Supersymmetry and fermionic modes in an oscillon background

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A B S T R A C T

The excitations referred to as oscillons are long-lived time-dependent field configurations which emerge dynamically from non-linear field theories. Such long-lived solutions are of interest in applications that include systems of Condensed Matter Physics, the Standard Model of Particle Physics, Lorentz-symmetry violating scenarios and Cosmology. In this work, we show how oscillons may be accommodated in a supersymmetric scenario. We adopt as our framework simple ($\mathcal{N} = 1$) supersymmetry in $D = 1 + 1$ dimensions. We focus on the bosonic sector with oscillon configurations and their (classical) effects on the corresponding fermionic modes, (supersymmetric) partners of the oscillons. The particular model we adopt to pursue our investigation displays cubic superfield which, in the physical scalar sector, corresponds to the usual quartic self-coupling.

1. Introduction

The study of field-theoretic non-linear systems is an area of increasing interest over the past few decades [1–3]. Evidences of non-linear behavior is, nowadays, found in a considerable part of physical systems. This includes systems of Condensed Matter Physics, field-theoretic models related to particle phenomenology, modern Cosmology and a large number of other domains of the physical sciences [4–34]. As examples, we can cite studies on non-linear acoustic metamaterials (NAMs) [5], in the resonances of two-dimensional (2D) materials [6], in nonlinear sigma models [7], in hard x-ray pulses in free-electron lasers (FELs) [8], and in supersymmetric quantum field theories with kink excitations [9].

In classical field theory, there is a class of configuration quite common and important called solitons [10], which are solutions of a set of classical relativistic non-linear field equations. Such configurations are characterized by some topological index, related to their behavior at spatial infinity. Solitons have the important feature of having energy density localized at space and having its profiles restored to their original shapes and velocities after collisions. Nowadays, those configurations is well understood in a broad number of scenarios. For instance, one can find investigations regarding monopoles, textures, strings and kinks [11].

Within the universe of non-linear field theories, it is important to highlight a specially important class of time-dependent stable solutions referred to in the literature as breathers. This configurations comes from Sine-Gordon like models. Another time-dependent field configuration whose stability is granted for by charge conservation are the Q-balls as named by Coleman [31] or non-topological solitons, according to Lee [32]. On the other hand, besides these physical systems that exhibit a metastable behavior, a new class of solutions was discovered in the years 1970 by Bogolyubsky and Makhankov [33], and re-assessed afterwards by Gleiser [36]. These solutions were identified during the study of the dynamics of first-order phase transitions, where oscillons arise from collapsing unstable spherically-symmetric bubbles in models with symmetric and asymmetric double-well potentials. Since that work, a series of investigations is addressed to study these objects [37–59]. For instance, we can find interesting investigations and consequences in Abelian–Higgs models [60,61], in massive Yang–Mills theories [62], and when there are nonlinear Schrodinger equations [63].

Oscillons do not keep their shape indefinitely. Indeed, for a ϕ^4 theory in $(1 + 1)$ dimension, Segur and Kruskal [34] have shown that oscillon configurations slowly emit their energy. A similar conclusion was obtained in the cases of $(2+1)$ and $(3+1)$ dimensions [35]. Also, the radiation rate for quantized oscillons was worked out by Hertzberg [59], who showed that there is a significant difference between the classical and quantum decays.

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The most common oscillon profile is a bell-like shape with a sinusoidal oscillation. A quite different oscillon configuration may be found in Ref. [53]. In that work, one analyzes the properties of oscillons in an expanding Universe. Interestingly, a new kind of oscillon which presents a plateau at its top is found, and cosmological applications can be implemented. It is also shown that these configurations present a stable behavior against localized perturbations.

In this contribution, we endeavor to present a $(1+1)$-dimensional supersymmetric framework where oscillon configurations show up and it is our interest to focus on the oscillon fermionic partner, that we may refer to as oscillino. The reason to consider oscillons in connection with supersymmetry (from now on, SUSY) is not of a purely academic interest. We take the viewpoint that fermions are the truly elementary matter excitations of space-time. Associating SUSY with oscillons is a natural way to couple oscillons to fermions if we wish to inspect their interaction. In so doing, besides the interest in identifying the profile of the fermionic configurations coupled to oscillons (which we do by building up oscillino-type solutions), it would be interesting to consider the behavior of oscillino condensates to find out whether it is possible to identify them with scalar or vector bosonic excitations present in lower-dimensional systems. Also, in a SUSY scenario, oscillino condensates, in turn, induce correction effects on the oscillons, and it might be interesting to assess how these effects may interfere on the stability of the oscillons. These are the primary motivations underneath our proposal to inspect oscillons in connection with SUSY.

