phi_{12}+i\phi_{23}} - \rho - i\eta\right) & -A\lambda^2 e^{i\phi_{23}} & 1 \end{bmatrix} \tag{5.27}$$

5.3.2.1 The Standard parametrization

The concerned PMNS matrix (motivated by **Type-III** U_l) predicts,

$$s_{13}^2 \approx \lambda^2\left(1 - 2s'\cos(\delta-\phi_{12}) + s'^2\right), \tag{5.28}$$

$$s_{12}^2 \approx s^2 - 2csc'\lambda\cos\phi_{12} + \lambda^2\left(c^2c'^2 - s^2 + s^2s'^2\right), \tag{5.29}$$

$$\begin{aligned} s_{23}^2 \approx\; & s'^2 + \lambda^2(s'^4 - 2Ac's'\cos\phi_{23} - 2s'^3\cos(\delta-\phi_{12}) \\ & +2s'\cos(\delta-\phi_{12}) - s'^2). \end{aligned} \tag{5.30}$$

Also, we have the expression for Dirac CP phase as in the following,

$$\begin{aligned}\delta_{CP} \approx\;& n\pi - \tan^{-1}\left(\frac{\sin\delta - s'\sin\phi_{12}}{\cos\delta - s'\cos\phi_{12}}\right)\\ &- \frac{\lambda^2}{\frac{(\sin\delta - s'\sin\phi_{12})^2}{1+(\cos\delta - s'\cos\phi_{12})^2}}\{\frac{2Ac'\sin(\phi_{12}+\phi_{23}) - 2A\eta c' - \sin\delta + s'\sin\phi_{12}}{2\cos\delta - 2s'\cos\phi_{12}}\\ &+ \frac{(2\sin\delta - 2s'\sin\phi_{12})(2A\rho c' - 2Ac'\cos(\phi_{12}+\phi_{23}) + \cos\delta - s'\cos\phi_{12})}{4(\cos\delta - s'\cos\phi_{12})^2}\}\\ &, \end{aligned} \tag{5.31}$$

and, the J_{CP} factor appears as shown below,

$$\begin{aligned}J_{CP} \approx\;& \tilde{J}_{CP} + \lambda csc's'^2\sin\phi_{12} + \lambda^2\left(c^2s' - c^2s'^3 - s^2c'^2s'\right)\sin(\delta-\phi_{12}) + Acss'c'^2\eta\lambda^3\\ &+\lambda^3\{Acsc'^2\sin(\delta-\phi_{23}) - Acsc'^2s'\sin(\phi_{12}-\phi_{23}) - Acsc'^2s'\sin(\phi_{12}+\phi_{23})\\ &-Acss'^2\sin(\delta+\phi_{23}) + Acss'^3\sin(\phi_{12}+\phi_{23}) - csc's'^2\sin(2\delta-\phi_{12})\\ &+3csc's'\sin\delta - csc's'^2\sin\phi_{12} - csc'\sin\phi_{12}\}.\end{aligned} \tag{5.32}$$

Note, if U_l is V_{CKM} (or, $V_{CKM}^\dagger$), only neutrino sector contributes towards δ_{CP} (or, J_{CP}). But on the contrary, if U_l is *CKM-like*, a nonzero δ_{CP} is obtainable even if there is no contribution from the neutrino sector . In the light $\delta = 0$, we concentrate on the **Type-III$\tilde{\mathbf{B}}$** scenario, where $\tilde{\mathbf{B}}$ designates the sign convention shown in Eq.(5.8)), followed by the choice, $\delta = 0$. In the absence of δ, the prediction for θ_{13} and θ_{12} are dependent on ψ and ϕ_{12} only, whereas for θ_{23} and δ_{CP}, the phase angle, ϕ_{23} plays a dominant role. On choosing $\psi \simeq 2.96$ and $\phi_{12} \simeq 0.233\,\pi$, we obtain $\sin^2\theta_{13} \approx 0.023$ and $\sin^2\theta_{12} \approx 0.323$. If, $\phi_{23} \approx \pi$, we obtain $\theta_{23} \approx 45^0$, $\delta_{CP} \approx 1.23\pi$ and $J_{CP} \approx 0.021$ (See Fig. (5.2) and Table. (5.2)).

