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A Spin Field Interacting with Gravity in RW Space-Time Cannot Have a "Standard-Like" Form

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Abstract

It is shown that if the Einstein field equation is coupled to the massive spinor field equation of spin $s > 1/2$, then the spin field cannot have the form of a standard solution nor to be superposition of standard solutions in Robertson Walker space-time. For standard solutions are meant the wide class of solutions of the spinor field equations previously determined by variable separation in RW space-time. The result is an extension to the fermion field case of a similar result previously obtained for massive bosons and for the Einstein-Dirac equation in RW space time.

Keywords: Robertson-Walker space-time; coupled Einstein-massive spin field equation; variable separation; solutions; no go solutions

1 Introduction

The study of the interaction of fields has always been a central point in physics. Among all case the study of the interaction of a spin field with gravity has become of more and more interest. In this connection a good context where to formulate the problem seems to be that of the General Relativity. In such case

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the gravity content is represented by the metric tensor $g_{\mu\nu}$ whose values result from the mutual assessment with the interacting field. Accordingly a possible formulation of the problem of studying a massive field Φ of spin s coupled to gravity can be formulated by the equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R_\alpha^{\ \alpha} = 8\pi T_{\mu\nu}(\Phi) \tag{1}$$

$$D(\Phi) = 0 \tag{2}$$

The equation (1) is the Einstein field equation in the metric $g_{\mu\nu}$ whose source $T_{\mu\nu}$ is a function of the interacting field Φ. The equation (2) is the differential equation governing the time evolution of the field Φ in the metric $g_{\mu\nu}$. In the following it is assumed that $T_{\mu\nu} = E_{\mu\nu}(\Phi) + \bar{E}_{\mu\nu}(\Phi)$ where $E_{\mu\nu}(\Phi)$ is the energy momentum tensor of the field Φ. It immediately appears the complexity of the problem. Possible situations run from the general case in which $g_{\mu\nu}$ itself has to be determined to the extreme case of pure compatibility of the equations when $g_{\mu\nu}$ is, a priori, completely given. An intermediate situation of interest is that of the Robertson-Walker (RW) space-time. In that case $g_{\mu\nu}$ has a sufficiently determined structure derived from general cosmological like principles [1]. An advantage of the choice of the Robertson-Walker space-time is that a class of solutions (the standard solution in the following) of massive field equation (2) are known. They have been obtained by variable separation [2] by means of the two spinor formalism of Newman and Penrose [3]. Their superposition generates a very large class of solutions. Therefore it makes sense to look for solutions of the system (1) (2) by choosing Φ in the class of the standard solutions. Another advantage is that $T_\alpha^\beta(\Phi)$ has an expression that is well known for arbitrary spin s. The study of the coupled equation (1), (2) has been already performed in case the spin s of the field Φ is $s = 1/2$ [7] and in case s is a non zero integer [6].

In the present paper it is shown that no solutions of (1), (2) are possible in RW metric with Φ in the class of the standard solutions of (2). The proof is done for the general case $s > 1/2$, in a more complete form than what done for the bosons case [7]. (The case $s = 1/2$ was considered in [7]). The negative result holds also for a large class of Φ solutions that are superposition of standard solutions. The demonstration is essentially a matter of verification. The main role in that is played by the results obtained in Ref. [8, 4] and by the knowledge of the explicit form of the standard solutions.

2 Preliminary assumptions and results

The field equations considered here are the wave equations for the spinor field $\Phi \equiv (\phi, \chi)$ [9, 10]

$$\nabla^{A}_{\dot{X}}\phi_{AA_1A_2\ldots A_n} + i\mu_*\chi_{A_1A_2\ldots A_n\dot{Z}} = 0 \tag{3}$$

$$\nabla^{\dot{Z}}_{(A}\chi_{A_1A_2\ldots A_n)\dot{Z}} - i\mu_*\phi_{AA_1A_2\ldots A_n} = 0 \tag{4}$$

with $\phi_{AA_1A_2\ldots A_n} = \phi_{(AA_1A_2\ldots A_n)}$, $\chi_{A_1A_2\ldots A_n\dot{Z}} = \chi_{(A_1A_2\ldots A_n)\dot{Z}}$, $\mu_* = m_o/\sqrt{2}$. Equations (3), (4) represent the field equations of the massive field Φ whose particles have mass m_o and spin $s = \frac{n+1}{2}$, $n > 0$. They can be obtained as the Euler-Lagrange equations of a suitable variational a principle [8] that includes both bosons and fermions.