This work is organized as follows: in Section 2, we introduce the general features of the supersymmetric model. After that, in Section 3, we review the oscillon configurations in the purely bosonic sector, with SUSY not yet switched on. Next, in Section 4, we investigate the fermionic partner's solutions in an oscillon background and we discuss on two different paths to get oscillino solutions. Finally, we present our Concluding Comments in Section 5. An Appendix follows, where we set up conventions and present the off-shell version of the SUSY action we adopt to carry out our inspections.

2. The supersymmetric model

The investigation of supersymmetric models in the context of nontrivial classical field configurations finds a remarkable result in the paper by Witten and Olive [64], where a connection between topological configurations and central charges of the SUSY algebra is established. Furthermore, many other field-theoretic models have been discussed in this context. Specifically, in $D = 1 + 1$ dimensions, we highlight a number of seminal works [65–67], which explored non-perturbative classical solutions and non-linear sigma-models. We also point out some works [68–75], where the quantum aspects of (non-)topological solutions in a supersymmetric scenario are discussed. However, one notices a lack of the attention related to oscillon configurations and supersymmetry. Here, we present a first contribution to this topic. In this Section, the Lagrangian density describing a theory in $D = 1 + 1$ dimensions in a supersymmetric framework is introduced. Our aim in working with a supersymmetric theory comes from the fact that, in its linear realization, SUSY provides the existence of a new class of field configurations, which correspond to the supersymmetric partner of the oscillons, named from now on *oscillino*. The on-shell $\mathcal{N} = 1$ supersymmetric Lagrangian density is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{i}{2}\bar{\psi}\gamma^\mu \partial_\mu \psi - \frac{[V_\varphi(\varphi)]^2}{2} - \frac{\bar{\psi}\psi}{2}V_{\varphi\varphi}(\varphi), \tag{1}$$

where φ and ψ are, respectively, a real scalar field and a Majorana spinor. Here $V(\varphi)$, an arbitrary function, is referred to as the prepotential, and we denote $V_\varphi \equiv \partial V/\partial \varphi$ and $V_{\varphi\varphi} \equiv \partial^2 V/\partial \varphi^2$. Moreover, we highlight that the scalar potential is related to $V(\varphi)$ as below:

$$U(\varphi) = \frac{1}{2}[V_\varphi(\varphi)]^2. \tag{2}$$

In the Appendix A, we present our conventions as well as the off-shell Lagrangian density obtained in the superspace formulation.

One can check that the action is invariant under the transformations

$$\delta\varphi = \bar{\xi}\psi, \tag{3}$$

$$\delta\psi = -i\gamma^\mu \xi \partial_\mu \varphi - \xi V_\varphi, \tag{4}$$

where ξ is a Majorana spinor parameter.

From the Lagrangian density (1), we obtain the following coupled equations of motion

$$\Box\varphi + V_\varphi V_{\varphi\varphi} + \frac{\bar{\psi}\psi}{2}V_{\varphi\varphi\varphi} = 0, \tag{5}$$

$$i\gamma^\mu \partial_\mu \psi - \psi V_{\varphi\varphi} = 0. \tag{6}$$

In this work, we investigate the particular case

$$V(\varphi) = \frac{a}{3}\varphi^3 - \frac{b}{2}\varphi^2, \tag{7}$$

where a and b are real (positive) parameters. $V(\varphi)$ is cubic, so that the scalar self-interaction becomes quartic, as given in Eq. (2).

In the sequel, we turn our attention to the problem of how to proceed to decouple the equations. In order to solve the equations to obtain oscillon-type configurations, we consider, as our initial step, the approximation that the interaction with the fermionic condensate $(\bar{\psi}\psi)$ can be neglected. One possible situation where this consideration is not an approximation is the case of half-SUSY, where the spinor field is Majorana–Weyl; but, this is not the case here. We start with the scalar field behaving like a classical background. A non-trivial fermionic solution shall be obtained by solving Eq. (6) in the oscillon background or, equivalently, by perturbing the oscillon configuration by means of a SUSY transformation, (Eqs. (3) and (4)). Thus, Eq. (5) becomes

$$\Box\varphi + V_\varphi V_{\varphi\varphi} = 0. \tag{8}$$

Using the Eq. (7), we can rewrite the Eq. (8) as follows below:

$$\Box\varphi + 2a^2\varphi^3 - 3ab\varphi^2 + b^2\varphi = 0. \tag{9}$$

As one notices, the equation above involves only the scalar sector. In the next Section, we shall work out the oscillons configurations from this scenario.