The corresponding PMNS matrix is shown as in the following,

$$U \approx \begin{bmatrix} U_{e1} & U_{e2} & U_{e3}\\ U_{\mu1} & U_{\mu2} & U_{\mu3}\\ U_{\tau1} & U_{\tau2} & U_{\tau3}\end{bmatrix} \tag{5.33}$$

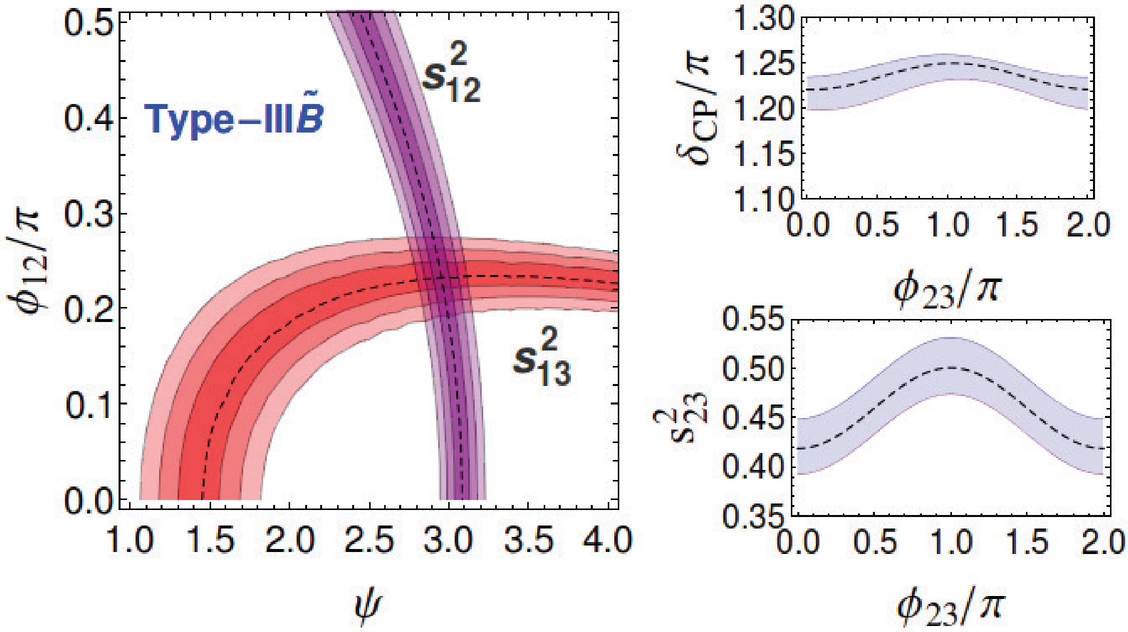

FIGURE 5.2: The **Type-III$\tilde{\textbf{B}}$** corresponds to the case when U_l is *CKM-like* and the $s = s' = \psi\lambda$, followed by $\delta = 0$, (No CP contribution from neutrino sector.) Prediction of s_{13}, s_{12} and s_{23} depends on the free parameters ψ and δ. Again, s_{23} and δ_{CP} are also dependent on ϕ_{23}. We calibrate ψ and δ in terms of the overlapping of the contour-plots corresponding to central values, 1σ, 2σ and 3σ ranges of s^2_{12} and s^2_{13} (**Left**). We use these output to predict s_{23} and δ_{CP}. But, s_{23} and δ_{CP} are dependent on ϕ_{23} also. We study the variation for prediction of s^2_{23} (**Right-Bottom**) and δ_{CP} (**Right-Top**) with respect to ϕ_{23}.