The considerations of the following Sections will be developed within the Robertson-Walker (RW) space-time of metric tensor $g_{\mu\nu}$ given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - R(t)^2\Big[\frac{dr^2}{1-ar^2} + r^2(d\theta^2 + \sin\theta^2 d\varphi^2)\Big], \tag{5}$$

Since the Weyl spinor vanishes in RW space-time, the symmetrization can be omitted in (3) (See, e.g., [8]). By the definitions

$$\begin{array}{lll} \phi_h & \equiv \phi_{AA_1A_2\ldots A_n} \Leftrightarrow A + A_1 + \ldots + A_n = h, & h = 0,1,2,\ldots,n+1 \\ \chi_{j\dot{X}} & \equiv \chi_{A_1A_2\ldots A_n\dot{X}} \Leftrightarrow A_1 + A_2 + \ldots A_n = j, & j = 0,1,\ldots,n \end{array} \tag{6}$$

a wide class of solutions of (3) in RW metric (4) can be obtained by variable separation by setting [2, 12]

$$\begin{array}{l} \phi_j(t,r,\theta,\varphi) = \alpha_\kappa(t)\psi_{j,l\kappa}(r)S_{j,lm}(\theta)\exp(im\varphi), \quad j = 0,1,\ldots,n+1 \\ \chi_{h\dot{0}}(t,r,\theta,\varphi) = A_\kappa(t)\psi_{h+1,l\kappa}(r)S_{h+1,lm}(\theta)\exp(im\varphi), \\ \chi_{h\dot{1}}(t,r,\theta,\varphi) = -A_\kappa(t)\psi_{h,\kappa l}(r)S_{h,lm}(\theta)\exp(im\varphi), \quad h = 0,1,\ldots,n \end{array} \tag{7}$$

$(m = 0, \pm1, \pm2, ...)$. For $m \geq 0$ one has

$$\begin{array}{l} S_{j,lm} = (1-\xi)^{\frac{m+s-j}{2}}(1+\xi)^{\frac{m-s+j}{2}}P_{l-m}^{(m+j-s,m-j+s)}(\xi), \quad \xi = \cos\theta \\ \lambda^2 = l(l+1) - s(s-1), \qquad l = m, m+1, m+2, .. \end{array} \tag{8}$$

with $P_l^{\alpha,\beta)}$ the Jacobi polynomials [13] and λ is the constant relative to the separation of the angular dependence. If $m \leq 0$ the angular solutions are obtained from (7) by the substitution $m \to |m|$, $\xi \to -\xi$.

For what concerns the radial functions $\psi_{j,l\kappa}(r)$ they are solution of the radial equations

$$\begin{array}{l} r(1-ar^2)\psi_j'' + \big[2s+2-(2s+3)ar^2\big]\psi_j' + \Big\{r\big[\kappa^2 - a(j+1)(2s+1-j)\big] + \\ +2i\kappa(s-j)\sqrt{1-ar^2} + \dfrac{2s-\lambda^2}{r}\Big\}\psi_j = 0 \end{array} \tag{9}$$

where $a = 0, \pm 1$; κ is the separation constant relative to the separation of time and radial dependence. Note that the substitution $j \to 2s - j$ changes (9) in its complex conjugate. Therefore $\psi_{n+1-j} \propto \bar{\psi}_j$. The equation (9) can be solved in terms of confluent hyper geometric functions ($a = 0$) or in terms of Heun functions ($a = \pm 1$) [12].