3. Bosonic sector: the usual oscillons

Since our first interest is to find periodic and localized solutions, it is useful to introduce a scale transformations in x and t, given by

$$\tau = \omega t, \quad y = \epsilon x, \tag{10}$$

with $\omega = \sqrt{1 - \epsilon^2}$ and $0 < \epsilon \leq 1$. Thus, Eq. (9) becomes

$$\omega^2 \partial_\tau^2 \varphi - \epsilon^2 \partial_y^2 \varphi + 2a^2\varphi^3 - 3ab\varphi^2 + b^2\varphi = 0. \tag{11}$$

From this equation, it is possible to find a oscillon configuration, which are localized in the central vacuum $\varphi_v = 0$ of the model described in Eq. (7). In this case, the classical scalar field φ is spatially localized and periodic in time. The usual procedure to obtain oscillon configurations in $D = 1 + 1$ dimensions consists in applying a small amplitude expansion of the scalar field in powers of ϵ in the form that follows:

$$\varphi(y, \tau) = \sum_{j=1}^{\infty} \epsilon^j \varphi_j(y, \tau). \tag{12}$$

Let us replace this expansion of the scalar field into the field equation (11). Doing that yields:

$$\epsilon \left(\partial_\tau^2 \varphi_1 + b^2 \varphi_1 \right) + \epsilon^2 \left(\partial_\tau^2 \varphi_2 + b^2 \varphi_2 - 3ab\varphi_1 \right)$$
$$+ \epsilon^3 \left(\partial_\tau^2 \varphi_3 + b^2 \varphi_3 - \partial_\tau^2 \varphi_1 - \partial_y^2 \varphi_1 - 6ab\varphi_1\varphi_2 + 2a^2\varphi_1^3 \right) + ...$$
$$= 0. \tag{13}$$

We notice that the procedure of performing a small amplitude expansion shows that the scalar field solution φ can be obtained from a set of scalar fields which satisfy coupled non-linear differential equations. This set of differential equations is found by taking the terms to all orders in ϵ in the above equation. Thus, it becomes immediately clear that, up to ϵ^3, this procedure leads to

$$\partial_\tau^2 \varphi_1 + b^2 \varphi_1 = 0, \tag{14}$$

$$\partial_\tau^2 \varphi_2 + b^2 \varphi_2 - 3ab\varphi_1^2 = 0, \tag{15}$$

$$\partial_\tau^2 \varphi_3 + b^2 \varphi_3 - \partial_\tau^2 \varphi_1 - \partial_y^2 \varphi_1 - 6ab\varphi_1\varphi_2 + 2a^2\varphi_1^3 = 0. \tag{16}$$

From Eq. (14), we can propose that

$$\varphi_1(y, \tau) = \Phi(y) \cos(b\tau), \tag{17}$$

where $\Phi(y)$ is function of the spatial variable y. We notice that the lowest order term of the solution is just a harmonic oscillator in time, with frequency b.

On the other hand, the solution to the Eq. (15), which is a linear inhomogeneous equation, can be found by considering the Eq. (17). Thus, the solution for $\varphi_2(y, \tau)$ is written as below:

$$\varphi_2(y, \tau) = \frac{a\Phi(y)^2}{2b} [3 - \cos(2b\tau)]. \tag{18}$$

From these solutions, Eqs. (17) and (18), we can obtain $\varphi_3(y, \tau)$. Then, after straightforward calculations, one can verify that Eq. (16) takes the form

$$\partial_\tau^2 \varphi_3 + b^2 \varphi_3 = \left(\Phi'' - b^2 \Phi + 6a^2\Phi^3 \right) \cos(b\tau) - 2a^2\Phi^3 \cos(3b\tau), \tag{19}$$

where $\Phi'' = d^2\Phi/dy^2$.

Our aim is to get configurations which are periodical in time. Then, if we solve the above partial differential equation in the presented form, we will have a term linear in τ. As a consequence, the solution for φ_3 is neither periodical nor localized. This result comes from the contribution of the function $\cos(b\tau)$ in the right-hand side of the partial differential equation (19). However, we can build up solutions for φ_3 which are periodical in time if we impose that

$$\Phi'' - b^2 \Phi + 6a^2\Phi^3 = 0. \tag{20}$$

In this case one gets

$$\Phi(y) = \frac{b}{a\sqrt{3}} \operatorname{sech}(by). \tag{21}$$

From the above results, as one can see, the leading order corresponding solution for the classical field is given by

$$\varphi(x, t) \simeq \frac{\epsilon b}{a\sqrt{3}} \operatorname{sech}(\epsilon bx) \cos\left(b\sqrt{1 - \epsilon^2}t \right). \tag{22}$$

We note that the bosonic sector accommodates usual oscillons configurations with small amplitude.