Type-III$\tilde{\textbf{B}}$	
ψ	$2.955^{+0.114}_{-0.103}$
δ/π	$0.233^{+0.043}_{-0.0436}$
s^2_{23}	$0.460^{+0.030}_{-0.026} - 0.0411^{+0.0002}_{-0.0004} \cos\phi_{23}$
δ_{CP}/π	$1.235^{+0.0117}_{-0.0202} - 0.0017^{+0.0036}_{-0.0024} \sin\,\phi_{23} - 0.0145^{+0.0017}_{-0.0021} \cos\,\phi_{23}$
J_{CP}	$0.0211^{+0.0054}_{-0.0051}$

TABLE 5.2: The results corresponding to **Type-III$\tilde{\textbf{B}}$** are summarized.

, where,

$$
\begin{aligned}
U_{e1} &\approx c - c\lambda^2 + ce^{-i\phi_{12}}s\lambda^2 + ce^{-i\phi_{12}}s\lambda, && (5.34)\\
U_{e2} &\approx s - e^{-i\phi_{12}}\lambda c^2 + e^{-i\phi_{12}}s^2\lambda^2 - s\lambda^2, && (5.35)\\
U_{e3} &\approx \lambda - e^{-i\phi_{12}}s\lambda, && (5.36)\\
U_{\mu 1} &\approx As^2\lambda^2 + \frac{1}{2}cs\lambda^2 + ce^{i\phi_{12}}\lambda - cs\lambda - cs, && (5.37)\\
U_{\mu 2} &\approx s\lambda e^{i\phi_{12}} - \frac{1}{2}\lambda^2 c^2 + c^2 - As\lambda^2 c - s^2\lambda, && (5.38)\\
U_{\mu 3} &\approx e^{i\phi_{12}}\lambda^2 + Ac\lambda^2 - s\lambda^2 + s, && (5.39)\\
U_{\tau 1} &\approx -\lambda c^2 + As\lambda^2 c + s^2, && (5.40)\\
U_{\tau 2} &\approx -Ac^2\lambda^2 - cs\lambda - cs, && (5.41)\\
U_{\tau 3} &\approx -As\lambda^2 - \frac{c\lambda^2}{2} + c. && (5.42)
\end{aligned}
$$

Again, a similar possibility **Type-III$\tilde{\mathbf{D}}$** (where, $s = s' = -\psi\lambda$ and $\delta = 0$) gives rise to the same result if we perturb the previous choices of ϕ_{12} and ϕ_{23} by an amount of π. For simple visualization, we present the texture of *CKM-like* U_ls associated with **Type-III$\tilde{\mathbf{B}}$** and **Type-III$\tilde{\mathbf{D}}$** scenarios in their respective order as shown in the following (for, $\phi_{23} = \pi$ and 2π respectively.),

$$
\begin{bmatrix} 1-\frac{\lambda^2}{2} & \lambda e^{-i\phi_{12}} & 0 \\ -\lambda e^{i\phi_{12}} & 1-\frac{\lambda^2}{2} & -A\lambda^2 \\ 0 & A\lambda^2 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1-\frac{\lambda^2}{2} & -\lambda e^{-i\phi_{12}} & 0 \\ \lambda e^{i\phi_{12}} & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ 0 & -A\lambda^2 & 1 \end{bmatrix} \qquad (5.43)
$$

The possibilities like **Type-III$\tilde{\mathbf{A}}$** and **Type-III$\tilde{\mathbf{C}}$** schemes fail to fit within the 3σ error of oscillation data.

If δ is not equal to zero, then we have four unknown parameters: ψ, δ, ϕ_{12} and ϕ_{23}, which are connected by four equations (see Eqs.(5.28)-(5.31)). It seems that the parametrization is least predictive.