Finally the time separated dependence $\alpha_\kappa(t)$, $A_\kappa(t)$ of the wave spinor is governed by the equations

$$\begin{aligned} \dot{\alpha} R + (s+1)\,\alpha \dot{R} - i m_0\, A R &= -i\kappa\alpha \\ \dot{A} R + (2-s)\,A\,\dot{R} - i m_o \alpha\, R &= \,i\kappa A \end{aligned} \tag{10}$$

that depend both on the value of the spin s and on the cosmological background $R(t)$.

In the following the solutions of the form (7) will be called standard solutions.

3 Gravity-spin field interaction in RW metric

It is possible to show that in the scheme (1), (2), Φ does not admit of standard solution in the Robertson-Walker space-time when the equation (2) represents a massive field equation of arbitrary spin. To show this we refer to the uniform Lagrangian formulation of Ref. [8, 4] that allows to describe massive fields of spin $s \geq 1/2$. Besides obtaining the field equations, that formalism furnishes the expression of the energy momentum tensor in two spinor form. In that scheme, the field Φ is represented by four spinor field ϕ, χ, ξ, θ with ϕ, χ that satisfy equations (3), (4) without symmetrization, while ξ, θ (with $\xi_{\dot{A}\dot{A}_1\ldots\dot{A}_n} = \xi_{(\dot{A}\dot{A}_1\ldots\dot{A}_n)}$, $\theta_{A\dot{X}_1\ldots\dot{X}_n} = \theta_{A(\dot{X}_1\ldots\dot{X}_n)}$) satisfy their complex conjugate equations respectively. The general expression of the energy-momentum tensor in terms of the solutions of the field equation, can be found in [4]. In the present paper the interest is in the trace of the energy momentum tensor $E^\alpha_\alpha = E(\phi, \chi, \xi, \theta)$ that is given by [4]

$$E = -i\mu_*(a+b)\big\{(n+1)\chi_{A_1\ldots A_n\dot{Z}}\bar{\theta}^{A_1\ldots A_n\dot{Z}} + (n-1)\phi_{AA_1\ldots A_n}\bar{\xi}^{AA_1\ldots A_n}\big\} \tag{11}$$

A central point for the following is that by taking trace of (1), $T^\alpha_\alpha(\Phi)$ must depend only on the variable t since so is the Ricci scalar in RW metric.

Since the equations for ξ, θ are the complex conjugate equations (3), (4) respectively, the class of solutions for $\bar{\xi}$, $\bar{\theta}$ are the standard solutions for ϕ, χ that is $\bar{\theta} = \chi_{\kappa' l' m'}$ and $\bar{\xi} = \phi_{\kappa' l' m'}$. To calculate E on the standard solutions one must choose $(\kappa,\, l\, m) = (\kappa',\, l'\, m')$ because the particle cannot simultaneously have different sets of quantum numbers. Accordingly, by considering the analog of (5),

$$\begin{array}{ll}
\phi^j \equiv \phi^{AA_1A_2\ldots A_n} & \Leftrightarrow A + A_1 + \ldots + A_n = j, \quad j = 0,1,2,\ldots,n+1 \\
\chi^{h\dot{X}} \equiv \chi^{A_1A_2\ldots A_n\dot{X}} & \Leftrightarrow A_1 + A_2 + \ldots A_n = h, \quad h = 0,1,\ldots,n \\
\phi_{AA_1A_2\ldots A_n}\phi^{AA_1A_2\ldots A_n} & \equiv \sum_{r=0}^{n+1}(-1)^r\binom{n+1}{r}\phi_r\phi_{n+1-r} \\
\chi_{A_1A_2\ldots A_n\dot{Z}}\chi^{A_1A_2\ldots A_n\dot{Z}} & \equiv \sum_{r=0}^{n}(-1)^r\binom{n}{r}\left[\chi_{r\dot{0}}\chi_{n-r\dot{1}} - \chi_{r\dot{1}}\chi_{n-r\dot{0}}\right]
\end{array} \tag{12}$$

the trace (11) of the energy momentum tensor, in terms of the standard solutions, becomes

$$\begin{aligned}
E_{\kappa lm} = -i\mu_*(a+b)e^{2im\varphi} \quad & \left[(n+1)A^2 + (n-1)\alpha^2\right] \times \\
& \times \sum_{j=0}^{n+1}(-1)^j\binom{n+1}{j}\psi_j\bar{\psi}_j S_j S_{n+1-j}
\end{aligned} \tag{13}$$