Finally, it is worthy to emphasize that we have expanded the scalar field up to 3th order in ϵ, obtaining thereby a system of equations (Eqs. (14)–(16)). The solutions to this system have a localized spatial profile with a sech(by)-functional dependence; the leading contribution is therefore the term proportional to ϵ. Notice that it is necessary to use the expansion up to 3th order in ϵ to obtain Eq. (20), whose solution determines the function $\Phi(y)$, which is responsible for the spatial profile of the oscillons. We then present the leading-order contribution for the scalar, given by Eq. (22). Also, in Ref. [59], it is shown that the leading contribution for the oscillon radiation is of order ϵ, so that the other terms are subleading for the analysis of the oscillon stability. A similar procedure shall be adopted in the fermionic case, in dealing with the oscillino solution.

In the next Section, we shall study the fermionic sector in the background of this oscillon.

4. The SUSY partners of oscillons

Having established the oscillon configuration, we now turn our attention to the fermionic sector. Our goal is to find the supersymmetric partner of the oscillon, which we shall refer to as *oscillino*. To do this, let us start by rewriting the Dirac equation (6) in the presence of the potential (7). In this case, we have

$$i\gamma^\mu \partial_\mu \psi - (2a\varphi - b)\psi = 0. \tag{23}$$

As we are working with two dimensions, the fermions are described by two-component spinors,

$$\psi(x, t) = \begin{pmatrix} \psi_1(x, t) \\ \psi_2(x, t) \end{pmatrix}. \tag{24}$$

As already mentioned in the Appendix, we adopted the Majorana representation of the gamma matrices ($\gamma^0 = \sigma_y$, $\gamma^1 = i\sigma_x$). Therefore, by using this representation and Eqs. (23) and (24) one can arrive at following coupled pair of first order differential equations:

$$\partial_t \psi_2 + \partial_x \psi_2 - S(\varphi)\psi_1 = 0, \tag{25}$$

$$-\partial_t \psi_1 + \partial_x \psi_1 - S(\varphi)\psi_2 = 0, \tag{26}$$

where $S(\varphi) \equiv 2a(\varphi - b/2a)$.

Now, with the purpose of decouple these equations, it is necessary to substitute the Eq. (25) into (26) and vice versa, which leads to the corresponding second order differential equations:

$$\Box\psi_1 + \frac{S_\varphi(\varphi)}{S(\varphi)} [(\partial_t\varphi + \partial_x\varphi)(-\partial_t\psi_1 + \partial_x\psi_1)] + S^2(\varphi)\psi_1 = 0,$$
$$\tag{27}$$

$$\Box\psi_2 - \frac{S_\varphi(\varphi)}{S(\varphi)} [(\partial_t\varphi - \partial_x\varphi)(\partial_t\psi_2 + \partial_x\psi_2)] + S^2(\varphi)\psi_2 = 0.$$
$$\tag{28}$$

As we are interested in obtaining the supersymmetric oscillons, it is natural to apply the scales transformation in x and t introduced in Eq. (10). Thus, the above equations can be rewritten in the form

$$\omega^2 \partial_\tau^2 \psi_1 - \epsilon^2 \partial_y^2 \psi_1 + \frac{S_\varphi}{S} \left[(\omega \partial_\tau \varphi + \epsilon \partial_y \varphi)(-\omega \partial_\tau \psi_1 + \epsilon \partial_y \psi_1) \right]$$
$$+ S^2(\varphi) \psi_1 = 0, \tag{29}$$

$$\omega^2 \partial_\tau^2 \psi_2 - \epsilon^2 \partial_y^2 \psi_2 - \frac{S_\varphi}{S} \left[(\omega \partial_\tau \varphi - \epsilon \partial_y \varphi)(\omega \partial_\tau \psi_2 + \epsilon \partial_y \psi_2) \right]$$
$$+ S^2(\varphi) \psi_2 = 0. \tag{30}$$

