On nets of local algebras on the Minkowski lattice $\mathbf{Z}^4$ *

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The results presented refer to the construction of causal nets of local algebras on the Minkowski lattice $\mathbf{Z}^4$, considered as subnets of (non-causal) nets on the Minkowski space $\mathbf{R}^4$. The results are published in [1] , [2].

Let $\mathcal{B}(\cdot)$ denote an isotone and additive net of C^*-algebras, defined on the compact subsets $\mathcal{M}$ of the Minkowski space $\mathbf{R}^4$ and which is generated by so-called atomic algebras $\mathcal{B}(\{x\})$, $x \in \mathbf{R}^4$. Let $\mathcal{B} := \operatorname{clo} \bigcup_{\mathcal{M}\subset\mathbf{R}^4} \mathcal{B}(\mathcal{M})$ denote the corresponding quasi local algebra. The net $\mathcal{B}(\cdot)$ is assumed to be $\mathbf{R}^4$-covariant, i.e. there is a representation $\mathbf{R}^4 \ni x \longrightarrow \beta_x \in \operatorname{aut}\mathcal{B}$ such that $\mathcal{B}(\mathcal{M}+x) = \beta_x\mathcal{B}(\mathcal{M})$. Furthermore, it is assumed that there is a $\mathbf{R}^4$-invariant state ω of $\mathcal{B}$ which satisfies a positivity condition (i.e. the strongly continuous representation of $\mathbf{R}^4$ on the GNS-Hilbert space of ω implied by the GNS-representation of $\mathcal{B}$ with respect to ω is spectral).

Let $\mathcal{P}$ denote the discrete Poincare group $\mathcal{P} := \mathbf{Z}^4 \circledS \mathcal{G}$ ($\circledS$ semidirect product), where $\mathcal{G}$ denotes the subgroup of SL(2,$\mathbf{Z} + i\mathbf{Z}$) consisting of all 2 2-matrices $\binom{a\,b}{c\,d}$ of SL(2,$\mathbf{Z} + i\mathbf{Z}$) where $|a|^2 + |b|^2 + |c|^2 + |d|^2 \equiv 0(\mathrm{mod}2)$. $\mathcal{G}$ was already considered by A.Schild [3].

Let $\mathcal{A}(\cdot)$ denote the subnet of $\mathcal{B}(\cdot)$, defined on the finite subsets $\mathcal{O} \subset \mathbf{Z}^4$. Let $\mathcal{A} := \operatorname{clo}\bigcup_{\mathcal{O}\subset\mathbf{Z}^4} \mathcal{A}(\mathcal{O})$. The net $\mathcal{A}(\cdot)$ is assumed to be $\mathcal{P}$-covariant, i.e. there is a representation $\mathcal{P} \ni g \longrightarrow \alpha_g \in \operatorname{aut}\mathcal{A}$ such that $\mathcal{A}(g\mathcal{O}) = \alpha_g\mathcal{A}(\mathcal{O})$ (note that $g = \{x, \Lambda\}$, $x \in \mathbf{Z}^4$, $\Lambda \in \mathcal{G}$). $\mathcal{A}$ is assumed to be invariant under β_x, $x \in \mathbf{Z}^4$ and $\alpha_{\{x,1\}} = \beta_x, \omega \restriction \mathcal{A}$ is $\mathcal{P}$-invariant.

Inclusions $\mathcal{A} \subset \mathcal{B}$ of quasilocal algebras just described are called admissible. The quasilocal algebra $\mathcal{A}$ (resp. a corresponding inclusion) is called causal if $\mathcal{O}_1 \perp \mathcal{O}_2$ (causal disjointness in the Minkowski sense) implies that $\mathcal{A}(\mathcal{O}_1)$and $\mathcal{A}(\mathcal{O}_2)$ commute by elements.

