Non-perturbative Studies in Supersymmetric Field Theories via String Theory

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Abstract of the Dissertation

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The strongly coupled regime of gauge theories is of great interest in high energy physics, with quantum chromodynamics at low energies being the prime example. Non-perturbative effects become important in this regime and it is necessary to understand their contribution to the observables of interest. Supersymmetry goes a long way in constraining the structure of these effects and makes their calculation tractable. In the past few decades, phenomenal progress has been achieved in this direction by exploiting the many rigid symmetries (spacetime and internal) that are usually present in a supersymmetric field theory. Novel infinite dimensional symmetries that act on field space have also been uncovered and summarised in the very general program of the BPS/CFT correspondence. These novel symmetries offer a deeper explanation for the highly constrained nature of non-perturbative effects in supersymmetric field theories.

Superstring theory has provided us with new and powerful ways of interpreting field theoretic non-perturbative objects such as instantons, monopoles and so on. Supersymmetric field theories and their non-perturbative effects can be realised in string theory by studying the low-energy dynamics of collections of Dirichlet branes. In this thesis, we study bound states of Dirichlet branes of various dimensionalities. The underlying

theme of the thesis is the rich interplay between physics in diverse dimensions and how superstring theory addresses them all in one go.

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[1] Anything more specific would require more space and time than provided even by the eleven dimensions that this thesis is based on.

Chapter 1

Introduction

Non-perturbative effects in field theories are of immense importance in understanding the full quantum structure of the theory. Many physically relevant field theories become strongly coupled at low energies in which case perturbation theory breaks down and it is necessary to include non-perturbative effects. In gauge theories, instantons are examples of non-perturbative effects because their contribution is beyond all orders in perturbation theory. Indeed, the classical contribution to the partition function of a single instanton in euclidean SU(n) gauge theory is

$$\exp\left(-\frac{8\pi^2}{g^2}\right) \ , \tag{1.1}$$

where g is the coupling constant of the gauge theory, assumed to be much less than 1. As we can see, the above term is beyond all orders in perturbation theory in g and it becomes $\mathcal{O}(1)$ when $g \to \infty$. Let us derive the above formula as a warm-up exercise. This will also help us set notation. The euclidean action for an SU(n) gauge field A_μ with field strength $F_{\mu\nu}$ is

$$\mathcal{S}_{\rm YM}[A] = \int \mathrm{d}^4x \ \mathrm{Tr}\left[\frac{1}{2g^2}F_{\mu\nu}F_{\mu\nu} + \frac{\mathrm{i}\theta}{16\pi^2}F_{\mu\nu}{}^\star F_{\mu\nu}\right] \ , \tag{1.2}$$

where ${}^\star F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$ is the dual field strength and θ is the microscopic θ-angle. Our conventions are such that the generators T_α are hermitian and the Killing form $\mathrm{Tr}\, T_\alpha T_\beta = \frac{1}{2}\delta_{\alpha\beta}$ is positive-definite. The partition function of the gauge theory is given by the path integral

$$\mathcal{Z} = \int [\mathrm{d}A] \exp(-\mathcal{S}_{\rm YM}[A]/\hbar) \ . \tag{1.3}$$

We have omitted gauge fixing terms and ghosts in the exponent but they are necessary to obtain the correct number of physical degrees of freedom in the path integral. We are interested in the semi-classical limit $\hbar \to 0$ in which case the path integral is dominated by the minima of $\mathcal{S}_{\rm YM}$. The action can be re-written as

$$\begin{aligned} \mathcal{S}_{\rm YM}[A] &= \int \mathrm{d}^4x\ \mathrm{Tr}\left(\frac{1}{4g^2}F^+_{\mu\nu}F^+_{\mu\nu} + \frac{\mathrm{i}\tau}{8\pi}F_{\mu\nu}{}^\star F_{\mu\nu}\right)\ , \\ &= \int \mathrm{d}^4x\ \mathrm{Tr}\left(\frac{1}{4g^2}F^-_{\mu\nu}F^-_{\mu\nu} + \frac{\mathrm{i}\bar\tau}{8\pi}F_{\mu\nu}{}^\star F_{\mu\nu}\right)\ , \end{aligned} \tag{1.4}$$

where $F^\pm_{\mu\nu} = F_{\mu\nu} \pm {}^\star F_{\mu\nu}$ is the self-dual (anti self-dual) part of the field strength and τ is the complexified coupling constant

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi\mathrm{i}}{g^2}\ . \tag{1.5}$$

The second term in the action above is a boundary term which captures the topological winding number of the gauge field configuration and is insensitive to infinitesimal variations of the gauge field. To proceed, we consider the sector of gauge fields which have a fixed winding number k:

$$c_2(F) := -\frac{1}{16\pi^2}\int \mathrm{d}^4x\ \mathrm{Tr}\ F_{\mu\nu}{}^\star F_{\mu\nu} = k \in \mathbf{Z}\ . \tag{1.6}$$

The first term in the action is a positive definite object since it is a sum of squares. Thus, the minima of the action are captured by configurations which satisfy

$$F^\pm_{\mu\nu} = 0 \quad \text{with} \quad c_2(F) = k\ . \tag{1.7}$$

It is evident that self-dual fields ($F^- = 0$) have $c_2(F) < 0$ and anti self-dual fields ($F^+ = 0$) have $c_2(F) > 0$. The contribution of such a configuration to the partition function is then

$$\mathrm{e}^{-\mathcal{S}_{\rm YM}/\hbar} = \mathrm{e}^{\mathrm{i}k\theta}\mathrm{e}^{-8\pi^2|k|/g^2} = \begin{cases} \mathrm{e}^{2\pi\mathrm{i}k\tau} & k > 0 \\ \mathrm{e}^{2\pi\mathrm{i}k\bar\tau} & k < 0 \end{cases} \tag{1.8}$$

One would like to study the space of solutions of the equations $F^+_{\mu\nu} = 0$ with $c_2(F) = k$ modulo the gauge invariance $\delta A_\mu = \mathrm{D}_\mu\lambda$. This is the moduli space $\mathcal{M}_{n,k}$ of instantons with winding k in $\mathrm{SU}(n)$ gauge theory. The ADHM construction utilises the algebraic properties of the solutions to specify the moduli space in terms of equations on finite dimensional matrices. We shall take an alternate route following [CG] by studying the solutions to the massless Dirac equation in the k-instanton background

$$\not{D}\psi = 0 \ , \quad \psi \text{ is in the } \boldsymbol{n} \text{ of } \mathrm{SU}(n) \ . \tag{1.9}$$

It can be shown that there are no positive chirality solutions to the equation. Then, using the index theorem Index $\not{D} = -c_2(F) = -k$ one can show that there are k negative chirality solutions to the Dirac equation. More details can be found in the review [BVvN]. Choose the following basis for γ-matrices and complex structure for $\mathbf{R}^4$:

$$\begin{aligned} &\gamma^1 = \sigma_1 \otimes \mathbb{1} \ , \quad \gamma^2 = \sigma_2 \otimes \mathbb{1} \ , \quad \gamma^3 = \sigma_3 \otimes \sigma_2 \ , \quad \gamma^4 = -\sigma_3 \otimes \sigma_1 \ , \quad \gamma_c = -\sigma_3 \otimes \sigma_3 \ , \\ &z_1 = x_1 + \mathrm{i}x_2 \ , \quad z_2 = x_3 - \mathrm{i}x_4 \ , \quad \partial_1 = \tfrac{1}{2}(\partial_{x_1} - \mathrm{i}\partial_{x_2}) \ , \quad \partial_2 = \tfrac{1}{2}(\partial_{x_3} + \mathrm{i}\partial_{x_4}) \ . \end{aligned} \tag{1.10}$$

Negative chirality spinors have two components $\psi = -\mathrm{i}\psi_{--}|-,-\rangle + \psi_{++}|+,+\rangle$. The signs in $|\pm,\pm\rangle$ are the eigenvalues of $\sigma_3 \otimes \mathbb{1}$ and $\mathbb{1} \otimes \sigma_3$ respectively. The Dirac equation then becomes

$$\mathrm{D}_2\psi_{++} = \mathrm{D}_1\psi_{--} \ , \quad \overline{\mathrm{D}}_1\psi_{++} = -\overline{\mathrm{D}}_2\psi_{--} \ . \tag{1.11}$$

We arrange the k solutions $\psi^i_{\pm\pm}$, $i = 1, \ldots, k$, into two $n \times k$ matrices $\Psi_{\pm\pm}$ as follows:

$$\Psi_{\pm\pm} = \begin{pmatrix} \psi^1_{\pm\pm} & \psi^2_{\pm\pm} & \cdots & \psi^k_{\pm\pm} \end{pmatrix} \ . \tag{1.12}$$

Given an instanton solution A_μ of winding k, we have

$$A_\mu \to g^{-1}\partial_\mu g \quad \text{as} \quad |x| \to \infty \ , \tag{1.13}$$

where g is an element of the gauge group with winding number k. We then have, in the limit $|x| \to \infty$,

$$\Psi_{--} \to -g^{-1}\frac{-\mathrm{i}I^\dagger z_1 + \mathrm{i}J\bar{z}_2}{(|z_1|^2 + |z_2|^2)^2} \ , \quad \Psi_{++} \to -g^{-1}\frac{\mathrm{i}J\bar{z}_1 + \mathrm{i}I^\dagger z_2}{(|z_1|^2 + |z_2|^2)^2} \ . \tag{1.14}$$

for constant $n \times k$ matrices $I^\dagger$ and J. Next, we assume that the solutions are normalised:

$$\int d^4x \Psi^\dagger(x)\Psi(x) = \pi^2 \mathbb{1}_k \quad \text{where} \quad \Psi = -i\Psi_{--}|-,-\rangle + \Psi_{++}|+,+\rangle \ . \tag{1.15}$$

Given this, we define the $k \times k$ complex matrices

$$B_a := \frac{1}{\pi^2} \int d^4x \, z_a \Psi^\dagger(x)\Psi(x) \ , \quad a = 1, 2 \ . \tag{1.16}$$

Using the properties of the solutions $\Psi_{\pm\pm}$, one can then derive the following identities satisfied by the matrices B_1, B_2, I and J:

$$\begin{aligned} \mu^{\mathbf{C}} &:= [B_1, B_2] + IJ = 0 \ , \\ \mu^{\mathbf{R}} &:= [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0 \ . \end{aligned} \tag{1.17}$$

First, we observe that there is a $\mathrm{U}(k)$ symmetry acting on the solutions $\Psi \to \Psi h^{-1}$ with $h \in \mathrm{U}(k)$. Under this symmetry, the matrices transform as

$$B_a \to h B_a h^{-1} \ , \quad I \to hI \ , \quad J \to Jh^{-1} \ . \tag{1.18}$$

Solutions that differ by $\mathrm{U}(k)$ arise from the same instanton solution. Hence, $\mathrm{U}(k)$ is a gauge invariance and we call it the *reciprocal* gauge group. Hence, to establish a one-to-one correspondence between instanton solutions and the matrices (B_1, B_2, I, J), we must divide the space of solutions to (1.17) by $\mathrm{U}(k)$. This is precisely the ADHM description of the moduli space of instantons!

$$\mathcal{M}_{n,k} = \left\{ B_1, B_2, I, J \mid \mu^{\mathbf{R}} = 0, \ \mu^{\mathbf{C}} = 0 \right\} \ / \ \mathrm{U}(k) \ . \tag{1.19}$$

A quick calculation provides the dimension of the tangent space at a sufficiently generic point in the moduli space. The matrices contain $4k^2 + 4kn$ real degrees of freedom while the equations give $3k^2$ real constraints. The $\mathrm{U}(k)$ transformations fix an additional k^2 real degrees of freedom. Thus, at the points where the above reasoning holds, the dimension of the tangent space is

$$4k^2 + 4nk - 3k^2 - k^2 = 4nk \ . \tag{1.20}$$

This reasoning fails for those points where the configurations preserve a proper subgroup of U(k). Let us list the symmetries that act on the instanton moduli space.

1. **U$(\boldsymbol{k})$ gauge invariance**:

$$B_a \to h B_a h^{-1} \ , \quad I \to h I \ , \quad J \to J h^{-1} \ , \quad h \in \mathrm{U}(k) \ . \tag{1.21}$$

2. **PSU$(\boldsymbol{n})$ framing rotations**: The asymptotic form of the gauge field is $A_\mu \to g^{-1}\partial_\mu g$ where $g(x)$ is a gauge transformation with winding number k. The above form is invariant under $g \to \alpha g$ with $\alpha \in \mathrm{PSU}(n)$. These are the *framing rotations*. We fix a particular PSU(n) equivalence class so that we have instanton solutions with a fixed *framing* at infinity.

   Demanding that the solutions $\Psi_{\pm\pm}$ in (1.14) are invariant under framing rotations, we see that PSU(n) acts on the matrices as

$$I \to I \alpha^{-1} \ , \quad J \to \alpha J \ , \quad B_a \to B_a \ . \tag{1.22}$$

3. **Rotational invariance**: Under mutually commuting rotations of $\mathbf{C}^2$ specified by $z_a \to \mathrm{e}^{\mathrm{i}\theta_a} z_a$, the solutions $\Psi_{\pm\pm}$ transform as $\Psi_{\pm\pm} \to \mathrm{e}^{\mp\frac{\mathrm{i}}{2}(\theta_1-\theta_2)}\Psi_{\pm\pm}$. Demanding that the asymptotic solutions in (1.14) transform in the same manner gives the following rules for I and J, and similarly for B_a from (1.16):

$$I \to \mathrm{e}^{\frac{\mathrm{i}}{2}(\theta_1+\theta_2)} I \ , \quad J \to \mathrm{e}^{\frac{\mathrm{i}}{2}(\theta_1+\theta_2)} J \ , \quad B_a \to \mathrm{e}^{\mathrm{i}\theta_a} B_a \ . \tag{1.23}$$

The ADHM equations are invariant under the rigid symmetries (1.22) and (1.23) and they commute with the U(k) action, so they persist as rigid symmetries on the moduli space $\mathcal{M}_{n,k}$.

The ADHM construction provides the opposite map: given a specific 4-tuple of matrices in $\mathcal{M}_{n,k}$, one writes down the instanton solution. Thus, the matrix moduli space provides a complete description of anti self-dual gauge fields.

We are interested in studying the collective dynamics of the k-instanton solution. In four euclidean dimensions, there is no room for the instantons to move. Hence, we embed the instantons as time independent solutions of five dimensional $\mathrm{SU}(n)$ gauge theory. This theory is ill-defined in the ultraviolet, but one can imagine (and indeed there exists, in string theory,) a suitable ultraviolet completion which then has these instantons as time independent solitonic solutions.

The collective dynamics can then be described by giving the ADHM matrices a time dependence and writing down the canonical kinetic energies for the matrices. The solutions to (1.17) are then interpreted as static solutions to the equations of motion. In order to preserve the $\mathrm{U}(k)$ invariance at various times, the $\mathrm{U}(k)$ transformations have to be made time dependent and the time derivative ∂_t has to be promoted to a covariant derivative $\mathrm{D}_t = \partial_t + \mathrm{i}a_t$. Here, a_t transforms under $\mathrm{U}(k)$ as a gauge field:

$$\mathrm{i}a_t \to h(t)(\partial_t + \mathrm{i}a_t)h(t)^{-1} \ , \quad h(t) \in \mathrm{U}(k) \ . \tag{1.24}$$

The action governing the collective dynamics is then

$$\mathcal{S}_{\mathrm{1d}} = \int \mathrm{d}t \ \mathrm{Tr}_k \Big(|\mathrm{D}_t B_1|^2 + |\mathrm{D}_t B_2|^2 + |\mathrm{D}_t I|^2 + |\mathrm{D}_t J|^2 - |\mu^{\mathbf{C}}|^2 - (\mu^{\mathbf{R}})^2 \Big) \ . \tag{1.25}$$

What we have achieved is that the collective quantum dynamics of instantons can be described by a one dimensional gauged linear sigma model with the above action.

Now, the question is where is the above linear sigma model relevant? Since instantons are classical minima of the action, the partition function is approximated in the $\hbar \to 0$ limit by a sum over the partition functions $\mathcal{Z}_k$ corresponding to the instanton sector with instanton number k:

$$\mathcal{Z} \ \approx \ \sum_{k\geq 0} q^k \mathcal{Z}_k + \sum_{k<0} \bar{q}^k \mathcal{Z}'_k \quad \text{with} \quad q = \mathrm{e}^{2\pi \mathrm{i}\tau} \ . \tag{1.26}$$

Here, q^k and $\bar{q}^k$ are the classical contributions to the path integral that we calculated earlier in (1.8). In principle there are contributions from both instantons and anti-instantons.

In $\mathcal{N}=2$ theories in four dimensions, the anti-instanton contribution turns out to be zero due to holomorphy of the partition function in τ [Sei1, Sei2]. The above equation (with $\mathcal{Z}'_k=0$) becomes exact and moreover, the k-instanton partition function $\mathcal{Z}_k$ is given by the Euler characteristic of instanton moduli space. The Euler characteristic on $\mathcal{M}_{n,k}$ can then be obtained by considering the appropriate supersymmetric version of the gauged linear sigma model in (1.25) and calculating its Witten index, first defined by Witten in [W4]. That is,

$$\mathcal{Z}_k = \mathrm{Tr}_{\mathcal{H}_k}\,(-1)^F \mathrm{e}^{-\beta H}\ . \tag{1.27}$$

Here, $\mathcal{H}_k$ is the Hilbert space of the supersymmetric quantum mechanics in (1.25) and H is its Hamiltonian. It is easy to observe that the moduli space $\mathcal{M}_{n,k}$ is singular and non-compact and hence the definition of Euler characteristic has to be regularised. This can be performed in a way that is consistent with the rigid symmetries in the problem. The regularisation proceeds in two steps.

First, we deform the right hand side of $\mu^{\mathbf{R}}=0$ to

$$\mu^{\mathbf{R}} = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = r\cdot \mathbb{1}_k\ , \quad r>0\ . \tag{1.28}$$

We observe that the above deformation preserves all the symmetries acting on $\mathcal{M}_{n,k}$. In what sense is this is a regularisation? It turns out that the above equation cuts out a slice in (B_1, B_2, I, J) space which avoids the singular points which preserve a proper subgroup of $\mathrm{U}(k)$. Physically, this procedure deforms the four dimensional space to non-commutative space with parameter r [NSc]. This has the effect of curing the moduli space from point-like instantons since point-like objects are no longer well-defined in non-commutative space.

Next, we choose a pair of supercharges Q, $\overline{\mathrm{Q}}$ in the quantum mechanics such that $\{\mathrm{Q}\,,\overline{\mathrm{Q}}\}=2H$ and consider all the rigid symmetries which commute with this subalgebra. Generically, the framing rotations in (1.22) and the spatial rotations in (1.23) commute with a suitably chosen Q once we also perform a compensating R-symmetry transformation.

Then, we can consider the deformed index

$$\mathcal{Z}_k(a_1,\ldots,a_n;\epsilon_1,\epsilon_2)=\mathrm{Tr}_{\mathcal{H}_k}\,(-1)^F\mathrm{e}^{\mathrm{i}a_\alpha T_\alpha}\,\mathrm{e}^{\mathrm{i}\epsilon_a J_a+\mathrm{i}\xi R}\,\mathrm{e}^{-\beta H}\ . \tag{1.29}$$

Here, T_α are generators of the maximal torus of $\mathrm{U}(n)$ and $\sum_\alpha a_\alpha=0$ so that $\mathrm{e}^{\mathrm{i}a_\alpha T_\alpha}$ is in (the maximal torus of) $\mathrm{PSU}(n)$. The J_a, $a=1,2$, are generators of rotations $z_a\to\mathrm{e}^{\mathrm{i}\theta_a}z_a$. Finally, ξ is a linear function of the ϵ_a and R is a generator in the Cartan subalgebra of the R-symmetry algebra. Let the overall torus group generated by T_α, J_a and R be denoted T. The deformation due to the torus of spatial rotations is called the Ω-*deformation*.

By standard index lore, the above index now receives contributions only from the fixed points under the various rigid symmetries in the trace above. The key point is that since spatial rotations are involved, the instanton configurations that now contribute are fixed points of spatial rotations. In the presence of the non-commutative deformation, these fixed points correspond to k isolated single instantons which are not exactly point-like but fuzzy due to the non-commutativity. Thus, the deformed index becomes a *finite* sum over the fixed points π_k of the torus group T!

$$\mathcal{Z}_k(a_1,\ldots,a_n;\epsilon_1,\epsilon_2)=\sum_{\pi_k}\mu_{\pi_k}(a_1,\ldots,a_n;\epsilon_1,\epsilon_2)\ , \tag{1.30}$$

where μ_{π_k} is a trigonometric function of the various parameters that one obtains by calculating the path integral of fluctuations about the fixed point π. The above index must be suitably generalised to include bare masses when there are matter multiplets in the theory. The tools for such calculations have been developed in [MNS, LNS] and applied to $\mathcal{N}=2$ theories in four dimensions in [N2, NO1] and others. The five dimensional perspective is developed in [NSh, N5, LN]. The topological version of the five dimensional theory which calculates the above partition function has also been discussed in [BLN].

1.1 Spiked Instantons

The lesson to take away from the above discussion is the following:

The supersymmetries along with the various rigid and gauge symmetries present in the theory are strong enough to allow us to calculate the full non-perturbative partition function as a sum over contributions from isolated point-like instantons.

In theories with eight supercharges, there are a host of other observables apart from the partition function that can be exactly calculated in a manner similar to above. These are the *BPS observables*. They are normalised expectation values of operators in the deformed theory which are invariant under four of the eight supersymmetries. A salient example [N3] is the following gauge invariant operator inserted at the origin of four dimensional space

$$\mathcal{Y}(x) = x^n \exp\left(-\sum_{\ell=1}^{\infty} \frac{1}{\ell x^\ell} \mathrm{Tr}\, \Phi^\ell\,|_0\right) , \tag{1.31}$$

where Φ is the adjoint complex scalar in the $\mathcal{N}=2$ supersymmetric gauge theory and x is a parameter. As was pointed out in [N3], the correct BPS observables to consider in the non-commutative theory is a deformed version of the above.

It is of interest to consider transitions in the gauge theory between configurations of different instanton number. Such transitions become amenable to a quantitative study since only point-like isolated instantons contribute to the BPS observables and the transitions are now discrete processes corresponding to adding or removing several point-like instantons.

Observables which encode information about these non-perturbative transitions, the *qq-characters* $\mathcal{X}(x)$, can be expressed as rational functions of the $\mathcal{Y}$-observables with shifted arguments. See [N3] for a number of examples. In the simplest of cases, the $\mathcal{X}$-observable can be seen to be the partition function of an auxiliary four dimensional supersymmetric gauge theory. Since the $\mathcal{X}$-observable is inserted at the origin of $\mathbf{C}^2$, the auxiliary gauge theory can be thought of as living on a second $\mathbf{C}^2$ that intersects the first at the origin. Then, integrating out the degrees of freedom of the auxiliary gauge theory would correspond to the insertion of an operator at the origin in the original $\mathbf{C}^2$. In this picture, instanton number transitions would correspond to the point-like instantons hopping between the two $\mathbf{C}^2$'s via the origin.

One can generalise and look at another auxiliary gauge theory on a third $\mathbf{C}^2$ which intersects the original $\mathbf{C}^2$ on a complex line $\mathbf{C}$. These would give rise to surface defects in the original gauge theory which can change instanton number. In fact, the most general setup of such intersecting four dimensional worlds which preserves a few supersymmetries consists of six such $\mathbf{C}^2$'s intersecting at the origin of $\mathbf{C}^4$ with pairwise intersections of complex dimension 0 and 1.

The moduli space of instantons bound to some or all six stacks of $\mathbf{C}^2$ is known as the *moduli space of spiked instantons*, first considered in [N3]. This moduli space is described as follows. Let $\underline{\mathbf{4}} = \{1,2,3,4\}$ be the set of coordinate labels of the $\mathbf{C}^4$. The six two-planes $\mathbf{C}^2_A$ that sit inside the $\mathbf{C}^4$ are labelled by the index $A \in \underline{\mathbf{6}} = \binom{\underline{\mathbf{4}}}{2}$ i.e. the set of unordered pairs of numbers in $\underline{\mathbf{4}}$. Explicitly, $\underline{\mathbf{6}} = \{(12),(13),(14),(23),(24),(34)\}$. We also start with positive integers k and n_A which denote the total instanton number and the rank of the unitary gauge groups on each of the six $\mathbf{C}^2$'s. Define the following matrices:

$$\begin{aligned} &B_1, B_2, B_3, B_4 \ : \ \text{in the adjoint of } \mathrm{U}(k) \ , \\ &I_A,\ J_A \ : \ \text{in the } \boldsymbol{k} \times \overline{\boldsymbol{n}}_{\boldsymbol{A}} \text{ and } \overline{\boldsymbol{k}} \times \boldsymbol{n}_{\boldsymbol{A}} \text{ of } \mathrm{U}(k) \times \mathrm{U}(n_A) \text{ for } A \in \underline{\mathbf{6}} \ . \end{aligned} \tag{1.32}$$

The equations are then

1. The real moment map:

$$\mu_{\mathbf{R}} - r \cdot \mathbb{1}_k := \sum_{a \in \underline{\mathbf{4}}} [B_a, B_a^\dagger] + \sum_{A \in \underline{\mathbf{6}}} (I_A I_A^\dagger - J_A^\dagger J_A) - r \cdot \mathbb{1}_k = 0 \ . \tag{1.33}$$

2. For $A = (ab) \in \underline{\mathbf{6}}$ with $a < b$,

$$\mu_A^{\mathbf{C}} := [B_a, B_b] + I_A J_A = 0 \ . \tag{1.34}$$

3. For $A \in \underline{\mathbf{6}}$, $\bar{A} = \underline{\mathbf{4}} \smallsetminus A$ and $\bar{a} \in \bar{A}$,

$$\sigma_{\bar{a}A}^{\mathbf{C}} := B_{\bar{a}} I_A = 0 \ , \quad \tilde{\sigma}_{\bar{a}A}^{\mathbf{C}} := J_A B_{\bar{a}} = 0 \ . \tag{1.35}$$

4. For $A \in \underline{\mathbf{6}}$, $\bar{A} = \underline{\mathbf{4}} \setminus A$,

$$\Upsilon_A^{\mathbf{C}} := J_{\bar{A}} I_A = 0 \ . \tag{1.36}$$

5. For $A, B \in \underline{\mathbf{6}}$ such that $A \cap B = \{c\} \in \underline{\mathbf{4}}$, and $j = 1, 2, \ldots$

$$\Upsilon_{A,B,j} := J_A (B_c)^{j-1} I_B = 0 \ . \tag{1.37}$$

The first and second sets of equations are the analogues of the ADHM equations for ordinary instantons in four dimensions. The other three sets relate instanton configurations in different $\mathbf{C}^2$'s.

1.2 Enter superstrings

String theory provides more than one way of constructing large classes of supersymmetric gauge theories with eight supercharges [DM, KKV, W1]. One such class is the class of quiver gauge theories [DM] which can be engineered by considering the gauge theory on a stack of D4-branes located at a singularity of ADE type. Instantons in this gauge theory have an alternate description as D0-branes bound to the D4-branes [D1]. Let us demonstrate this fact by studying the coupling of k D0-branes along $\mathbf{R}_t$ with n D4-branes along $\mathbf{R}_t \times \mathbf{C}^2$. There is a $\mathrm{U}(k)$ gauge theory on the D0-branes and a $\mathrm{U}(n)$ gauge theory on the D4-branes with additional matter fields in the bifundamental of $\mathrm{U}(k) \times \mathrm{U}(n)$.

The low-energy effective action for the D4-branes contains the following coupling to the (pullback of the) RR one-form gauge field C_1:

$$\frac{e_4}{2} \int_{\mathbf{R}_t \times \mathbf{C}^2} C_1 \wedge \mathrm{Tr}\left(2\pi\alpha' F \wedge 2\pi\alpha' F\right) \ , \tag{1.38}$$

where F is the $\mathrm{U}(n)$ field strength on the stack of D4-branes. The RR one-form C_1 is a background field that arises from the low-energy spectrum of closed superstrings. The $\mathrm{U}(n)$ field strength arises from open strings ending on the stack of D3-branes. The charge quantum e_4 is related to the D4-brane tension as $e_4 = T_4$ by virtue of its BPS nature and is given by

$$e_4 = \frac{1}{g_s\sqrt{\alpha'}\,(2\pi\sqrt{\alpha'}\,)^4}\ , \tag{1.39}$$

where g_s is the string coupling constant and α' is related to the string length as $\ell^2 = 2\alpha'$. Consider a situation in which the gauge field on the D4-brane is time-independent and C_1 is independent of the $\mathbf{C}^2$ directions. The above coupling becomes

$$-e_0 k \int_{\mathbf{R}_t} C_1 \quad \text{with} \quad k = -\frac{1}{8\pi^2}\int_{\mathbf{C}^2} \mathrm{Tr}\ F\wedge F\ . \tag{1.40}$$

Here, $e_0 = (g_s\sqrt{\alpha'})^{-1}$ is the D0 charge quantum and k is the familiar instanton number of a U(n) instanton in $\mathbf{C}^2$. The above coupling implies that instantons of charge k in the U(n) gauge theory on the D4-branes induce D0-branes of charge charge $-e_0 k$ on the worldvolume. This was first realised in [D1]. In fact, the worldvolume U(k) gauge theory on the D0-branes is precisely the supersymmetric quantum mechanics we have been looking at previously!

The spiked instanton scenario is then obtained by adding the appropriate extra stacks of D4-branes according to the description previously. In this thesis, we consider the following setup. Write the ten dimensional spacetime $\mathbf{R}^{1,9} \simeq \mathbf{R}^{1,1}\times\mathbf{R}^8$ as $\mathbf{R}^{1,1}\times\mathbf{C}^4$ by choosing a complex structure on the $\mathbf{R}^8$. Let $\underline{\mathbf{4}} = \{1,2,3,4\}$ be the set of coordinate labels of the $\mathbf{C}^4$. Consider a system of D-branes which consists of k D1-branes spanning $\mathbf{R}^{1,1}$ and n_A D5-branes spanning $\mathbf{R}^{1,1}\times\mathbf{C}^2_A$ with $A\in\underline{\mathbf{6}}$. Here onwards, $\mathbf{R}^{1,1}$ refers to the common $1+1$ dimensional intersection of the D-brane configuration and is taken to be along the x^0, x^9 directions.

We would like the above setup to preserve some supersymmetries. Type IIB string theory has two supersymmetry parameters ϵ and $\tilde\epsilon$ which are Majorana-Weyl spinors of the same chirality (say left-handed). That is,

$$\Gamma_c\epsilon = \epsilon\ , \quad \Gamma_c\tilde\epsilon = \tilde\epsilon \quad \text{where} \quad \Gamma_c = \Gamma^1\cdots\Gamma^9\Gamma^0 \quad \text{and} \quad (\Gamma_c)^2 = \mathbb{1}\ . \tag{1.41}$$

The presence of a Dp-brane gives the following constraint on the supersymmetry parame-

Table 1.1: The intersecting D1-D5 system for spiked instantons. Crosses indicate worldvolume directions.

$\mathbf{R}^{1,9}$	1	2	3	4	5	6	7	8	9	0
$\mathbf{C}^4 \times \mathbf{R}^{1,1}$	z^1		z^2		z^3		z^4		x	t
D1									×	×
$\text{D5}_{(12)}$	×	×	×	×					×	×
$\text{D5}_{(13)}$	×	×			×	×			×	×
$\text{D5}_{(14)}$	×	×					×	×	×	×
$\text{D5}_{(23)}$			×	×	×	×			×	×
$\text{D5}_{(24)}$			×	×			×	×	×	×
$\text{D5}_{(34)}$					×	×	×	×	×	×

ters:

$$\tilde{\epsilon} = \frac{1}{(p+1)!} \varepsilon_{\mu_0\mu_1\cdots\mu_p} \Gamma^{\mu_0\mu_1\cdots\mu_p} \epsilon \ . \tag{1.42}$$

Here, $\mu_0, \ldots, \mu_p$ take $p+1$ values corresponding to the spacetime extent of the Dp-brane and $\Gamma^{\mu_0\mu_1\cdots\mu_p}$ is the totally antisymmetrised product of $p+1$ Γ-matrices. Suppose the spatial extent of the Dp-brane is along $\{x^{i_1}, \ldots, x^{i_p}\}$ with $i_1 < i_2 < \cdots < i_p$. Then, the Levi-Civita symbol ε is normalised such that $\varepsilon_{i_1 i_2 \cdots i_p 0} = +1$. In the presence of $\overline{\text{D1}}$-branes along $\mathbf{R}^{1,1}$ and D5-branes along $\mathbf{R}^{1,1} \times \mathbf{C}^2_{(12)}$, the constraints are $\tilde{\epsilon} = -\Gamma^{90}\epsilon$ and $\tilde{\epsilon} = \Gamma^{123490}\epsilon$ which give

$$\Gamma^{1234}\epsilon = -\epsilon \ . \tag{1.43}$$

Since Γ^{1234} squares to identity and is traceless, half of the sixteen real components of ϵ are set to zero. This leaves us with a total of eight independent supersymmetry parameters for the D1-$\text{D5}_{(12)}$ system. In order to preserve some supersymmetry when we include all six stacks of D5-branes, we choose the following signs for the constraints on ϵ:

$$\Gamma^{1234}\epsilon = -\epsilon \ , \quad \Gamma^{1256}\epsilon = -\epsilon \ , \quad \Gamma^{1278}\epsilon = -\epsilon \ , \quad \Gamma^{3456}\epsilon = -\epsilon \ , \quad \Gamma^{3478}\epsilon = -\epsilon \ , \quad \Gamma^{5678}\epsilon = -\epsilon \ .$$

Only three of the above six constraints are independent, preserving one-sixteenth of the 32 supercharges. Thus, a configuration of $\overline{\text{D1}}$-branes with six stacks of D5-branes preserves **two supercharges**. The above constraints also give $\Gamma^{90}\epsilon = \epsilon$ which means that the two preserved supercharges are chiral in $\mathbf{R}^{1,1}$. Thus, the low-energy effective theory in $\mathbf{R}^{1,1}$ will be a $\mathcal{N} = (0,2)$ supersymmetric theory.

We need one last ingredient to match the field theory story, and that is to find a way to bind the $\overline{\text{D1}}$-branes to the worldvolume of D5-branes to form a stable bound state. This is when the above $\overline{\text{D1}}$-D5 system truly represents the spiked instanton scenario. Fortunately, there exists a way to achieve this:

One has to turn on a constant NSNS B-field along the $\mathbf{C}^4$ that is consistent with the rotational symmetries of the intersecting D-brane system.