Let us discuss the **Type-IIIB** scenario in details. A choice of $\psi = 3.01544$ (3.08379), $\delta/\pi = 0.405362\,(0.259755)$, $\phi_{12}/\pi = 0.171825\,(0.0256385)$ and $\phi_{23}/\pi =$ 1.05303 (0.963141), set $s_{12}^2 \approx 0.323$, $s_{13}^2 \approx 0.023$ and $\delta_{CP}/\pi \approx 1.34\,(1.48)$ and sets $s_{23}^2 \approx 0.518\,(0.539)$. It is found that the prediction of s_{23}^2 is little constrained. A similar discussion holds good for other three (*CKM-like*) **Type-III** U_l motivated scenarios also. It is found that for all the three cases, $\psi \simeq 3$. Choosing the parameters, ψ, δ, ϕ_{12} and ϕ_{23} corresponding to the 1σ or 3σ range of the observational

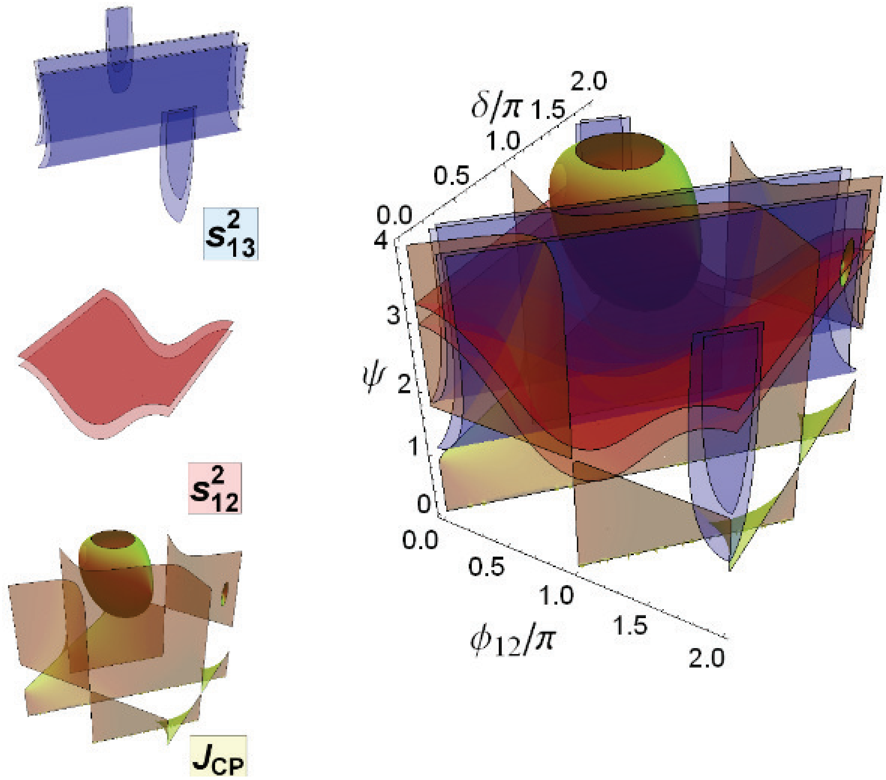

FIGURE 5.3: For the **Type-III** U_l (having the general *CKM-like* texture), we see that the prediction of s_{13}, s_{12} and J_{CP} (neglecting the $\mathcal{O}(\lambda^3)$ contribution) are dependent on the three parameters ψ, ϕ_{12} and δ. We see that the 3D contour-plots corresponding to the 3σ range of s_{13}^2(**Left-Top**), s_{12}^2 (**Left-middle**) and J_{CP} (**Left-Bottom**) in the three dimensional plane, ψ-δ-ϕ_{12}, overlap in a very complicated way (**Right**). We can not include the observational parameter s_{23}^2, because it depends on four parameters including ϕ_{23}. The above complexity in the overlapping makes it difficult to obtain a range of the parameters (ψ, δ, ϕ_{12})

parameters is little difficult, and we keep aside this task in the present discussion (See Fig.(5.3)).

5.3.2.2 The Symmetric parametrization

Till now we are dealing with the parametrization from standard parametrization point of view. We shall look into the general phase texture in the light of symmetric parametrization [17].