After that, let us use the small amplitude expansion for the fields ψ_1 and ψ_2. Here, we will assume that

$$\psi_1(y, \tau) = \sum_{j=1}^{\infty} \epsilon^j \sigma_j(y, \tau), \tag{31}$$

$$\psi_2(y, \tau) = \sum_{j=1}^{\infty} \epsilon^j \rho_j(y, \tau). \tag{32}$$

By replacing the expansion given above into Eqs. (29) and (30), we obtain

$$\epsilon(\partial_\tau^2 \sigma_1 + b^2 \sigma_1) + \epsilon^2 \left[\partial_\tau^2 \sigma_2 + b^2 \sigma_2 - \frac{4b^2 \cos(b\tau) \operatorname{sech}(by) \sigma_1}{\sqrt{3}} \right.$$
$$\left. - \frac{2b \operatorname{sech}(by) \sin(b\tau) \partial_\tau \sigma_1}{\sqrt{3}} \right]$$
$$+ \epsilon^3 \left[\partial_\tau^2 \sigma_3 + b^2 \sigma_3 + \frac{4}{3} b^2 \cos^2(b\tau) \operatorname{sech}^2(by) \sigma_1 \right.$$
$$- \frac{4b^2 \cos(b\tau) \operatorname{sech}(by) \sigma_2}{\sqrt{3}}$$
$$- \frac{4}{3} b \cos(b\tau) \operatorname{sech}^2(by) \sin(b\tau) \partial_\tau \sigma_1$$
$$+ \frac{2b \cos(b\tau) \operatorname{sech}(by) \tanh(by) \partial_\tau \sigma_1}{\sqrt{3}}$$
$$- \frac{2b \operatorname{sech}(by) \sin(b\tau) \partial_\tau \sigma_2}{\sqrt{3}} - \partial_\tau^2 \sigma_1$$
$$\left. + \frac{2b \operatorname{sech}(by) \sin(b\tau) \partial_y \sigma_1}{\sqrt{3}} - \partial_y^2 \sigma_1 \right] + O(\epsilon^4) = 0, \tag{33}$$

$$\epsilon(\partial_\tau^2 \rho_1 + b_1^2 \rho) + \epsilon^2 \left[\partial_\tau^2 \rho_2 + b^2 \rho_2 - \frac{4b^2 \cos(b\tau) \operatorname{sech}(by) \rho_1}{\sqrt{3}} \right.$$
$$\left. - \frac{2b \operatorname{sech}(by) \sin(b\tau) \partial_\tau \rho_1}{\sqrt{3}} \right]$$
$$+ \epsilon^3 \left[\partial_\tau^2 \rho_3 + b_3^2 \rho_3 + \frac{4}{3} b^2 \cos^2(b\tau) \operatorname{sech}^2(by) \rho_1 \right.$$
$$- \frac{4b^2 \cos(b\tau) \operatorname{sech}(by) \rho_2}{\sqrt{3}}$$
$$- \frac{4}{3} b \cos(b\tau) \operatorname{sech}^2(by) \sin(b\tau) \partial_\tau \rho_1$$
$$+ \frac{2b \cos(b\tau) \operatorname{sech}(by) \tanh(by) \partial_\tau \rho_1}{\sqrt{3}}$$
$$- \frac{2b \operatorname{sech}(by) \sin(b\tau) \partial_\tau \rho_2}{\sqrt{3}} - \partial_\tau^2 \rho_1$$
$$\left. - \frac{2b \operatorname{sech}(by) \sin(b\tau) \partial_y \rho_1}{\sqrt{3}} - \partial_y^2 \rho_1 \right] + O(\epsilon^4) = 0. \tag{34}$$

The procedure of performing a small amplitude expansion shows that the fields ψ_1- and ψ_2-fields can be obtained from a set of fields which satisfy coupled non-linear differential equations. This set of differential equations is found by taking the terms in all orders of ϵ in the above equation. Thus, one can check that up to ϵ^3 the above supposition leads to the following set of equations

$$\partial_\tau^2 \sigma_1 + b^2 \sigma_1 = 0, \tag{35}$$

$$\partial_\tau^2 \rho_1 + b^2 \rho_1 = 0, \tag{36}$$

$$\partial_\tau^2 \sigma_2 + b^2 \sigma_2 - \frac{4b^2 \cos(b\tau) \operatorname{sech}(by) \sigma_1}{\sqrt{3}}$$
$$- \frac{2b \operatorname{sech}(by) \sin(b\tau) \partial_\tau \sigma_1}{\sqrt{3}} = 0, \tag{37}$$

$$\partial_\tau^2 \rho_2 + b^2 \rho_2 - \frac{4b^2 \cos(b\tau) \operatorname{sech}(by) \rho_1}{\sqrt{3}}$$
$$- \frac{2b \operatorname{sech}(by) \sin(b\tau) \partial_\tau \rho_1}{\sqrt{3}} = 0, \tag{38}$$