A constant NSNS B-field changes the boundary conditions obeyed by an open string and this changes the spectrum of open strings with ends attached to the D-branes. We first study open strings propagating in a constant B-field background and study its consequences for the D-brane spectra in Chapter 2. Next, we set up the formalism of $\mathcal{N} = (0,2)$ superspace in Chapter 3. This allows us to succinctly write down the form of the couplings of the $\mathcal{N} = (0,2)$ supersymmetric gauged linear sigma model in $\mathbf{R}^{1,1}$.

In Chapter 4, we get to work. The low-energy effective theory of open strings in the above D-brane setup corresponds to a specific $\mathcal{N} = (0,2)$ gauged linear sigma model. The couplings of the model be obtained by studying the scattering amplitudes of the corresponding string states. Using the formalism developed in Chapter 2, we compute these amplitudes and ergo the low-energy couplings. When cast into the language of superspace, these couplings directly give us the spiked instanton equations!

In Chapter 5, we compute the equivariant elliptic genus of the spiked instanton moduli space. This is a version of the twisted index we considered in (1.29) for two dimensional supersymmetric models. We infer some properties of spiked instantons by studying the expression for the equivariant elliptic genus. Then, we conclude with some speculations and possible directions for future research.

Chapter 2

Open Strings in a constant B-field

We follow the treatment of background gauge fields in [ACNY]. Consider a open string propagating in ten dimensional flat spacetime with metric $g_{\mu\nu}$ in the presence of a constant B-field $B_{\mu\nu}$. The $\mathcal{N} = (1,1)$ superconformal worldsheet theory is formulated in terms of the superfield X^μ with components

$$X^\mu := X^\mu| \;, \quad \mathrm{i}\psi^\mu_\pm := (\mathrm{D}_\pm X^\mu)| \;, \quad \mathrm{i}F^\mu = (\mathrm{D}_+\mathrm{D}_- X^\mu)| \;, \tag{2.1}$$

where the $|$ sets all the Grassmann coordinates to zero. Our conventions are such that the supersymmetry derivatives satisfy $\mathrm{D}^2_\pm = \mathrm{i}\partial_{\pm\pm}$, $\{\mathrm{D}_+, \mathrm{D}_-\} = 0$ with $\partial_{\pm\pm} = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$. The action is given by

$$\begin{aligned}
\mathcal{S} &= \frac{1}{\pi\alpha'}\int \mathrm{d}\tau\mathrm{d}\sigma\, \mathrm{D}_+\mathrm{D}_- \left\{(g_{\mu\nu} + 2\pi\alpha' B_{\mu\nu})\,\mathrm{D}_+X^\mu \mathrm{D}_-X^\nu\right\} \;, \\
&= \frac{1}{\pi\alpha'}\int \mathrm{d}\tau\mathrm{d}\sigma\, g_{\mu\nu}\left(\partial_{++}X^\mu\partial_{--}X^\nu + F^\mu F^\nu - \mathrm{i}\psi^\mu_-\partial_{++}\psi^\nu_- - \mathrm{i}\psi^\mu_+\partial_{--}\psi^\nu_+\right) + \\
&\quad - \frac{1}{2}\int \mathrm{d}\tau\, B_{\mu\nu}\Big[(\partial_\tau X^\mu)X^\nu - \mathrm{i}\psi^\mu_-\psi^\nu_- - \mathrm{i}\psi^\mu_+\psi^\nu_+\Big]^{\sigma=\pi}_{\sigma=0} + \text{total } \tau\text{-derivative}\;.
\end{aligned} \tag{2.2}$$

The boundary terms in the Euler-Lagrange variation of the above action are

$$\begin{aligned}
\delta\mathcal{S} = -\frac{1}{2\pi\alpha'}\int \mathrm{d}\tau\Big[&\partial_{++}X^\mu(g_{\mu\nu} + 2\pi\alpha' B_{\mu\nu})\delta X^\nu - \partial_{--}X^\mu(g_{\mu\nu} - 2\pi\alpha' B_{\mu\nu})\delta X^\nu + \\
&- \mathrm{i}\psi^\mu_+(g_{\mu\nu} + 2\pi\alpha' B_{\mu\nu})\delta\psi^\nu_+ + \mathrm{i}\psi^\mu_-(g_{\mu\nu} - 2\pi\alpha' B_{\mu\nu})\delta\psi^\nu_-\Big]^{\sigma=\pi}_{\sigma=0}\;.
\end{aligned} \tag{2.3}$$

The boundary conditions that set the above variation to zero are then given by

$$\begin{aligned}
\partial_{++}X^\mu(g_{\mu\nu} + 2\pi\alpha' B_{\mu\nu})\delta X^\nu &= \partial_{--}X^\mu(g_{\mu\nu} - 2\pi\alpha' B_{\mu\nu})\delta X^\nu\;. \\
\psi^\mu_-(g_{\mu\nu} - 2\pi\alpha' B_{\mu\nu})\delta\psi^\nu_- &= \psi^\mu_+(g_{\mu\nu} + 2\pi\alpha' B_{\mu\nu})\delta\psi^\nu_+\;.
\end{aligned} \tag{2.4}$$

These must hold separately for $\sigma = 0$ and $\sigma = \pi$. A solution of these boundary conditions is given by

$$
\begin{aligned}
&\textbf{Bosons } X^\mu : \quad \delta X^\mu = 0 \ , \quad \text{or} \quad \partial_{--} X^\mu = \left(\frac{g + 2\pi\alpha' B}{g - 2\pi\alpha' B}\right)^\mu{}_\nu \partial_{++} X^\nu \ , \\
&\textbf{Fermions } \psi^\mu_\pm : \quad \psi^\mu_- = R^\mu{}_\nu \psi^\nu_+ \ , \quad \text{or} \quad \psi^\mu_- = \left(\frac{g + 2\pi\alpha' B}{g - 2\pi\alpha' B}\right)^\mu{}_\nu (R\psi_+)^\nu \ . \qquad (2.5)
\end{aligned}
$$

where $R^\mu{}_\nu$ is an O(1,9) matrix which flips the sign of $B_{\mu\nu}$ i.e. $R_\mu{}^\rho B_{\rho\sigma} R^\sigma{}_\nu = -B_{\mu\nu}$. Any combination of the above boundary conditions for the bosons and fermions solve the Euler-Lagrange boundary conditions. However, these boundary conditions are not consistent with supersymmetry as is evident from the standard supersymmetry transformations $\delta\varphi = (\epsilon^+ \mathrm{D}_+ + \epsilon^- \mathrm{D}_-)\varphi_|$:

$$
\begin{aligned}
&\delta X^\mu = \mathrm{i}\epsilon^+ \psi^\mu_+ + \mathrm{i}\epsilon^- \psi^\mu_- \ , \qquad \delta F^\mu = \mathrm{i}\epsilon^+ \partial_{++} \psi^\mu_- - \mathrm{i}\epsilon^- \partial_{--} \psi^\mu_+ \ , \\
&\delta \psi^\mu_+ = \partial_{++} X^\mu \epsilon^+ - F^\mu \epsilon^- \ , \quad \delta \psi^\mu_- = \partial_{--} X^\mu \epsilon^- + F^\mu \epsilon^+ \ . \qquad (2.6)
\end{aligned}
$$

Incidentally, the Euler-Lagrange variations in each column in (2.5) transform into each other under the following modified transformation rules once we impose the constraint $\epsilon^+ = \epsilon^-$:

$$
\begin{aligned}
&\delta_R X^\mu = \mathrm{i}\epsilon^+ (R\psi_+)^\mu + \mathrm{i}\epsilon^- \psi^\mu_- \ , \qquad \delta_R F^\mu = \mathrm{i}\epsilon^+ \partial_{++} \psi^\mu_- - \mathrm{i}\epsilon^- \partial_{--} (R\psi_+)^\mu \ , \\
&\delta_R \psi^\mu_+ = \partial_{++} (R^{-1} X)^\mu \epsilon^+ - (R^{-1} F)^\mu \epsilon^- \ , \quad \delta_R \psi^\mu_- = \partial_{--} X^\mu \epsilon^- + F^\mu \epsilon^+ \ . \qquad (2.7)
\end{aligned}
$$

With the boundary conditions in either the first or the second column of (2.5), we see that the variation of the action in (2.3) is zero under the supersymmetry of (2.7). One gets an alternate viewpoint by transporting the R matrix into the action by writing $\psi^\mu_+ = R^{-1} \psi'^\mu_+$. The action then becomes

$$
\begin{aligned}
\mathcal{S} = &\frac{1}{\pi\alpha'} \int \mathrm{d}\tau \mathrm{d}\sigma\, g_{\mu\nu} \left(\partial_{++} X^\mu \partial_{--} X^\nu + F^\mu F^\nu - \mathrm{i}\psi^\mu_- \partial_{++} \psi^\nu_- - \mathrm{i}\psi'^\mu_+ \partial_{--} \psi'^\nu_+ \right) + \\
&- \frac{1}{2} \int \mathrm{d}\tau\, B_{\mu\nu} \Big[(\partial_\tau X^\mu) X^\nu - \mathrm{i}\psi^\mu_- \psi^\nu_- + \mathrm{i}\psi'^\mu_+ \psi'^\nu_+ \Big]^{\sigma=\pi}_{\sigma=0} + \text{total } \tau\text{-derivative} \ . \qquad (2.8)
\end{aligned}
$$

The Euler-Lagrange variations for this action are compatible with the supersymmetry transformations in (2.6) with ψ'^{μ}_{+} instead of ψ^{μ}_{+}.

Another point of view is to add the following boundary term to the original action in (2.2):

$$\Delta\mathcal{S} = -\mathrm{i}\int \mathrm{d}\tau B_{\mu\nu}\psi^{\mu}_{+}\psi^{\nu}_{+} \ . \tag{2.9}$$

This new action takes the same form as the action with ψ'^{μ}_{+} in (2.8).

Another solution is to add a boundary term which cancels the fermionic part of the boundary term in the original action:

$$\Delta'\mathcal{S} = -\frac{\mathrm{i}}{2}\int \mathrm{d}\tau B_{\mu\nu}(\psi^{\mu}_{-}\psi^{\nu}_{-} + \psi^{\mu}_{+}\psi^{\nu}_{+}) \ . \tag{2.10}$$

This was done in [ALZ]. The authors claim that the above boundary term is the correct term that extends to the case of general, non-constant B-field. In other words, the fermionic boundary terms in the original action (2.2) have to be dropped.

Superconformal variation

The terms in the action (2.2) that are proportional to the metric $g_{\mu\nu}$ are invariant under off-shell superconformal transformations (parameters satisfy $\partial_{\mp\mp}\epsilon^{\pm} = 0$) provided

- the constraint $\epsilon^{+} = \pm\epsilon^{-}$ is imposed at the boundaries.
- Extra boundary terms of the form $\frac{1}{2}X^{\mu}F_{\mu} - \frac{1}{4}\partial_{\sigma}(X^2)$ must be added to cancel variations from the bulk.

This is the standard story and has been dealt with in great detail in the lecture notes [RvN]. Work of a similar spirit has been done in [LRvN].

In the case of a constant B-field, once we add the boundary term in (2.10) to cancel the fermionic boundary terms, only the bosonic term $\int \mathrm{d}\tau B_{\mu\nu}\partial_{\tau}X^{\mu}X^{\nu}$ contributes to the superconformal variation. Clearly this cannot cancel on its own and extra terms must be added. We leave this as an open question and proceed further.

For the remainder of this section, we assume that the metric $g_{\mu\nu}$ of flat spacetime is the standard Minkowski metric and choose a coordinate system such that the constant spatial B-field is in block diagonal form:

$$2\pi\alpha' B = \begin{pmatrix} 0 & b_1 & & & \\ -b_1 & 0 & & & \\ & & 0 & b_2 & \\ & & -b_2 & 0 & \\ & & & & \ddots \end{pmatrix} . \tag{2.11}$$

If the metric contains off-diagonal components, it is in general not possible to cast the B-field in the above form since the metric and B-field preserve different subgroups of $\mathrm{GL}(1,9)$. In such a coordinate system, the above analysis reduces to that of an open string in $\mathbf{R}^2$ with a constant B-field $B_{12} = -B_{21} = b/2\pi\alpha'$. We study the worldsheet bosons and fermions separately next.

2.1 Worldsheet bosons

In terms of $Z := \frac{1}{\sqrt{2}}(X^1 + \mathrm{i}X^2)$ the boundary condition becomes

$$(\partial_\sigma Z + 2\pi\mathrm{i}\alpha' B\, \partial_\tau Z)\delta\overline{Z}\Big|_{\sigma=0} = 0 . \tag{2.12}$$

Thus, we can have two types of boundary conditions at each end:

$$\begin{aligned} &\textbf{Dirichlet (D)} : \quad \delta Z = 0 \;\text{ i.e. }\; Z = z_0 \in \mathbf{C} , \\ &\textbf{Mixed (M)} : \quad \partial_\sigma Z + 2\pi\mathrm{i}\alpha' B\, \partial_\tau Z = 0 \;\text{ or }\; \partial_{++} Z = e^{-2\pi\mathrm{i}v}\partial_{--} Z . \end{aligned} \tag{2.13}$$

with $2\pi\alpha' B = \tan \pi v$. Note that Dirichlet boundary conditions are realised by taking $v \to \infty$. In order to accommodate all types of boundary conditions at both ends, we introduce the more general boundary conditions

$$\partial_{++}Z = e^{-2\pi \mathrm{i}\nu}\partial_{--}Z \ , \quad \text{at } \sigma = 0 \ ,$$
$$\partial_{++}Z = e^{-2\pi \mathrm{i}\mu}\partial_{--}Z \ , \quad \text{at } \sigma = \pi \ . \tag{2.14}$$

The boundary conditions with B-field can be realised by taking $\nu = \upsilon$, $\mu = \frac{1}{2}$ for the **MD** case and $\nu = \frac{1}{2}$, $\mu = \upsilon$ for the **DM** case. The solution to the Z field equation consists of independent left-moving and right-moving waves:

$$Z(\tau,\sigma) = \tfrac{1}{2}Z_L(\tau+\sigma) + \tfrac{1}{2}Z_R(\tau-\sigma) \ , \tag{2.15}$$

with the mode expansions

$$Z_L = z_L + \ell^2 p_L(\tau+\sigma) + \ell \sum_{k\neq 0} \frac{\alpha_{L,k}}{k}\, e^{-\mathrm{i}k(\tau+\sigma)} \ ,$$
$$Z_R = z_R + \ell^2 p_R(\tau-\sigma) + \ell \sum_{k\neq 0} \frac{\alpha_{R,k}}{k}\, e^{-\mathrm{i}k(\tau-\sigma)} \ . \tag{2.16}$$

Here, ℓ is the string length. The boundary conditions relate the modes in Z_L and Z_R as

$$p_L = e^{-2\pi\mathrm{i}\nu} p_R \ , \quad \alpha_{L,k} = e^{-2\pi\mathrm{i}\nu}\alpha_{R,k} \ ,$$
$$p_L = e^{-2\pi\mathrm{i}\mu} p_R \ , \quad \alpha_{L,k}\, e^{-\mathrm{i}k\pi} = e^{-2\pi\mathrm{i}\mu} e^{\mathrm{i}k\pi}\, \alpha_{R,k} \ . \tag{2.17}$$

For $\nu \neq \mu$ we get $p_L = p_R = 0$ and

$$e^{2\pi\mathrm{i}(k-\mu+\nu)} = 1 \Longrightarrow k \in \mathbf{Z} + \mu - \nu \ . \tag{2.18}$$

Let $z = \frac{1}{2}(z_L + z_R)$, $\theta = \mu - \nu$ and $\theta_n = n + \theta$. The mode expansion for Z becomes

$$Z(\tau,\sigma) = z + \ell \left[\sum_{m=1}^{\infty} \frac{\alpha_m}{\theta_m}\, f_m(\tau,\sigma) + \sum_{n=0}^{\infty} \frac{\beta_n^\dagger}{\theta_{-n}}\, f_{-n}(\tau,\sigma) \right] \ ,$$
$$\text{with} \quad f_n(\tau,\sigma) = e^{-\mathrm{i}\pi\nu} e^{-\mathrm{i}\theta_n \tau} \cos[\theta_n \sigma + \pi\nu] \ . \tag{2.19}$$

The oscillators α_m, β_n are defined as $\alpha_{R,m} = \alpha_m$ for $m \geq 1$ and $\alpha_{R,n} = \beta_n^\dagger$ for $n \geq 0$.

Note: For $\theta = 0$, there will be no $\beta_0^\dagger$ term above but there will be a momentum zero-mode $\ell^2 p_R e^{-\mathrm{i}\pi\nu} (\tau \cos \pi\nu - \mathrm{i}\sigma \sin \pi\nu)$. We handle this case separately below. We introduce the notation $b = \tan \pi\nu$ and $b' = \tan \pi\mu$. The functions $\varphi_n(\sigma) := \cos[\theta_n \sigma + \pi\nu]$ satisfy the completeness relation:

$$\int_0^\pi \mathrm{d}\sigma \, \big[(\theta_m + \theta_n) + b\;\delta(\sigma) - b'\;\delta(\pi - \sigma)\big] \varphi_m(\sigma)\varphi_n(\sigma) = \pi\theta_m\delta_{mn} \; . \tag{2.20}$$

Next, we explore the completeness relations for $f_n(\tau, \sigma)$. We have

$$\int_0^\pi \mathrm{d}\sigma \;\bar{f}_m \big[\mathrm{i}\overleftrightarrow{\partial_\tau} + b\;\delta(\sigma) - b'\;\delta(\pi - \sigma)\big] f_n = \pi\theta_m\delta_{mn} \; , \tag{2.21}$$

The f_n are orthogonal to the constant mode 1:

$$\int_0^\pi \mathrm{d}\sigma \, \big[\mathrm{i}\overleftrightarrow{\partial_\tau} + b\;\delta(\sigma) - b'\;\delta(\pi - \sigma)\big] f_n = 0 \; . \tag{2.22}$$

Using the above relations one can invert the formula for Z to obtain

$$\begin{aligned} z &= \frac{1}{b - b'} \int \mathrm{d}\sigma \left[\mathrm{i}\partial_\tau Z + (b\,\delta(\sigma) - b'\,\delta(\pi - \sigma))\, Z\right] \; , \\ \ell\, \alpha_m &= \int \frac{\mathrm{d}\sigma}{\pi} \left[\mathrm{i}\bar{f}_m \partial_\tau Z + (\theta_m + b\,\delta(\sigma) - b'\,\delta(\pi - \sigma))\, \bar{f}_m Z\right] \; , \\ \ell\, \beta_n^\dagger &= \int \frac{\mathrm{d}\sigma}{\pi} \left[\mathrm{i}\bar{f}_{-n} \partial_\tau Z + (\theta_{-m} + b\,\delta(\sigma) - b'\,\delta(\pi - \sigma))\, \bar{f}_{-n} Z\right] \; . \end{aligned} \tag{2.23}$$

To quantise the system, we impose the following equal-time commutation relations which are valid except possibly at the boundaries where there can be finite discontinuities:

$$\begin{aligned} [P(\tau, \sigma) \, , Z(\tau, \sigma')] = -\mathrm{i}\hbar\delta(\sigma, \sigma') \quad , \quad [\bar{P}(\tau, \sigma) \, , \bar{Z}(\tau, \sigma')] = -\mathrm{i}\hbar\delta(\sigma, \sigma') \; , \\ [P(\tau, \sigma) \, , \bar{P}(\tau, \sigma')] = 0 \; , \quad [Z(\tau, \sigma) \, , \bar{Z}(\tau, \sigma')] = 0 \; . \end{aligned} \tag{2.24}$$

The conjugate momentum $P(\tau, \sigma)$ is given by

$$P(\tau, \sigma) = \frac{\partial L}{\partial(\partial_\tau Z(\tau, \sigma))} = \frac{1}{2\pi\alpha'} \left[\partial_\tau \bar{Z}(\tau, \sigma) - \frac{\mathrm{i}b'}{2}\, \bar{Z}(\tau, \pi)\delta(\pi - \sigma) + \frac{\mathrm{i}b}{2}\, \bar{Z}(\tau, 0)\delta(\sigma)\right] \; . \tag{2.25}$$

In terms of $Z(\tau,\sigma)$ and $P(\tau,\sigma)$ the zero mode and oscillators are given by

$$
\begin{aligned}
z &= \frac{1}{b-b'}\int \mathrm{d}\sigma \left[2\pi \mathrm{i}\alpha' \,\overline{P} + \left(\tfrac{b}{2}\,\delta(\sigma) - \tfrac{b'}{2}\,\delta(\pi-\sigma)\right) Z\right] \ , \\
\ell\,\alpha_m &= \int \frac{\mathrm{d}\sigma}{\pi}\left[2\pi \mathrm{i}\alpha' \bar{f}_m\,\overline{P} + \left(\theta_m + \tfrac{b}{2}\,\delta(\sigma) - \tfrac{b'}{2}\,\delta(\pi-\sigma)\right)\bar{f}_m Z\right] \ , \\
\ell\,\beta_n^\dagger &= \int \frac{\mathrm{d}\sigma}{\pi}\left[2\pi \mathrm{i}\alpha' \bar{f}_{-n}\,\overline{P} + \left(\theta_{-n} + \tfrac{b}{2}\,\delta(\sigma) - \tfrac{b'}{2}\,\delta(\pi-\sigma)\right)\bar{f}_{-n} Z\right] \ .
\end{aligned}
\tag{2.26}
$$

Setting $2\alpha' = \ell^2$ and using the above completeness relations, we get

$$
\boxed{[z,\bar{z}] = \frac{\pi\ell^2}{b-b'} \ , \quad [\alpha_m\,,\alpha^\dagger_{m'}] = (m+\theta)\delta_{mm'} \ , \quad [\beta_n\,,\beta^\dagger_{n'}] = (n-\theta)\delta_{nn'} \ .}
\tag{2.27}
$$

Note: In obtaining the above commutation relations, one has to evaluate the integral

$$
\int \mathrm{d}\sigma \mathrm{d}\sigma' (\cdots)[Z(\tau,\sigma), \overline{Z}(\tau,\sigma')] \ .
$$

This integral has been set to zero since, according to our ansatz in (2.24), the commutator $[Z,\overline{Z}]$ is non-zero only at isolated points in the interval $(\sigma,\sigma') \in [0,\pi]\times[0,\pi]$ and the integral is not affected by these jumps in the value of $[Z,\overline{Z}]$.

We now verify that our ansatz for the canonical commutation relations in (2.24) is correct. Define ε_m such that $\varepsilon_m = 1$ for $m=0$ and $\varepsilon_m = 2$ for $m\geq 1$. We need the following series expansions from the Appendix of [MO]. Let $2\alpha\pi \leq x \leq (2\alpha+2)\pi$. Then, we have

$$
\begin{aligned}
\sum_0^\infty \frac{\varepsilon_m \cos(mx)}{m^2-\theta^2} &= -\frac{\pi\cos[((2\alpha+1)\pi - x)\theta]}{\theta\sin\pi\theta} \ , \\
\sum_0^\infty \frac{m\sin(mx)}{m^2-\theta^2} &= \frac{\pi\sin[((2\alpha+1)\pi - x)\theta]}{2\sin\pi\theta} \ .
\end{aligned}
\tag{2.28}
$$

Similarly, for $(2\alpha-1)\pi \leq x \leq (2\alpha+1)\pi$, we have

$$
\begin{aligned}
\sum_0^\infty \frac{(-1)^m\varepsilon_m \cos(mx)}{m^2-\theta^2} &= -\frac{\pi\cos[(2\alpha\pi - x)\theta]}{\theta\sin\pi\theta} \ , \\
\sum_0^\infty \frac{(-1)^m m\sin(mx)}{m^2-\theta^2} &= \frac{\pi\sin[(2\alpha\pi - x)\theta]}{2\sin\pi\theta} \ .
\end{aligned}
\tag{2.29}
$$

$[\boldsymbol{Z(\tau,\sigma)}\,,\boldsymbol{\bar{Z}(\tau,\sigma')}]$

$$
\begin{aligned}
[Z(\tau,\sigma)\,,\bar{Z}(\tau,\sigma')] &= [z\,,\bar{z}] + \ell^2 \sum_{-\infty}^{\infty} \frac{1}{\theta_n} \cos(\theta_n\sigma + \pi\nu)\cos(\theta_n\sigma' + \pi\nu)\ , \\
&= \frac{\pi\ell^2}{b-b'} + \frac{\ell^2}{2} s_1(\sigma+\sigma') + \frac{\ell^2}{2} s_2(\sigma-\sigma')\ . \qquad (2.30)
\end{aligned}
$$

The terms s_1 and s_2 arise from writing the product of cosines as a sum of two cosines. We focus on the two series next. let $a_1 = \theta(\sigma+\sigma') + 2\pi\nu$ and $a_2 = \theta(\sigma-\sigma')$. We have

$$
\begin{aligned}
s_1(\sigma+\sigma') &= \sum_{-\infty}^{\infty} \frac{1}{\theta_n} \cos\left[\theta_n(\sigma+\sigma') + 2\pi\nu\right]\ , \\
&= -\theta\cos a_1 \sum_{0}^{\infty} \frac{\varepsilon_n \cos(n(\sigma+\sigma'))}{n^2-\theta^2} - 2\sin a_1 \sum_{0}^{\infty} \frac{n\sin(n(\sigma+\sigma'))}{n^2-\theta^2}\ , \\
&= \frac{\pi\theta\cos[\theta(\pi-(\sigma+\sigma'))]\cos a_1}{\theta\sin\pi\theta} - \frac{2\sin[\theta(\pi-(\sigma+\sigma'))]\sin a_1}{2\sin\pi\theta}\ , \\
&= \pi\frac{\cos[(\mu+\nu)\pi]}{\sin[(\mu-\nu)\pi]}\ . \qquad (2.31)
\end{aligned}
$$

In going to the third step, we have used the formulas in (2.28) with $\alpha = 0$ since the requirement $0 \leq \sigma+\sigma' \leq 2\pi$ is satisfied. Similarly, we have

$$
\begin{aligned}
s_2(\sigma-\sigma') &= \sum_{-\infty}^{\infty} \frac{1}{\theta_n} \cos\left[\theta_n(\sigma-\sigma')\right]\ , \\
&= -\theta\cos a_2 \sum_{0}^{\infty} \frac{\varepsilon_n \cos(n(\sigma-\sigma'))}{n^2-\theta^2} - 2\sin a_2 \sum_{0}^{\infty} \frac{n\sin(n(\sigma-\sigma'))}{n^2-\theta^2}\ , \\
&= \begin{cases} \pi\cos[\theta(\pi+(\sigma-\sigma'))-a_2]/\sin\pi\theta & \sigma-\sigma'<0 \\ \pi\cos[\theta(\pi-(\sigma-\sigma'))+a_2]/\sin\pi\theta & \sigma-\sigma'>0 \end{cases}\ , \\
&= \pi\frac{\cos[(\mu-\nu)\pi]}{\sin[(\mu-\nu)\pi]} \quad \text{for} \quad -\pi \leq \sigma-\sigma' \leq \pi\ . \qquad (2.32)
\end{aligned}
$$

In the third step, we have split the range of $\sigma-\sigma'$ into $\sigma-\sigma'>0$ and $\sigma-\sigma'<0$ and used the formulas in (2.28) with $\alpha=0$ and $\alpha=-1$ respectively. Putting the above two results together, we get

$$\frac{\ell^2}{2}(s_1+s_2) = \frac{\pi\ell^2}{2}\frac{\cos((\mu+\nu)\pi)+\cos((\mu-\nu)\pi)}{\sin((\mu-\nu)\pi)} = \pi\ell^2\frac{\cos\pi\mu\cos\pi\nu}{\sin((\mu-\nu)\pi)} = \frac{\pi\ell^2}{b'-b} \ . \tag{2.33}$$

Plugging this back in (2.30), we get

$$[Z(\tau,\sigma)\,,\overline{Z}(\tau,\sigma')] = \frac{\pi\ell^2}{b-b'}+\frac{\pi\ell^2}{b'-b} = 0 \ . \tag{2.34}$$

$[\boldsymbol{P}(\boldsymbol{\tau},\boldsymbol{\sigma})\,,\boldsymbol{Z}(\boldsymbol{\tau},\boldsymbol{\sigma}')]$

$$2\pi\alpha'[P(\sigma)\,,Z(\sigma')] = \left[\partial_\tau\overline{Z}(\sigma)-\tfrac{ib'}{2}\,\overline{Z}(\pi)\delta(\pi-\sigma)+\tfrac{ib}{2}\,\overline{Z}(0)\delta(\sigma)\,,Z(\sigma')\right] \ . \tag{2.35}$$

We calculate each piece individually. First, we have

$$[\overline{Z}(\pi)\,,Z(\sigma')] = \frac{\pi\ell^2}{b'-b}-\ell^2\sum_{-\infty}^{\infty}\frac{(-1)^m\cos\pi\mu}{\theta_m}\cos(\theta_m\sigma'+\pi\nu) \ , \tag{2.36}$$

$$[\overline{Z}(0)\,,Z(\sigma')] = \frac{\pi\ell^2}{b'-b}-\ell^2\sum_{-\infty}^{\infty}\frac{\cos\pi\nu}{\theta_m}\cos(\theta_m\sigma'+\pi\nu) \ . \tag{2.37}$$

Let us write $a'=\theta\sigma'+\pi\nu$. Then the infinite series part of the commutators in (2.36) can be simplified using the formulas in (2.29) with $\alpha=0$.

$$\begin{aligned}
&\ell^2\cos\pi\mu\left[\cos a'\sum_{-\infty}^{\infty}\frac{(-1)^m\cos(m\sigma')}{m+\theta}-\sin a'\sum_{-\infty}^{\infty}\frac{(-1)^m\sin(m\sigma')}{m+\theta}\right] \ , \\
&= -\ell^2\cos\pi\mu\left[\theta\cos a'\sum_{0}^{\infty}\frac{(-1)^m\varepsilon_m\cos(m\sigma')}{m^2-\theta^2}+2\sin a'\sum_{0}^{\infty}\frac{(-1)^m m\sin(m\sigma')}{m^2-\theta^2}\right] \ , \\
&= \pi\ell^2\cos\pi\mu\,\frac{\cos(\theta\sigma'-a')}{\sin\theta\pi} = \pi\ell^2\frac{\cos\pi\mu\cos\pi\nu}{\sin\pi\theta} = \frac{\pi\ell^2}{b'-b} \ .
\end{aligned} \tag{2.38}$$

We get the same answer as above for the commutators in (2.37). Thus, we get

$$[\overline{Z}(0)\,,Z(\sigma')] = [\overline{Z}(\pi)\,,Z(\sigma')] = 0 \ . \tag{2.39}$$

Next, we have, using the commutators in (2.27),

$$[\partial_\tau \overline{Z}(\sigma)\,, Z(\sigma')] = -\mathrm{i}\,\ell^2 \sum_{m=-\infty}^{\infty} \overline{f}_m(\sigma) f_m(\sigma') = -\mathrm{i}\,\ell^2 \sum_{m=-\infty}^{\infty} \varphi_m(\sigma)\varphi_m(\sigma')\ . \tag{2.40}$$

We write the above series as the derivative of another series which has better convergence properties:

$$\begin{aligned}
\sum_{-\infty}^{\infty} \varphi_m(\sigma)\varphi_m(\sigma') &= \partial_\sigma \sum_{-\infty}^{\infty} \frac{1}{\theta_m} \sin(\theta_m\sigma + \pi\nu)\cos(\theta_m\sigma' + \pi\nu)\ , \\
&= \frac{1}{2}\partial_\sigma \sum_{-\infty}^{\infty} \frac{1}{\theta_m} \left(\sin(\theta_m(\sigma+\sigma') + 2\pi\nu) + \sin(\theta_m(\sigma-\sigma'))\right)\ , \\
&= \frac{1}{2}\partial_\sigma \left[t_1(\sigma+\sigma') + t_2(\sigma-\sigma')\right]\ .
\end{aligned} \tag{2.41}$$

Recall that $a_1 = (\sigma+\sigma')\theta + \pi\nu$ and $a_2 = (\sigma-\sigma')\theta$. We have

$$\begin{aligned}
t_1(\sigma+\sigma') &= \sum_{-\infty}^{\infty} \frac{1}{\theta_m} \sin\left[m(\sigma+\sigma') + a_1\right]\ , \\
&= -2\cos(a_1) \sum_{-\infty}^{\infty} \frac{\sin\left[m(\sigma+\sigma')\right]}{\theta^2 - m^2} + \sin(a_1) \sum_{-\infty}^{\infty} \frac{1}{\theta_m} \cos\left[m(\sigma+\sigma')\right]\ , \\
&= \frac{\pi}{\sin\pi\theta} \left(\cos a_1 \sin\left[(\pi - (\sigma+\sigma'))\theta\right] + \sin a_1 \cos\left[(\pi - (\sigma+\sigma'))\theta\right]\right)\ , \\
&= \pi\, \frac{\sin[\pi(\mu+\nu)]}{\sin[\pi(\mu-\nu)]}\ .
\end{aligned} \tag{2.42}$$

$$\begin{aligned}
t_2(\sigma-\sigma') &= \sum_{-\infty}^{\infty} \frac{1}{\theta_m} \sin\left[\theta_m(\sigma-\sigma')\right]\ , \\
&= -\theta \sin a_2 \sum_{0}^{\infty} \frac{\varepsilon_n \cos(n(\sigma-\sigma'))}{n^2 - \theta^2} + 2\cos a_2 \sum_{0}^{\infty} \frac{n \sin(n(\sigma-\sigma'))}{n^2-\theta^2}\ , \\
&= \pi\,\mathrm{sgn}(\sigma-\sigma')\ .
\end{aligned} \tag{2.43}$$

This gives, using $\ell^2 = 2\alpha'$,

$$[P(\sigma)\,, Z(\sigma')] = -\frac{\mathrm{i}}{2\pi}\partial_\sigma \left[t_1(\sigma+\sigma') + t_2(\sigma-\sigma')\right] = -\mathrm{i}\delta(\sigma,\sigma')\ . \tag{2.44}$$

Alternate derivation: Let us expand the Dirac delta function $\delta(\sigma-\sigma')$ in terms of the mode functions f_n. In general, we can write

$$A(\tau,\sigma)=\widetilde{a}_0+\sum_{n=-\infty}^{\infty}\frac{a_m}{\theta_m}f_m(\tau,\sigma)\ . \tag{2.45}$$

The coefficients a_n and $\widetilde{a}_0$ are extracted using the completeness relations in (2.21) and (2.22):

$$\begin{aligned}
a_m &= \int\frac{\mathrm{d}\sigma}{\pi}\bar{f}_n(\tau,\sigma)\Big[\mathrm{i}\overleftrightarrow{\partial_\tau}+b\delta(\sigma)-b'\delta(\pi-\sigma)\Big]A(\tau,\sigma)\ ,\\
\widetilde{a}_0 &= \frac{1}{b-b'}\int\mathrm{d}\sigma\Big[\mathrm{i}\overleftrightarrow{\partial_\tau}+b\delta(\sigma)-b'\delta(\pi-\sigma)\Big]A(\tau,\sigma)\ .
\end{aligned}\tag{2.46}$$