We present U_{BL} in the following way,

$$U_{BL} \approx \begin{bmatrix} c\left(1-\frac{\lambda^2}{2}\right) & s\left(1-\frac{\lambda^2}{2}\right)e^{-i\delta_{12}} & \lambda e^{-i\delta_{13}} \\ -sc'e^{i\delta_{12}} - s'c\lambda e^{i\delta_{13}-i\delta_{23}} & cc' - ss'\lambda e^{-i\delta_{12}+i\delta_{13}-i\delta_{23}} & s'\left(1-\frac{\lambda^2}{2}\right)e^{-i\delta_{23}} \\ ss'e^{i\delta_{12}+i\delta_{23}} - cc'\lambda e^{i\delta_{13}} & -sc'\lambda e^{i\delta_{13}-i\delta_{12}} - cs'e^{i\delta_{23}} & c'\left(1-\frac{\lambda^2}{2}\right) \end{bmatrix} \tag{5.44}$$

$$\tilde{J}_{CP} \approx csc's'\lambda \sin\left(\delta_{12} - \delta_{13} + \delta_{23}\right). \tag{5.45}$$

The mixing angles are,

$$s_{13}^2 \approx \lambda^2\left(1 - 2s'\cos\left(\delta_{13} - \delta_{23} - \phi_{12}\right) + s'^2\right), \tag{5.46}$$

$$s_{12}^2 \approx s^2 - 2csc'\lambda\cos\left(\delta_{12} - \phi_{12}\right) + \lambda^2\left(c^2c'^2 - s^2 + s^2s'^2\right), \tag{5.47}$$

$$\begin{aligned} s_{23}^2 \approx\ & s'^2 + \lambda^2\{-2Ac's'\cos\left(\delta_{23} - \phi_{23}\right) - 2s'^3\cos\left(\delta_{13} - \delta_{23} - \phi_{12}\right) \\ & +2s'\cos\left(\delta_{13} - \delta_{23} - \phi_{12}\right) + s'^4 - s'^2\}. \end{aligned} \tag{5.48}$$

But for the present case, it is difficult to have an simplified expression for δ_{CP}. We put forward the following expression for J_{CP},

$$\begin{aligned} J_{CP} \approx\ & \tilde{J}_{CP} - csc'\lambda s'^2 \sin\left(\delta_{12} - \phi_{12}\right) \\ & +\lambda^2\left(-c^2s'^3 + c^2s' - s^2c'^2s'\right)\sin\left(\delta_{13} - \delta_{23} - \phi_{12}\right). \end{aligned} \tag{5.49}$$

Along-with the parameter ψ, we have other five unknowns: ϕ_{12}, ϕ_{23}, δ_{12}, δ_{13} and δ_{23}. But we see that the observational parameters involve the following four combinations,

$$\delta_{12} - \delta_{13} + \delta_{23}, \tag{5.50}$$

$$\delta_{13} - \delta_{23} - \phi_{12}, \tag{5.51}$$

$$\delta_{12} - \phi_{12}, \tag{5.52}$$

$$\delta_{23} - \phi_{23}, \tag{5.53}$$

out of which only following the three combinations are independent,

$$\delta_{12} - \delta_{13} + \delta_{23}, \tag{5.54}$$

$$\delta_{12} - \phi_{12}, \tag{5.55}$$

$$\delta_{23} - \phi_{23}. \tag{5.56}$$

Equivalence of both "standard" and "symmetric" parametrization schemes (See (5.46)-(5.48) and Eqs.(5.28)-(5.30)) suggests,

$$\delta \longleftrightarrow \delta_{12} - \delta_{13} + \delta_{23}, \tag{5.57}$$

$$(\textbf{Standard}) \qquad \phi_{12} \longleftrightarrow \phi_{12} - \delta_{12}, \qquad (\textbf{Symmetric}) \tag{5.58}$$

$$\phi_{23} \longleftrightarrow \phi_{23} - \delta_{23}. \tag{5.59}$$

So comparing Eq. (5.31) and Eq. (5.60), we can derive an equivalent expression of δ_{CP} for symmetric parametrization scheme also,