$$\partial_\tau^2 \sigma_3 + b^2 \sigma_3 + \frac{4}{3} b^2 \cos^2(b\tau) \operatorname{sech}^2(by) \sigma_1$$
$$- \frac{4b^2 \cos(b\tau) \operatorname{sech}(by) \sigma_2}{\sqrt{3}}$$
$$- \frac{4}{3} b \cos(b\tau) \operatorname{sech}^2(by) \sin(b\tau) \partial_\tau \sigma_1$$
$$+ \frac{2b \cos(b\tau) \operatorname{sech}(by) \tanh(by) \partial_\tau \sigma_1}{\sqrt{3}}$$
$$- \frac{2b \operatorname{sech}(by) \sin(b\tau) \partial_\tau \sigma_2}{\sqrt{3}} - \partial_\tau^2 \sigma_1$$
$$+ \frac{2b \operatorname{sech}(by) \sin(b\tau) \partial_y \sigma_1}{\sqrt{3}} - \partial_y^2 \sigma_1 = 0, \tag{39}$$

$$\partial_\tau^2 \rho_3 + b_3^2 \rho_3 + \frac{4}{3} b^2 \cos^2(b\tau) \operatorname{sech}^2(by) \rho_1$$
$$- \frac{4b^2 \cos(b\tau) \operatorname{sech}(by) \rho_2}{\sqrt{3}}$$
$$- \frac{4}{3} b \cos(b\tau) \operatorname{sech}^2(by) \sin(b\tau) \partial_\tau \rho_1$$
$$+ \frac{2b \cos(b\tau) \operatorname{sech}(by) \tanh(by) \partial_\tau \rho_1}{\sqrt{3}}$$
$$- \frac{2b \operatorname{sech}(by) \sin(b\tau) \partial_\tau \rho_2}{\sqrt{3}} - \partial_\tau^2 \rho_1$$
$$- \frac{2b \operatorname{sech}(by) \sin(b\tau) \partial_y \rho_1}{\sqrt{3}} - \partial_y^2 \rho_1 = 0. \tag{40}$$

From now on, we shall solve these equations. For the sake of simplicity, let us propose a particular solution for the first two equations, namely,

$$\sigma_1(y, \tau) = \sigma(y) \cos(b\tau), \tag{41}$$

$$\rho_1(y, \tau) = \rho(y) \sin(b\tau). \tag{42}$$

Plugging the solutions (41) and (42) into Eqs. (37) and (38), we have

$$\sigma_2(y, \tau) = -\frac{2\sigma(y) [-3 + \cos(2b\tau)] \operatorname{sech}(by)}{3\sqrt{3}}, \tag{43}$$

$$\rho_2(y, \tau) = -\frac{\rho(y)}{\sqrt{3}} \operatorname{sech}(by) \sin(2b\tau). \tag{44}$$

Using the results (41)–(44), the Eqs. (39) and (40) can be rewritten as

$$\partial_\tau^2 \sigma_3 + b^2 \sigma_3 = -\frac{8b^2 \sigma(y)\cos(b\tau)[-3+\cos(2b\tau)]\operatorname{sech}^2(bx)}{9}$$

$$-\frac{b\operatorname{sech}(by)\sin(2b\tau)\sigma'(y)}{\sqrt{3}} + \cos(b\tau)\sigma''(y)\sigma(y) \qquad (45)$$

$$-\frac{1}{18}b^2\cos(b\tau)\operatorname{sech}^2(by)\big[17+16\cos(2b\tau)+9\cosh(2by)$$

$$-12\sqrt{3}\sin(b\tau)\sinh(by)\big]\sigma(y),$$

$$\partial_\tau^2 \rho_3 + b^2 \rho_3 = -b^2\rho(y)\sin(b\tau)$$

$$-b^2\rho(y)\cos(2b\tau)\operatorname{sech}^2(by)\sin(b\tau)$$

$$-\frac{4}{3}b^2\rho(y)\cos(b\tau)\operatorname{sech}^2(by)\sin(2b\tau) \qquad (46)$$

$$-\frac{2}{\sqrt{3}}b^2\rho(y)\cos^2(b\tau)\operatorname{sech}(by)\tanh(by)\sin(b\tau)$$

$$+\rho''(y)\sin(b\tau)$$

$$+\frac{2}{\sqrt{3}}b^2\rho'(y)\operatorname{sech}(by)\sin^2(b\tau).$$

We remember that our aim is to get configurations which are periodical in time. In this sense, it is necessary to impose that the contribution of the functions $\cos(b\tau)$ and $\sin(b\tau)$ in the right-hand side of the above partial differential equations should be annulled. Therefore, we can obtain the functions $\sigma(y)$ and $\rho(y)$, which are given by

$$\sigma(y) = \operatorname{sech}(by), \ \rho(y) = -\operatorname{sech}(by). \qquad (47)$$