Taking $A(\tau,\sigma)=\delta(\sigma,\sigma')$ (independent of τ), we get

$$\begin{aligned}
\widetilde{a}_0 &= \frac{1}{b-b'}\int\mathrm{d}\sigma\Big[\mathrm{i}\overleftrightarrow{\partial_\tau}+b\delta(\sigma)-b'\delta(\pi-\sigma)\Big]\delta(\sigma,\sigma')=\frac{b\delta(\sigma')-b'\delta(\pi-\sigma')}{b-b'}\ ,\\
a_n &= \int\frac{\mathrm{d}\sigma}{\pi}\bar{f}_n(\tau,\sigma)\big[\theta_n+b\delta(\sigma)-b'\delta(\pi-\sigma)\big]\delta(\sigma,\sigma')\ ,\\
&= \frac{\theta_n}{\pi}\bar{f}_n(\tau,\sigma')+\int\frac{\mathrm{d}\sigma}{\pi}\big[b\delta(\sigma)\bar{f}_n(\tau,0)-b'\delta(\pi-\sigma)\bar{f}_n(\tau,\pi)\big]\delta(\sigma,\sigma')\ ,\\
&= \frac{\theta_n}{\pi}\bar{f}_n(\tau,\sigma')+\frac{e^{\mathrm{i}\theta_n\tau}}{\pi}\big[b\delta(\sigma')\cos\pi\nu-b'\delta(\pi-\sigma')(-1)^n\cos\pi\mu\big]\ .
\end{aligned}\tag{2.47}$$

Thus, we have

$$\begin{aligned}
\delta(\sigma,\sigma') &= \frac{1}{\pi}\sum_{m=-\infty}^{\infty}\varphi_m(\sigma)\varphi_m(\sigma')+\frac{b\delta(\sigma')-b'\delta(\pi-\sigma')}{b-b'}+\\
&\quad+\frac{1}{\pi}\sum_{m=-\infty}^{\infty}\frac{\varphi_m(\sigma)}{\theta+m}\big[b\delta(\sigma')\cos\pi\nu-b'\delta(\pi-\sigma')(-1)^m\cos\pi\mu\big]\ ,\\
&= \frac{1}{\pi}\sum_{m=-\infty}^{\infty}\varphi_m(\sigma)\varphi_m(\sigma')+\\
&\qquad+\frac{b\delta(\sigma')-b'\delta(\pi-\sigma')}{b-b'}+\frac{1}{\pi}\Big[\frac{b\pi}{b'-b}\delta(\sigma')-\frac{b'\pi}{b'-b}\delta(\pi-\sigma')\Big]\ ,\\
&\overset{!}{=}\frac{1}{\pi}\sum_{m=-\infty}^{\infty}\varphi_m(\sigma)\varphi_m(\sigma')\ .
\end{aligned}\tag{2.48}$$

Finally, using $\ell^2 = 2\alpha'$, we get

$$[P(\sigma)\,,Z(\sigma')] = -\mathrm{i}\delta(\sigma,\sigma')\ . \tag{2.49}$$

$[\boldsymbol{P(\tau,\sigma)}\,,\boldsymbol{\overline{P}(\tau,\sigma')}]$

The formula for the momentum $P(\tau,\sigma)$ is

$$2\pi\alpha' P(\tau,\sigma) = \partial_\tau \overline{Z}(\tau,\sigma) - \frac{\mathrm{i}b'}{2}\,\overline{Z}(\tau,\pi)\delta(\pi-\sigma) + \frac{\mathrm{i}b}{2}\,\overline{Z}(\tau,0)\delta(\sigma)\ . \tag{2.50}$$

We get

$$\begin{aligned}(2\pi\alpha')^2[P(\sigma)\,,\overline{P}(\sigma')] = {}& [\partial_\tau\overline{Z}(\sigma)\,,\partial_\tau Z(\sigma')] + \\ & + \frac{\mathrm{i}b'}{2}\delta(\pi-\sigma')[\partial_\tau\overline{Z}(\sigma)\,,Z(\pi)] - \frac{\mathrm{i}b}{2}\delta(\sigma')[\partial_\tau\overline{Z}(\sigma)\,,Z(0)] + \\ & - \frac{\mathrm{i}b'}{2}\delta(\pi-\sigma)[\overline{Z}(\pi)\,,\partial_\tau Z(\sigma')] + \frac{\mathrm{i}b}{2}\delta(\sigma)[\overline{Z}(0)\,,\partial_\tau Z(\sigma')]\ . \end{aligned} \tag{2.51}$$

The first term simplifies as follows:

$$\begin{aligned}[\partial_\tau\overline{Z}(\sigma)\,,\partial_\tau Z(\sigma')] &= \sum_{-\infty}^{\infty}\theta_m\varphi_m(\sigma)\varphi_m(\sigma') = \partial_\sigma\partial_{\sigma'}\sum_{-\infty}^{\infty}\frac{1}{\theta_m}\sin(\theta_m\sigma+\pi\nu)\sin(\theta_m\sigma+\pi\nu)\ , \\ &= \partial_\sigma\partial_{\sigma'}[s_1(\sigma+\sigma') - s_2(\sigma-\sigma')] = 0\ . \end{aligned} \tag{2.52}$$

Next we look at $[\partial_\tau\overline{Z}(\sigma)\,,Z(\pi)]$. Let $a = \theta\sigma + \pi\nu$. Then, the commutator equals

$$\begin{aligned} &= \cos\pi\mu\sum_{-\infty}^{\infty}(-1)^m\cos(\theta_m\sigma+\pi\nu) = \cos\pi\mu\ \partial_\sigma\sum_{-\infty}^{\infty}\frac{(-1)^m\sin(\theta_m\sigma+\pi\nu)}{\theta_m}\ , \\ &= \cos\pi\mu\ \partial_\sigma\left[\cos a\sum_{-\infty}^{\infty}\frac{(-1)^m\sin(m\sigma)}{m+\theta} + \sin a\sum_{-\infty}^{\infty}\frac{(-1)^m\cos(m\sigma)}{m+\theta}\right]\ , \\ &= \cos\pi\mu\ \partial_\sigma\left[2\cos a\sum_{0}^{\infty}\frac{(-1)^m m\sin(m\sigma)}{m^2-\theta^2} - \theta\sin a\sum_{0}^{\infty}\frac{(-1)^m\varepsilon_m\cos(m\sigma)}{m^2-\theta^2}\right]\ , \\ &= \pi\cos\pi\mu\ \partial_\sigma\frac{\sin(a-\theta\sigma)}{\sin\theta\pi} = \pi\partial_\sigma\frac{\cos\pi\mu\sin\pi\nu}{\sin\pi\theta} = 0\ . \end{aligned} \tag{2.53}$$

Similarly, the other three commutators with delta functions in (2.51) are zero. Thus, we have

$$[P(\sigma)\,,\overline{P}(\sigma')] = 0\ . \tag{2.54}$$

The case $\mu = \nu$

Let $\mu = \nu$ i.e. $\theta = 0$. From (2.18), we see that the mode numbers are integers. Let $\bar{p} := p_R \cos\pi v\, e^{-\mathrm{i}\pi\nu}$. Recall that $b = \tan\pi\nu$. The mode expansion for Z becomes

$$\begin{aligned} Z(\tau,\sigma) &= z + \ell^2\bar{p}(\tau - \mathrm{i}b\sigma) + \ell\sum_{m=1}^{\infty}\left[\frac{\alpha_m}{m}\ f_m(\tau,\sigma) - \frac{\beta_m^\dagger}{m}\ f_{-m}(\tau,\sigma)\right]\ , \\ \text{with}\quad & f_n(\tau,\sigma) = e^{-\mathrm{i}\pi\nu}e^{-\mathrm{i}n\tau}\cos[n\sigma+\pi\nu]\ . \end{aligned} \tag{2.55}$$

The functions $\varphi_n(\sigma) = \cos[n\sigma + \pi\nu]$ satisfy the completeness relation in (2.20) with $\theta = 0$:

$$\int_0^\pi \mathrm{d}\sigma\ \big[(m+n) + b\ \delta(\sigma) - b\ \delta(\pi-\sigma)\big]\varphi_m(\sigma)\varphi_n(\sigma) = \pi m\delta_{mn}\ , \tag{2.56}$$

which can be written as

$$\int_0^\pi \mathrm{d}\sigma\ \big[(m+n) - b\,\partial_\sigma\big]\varphi_m(\sigma)\varphi_n(\sigma) = \pi m\delta_{mn}\ . \tag{2.57}$$

Given two functions $f(x)$ and $g(x)$, define the operator $\widetilde{\partial}$ to satisfy $\widetilde{\partial}(f\cdot g) = f\cdot\partial g - (\partial f)\cdot g$. Note that if f, g, p, q are such that $f\cdot g = p\cdot q$, then $\widetilde{\partial}(f\cdot g) \neq \widetilde{\partial}(p\cdot q)$ in general. For example, take $f = g = x$, $p = 1$, $q = x^2$. Then, we have $\widetilde{\partial}(f\cdot g) = 0$ whereas $\widetilde{\partial}(p\cdot x) = 2x$. Then, we have

$$\int_0^\pi \mathrm{d}\sigma\ \big[\mathrm{i}\widetilde{\partial}_\tau - b\,\partial_\sigma\big](\overline{f}_m\cdot f_n) = \pi m\delta_{mn}\ , \tag{2.58}$$

for f_n with the constant mode 1:

$$\int_0^\pi \mathrm{d}\sigma\ \big[\mathrm{i}\widetilde{\partial}_\tau - b\,\partial_\sigma\big](1\cdot f_n) = 0\ , \tag{2.59}$$

and for the momentum mode $\bar{p}$

$$
\begin{aligned}
&\text{with itself}: \int_0^\pi \mathrm{d}\sigma\ \left[\mathrm{i}\widetilde{\partial}_\tau - b\,\partial_\sigma\right]\big((\tau+\mathrm{i}b\sigma)\cdot(\tau-\mathrm{i}b\sigma)\big) = -\pi^2 b(1+b^2)\ ,\\
&\text{with } f_n: \quad \int_0^\pi \mathrm{d}\sigma\ \left[\mathrm{i}\widetilde{\partial}_\tau - b\,\partial_\sigma\right]\big((\tau+\mathrm{i}b\sigma)\cdot f_n\big) = 0\ ,\\
&\text{with } 1: \quad \int_0^\pi \mathrm{d}\sigma\ \left[\mathrm{i}\widetilde{\partial}_\tau - b\,\partial_\sigma\right]\big((\tau+\mathrm{i}b\sigma)\cdot 1\big) = -\mathrm{i}\pi(1+b^2)\ .
\end{aligned}
\tag{2.60}
$$

Using the above relations one can invert the formula for Z to obtain

$$
\begin{aligned}
\bar{p} &= \frac{1}{\mathrm{i}\ell^2(1+b^2)}\int\frac{\mathrm{d}\sigma}{\pi}\left[(\mathrm{i}\partial_\tau - b\,\partial_\sigma)\,Z\right]\ ,\\
\ell\,\alpha_m &= \int\frac{\mathrm{d}\sigma}{\pi}\left[\mathrm{i}\overline{f}_m\partial_\tau Z + (m - b\,\partial_\sigma)\,\overline{f}_m Z\right]\ ,\\
\ell\,\beta_n^\dagger &= \int\frac{\mathrm{d}\sigma}{\pi}\left[\mathrm{i}\overline{f}_{-n}\partial_\tau Z + (-n - b\,\partial_\sigma)\,\overline{f}_{-n} Z\right]\ .
\end{aligned}
\tag{2.61}
$$

The conjugate momentum $P(\tau,\sigma)$ is given by

$$
P(\tau,\sigma) = \frac{\partial L}{\partial(\partial_\tau Z(\tau,\sigma))} = \frac{1}{2\pi\alpha'}\left[\partial_\tau\overline{Z}(\tau,\sigma) - \mathrm{i}b\,\partial_\sigma\overline{Z}(\tau,\sigma)\right]\ .
\tag{2.62}
$$

In terms of $Z(\tau,\sigma)$ and $P(\tau,\sigma)$ the zero modes and oscillators are given by

$$
\begin{aligned}
z &= \frac{-1}{1+b^2}\int\frac{\mathrm{d}\sigma}{\pi}2\pi\alpha'(\tau+\mathrm{i}b(\sigma-\pi))\overline{P} + \int\frac{\mathrm{d}\sigma}{\pi}Z\ ,\\
\bar{p} &= \frac{1}{\ell^2(1+b^2)}\int\frac{\mathrm{d}\sigma}{\pi}\,2\pi\alpha'\,\overline{P}\ ,\\
\ell\,\alpha_m &= \int\frac{\mathrm{d}\sigma}{\pi}\left[2\pi\mathrm{i}\alpha'\overline{f}_m\,\overline{P} + \big(m - b\,(\partial_\sigma\overline{f}_m)\big)\,Z\right]\ ,\\
\ell\,\beta_n^\dagger &= \int\frac{\mathrm{d}\sigma}{\pi}\left[2\pi\mathrm{i}\alpha'\overline{f}_{-n}\,\overline{P} + \big(-n - b\,(\partial_\sigma\overline{f}_{-n})\big)\,Z\right]\ .
\end{aligned}
\tag{2.63}
$$

Setting $2\alpha' = \ell^2$ and using the above completeness relations, we get

$$
\boxed{[z,\bar{z}] = \pi\alpha'\sin 2\pi v\ ,\quad [z,p] = \mathrm{i}\cos^2\pi v\ ,\quad [\alpha_m\,,\alpha_{m'}^\dagger] = [\beta_m\,,\beta_{m'}^\dagger] = m\delta_{mm'}\ .}
\tag{2.64}
$$

Next, we compute

$$
\begin{aligned}
[Z(\tau,\sigma)\,,\overline{Z}(\tau,\sigma')] = & \frac{\ell^2 b}{1+b^2}(\pi-\sigma-\sigma')+ \\
& + \quad \ell^2 \sum_{m=1}^{\infty} \frac{1}{m}[\bar{f}_m(\tau,\sigma) f_m(\tau,\sigma') - \bar{f}_{-m}(\tau,\sigma) f_{-m}(\tau,\sigma')] \ , \\
& = \frac{\ell^2 b}{1+b^2}\left[\pi - (\sigma+\sigma') - \sum_{m=1}^{\infty} \frac{2}{m} \sin\left(m(\sigma+\sigma')\right)\right] \ .
\end{aligned} \tag{2.65}
$$

In the last line, we recognise the Fourier series of the sawtooth wave $g(\omega) = \omega$ for $\omega \in (0, 2\pi)$:

$$
g(\omega) = \pi - \sum_{m \neq 0}^{\infty} \frac{1}{m} \sin m\omega = \omega \ . \tag{2.66}
$$

Thus, we have, with $\vartheta = \pi\alpha' \sin 2\pi v$,

$$
[Z(\tau,\sigma)\,,\overline{Z}(\tau,\sigma')] = \begin{cases} +\vartheta & \sigma = \sigma' = 0 \ , \\ -\vartheta & \sigma = \sigma' = \pi \ , \\ 0 & \text{otherwise} \ . \end{cases} \tag{2.67}
$$

This is consistent with our ansatz that $[Z(\tau,\sigma)\,,\overline{Z}(\tau,\sigma')] = 0$ except at a few isolated points.

Comments: Our analysis above agrees with that in [CH1, CH2]. In [CH1], the authors define the following time-averaged symplectic form on phase space described by $(Z, \overline{Z}, P, \overline{P})$:

$$
\Omega = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathrm{d}\tau \int_0^{\pi} \mathrm{d}\sigma \ \delta P(\tau,\sigma) \wedge \delta\overline{Z}(\tau,\sigma) + \text{c.c.} \ . \tag{2.68}
$$

By plugging in the mode expansions, they read off the various Poisson brackets between the modes and obtain the Poisson brackets in (2.64). In [CH2], the boundary conditions are considered as constraints in phase space and the Dirac bracket is computed. It turns out that there are an infinite number of second class constraints. The authors directly arrive at the commutation relations (2.24). Similar work has been done in [AAS1, AAS2, SS] but with differing results.

2.2 Worldsheet fermions

In 1+1 dimensions, right(left)-handed spinors are left(right)-moving on-shell and superconformal symmetry relates left-movers to left-movers and right-movers to right-movers:

$$\begin{aligned} \delta Z &= \mathrm{i}\epsilon^{+}\Psi_{+} + \mathrm{i}\epsilon^{-}\Psi_{-}\ , \qquad & \delta\mathcal{F} &= \mathrm{i}\epsilon^{+}\partial_{++}\Psi_{-} - \mathrm{i}\epsilon^{-}\partial_{--}\Psi_{+}\ , \\ \delta\Psi_{-} &= \epsilon^{-}\partial_{--}Z + \epsilon^{+}\mathcal{F}\ , & \delta\Psi_{+} &= \epsilon^{+}\partial_{++}Z - \epsilon^{-}\mathcal{F}\ , \end{aligned} \tag{2.69}$$

where we have introduced the complex combinations $\Psi_{\pm} = \frac{1}{\sqrt{2}}(\psi^1_{\pm} + \mathrm{i}\psi^2_{\pm})$ and $\mathcal{F} = \frac{1}{\sqrt{2}}(F^1 + \mathrm{i}F^2)$. The presence of a boundary reduces the superconformal symmetry by half by imposing a relation between the parameters: $\epsilon^{+} = \pm\epsilon^{-}$. We impose $\epsilon^{+} = \epsilon^{-}$ at one end, say $\sigma = 0$. On the other end, two choices are possible and they correspond to the R and NS sectors:

$$\sigma = \pi : \qquad \begin{cases} \epsilon^{+} = \epsilon^{-} & \text{Ramond}\ , \\ \epsilon^{+} = -\epsilon^{-} & \text{Neveu-Schwarz}\ . \end{cases} \tag{2.70}$$

It is evident that rigid supersymmetry is present only in the R sector and that it has only one parameter $\epsilon = \epsilon^{+} = -\epsilon^{-}$. The boundary condition on $\Psi_{\pm}$ corresponding to $\partial_{++}Z = e^{-2\pi\mathrm{i}\nu}\partial_{--}Z$ at $\sigma = 0$ is given by

$$\Psi_{+} = e^{-2\pi\mathrm{i}\nu}\,\Psi_{-} \quad \text{at} \quad \sigma = 0\ . \tag{2.71}$$

Similarly, the boundary condition at $\sigma = \pi$ is

$$\text{At } \sigma = \pi : \quad \begin{cases} \Psi_{+} = e^{-2\pi\mathrm{i}\mu}\,\Psi_{-} & \text{R sector}\ , \\ \Psi_{+} = -e^{-2\pi\mathrm{i}\mu}\,\Psi_{-} & \text{NS sector}\ . \end{cases} \tag{2.72}$$

In order to write down the mode expansions, we combine $\Psi_{+}(\tau+\sigma)$ and $\Psi_{-}(\tau-\sigma)$ on $0 \le \sigma \le \pi$ into one field Ψ on the double interval $-\pi \le \sigma \le \pi$ such that

$$\Psi(\tau+\sigma) = \begin{cases} \Psi_{-}(\tau+\sigma) & -\pi \le \sigma \le 0\ , \\ e^{2\pi\mathrm{i}\nu}\,\Psi_{+}(\tau+\sigma) & 0 \le \sigma \le \pi\ . \end{cases} \tag{2.73}$$

Treating $-\pi \leq \sigma \leq \pi$ as an angular variable we see that Ψ is continuous at $\sigma = 0$ by virtue of (2.71) and twisted-periodic across $\sigma = \pi$ due to (2.72): $\Psi(\tau+\pi) = \pm e^{2\pi \mathrm{i}(\nu-\mu)}\Psi(\tau-\pi)$. The mode expansion for $\Psi(\tau+\sigma)$ in the R sector is

$$\Psi_{\mathrm{R}}(\tau+\sigma) = \frac{\ell}{2}\left[\sum_{m=1}^{\infty} a_m\, e^{-\mathrm{i}\theta_m(\tau+\sigma)} + \sum_{n=0}^{\infty} b_n^\dagger\, e^{-\mathrm{i}\theta_{-n}\,(\tau+\sigma)}\right] \ , \tag{2.74}$$

and in the NS sector is

$$\Psi_{\mathrm{NS}}(\tau+\sigma) = \frac{\ell}{2}\left[\sum_{r=1}^{\infty} c_r\, e^{-\mathrm{i}\epsilon_r(\tau+\sigma)} + \sum_{s=0}^{\infty} d_s^\dagger\, e^{-\mathrm{i}\epsilon_{-s}\,(\tau+\sigma)}\right] \ , \tag{2.75}$$

where $\epsilon = \theta + \frac{1}{2} = \mu - \nu + \frac{1}{2}$ and $\epsilon_n = \epsilon + n$. The action for the doubled $\Psi(\tau+\sigma)$ is

$$\mathcal{S}[\Psi] = \frac{2\mathrm{i}}{\pi\alpha'}\int \mathrm{d}\tau \int_{-\pi}^{\pi} \mathrm{d}\sigma\ \overline{\Psi}\,\partial_{--}\Psi \ . \tag{2.76}$$

The boundary term for the fermions in (2.2) measures the jump in $\overline{\Psi}_+\Psi_+ + \overline{\Psi}_-\Psi_-$ between two boundary components ($\sigma = 0, \pi$) of the worldsheet. Since $\overline{\Psi}\Psi$ is periodic on the double interval, such a boundary term is absent in the above action. The conjugate momentum is then

$$\Pi(\tau+\sigma) = \frac{\partial\mathcal{L}}{\partial\dot{\Psi}} = -\frac{2\mathrm{i}}{\pi\alpha'}\,\overline{\Psi}(\tau+\sigma) \ . \tag{2.77}$$

The correct equal-time anticommutation relation follows from Dirac's constrained Hamiltonian formalism:

$$\{\Pi(\tau+\sigma)\,,\Psi(\tau+\sigma')\} = -\frac{\mathrm{i}}{2}\delta(\sigma-\sigma') \implies \{\overline{\Psi}(\tau+\sigma)\,,\Psi(\tau+\sigma')\} = \frac{\pi\alpha'}{4}\delta(\sigma-\sigma') \ . \tag{2.78}$$

Using the completeness relations and $2\alpha' = \ell^2$, we get

$$\{a_m\,, a_{m'}^\dagger\} = \delta_{mm'} \ , \quad \{b_n\,, b_{n'}^\dagger\} = \delta_{nn'} \ , \quad \{c_r\,, c_{r'}^\dagger\} = \delta_{rr'} \ , \quad \{d_s\,, d_{s'}^\dagger\} = \delta_{ss'} \ . \tag{2.79}$$

The expression for L_0 in the R and NS sectors is given by

$$L_0^{(\mathrm{R})} = \sum_{m=1}^{\infty} \left[\alpha_m^\dagger \alpha_m + (m+\theta) a_m^\dagger a_m \right] + \sum_{n=0}^{\infty} \left[: \beta_n^\dagger \beta_n : + (n-\theta) : b_n^\dagger b_n : \right] ,$$

$$L_0^{(\mathrm{NS})} - E_0 = \sum_{m=1}^{\infty} \alpha_m^\dagger \alpha_m + \sum_{r=1}^{\infty} (r+\epsilon) c_r^\dagger c_r + \sum_{n=0}^{\infty} : \beta_n^\dagger \beta_n : + \sum_{s=0}^{\infty} (s-\epsilon) : d_s^\dagger d_s : \quad . \tag{2.80}$$

Recall that $\epsilon = \mu - \nu + \frac{1}{2}$. Since $|\mu|, |\nu| < \frac{1}{2}$, we have $-\frac{1}{2} < \epsilon < \frac{3}{2}$. The first few states of the spectrum in the NS sector for different ranges of ϵ are as in Table 2.1. Observe

Table 2.1: Spectral flow in the NS sector

(a) $-\frac{1}{2} < \epsilon < 0$

$E - E_0$	NS
$-\epsilon$	$d_0^\dagger$
$\epsilon + 1$	$c_1^\dagger$
$-\epsilon + 1$	$d_1^\dagger$
$\epsilon + 2$	$c_2^\dagger$

(b) $0 < \epsilon < \frac{1}{2}$

$E - E_0$	NS
ϵ	d_0
$-\epsilon + 1$	$d_1^\dagger$
$\epsilon + 1$	$c_1^\dagger$
$-\epsilon + 2$	$d_2^\dagger$

(c) $\frac{1}{2} < \epsilon < 1$

$E - E_0$	NS
$-\epsilon + 1$	$d_1^\dagger$
ϵ	d_0
$-\epsilon + 2$	$d_2^\dagger$
$\epsilon + 1$	$c_1^\dagger$

(d) $1 < \epsilon < \frac{3}{2}$

$E - E_0$	NS
$\epsilon - 1$	d_1
$-\epsilon + 2$	$d_2^\dagger$
ϵ	d_0
$-\epsilon + 3$	$d_3^\dagger$

that as we dial up ϵ, negative energy states from the Dirac sea cross the zero-point energy and become positive energy states. The first excited state in the NS sector has energy $|\epsilon|$ or $|1-\epsilon|$ depending on whether $-\frac{1}{2} < \epsilon < \frac{1}{2}$ or $\frac{1}{2} < \epsilon < \frac{3}{2}$. A similar analysis can be made for the R sector and the results are in Table 2.2.

Table 2.2: Spectral flow in the R sector

(a) $-1 < \theta < -\frac{1}{2}$

E	R
$\theta + 1$	$a_1^\dagger$
$-\theta$	$b_0^\dagger$
$\theta + 2$	$a_2^\dagger$
$-\theta + 1$	$b_1^\dagger$

(b) $-\frac{1}{2} < \theta < 0$

E	R
$-\theta$	$b_0^\dagger$
$\theta + 1$	$a_1^\dagger$
$-\theta + 1$	$b_1^\dagger$
$\theta + 2$	$a_2^\dagger$

(c) $0 < \theta < \frac{1}{2}$

E	R
θ	b_0
$-\theta + 1$	$b_1^\dagger$
$\theta + 1$	$a_1^\dagger$
$-\theta + 2$	$b_2^\dagger$

(d) $\frac{1}{2} < \theta < 1$

E	R
$-\theta + 1$	$b_1^\dagger$
θ	b_0
$-\theta + 2$	$b_2^\dagger$
$\theta + 1$	$a_1^\dagger$

2.3 State space

The zero-point energies for a complex boson and a complex fermion with moding $\mathbf{Z}+v+\frac{1}{2}$, $|v| \leq \frac{1}{2}$ are

$$\frac{1}{24} - \frac{v^2}{2} \,, \quad -\frac{1}{24} + \frac{v^2}{2} \quad \text{respectively} \;. \tag{2.81}$$

The complex boson Z has moding $\mathbf{Z}+\theta$ and so do the fermions in the R sector. This is a consequence of rigid supersymmetry on the worldsheet in the R sector. Thus, the zero-point energy in the R sector vanishes. The fermions in the NS sector have moding $\mathbf{Z}+\epsilon$ and the total zero-point energy in the NS sector is given by

$$\begin{aligned} E_0 &= \frac{1}{24} - \frac{\left(|\theta| - \frac{1}{2}\right)^2}{2} - \frac{1}{24} + \frac{\left(\left||\epsilon - \frac{1}{2}| - \frac{1}{2}\right| - \frac{1}{2}\right)^2}{2} \,, \\ &= \frac{1}{8} - \frac{1}{2}\left||\theta| - \tfrac{1}{2}\right| \;. \end{aligned} \tag{2.82}$$

Since $[z\,, \bar{z}] = \frac{\ell^2}{b-b'}$ and $[z\,, L_0] = 0$, we can build an infinite tower of states (given that the Z direction is non-compact) from each L_0 eigenstate without z, $\bar{z}$.

$\boldsymbol{\theta = 0}$:

The complex boson Z and the R sector fermions have moding $\mathbf{Z}$ and consequently the zero-point energy vanishes in the R sector. The fermions in the NS sector have moding $\mathbf{Z}+\frac{1}{2}$ and the total zero-point energy in the NS sector is given by

$$E_0 = \frac{1}{24} - \frac{1}{2}\left(\frac{1}{2}\right)^2 - \frac{1}{24} = -\frac{1}{8} \;. \tag{2.83}$$

The Fock space R vacuum $|\mathrm{R}\rangle$ is defined by

$$\alpha_m|\mathrm{R}\rangle = \beta_m|\mathrm{R}\rangle = a_m|\mathrm{R}\rangle = b_m|\mathrm{R}\rangle = 0 \quad \text{for} \quad m \geq 1 \quad \text{and} \quad p|\mathrm{R}\rangle = \bar{p}|\mathrm{R}\rangle = b_0|\mathrm{R}\rangle = 0 \;. \tag{2.84}$$

Since b_0 and $b_0^\dagger$ do not occur in L_0, the R vacuum is doubly degenerate with basis:

$$|\mathrm{R}\rangle \quad \text{and} \quad b_0^\dagger|\mathrm{R}\rangle \;. \tag{2.85}$$

The Fock space NS vacuum $|\mathrm{NS}\rangle$ is defined by

$$\begin{aligned} \alpha_m|\mathrm{NS}\rangle = \beta_m|\mathrm{NS}\rangle = c_m|\mathrm{NS}\rangle = d_m|\mathrm{NS}\rangle = 0 \quad \text{for} \quad m \geq 1 \ , \\ p|\mathrm{NS}\rangle = \bar{p}|\mathrm{NS}\rangle = d_0^\dagger|\mathrm{NS}\rangle = 0 \ . \end{aligned} \tag{2.86}$$

There is an additional infinite degeneracy in both the NS and R sectors from the bosonic zero modes z, $\bar{z}$, p and $\bar{p}$ which satisfy

$$[z,\bar{z}] = \frac{2\pi\alpha' b}{1+b^2} = \vartheta \ , \quad [z,p] = [\bar{z},\bar{p}] = \frac{\mathrm{i}}{1+b^2} \ , \quad [p,\bar{p}] = 0 \ . \tag{2.87}$$

Define the normalised oscillators

$$z = \sqrt{\vartheta}\ \hat{z} \ , \quad \bar{z} = \sqrt{\vartheta}\ \hat{z}^\dagger \ , \quad p = \frac{\hat{p}}{\sqrt{\vartheta}(1+b^2)} \ , \quad \bar{p} = \frac{\hat{p}^\dagger}{\sqrt{\vartheta}(1+b^2)} \ . \tag{2.88}$$

which satisfy the algebra

$$[\hat{z},\hat{z}^\dagger] = 1 \ , \quad [\hat{z},\hat{p}] = [\hat{z}^\dagger,\hat{p}^\dagger] = \mathrm{i} \ , \quad [\hat{p},\hat{p}^\dagger] = 0 \ . \tag{2.89}$$

The expression for L_0 becomes $L_0 - E_0 = \frac{\hat{p}\hat{p}^\dagger}{\pi b(1+b^2)} + \cdots$. We construct a basis of states which have definite value of $\hat{p}$, $\hat{p}^\dagger$:

$$|\lambda,\bar{\lambda}\rangle := \exp\left(\mathrm{i}\lambda\hat{z}^\dagger + \mathrm{i}\bar{\lambda}\hat{z}\right)|0\rangle \quad \text{with} \quad \hat{p}|\lambda,\bar{\lambda}\rangle = \bar{\lambda}|\lambda,\bar{\lambda}\rangle \quad \text{and} \quad \hat{p}^\dagger|\lambda,\bar{\lambda}\rangle = \lambda|\lambda,\bar{\lambda}\rangle \ . \tag{2.90}$$

The L_0 eigenvalue is then $\frac{|\lambda|^2}{\pi b(1+b^2)}$. Let $\mathcal{C}(u)$ be the contour $|\lambda| = \sqrt{u}$ in the λ-plane. We then define the following set of states:

$$|n,u\rangle := \frac{1}{2\pi\mathrm{i}}\int_{\mathcal{C}(u)} \frac{\mathrm{d}\lambda}{\lambda^{n+1}} \exp\left(\mathrm{i}\lambda\hat{z}^\dagger + \mathrm{i}\bar{\lambda}\hat{z}\right)|0\rangle \quad \text{for} \quad n \in \mathbf{Z}_{\geq 0} \ , \quad u > 0 \ . \tag{2.91}$$

We see that all states for a given u are degenerate with L_0 eigenvalue u. In the power series expansion in the operators $\hat{z}$, $\hat{z}^\dagger$, we see that the leading power of $\hat{z}^\dagger$ is n. In the $\lambda \to 0$ limit, these states persist and have $L_0 = 0$. There are similar states with leading power of $\hat{z}$ equal to n. The $L_0 = 0$ states can also be obtained by taking wavepackets associated with the following states:

$$|n+\rangle = \hat{z}^n|0\rangle \ , \quad |n-\rangle = (\hat{z}^\dagger)^n|0\rangle \ . \tag{2.92}$$

The $|n\pm\rangle$ are simpler to handle in evaluating string amplitudes.

2.4 Boundary condition changing operators

We map the strip $-\infty < \tau < \infty$, $0 \le \sigma \le \pi$ to the upper half plane $\mathbf{H} = \{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\}$ by first Wick-rotating $\tau = -\mathrm{i}t$ and using the map $z = \exp(t + \mathrm{i}\sigma)$. In particular, the boundary at $\sigma = 0, \pi$ is mapped to $z = \bar{z} > 0$ and < 0 respectively. We use this form of the open string worldsheet to compute string amplitudes. The vertex operators corresponding to the states $|n, u\rangle$, $|n\pm\rangle$ are

$$\begin{aligned}
V(\lambda, \bar{\lambda}; x) &= \boldsymbol{:} \exp\left(\tfrac{1}{\sqrt{\vartheta}}(\mathrm{i}\lambda \overline{Z} + \mathrm{i}\bar{\lambda} Z)\right)(x) \boldsymbol{:} \ , \\
V(n, u; x) &= \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}(u)} \frac{\mathrm{d}\lambda}{\lambda^{n+1}} \boldsymbol{:} \exp\left(\tfrac{1}{\sqrt{\vartheta}}(\mathrm{i}\lambda \overline{Z} + \mathrm{i}\bar{\lambda} Z)\right)(x) \boldsymbol{:} \ , \quad x \in \partial\mathbf{H} \ , \\
V(n+; x) &= \frac{1}{\vartheta^{n/2}} \boldsymbol{:} Z^n(x) \boldsymbol{:} \ , \quad V(n-; x) = \frac{1}{\vartheta^{n/2}} \boldsymbol{:} \overline{Z}^n(x) \boldsymbol{:} \ ,
\end{aligned} \tag{2.93}$$

where $\vartheta = \pi\alpha' \sin 2\pi v$ is the non-commutativity parameter in (2.67).