It is understood $\psi_j \equiv \psi_{j,\kappa l}$, $S_j \equiv S_{j,lm}$. It has also been used the relation $\binom{n}{j-1} + \binom{n}{j} = \binom{n+1}{j}$ and it has been chosen $\psi_{n+1-j} = \bar{\psi}_j$. Since $T = E + \bar{E}$ cannot depend on φ it must be $m = 0$. By the relation $S_{n+1-j,lm}(\xi) = (-1)^l S_{j,lm}(-\xi)$, the expression (13), for $m = 0$, is in fact a polynomial of degree $2l$ in $\cos\theta$ because all the involved polynomials have degree l. The condition $\partial_\theta(E_{\kappa l0} + \bar{E}_{\kappa l0}) = 0$ implies then (unless $l = 0$), $\theta = 0, \pi$ or $\theta = \theta(r,t)$. This is not possible θ being an independent variable. Then the expression of E is reduced to

$$\begin{array}{l}
E_{\kappa 00} = c\,\tau(t)\,G(r), \qquad c = -i\mu_*(a+b) \\
\tau = \left[(n+1)A^2(t) + (n-1)\alpha^2(t)\right], \quad G = \sum_{j=0}^{n+1}(-1)^j\binom{n+1}{j}\psi_j\bar{\psi}_j
\end{array} \tag{14}$$

One has to distinguish according to the parity of n.

i) Fermion case: n is an even number. $G(r)$ consists of $n+2$ terms that, on account of the property $\bar{\psi}_j = \psi_{n+1-j}$, can be recast into the form

$$G(r) = \sum_{k=0}^{s-\frac{1}{2}} \psi_k\bar{\psi}_k\left[(-1)^k\binom{n+1}{k} + (-1)^{n+1-k}\binom{n+1}{n+1-k}\right] \tag{15}$$

that identically vanish on account of the relation $\binom{n+1}{k} = \binom{n+1}{n+1-k}$ and the fact that n is even.

ii) Boson case: n is an odd number. Here $G(r)$ can be written into the form

$$G(r) = (-1)^s\psi_s\bar{\psi}_s\binom{n+1}{s} + \sum_{j=0}^{s-1}\psi_j\bar{\psi}_j\left[(-1)^j\binom{n+1}{j} + (-1)^{n+1-j}\binom{n+1}{n+1-j}\right] \tag{16}$$

that is a real expression. The condition $\partial_r(E + \bar{E}) = 0$ implies then that $E_{\kappa 00}$ must vanish identically. Suppose indeed that $c\tau + \bar{c}\bar{\tau}$ is not zero for some t. Than $G(r)$ in (16) must be constant, say c_o. Since for large r it holds $\psi_j\bar{\psi}_j \sim 1/r^{2+2j}$ for $j < s$ and $\psi_s\bar{\psi}_s \sim (\log r)^2/r^{2+2s}$ (cf.[12]), there follows $c_o = 0$.

4 Generalization

The previous negative result can be extended heuristically to hold for superposition of a wide class of standard solutions. With reference to the previous general scheme, let us first choose in $(\bar{\xi}, \bar{\theta}) \equiv (\phi, \psi)$ in (11). Then the trace of the energy momentum tensor $E = E(\phi, \chi, \xi, \theta)$ reads

$$E = c(n+1) \sum\nolimits_{j=0}^{n} \binom{n}{j} \qquad (-1)^j \left[\chi_{j\dot{0}}\chi_{n-j\,\dot{1}} - \chi_{j\dot{1}}\chi_{n-j\,\dot{0}}\right] + \\ + c(n-1) \sum\nolimits_{j=0}^{n+1} \binom{n+1}{j} (-1)^j \phi_j \phi_{n+1-j} \quad (17)$$