Then, using the solutions (43), (44), and (47), the fermionic sector can be written as

$$\psi(x,t) \sim \begin{pmatrix} \epsilon\operatorname{sech}(b\epsilon x)\cos(b\sqrt{1-\epsilon^2}t) \\ -\epsilon\operatorname{sech}(b\epsilon x)\sin(b\sqrt{1-\epsilon^2}t) \end{pmatrix}. \qquad (48)$$

Hence, it was also possible to find oscillon-type configuration in the fermionic sector. This fermionic oscillon is the supersymmetric partner of the oscillon in the bosonic sector. It is worthy to highlight that this fermionic solution is consistent with the idea of a supersymmetric multiplet. Indeed, by applying the transformation (4) as a perturbation on the initial configuration (oscillon, Eq. (22), and $\psi = 0$), one generates solutions with non-trivial fermionic sector, which at order-ϵ read as follows

$$\psi(x,t)$$
$$\simeq \epsilon\frac{b^2}{a\sqrt{3}}\operatorname{sech}(\epsilon bx)\begin{pmatrix} \xi_1\cos(b\sqrt{1-\epsilon^2}t)+\xi_2\sin(b\sqrt{1-\epsilon^2}t) \\ \xi_2\cos(b\sqrt{1-\epsilon^2}t)-\xi_1\sin(b\sqrt{1-\epsilon^2}t) \end{pmatrix}. \qquad (49)$$

In the particular case $\xi_2 = 0$, we recover the result in Eq. (48) (except for some factors in the amplitude). Here, we re-inforce that Eq. (49) shall be obtained as a solution to the field equations, if one considers a general solution with both $\sin(b\tau)$- and $\cos(b\tau)$-contributions in Eqs. (41) and (42).

The oscillon and oscillino solutions have the same functional behavior at order-ϵ, namely, spatial-localized and periodical-time dependences, given by $\operatorname{sech}(\epsilon bx)$ and $\sin(b\sqrt{1-\epsilon^2}t)$ or $\cos(b\sqrt{1-\epsilon^2}t)$, respectively. However, if we do not restrict ourselves to order-ϵ in the transformation (4), the contributions coming from the non-linear part in ξV_φ and $i\gamma^\mu \xi \partial_\mu \varphi$ will introduce higher powers in ϵ leading then to different behavior for the oscillino.

5. Concluding comments and further steps

In this work, we have shown that oscillon configurations in $D = 1+1$ dimensions, which in this context are also dubbed small amplitude oscillons, can be accommodated and suitably in a supersymmetric framework. We have analyzed both the bosonic and fermionic sectors and noticed that the oscillon and oscillino solutions display similar features at the ϵ-order; by that, we mean the same oscillation frequency and functional behavior.

As we have already stated in the Introduction of our work, there are several reasons that support our proposal of investigating oscillons in connection with SUSY. The main motivation is based on the fact that, if we wish to consider the interaction between fermions and oscillons, SUSY provides a natural set-up. SUSY transformations on oscillon configurations yield the fermionic partners of the oscillons that lie in the same multiplet. The fermionic excitations induced as SUSY perturbations on the oscillons propagate and interact with the latter. In turn, the condensates of the (fermionic) SUSY partners directly affect the oscillons' propagation, so that, it is an interesting issue, for a forthcoming contribution, to compute how the oscillon stability changes by virtue of the presence of the fermionic excitations.

On the other hand, by taking the viewpoint that fermions are the most elementary matter excitations in Nature, we intend to go ahead in our endeavor to search for possible quasi-particle excitations in low-dimensional charged systems that we might identify as oscillino-like (if the modes are fermionic) or as oscillino condensates, in the case of bosonic modes. For that, it is mandatory that we focus on the study of charged oscillons in the framework of an Abelian gauge model to, afterwards, embed the system of charged matter-gauge bosons in a SUSY context. We shall be reporting on that elsewhere in a forthcoming contribution.

Appendix A. Conventions and superspace formulation

Initially, let us fix our conventions. In two-dimensional Minkowski space-time, we adopt the metric $\eta^{\mu\nu} = \operatorname{diag}(+1,-1)$. One possible choice to satisfy the Clifford algebra, $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, is given by $\gamma^0 = \sigma_y$ and $\gamma^1 = i\sigma_x$, where σ_x and σ_y denote the usual Pauli matrices. This choice is known as Majorana representation of the gamma matrices.