Worldsheet bosons

Consider a complex boson Z with $\mathbf{MM'}$ boundary conditions. Using $\partial_{++} = \mathrm{i}z\partial$ and $\partial_{--} = \mathrm{i}\bar{z}\bar{\partial}$, we can write the corresponding boundary conditions on $\mathbf{H}$:

$$\partial Z = \mathrm{e}^{-2\pi \mathrm{i}\nu}\, \bar{\partial} Z \quad \text{for} \quad z = \bar{z} > 0 \ , \qquad \partial Z = \mathrm{e}^{-2\pi \mathrm{i}\mu}\, \bar{\partial} Z \quad \text{for} \quad z = \mathrm{e}^{2\pi \mathrm{i}} \bar{z} < 0 \ . \tag{2.94}$$

with $-\frac{1}{2} \le \mu, \nu \le \frac{1}{2}$. We define bulk chiral currents $J(z) = \mathrm{i}\partial Z$, $\bar{J}(\bar{z}) = \mathrm{i}\bar{\partial} Z$ using the modes in (2.19):

$$J(z) = \mathrm{i}\partial Z(z) = -\frac{\mathrm{i}\ell}{2}\sum_{n=1}^{\infty}\alpha_n\,\mathrm{e}^{-2\pi\mathrm{i}\nu}z^{-1-\theta_n} - \frac{\mathrm{i}\ell}{2}\sum_{m=0}^{\infty}\beta_m^\dagger\,\mathrm{e}^{-2\pi\mathrm{i}\nu}z^{-1-\theta_{-m}} \ ,$$

$$\bar{J}(\bar{z}) = \mathrm{i}\bar{\partial} Z(\bar{z}) = -\frac{\mathrm{i}\ell}{2}\sum_{n=1}^{\infty}\alpha_n\,\bar{z}^{-1-\theta_n} - \frac{\mathrm{i}\ell}{2}\sum_{m=0}^{\infty}\beta_m^\dagger\,\bar{z}^{-1-\theta_{-m}} \ , \tag{2.95}$$

where $\theta = \mu - \nu$ and $\theta_n = \theta + n$. Since the modes are not integers, we need to specify a branch cut: we choose it to be at $-\infty < z \le 0$. We also define the hermitian conjugate currents

$$J^*(z) := z^{-2}\,\overline{J(\bar{z}^{-1})} \ , \quad \bar{J}^*(\bar{z}) := \bar{z}^{-2}\,\overline{\bar{J}(z^{-1})} \ . \tag{2.96}$$

The gluing conditions for the currents are then:

$$J(z) = \mathrm{e}^{-2\pi\mathrm{i}\nu}\bar{J}(\bar{z}) \quad \text{for} \quad z = \bar{z} > 0 \ , \qquad J(z) = \mathrm{e}^{2\pi\mathrm{i}\mu}\bar{J}(\bar{z}) \quad \text{for} \quad z = \mathrm{e}^{2\pi\mathrm{i}}\bar{z} < 0 \ ,$$
$$J^*(z) = \mathrm{e}^{2\pi\mathrm{i}\nu}\bar{J}^*(\bar{z}) \quad \text{for} \quad z = \bar{z} > 0 \ , \qquad J^*(z) = \mathrm{e}^{-2\pi\mathrm{i}\mu}\bar{J}^*(\bar{z}) \quad \text{for} \quad z = \mathrm{e}^{2\pi\mathrm{i}}\bar{z} < 0 \ . \tag{2.97}$$

The gluing conditions allow us to extend the domain of definition of the currents to the full z-plane by employing the *doubling trick*:

$$\mathsf{J}(z) = -\frac{\mathrm{i}\ell}{2}\sum_{n\ge 1}\alpha_m\,\mathrm{e}^{-2\pi\mathrm{i}\nu}z^{-1-\theta_n} - \frac{\mathrm{i}\ell}{2}\sum_{m\ge 0}\beta_m^\dagger\,\mathrm{e}^{-2\pi\mathrm{i}\nu}z^{-1-\theta_{-m}} \ ,$$

$$\mathsf{J}^*(z) = \frac{\mathrm{i}\ell}{2}\sum_{n\ge 1}\alpha_n^\dagger\,\mathrm{e}^{2\pi\mathrm{i}\nu}z^{-1+\theta_n} + \frac{\mathrm{i}\ell}{2}\sum_{m\ge 0}\beta_m\,\mathrm{e}^{2\pi\mathrm{i}\nu}z^{-1+\theta_{-m}} \ . \tag{2.98}$$

The doubled stress tensor $\mathsf{T}(z)$ is given by

$$\mathsf{T}(z) = \lim_{w\to z}\frac{4}{\ell^2}\left(\mathsf{J}(w)\mathsf{J}^*(z) - \frac{\ell^2}{4(w-z)^2}\right) \ . \tag{2.99}$$

The change in boundary conditions from μ to ν at $z = 0$ and vice-versa at $z = \infty$ can be interpreted as there being present *boundary condition changing operators* (BCC) $\sigma(0)$ and $\sigma^+(\infty)$ where σ^+ is the operator conjugate to σ. The conformal dimension of σ is obtained from the one-point function of $\mathsf{T}(z)$. Following the treatment in [DFMS, FGRS] we first define $\mathsf{J} = \mathsf{J}_> + \mathsf{J}_<$ where $\mathsf{J}_>$ contains only annihilation operators. We have, for $0 < \theta < 1$,

$$\mathsf{J}_>(w) = -\frac{\mathrm{i}\ell}{2}\sum_{n\geq 1}\alpha_n\,\mathrm{e}^{-2\pi\mathrm{i}\nu}w^{-1-\theta_n} - \frac{\mathrm{i}\ell}{2}\,\beta_0^\dagger\,\mathrm{e}^{-2\pi\mathrm{i}\nu}w^{-\theta-1}\ , \tag{2.100}$$

and for $-1 < \theta < 0$ the last term is absent. Next, we compute

$$\begin{aligned}\mathsf{J}(w)\mathsf{J}^*(z) - \frac{\ell^2}{4(w-z)^2} &= \mathsf{J}_<(w)\mathsf{J}^*(z) + \mathsf{J}^*(z)\mathsf{J}_>(w) + [\mathsf{J}_>(w),\mathsf{J}^*(z)] - \frac{\ell^2}{4(w-z)^2}\ ,\\ &= \mathsf{J}_<(w)\mathsf{J}^*(z) + \mathsf{J}^*(z)\mathsf{J}_>(w) + \frac{\ell^2}{4}\,\partial_z\left[\left(\frac{z}{w}\right)^{(\star)}\frac{1}{w-z}\right]\ ,\end{aligned} \tag{2.101}$$

where the exponent $(\star)$ is θ for $0 < \theta < 1$ and $1+\theta$ for $-1 < \theta < 0$. This finally gives

$$\mathsf{T}(z) = \frac{|\theta|\,(1-|\theta|)}{2z^2} + \frac{4}{\ell^2}\mathsf{J}_<(z)\mathsf{J}^*(z) + \frac{4}{\ell^2}\mathsf{J}^*(z)\mathsf{J}_>(z)\ , \tag{2.102}$$

which gives the one-point function

$$\langle\mathsf{T}(z)\rangle = \frac{|\theta|\,(1-|\theta|)}{2z^2}\ . \tag{2.103}$$

This can be interpreted as there being two BCC operators σ, σ^+ inserted resp. at $z = 0$ and $z = \infty$ with conformal weight $h_\sigma = \frac{|\theta|(1-|\theta|)}{2}$. Their two-point function is

$$\langle\sigma(x_1)\sigma^+(x_2)\rangle = \frac{1}{|x_1 - x_2|^{2h_\sigma}}\ . \tag{2.104}$$

To get a more complete understanding of the boundary condition changing operators, we explore their OPE with the currents $\mathsf{J}(z)$, $\mathsf{J}^*(z)$. The in-vacuum $|\theta\rangle$ for the worldsheet bosons Z, $\overline{Z}$ is defined by the relations:

$$\begin{aligned}\text{For}\quad -1 < \theta < 0: &\quad \alpha_m|\theta\rangle = 0\ ,\quad \beta_{m-1}|\theta\rangle = 0\ ,\quad m = 1,2,\dots\ ,\\ \text{For}\quad 0 < \theta < 1: &\quad \alpha_m|\theta\rangle = 0\ ,\quad \beta_m|\theta\rangle = 0\ ,\quad m = 1,2,\dots\ .\end{aligned} \tag{2.105}$$

The in-vacuum is the vacuum of the Hilbert space at $t = -\infty$, or using the map $z = \exp(t + \mathrm{i}\sigma)$, at $z = 0$. It can be interpreted as the state obtained by acting on the

SL(2, $\mathbf{R}$)-invariant vacuum $|\Omega\rangle$ by the operator $\sigma(0)$:

$$|\theta\rangle := \sigma(0)|\Omega\rangle \ . \tag{2.106}$$

The effect of inserting $\sigma(0)$ is to introduce the branch point at $z = 0$ with $\mathsf{J}(z)$ having a monodromy $\mathrm{e}^{-2\pi\mathrm{i}\theta}$ around it and the appropriate branch cut (here $-\infty < z \leq 0$). Let us focus on the case $-1 \leq \theta \leq 0$. From the mode expansion of $\mathsf{J}(z)$ and $\mathsf{J}^*(z)$, we have

$$\lim_{z\to 0}\ \mathsf{J}(z)|\theta\rangle \sim -\frac{\mathrm{i}\ell}{2}\mathrm{e}^{-2\pi\mathrm{i}\nu}\ z^{-1-\theta}\beta_0^\dagger|\theta\rangle \ , \quad \lim_{z\to 0}\ \mathsf{J}^*(z)|\theta\rangle \sim \frac{\mathrm{i}\ell}{2}\mathrm{e}^{2\pi\mathrm{i}\nu}\ z^{\theta}\alpha_1^\dagger|\theta\rangle \ . \tag{2.107}$$

The $\sim$ indicates that we have suppressed less singular terms on the right-hand side. From this, we infer the following OPE:

$$\mathsf{J}(z)\sigma(0) \sim z^{-1-\theta}\,\tau_1(0) \ , \quad \mathsf{J}^*(z)\sigma(0) \sim z^{\theta}\,\tau_2(0) \ . \tag{2.108}$$

The operators $\tau_1(0)$ and $\tau_2(0)$ are excited BCC operators corresponding to the excitations $\beta_0^\dagger|\theta\rangle$ and $\alpha_1^\dagger|\theta\rangle$. Similarly, for the case $0 \leq \theta \leq 1$, we get

$$\mathsf{J}(z)\sigma(0) \sim z^{-\theta}\,\tau_3(0) \ , \quad \mathsf{J}^*(z)\sigma(0) \sim z^{-1+\theta}\,\tau_4(0) \ . \tag{2.109}$$

Here, $\tau_3(0)$ and $\tau_4(0)$ correspond to the excited states $\beta_1^\dagger|\theta\rangle$ and $\beta_0|\theta\rangle$ respectively. Note that β_0 is a creation operator for this range of θ.

Worldsheet fermions

We now describe BCC operators for the worldsheet fermions $\Psi^\pm$. Since the fermions have conformal dimension $\frac{1}{2}$, we should include a Jacobian factor $z^{-1/2}$ while mapping them from the strip to the upper half-plane. We employ the doubling trick to directly write the

R and NS fermions on the full z-plane in the R and NS sectors:

$$\Psi_{\rm R}(z) = \frac{\ell}{2}\sum_{n\geq 1} a_n\, z^{-\epsilon_n} + \frac{\ell}{2}\sum_{m\geq 0} b_m^\dagger\, z^{-\epsilon_{-m}} \ ,$$
$$\Psi_{\rm NS}(z) = \frac{\ell}{2}\sum_{r\geq 1} c_r\, z^{-\theta_r - 1} + \frac{\ell}{2}\sum_{s\geq 0} d_s^\dagger\, z^{-\theta_{-s}-1} \ . \tag{2.110}$$

with $\epsilon = \theta + \frac{1}{2}$. In order to describe the BCC operators, we first bosonise $\Psi(z)$ by introducing an antihermitian scalar $H(z)$:

$$\Psi(z) = \mathrm{e}^{H(z)}\ , \quad \Psi^*(z) = \mathrm{e}^{-H(z)} \quad \text{with} \quad \langle H(w)H(z)\rangle = \log(w-z)\ . \tag{2.111}$$

The normal ordering symbol : : is omitted in the above definition for the sake of brevity. Now, consider the OPE of the operator $\mathrm{e}^{-\theta H(x)}$ with Ψ:

$$\Psi(z)\mathrm{e}^{-\theta H(0)} \sim z^{-\theta}\mathrm{e}^{(1-\theta)H(0)} + \cdots\ . \tag{2.112}$$

First, we notice that as $z \to \mathrm{e}^{2\pi\mathrm{i}}z$, Ψ picks up a phase $\mathrm{e}^{-2\pi\mathrm{i}\theta}$, which matches with the monodromy of $\Psi_{\rm NS}$. Further, we observe that the right-hand side of the first OPE is regular as $z \to 0$ for $\theta < 0$. This requires that $\Psi_{\rm NS}(z)$ annihilate the state $\mathrm{e}^{-\theta H(0)}|0\rangle$ in the limit $z \to 0$ where the state $|\Omega\rangle$ is the $\mathrm{SL}(2,\mathbf{R})$-invariant vacuum. From Table 2.1, we see that the NS vacuum $|\mathrm{NS}\rangle$ has the same properties for $-\frac{1}{2} < \theta < 0$. Thus, we can identify the state $\mathrm{e}^{-\theta H(0)}|0\rangle$ with the NS vacuum for this range of θ:

$$\mathrm{e}^{-\theta H(0)}|0\rangle = |\mathrm{NS}\rangle \quad \text{for} \quad -\frac{1}{2} < \theta < 0\ . \tag{2.113}$$

For the range $-1 < \theta < -\frac{1}{2}$, we see from Table 2.1 that $\Psi_{\rm NS}$ annihilates the excited state $d_0^\dagger|\mathrm{NS}\rangle$. Thus, for this range of θ one has to identify $\mathrm{e}^{-\theta H(0)}|0\rangle$ with the first excited state:

$$\mathrm{e}^{-\theta H(0)}|0\rangle = d_0^\dagger|\mathrm{NS}\rangle \quad \text{for} \quad -1 < \theta < -\frac{1}{2}\ . \tag{2.114}$$

The NS vacuum is obtained by applying d_0 to the above and d_0 is contained in the Hermitian conjugate field $\Psi^*(z)$, defined as

$$\Psi^*(z) := z^{-1}\ \overline{\Psi(\bar{z}^{-1})}\ ,$$

with mode expansion

$$\Psi^*_{\mathrm{NS}}(z) := \frac{\ell}{2}\sum_{r\geq 1} c_r^\dagger\, z^{\theta_r} + \frac{\ell}{2}\sum_{s\geq 0} d_s\, z^{\theta_{-s}}\ . \tag{2.115}$$

The operator corresponding to the NS vacuum for this range of θ is then obtained by fusing Ψ^* with $e^{-\theta H(x)}$:

$$\Psi^*(z)\mathrm{e}^{-\theta H(0)} \sim z^{\theta}\mathrm{e}^{-(1+\theta)H(0)} + \cdots\ . \tag{2.116}$$

Thus the NS vacuum is to be identified with the operator on the right hand side:

$$\mathrm{e}^{-(1+\theta)H(0)}|0\rangle = |\mathrm{NS}\rangle \quad \text{for} \quad -1<\theta<-\frac{1}{2}\ . \tag{2.117}$$

Similarly, for $\theta>0$ there are two cases $0<\theta<\frac{1}{2}$ and $\frac{1}{2}<\theta<1$ for which the operators corresponding to $|\mathrm{NS}\rangle$ are $\mathrm{e}^{-\theta H(x)}$ and $\mathrm{e}^{(1-\theta)H(x)}$ respectively. The same analysis can be made for the R sector as well. We summarise the results in Table 2.3. We designate the operator corresponding to the NS and R vacua as $s^{\mathrm{NS}}(x)$ and $s^{\mathrm{R}}(x)$ respectively. These shall be the BCC operators for the respective sectors. Also observe that the operators

Table 2.3: Ground BCC operators for the NS and R sectors

Ground BCC operator	$-1<\theta<-\frac{1}{2}$	$-\frac{1}{2}<\theta<0$	$0<\theta<\frac{1}{2}$	$\frac{1}{2}<\theta<1$
$s^{\mathrm{NS}}(x)$	$\mathrm{e}^{-(1+\theta)H(x)}$	$\mathrm{e}^{-\theta H(x)}$	$\mathrm{e}^{-\theta H(x)}$	$\mathrm{e}^{(1-\theta)H(x)}$
$s^{\mathrm{R}}(x)$	$\mathrm{e}^{-\epsilon H(x)}$	$\mathrm{e}^{-\epsilon H(x)}$	$\mathrm{e}^{(1-\epsilon)H(x)}$	$\mathrm{e}^{(1-\epsilon)H(x)}$

$s^{\mathrm{NS,R}}$ always have the smallest conformal dimension in each range of θ. The operators corresponding to the excited states can be inferred in a similar fashion and are summarised in Table 2.4. Let us study the limiting cases of **NN**, **DD**, **DN**. For both **NN** and **DD**

Table 2.4: Excited BCC operators for the NS and R sectors

Excited BCC operator	$-1 < \theta < -\frac{1}{2}$	$-\frac{1}{2} < \theta < 0$	$0 < \theta < \frac{1}{2}$	$\frac{1}{2} < \theta < 1$
$t^{\rm NS}(x)$	$e^{-\theta H(x)}$	$e^{-(1+\theta)H(x)}$	$e^{(1-\theta)H(x)}$	$e^{-\theta H(x)}$
$\tilde{t}^{\rm NS}(x)$	$e^{-(2+\theta)H(x)}$	$e^{(1-\theta)H(x)}$	$e^{-(1+\theta)H(x)}$	$e^{(2-\theta)H(x)}$
$t^{\rm R}(x)$	$e^{-(1+\epsilon)H(x)}$	$e^{(1-\epsilon)H(x)}$	$e^{-\epsilon H(x)}$	$e^{(2-\epsilon)H(x)}$
$\tilde{t}^{\rm R}(x)$	$e^{(1-\epsilon)H(x)}$	$e^{-(1+\epsilon)H(x)}$	$e^{(2-\epsilon)H(x)}$	$e^{-\epsilon H(x)}$

we have $\theta = 0$. From Table 2.3, we see that for either of the two limits $\theta \to 0^+$ or $\theta \to 0^-$, the NS vacuum is the SL(2, **R**)-invariant vacuum $|\Omega\rangle$. The first two excited states corresponding to $e^{\pm H}$ are degenerate. In the R sector, for $\theta \to 0^\pm$, the vacuum corresponds to $e^{\pm H/2}$ and the first excited state to $e^{\mp H/2}$. The two states are degenerate, so either limit gives the same spectrum.

For **DN** boundary conditions, we have $\mu = 0$ and $\nu = \frac{1}{2}$ giving $\theta = -\frac{1}{2}$ and $\epsilon = 0$. In the NS sector, the ground state and the first excited state are degenerate, corresponding to the operators $e^{\pm H/2}$. In the R sector, the ground state is the SL(2, **R**)-invariant vacuum and the first two excited states corresponding to $e^{\pm H}$ are degenerate.

For **ND** boundary conditions, we have $\mu = \frac{1}{2}$ and $\nu = 0$ giving $\theta = \frac{1}{2}$ and $\epsilon = 1$. The discussion on the NS and R sector states is identical to the **DN** case.

For the **MD** case, we have $\mu = \frac{1}{2}$ and $\nu = v$ giving $\theta = \frac{1}{2} - v$ and $\epsilon = 1 - v$. The range of θ is $0 \leq \theta \leq 1$, giving the ground BCC operator $e^{(1-\epsilon)H(x)}$ in the R sector and $e^{-\theta H(x)}$, $e^{(1-\theta)H(x)}$ in the NS sector for $0 \leq \theta \leq \frac{1}{2}$ and $\frac{1}{2} \leq \theta \leq 1$ respectively.

For the **DM** case, we have $\mu = v$ and $\nu = \frac{1}{2}$ giving $\theta = v - \frac{1}{2}$ and $\epsilon = v$. The ground BCC operators are $e^{-\epsilon H(x)}$ in the R sector and $e^{-(1+\theta)H(x)}$, $e^{-\theta H(x)}$ in the NS sector for $-1 \leq \theta \leq -\frac{1}{2}$ and $-\frac{1}{2} \leq \theta \leq 0$ respectively.

2.5 The covariant lattice

Consider the following linear combinations of the holomorphic (left-moving) part of the worldsheet fermions:

$$\Psi^{\pm e_a} = \frac{\psi^{2a-1} \pm \mathrm{i}\psi^{2a}}{\sqrt{2}} \ , \ a \in \underline{\mathbf{4}} \ , \quad \Psi^{\pm e_5} = \frac{\psi^9 \pm \psi^0}{\sqrt{2}} \ . \tag{2.118}$$

Here e_m, $m = 1, \ldots, 5$, are unit vectors $(e_m)_i = \delta_{im}$ of the D_5 weight lattice. Along with their negatives, they form the weights of the vector representation of so$(1, 9)$. Under complex conjugation the fermions behave as follows:

$$(\Psi^{+e_a})^* = \Psi^{-e_a} \ , \quad (\Psi^{\pm e_5})^* = \Psi^{\pm e_5} \ . \tag{2.119}$$

In order to bosonize these with the correct properties under complex conjugation, we introduce antihermitian scalars $H_a(z)$ and a hermitian scalar $H_5(z)$ which satisfy $\langle H_m(z)H_n(w)\rangle = \delta_{mn} \log(z - w)$, $m, n = 1, \ldots, 5$. The bosonised versions of the fermions are

$$\Psi^{\pm e_m}(z) := \mathrm{e}^{\pm H_m(z)} c_{\pm e_m} \ , \quad m = 1, \ldots, 5 \ . \tag{2.120}$$

The object $c_{\pm e_m}$ is a *cocycle operator* which is defined [KLLSW] in terms of the fermion number operators N_m as

$$c_{\pm e_m} := (-)^{N_1 + \cdots + N_{m-1}} \ . \tag{2.121}$$

These ensure that the fermions $\Psi^{\pm e_m}$, $\Psi^{\pm e_n}$ for $m \neq n$ anticommute after bosonisation. In a broader context, these fermions are used to construct the currents whose modes satisfy the commutation relations of the affine Kač-Moody algebra $\hat{D}_5$. The commutation relations between the generators corresponding to the roots of the Lie algebra involve certain 2-cocycles. In order to obtain these 2-cocycles correctly via bosonised vertex operators, we need to include the above *cocycle operators* in the definition of the vertex operators. These 2-cocycles were first treated by Bardakci and Halpern [BH] in the physics literature and by Frenkel and Kač [FK], and Segal [S] in the mathematics literature.

In terms of the bosons H_m, the number operators are given by $N_m := (\partial H_m)_0$ where $(\partial H_m)_0$ are the zero modes of ∂H_m. The bosons H_m have the following mode expansion [PR]:

$$H_m(z) = h_m + N_m \log z + \sum_{k\neq 0} \frac{\alpha_{m,k}}{k} z^{-k} \ . \tag{2.122}$$

The Hermitian conjugate field H_m^* is defined as follows:

$$H_m^*(z) := \overline{H_m(\bar{z}^{-1})} \ . \tag{2.123}$$

Since H_a are antihermitian and H_5 is hermitian, the modes satisfy

$$\begin{aligned}(h_a)^\dagger &= -h_a \ , \quad (N_a)^\dagger = N_a \ , \quad (\alpha_{a,k})^\dagger = \alpha_{a,k} \ , \ a \in \underline{\mathbf{4}} \ , \\ (h_5)^\dagger &= h_5 \ , \quad (N_5)^\dagger = -N_5 \ , \quad (\alpha_{5,k})^\dagger = -\alpha_{5,k} \ .\end{aligned} \tag{2.124}$$

These properties will be required in the discussion on the cocycle operators. Next, we discuss superconformal ghosts. The contribution due to these have to be included appropriately in each vertex operator to ensure that operator products are mutually local. One bosonises the superconformal ghosts β, γ using a hermitian scalar field φ with $\langle \varphi(z)\varphi(w)\rangle = -\log(z-w)$ and two fermions $\xi(z)$ and $\eta(z)$ with $\langle \xi(z)\zeta(w)\rangle = (z-w)^{-1}$:

$$\beta(z) := \mathrm{e}^{-\varphi(z)}\partial\xi(z) \ , \quad \gamma(z) := \eta(z)\mathrm{e}^{\varphi(z)} \ . \tag{2.125}$$

The fermions ξ and η are further bosonised as

$$\xi(z) = \mathrm{e}^{\zeta(z)} \ , \quad \eta(z) = \mathrm{e}^{-\zeta(z)} \ . \tag{2.126}$$

with $\zeta(z)$ a hermitian scalar with $\langle \zeta(z)\zeta(w)\rangle = \log(z-w)$. The superghost picture number operator N_6 is given by the zero mode of $\partial\zeta - \partial\varphi$. Under conjugation, it satisfies

$$(N_6)^\dagger = -N_6 - Q = -N_6 - 2 \ , \tag{2.127}$$

where $Q = 2$ is the background charge of the $\beta\gamma$ CFT. The picture charge of $\mathrm{e}^{q\varphi(z)}$ is q and its conformal dimension is $-\frac{1}{2}q(q+Q)$. The operator conjugate to $\mathrm{e}^{q\varphi}$ is $\mathrm{e}^{-(q+Q)\varphi}$ and it also has conformal dimension $-\frac{1}{2}q(q+Q)$.

In the canonical ghost picture, vertex operators in the NS sector acquire a factor of $\mathrm{e}^{-\varphi}$ and those in the R sector a factor of $\mathrm{e}^{-\varphi/2}$. The integer and half-integer exponents are correlated with the integer and half-integer modes for the NS and R fermions on the doubled plane. The integer and half-integer ghost numbers can be interpreted as belonging to a D_1 weight lattice which can then be combined with the spacetime D_5 weights to get a *covariant lattice* $\Gamma_{5,1}$. The lattice $\Gamma_{5,1}$ is Lorentzian since $\langle\varphi(z)\varphi(w)\rangle = -\log(z-w)$ as opposed to $\langle H_m(z)H_n(w)\rangle = \delta_{mn}\log(z-w)$. Writing $H_6 := -\varphi$, we have $\langle H_\mu(z)H_\nu(w)\rangle = \eta_{\mu\nu}\log(z-w)$ with $\eta_{66} = -1$, $\eta_{6m} = 0$ and $\eta_{mn} = \delta_{mn}$. A general vertex operator in the (worldsheet) fermionic sector is then given by

$$\mathrm{e}^{\lambda\cdot H(z)}c_\lambda \ . \tag{2.128}$$

where λ is a weight in the covariant lattice $\Gamma_{5,1}$, c_λ is the cocycle operator corresponding to λ and the dot product $\lambda\cdot H$ is with respect to the Lorentzian metric $\eta_{\mu\nu}$. We give a formula for c_λ below. The $\Gamma_{5,1}$ weights λ with $\lambda_6 = -1, -\frac{1}{2}, -\frac{3}{2}$ directly correspond to physical states whose mass-squared is given by

$$\alpha' m^2 = \frac{1}{2}\lambda^2 + \lambda\cdot e_6 - 1 \ , \tag{2.129}$$

where e_6 is the unit vector $(0,0,0,0,0;1)$. The term $\lambda\cdot e_6$ arises due to the background charge of the $\beta\gamma$ CFT. The other $\Gamma_{5,1}$ weights do not correspond directly to physical states but linear combinations of the corresponding vertex operators correspond to physical operators with picture charge different from the canonical ones.

2.5.1 The D1-D5$_A$-D5$_{\bar{A}}$ system

Consider the D1-D5$_A$-D5$_{\bar{A}}$ system. The spacetime Lorentz symmetry $\mathrm{SO}(1,9)$ is broken down to $\mathrm{SO}(4)\times\mathrm{SO}(4)'\times\mathrm{SO}(1,1)$ with spacetime now being the $1+1$ dimensional intersection $\mathbf{R}^{1,1}$ of the D-branes. The worldsheet fermion contribution to the total vertex operator can thus be described by (2.128) where λ is now a weight in the covariant lattice

$D_2 \oplus D_2 \oplus \Gamma_{1,1}$. In the presence of a constant B-field, the weights λ have to be generalised to include entries which are neither integral nor half-integral. The precise weights can be obtained by following the procedure outlined in the previous sections.

An (unintegrated) open string vertex operator with only fermionic oscillators then has the form

$$V_\lambda(k,z) = \omega(\lambda)\, c(z)\, \mathcal{B}(z)\, \mathrm{e}^{\lambda\cdot H(z)}\, \mathrm{e}^{2\mathrm{i}k\cdot X(z)} c_\lambda \ , \tag{2.130}$$

where $\omega(\lambda)$ is an arbitrary c-number phase, $c(z)$ is the coordinate ghost, $\mathcal{B}(z)$ is the product of the appropriate BCC operators for the worldsheet bosons and k is the $1+1$ dimensional spacetime momentum. We have suppressed Chan-Paton factors. The mass formula for a state with weight λ becomes

$$\alpha' m^2(\lambda) = -\alpha' k^2 = \frac{1}{2}\lambda^2 + \lambda\cdot e_6 - 1 + \sum_{\sigma|\mathcal{B}} h_\sigma \ . \tag{2.131}$$

The notation $\sum_{\sigma|\mathcal{B}}$ indicates the summation of the conformal dimensions h_σ of bosonic BCC operators $\sigma(z)$ present in $\mathcal{B}(z)$ above.

2.5.2 Cocycle operators

We follow the treatment in [KLLSW] and write the cocycle operators c_λ as follows:

$$c_\lambda := \exp\left(\mathrm{i}\pi M_{\rho\sigma}\lambda^\rho N^\sigma\right) \ , \tag{2.132}$$

where N_σ is the vector of number operators $(N_1, \ldots, N_6)$ and $M_{\mu\nu}$ is the matrix

$$M_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & 0 & 0 \\ +1 & +1 & 0 & 0 & 0 & 0 \\ -1 & +1 & -1 & 0 & 0 & 0 \\ +1 & +1 & +1 & +1 & 0 & 0 \\ -1 & -1 & -1 & -1 & +1 & 0 \end{pmatrix} \ . \tag{2.133}$$

The indices σ, ρ are raised and lowered using the indefinite metric $\eta_{\mu\nu}$. The OPE between two vertex operators $V_\lambda(z)$ and $V_{\lambda'}(z)$ then becomes

$$V_\lambda(z)V_{\lambda'}(w) \sim (z-w)^{\lambda\cdot\lambda'} \mathrm{e}^{\mathrm{i}\pi\lambda\cdot M\cdot\lambda'} V_{\lambda+\lambda'}(w) + \cdots \ . \tag{2.134}$$

The signs in the matrix M are chosen to obtain the correct charge conjugation matrices in the OPEs

$$S^A(z)S^B(w) \sim (z-w)^{-1}C^{AB} + \cdots \ , \quad S^{\dot{A}}(z)S^{\dot{B}}(w) \sim -\mathrm{i}(z-w)^{-1}C^{\dot{A}\dot{B}} + \cdots \ , \tag{2.135}$$

where S^A, $S^{\dot{B}}$ are the $9+1$ dimensional left- and right-handed spinor vertex operators in the canonical ghost picture. They are given by the $D_{5,1}$ weights $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}; -\frac{1}{2})$ with even and odd number of minus signs respectively. The corresponding Γ-matrices are obtained from the OPE

$$\psi^\mu(z)S^A(w) \sim (z-w)^{-1}(\Gamma^\mu)^A{}_{\dot{B}}\, S^{\dot{B}}_{(-3/2)}(w) + \cdots \ , \tag{2.136}$$

where $\psi^\mu(z)$ is the $9+1$ dimensional vector vertex operator from the NS sector with $D_{5,1}$ weight $(0, \ldots, 0, \pm 1, 0, \ldots, 0; -1)$ and $S^{\dot{B}}_{(-3/2)}$ is the operator that is conjugate to the operator $S^{\dot{B}}_{(-1/2)}$ in the canonical ghost picture. We obtain the following helicity representation for the Γ-matrices and the charge conjugation matrix from the above OPEs [KLLSW]:

$$\begin{aligned}
\Gamma^1 &= \sigma_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \ , & \Gamma^7 &= -\sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_2 \otimes \mathbb{1} \ , \\
\Gamma^2 &= \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \ , & \Gamma^8 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \mathbb{1} \ , \\
\Gamma^3 &= \sigma_3 \otimes \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \ , & \Gamma^9 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \ , \\
\Gamma^4 &= -\sigma_3 \otimes \sigma_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \ , & \Gamma^0 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes (-\mathrm{i}\sigma_2) \ , \\
\Gamma^5 &= -\sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \mathbb{1} \otimes \mathbb{1} \ , & \Gamma_c &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \ , \\
\Gamma^6 &= -\sigma_3 \otimes \sigma_3 \otimes \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1} \ , & C_- &= \mathrm{e}^{3\pi\mathrm{i}/4}\sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \ .
\end{aligned} \tag{2.137}$$

2.5.3 CPT conjugate vertex operators

In the calculation of the Yukawa couplings arising from the various E-terms and J-terms, the J-term couplings involve the right-moving fermions while the E-term couplings involve the conjugate right-moving fermions. Hence we need vertex operators for CPT conjugate states. The transformation of the cocycle operators under CPT are quite intricate and must be handled with care.