Proposition 1 *There are admissible inclusions $\mathcal{A} \subset \mathcal{B}$.*

For the construction of admissible inclusions CCR-Weyl algebras over suitable phase spaces (pre-Hilbert spaces) can be used. For example, the following construction is possible: Put $H := L_{\mathrm{fin}}(\mathcal{G}, M(\mathbf{R}^4))$, $H_0 := L_{\mathrm{fin}}(\mathcal{G}, M(\mathbf{Z}^4))$, where $M(\mathbf{R}^4)$ denotes the linear space of all complex-valued Borel measures ν on $\mathbf{R}^4$ with compact support and $M(\mathbf{Z}^4) \subset M(\mathbf{R}^4)$ the subspace where supp $\nu \subset \mathbf{Z}^4$. By (,) one denotes the semi-scalar product

$$
\begin{aligned}
(\nu_1, \nu_2) &:= \sum_{\Lambda\in\mathcal{G}} \int_{\mathbf{R}^4} \overline{\hat{\nu}_1(p,\Lambda)}\hat{\nu}_2(p,\Lambda)\mu(\Lambda dp), \\
\|\nu\| &:= (\nu,\nu)^{1/2}
\end{aligned}
$$

*Talk given at the XVI. International Colloquium on Group Theoretical Methods in Physics, Varna, Bulgaria, June 15-21, 1987 where the parameter μ denotes a positive finite Borel measure on $\mathbf{R}^4$ with supp $\mu \subseteq$ clo V_+ (closed forward cone), $\mu(\text{clo } V_+) = 1$. Then the phase spaces are defined by $f_{\mathcal{B}} := H \bmod \ker \| \cdot \|$ and $f_{\mathcal{A}} := H_0 \bmod \ker \| \cdot \|$, $f_{\mathcal{A}} \subset f_{\mathcal{B}}$.The action of $\mathcal{P}$ on H_0 is defined by $(V_{\{a,\Lambda\}}f)(x,\Lambda\prime) := f(\Lambda^{-1}(x-a),\Lambda'\Lambda)$. The required invariant state ω can be chosen as the Fock state on $\mathcal{B}$ (the CCR-Weyl algebra over $f_{\mathcal{B}}$). The nets are defined as usual by localization in the phase spaces $f_{\mathcal{A}}, f_{\mathcal{B}}$.

If the parameter measure μ satisfies the condition

$$\int_{\mathbf{R}^4} \sin < a,p > \mu(dp) = 0 \, , a \in \mathbf{Z}^4, a \, \text{spacelike}, \tag{1}$$

then the net $\mathcal{A}(\cdot)$ is causal ($< \cdot,\cdot >$ means the Minkowski scalar product).

Let $m_0 \geq 0$ and let $H_{m_0} \subset \mathbf{R}^4$ denote the corresponding mass hyperboloid

$$p_0^2 = m_0^2 + \mid g \mid^2 , p_0 > 0, \; p = \{p_0, g\}.$$

Proposition 2 *There is a positive finite Borel measure μ with supp $\mu = H_{m_0}$, $\mu(H_{m_0}) = 1$, such that (1) is satisfied, i.e. the corresponding inclusion is causal.*

A proof is given in [2].

Further absolutely continuous measures μ are considered, which lead to admissible nets. In this case, if supp $\mu = H_{m_0}$, μ is quasiinvariant with respect to $\mathcal{G}$. This leads to a representation of the phase space $f_{\mathcal{A}}$ by $l^2(\mathcal{G}) \otimes L^2(\mathbf{R}^4, d\gamma_{m_0})$, where $d\gamma_{m_0}$ denotes the Lorentz invariant measure on H_{m_0} (decoupling of shift and geometric action of discrete Lorentz transformations). Using this decoupling and former results on asymptotic constants which commute with Lorentz transformations (see [4]), for the Fock representation of the inclusion one can establish a $\mathcal{P}$-covariant perturbation theory which leads to the construction of 'interacting nets' given on the Fock space, with prescribed scattering operator via LSZ-scattering.

References

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