$$\begin{aligned}
\delta_{CP} \approx\; & n\pi - \tan^{-1}\left(\frac{\sin(\delta_{12}-\delta_{13}+\delta_{23}) - s'\sin(\phi_{12}-\delta_{12})}{\cos(\delta_{12}-\delta_{13}+\delta_{23}) - s'\cos(\phi_{12}-\delta_{12})}\right) \\
& - \frac{\lambda^2}{\frac{(\sin(\delta_{12}-\delta_{13}+\delta_{23})-s'\sin(\phi_{12}-\delta_{12}))^2}{1+(\cos(\delta_{12}-\delta_{13}+\delta_{23})-s'\cos(\phi_{12}-\delta_{12}))^2}} \times \\
& \{\frac{2Ac'\sin(\phi_{12}-\delta_{12}+\phi_{23}-\delta_{23}) - 2A\eta c' - \sin(\delta_{12}-\delta_{13}+\delta_{23}) + s'\sin(\phi_{12}-\delta_{12})}{2\cos(\delta_{12}-\delta_{13}+\delta_{23}) - 2s'\cos(\phi_{12}-\delta_{12})} \\
& + \frac{(2\sin(\delta_{12}-\delta_{13}+\delta_{23}) - 2s'\sin(\phi_{12}-\delta_{12}))(2A\rho c' - 2Ac'\cos(\phi_{12}-\delta_{12}+\phi_{23}-\delta_{23}))}{4(\cos(\delta_{12}-\delta_{13}+\delta_{23}) - s'\cos(\phi_{12}-\delta_{12}))^2} \\
& + \frac{(2\sin(\delta_{12}-\delta_{13}+\delta_{23}) - 2s'\sin(\phi_{12}-\delta_{12}))(\cos(\delta_{12}-\delta_{13}+\delta_{23}) - s'\cos(\phi_{12}-\delta_{12}))}{4(\cos(\delta_{12}-\delta_{13}+\delta_{23}) - s'\cos(\phi_{12}-\delta_{12}))^2}\}
\end{aligned} \tag{5.60}$$

As before, we have four unknowns: ψ, $\delta_{12} - \delta_{13} + \delta_{23}$, $\delta_{12} - \phi_{12}$, and $\delta_{23} - \phi_{23}$. On choosing, $\psi = 3.01544\,(3.08379)$, $(\delta_{12} - \delta_{13} + \delta_{23})/\pi = 0.405362\,(0.259755)$, $(\delta_{12} - \phi_{12})/\pi = 0.171825\,(0.0256385)$ and $(\delta_{23} - \phi_{23})/\pi = 1.05303\,(0.963141)$, we obtain $s_{12}^2 \approx 0.323$, $s_{13}^2 \approx 0.023$ and $\delta_{CP}/\pi \approx 1.34\,(1.48)$ and $s_{23}^2 \approx 0.518\,(0.539)$.

5.4 Discussion

In summary we proposed a generalized fermion mixing ansatz where the neutrino mixing is Bi-Large, while the charged lepton mixing matrix is "CKM" or " *CKM-like*". Inspired by SO(10) and SU(5) unification, we select three charged lepton diagonalizing matrices, 'U_l's (**Type-I, II, III**) and discuss the phenomenological viability of the resulting schemes. All the four models are congruous with best-fit solar and reactor angles, making definite predictions for the atmospheric angle and CP phase, which may be further tested in upcoming neutrino experiments. In particular the Type-4 BL model appears interesting in the sense that it extends the original BL model to encompass maximal atmospheric mixing. Ours is a "theory-inspired" bottom-up approach to the flavour problem, that highlights the role of θ_c as the universal seed of quark and lepton mixings and incorporates the main characteristic features of unification models. We have shown how this generalizes the original Bi-Large ansatz [87] to make it fully realistic. Further investigation on the physics underlying this ansatz may bring new insights on both fermion mixing and unification.