Recall that the H_a are antihermitian and H_5, H_6 are hermitian. The number operators satisfy:

$$(N_a)^\dagger = N_a \ , \ a \in \underline{\mathbf{4}} \ , \quad (N_5)^\dagger = -N_5 \ , \quad (N_6)^\dagger = -N_6 - 2 \ . \tag{2.138}$$

Spacetime CPT is implemented as Hermitian conjugation on the vertex operators. The operator $\mathrm{e}^{\lambda \cdot H}$ thus transforms as

$$(\mathrm{e}^{\lambda \cdot H})^\dagger = (\mathrm{e}^{\lambda_a H_a + \lambda_5 H_5 - \lambda_6 H_6})^\dagger = \mathrm{e}^{-\lambda_a H_a + \lambda_5 H_5 - \lambda_6 H_6} =: \mathrm{e}^{\lambda^\star \cdot H} \ , \tag{2.139}$$

where we have defined $\lambda^\star := (-\lambda_a, \lambda_5; \lambda_6)$ to be the CPT conjugate weight. The cocycle operator c_λ transforms as

$$\begin{aligned}
(c_\lambda)^\dagger &= \exp\left[-\mathrm{i}\pi \lambda \cdot M \cdot N^\dagger\right] = \exp\left[-\mathrm{i}\pi(\lambda \cdot M)_b N_b + \mathrm{i}\pi(\lambda \cdot M)_5 N_5\right] \ , \\
&= \exp\left[\mathrm{i}\pi(-\lambda_a M_{ab} N_b - \lambda_5 M_{5a} N_a + \lambda_6 M_{6a} N_a - \lambda_6 M_{65} N_5)\right] \ , \\
&= \exp\left[\mathrm{i}\pi(\lambda^\star \cdot M)_a N_a + \mathrm{i}\pi(\lambda^\star \cdot M)_5 N_5\right] \times \exp\left[-2\pi\mathrm{i}(\lambda_5 M_{5a} N_a - \lambda_6 M_{6a} N_a)\right] \ , \\
&= c_{\lambda^\star} \ \mathrm{e}^{-\pi\mathrm{i}\left[2(\lambda_5+\lambda_6)\sum_b N_b\right]} =: c_{\lambda'} \ ,
\end{aligned} \tag{2.140}$$

where we have defined $\lambda' = (-\lambda_a, -\lambda_5 - 2\lambda_6; \lambda_6) = \lambda^\star - 2(\lambda_5 + \lambda_6) e_5$. Thus, the CPT conjugate of the operator $V_\lambda(k, z)$ in (2.130) is given by

$$\begin{aligned}
\widetilde{V}_\lambda(k, z) &= \omega(\lambda)^\dagger \, c(z) \, \mathcal{B}^\dagger(z) \, (c_\lambda)^\dagger \, (\mathrm{e}^{\lambda \cdot H(z)})^\dagger \, \mathrm{e}^{-2\mathrm{i}k \cdot X(z)} \\
&= \omega(\lambda)^\dagger \, \mathrm{e}^{\mathrm{i}\pi \lambda' \cdot M \cdot \lambda^\star} \, c(z) \, \mathcal{B}^\dagger(z) \, \mathrm{e}^{\lambda^\star \cdot H(z)} c_{\lambda'} \, \mathrm{e}^{-2\mathrm{i}k \cdot X(z)} \ .
\end{aligned} \tag{2.141}$$

□

This concludes our exposition of supersymmetric open strings in a constant B-field. We have constructed the vertex operators for various string states which shall be used in the sequel to calculate the appropriate amplitudes. Next, we describe $\mathcal{N} = (0, 2)$ superspace.

Chapter 3

$\mathcal{N} = (0,2)$ superspace

A 1+1 dimensional theory with $\mathcal{N} = (0,2)$ supersymmetry has two supercharges $(\mathrm{Q}_+, \overline{\mathrm{Q}}_+)$ in the left-moving sector. The corresponding supersymmetry parameters are left-handed spinors $(\epsilon^+, \bar{\epsilon}^+)$.

The Dirac equation in $1+1$ dimensions imposes that left(right)-handed fermions are right(left)-movers on-shell. A scalar has both left- and right-moving parts. The left-moving part of the scalar will then have a superpartner fermion which is left-moving on-shell and hence right-handed. Thus, a scalar multiplet has a scalar and a right-handed fermion as its on-shell degrees of freedom. A fermion which is right-moving on-shell (and hence left-handed) can form a multiplet on its own under such a supersymmetry. We next describe these multiplets and their gauged versions in superspace.

$\mathcal{N} = (0,2)$ superspace is described by coordinates $(x^{\pm\pm}, \theta^+, \bar{\theta}^+)$ where θ^+ and $\bar{\theta}^+$ are left-handed spinors. The corresponding supercovariant derivatives are denoted by $(\partial_{++}, \mathrm{D}_+, \overline{\mathrm{D}}_+)$. They satisfy the algebra

$$\mathrm{D}_+^2 = \overline{\mathrm{D}}_+^2 = 0 \ , \quad \{\mathrm{D}_+ \, , \overline{\mathrm{D}}_+\} = 2i\partial_{++} \ . \tag{3.1}$$

We would like to study constrained superfields of the form $\overline{\mathrm{D}}_+(\cdot) = 0$. The natural complex structure of the supercovariant derivatives then imposes a complex structure on the space of constrained superfields. There are three kinds of $\mathcal{N} = (0,2)$ superfields that will be important for us: Vector, Chiral, Fermi. Before we study these representations, let us briefly discuss the representation theory of SO(1,1): the Lorentz group in 1+1 dimensions.

3.1 Representations of SO(1, 1)

The group $\mathrm{SO}(1,1)$ is abelian and has a single boost generator T_{01} with group element

$$g(\lambda) = \exp(\lambda T_{01}) \ , \quad \lambda \in \mathbf{R} \ .$$

All irreducible representations are one dimensional. The representation on coordinates x^μ, $\mu = 0, 1$, is given by $T_{01} = -x_0\partial_1 + x_1\partial_0$. In terms of lightcone coordinates $x^{\pm\pm} = \frac{1}{2}(x^0 \pm x^1)$ where x^{++} is left-moving and x^{--} is right-moving, we have

$$g(\lambda) \cdot x^{\pm\pm} = \mathrm{e}^{\pm\lambda} x^{\pm\pm} \ . \tag{3.2}$$

The vector representation mimics the above transformation rule: $v^{\pm\pm} \to \mathrm{e}^{\mp\lambda} v^{\pm\pm}$.

The spinor representations are the basic representations of the double cover $\mathrm{Spin}(1,1)$ with $T_{01} = \frac{1}{2}\rho_0\rho_1$ where ρ^μ are $1+1$ dimensional Dirac matrices. Let $\rho_c = -\rho^0\rho^1 = \rho_0\rho_1$ and define *left-handed* and *right-handed* spinors v^+ and v^- to satisfy $\rho_c v^\pm = \pm v^\pm$. We then have $T_{01} = \frac{1}{2}\rho_c$ and $v^\pm$ transform as

$$g(\lambda) \cdot v^\pm = \mathrm{e}^{\pm\frac{1}{2}\lambda}\, v^\pm \ . \tag{3.3}$$

(Observe that the product $v^\pm w^\pm$ transforms in the same way as $x^{\pm\pm}$.) We raise and lower the indices using the totally antisymmetric ε-symbol with $\varepsilon_{+-} = +1 = \varepsilon^{+-}$:

$$v^+ = \varepsilon^{+-} v_- = v_- \ , \quad v^- = \varepsilon^{-+} v_+ = -v_+ \ . \tag{3.4}$$

We thus conclude that an irreducible representation of $\mathrm{SO}(1,1)$ is an object with some number of $+$ signs $v^{++\cdots+}$ (left-moving) or some number of $-$ signs $w^{--\cdots-}$ (right-moving).

Note: The Berezin differentials $\mathrm{d}\theta^+$, $\mathrm{d}\bar{\theta}^+$ transform as $\mathrm{d}\theta^+ \to \mathrm{e}^{-\frac{\lambda}{2}}\mathrm{d}\theta^+$, $\mathrm{d}\bar{\theta}^+ \to \mathrm{e}^{-\frac{\lambda}{2}}\mathrm{d}\bar{\theta}^+$. Thus, the most general superspace action is of the form

$$\int \mathrm{d}^2x\, \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ K_{--} + \int \mathrm{d}^2x \left(\mathrm{d}\theta^+ \mathcal{W}_- - \mathrm{h.c.}\right) \ , \tag{3.5}$$

where K_{--} and $\mathcal{W}_-$ are functions of the various superfields in the theory with K_{--} unconstrained and $\overline{\mathrm{D}}_+\mathcal{W}_- = 0$. Equivalently, one can write

$$\int \mathrm{d}^2x\,\mathrm{D}_+\overline{\mathrm{D}}_+K_{--} + \int \mathrm{d}^2x\,(\mathrm{D}_+\mathcal{W}_- - \mathrm{h.c.})\ , \tag{3.6}$$

upto total ∂_{++} derivative terms.

3.2 Chiral

A chiral superfield Φ is a Lorentz scalar and satisfies $\overline{\mathrm{D}}_+\Phi = 0$ and has components

$$\phi := \Phi_| \ , \quad \sqrt{2}\,\zeta_+ := (\mathrm{D}_+\Phi)_| \ . \tag{3.7}$$

The object $\overline{\mathrm{D}}_+\mathrm{D}_+\Phi$ contains nothing new: $(\overline{\mathrm{D}}_+\mathrm{D}_+\Phi)_| = 2\mathrm{i}\partial_{++}\phi$. Thus, this multiplet contains a scalar ϕ and a right-handed fermion ζ. The free action is

$$\mathcal{S}_{\text{chiral}} = -\frac{\mathrm{i}}{2}\int \mathrm{d}^2x\,\mathrm{D}_+\overline{\mathrm{D}}_+\ \overline{\Phi}\,\partial_{--}\Phi = \int \mathrm{d}^2x\left(-\overline{\partial^\mu\phi}\,\partial_\mu\phi - \mathrm{i}\bar{\zeta}_+\partial_{--}\zeta_+\right)\ . \tag{3.8}$$

3.3 Fermi

A Fermi superfield Ψ_- is a left-handed spinor and satisfies $\overline{\mathrm{D}}_+\Psi_- = \sqrt{2}E(\Phi)$ where $E(\Phi)$ is a holomorphic function of the chiral multiplets Φ_i in the theory. Ψ_- has components

$$\psi_- := (\Psi_-)_| \ , \quad -\sqrt{2}\,G := (\mathrm{D}_+\Psi_-)_| \ , \quad (\mathrm{D}_+\overline{\mathrm{D}}_+\Psi_-)_| = 2\frac{\partial E}{\partial\phi_i}\,\zeta_{+,i}\ . \tag{3.9}$$

The two-derivative action for Ψ_- is

$$\begin{aligned}\mathcal{S}_{\text{Fermi}} &= \frac{1}{2}\int \mathrm{d}^2x\,\mathrm{D}_+\overline{\mathrm{D}}_+\ \overline{\Psi}_-\Psi_-\ ,\\ &= \int \mathrm{d}^2x\left(-\mathrm{i}\bar{\psi}_-\partial_{++}\psi_- + |G|^2 - |E(\phi)|^2 + \bar{\psi}_-\frac{\partial E}{\partial\phi_i}\,\zeta_{i+} + \frac{\partial\overline{E}}{\partial\bar{\phi}^i}\,\bar{\zeta}^i_+\,\psi_-\right)\ . \end{aligned}\tag{3.10}$$

We see that the left-handed fermion ψ_- satisfies $\partial_{++}\psi_- = 0$ for $E = 0$ and hence is right-moving on-shell.

3.4 Potential terms

Let Φ_i collectively denote all the $(0,2)$ chiral multiplets in the theory and Ψ_a the $(0,2)$ Fermi multiplets (we suppress the Lorentz index on Ψ_a). We have already seen the E-term previously when we discussed kinetic terms. We can also write a superpotential, also known as "J-term" in $(0,2)$ literature:

$$\begin{aligned} \mathcal{S}_J &= -\frac{1}{\sqrt{2}} \int \mathrm{d}^2 x \, \mathrm{D}_+ \left(J^a(\Phi_i)\Psi_a \right) - \text{h.c.} \ , \\ &= \int \mathrm{d}^2 x \left(J^a(\phi_i) G_a + \overline{G}^a \overline{J}_a(\bar{\phi}) - \frac{\partial J^a}{\partial \phi_j} \zeta_{j+} \psi_{a-} - \bar{\psi}^a_- \frac{\partial \overline{J}_a}{\partial \bar{\phi}^j} \bar{\zeta}^j_+ \right) \ . \end{aligned} \tag{3.11}$$

Invariance of the above term under $\mathcal{N} = (0,2)$ supersymmetry requires $\overline{\nabla}_+(\Psi_a J^a) = 0$. This implies

$$0 = E_a J^a =: E \cdot J \ . \tag{3.12}$$

This constraint is necessary for the action to be $\mathcal{N} = (0,2)$ supersymmetric. If the action for a theory can be written in $(0,2)$ superspace but the above constraint in violated, then the theory is only $(0,1)$ supersymmetric.

3.5 Vector

Suppose we have some matter fields Υ transforming under a rigid symmetry $\Upsilon \to \mathrm{e}^{\mathrm{i}K}\Upsilon$ with $K = K^a T_a$ an hermitian parameter. We choose hermitian generators T_a with $\mathrm{Tr}(T_a T_b) = \frac{1}{2}\delta_{ab}$ in the fundamental representation. We gauge this symmetry by introducing gauge-covariant supercovariant derivatives ∇_+, $\overline{\nabla}_+$ and $\nabla_{\pm\pm}$ which transform as $\nabla \longrightarrow \mathrm{e}^{\mathrm{i}K}\nabla\mathrm{e}^{-\mathrm{i}K}$ under gauge transformations. The superspace constraints are

$$\nabla_+^2 = 0 \ , \quad \overline{\nabla}_+^2 = 0 \ , \quad \text{and} \quad \{\nabla_+ \, , \overline{\nabla}_+\} = 2\mathrm{i}\nabla_{++} \ . \tag{3.13}$$

The non-trivial curvatures are given by

$$-2\mathrm{i}v_{01} = [\nabla_{++}, \nabla_{--}] \ , \quad -2\mathrm{i}\mathcal{F}_{-} = [\overline{\nabla}_{+}, \nabla_{--}] \ , \quad 2\mathrm{i}\overline{\mathcal{F}}_{-} = [\nabla_{+}, \nabla_{--}] \ . \tag{3.14}$$

The Bianchi identities give

$$\nabla_{+}\mathcal{F}_{-} = \nabla_{+}\overline{\mathcal{F}}_{-} = 0 \ , \quad \nabla_{+}\mathcal{F}_{-} - \overline{\nabla}_{+}\overline{\mathcal{F}}_{-} = 2\mathrm{i}v_{01} \ . \tag{3.15}$$

The components of the above field strengths are given by

$$\lambda_{-} := -(\mathcal{F}_{-})_{|} \ , \quad D + \mathrm{i}v_{01} := (\nabla_{+}\mathcal{F}_{-})_{|} \ . \tag{3.16}$$

The gauge action is given by

$$\begin{aligned} \mathcal{S}_{\text{gauge}} &= \frac{1}{2g^2}\int \mathrm{d}^2x\, \mathrm{D}_{+}\overline{\mathrm{D}}_{+}\, \mathrm{Tr}\, \overline{\mathcal{F}}_{-}\mathcal{F}_{-} \ , \\ &= \frac{1}{g^2}\int \mathrm{d}^2x \ \mathrm{Tr}\left(\frac{1}{2}v_{01}^2 - \mathrm{i}\bar{\lambda}_{-}\mathrm{D}_{++}\lambda_{-} + \frac{1}{2}D^2\right) \ . \end{aligned} \tag{3.17}$$

The chirality constraint for a chiral superfield Φ in a complex representation of the gauge group becomes $\overline{\nabla}_{+}\Phi = 0$ and the components are defined to be

$$\phi := \Phi_{|} \ , \quad \sqrt{2}\,\zeta_{+} := (\nabla_{+}\Phi)_{|} \ . \tag{3.18}$$

The minimally coupled action is

$$\begin{aligned} \mathcal{S}_{\text{chiral}} &= -\frac{i}{2}\int \mathrm{d}^2x\, \mathrm{D}_{+}\overline{\mathrm{D}}_{+}\ \bar{\Phi}\,\nabla_{--}\Phi \ , \\ &= \int \mathrm{d}^2x \left(-\overline{\mathrm{D}^{\mu}\phi}\,\mathrm{D}_{\mu}\phi - \mathrm{i}\bar{\zeta}_{+}\mathrm{D}_{--}\zeta_{+} + \mathrm{i}\sqrt{2}\bar{\phi}\lambda_{-}\zeta_{+} - \mathrm{i}\sqrt{2}\bar{\zeta}_{+}\bar{\lambda}_{-}\phi - \bar{\phi}D\phi\right) \ . \end{aligned} \tag{3.19}$$

Similarly, the constraint for a Fermi superfield Ψ_{a-} in some representation of the gauge group becomes $\overline{\nabla}_+\Psi_a = \sqrt{2}E_a(\Phi)$. The minimally coupled action is

$$\begin{aligned}\mathcal{S}_{\text{Fermi}} &= \frac{1}{2}\int \mathrm{d}^2x\, \mathrm{D}_+\overline{\mathrm{D}}_+\, \overline{\Psi}^a\Psi_a \ , \\ &= \int \mathrm{d}^2x \left(-\mathrm{i}\bar{\psi}^a_-\mathrm{D}_{++}\psi_{a-} + \overline{G}^aG_a - E_a\overline{E}^a + \bar{\psi}^a_- \frac{\partial E_a}{\partial\phi_j}\,\zeta_{j+} + \frac{\partial\overline{E}^a}{\partial\bar{\phi}^j}\,\bar{\zeta}^j_+\,\psi_{a-}\right) \ .\end{aligned} \tag{3.20}$$

3.6 Holomorphic representation

The constraints $\nabla_+^2 = \overline{\nabla}_+^2 = 0$ can be solved by introducing a complex Lie algebra valued superfield $\Omega = \Omega^aT_a$ called the *prepotential*:

$$\nabla_+ = \mathrm{e}^{-\mathrm{i}\Omega}\mathrm{D}_+\mathrm{e}^{\mathrm{i}\Omega} := \mathrm{D}_+ + \mathrm{i}\Gamma_+ \ , \quad \overline{\nabla}_+ = \mathrm{e}^{-\mathrm{i}\overline{\Omega}}\,\overline{\mathrm{D}}_+\mathrm{e}^{\mathrm{i}\overline{\Omega}} := \overline{\mathrm{D}}_+ - \mathrm{i}\overline{\Gamma}_+ \ . \tag{3.21}$$

where we have defined the spinor connections Γ_+ and $\overline{\Gamma}_+$. We also define $\nabla_{\pm\pm} := \mathrm{D}_{\pm\pm} + \mathrm{i}\Gamma_{\pm\pm}$. The gauge transformation $\nabla \to \mathrm{e}^{\mathrm{i}K}\nabla\mathrm{e}^{-\mathrm{i}K}$ can be reproduced by assigning the following transformation rule for Ω:

$$\mathrm{e}^{\mathrm{i}\Omega} \to \mathrm{e}^{\mathrm{i}\Omega}\mathrm{e}^{-\mathrm{i}K} \ , \quad \mathrm{e}^{\mathrm{i}\overline{\Omega}} \to \mathrm{e}^{\mathrm{i}\overline{\Omega}}\mathrm{e}^{-\mathrm{i}K} \ . \tag{3.22}$$

The above solution has additional gauge invariances:

$$\mathrm{e}^{\mathrm{i}\Omega} \to \mathrm{e}^{\mathrm{i}\overline{\Lambda}}\mathrm{e}^{\mathrm{i}\Omega} \ , \quad \mathrm{e}^{\mathrm{i}\overline{\Omega}} \to \mathrm{e}^{\mathrm{i}\Lambda}\mathrm{e}^{\mathrm{i}\overline{\Omega}} \ . \tag{3.23}$$

where Λ is a Lie algebra valued chiral superfield $\overline{\mathrm{D}}_+\Lambda = 0$. One can use the hermitian K to gauge away the hermitian part of Ω. Equivalently, we look at the K-inert hermitian object

$$\mathrm{e}^V := \mathrm{e}^{\mathrm{i}\Omega}\mathrm{e}^{-\mathrm{i}\overline{\Omega}} \ , \quad \text{with} \quad \mathrm{e}^V \longrightarrow \mathrm{e}^{\mathrm{i}\overline{\Lambda}}\,\mathrm{e}^V\,\mathrm{e}^{-\mathrm{i}\Lambda} \ . \tag{3.24}$$

(In the gauge where $\Omega = -\overline{\Omega}$, we have $V = 2\mathrm{i}\Omega$.)

One can go to the *holomorphic representation* via a (non-unitary) change of basis $\nabla \to \mathrm{e}^{\mathrm{i}\overline{\Omega}}\nabla\mathrm{e}^{-\mathrm{i}\overline{\Omega}}$, $\Upsilon \to \mathrm{e}^{\mathrm{i}\overline{\Omega}}\Upsilon$ for a general matter superfield Υ. The spinor derivatives become $\nabla_+ = \mathrm{e}^{-V}\mathrm{D}_+\mathrm{e}^{V}$, $\overline{\nabla}_+ = \overline{\mathrm{D}}_+$ which gives

$$\mathrm{i}\Gamma_+ = \mathrm{e}^{-V}(\mathrm{D}_+\mathrm{e}^{V}) \ , \quad \overline{\Gamma}_+ = 0 \ , \tag{3.25}$$

thus justifying the name *holomorphic*. In this representation, the chirality constraint becomes $\overline{\mathrm{D}}_+\Upsilon = 0$. All the derivatives are K-inert but transform under Λ as $\nabla \to \mathrm{e}^{\mathrm{i}\Lambda}\nabla\mathrm{e}^{-\mathrm{i}\Lambda}$ with $\overline{\mathrm{D}}_+\Lambda = 0$ and the connections transform as

$$\mathrm{i}\delta\Gamma_+ = -\mathrm{i}\nabla_+\Lambda \ , \quad \delta\overline{\Gamma}_+ = 0 \ . \tag{3.26}$$

The components of Γ_+ are $\gamma_+ := (\Gamma_+)|$, $v_{++} := \frac{1}{2}(\overline{\nabla}_+\Gamma_+)|$ of which γ_+ can be set to zero using the gauge transformation above. The same gauge freedom gives

$$\delta v_{++} = \frac{1}{2}(\overline{\nabla}_+\delta\Gamma_+)| = -\frac{1}{2}(\{\overline{\nabla}_+, \nabla_+\}\Lambda)| = -\mathrm{i}\nabla_{++}\lambda \ , \tag{3.27}$$

which is the usual transformation for a non-abelian gauge field v_{++}. The final constraint $\{\overline{\nabla}_+, \nabla_+\} = 2\mathrm{i}\nabla_{++}$ gives $2\mathrm{i}\Gamma_{++} = \overline{\nabla}_+\Gamma_+$ whose bosonic part is $2v_{++}$. The curvatures are given by

$$\mathcal{F}_- = [\overline{\mathrm{D}}_+ \, , \nabla_{--}] \ , \quad \widetilde{\mathcal{F}}_- = -[\nabla_+, \nabla_{--}] \ . \tag{3.28}$$

The superspace Lagrangians for the chiral, Fermi and vector multiplets in the holomorphic representation are $\overline{\Phi}\mathrm{e}^{V}\nabla_{--}\Phi$, $\overline{\Psi}\mathrm{e}^{V}\Psi$ and $\mathrm{Tr}\,\mathcal{F}_-\widetilde{\mathcal{F}}_-$ respectively.

3.7 Duality exchanging $E \leftrightarrow J$

Consider a Fermi superfield Ψ_a satisfying $\overline{\nabla}_+\Psi_a = \sqrt{2}E_a$. The most general action with J-term is

$$\mathcal{S}[\Psi_a] = -\frac{1}{2}\int \mathrm{d}^2x\, \mathrm{D}_+\overline{\mathrm{D}}_+\, \overline{\Psi}^a\Psi_a - \frac{1}{\sqrt{2}}\int \mathrm{d}^2x \left\{\mathrm{D}_+\, \Psi_a J^a + \overline{\mathrm{D}}_+\overline{\Psi}^a\overline{J}_a\right\} \ . \tag{3.29}$$

The kinetic term can be reproduced from the following first order action for Ψ_a by integrating out the unconstrained Grassmann superfield Λ_a:

$$\mathcal{S}[\Lambda_a, \Psi_a] = \frac{1}{2}\int \mathrm{d}^2x\, \mathrm{D}_+\overline{\mathrm{D}}_+ \left\{\overline{\Lambda}^a\Lambda_a - \Psi_a\overline{\Lambda}^a - \Lambda_a\overline{\Psi}^a\right\} - \frac{1}{\sqrt{2}}\int \mathrm{d}^2x \left\{\mathrm{D}_+\Psi_a J^a + \overline{\mathrm{D}}_+\overline{\Psi}^a \overline{J}_a\right\} \ . \tag{3.30}$$

Instead, we could integrate out Ψ_a. To do this, we push in $\overline{\mathrm{D}}_+$ in the Lagrange multiplier term $\Psi_a\overline{\Lambda}^a$ (and appropriately for its complex conjugate) to get

$$\mathcal{S}[\Lambda_a, \Psi_a] = \frac{1}{2}\int \mathrm{d}^2x\, \mathrm{D}_+ \left\{-\sqrt{2}E_a\overline{\Lambda}^a + \Psi_a\overline{\nabla}_+\overline{\Lambda}^a - \sqrt{2}\Psi_a J^a\right\} - \text{h.c.} \ . \tag{3.31}$$

Integrating out Ψ_a gives $\overline{\nabla}_+\overline{\Lambda}^a = \sqrt{2}J^a$. Relabelling $\overline{\Lambda}^a = \Psi'^a$, we have $\overline{\nabla}_+\Psi'^a = \sqrt{2}J^a$ and the action

$$\mathcal{S}[\Psi'_a] = -\frac{1}{2}\int \mathrm{d}^2x\, \mathrm{D}_+\overline{\mathrm{D}}_+\, \overline{\Psi}'_a\Psi'^a - \frac{1}{\sqrt{2}}\int \mathrm{d}^2x \left\{\mathrm{D}_+(\Psi'^a E_a) + \overline{\mathrm{D}}_+(\overline{\Psi}'_a\overline{E}^a)\right\} \ . \tag{3.32}$$

Note: The new Fermi multiplet Ψ'^a transforms in the conjugate representation of the various symmetry groups in the theory as compared to Ψ_a.

3.8 $(2,2) \to (0,2)$

We shall be schematic here and details can be found in Section 6 of [W2]. A twisted chiral superfield Σ satisfies $\overline{\nabla}_+\Sigma = \nabla_-\Sigma = 0$. The $(0,2)$ decomposition is then a chiral and a Fermi multiplet:

$$\begin{aligned} \Sigma &:= \Sigma_| \ , \quad \text{with} \quad \overline{\nabla}_+\Sigma = 0 \ , \\ \widetilde{\Sigma}_- &:= \frac{1}{\sqrt{2}}(\overline{\nabla}_-\Sigma)_| \ , \quad \text{with} \quad \overline{\nabla}_+\widetilde{\Sigma}_- = 0 \ , \end{aligned} \tag{3.33}$$

where $|$ indicates that we have set $\theta^- = \bar{\theta}^- = 0$. The (2,2) field strength is a twisted chiral multiplet: $2\sqrt{2}\,\Sigma = \{\overline{\nabla}_+, \nabla_-\}$. The complex scalar σ now sits in a separate (0,2) chiral multiplet Σ and $\widetilde{\Sigma}_-$ is the familiar (0,2) field strength $\mathcal{F}_-$ (upto a factor of $-\mathrm{i}/2$).

A (2,2) chiral superfield Φ satisfies $\overline{\nabla}_+\Phi = \overline{\nabla}_-\Phi = 0$. The (0,2) decomposition is then

$$\begin{aligned} \Phi &:= \Phi| \ , \quad \text{with} \quad \overline{\nabla}_+\Phi = 0 \ , \\ \Phi_- &:= \frac{1}{\sqrt{2}}(\nabla_-\Phi)| \ , \quad \text{with} \quad \overline{\nabla}_+\Phi_- = \frac{1}{\sqrt{2}}\{\overline{\nabla}_+, \nabla_-\}\Phi = 2\Sigma\Phi \ , \end{aligned} \tag{3.34}$$

where Σ is the (0,2) chiral multiplet that contains the complex scalar σ. Thus, a (2,2) chiral multiplet Φ splits into a (0,2) chiral Φ and a Fermi Φ_- which has an E-term $E_{\Phi_-} = \sqrt{2}\Sigma\Phi$.

A (2,2) superpotential $W(\Phi_i)$ gives rise to (0,2) superpotential $\mathcal{W}(\Phi_i, \Phi_{i-})$ after the D_- in the measure has been pushed into the action:

$$\int \mathrm{D}_+\mathrm{D}_- W(\Phi_i) = \int \mathrm{D}_+ \frac{\partial W}{\partial \Phi_i} \nabla_-\Phi_i = \sqrt{2} \int \mathrm{D}_+ \frac{\partial W}{\partial \Phi_i} \Phi_{i-} \ , \tag{3.35}$$

giving a J-term $J^i = -2\frac{\partial W}{\partial \Phi_i}$. The constraint $\overline{\nabla}_+(J^i\Phi_{i-}) = 0$ becomes

$$\frac{\partial W}{\partial \Phi_i} \Sigma\Phi_i = 0 \ , \tag{3.36}$$

which is nothing but the condition of gauge invariance of $W(\Phi)$. □

Chapter 4

The spiked instanton gauged linear sigma model

4.1 Supersymmetry in a constant B-field background

Consider a constant NSNS B-field background of the form:

$$2\pi\alpha' B_{12} = b_1 \ , \quad 2\pi\alpha' B_{34} = b_2 \ , \quad 2\pi\alpha' B_{56} = b_3 \ , \quad 2\pi\alpha' B_{78} = b_4 \ . \tag{4.1}$$

This choice of B-field preserves the $\mathrm{SO}(2)^4$ rotational symmetry of the above intersecting D-brane system. Such a symmetry is essential for eventually considering the generalisation to the Ω-background.

We first state our conventions and introduce some notation.

- Introduce the variables v_a, $a \in \underline{\mathbf{4}}$ with

$$\mathrm{e}^{2\pi \mathrm{i} v_a} = \frac{1+\mathrm{i}b_a}{1-\mathrm{i}b_a} \ , \quad b_a = \tan \pi v_a \ , \quad -\frac{1}{2} < v_a < \frac{1}{2} \ . \tag{4.2}$$

  The limits $v_a \to \pm\frac{1}{2}$ correspond to $b_a \to \pm\infty$.

- For each $A \in \underline{\mathbf{6}}$, let $\Gamma_A = \Gamma^{2a-1}\Gamma^{2a}\Gamma^{2b-1}\Gamma^{2b}$ for $A = (ab)$.

- Choose the following representation for the Γ-matrices. This representation corresponds to a particular choice of the cocycle operators for open string vertex operators given in [KLLSW]. See Chapter 2 for more details.

$$\begin{aligned}
\Gamma^1 &= \sigma_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \ , & \Gamma^7 &= -\sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_2 \otimes \mathbb{1} \ , \\
\Gamma^2 &= \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \ , & \Gamma^8 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \mathbb{1} \ , \\
\Gamma^3 &= \sigma_3 \otimes \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \ , & \Gamma^9 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \ , \\
\Gamma^4 &= -\sigma_3 \otimes \sigma_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \ , & \Gamma^0 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes (-\mathrm{i}\sigma_2) \ , \\
\Gamma^5 &= -\sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \mathbb{1} \otimes \mathbb{1} \ , & \Gamma_c &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \ , \\
\Gamma^6 &= -\sigma_3 \otimes \sigma_3 \otimes \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1} \ , & C_- &= \mathrm{e}^{3\pi \mathrm{i}/4} \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \ .
\end{aligned} \tag{4.3}$$

The chirality matrices in $\mathbf{C}^2_A$ are chosen to be $\Gamma_c(\mathbf{C}^2_A) = \Gamma_A$ where Γ_A is defined above and the chirality matrix in $\mathbf{R}^{1,1}$ is $\Gamma_c(\mathbf{R}^{1,1}) = -\Gamma^0\Gamma^9$.

- The $32_{\mathbf{C}}$ dimensional spinor representation can then be constructed by considering simultaneous eigenvectors $|\pm,\pm,\pm,\pm,\pm\rangle$ of $-\frac{\mathrm{i}}{2}\Gamma^{12}$, $-\frac{\mathrm{i}}{2}\Gamma^{34}$, $-\frac{\mathrm{i}}{2}\Gamma^{56}$, $-\frac{\mathrm{i}}{2}\Gamma^{78}$ and $-\frac{1}{2}\Gamma^{09}$, and using the linear combinations $-\mathrm{i}\Gamma^1 \pm \Gamma^2, \ldots, -\mathrm{i}\Gamma^7 \pm \Gamma^8$, $\Gamma^0 \pm \Gamma^9$ as raising and lowering operators respectively. The basis of the representation is then given by the 32 vectors $|\pm,\pm,\pm,\pm,\pm\rangle$. The left-handed (right-handed) $16_{\mathbf{C}}$ subspace of Γ_c is then spanned by the subset of the above with even (odd) number of negative signs.

Next, we study the amount of supersymmetry preserved in the presence of a constant B-field. In the presence of a constant B-field of the form (4.1), the constraint arising from the stack of $\mathrm{D}5_A$ branes becomes

$$\tilde{\epsilon} = \Gamma^{90} R_A \, \epsilon \ , \tag{4.4}$$

where R_A is given by

$$R_A = \exp\left(\sum_{a \in A} \pi \theta_a \Gamma^{2a-1} \Gamma^{2a} \right) \ , \tag{4.5}$$

with $\theta_a := \frac{1}{2} - v_a$. Combining this with the constraint $\tilde{\epsilon} = -\Gamma^{90}\epsilon$ from the $\overline{\mathrm{D1}}$-branes, we get

$$R_A \, \epsilon = -\epsilon \quad \text{for every} \quad A \in \underline{\mathbf{6}} \ . \tag{4.6}$$

Let $r(\theta) := \exp(\mathrm{i}\sigma_3\pi\theta)$. Then, we have

$$\begin{aligned} R_{(12)} &= r(\theta_1)\otimes r(\theta_2)\otimes \mathbb{1}\otimes \mathbb{1}\otimes \mathbb{1}\ , \quad R_{(13)} = r(\theta_1)\otimes \mathbb{1}\otimes r(\theta_3)\otimes \mathbb{1}\otimes \mathbb{1}\ , \\ R_{(14)} &= r(\theta_1)\otimes \mathbb{1}\otimes \mathbb{1}\otimes r(\theta_4)\otimes \mathbb{1}\ , \quad R_{(23)} = \mathbb{1}\otimes r(\theta_2)\otimes r(\theta_3)\otimes \mathbb{1}\otimes \mathbb{1}\ , \\ R_{(24)} &= \mathbb{1}\otimes r(\theta_2)\otimes \mathbb{1}\otimes r(\theta_4)\otimes \mathbb{1}\ , \quad R_{(34)} = \mathbb{1}\otimes \mathbb{1}\otimes r(\theta_3)\otimes r(\theta_4)\otimes \mathbb{1}\ . \end{aligned} \tag{4.7}$$

The equations $R_A\epsilon = -\epsilon$ have a solution if, for some choice of signs,

$$\exp\left(\pm\mathrm{i}\pi\theta_a \pm \mathrm{i}\pi\theta_b\right) = -1 \quad \text{with} \quad 0\le\theta_a\le 1 \quad \forall \quad a\in\underline{\mathbf{4}}\ . \tag{4.8}$$

These equations have solutions corresponding to finite B only when $\theta_a = \frac{1}{2}$ for all $a\in\underline{\mathbf{4}}$ with all plus or all minus signs. This corresponds to $v_a = 0$ which is the zero B-field point. Thus, turning on a finite B-field of the above form does not make the brane configuration supersymmetric.

Stability

First we observe that supersymmetry is completely lost about the original vacuum for a non-zero finite value of the constant B-field. Thus, stability is no longer guaranteed. Secondly, a constant B-field background typically introduces instability in the form of tachyons in the D-brane spectrum.

In some situations, e. g. the D1-D5 system, the effects of the B-field can be accommodated by turning on a Fayet-Iliopoulos parameter in the low energy effective action. The tachyon instability leads to the system transitioning to a nearby vacuum at which point supersymmetry is restored.