Suppose now ϕ, χ be solutions of the form

$$\begin{aligned} \phi_j &= \sum_{lm} \int d\kappa\ c^j_{lm}(\kappa)\alpha_\kappa(t) S_{j,lm}(\theta)\psi_{j,\kappa l}(r)\, e^{im\varphi}, \quad j = 0, 1, ..., n+1 \\ \chi_{h\dot{0}} &= \sum_{lm} \int d\kappa\ c^h_{lm}(\kappa) A_\kappa(t) S_{h+1,lm}(\theta)\psi_{h+1,\kappa l}(r)\, e^{im\varphi} \qquad (18) \\ -\chi_{h\dot{1}} &= \sum_{lm} \int d\kappa\ c^h_{lm}(\kappa) A_\kappa(t) S_{h,lm}(\theta)\psi_{h,\kappa l}(r)\, e^{im\varphi}, \quad h = 0, 1, ..., n \end{aligned}$$

The assertion can be justified by selecting, between the two possible j-independent behavior of ψ_j for $r \to 0$, the one of the form r^{l-s} (see, e.g., [12]). When the limit $r \to 0$ is considered in (17) with the assumptions (18), the dominant term of the series is for $l = 0$ and hence $m = 0$. Then the first sum in (17) identically vanish and one is left with

$$\begin{aligned} &E \overset{r\to 0}{\sim} c(n-1)\, r^{-2s} \times T(t), \\ &T(t) = \textstyle\sum_0^{n+1} (-1)^j \binom{n+1}{j} \left(\int dk\, c^j_{00}(k)\alpha_k(t)\right)\left(\int dk\, c^{n+1-j}_{00}(k)\alpha_k(t)\right) \end{aligned} \qquad (19)$$

The condition $\partial_r(E+\bar{E})(t) = 0$ is then not possible for those $c^j_{00}(k)$'s for which there exists a time t such that $(cT + \bar{c}\bar{T}) \neq 0$. In particular, if $c^{j'}_{00} = c^{n+1-j'}_{00} \neq 0$ and $c^j_{00} = 0$ for $j \neq j'$, $n+1-j'$, that vanishing condition is not possible by requiring $\Re\, c \int \alpha_\kappa(t) c^{j'}_{00}(\kappa) \neq 0$ for at least a time t. Such request is a weak condition if one considers the generality of the possible choice of the c^j_{lm}'s.

5 Remarks and comments

In the present paper the problem of determining solutions of the Einstein field equation coupled to massive field Φ of spin s in RW space-time has been considered. What has been found is that Φ does not admit solution within the class of the standard solutions of the field equation of spin $s \geq 1$. It has also been shown that the same holds for a wide class of superpositions of standard solutions. The result is an extension and a completion of the previous study with the same negative results for spin s a positive integer and $s = 1/2$ [6, 7]. [Massless fields are not considered in the paper. The choice of the parameters of the theory of Ref. [8], that has been done to have the field equations (3), (4), is such that the expression of the trace of the energy momentum tensor (11) results proportional to the mass of the particles of the field].

The result of the paper is not conclusive because only solutions of the spin field equation of factorized form have been considered. (Factorized solutions of field equations, that are non standard, also exist (e.g., [6, 5]). Accordingly one is faced with the preliminary open problem of determining solutions of the spinor field equation that are factorized but non standard or generically non standard or that are not superpositions of standard solutions, or that do not have a factorized form. Moreover they should give rise to an energy momentum tensor whose trace has to depend only on the variable t, a strong condition indeed. (A possibility could be that of searching within the pure time dependent solutions of the spin field equation).

One can note that the above considerations hold also if one adds a "cosmological term" $\Lambda g_{\mu\nu}$ to the Einstein equation (1) of the interacting scheme. Indeed this formally amounts to modify $T_\alpha^\alpha(\Phi)$ by the addition of a numerical constant. Again, by the argument of the previous Sections, the standard solutions are not suitable to solve the problem.

One could also try to further generalize the interacting scheme by adding a "back reaction-like" term to the equation (1). In the opinion of the author this would require then a modification of the spin field equation too. But this does not seem correct because the mutual assessment of both gravity and spin field is already contained in coupling the equations.

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