Chapter 6

Summary and conclusion

What we have covered

In Chapter 1, we have displayed a summarized picture of neutrinos. Here, we have highlighted why the research in neutrino physics is important. With a brief historical introduction to neutrinos, we have discussed different theoretical prospects related with mass generation of neutrinos which includes the discussion of Dirac and Majorana behavior of neutrinos, and See-Saw mechanism. We have discussed "Neutrino-mixing" and reviewed the parametrization of PMNS matrix. We have highlighted briefly the diversity of neutrino oscillation processes involving neutrino-neutrino oscillation, discussed in short about matter effect and have highlighted the present status of the observable parameters related with the oscillation experiments. We have reviewed in short on beta-decay, neutrino-less double beta-decay experiments, cosmological bounds on sum of the three neutrino masses in context with the measurement of absolute neutrino mass scale. The CP violation arising due to the Dirac and Majorana nature is discussed in terms of Jarlskog (significant for neutrino-neutrino oscillation) and Jarlskog-like parameters (significant for neutrino-anti neutrino oscillation) respectively. We have underlined two strategies called "Top-down" and "Bottom-up", usually followed by the theorists to answer the unsolved riddles related with neutrinos. And, finally we have featured the scope of the thesis.

In Chapter 2, we have discussed the phenomenological aspects and the importance of the so called μ-τ symmetry of neutrino mass matrix. In fact, starting from different discrete flavour symmetry groups like A_4 and S_4, one can reach this symmetry. The important feature of latter is that it keeps θ_{12} free. It is consistent with maximal θ_{23} and vanishing θ_{13}. Once, we switch on to the basis where the charged lepton mass matrix is non-diagonal, all the mixing angles get deviated. We assign an unitary texture to the charged lepton diagonalizing matrix in terms of λ, the Wolfenstein parameter such that it drifts θ_{13} and θ_{23} from zero and 45^0 respectively. And, at the same time, θ_{12} is kept untouched. In this way, θ_{12} is controlled solely from neutrino sector and, θ_{13} and θ_{23} are dictated by that of charged leptons. We put forward six different cases of Quasi-degenerate neutrino mass matrix, such as $QDNH$-$Type{-}IA$, IB, IC, and $QDIH$-$Type{-}IA$, IB, IC, where we have used only three parameters: α, η and λ in order to parametrize the mass matrices. The number of free parameters are lesser than that of observational ones. The parametrization respects μ-τ symmetry, if λ is absent. The work in this chapter highlights the existence of four independent building block textures, I_0, I_1, I_2 and I_3, a linear superposition of which begets a general μ-τ symmetric texture. The present parametrization of degenerate neutrino scenario, supports both TBM and TBM deviated scenarios, but not the BM mixing case. In order to keep the discussion simple we have not dealt with the Dirac CP violating phase δ_{CP}. But in fact it can enter the parametrization as an input through the charged lepton diagonalizing matrix. The present parametrization supports for $\theta_{23} \approx 39^0$. But on choosing $\tau = -\lambda/2$, we can propel θ_{23} to acquire a value greater than 45^0 (This possibility is not discussed in Chapter 2). But in Chapter 2, we hardly found a possibility to discriminate the six QDN cases.

In Chapter 3, we have tried to look into other possibilities to parametrize the neutrino mass matrix from a different perspective. This approach takes us beyond the territory of μ-τ symmetry. The quark sector is completely different from that of the leptons. The quark mixing angles can be expressed in terms of the corresponding quark mass ratio. In this chapter we have tried to see what will happen if we consider similar happenings in neutrino sector also. We have found that this is not impossible in neutrino sector but realization of a similar ansatz is partial. We can experience only two possibilities, $\sqrt{m_1 : m_3} \simeq \sin\theta_{13}$ and $\sqrt{m_2 : m_3} \simeq \sin\theta_{23}$, but not simultaneously. This ansatz although cannot answer to the hierarchy issue, but constrains the mass spectrum in several ways. We put forward two different parametrization of U_{PMNS}. But more interestingly we found five different *hierarchy dependent*, *exact* textures of neutrino mass matrix, where the matrix elements are connected by certain sum rules. For example, we experience a texture which is a combination of a Fritzsch-like and $\mu-\tau$ symmetric textures. In this chapter, we have considered the basis where charged lepton mass matrix is diagonal and for simplicity we have considered all the parameters in the mass matrix as real.