We shall see that something similar happens in the spiked scenario as well, with some differences. To study the stability we need to derive the spectrum of open strings in the presence of D-branes in a constant B-field background.

4.2 Spectrum of Dp-Dp' strings

The boundary conditions for an open string are modified in the presence of a B-field. Let the worldsheet bosons and fermions along $\mathbf{C}^4$ be resp. $Z^a(\sigma,\tau)$ and $\Psi^{\pm,a}(\sigma,\tau)$, $a \in \underline{\mathbf{4}}$. Neumann boundary conditions along $\mathbf{C}_a$ are modified to (cf. Chapter 2):

$$\textbf{Mixed (M):} \qquad \partial_{++}Z^a = \mathrm{e}^{-2\pi \mathrm{i} v_a}\partial_{--}Z^a \ , \qquad \Psi^{a+} = \mathrm{e}^{2\pi \mathrm{i} v_a}\Psi^{a-} \ . \tag{4.9}$$

Neumann and Dirichlet boundary conditions are obtained by setting $v_a = 0$ and $v_a \to \frac{1}{2}$ respectively. Sending one of the v_a's to $-\frac{1}{2}$ would give Dirichlet boundary conditions on an anti D-brane. Consider the more general boundary conditions with $-\frac{1}{2} \leq \mu, \nu \leq \frac{1}{2}$:

$$\begin{aligned} \partial_{++}Z &= \mathrm{e}^{-2\pi \mathrm{i}\nu}\partial_{--}Z \ , \quad \Psi^+ = \mathrm{e}^{2\pi \mathrm{i}\nu}\Psi^- \quad \text{at} \quad \sigma = 0 \ , \\ \partial_{++}Z &= \mathrm{e}^{-2\pi \mathrm{i}\mu}\partial_{--}Z \ , \quad \Psi^+ = \pm\mathrm{e}^{2\pi \mathrm{i}\mu}\Psi^- \quad \text{at} \quad \sigma = \pi \ , \end{aligned} \tag{4.10}$$

The low-energy spectrum for this system has been worked out in Chapter 2. We summarise the results here.

1. **Non-integer modes:** The worldsheet boson Z has moding $\mathbf{Z} + \theta$ with $\theta = \mu - \nu$. The R sector fermions have the same moding as Z due to rigid supersymmetry on the worldsheet and the NS sector fermions have moding $\mathbf{Z} + \epsilon$ with $\epsilon = \theta + \frac{1}{2} = \mu - \nu + \frac{1}{2}$.

2. **Excitations:** The zero-point energy in the NS sector is given by

$$E_0 = \frac{1}{8} - \frac{1}{2}\big||\theta| - \tfrac{1}{2}\big| \ . \tag{4.11}$$

   The first excited state in the NS sector has energy $E_0 + |\epsilon|$ or $E_0 + |1-\epsilon|$ when $-\frac{1}{2} \leq \epsilon \leq \frac{1}{2}$ and $\frac{1}{2} < \epsilon < \frac{3}{2}$ respectively.

   The zero-point energy in the R sector vanishes due to rigid supersymmetry on the worldsheet. The first excited state in the R sector has energy $|\theta|$ for $0 \leq |\theta| \leq \frac{1}{2}$ and $1 - |\theta|$ for $\frac{1}{2} \leq |\theta| \leq 1$.

3. **Spectral flow:** When ϵ crosses the integer s from the left ($s = 0$ or 1), the state

with energy $s-\epsilon$ becomes negative and enters the Dirac sea and the state $\epsilon - s$ crosses into the positive energy region. The raising and lowering roles of the NS fermion operators d_s and $d_s^\dagger$ are interchanged. Using $d_s^\dagger d_s = -d_s d_s^\dagger + 1$, we see that the number operator changes by one unit $N_d \to N_d + 1$. This changes the sign of the parity operator $(-)^{F_{\rm NS}} := (-1)^{N_d}$ and the GSO projectors $\frac{1}{2}(1 \pm (-)^{F_{\rm NS}})$ are consequently interchanged. A similar phenomenon occurs in the R sector when θ crosses 0.

4.2.1 $\overline{\mathbf{D1}}$-$\overline{\mathbf{D1}}$ strings

The open strings satisfy **NN** boundary conditions along $\mathbf{R}^{1,1}$ and **DD** boundary conditions along $\mathbf{C}^4$. The worldsheet bosons have momentum zero modes along $\mathbf{R}^{1,1}$ and none along $\mathbf{C}^4$ and hence all the states are supported along $\mathbf{R}^{1,1}$.

NS sector: There are no zero modes for the NS fermions and the NS zero-point energy is $-\frac{1}{2}$. The NS fermion oscillators $d_1^{\mu\dagger}$, $\mu = 0, 9$ and $d_1^{a\dagger}$, $a \in \underline{\mathbf{4}}$ raise the energy by $\frac{1}{2}$. The oscillators $d_1^{\mu\dagger}$ gives rise to two states which are the components of a gauge field $v_{\pm\pm}(x,t)$ while the four complex oscillators $d_1^{a\dagger}$ create four states in the adjoint of U(k) corresponding to complex scalars $B_a(x,t)$. Assigning the NS vacuum a fermion number $F_{\rm NS} = -1$, the GSO projection with projector $\frac{1}{2}(1 + (-)^{F_{\rm NS}})$ projects out the vacuum while retaining the zero-energy states.

R sector: The R sector has ten zero modes thus giving a real 32 dimensional ground state transforming in the adjoint of U(k). The fermion parity $(-)^{F_{\rm R}}$ on the zero modes is then $(-)^{F_{\rm R}} = \Gamma^{1\cdots 8}\Gamma^{90} = \Gamma_c(\mathbf{R}^{1,9})$. The GSO projection with $\frac{1}{2}(1 + (-)^{F_{\rm R}})$ gives a left-handed fermion in $1+9$ dimensions which splits up into eight right-handed and eight left-handed fermions in $1+1$ dimensions.

We decompose the spacetime scalars and fermions into representations of SO$(\mathbf{R}^4_A) \times$ SO$(\mathbf{R}^4_{\overline{A}})$ using $\Gamma_c(\mathbf{R}^{1,9}) = \Gamma_c(\mathbf{R}^{1,1})\Gamma_c(\mathbf{C}^2_A)\Gamma_c(\mathbf{C}^2_{\overline{A}})$. Writing each SO(4) as SU(2) $\times$ SU(2) with $\alpha, \dot\alpha, \alpha', \dot\alpha'$ denoting the fundamentals of the four SU(2)'s, we have

$$\begin{aligned} \text{Scalars}: &\quad X^{\alpha\dot\alpha} \oplus X^{\alpha'\dot\alpha'} , \\ \text{Fermions}: &\quad \lambda_-^{\alpha\alpha'} \oplus \lambda_-^{\dot\alpha\dot\alpha'} \oplus \zeta_+^{\alpha\dot\alpha'} \oplus \zeta_+^{\alpha'\dot\alpha} , \end{aligned} \tag{4.12}$$

with reality conditions $\lambda_-^{\alpha\alpha'} = -\varepsilon^{\alpha\beta}\varepsilon^{\alpha'\beta'}\overline{\lambda_-^{\beta\beta'}}$ and so on for the fermions.

4.2.2 $\overline{\text{D1}}$-D5$_A$ strings

For a $\overline{\text{D1}}$-D5$_A$ string the boundary conditions are **DM** for $a \in A$ and **DD** for $a \in \bar{A}$. These boundary conditions imply $\mathbf{Z} + v_a - \frac{1}{2}$ moding for the bosons Z^a with $a \in A$ and $\mathbf{Z}$ moding for $a \in \bar{A}$. The R fermions have the same moding as the bosons and the NS fermions have moding $\mathbf{Z} + v_a$ for $a \in A$ and $\mathbf{Z} + \frac{1}{2}$ for $a \in \bar{A}$. Since the string is orientable, states from different orientations are distinct and have to be combined together in order to form a CPT invariant spectrum.

NS sector: Let $A = (ab)$. The NS zero-point energy is given by $-\frac{1}{2}(|v_a| + |v_b|)$. For v_a and v_b close to zero, the oscillators with lowest positive energy are from the NS fermions and increase energy by $|v_a|$ and $|v_b|$. The first four states in the NS sector have the energies

$$\frac{1}{2}(\pm|v_a| \pm |v_b|) \;\text{ or equivalently, }\; \frac{1}{2}(\pm v_a \pm v_b) \; . \tag{4.13}$$

When either of v_a and v_b crosses zero, the sign of $(-)^{F_{\text{NS}}}$ is flipped (cf. point 3 above). It is then easy to see that states which have definite values of $(-)^{F_{\text{NS}}}$ are $\frac{1}{2}(\pm v_a \pm v_b)$ rather than $\frac{1}{2}(\pm|v_a| \pm |v_b|)$.

We assign $(-)^{F_{\text{NS}}} = -1$ to the state with energy $-\frac{1}{2}(v_a + v_b)$ and choose the GSO projector to be $\frac{1}{2}(1 - (-)^{F_{\text{NS}}})$. The states with energies $\pm\frac{1}{2}(v_a - v_b)$ are projected out and the states that remain are

$$+\frac{1}{2}(v_a + v_b) \; , \; -\frac{1}{2}(v_a + v_b) \; . \tag{4.14}$$

These states transform in the $(\boldsymbol{k}, \overline{\boldsymbol{n}}_{\boldsymbol{A}})$ of $\text{U}(k) \times \text{U}(n_A)$. The string with opposite orientation furnishes two more states with the same energy and which transform in the $(\bar{\boldsymbol{k}}, \boldsymbol{n_A})$ of $\mathrm{U}(k) \times \mathrm{U}(n_A)$. Thus, we get two complex scalars ϕ^1 and ϕ^2 in the bifundamental of $\mathrm{U}(k) \times \mathrm{U}(n_A)$ with masses given by

$$m^2 = \mp \frac{1}{2\alpha'}(v_a + v_b) \ . \tag{4.15}$$

In the limit $v_a, v_b \to 0$, the two states become degenerate. We also have $(-)^{F_{\mathrm{NS}}} = \Gamma_A = \Gamma_c(\mathbf{C}^2_A)$ which implies that the above GSO projection results in a right-handed spinor ϕ^α in $\mathbf{C}^2_A$. These constitute the two complex scalars of a $\mathcal{N} = (4,4)$ bifundamental hypermultiplet in $\mathbf{R}^{1,1}$.

R sector: The zero-point energy vanishes in the R sector. There are six zero modes from fermions along $\mathbf{R}^{1,1} \times \mathbf{C}^2_{\bar{A}}$ which give an eight dimensional ground state consisting of spinors $|\alpha', \pm\rangle$ and $|\dot{\alpha}', \pm\rangle$ where $+(-)$ indicates left(right)-handed spinors in $\mathbf{R}^{1,1}$ and $\alpha'(\dot{\alpha}')$ right(left)-handed spinors in $\mathbf{C}^2_{\bar{A}}$. The fermion parity operator $(-)^{F_{\mathrm{R}}}$ is given by $(-)^{F_{\mathrm{R}}} = \Gamma_{\bar{A}}\Gamma^{90} = \Gamma_c(\mathbf{R}^{1,1})\Gamma_c(\mathbf{C}^2_{\bar{A}})$. The GSO projection with $\frac{1}{2}(1+(-)^{F_{\mathrm{R}}})$ retains the states that satisfy $\Gamma_c(\mathbf{R}^{1,1}) = \pm 1$, $\Gamma_c(\mathbf{C}^2_{\bar{A}}) = \pm 1$. Together with the states from the oppositely oriented string, we thus have spinors $\zeta^{\alpha' -} = -\zeta^{\alpha'}_+$ and $\lambda^{\dot{\alpha}' +} = \lambda^{\dot{\alpha}'}_-$. They transform in the $(\boldsymbol{k}, \bar{\boldsymbol{n}}_{\boldsymbol{A}})$ of $\mathrm{U}(k) \times \mathrm{U}(n_A)$ and constitute the fermionic part of the $\mathcal{N} = (4,4)$ bifundamental hypermultiplet in $\mathbf{R}^{1,1}$.

4.2.3 $\mathbf{D5_A}$-$\mathbf{D5_{\bar{A}}}$ strings

The boundary conditions are **MD** for $a \in A$ and **DM** for $a \in \bar{A}$. These imply the following modings for the bosons and R fermions:

$$\mathbf{Z} + \frac{1}{2} - v_a \ \text{ for } \ a \in A \quad \text{and} \quad \mathbf{Z} + v_{\bar{a}} - \frac{1}{2} \ \text{ for } \ \bar{a} \in \bar{A} \ . \tag{4.16}$$

The NS fermions have $\mathbf{Z} - v_a$ and $\mathbf{Z} + v_{\bar{a}}$ moding respectively.

NS sector: The zero point energy in the NS sector is then

$$\frac{1}{2} - \frac{1}{2} \sum_{a \in \underline{\mathbf{4}}} |v_a| \ . \tag{4.17}$$

The lowest excitation energies in the NS sector are $|v_a|$ for $a \in \underline{\mathbf{4}}$. The first few states are then

$$\frac{1}{2}(1 \pm v_1 \pm v_2 \pm v_3 \pm v_4) \ . \tag{4.18}$$

We assign $(-)^{F_{\rm NS}} = -1$ to the state with energy $\frac{1}{2}(1 - (v_1 + v_2 + v_3 + v_4))$. GSO projection with $\frac{1}{2}(1 + (-)^{F_{\rm NS}})$ removes states with an even number of negative signs. The remaining states are

$$\begin{aligned} &\frac{1}{2}[1 \pm (v_1 - v_2 - v_3 - v_4)] \ , \ \frac{1}{2}[1 \pm (v_1 + v_2 + v_3 - v_4)] \ , \\ &\frac{1}{2}[1 \pm (v_1 + v_2 - v_3 + v_4)] \ , \ \frac{1}{2}[1 \pm (v_1 - v_2 + v_3 + v_4)] \ . \end{aligned} \tag{4.19}$$

For small enough $|v_a|$, the above energies are all positive: there is no tachyon or massless state in the NS sector. There is another copy of these states from the string with opposite orientation. Together, they form eight massive complex scalars that transform in the $(\boldsymbol{n_A}, \overline{\boldsymbol{n}}_{\overline{\boldsymbol{A}}})$ of $\mathrm{U}(n_A) \times \mathrm{U}(n_{\bar{A}})$.

R sector: The ground state energy in the R sector is zero as always. The only zero modes are the ones along $\mathbf{R}^{1,1}$ and we denote them by Γ^0 and Γ^9. We have $(-)^{F_{\rm R}} = \Gamma^{90} = \Gamma_c(\mathbf{R}^{1,1})$. Assign $(-)^{F_{\rm R}} = -1$ for the ground state $|\mathrm{R}\rangle$ and define

$$g = \frac{\Gamma^9 + \Gamma^0}{\sqrt{2}} \ , \quad g^\dagger = \frac{\Gamma^9 - \Gamma^0}{\sqrt{2}} \ . \tag{4.20}$$

Acting on $|\mathrm{R}\rangle$ with $g^\dagger$ provides another state of zero energy but with $(-)^{F_{\rm R}} = +1$. The GSO projection with $\frac{1}{2}(1 + (-)^{F_{\rm R}})$ retains $g^\dagger|\mathrm{R}\rangle$ which is a left-handed fermion. Together with a similar state from the oppositely oriented string, this fermion transforms in the bifundamental of $\mathrm{U}(n_A) \times \mathrm{U}(n_{\bar{A}})$.

For small v_a, the first two sets of single-oscillator excitations for the worldsheet bosons and R fermions come from the $\mathbf{C}^4$ directions and have energy $\frac{1}{2} \mp v_a$. The GSO projection keeps the eight states obtained from the worldsheet bosons acting on $g^\dagger|\mathrm{R}\rangle$ and the eight states from R fermions acting on $|\mathrm{R}\rangle$. Together with states from the oppositely oriented string, they form four right- and left-moving fermions with mass-squared $\frac{1}{2}+v_a$ and four right- and left-moving fermions with mass-squared $\frac{1}{2}-v_a$.

In the limit $v_a \to 0$, the eight right-moving and eight left-moving fermions become degenerate and the eight right-movers are in fact the superpartners of the scalars from the NS sector.

4.2.4 $\mathbf{D5_{(ca)}}$-$\mathbf{D5_{(cb)}}$ strings

Here $\mathbf{C}^2_{(ca)}$ and $\mathbf{C}^2_{(cb)}$ share a common $\mathbf{C}_c$. Let the remaining direction be $\mathbf{C}_d$. The boundary conditions are now **MM** for Z^c, **MD** for Z^a, **DM** for Z^b and **DD** for Z^d. The modings are $\mathbf{Z}$ for Z^c and Z^d, $\mathbf{Z}+\frac{1}{2}-v_a$ for Z^a and $\mathbf{Z}+v_b-\frac{1}{2}$ for Z^b. The R fermions have the same modings and the NS fermions have the modings shifted by $\frac{1}{2}$. The modings are the same as for a $\overline{\mathrm{D1}}$-$\mathrm{D5}_{(ab)}$ system. The worldsheet bosons have momentum and position zero modes along $\mathbf{R}^{1,1}\times\mathbf{C}_c$. Hence all the states will be supported on the four dimensional space $\mathbf{R}^{1,1}\times\mathbf{C}_c$.

NS sector: The zero-point energy is $-\frac{1}{2}(|v_a|+|v_b|)$ and the lowest-lying excitation energies are $|v_a|$ and $|v_b|$. Thus, the lowest energy states are $\frac{1}{2}(\pm v_a \pm v_b)$. We assign $(-)^{F_{\mathrm{NS}}}=-1$ to the state $-\frac{1}{2}(v_a+v_b)$ and perform GSO projection with $\frac{1}{2}(1-(-)^{F_{\mathrm{NS}}})$ to get the states

$$-\frac{1}{2}(v_a+v_b)\ ,\ \frac{1}{2}(v_a+v_b)\ . \qquad (4.21)$$

After including states from the oppositely oriented string, these give two complex scalars σ^1, σ^2 which transform as $(\boldsymbol{n_{(ca)}},\overline{\boldsymbol{n}}_{\boldsymbol{(cb)}})$ with masses $m^2=\pm\frac{1}{2\alpha'}(v_a+v_b)$. In the limit $v_a, v_b \to 0$, the two scalars are massless and combine into a right-handed spinor in $\mathbf{C}^2_{(ab)}$ since $(-)^{F_{\mathrm{NS}}}=\Gamma_c(\mathbf{C}^2_{(ab)})$. These constitute the bosonic part of a $\mathcal{N}=2$ hypermultiplet in $\mathbf{R}^{1,1}\times\mathbf{C}_c$.

R sector: The worldsheet fermions along $\mathbf{R}^{1,1}\times\mathbf{C}^2_{(cd)}$ are integer moded, giving six zero modes and an eight dimensional ground state. The fermion parity operator is given by $(-)^{F_{\rm R}} = \Gamma_c(\mathbf{C}^2_{(cd)})\Gamma^{90} = \Gamma_c(\mathbf{R}^{1,1}\times\mathbf{C}_c)\Gamma_c(\mathbf{C}_d)$ where $\Gamma_c(\mathbf{R}^{1,1}\times\mathbf{C}_c) = \mathrm{i}\Gamma^{2c-1}\Gamma^{2c}\Gamma^9\Gamma^0$ and $\Gamma_c(\mathbf{C}_d) = -\mathrm{i}\Gamma^{2d-1}\Gamma^{2d}$. We use the GSO projector $\frac{1}{2}(1+(-)^{F_{\rm R}})$ to get a left-handed fermion λ and a right-handed fermion $\bar{\zeta}$ in $\mathbf{R}^{1,1}\times\mathbf{C}_c$ with $\Gamma_c(\mathbf{C}_d) = \pm 1$ respectively. These constitute the fermionic part of a $\mathcal{N}=2$ hypermultiplet.

4.3 Crossed instantons

We first consider the simpler configuration of *crossed instantons*: k $\overline{\rm D1}$-branes along $\mathbf{R}^{1,1}$, n D5-branes along $\mathbf{R}^{1,1}\times\mathbf{C}^2_{(12)}$ and n' D5-branes along $\mathbf{R}^{1,1}\times\mathbf{C}^2_{(34)}$. This setup preserves four supercharges organised into $\mathcal{N}=(0,4)$ supersymmetry on the two dimensional intersection $\mathbf{R}^{1,1}$. This setup has been studied in the context of $\mathbf{AdS}_3$ holography by [To, GMMS] and others. Another place where $\mathcal{N}=(0,4)$ supersymmetry appears is the ADHM sigma model [W3] which has a stringy realisation as a D1-D5-D9 brane system [D2]. More recently, the authors in [PSY] explore a class of $\mathcal{N}=(0,4)$ superconformal theories obtained by compactifying M5-branes on four-manifolds of the form $\mathbf{P}^1\times\mathcal{C}$ where $\mathcal{C}$ is a Riemann surface with punctures.

We are interested in studying the bound states of $\overline{\rm D1}$-branes with the crossed D5-branes above with the constant B-field background in (4.1). As we have seen in the previous section, there are generically tachyons in the spectrum and supersymmetry is broken. We are interested in the end point of tachyon condensation [Sen1, A, GS] and the all-important question: is supersymmetry restored at the end point of the condensation?

We shall find that for a particular locus in the space of B-fields, the supersymmetry breaking can be described by a Fayet-Iliopoulos term in the low-energy theory. For small values of B-field, we can then study the condensation of the tachyons in the low-energy effective theory. The relevant low-energy degrees of freedom are those of a supersymmetric U(k) gauge theory interacting with various matter multiplets supported on $\mathbf{R}^{1,1}$. In particular, we freeze the supersymmetric gauge degrees of freedom supported on the D5-branes to their classical vacuum expectation values.

Note: The above D5-brane system without the D1-branes has been studied in great detail by many authors, notably by [IKS]. There are chiral fermions (the field λ below) in the $1+1$ dimensional intersection arising from the strings stretching between the two stacks of D5-branes. The chiral fermions render the gauge theories on the intersection anomalous and the degrees of freedom in the bulk of the D5-branes are necessary to cancel these anomalies via the anomaly inflow mechanism. Since we have frozen these gauge degrees of freedom, these issues are not immediately relevant to our analysis below. In our case, the low-energy theory on the intersection has $\mathrm{U}(n)\times\mathrm{U}(n')$ as rigid symmetries.

The spacetime Lorentz group $\mathrm{SO}(1,9)$ is broken down to $\mathrm{SO}(1,1)\times\mathrm{SO}(4)\times\mathrm{SO}(4)'$. The low energy theory on $\mathbf{R}^{1,1}$ has the internal rigid symmetry group $\mathrm{SO}(4)\times\mathrm{SO}(4)'\times\mathrm{U}(n)\times\mathrm{U}(n')$. *It will be useful to write* $\mathrm{SO}(4)\times\mathrm{SO}(4)'=\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R\times\mathrm{SU}(2)'_L\times\mathrm{SU}(2)'_R$ *with the indices* $(\dot\alpha,\alpha,\dot\alpha',\alpha')$ *denoting the fundamental representations of the respective* $\mathrm{SU}(2)$*s*. The sixteen components of the left-handed spinor ϵ can be written in terms of spinors which have definite chirality under each of $\mathrm{SO}(1,1)$, $\mathrm{SO}(4)$ and $\mathrm{SO}(4)'$ as follows:

$$\epsilon=\eta_L^{\alpha\alpha'}\oplus\eta_R^{\alpha\dot\alpha'}\oplus\eta_R^{\dot\alpha\alpha'}\oplus\eta_L^{\dot\alpha\dot\alpha'}\ . \tag{4.22}$$

The subscripts indicate chirality in $1+1$ dimensions. We see that the product of the three chiralities is $+1$ which agrees with ϵ being left-handed in $9+1$ dimensions. There must also be a reality condition on each of the η's that arises from the Majorana condition on ϵ. Since the fundamental representation of $\mathrm{SU}(2)$ is pseudoreal, the η's are in a real representation of the corresponding $\mathrm{SU}(2)\times\mathrm{SU}(2)$. In other words, we have

$$\eta_R^{\alpha\alpha'}=-\varepsilon^{\alpha\beta}\varepsilon^{\alpha'\beta'}\,\overline{\eta_R^{\beta\beta'}}\quad\text{and so on.} \tag{4.23}$$

(To check this, write $\eta^{\alpha\alpha'}=\eta_m(\sigma^m)^{\alpha\alpha'}$ for some dummy real 4-vector η_m with $\sigma^m=(\sigma^1,\sigma^2,\sigma^3,\mathrm{i}\mathbb{1})$, $\varepsilon^{12}=\varepsilon^{1'2'}=+1$ and $\varepsilon^{\alpha\beta}\varepsilon_{\beta\gamma}=-\delta^\alpha{}_\gamma$, $\varepsilon^{\dot\alpha\dot\beta}\varepsilon_{\dot\beta\dot\gamma}=-\delta^{\dot\alpha}{}_{\dot\gamma}$.)

The constraints on ϵ due to the above configuration of branes are $\Gamma^{1234}\epsilon=-\epsilon$ and $\Gamma^{5678}\epsilon=-\epsilon$ which means ϵ has to be right-handed in both $\mathbf{C}^2_{(12)}$ and $\mathbf{C}^2_{(34)}$ and hence left-handed in $\mathbf{R}^{1,1}$. Thus, there are four real left-handed supersymmetry parameters $\eta_L^{\alpha\alpha'}$ corresponding to supersymmetry in the left-moving sector: *we have* $\mathcal{N}=(0,4)$ *supersymmetry in the* $1+1$ *dimensional intersection* $\mathbf{R}^{1,1}$. The R-symmetry of the

$\mathcal{N} = (0,4)$ supersymmetry algebra is $\mathrm{SU}(2)_R \times \mathrm{SU}(2)'_R$ and the parameters $\eta_L^{\alpha\alpha'}$ transform as a bispinor under this R-symmetry. We denote $\eta_L^{\alpha\alpha'}$ as $\eta^{\alpha\alpha'+}$ or equivalently $\eta_-^{\alpha\alpha'}$ in the sequel.

4.3.1 Low-energy spectrum and $\mathcal{N} = (0,2)$ decomposition

We write the low-energy action in $\mathcal{N} = (0,2)$ superspace by choosing a particular $\mathcal{N} = (0,2)$ subalgebra of the $\mathcal{N} = (0,4)$ supersymmetry algebra. See Chapter 3 for a description of $\mathcal{N} = (0,2)$ superspace.

We choose the $\mathcal{N} = (0,2)$ subalgebra generated by $\eta^{11'+} := \eta^+$ and $-\eta^{22'+} = \overline{\eta^{11'+}} = \bar{\eta}^+$ (this will be the subalgebra preserved by the spiked instanton configuration). The supercoordinates are θ^+ and $\bar{\theta}^+$. The R-symmetry $U(1)_\ell$ of the left-moving supersymmetry is generated by $F_\ell := F_L + F_R + F'_L + F'_R = F_{34} + F_{78}$ where $F_L = \frac{1}{2}(-F_{12} + F_{34})$, $F_R = \frac{1}{2}(F_{12} + F_{34})$, $F'_L = \frac{1}{2}(-F_{56} + F_{78})$ and $F'_R = \frac{1}{2}(F_{56} + F_{78})$. In our conventions, $\eta^+ = \eta^{11'+}$ has charges $F_{12} = F_{56} = \frac{1}{2}$ and $F_{34} = F_{78} = \frac{1}{2}$ giving $F_R = F'_R = +1/2$ and $F_L = F'_L = 0$ and hence a charge of $+1$ under $U(1)_\ell$. The $\mathcal{N} = (0,2)$ content of the various multiplets from Dp-Dp' strings are summarised in Table 4.1. The various fields are displayed with indices that indicate their $\mathrm{SO}(4) \times \mathrm{SO}(4)'$ representations.

Note: In order to avoid too many indices on the fields, the scalar component of a chiral multiplet Φ will be denoted by the same letter and the right-handed spin-$\frac{1}{2}$ component by ζ_Φ in the sequel. Also, the left-handed spin-$\frac{1}{2}$ component of a Fermi superfield Λ_a will be denoted as λ_a where a is an index that runs over all Fermi superfields in the theory. For example, the chiral multiplet $\widetilde{J}$ in Table 4.1b has components $\widetilde{\phi}^{2'\dagger}$ and $\widetilde{\zeta}_+^{1\dagger}$ which will be alternatively referred to as $\widetilde{J}$ and $\zeta_{\widetilde{J}}$ respectively. The left-moving fermionic component Fermi superfield $\Lambda_{\widetilde{J}}$ will be denoted as $\lambda_{\widetilde{J}}$.

4.3.2 Tachyons and Fayet-Iliopoulos terms

We are interested in generalising the above setup to one with a constant B-field of the form (4.1). We have seen in the analysis of the open string spectrum that there are

Table 4.1: Various $\mathcal{N} = (0,2)$ multiplets for the crossed instanton system.

(a) **D1-D1 strings**

$(0,4)$ multiplet	Fields	$(0,2)$ multiplets
Vector	v_{--} ; $\lambda_-^{\alpha\alpha'}$	Vector $V = (v_{--}\,;\,\lambda_-^{11'})$, Fermi $\Lambda_2 = (\lambda_-^{12'})$
Standard hyper	$X^{\alpha\dot{\alpha}}$; $\zeta_+^{\alpha'\dot{\alpha}}$	Chiral $B_1 = (X^{1\dot{1}}\,;\,\zeta_+^{2'\dot{1}})$, Chiral $B_2 = (X^{1\dot{2}}\,;\,\zeta_+^{2'\dot{2}})$
Twisted hyper	$X^{\alpha'\dot{\alpha}'}$; $\zeta_+^{\alpha\dot{\alpha}'}$	Chiral $B_3 = (X^{1'\dot{1}'}\,;\,\zeta_+^{2\dot{1}'})$, Chiral $B_4 = (X^{1'\dot{2}'}\,;\,\zeta_+^{2\dot{2}'})$
Fermi	$\lambda_-^{\dot{\alpha}\dot{\alpha}'}$	Fermi $\Lambda_3 = (\lambda_-^{\dot{1}\dot{1}'})$, Fermi $\Lambda_4 = (\lambda_-^{\dot{1}\dot{2}'})$

(b) **D1-D5$_{(12)}$ strings**
I, Λ_I transform in the $(\boldsymbol{k},\overline{\boldsymbol{n}})$ of U$(k)\times$U(n) while J, Λ_J transform in the $(\overline{\boldsymbol{k}},\boldsymbol{n})$.

$(0,4)$ multiplet	Fields	$(0,2)$ multiplets
Standard hyper	ϕ^α ; $\zeta_+^{\alpha'}$	Chiral $I = (\phi^1\,;\,\zeta_+^{2'})$, Chiral $J = (\phi^{2\dagger}\,;\,\zeta_+^{1'\dagger})$
Fermi	$\lambda_-^{\dot{\alpha}'}$	Fermi $\Lambda_I = (\lambda_-^{\dot{2}'})$, Fermi $\Lambda_J = (\lambda_-^{\dot{1}'\dagger})$

(c) **D1-D5$_{(34)}$ strings**
$\widetilde{I}$, $\widetilde{\Lambda}_I$ transform in the $(\boldsymbol{k},\overline{\boldsymbol{n}}')$ of U$(k)\times$U(n') while $\widetilde{J}$, $\widetilde{\Lambda}_J$ transforms in the $(\overline{\boldsymbol{k}},\boldsymbol{n}')$.

$(0,4)$ multiplet	Fields	$(0,2)$ multiplets
Twisted hyper	$\widetilde{\phi}^{\alpha'}$; $\widetilde{\zeta}_+^{\alpha}$	Chiral $\widetilde{I} = (\widetilde{\phi}^{1'}\,;\,\widetilde{\zeta}_+^{2})$, Chiral $\widetilde{J} = (\widetilde{\phi}^{2'\dagger}\,;\,\widetilde{\zeta}_+^{1\dagger})$
Fermi	$\widetilde{\lambda}_-^{\dot{\alpha}}$	Fermi $\widetilde{\Lambda}_I = (\lambda_-^{\dot{2}})$, Fermi $\widetilde{\Lambda}_J = (\lambda_-^{\dot{1}\,\dagger})$

(d) **D5$_{(12)}$-D5$_{(34)}$ strings**
Λ transforms in the $(\boldsymbol{n},\overline{\boldsymbol{n}}')$ of U$(n)\times$U(n').

$(0,4)$ multiplet	Fields	$(0,2)$ multiplets
Fermi	λ_-	Fermi $\Lambda = (\lambda_-)$

tachyons of mass-squared

$$-\frac{1}{2\alpha'}|v_1+v_2| \ , \quad -\frac{1}{2\alpha'}|v_3+v_4| \ , \tag{4.24}$$

in the $\overline{\mathrm{D1}}$-D5 and $\overline{\mathrm{D1}}$-D5$'$ spectra. In our conventions these correspond to the fields I, $\widetilde{I}$ for $(v_1+v_2),(v_3+v_4)>0$ and J, $\widetilde{J}$ for $(v_1+v_2),(v_3+v_4)<0$. The system is no longer supersymmetric about the original vacuum (where all the vacuum expectation values are set to zero) due to the presence of tachyons. Can this supersymmetry breaking be interpreted as an F-term or D-term breaking?

Let us study the simpler problem k $\overline{\mathrm{D1}}$-branes along $\mathbf{R}^{1,1}$ and n D5-branes along $\mathbf{R}^{1,1}\times\mathbf{C}^2_A$. The low-energy effective action for the D5-branes contains the following coupling to the (pullback of the) 2-form RR gauge field C_2:

$$\frac{e_5}{2}\int_{\mathbf{R}^{1,1}\times\mathbf{C}^2_A} C_2\wedge \mathrm{Tr}_n\,(\mathcal{F}\wedge\mathcal{F}) \ , \tag{4.25}$$

where $\mathcal{F}:=2\pi\alpha'(F-B)$ with F the U(n) field strength on the stack of D5-branes and B the (pullback of the) NSNS B-field. The charge quantum e_5 is given by

$$e_5=\frac{1}{g_s\sqrt{\alpha'}\,(2\pi\sqrt{\alpha'}\,)^5} \ . \tag{4.26}$$

We know that instantons of charge k in the U(n) gauge theory on the D5-branes induce D1-brane charge e_1k on the worldvolume. A constant B-field along $\mathbf{C}^2_A$ does a similar job and induces a D1-brane charge density

$$\mathcal{J}_1=\frac{ne_1}{8\pi^2}\,B\wedge B=\frac{ne_1}{8\pi^2}\,\frac{b_ab_b}{(2\pi\alpha')^2}\,\mathrm{dVol}(\mathbf{C}^2_A) \ , \tag{4.27}$$

The instability is qualitatively different for different ranges of the B-field values [SW3]. Let $\mathbf{C}^2_A$ have the standard orientation. When v_a and v_b have opposite signs, $\mathcal{J}_1$ is negative and corresponds to induced $\overline{\mathrm{D1}}$-branes. For $v_a+v_b\neq 0$, tachyon condensation corresponds to the external $\overline{\mathrm{D1}}$-brane dissolving into the D5-brane and forming a bound state with the induced $\overline{\mathrm{D1}}$-branes (the *Higgs branch* of the D1-D5 system). The point with $v_a=-v_b\neq 0$ corresponds to a anti self-dual B-field in which case the tachyon disappears and the $\overline{\mathrm{D1}}$-D5 system forms a bound state at threshold.