In order to define a Majorana spinor, we introduce the charge conjugation matrix, C, which is antisymmetric ($C^t = -C$), unitary ($C^\dagger = C^{-1}$) and satisfies $C\gamma^{\mu\,t}C^{-1} = -\gamma^\mu$. In the Majorana representation, we have $C = -\gamma^0$. The charge conjugation operation is then defined by $\psi^c \equiv C\bar\psi^{\,t}$, where $\bar\psi = \psi^\dagger \gamma^0$. A Majorana spinor satisfies the constraint condition $\psi^c = \psi$ and, particularly, in the Majorana representation, we obtain that $\psi^c = \psi \Rightarrow \psi^* = \psi$, i.e., the spinor has real components.

In what follows, we present the Superspace Formulation. We consider the Supersymmetry $\mathcal{N} = 1$ in which the superspace is given by the coordinates (x^μ, θ), where θ is a Majorana spinor parameter.

We implement the superspace coordinate transformation as a translation, namely,

$$x'^\mu = x^\mu + i\bar\xi\gamma^\mu\theta, \qquad (A.1)$$

$$\theta'_\alpha = \theta_\alpha + \xi_\alpha \,, \tag{A.2}$$

where $\alpha = 1, 2$ and ξ is also a Majorana spinor parameter.

The simplest superfield, including two real scalar fields (φ, F) and a Majorana spinor field (ψ), is given by

$$\Phi(x, \theta) = \varphi(x) + \bar{\theta}\,\psi(x) + \bar{\theta}\theta\,F(x) \,. \tag{A.3}$$

With the aforementioned transformations, Eqs. (A.1) and (A.2), one may obtain the supersymmetric charge operator Q, by considering the variation

$$\delta\Phi \equiv \Phi(x', \theta') - \Phi(x, \theta) = \bar{\xi}\,Q\,\Phi(x, \theta) \,, \tag{A.4}$$

which reads

$$Q_\alpha = -C_{\alpha\beta}\,\frac{\partial}{\partial\theta_\beta} + i\left(\gamma^\mu\theta\right)_\alpha\,\frac{\partial}{\partial x^\mu} \,. \tag{A.5}$$

By comparing Eq. (A.4) with $\delta\Phi = \delta\varphi + \bar{\theta}\,\delta\psi + \bar{\theta}\theta\,\delta F$, one may conclude that

$$\delta\varphi = \bar{\xi}\psi \,, \tag{A.6}$$

$$\delta\psi = -i\,\gamma^\mu\xi\,\partial_\mu\varphi + 2\,F\xi \,, \tag{A.7}$$

$$\delta F = \partial_\mu\left(-\frac{i}{2}\,\bar{\xi}\gamma^\mu\psi\right) \,. \tag{A.8}$$

Having established the supercharge and transformations, we introduce the supersymmetric covariant derivative,

$$D_\alpha = -C_{\alpha\beta}\,\frac{\partial}{\partial\theta_\beta} - i\left(\gamma^\mu\theta\right)_\alpha\,\frac{\partial}{\partial x^\mu} \,, \tag{A.9}$$

which satisfies $\left\{D_\alpha, Q_\beta\right\} = 0$.

We propose the following action in terms of the superfield and covariant derivative,

$$S = \int d^2x\,d^2\theta\left[-\frac{1}{4}\,\bar{D}\Phi\,D\Phi + V(\Phi)\right] \,, \tag{A.10}$$

where $\int d^2\theta \equiv i\int d\theta_2 d\theta_1$ and $V(\Phi)$ denotes the superpotential, namely, an arbitrary function of the superfield Φ.

After using some Fierz rearrangements, such as $\bar{\theta}\psi\,\bar{\theta}\psi = -\frac{1}{2}\bar{\theta}\theta\,\bar{\psi}\psi$, and carrying out the Grassmann integral, one can obtain the off-shell Lagrangian density

$$\mathcal{L}_{\text{off-shell}} = \frac{1}{2}(\partial_\mu\varphi)^2 + \frac{i}{2}\,\bar{\psi}\gamma^\mu\partial_\mu\psi + 2F^2 + 2F\,V_\varphi - \frac{\bar{\psi}\psi}{2}V_{\varphi\varphi} \,. \tag{A.11}$$

Finally, we notice that F is an auxiliary field. Then, we work out its equation of motion and conclude that

$$\frac{\delta\mathcal{L}_{\text{off-shell}}}{\delta F} = 0 \;\Rightarrow\; F = -\frac{V_\varphi}{2} \,. \tag{A.12}$$

Hence, if we substitute this constraint in Eqs. (A.11) and (A.6)–(A.7), we arrive at the on-shell Lagrangian density (1) and the supersymmetry transformations (3)–(4), respectively.

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