In Chapter 4, we have highlighted different possibilities to modify the TBM mixing scenario. The TBC mixing is one of the interesting mixing schemes next to TBM mixing which differs from the former in view of the prediction of θ_{13} ($\theta_{13} = \theta_C/\sqrt{2}$). We have tried to mend the existing TBC mixing scheme by considering the contribution from charged lepton sector and have extended the original scheme to encompass the cases: $\theta_{12} < \sin^{-1}(1/\sqrt{3})$, and $\theta_{23} \leq 45^0$ or $\theta_{23} \geq 45^0$. The Chapter 4 is related to Chapter 2. We have used the unique feature of μ-τ symmetric mass matrix to control θ_{12} from neutrino sector. Whether $\theta_{23} > 45^0$, or $\theta_{23} < 45^0$, depends on the choice of a CKM like correction either, $R_{23} = R_{23}^{-1}(A\,\lambda^2)$ or $R_{23} = R_{23}(A\,\lambda^2)$ respectively. But in this approach, δ, the Dirac CP violating phase remains as an input. In the next part, we have discussed two scenarios, where, either $U_l = V_{CKM-like}$, or $U_l = V^{\dagger}_{CKM-like}$ and all the angles are now controlled from charged lepton sector. The present approach leads to a prediction of the Dirac CP phase, $\delta_{CP} \sim 1.5\,\pi$. The present analysis leads to more realistic textures of U_{PMNS}, deviated from TBM mixing scenario.

In Chapter 5, we have looked into the possibilities to unearth certain framework(s), where we can find some similarities between the lepton and the quark sectors. The Bilarge neutrino mixing hints for $\theta_{13} = \theta_c$. Whereas the SO(10)and SU(5) GUTs indicate $\theta^l_{12} = \theta_c$. We have tried to construct a framework in symmetry basis, $U_{PMNS} = U_l^{\dagger}.U_{BL}$. We have assumed that neutrinos mix in the Bi-large fashion and, have opted for the options: $U_l = V_{CKM}$, $V^{\dagger}_{CKM}$, $V_{general-CKM-like}$. The aim of this parametrization is to find out one unknown parameter ψ, in the neutrino sector, and Dirac-CP phase δ. We have discussed all the situations when δ_{CP} can enter the frame-work either through neutrino or charged lepton sector, and the scenarios where both the sectors contribute towards the CP phase. The analysis predicts certain possibilities covering the cases: $\theta_{23} \geq 45^0$, $\theta_{23} \leq 45^0$ and $\delta_{CP} \sim 1.5\pi$ or 1.3π etc. The analysis predicts the parameter ψ (which is an unphysical parameter in our analysis) to be approximately 3. This prediction coincides with what is worked out in the original reference. In concern with the case when U_l is a general *CKM-like* matrix, we have highlighted scenarios encompassing both "Standard" and "Symmetric" parametrization.

What we have not covered

- In Chapter 2, we are unable to discriminate the six possible cases of QDN models. But we expect that an extension of the work to study Baryogenesis can make this task possible. At the same time, we expect that the analysis will help to get some idea about the interval of the three CP violating phases.

- We have derived the five different mass matrix textures from a model independent perspective. We expect that may be these textures are related to certain flavor symmetry groups and a similar analysis in this line can be extended in future. Also, the Dirac and Majorana phases are to be taken into consideration.

- In Chapter 2 and Chapter 4, we have pointed out certain textures of left handed unitary charged lepton diagonalizing matrices in the light of present lepton mixing observational parameters. We can extend the work to find out what could be the texture of left handed charged lepton mass matrix, M_l or $M_l^\dagger M_l$ and to relate the same to certain horizontal symmetries.

- In Chapter 4, we have highlighted the relevance of the ansatz, $\theta_{13}^{\nu} = \theta_{12}^{l} = \theta_c$, which seems important from the unification point of view. As a future scope, we can try to obtain the same frame-work from a top-down perspective. The present analysis make certain predictions on δ_{CP}. The extension of the same to Baryogenesis scenario, can provide us the information of Majorana CP phases.

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