When v_a and v_b have the same sign, the charge density is negative and corresponds to induced D1-branes. The tachyon in the NS sector then corresponds to the standard $\overline{\text{D1}}$-D1 tachyon. The condensation of this tachyon results in the annihilation of part of the D1 charge density and results in an excited state of the D5-brane with excitation energy proportional to the tachyon mass $m^2 = -\frac{1}{2\alpha'}|v_a + v_b|$.

In either of these scenarios, one can describe these tachyon masses as arising from FI terms in the low energy effective action, at least for small values of $v_a + v_b$.

In the present situation of crossed instantons, Fayet-Iliopoulos terms arise as vacuum expectation values of auxiliary fields in the adjoint representation of $\mathrm{U}(k)$. We have one real auxiliary field D and one complex auxiliary field G_2 in the $\mathcal{N} = (0,4)$ vector multiplet, two complex auxiliary fields G_3 and G_4 from the $\mathcal{N} = (0,4)$ Fermi multiplets Λ_3 and Λ_4. The FI terms then correspond to the following J-terms in the $\mathcal{N} = (0,2)$ action:

$$\begin{aligned}\mathcal{S}_{\mathrm{FI}} &= -\frac{1}{\sqrt{2}}\mathrm{Im}\int \mathrm{d}^2x\, \mathrm{D}_+ \mathrm{Tr}\left\{-\sqrt{2}t\mathcal{F}_- + b_2\Lambda_2 + b_3\Lambda_3 + b_4\Lambda_4\right\} \;, \\ &= \int \mathrm{d}^2x\, \mathrm{Tr}\left\{\frac{\theta}{2\pi}v_{01} + rD + \mathrm{Re}(b_2G_2 + b_3G_3 + b_4G_4)\right\} \;. \end{aligned} \tag{4.28}$$

$t = \dfrac{\theta}{2\pi} + \mathrm{i}r$ is the complexified Fayet-Iliopoulos parameter where θ is the two dimensional θ-angle and r is the real FI parameter. The components of the field strength Fermi multiplet $\mathcal{F}_-$ are given by

$$\lambda_-^{11'} := -(\mathcal{F}_-)_| \;, \quad D + \mathrm{i}v_{01} := (\nabla_+\mathcal{F}_-)_| \;. \tag{4.29}$$

From the $\mathrm{SO}(4)\times\mathrm{SO}(4)'$ properties of the Fermi multiplets in table 4.1a, it is easy to see that all FI terms except r break the $\mathrm{SO}(2)^4$ rotational symmetry that is preserved by the B-field in (4.1). Hence, only a non-zero r could possibly account for the effect of such a

B-field. The terms in the action involving D are

$$\mathrm{Tr}_k\left(\frac{1}{2g^2}D^2-\sum_{a\in\underline{\mathbf{4}}}[B_a,B_a^\dagger]D-IDI^\dagger+J^\dagger DJ-\widetilde{I}D\widetilde{I}^\dagger+\widetilde{J}^\dagger D\widetilde{J}+rD\right)\ , \tag{4.30}$$

which gives the field equation

$$\frac{1}{g^2}D=\sum_{a\in\underline{\mathbf{4}}}[B_a,B_a^\dagger]+II^\dagger-J^\dagger J+\widetilde{I}\widetilde{I}^\dagger-\widetilde{J}^\dagger\widetilde{J}-r\cdot\mathbb{1}_k\ . \tag{4.31}$$

The contribution to the Lagrangian from the D-terms is $-\frac{1}{2g^2}\mathrm{Tr}_kD^2$ where D substituted with its field equation. There are various quartic interaction terms along with the following mass terms for I, $\widetilde{I}$, J and $\widetilde{J}$:

$$-\frac{g^2}{2}\mathrm{Tr}_k\left(-rII^\dagger-r\widetilde{I}\widetilde{I}^\dagger+rJ^\dagger J+r\widetilde{J}^\dagger\widetilde{J}\right)\ . \tag{4.32}$$

As we can see, the mass-squared of I and $\widetilde{I}$ are equal to $-\frac{g^2}{2}r$ and those of J and $\widetilde{J}$ are equal to $+\frac{g^2}{2}r$. Comparing this with (4.24), we see that the B-fields must be related to each other and to r as

$$v_1+v_2=v_3+v_4=\frac{g_s}{2\pi}r\ . \tag{4.33}$$

Here, we have used that the coupling constant g^2 is given in terms of α' and the closed string coupling g_s as $g^2=g_s/2\pi\alpha'$. Thus, for the low-energy effective action to be supersymmetric, the constant B-field must satisfy

$$v_1+v_2=v_3+v_4\ . \tag{4.34}$$

We restrict our attention to constant B-field backgrounds satisfying the above constraint. B-field backgrounds which do not satisfy the above constraint do not allow for a consistent low-energy limit where the gauge modes of the D5 branes are frozen. Since our requirement is to have a non-zero FI term r and the above constrained values do give such a term, we shall not pursue this more general case further. It would be interesting to understand how the decoupling of the D5-D5 modes actually takes place in the limit $v_1+v_2=v_3+v_4$.

4.3.3 Yukawa couplings

So far, we have determined the minimally coupled kinetic terms and the masses coming from D-term interactions in the low energy effective theory. The remaining terms describing the dynamics are the E-terms and J-terms for the various Fermi multiplets. A simple way to obtain these is to look at the Yukawa couplings in the theory. Recall from Chapter 3 that Yukawa terms for a Fermi superfield Ψ are of the general form

$$\mathcal{E}_\Psi = +\bar{\psi}^a_- \frac{\partial E_a}{\partial \phi_j} \zeta_{j+} \ , \quad \text{and} \quad \mathcal{J}^\Psi = -\frac{\partial J^a}{\partial \phi_j} \zeta_{j+} \psi_{a-} \ . \tag{4.35}$$

We obtain these terms in the low-energy effective action by computing 3-point string amplitudes on the disk. The idea is to look for non-zero amplitudes that involve only fields in the chiral multiplets but not their complex conjugates i.e. the fields in the chiral multiplets displayed in Table 4.1.

A general open string vertex operator in a constant B-field background has the form

$$V_\lambda(k,z) = \omega(\lambda)\, c(z)\, \mathcal{B}(z)\, \mathrm{e}^{\lambda\cdot H(z)}\, \mathrm{e}^{2\mathrm{i}k\cdot X(z)} c_\lambda \ . \tag{4.36}$$

Here, λ is a weight in the covariant lattice $D_2 \oplus D_2 \oplus \Gamma_{1,1}$ corresponding to the spacetime symmetry $\mathrm{SO}(1,1)\times\mathrm{SO}(4)\times\mathrm{SO}(4')$ and c_λ is the associated cocycle operator. $\mathcal{B}(z)$ is the appropriate product of boundary condition changing operators for the worldsheet bosons. The weights for the various fields and the boundary condition changing operators for the worldsheet bosons have been derived in Chapter 2 and summarised in Tables 4.2 and 4.3.

The rest of the notation is quite standard: $c(z)$ is the coordinate ghost, $H(z)$ is a 6-dimensional vector containing the five bosons that bosonise the ten worldsheet fermions and the sixth boson being the one that bosonises the superconformal ghosts, $k = (k^0, k^9)$ is the $1+1$ dimensional momentum and $X = (X^0, X^9)$ are the worldsheet bosons corresponding to the $1+1$ dimensional intersection. $\omega(\lambda)$ is an *a priori* undetermined c-number phase.

The general structure of a 3-pt function with open string vertex operators in the

Table 4.2: Covariant weights for the vertex operators arising from $\overline{\text{D1}}$-$\overline{\text{D1}}$ strings. In our conventions, a right-handed spinor ψ^α of SO(4) is specified by the weights $\psi^{\alpha=1} = (+,+)$, $\psi^{\alpha=2} = (-,-)$ and a left-handed spinor $\psi^{\dot\alpha}$ by $\psi^{\dot\alpha=1} = (+,-)$, $\psi^{\dot\alpha=2} = (-,+)$.

State	Field	$\mathrm{U}(1)_\ell$	$D_2 \oplus D_2 \oplus \Gamma_{1,1}$ weight
$\overline{\text{D1}}$-$\overline{\text{D1}}$ vector	$v_{\pm\pm}$	0	$0,\ 0,\ 0,\ 0,\ \mp1; -1$
$\overline{\text{D1}}$-$\overline{\text{D1}}$ scalars	$X^{1\dot1}$, B_1	0	$1,\ 0,\ 0,\ 0,\ 0; -1$
	$X^{1\dot2}$, B_2	1	$0,\ 1,\ 0,\ 0,\ 0; -1$
	$X^{1'\dot1'}$, B_3	0	$0,\ 0,\ 1,\ 0,\ 0; -1$
	$X^{1'\dot2'}$, B_4	1	$0,\ 0,\ 0,\ 1,\ 0; -1$
$\overline{\text{D1}}$-$\overline{\text{D1}}$ gauginos	$\lambda_-^{11'}$, f	1	$+,\ +,\ +,\ +,\ +; -$
	$\lambda_-^{12'}$, λ_2	0	$+,\ +,\ -,\ -,\ +; -$
	$\lambda_-^{\dot1 1'}$, λ_3	-1	$+,\ -,\ +,\ -,\ +; -$
	$\lambda_-^{\dot1 2'}$, λ_4	0	$+,\ -,\ -,\ +,\ +; -$
	$\zeta_+^{\dot1 2'}$, ζ_1	-1	$+,\ -,\ -,\ -,\ -; -$
	$\zeta_+^{\dot2 2'}$, ζ_2	0	$-,\ +,\ -,\ -,\ -; -$
	$\zeta_+^{2\dot1'}$, ζ_3	-1	$-,\ -,\ +,\ -,\ -; -$
	$\zeta_+^{2\dot2'}$, ζ_4	0	$-,\ -,\ -,\ +,\ -; -$

Table 4.3: Covariant weights for $\overline{\text{D1}}$-$\text{D5}_{(12)}$, $\overline{\text{D1}}$-$\text{D5}_{(34)}$ and $\text{D5}_{(12)}$-$\text{D5}_{(34)}$ strings.

State	Field	$\mathrm{U}(1)_\ell$	$D_2 \oplus D_2 \oplus \Gamma_{1,1}$ weight
$\overline{\text{D1}}$-$\text{D5}_{(12)}$ bosons	ϕ^1, I	$\frac{1}{2} - v_2$	$-v_1 + \frac{1}{2},\ -v_2 + \frac{1}{2},\ 0,\ 0,\ 0; -1$
	$\phi^{2\dagger}$, J	$\frac{1}{2} + v_2$	$+v_1 + \frac{1}{2},\ +v_2 + \frac{1}{2},\ 0,\ 0,\ 0; -1$
$\overline{\text{D1}}$-$\text{D5}_{(12)}$ fermions	$\zeta_+^{1'\dagger}$, ζ_J	$-\frac{1}{2} + v_2$	$+v_1,\ +v_2,\ -,\ -,\ -; -$
	$\zeta_+^{2'}$, ζ_I	$-\frac{1}{2} - v_2$	$-v_1,\ -v_2,\ -,\ -,\ -; -$
	$\lambda_-^{\dot1'\dagger}$, λ_J	$\frac{1}{2} + v_2$	$+v_1,\ +v_2,\ -,\ +,\ +; -$
	$\lambda_-^{\dot2'}$, λ_I	$\frac{1}{2} - v_2$	$-v_1,\ -v_2,\ -,\ +,\ +; -$
$\overline{\text{D1}}$-$\text{D5}_{(34)}$ bosons	$\widetilde\phi^{1'}$, $\widetilde I$	$\frac{1}{2} - v_4$	$0,\ 0,\ -v_3 + \frac{1}{2},\ -v_4 + \frac{1}{2},\ 0; -1$
	$\widetilde\phi^{2'\dagger}$, $\widetilde J$	$\frac{1}{2} + v_4$	$0,\ 0,\ +v_3 + \frac{1}{2},\ +v_4 + \frac{1}{2},\ 0; -1$
$\overline{\text{D1}}$-$\text{D5}_{(34)}$ fermions	$\widetilde\zeta_+^{1\dagger}$, $\widetilde\zeta_J$	$-\frac{1}{2} + v_4$	$-,\ -,\ +v_3,\ +v_4,\ -; -$
	$\widetilde\zeta_+^{2}$, $\widetilde\zeta_I$	$-\frac{1}{2} - v_4$	$-,\ -,\ -v_3,\ -v_4,\ -; -$
	$\widetilde\lambda_-^{\dot1\dagger}$, $\widetilde\lambda_J$	$\frac{1}{2} + v_4$	$-,\ +,\ +v_3,\ +v_4,\ +; -$
	$\widetilde\lambda_-^{\dot2}$, $\widetilde\lambda_I$	$\frac{1}{2} - v_4$	$-,\ +,\ -v_3,\ -v_4,\ +; -$
$\text{D5}_{(12)}$-$\text{D5}_{(34)}$ fermions	λ_-, λ	$v_2 - v_4$	$+v_1,\ +v_2,\ -v_3,\ -v_4,\ +; -$

canonical ghost picture is given by

$$
\begin{aligned}
\langle V_{\lambda_1}(k_1,x_1)V_{\lambda_2}(k_2,x_2)V_{\lambda_3}(k_3,x_3)\rangle &= \omega(\lambda_1)\omega(\lambda_2)\omega(\lambda_3)\,\langle \mathcal{B}_1(x_1)\mathcal{B}_2(x_2)\mathcal{B}_3(x_3)\rangle\times \\
&\quad\times \langle \mathrm{e}^{\lambda_1\cdot H(x_1)}c_{\lambda_1}\,\mathrm{e}^{\lambda_2\cdot H(x_2)}c_{\lambda_2}\,\mathrm{e}^{\lambda_3\cdot H(x_3)}c_{\lambda_3}\rangle \\
&\quad\times \langle c(x_1)c(x_2)c(x_3)\rangle\,\langle \mathrm{e}^{\mathrm{i}k_1\cdot X}(x_1)\mathrm{e}^{\mathrm{i}k_2\cdot X}(x_2)\mathrm{e}^{\mathrm{i}k_3\cdot X}(x_3)\rangle \\
&= \omega(\lambda_1)\omega(\lambda_2)\omega(\lambda_3)\times\langle \mathcal{B}_1(x_1)\mathcal{B}_2(x_2)\mathcal{B}_3(x_3)\rangle\times \\
&\quad\times \prod_{i<j}\mathrm{e}^{\mathrm{i}\pi\lambda_i\cdot M\cdot\lambda_j}(x_i-x_j)^{1+\lambda_i\cdot\lambda_j+2\alpha' k_i\cdot k_j}\,. \qquad (4.37)
\end{aligned}
$$

A few comments are in order:

1. The phase prefactor $\prod_{i<j}\mathrm{e}^{\mathrm{i}\pi\lambda_i\cdot M\cdot\lambda_j}$ in the last expression is due to the cocycle operators c_{λ_i} commuting across the vertex operators $\mathrm{e}^{\lambda_j\cdot H}$. Here, M is a 6×6 matrix whose form is given in Chapter 2. These phases are crucial for obtaining the correct low-energy Yukawa couplings.

2. For the case of crossed instantons, all the E-terms and J-terms turn out to be quadratic in the superfields. Looking at (4.35), it is easy to see that there will be two different amplitudes that arise from the same E- or J-term. We get relations between the phases $\omega(\lambda)$ by equating the coefficients of these two amplitudes.

3. The correlators are non-zero only when the spacetime momenta add up to zero, the $D_2\oplus D_2\oplus\Gamma_{1,1}$ weights add up to $(0,0,0,0,0;-2)$ with the first five entries signifying $\mathrm{SO}(4)\times\mathrm{SO}(4)'\times\mathrm{SO}(1,1)$ invariance and the -2 indicating that the superconformal anomaly on the disk is soaked up.

4. When the correlators are non-zero, it can be shown that the different contributions to the exponent of x_i-x_j coming from the coordinate ghosts, the BCC operators for the worldsheet bosons, the vertex operators for the worldsheet fermions and the vertex operators for the $\mathbf{R}^{1,1}$ directions all add up to zero. This shows that the correlator is independent of the points of insertion of the vertex operators as it should be due to $\mathrm{SL}(2,\mathbf{R})$ invariance.

After choosing suitable values for the phases [NP], the E-term and J-term Yukawa couplings for the various Fermi multiplets are as follows:

$$
\begin{aligned}
&J^{\Lambda_2} = [B_3, B_4] + \widetilde{I}\widetilde{J}\ , \quad E_{\Lambda_2} = [B_1, B_2] + IJ\ ,\\
&J^{\Lambda_3} = [B_2, B_4]\ , \quad E_{\Lambda_3} = -[B_1, B_3]\ , \quad J^{\Lambda_4} = [B_2, B_3]\ , \quad E_{\Lambda_4} = [B_1, B_4]\ ,\\
&J^{\Lambda_J} = B_3 I\ , \quad E_{\Lambda_J} = JB_4\ , \quad J^{\Lambda_I} = -JB_3\ , \quad E_{\Lambda_I} = B_4 I\ ,\\
&J^{\widetilde{\Lambda}_J} = -B_1\widetilde{I}\ , \quad E_{\widetilde{\Lambda}_J} = -\widetilde{J}B_2\ , \quad J^{\widetilde{\Lambda}_I} = \widetilde{J}B_1\ , \quad E_{\widetilde{\Lambda}_I} = -B_2\widetilde{I}\ ,\\
&J^{\Lambda} = \widetilde{J}I\ , \quad E_{\Lambda} = -J\widetilde{I}\ .
\end{aligned}
\tag{4.38}
$$

__The identity $\mathrm{Tr}_k\ \boldsymbol{J}\cdot\boldsymbol{E} = 0$:__

We have

$$
\begin{aligned}
\mathrm{Tr}_k\ \Big\{&([B_3, B_4] + \widetilde{I}\widetilde{J})([B_1, B_2] + IJ) - [B_2, B_4][B_1, B_3] + [B_2, B_3][B_1, B_4] +\\
&+ B_3 IJB_4 - B_4 IJB_3 + B_1\widetilde{I}\widetilde{J}B_2 - B_2\widetilde{I}\widetilde{J}B_1 - IJ\widetilde{I}\widetilde{J}\Big\} = 0\ .
\end{aligned}
\tag{4.39}
$$

Thus, $\mathrm{Tr}_k\ J\cdot E = 0$ is indeed satisfied and the low-energy effective action is indeed $\mathcal{N} = (0, 2)$ supersymmetric. The action is also covariant with respect to the diagonal SU(2) subgroup of $\mathrm{SU}(2)_R \times \mathrm{SU}(2)'_R$ R-symmetry due to the presence of the D5-D5′ fermis which mix standard and twisted hypermultiplets. Thus, it is $\mathcal{N} = (0, 4)$ supersymmetric as well. This is the same result that is obtained in [To] for the case of zero B-field.

4.3.4 The crossed instanton moduli space

The bosonic potential energy U is

$$
U = \frac{g^2}{2}\mathrm{Tr}\, D^2 + \sum_a |E_a|^2 + \sum_a |J^a|^2\ ,
\tag{4.40}
$$

with the auxiliary field D substituted with its field equation in (4.31). The minima of the potential can be obtained by solving the equations $D = 0$, $E_a = 0$ and $J^a = 0$. We relabel $I, J \to I_{12}, J_{12}$ and $\widetilde{I}, \widetilde{J} \to I_{34}, J_{34}$ in anticipation of the spiked instanton case. The vacuum moduli space is then defined by the following equations upto a $\mathrm{U}(k)$ gauge transformation:

$$\underline{D\text{-term:}} \tag{4.41}$$

$$\mu^{\mathbf{R}} - r \cdot \mathbb{1}_k = \sum_{a=1}^{4} [B_a, B_a^\dagger] + I_{12} I_{12}^\dagger - J_{12}^\dagger J_{12} + I_{34} I_{34}^\dagger - J_{34}^\dagger J_{34} - r \cdot \mathbb{1}_k = 0 \ .$$

$$\underline{J\text{-terms:}} \tag{4.42}$$

$$\begin{aligned}
&\mu_{34}^{\mathbf{C}} = [B_3, B_4] + I_{34} J_{34} = 0 \ , \quad \mu_{24}^{\mathbf{C}} = [B_2, B_4] = 0 \ , \quad \mu_{23}^{\mathbf{C}} = [B_2, B_3] = 0 \ , \\
&\sigma_{3,12}^{\mathbf{C}} = B_3 I_{12} = 0 \ , \quad \tilde{\sigma}_{3,12}^{\mathbf{C}} = -J_{12} B_3 = 0 \ , \quad \sigma_{1,34}^{\mathbf{C}} = -B_1 I_{34} = 0 \ , \\
&\tilde{\sigma}_{1,34}^{\mathbf{C}} = J_{34} B_1 = 0 \ , \quad \Upsilon_{12}^{\mathbf{C}} = J_{34} I_{12} = 0 \ .
\end{aligned}$$

$$\underline{E\text{-terms:}} \tag{4.43}$$

$$\begin{aligned}
&\mu_{12}^{\mathbf{C}} = [B_1, B_2] + I_{12} J_{12} = 0 \ , \quad \mu_{13}^{\mathbf{C}} = -[B_1, B_3] = 0 \ , \quad \mu_{14}^{\mathbf{C}} = [B_1, B_4] = 0 \ , \\
&\sigma_{4,12}^{\mathbf{C}} = B_4 I_{12} = 0 \ , \quad \tilde{\sigma}_{4,12}^{\mathbf{C}} = J_{12} B_4 = 0 \ , \quad \sigma_{2,34}^{\mathbf{C}} = -B_2 I_{34} = 0 \ , \\
&\tilde{\sigma}_{2,34}^{\mathbf{C}} = -J_{34} B_2 = 0 \ , \quad \Upsilon_{34}^{\mathbf{C}} = -J_{12} I_{34} = 0 \ .
\end{aligned}$$

Symmetries

Note that the above equations are invariant under $\mathrm{U}(k) \times \mathrm{U}(n) \times \mathrm{U}(n')$ transformations. The crossed instanton moduli space is then defined by the solutions of the above equations modulo $\mathrm{U}(k)$ gauge transformations. The group $\mathrm{P}\left(\mathrm{U}(n) \times \mathrm{U}(n')\right) \cong \frac{U(n) \times \mathrm{U}(n')}{\mathrm{U}(1)_c}$, where $\mathrm{U}(1)_c$ is the common centre of $\mathrm{U}(n) \times \mathrm{U}(n')$, remains a global symmetry on the moduli space. These are the *framing rotations* described in [N4].

There are additional symmetries from the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{SU}(2)'_L \times \mathrm{SU}(2)'_R$ arising from rotations of the transverse $\mathbf{R}^8$. To see how many of these symmetries are preserved by the vacuum moduli space, we first form real combinations of the holomorphic equations above:

$$\begin{aligned} s_A &:= \mu_A^{\mathbf{C}} + \varepsilon_{A\bar{A}} \left(\mu_{\bar{A}}^{\mathbf{C}}\right)^\dagger = 0 \ , \quad \text{for} \quad A \in \underline{\mathbf{6}} \ , \\ \sigma_{\bar{a}A} &:= \sigma_{\bar{a}A}^{\mathbf{C}} + \varepsilon_{\bar{a}\bar{b}A} \left(\tilde{\sigma}_{\bar{b}A}^{\mathbf{C}}\right)^\dagger = 0 \ , \quad \text{for} \quad A \in \underline{\mathbf{6}} \ , \quad \bar{a} \in \bar{A} \ , \\ \Upsilon_A &:= \Upsilon_A^{\mathbf{C}} - \varepsilon_{A\bar{A}} \left(\Upsilon_{\bar{A}}^{\mathbf{C}}\right)^\dagger = 0 \quad \text{for} \quad A \in \underline{\mathbf{6}} \ . \end{aligned} \tag{4.44}$$

Using the $SO(4) \times SO(4)'$ transformation properties of the fields in Table 4.1 it is easy to see that the equations with $r = 0$ preserve a diagonal subgroup $SU(2)_\Delta$ of the R-symmetry $SU(2)_R \times SU(2)'_R$. The equations $\mu^{\mathbf{R}}$, s_{12} and s_{34} form a triplet and the other real equations are invariant under $SU(2)_\Delta$.

For $r \neq 0$, the subgroup $SU(2)_\Delta$ is broken down to its maximal torus $U(1)_\Delta$ which is the R-symmetry $U(1)_\ell$ of the $\mathcal{N} = (0,2)$ subalgebra that was chosen above. The factors $SU(2)_L \times SU(2)'_L$ survive as spectator symmetries. Hence, the total global symmetry on the crossed instanton moduli space is

$$P\left(U(n) \times U(n')\right) \times SU(2)_L \times SU(2)'_L \times U(1)_\Delta \ . \tag{4.45}$$

Note: The vacuum moduli space for $r = 0$ splits up into many distinct branches corresponding to the Coulomb branch, the two Higgs branches (with the D1's binding to either of the D5-branes) and mixed branches [To]. Once a non-zero r is introduced, the D1-branes bind necessarily to some stack of D5-branes and the moduli space becomes connected. Turning on r also has the effect of reducing the global symmetries as we saw above. It would be interesting to repeat the R-charge analysis of [To] in this case.

4.4 Spiked instantons

Consider the crossed instanton setup of D1-$D5_{(12)}$-$D5_{(34)}$ branes. Let us choose the B-field such that $v_1 v_2 \geq 0$ and $v_3 v_4 \geq 0$. This ensures that the tachyons are of $\overline{D1}$-D1 type. In this region of the space of B-fields, the tachyon mass can never be zero unless the v's are zero.

Let us introduce a stack of $D5_{(23)}$-branes to the mix. In order to realise a symmetric situation where the instability here is also of $\overline{D1}$-D1 type, we need $v_2 v_3 \geq 0$. This implies that $v_1 v_3 \geq 0$ and $v_2 v_4 \geq 0$. Suppose we next add the two stacks of five branes along $\mathbf{R}^{1,1} \times \mathbf{C}^2_{(13)}$ and $\mathbf{R}^{1,1} \times \mathbf{C}^2_{(13)}$. The constraints $v_1 v_3 \geq 0$ and $v_2 v_4 \geq 0$ and the requirement that the tachyons should be $\overline{D1}$-D1 tachyons automatically force these stacks to be made of D5-branes. We thus have the following six stacks of D5-branes:

$$
D5_{(12)} \ , \ D5_{(34)} \ , \ D5_{(23)} \ , \ D5_{(14)} \ , \ D5_{(13)} \ , \ D5_{(24)} \ . \tag{4.46}
$$

This is the same configuration of six stacks of D5-branes which preserves two supercharges when the B-field is dialled to zero. One may again enquire as to whether an FI term in the low-energy effective action can accommodate the effect of the constant B-field of the form (4.1). The m^2 of the tachyons for the various $\overline{D1}$-D5 strings can be read off from the derivation of the open string spectrum in Section 4.2:

$$
\begin{aligned}
&-\frac{1}{2\alpha'}|v_1+v_2| \ , \quad -\frac{1}{2\alpha'}|v_3+v_4| \ , \quad -\frac{1}{2\alpha'}|v_2+v_3| \ , \\
&-\frac{1}{2\alpha'}|v_1+v_4| \ , \quad -\frac{1}{2\alpha'}|v_1+v_3| \ , \quad -\frac{1}{2\alpha'}|v_2+v_4| \ .
\end{aligned} \tag{4.47}
$$

Repeating the analysis in the crossed case, we see that the field equation for the auxiliary field D becomes

$$
D = \sum_{a\in\underline{\mathbf{4}}}[B_a, B_a^\dagger] + \sum_{A\in\underline{\mathbf{6}}}(I_A I_A^\dagger - J_A^\dagger J_A) - r\cdot \mathbb{1}_k \ . \tag{4.48}
$$

giving rise to the same mass-squared $-|r|$ to all the tachyons. Thus, the B-field values must satisfy

$$
\boxed{v_1 = v_2 = v_3 = v_4} \ , \tag{4.49}
$$

in order to be accounted for by the real FI parameter in the low-energy theory.

The presence of the extra four stacks of D5-branes gives rise to additional terms in the E-terms and J-terms for the Fermi multiplets Λ_3 and Λ_4. There are also additional Fermi multiplets from the open strings stretching between $\overline{D1}$-branes and these stacks of D5-branes. Repeating the disk amplitude calculation as above, one get the following equations:

1. The real moment map:

$$\mu_{\mathbf{R}} - r \cdot \mathbb{1}_k := \sum_{a\in\underline{\mathbf{4}}} [B_a, B_a^\dagger] + \sum_{A\in\underline{\mathbf{6}}} (I_A I_A^\dagger - J_A^\dagger J_A) - r \cdot \mathbb{1}_k = 0 \ . \tag{4.50}$$

2. For $A = (ab) \in \underline{\mathbf{6}}$ with $a < b$,

$$\mu_A^{\mathbf{C}} := [B_a, B_b] + I_A J_A = 0 \ . \tag{4.51}$$

3. For $A \in \underline{\mathbf{6}}$, $\bar{A} = \underline{\mathbf{4}} \smallsetminus A$ and $\bar{a} \in \bar{A}$,

$$\sigma_{\bar{a}A}^{\mathbf{C}} := B_{\bar{a}} I_A = 0 \ , \quad \tilde{\sigma}_{\bar{a}A}^{\mathbf{C}} := J_A B_{\bar{a}} = 0 \ . \tag{4.52}$$

4. For $A \in \underline{\mathbf{6}}$, $\bar{A} = \underline{\mathbf{4}} \smallsetminus A$,

$$\Upsilon_A^{\mathbf{C}} := J_{\bar{A}} I_A = 0 \ . \tag{4.53}$$

Symmetries

The symmetries of the above equations can be obtained in a similar way to the crossed instanton case. The total global symmetry is given by

$$\mathrm{P}\left(\bigtimes_{A\in\underline{\mathbf{6}}} \mathrm{U}(n_A) \right) \times \mathrm{U}(1)^3 \ , \tag{4.54}$$

where $\mathrm{U}(1)^3$ is a maximal torus of SU(4), the isometry group of the transverse $\mathbf{C}^4$ which preserves some fraction of supersymmetry.

4.4.1 Folded branes

The above equations arise from considering $\overline{\mathrm{D1}}$-$\overline{\mathrm{D1}}$ strings, D1-D5$_A$ strings and D5$_A$-D5$_{\bar{A}}$ strings. There are also additional equations that result from the interaction of D1-branes with states from open strings stretching between D5$_A$ and D5$_B$ with $A = (ac)$ and $B = (bc)$

i.e. two stacks of D5-branes that have a line $\mathbf{C}_c$ in common. This is the setup of *folded branes*. Once we throw in D1-branes, the classical moduli space of vacua is called the moduli space of *folded instantons*.

The open string spectrum for this case was analysed in Section 2 and there we saw that there were tachyons in the NS sector with $m^2 = -\frac{1}{2}|v_a + v_b|$. Thus, for the configuration of branes in (4.46) it is easy to see that the spectrum of tachyon masses is precisely the same as in (4.47). With the constraint in (4.49), all tachyons have the same m^2 which is equal to $-\frac{1}{\alpha'}|v_1|$.

All the states arising from such strings are supported over the four dimensional subspace $\mathbf{R}^{1,1} \times \mathbf{C}_c$ with a constant B-field $\tan \pi v_c$ along $\mathbf{C}_c$ which makes the space non-commutative. It has been conjectured in [N3, N4] that the interaction of these states with the states supported on $\mathbf{R}^{1,1}$ gives rise to an additional (infinite) set of equations of the form

$$\Upsilon_{A,B,j} = J_A(B_c)^n I_B = 0 \ , \quad \text{for} \quad n = 1, 2, \ldots \tag{4.55}$$

In this section, we derive the above equations by considering $n+3$-point amplitudes of the Yukawa type $\zeta_+\lambda_- f(\phi)$ where f is a polynomial in the scalars. Below, we consider the case $c = 2$, $A = (12)$ and $B = (23)$. The other equations follow from similar considerations.

We use the following setup of D-branes: k $\overline{\text{D1}}$-branes along $\mathbf{R}^{1,1}$, n D5-branes along $\mathbf{R}^{1,1} \times \mathbf{C}^2_{(12)}$ and n' D5-branes along $\mathbf{R}^{1,1} \times \mathbf{C}^2_{(23)}$. The spacetime isometry SO$(1,9)$ is now broken down to SO$(1,1) \times$ SO$(2)^4$. The constraints on ϵ are $\Gamma^{1234}\epsilon = \epsilon$ and $\Gamma^{3456}\epsilon = \epsilon$ which preserve the following spinors:

$$\begin{aligned} \text{Right-handed in } \mathbf{R}^{1,1} : \quad & \eta^- \leftrightarrow |+,-,+,+,-\rangle \ , \quad \bar{\eta}^- \leftrightarrow |-,+,-,-,-\rangle \ , \\ \text{Left-handed in } \mathbf{R}^{1,1} : \quad & \eta^+ \leftrightarrow |+,+,+,+,+\rangle \ , \quad \bar{\eta}^+ \leftrightarrow |-,-,-,-,+\rangle \ . \end{aligned} \tag{4.56}$$

The last entry in the above spinors corresponds to their chirality in the $1+1$ dimensional intersection. Left(right)-handed spinors generate left(right)-moving supersymmetry. Thus, we have $\mathcal{N} = (2,2)$ supersymmetry on $\mathbf{R}^{1,1}$. The R-symmetry group $U(1)_\ell \times U(1)_r$ is an appropriate subgroup of the internal symmetry $U(1)^4$. We choose the generators $F_{\ell,r}$ to be

$$F_\ell = F_{34} + F_{78} \ , \quad F_r = F_{34} - F_{78} \ , \tag{4.57}$$

where F_{34} and F_{78} are the generators of $U(1)_{34}$ and $U(1)_{78}$ respectively. The choice of left-moving R-charge F_ℓ matches that of the $(0,2)$ subalgebra in the crossed instanton case. In the spinor representation, we have

$$\begin{aligned} F_{34} &= -\frac{\mathrm{i}}{2}\Gamma^{34} = \mathbb{1} \otimes \frac{\sigma_3}{2} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \ , \\ F_{78} &= -\frac{\mathrm{i}}{2}\Gamma^{78} = \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \frac{\sigma_3}{2} \otimes \mathbb{1} \ . \end{aligned} \tag{4.58}$$

This gives $F_r[\theta^-] = +1$, $F_r[\bar\theta^-] = -1$, $F_\ell[\theta^+] = +1$, $F_\ell[\bar\theta^+] = -1$ and $F_\ell[\text{right-movers}] = F_r[\text{left-movers}] = 0$.

Low energy spectrum

The new types of strings are $\overline{\text{D1}}$-$\text{D5}_{(23)}$ strings and $\text{D5}_{(12)}$-$\text{D5}_{(23)}$ strings.

$\overline{\mathbf{D1}}$-$\mathbf{D5_{(23)}}$ strings: These give rise to a $\mathcal{N} = (4,4)$ hypermultiplet transforming in $(\boldsymbol{k}, \overline{\boldsymbol{n}}')$ of $U(k) \times \mathrm{U}(n')$ when the B-field is zero. The two complex scalars $(J'^\dagger, \widetilde{I}')$ of the hypermultiplet transform as a right-handed spinor in $\mathbf{C}^2_{(23)}$ i.e. $J'^\dagger$ and I' satisfy $F_{34} = F_{56} = \mp\frac{1}{2}$ respectively. This gives $F_\ell = \mp\frac{1}{2}$ respectively. When the constant B-field is turned on, the scalars I', J' obtain masses

$$\alpha' m^2 = \mp\frac{1}{2}(v_2 + v_3) \ . \tag{4.59}$$

The right-handed fermions $(\zeta^\dagger_{J'}, \zeta_{I'})$ transform as a right-handed spinor in $\mathbf{C}^2_{(14)}$ with $F_{12} = F_{78} = \pm\frac{1}{2}$ and R-charge $F_\ell = \pm\frac{1}{2}$. The left-handed fermions $(\lambda^\dagger_{J'}, \lambda_{I'})$ transform as a left-handed spinor in $\mathbf{C}^2_{(14)}$ with $F_{12} = -F_{78} = \mp\frac{1}{2}$ and $F_\ell = \mp\frac{1}{2}$ for $\widetilde{\lambda}^\dagger_J$ and $\widetilde{\lambda}_I$ respectively.

Thus, we have two chiral multiplets (I', J') and two Fermi multiplets $(\Lambda_{I'}, \Lambda_{J'})$ which transform in the $(\boldsymbol{k}, \overline{\boldsymbol{n}}')$ and $(\overline{\boldsymbol{k}}, \boldsymbol{n}')$ respectively with all multiplets carrying a left-moving R-charge of $+\frac{1}{2}$.

$\mathbf{D5_{(12)}}$-$\mathbf{D5_{(23)}}$ strings: These strings give a $\mathcal{N}=2$ hypermultiplet that is supported along the four dimensional intersection $\mathbf{R}^{1,1}\times\mathbf{C}_{a=2}$. We have two scalars (σ^1,σ^2) with masses

$$\alpha' m^2 = \pm\frac{1}{2}(v_1+v_3)\ . \tag{4.60}$$

For $v_1=v_3=0$, these transform as a right-handed spinor in $\mathbf{C}^2_{(13)}$. There is a left-handed fermion ξ^α which also satisfies $-\mathrm{i}\Gamma^{78}=1$ and a right-handed fermion $\tilde{\xi}^{\dot\alpha}$ which satisfies $-\mathrm{i}\Gamma^{78}=-1$ where α and $\dot\alpha$ are spinor indices in $\mathbf{R}^{1,1}\times\mathbf{C}_{a=2}$. These have zero mass. We give alternate names $S=\sigma^1$, $T=\sigma^{2\dagger}$, $\zeta_S=\tilde{\xi}^{\dot 1}$, $\zeta_T=\xi^{2\dagger}$, $\lambda_S=\xi^1$ and $\lambda_T=\tilde{\xi}^{\dot 2\dagger}$. The fields S, ζ_S and λ_S are in the $\boldsymbol{n}\times\boldsymbol{\bar n'}$ of $\mathrm{U}(n)\times\mathrm{U}(n')$ whereas T, ζ_T and λ_T are in the $\boldsymbol{\bar n}\times\boldsymbol{n'}$.

Next, we dimensionally reduce the above states to $\mathbf{R}^{1,1}$. The constant B-field makes the worldsheet bosonic zero modes z_2, $\bar z_2$ non-commutative:

$$[z_2,\bar z_2]=\vartheta_2=\pi\alpha'\sin 2\pi v_2\ ,\quad [z_2,p_2]=[\bar z_2,\bar p_2]=\mathrm{i}\cos^2\pi v_2\ ,\quad [p_2,\bar p_2]=0\ . \tag{4.61}$$

Let the normalised modes be

$$\hat z_2=\frac{z_2}{\sqrt{\vartheta_2}}\ ,\quad \hat z_2^\dagger=\frac{\bar z_2}{\sqrt{\vartheta_2}}\ ,\quad \hat p_2=\frac{\sqrt{\vartheta_2}}{\cos^2\pi v_2}\,p_2\ ,\quad \hat p_2^\dagger=\frac{\sqrt{\vartheta_2}}{\cos^2\pi v_2}\,\bar p_2\ . \tag{4.62}$$

We define the worldsheet NS and R vacua to satisfy $\hat p_2|\mathrm{NS}\rangle=\hat p_2^\dagger|\mathrm{NS}\rangle=\hat p_2|\mathrm{R}\rangle=\hat p_2^\dagger|\mathrm{R}\rangle=0$. The tower of states is defined as

$$|n+;\boldsymbol{\alpha}\rangle=(\hat z_2)^n|\boldsymbol{\alpha}\rangle\ ,\quad |n-;\boldsymbol{\alpha}\rangle=(\hat z_2^\dagger)^n|\boldsymbol{\alpha}\rangle\quad\text{where}\quad \boldsymbol{\alpha}=\mathrm{NS},\mathrm{R}\quad\text{and}\quad n\geq 0\ . \tag{4.63}$$

Thus, corresponding to each of the fields $\varphi=\{\sigma^1,\sigma^2,\xi^\alpha,\tilde\xi^{\dot\alpha}\}$ making up the hypermultiplet in $\mathbf{R}^{1,1}\times\mathbf{C}_a$, we have a doubly infinite tower of fields $\varphi^{n\pm}$ in $\mathbf{R}^{1,1}$ with $\mathrm{U}(1)_2$ eigenvalue $\pm n$. The L_0 eigenvalue of these states is zero they have one of p_2 or $\bar p_2$ equal to zero. Hence, the masses of these states are still at their four dimensional values.

The open string vertex operators for the φ^{n-} fields take the form

$$V(\varphi^{n-};x) = \frac{1}{(\vartheta_2)^{n/2}}\omega(\lambda)\, c(x)\, \mathcal{B}(x)\, \left(\overline{Z}_2(x)\right)^n \mathrm{e}^{\lambda\cdot H(x)}\, \mathrm{e}^{2ik\cdot X(x)} c_\lambda \; . \tag{4.64}$$

Here, $c(x)$ is the worldsheet coordinate ghost, λ is the covariant weight for φ in Table 4.4, k is the momentum in $\mathbf{R}^{1,1}$, c_λ is the associated cocycle operator and $\omega(\lambda)$ is a c-number phase factor. $\mathcal{B}(x)$ is the appropriate product of boundary condition changing operators for the worldsheet bosons. The weights of the D5$_{(12)}$-D5$_{(23)}$ strings mimic those of D1-$\overline{\text{D5}}_{(13)}$ strings. This can be observed in the relative sign between v_1 and v_3 in the vertex operators.

Table 4.4: Covariant weights for $\overline{\text{D1}}$-D5$_{(23)}$ and D5$_{(12)}$-D5$_{(23)}$ strings.

State	Field	$\mathrm{U}(1)_\ell$	$D_1 \oplus D_1 \oplus D_1 \oplus D_1 \oplus \Gamma_{1,1}$ weight
D1-D5$_{(23)}$ bosons	I'	$\frac{1}{2} - v_2$	$0,\ -v_2+\frac{1}{2},\ -v_3+\frac{1}{2},\ 0,\ 0;-1$
	J'	$\frac{1}{2} + v_2$	$0,\ +v_2+\frac{1}{2},\ +v_3+\frac{1}{2},\ 0,\ 0;-1$
D1-D5$_{(23)}$ fermions	$\zeta_{J'}$	$-\frac{1}{2} + v_2$	$-,\ +v_2,\ +v_3,\ -,\ -;-$
	$\zeta_{I'}$	$-\frac{1}{2} - v_2$	$-,\ -v_2,\ -v_3,\ -,\ -;-$
	$\lambda_{J'}$	$\frac{1}{2} + v_2$	$-,\ +v_2,\ +v_3,\ +,\ +;-$
	$\lambda_{I'}$	$\frac{1}{2} - v_2$	$-,\ -v_2,\ -v_3,\ +,\ +;-$
D5$_{(12)}$-D5$_{(23)}$ bosons	σ^1, S	0	$-v_1+\frac{1}{2},\ 0,\ -v_3+\frac{1}{2},\ 0,\ 0;-1$
	$\sigma^{2\dagger}$, T	0	$+v_1+\frac{1}{2},\ 0,\ +v_3+\frac{1}{2},\ 0,\ 0;-1$
D5$_{(12)}$-D5$_{(23)}$ fermions	$\tilde{\xi}^{1}$, ζ_S	-1	$-v_1,\ -,\ +v_3,\ -,\ -;-$
	$\xi^{2\dagger}$, ζ_T	-1	$+v_1,\ -,\ +v_3,\ -,\ -;-$
	ξ^1, λ_S	0	$+v_1,\ -,\ -v_3,\ +,\ +;-$
	$\tilde{\xi}^{2\dagger}$, λ_T	0	$-v_1,\ -,\ +v_3,\ +,\ +;-$

4.4.2 $(n+3)$-point amplitudes

We are interested in calculating amplitudes that give rise to J-terms of the form $\widetilde{J}(B_2)^n I$ in the low-energy theory. In the effective action they turn up as Yukawa couplings of the form

$$\mathcal{J}^\Psi = -\frac{\partial J_\psi}{\partial \phi}\, \zeta_+\, \psi_- \; , \tag{4.65}$$

Here, ζ_+ is the right-handed superpartner of ϕ and ψ_- is the left-handed fermi field whose J-term is J_ψ. The above J-term is in the $\overline{\boldsymbol{n}} \times \boldsymbol{n'}$ of $\mathrm{U}(n) \times \mathrm{U}(n')$. Such a term should then arise from an open string disk amplitude involving the insertion of λ_S^{n-} and the following $n+2$ vertex operators on the boundary of the disk:

$$V(\zeta_{J'};x_{-2}) \ , \quad V(\lambda_S^{n-};x_{-1}) \ , \quad V(I;x_0) \ , \quad V(B_2;x_1) \ , \quad V(B_2;x_2) \ , \ \ldots \ , \quad V(B_2;x_n) \ . \tag{4.66}$$

Under the map from the strip to the upper half plane, the boundary at $\sigma = 0$ is mapped to the positive real axis and the boundary at $\sigma = \pi$ is mapped to the negative real axis. As a consequence, the order of the Chan-Paton factors for the sequence in (4.66) should be in the reverse order. Indeed, the J-term would correspond to $\mathrm{Tr}_k((B_2)^n I \lambda_S^{n-} J') = \mathrm{Tr}_n(J'(B_2)^n I \lambda_S^{n-})$.

The total picture number of the above set of vertex operators is $-\frac{1}{2} - \frac{1}{2} - 1 - n = -n-2$. Since the total picture number has to be -2 on the disk, the above amplitudes must have n picture changing operators $\mathcal{X}(z_i)$, $i = 1, \ldots, n$, inserted at points z_i in the bulk as well.

Let $\mathcal{M}(g, b, n_C, n_O)$ be the moduli space of genus g Riemann surfaces with b boundaries, n_C bulk punctures (closed string insertions) and n_O boundary punctures (open string insertions). Its real dimension is

$$\dim_{\mathbf{R}} \mathcal{M}(g, b, n_C, n_O) = 6g + 3b - 6 + 2n_C + n_O \ . \tag{4.67}$$

At this stage, it is convenient to define the infinite dimensional space $\mathcal{P}(g, b, n_C, n_O)$ to be the moduli space of genus g Riemann surfaces with b boundaries, n_C closed string insertions and n_O open string insertions with a choice of local coordinates around each puncture.

Here, we have $g = 0$, $b = 1$, $n_C = 0$ and $n_O = n+3$, giving the dimension of $\mathcal{M}(0,1,0,n+3)$ to be n. The $(n+3)$-point amplitude is then given by the integration of an n-form over $\mathcal{M}(0,1,0,n+3)$. We now describe the construction of the above n-form following [Sen2, Z]. Let $\langle\Sigma|$ be the surface state corresponding to the disk with $n+3$ insertions and $|\Phi\rangle$ denote the particular state in the Hilbert space of the worldsheet superconformal theory corresponding to the $n+3$ insertions in (4.66). The n-form is then defined as follows:

$$\Omega_n(\Phi) := \langle\Sigma|\mathbf{B}_n|\Phi\rangle \quad \text{with} \quad \mathbf{B}_n = \sum_{r=0}^{n} K^{(r)} \wedge B_{n-r} \ . \tag{4.68}$$

The B_p are operator-valued p-forms defined as follows. If we adopt the presentation of tangent vectors of $\mathcal{P}(0,1,0,n+3)$ in terms of Schiffer variation, then each tangent vector is described by an $(n+3)$-tuple of vector fields on the disk: one vector field each for infinitesimal coordinate changes around the $(n+3)$ punctures.

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ be p such $(n+3)$-tuples of vector fields on the disk and let w_j be the local coordinate around the j-th puncture. Then,

$$B_p[\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p] := b(\vec{v}_1)b(\vec{v}_2)\cdots b(\vec{v}_p) \ , \quad \text{with} \quad b(\vec{v}) = \sum_{j=1}^{n+3} \oint \frac{\mathrm{d}w_j}{2\pi\mathrm{i}} v^{(j)}(w_j) b(w_j) \ . \tag{4.69}$$

Here, $b(z)$ is the doubled version of the reparametrisation antighost field on the worldsheet and the contour integral is carried out around a small contour encircling the j-th puncture.

The $K^{(r)}$ are r-forms on the worldsheet constructed out of the picture changing operators and the ξ fields which bosonise the superconformal ghosts. We have

$$K^{(r)} := [(\mathcal{X}(z_1) - \partial\xi(z_1)\mathrm{d}z_1) \wedge (\mathcal{X}(z_2) - \partial\xi(z_2)\mathrm{d}z_2) \wedge \cdots \wedge (\mathcal{X}(z_n) - \partial\xi(z_n)\mathrm{d}z_n)]^{(r)} \ , \tag{4.70}$$

where the superscript r on the right hand side indicates that we should take the degree r part of the inhomogeneous differential form.

Important note: The locations of the picture changing operators have to be chosen such that picture number is always conserved in any degeneration of the punctured disk.

For our situation with the vertex operators in (4.66), it turns out that the correct locations of the n picture changing operators are such that one is in the patch of the I insertion and the rest are in the patches corresponding to the last $n-1$ B_2 insertions:

$$z_1 = x_0 \ , \quad z_j = \alpha_j x_j \quad \text{for} \quad j \geq 2 \ , \quad \alpha_j \text{ to be taken to 1 at the end} \ . \tag{4.71}$$

Using the three conformal Killing vectors on the disk, we fix the positions $x_{-2} = \infty$, $x_{-1} = 0$ and $x_0 = 1$. The moduli are described by the x_j for $j = 1, \ldots, n$. Let z be the coordinate describing the upper half-plane $z \in \mathbf{C}$ with $\mathrm{Im}(z) > 0$. The coordinate w_j around the j-th puncture is given by

$$w_j = z - x_j \ , \quad v^{(j)}(w_j) = -1 \ . \tag{4.72}$$

where $v^{(j)}(w_j) = -1$ is the vector field that represents the change in $w_j \to w_j - \delta x_j$ under a change in the modulus $x_j \to x_j + \delta x_j$. Thus, we get

$$b(\vec{v}) = \sum_{j=1}^{n} \oint \frac{\mathrm{d}w_j}{2\pi \mathrm{i}}(-1) b(w_j) =: \sum_{j=1}^{n}(-b_{x_j}) \ . \tag{4.73}$$

Plugging the above in to the definition of the operator-valued n-form $\mathbf{B}_n$, we get

$$\begin{aligned} \mathbf{B}_n &= \mathcal{X}(x_0)\ (-b_{x_1})\ \mathcal{Y}(\alpha_2, x_2) \cdots \mathcal{Y}(\alpha_n, x_n)\ \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \ , \\ \text{with} \quad & \mathcal{Y}(\alpha_j, x_j) := [\mathcal{X}(\alpha_j x_j)(-b_{x_j}) - \alpha_j \partial \xi(\alpha_j x_j)] \ . \end{aligned} \tag{4.74}$$

The moduli space $\mathcal{M}(0,1,0,n+3)$ is the space of locations of the $n+3$ operators upto the action of $\mathrm{SL}(2,\mathbf{R})$. The $\mathrm{SL}(2,\mathbf{R})$ is soaked up by fixing the locations of x_{-2}, x_{-1} and x_0 to ∞, 0 and 1 respectively. Since $\mathrm{SL}(2,\mathbf{R})$ does not change the cyclic ordering of the operators, the moduli space is the union of the space of locations $x_1, \ldots, x_n$ with different cyclic orderings. Since the vertex operators $\zeta_{\tilde{J}}$, $\lambda_S^{(n-)}$ and I change boundary conditions from one D-brane to another, the permutation group acts only on the positions of the B_2 vertex operators. Thus, the amplitude is given by

$$\begin{aligned} A_{n+3} &= \int_{\mathcal{M}(0,1,0,n+3)} \langle \Sigma | \mathbf{B}_n | \Phi \rangle \ , \\ &= \int_1^{\infty} \mathrm{d}x_1 \cdots \mathrm{d}x_n \ \langle V(\zeta_{\tilde{J}}; x_{-2})\ V(\lambda_S^{n-}; x_{-1})\ \mathcal{X}(x_0) V(I; x_0) \times \\ & \quad \times (-b_{x_1}) V(B_2; x_1)\ \mathcal{Y}(\alpha_2, x_2) V(B_2; x_2)\ \cdots\ \mathcal{Y}(\alpha_n, x_n) V(B_2; x_n) \rangle \ . \end{aligned} \tag{4.75}$$

The vertex operators (sans the overall phase and cocycle operators) are given by

$$\begin{aligned}
V(\zeta_{J'};x_{-2}) &= c\ \sigma_2^+\sigma_3^+\ \mathrm{e}^{\lambda(\zeta_{J'})\cdot H}\ \mathrm{e}^{2\mathrm{i}k_{-2}\cdot X}(x_{-2})\ ,\\
V(\lambda_S^{n-};x_{-1}) &= \frac{1}{(\vartheta_2)^{n/2}}c\ \sigma_1^+\sigma_3\ (\overline{Z}_2)^n\ \mathrm{e}^{\lambda(\lambda_S)\cdot H}\ \mathrm{e}^{2\mathrm{i}k_{-1}\cdot X}(x_{-1})\ ,\\
V(I;x_0) &= c\ \sigma_1\sigma_2\ \mathrm{e}^{\lambda(I)\cdot H}\ \mathrm{e}^{2\mathrm{i}k_0\cdot X}(x_0)\ ,\\
V(B_2;x_j) &= c\ \mathrm{e}^{\lambda(B_2)\cdot H}\ \mathrm{e}^{2\mathrm{i}k_j\cdot X}(x_j)\ ,
\end{aligned} \tag{4.76}$$

with $\vartheta_2 = \pi\alpha' \sin 2\pi v_2$. We proceed by moving the picture changing operator at x_0 to x_{-1} by writing

$$\mathcal{X}(x_0) = \mathcal{X}(x_{-1}) + \mathcal{X}(x_0) - \mathcal{X}(x_{-1})\ .$$

(Such a trick was also used in [Sen2].) Using the identity $\mathcal{X}(x_0) - \mathcal{X}(x_{-1}) = \{\mathrm{Q_B}\,, \xi(x_0) - \xi(x_{-1})\}$, $\mathrm{Q_B}$ being the doubled BRST charge, the above amplitude can be written as the sum of two pieces $A_{n+3} = B_{n+3} + C_{n+3}$ with

$$\begin{aligned}
B_{n+3} = \int_1^\infty &\mathrm{d}x_1 \cdots \mathrm{d}x_n\ \langle V(\zeta_{\widetilde{J}};x_{-2})\ \mathcal{X}(x_{-1})V(\lambda_S^{n-};x_{-1})\ V(I;x_0)\times\\
&\times(-b_{x_1})V(B_2;x_1)\ \mathcal{Y}(\alpha_2,x_2)V(B_2;x_2)\ \cdots\ \mathcal{Y}(\alpha_n,x_n)V(B_2;x_n)\rangle\ ,\\
C_{n+3} = \int_1^\infty &\mathrm{d}x_1 \cdots \mathrm{d}x_n\ \langle V(\zeta_{\widetilde{J}};x_{-2})\ V(\lambda_S^{n-};x_{-1})\ \{\mathrm{Q_B}\,,\xi(x_0)-\xi(x_{-1})\}V(I;x_0)\times\\
&\times(-b_{x_1})V(B_2;x_1)\ \mathcal{Y}(\alpha_2,x_2)V(B_2;x_2)\ \cdots\ \mathcal{Y}(\alpha_n,x_n)V(B_2;x_n)\rangle\ .
\end{aligned} \tag{4.77}$$

Let us first evaluate B_{n+3}. The picture changing operator is given by

$$\mathcal{X} = c\partial\xi + \mathrm{e}^{\varphi}T_F - \frac{1}{4}\partial\eta\,\mathrm{e}^{2\varphi}b - \frac{1}{4}\partial(\eta\mathrm{e}^{2\varphi}b)\ . \tag{4.78}$$

We have, in the $\alpha_j \to 1$ limit,

$$\begin{aligned}
\lim_{\alpha_j\to 1}\mathcal{Y}(\alpha_j,x_j)V(B_2;x_j) &= \lim_{\alpha_j\to 1}[\mathcal{X}(\alpha_j x_j)(-b_{x_j}) - \alpha_j\partial\xi(\alpha_j x_j)]V(B_2;x_j)\ ,\\
&= (\mathrm{i}\partial Z_2 + (k_j\cdot\Psi)\Psi_2)\,\mathrm{e}^{2\mathrm{i}k_j\cdot X}(x_j)\ .
\end{aligned} \tag{4.79}$$

In the above, we see that the $\partial\xi$ term cancels the $c\partial\xi$ in $\mathcal{X}(x_j)$. Only the $\mathrm{e}^{\varphi}T_F$ term gives a first order pole and the last line is the corresponding residue. We have used $\Psi_2(z)\overline{\Psi}_2(w) \sim \alpha'(z-w)^{-1}$, $T_F = \frac{1}{\alpha'}\partial Z_2 \overline{\Psi}_2 + \cdots$, and so on.

The only term in $\mathcal{X}(x_{-1})V(\lambda_S^{(n-)};x_{-1})$ that contributes to the correlator in B_{n+3} are

$$\mathcal{X}(x_{-1})V(\lambda_S^{n-};x_{-1}) = \frac{n\alpha'}{(\vartheta_2)^{n/2}} \mathrm{e}^{-H_2-H_6}(x_{-1})V(\lambda_S^{(n-1)-};x_{-1}) \ . \tag{4.80}$$

Also, only the $\mathrm{i}\partial Z_2$ term in (4.79) contributes to the correlator. Thus, the integrand of B_{n+3} becomes

$$\begin{aligned}\frac{\mathrm{i}^{n-1}\, n\alpha'}{(\vartheta_2)^{n/2}} \langle V(\zeta_{J'};x_{-2})\ \mathrm{e}^{-H_2-H_6}(x_{-1})V(\lambda_S^{(n-1)-};x_{-1})\ V(I;x_0)\times \\ \times (-b_{x_1})V(B_2;x_1)\ \partial Z_2\, \mathrm{e}^{2\mathrm{i}k_2\cdot X}(x_2)\cdots \partial Z_2\, \mathrm{e}^{2\mathrm{i}k_n\cdot X}(x_n)\rangle \ ,\end{aligned} \tag{4.81}$$

which decomposes into the following product of correlators:

$$\begin{aligned}&\frac{\mathrm{i}^{n-1}\, n\alpha'}{(\vartheta_2)^{n/2}} \langle c(x_{-2})c(x_{-1})c(x_0)\rangle\ \langle \sigma_3^+(x_{-2})\sigma_3(x_{-1})\rangle\ \langle \sigma_1^+(x_{-1})\sigma_1(x_0)\rangle\times \\ &\times \langle \mathrm{e}^{\lambda(\zeta_{J'})\cdot H}(x_{-2})\, \mathrm{e}^{-H_2-H_6}(x_{-1})\, \mathrm{e}^{\lambda(\lambda_S)\cdot H}(x_{-1})\, \mathrm{e}^{\lambda(I)\cdot H}(x_0)\, \mathrm{e}^{\lambda(B_2)\cdot H}(x_1)\rangle\times \\ &\times\ \langle \sigma_2^+(x_{-2})\, \boldsymbol{:}(\overline{Z}_2)^{n-1}(x_{-1})\boldsymbol{:}\, \partial Z_2(x_2)\cdots \partial Z_2(x_n)\sigma_2(x_0)\rangle\ \times \\ &\times \langle \mathrm{e}^{2\mathrm{i}k_{-2}\cdot X}(x_{-2})\, \mathrm{e}^{2\mathrm{i}k_{-1}\cdot X}(x_{-1})\cdots \mathrm{e}^{2\mathrm{i}k_n\cdot X}(x_n)\rangle\ .\end{aligned} \tag{4.82}$$

All the correlators above are standard except $\langle \sigma_2^+ \cdots \sigma_2\rangle$ in the third line. To proceed, we study correlators of the form

$$G_n(\boldsymbol{z},\boldsymbol{w}) := \frac{(-2/\alpha')^n}{\langle \sigma^+(\infty)\sigma(0)\rangle}\ \langle \sigma^+(\infty)\mathsf{J}^*(w_1)\mathsf{J}^*(w_2)\cdots \mathsf{J}^*(w_n)\mathsf{J}(z_1)\mathsf{J}(z_2)\cdots \mathsf{J}(z_n)\sigma(0)\rangle\ , \tag{4.83}$$

where J and J^* are the doubled worldsheet currents. Let us study the $n=2$ case first with $0 \le \theta \le 1$ where $\theta = \frac{1}{2} - v$. We need the following OPEs:

$$\mathsf{J}(z)\sigma(0) \sim z^{-\theta}\, \tau_3(0)\ , \quad \mathsf{J}^*(z)\sigma(0) \sim z^{-1+\theta}\, \tau_4(0)\ . \tag{4.84}$$

Based on the above OPEs and the $\mathsf{J}\mathsf{J}^*$ OPE, we write down the following expression for

G_2:

$$G_2(\boldsymbol{z},\boldsymbol{w}) = \frac{(-2/\alpha')^2}{\langle\sigma^+(\infty)\sigma(0)\rangle}\ \langle\sigma^+(\infty)\mathsf{J}^*(w_1)\mathsf{J}^*(w_2)\mathsf{J}(z_1)\mathsf{J}(z_2)\sigma(0)\rangle\ ,$$
$$= z_1^{-\theta}z_2^{-\theta}w_1^{-1+\theta}w_2^{-1+\theta}\left[\frac{((1-\theta)w_1+\theta z_2)((1-\theta)w_2+\theta z_1)}{(z_2-w_1)^2(z_1-w_2)^2} + \{w_1 \leftrightarrow w_2\}\right]\ . \tag{4.85}$$

This expression has the correct properties in the various limits of the insertion points $\boldsymbol{z}$ and $\boldsymbol{w}$. The generalisation to G_n is given by

$$G_n(\boldsymbol{z},\boldsymbol{w}) = z_1^{-\theta}\cdots z_n^{-\theta}w_1^{-1+\theta}\cdots w_n^{-1+\theta}\times$$
$$\times\left[\frac{((1-\theta)w_1+\theta z_1)\cdots((1-\theta)w_n+\theta z_n)}{(z_1-w_1)^2\cdots(z_n-w_n)^2} + \text{ permutations of } w_1,\ldots,w_n\right]\ . \tag{4.86}$$

We obtain the correlation function with $:(\overline{Z})^n(w):$ by integrating G_n with respect to $w_1,\ldots,w_n$ and setting $w_j = w$ for all $j=1,\ldots,n$. The result is

$$\widetilde{G}_n(\boldsymbol{z};w) := \frac{(-2/\alpha')^n}{\langle\sigma^+(\infty)\sigma(0)\rangle}\langle\sigma^+(\infty):\overline{Z}^n(w):\partial Z(z_1)\cdots\partial Z(z_n)\sigma(0)\rangle$$
$$= \frac{n!\,z_1^{-\theta}z_2^{-\theta}\cdots z_n^{-\theta}\,w^{n\theta}}{(z_1-w)(z_2-w)\cdots(z_n-w)}\ . \tag{4.87}$$

The integrand of B_{n+3} becomes

$$\frac{\mathrm{i}^{n-1}\,n\alpha'}{(\vartheta_2)^{n/2}}\times(x_{-2}-x_{-1})(x_{-2}-x_0)(x_{-1}-x_0)\ \times\prod_{-2\le i<j\le 1}(x_i-x_j)^{\lambda_i\cdot\lambda_j}\ \times$$
$$\frac{1}{(x_{-2}-x_{-1})^{2h_3}}\frac{1}{(x_{-2}-x_0)^{2h_2}}\frac{1}{(x_{-1}-x_0)^{2h_1}}\times\prod_{-2\le i<j\le n}(x_i-x_j)^{2\alpha' k_i\cdot k_j}\times$$
$$\times\,(-\tfrac{\alpha'}{2})^{n-1}\widetilde{G}_{n-1}(x_2-x_0,\ldots,x_n-x_0;x_{-1}-x_0)\ . \tag{4.88}$$

Here, λ_i and k_i are the total covariant weight and momentum at x_i respectively. Recall that we fixed $x_{-2}=\infty$, $x_{-1}=0$ and $x_0=1$ using the $\mathrm{SL}(2,\mathbf{R})$ isometry. We set $k_j=0$

since that corresponds to the irreducible tree-level vertex part of the amplitude. We get

$$\frac{n(\alpha')^n}{(\vartheta_2)^{n/2}(2\mathrm{i})^{n-1}} \times (x_{-2})^{2+\lambda_{-2}\cdot(\lambda_{-1}+\lambda_0+\lambda_1)-2h_3-2h_2} \times \prod_{-1\le i<j\le 1}(x_i-x_j)^{\lambda_i\cdot\lambda_j} \times$$
$$\times \frac{(n-1)!\,(x_2-1)^{-\theta_2}(x_3-1)^{-\theta_2}\cdots(x_n-1)^{-\theta_2}}{x_2x_3\cdots x_n} \ . \qquad (4.89)$$

We have $2h_3 = \frac{1}{4} - v_3^2$, $2h_2 = \frac{1}{4} - v_2^2$ and

$$\lambda_{-2} = \lambda(\zeta_{J'}) \ , \quad \lambda_{-1} = \lambda(\lambda_S) + (0,-1,0,0,0;1) \ , \quad \lambda_0 = \lambda(I) \quad \lambda_1 = \lambda(B_2) \ . \qquad (4.90)$$

Using the above expressions, we see that the exponent of x_{-2} is

$$2+\lambda_{-2}\cdot(\lambda_{-1}+\lambda_0+\lambda_1)-2h_3-2h_2$$
$$= 2+(-\tfrac{1}{4}-v_2^2-v_3^2-\tfrac{1}{4}-\tfrac{1}{4}-\tfrac{3}{4})-(\tfrac{1}{4}-v_3^2)-(\tfrac{1}{4}-v_2^2) = 0 \ , \qquad (4.91)$$

which vanishes as it should. The integrand thus becomes

$$\frac{n!(\alpha')^n}{(\vartheta_2)^{n/2}(2\mathrm{i})^{n-1}} \times \frac{(x_1-1)^{-v_2-\frac{1}{2}}(x_2-1)^{v_2-\frac{1}{2}}\cdots(x_n-1)^{v_2-\frac{1}{2}}}{x_1x_2\cdots x_n} \ . \qquad (4.92)$$

Changing variables to $y_j = \frac{x_j-1}{x_j}$ for $j=1,\ldots,n$, we get

$$B_{n+3} = \mathrm{Tr}(B_2^n I \lambda_S^{n-}\zeta_{J'})\ \frac{n!(\alpha')^n}{(\vartheta_2)^{n/2}(2\mathrm{i})^{n-1}}\times$$
$$\times \int_0^1 \mathrm{d}y_1\ y_1^{-v_2-\frac{1}{2}}(1-y_1)^{v_2-\frac{1}{2}} \prod_{j=2}^{n}\int_0^1 \mathrm{d}y_j\ y_j^{v_2-\frac{1}{2}}(1-y_j)^{-v_2-\frac{1}{2}} \ ,$$
$$= \frac{n!(\alpha')^n}{(\vartheta_2)^{n/2}(2\mathrm{i})^{n-1}}\,\mathrm{B}(-v_2+\tfrac{1}{2}, v_2+\tfrac{1}{2})^n\ \mathrm{Tr}(B_2^n I \lambda_S^{n-}\zeta_{J'}) \ . \qquad (4.93)$$

We next evaluate C_{n+3}:

$$C_{n+3} = \int_1^\infty \mathrm{d}x_1\cdots\mathrm{d}x_n\ \langle V(\zeta_{J'};x_{-2})\ V(\lambda_S^{n-};x_{-1})\ \{\mathrm{Q_B}\,,\xi(x_0)-\xi(x_{-1})\}V(I;x_0)\times$$
$$\times(-b_{x_1})V(B_2;x_1)\ \mathcal{Y}(\alpha_2,x_2)V(B_2;x_2)\ \cdots\ \mathcal{Y}(\alpha_n,x_n)V(B_2;x_n)\rangle \ . \qquad (4.94)$$

We lasso the contour for the BRST charge around the rest of the punctures in the correlator. We need the action of $\mathrm{Q_B}$ on the various operators in the correlator:

$$[\mathrm{Q_B}\,,\mathcal{X}(z)]=0\ ,\quad \{\mathrm{Q_B}\,,\partial\xi(z)\}=\partial\mathcal{X}(z)\ ,\quad [\mathrm{Q_B}\,,V(z)]=0\ ,\quad [\mathrm{Q_B}\,,b_zV(z)]=\partial_zV(z)\ . \tag{4.95}$$

Recall that $\mathcal{Y}(\alpha_j,x_j)=\mathcal{X}(\alpha_jx_j)(-b_{x_j})-\alpha_j\partial\xi(\alpha_jx_j)$. We then have

$$\begin{aligned}[\mathrm{Q_B}\,,\mathcal{Y}(\alpha_j,x_j)V(B_2;x_j)] &= -\mathcal{X}(\alpha_jx_j)\partial_{x_j}V(B_2;x_j)-\alpha_j\partial\mathcal{X}(\alpha_jx_j)V(B_2;x_j)\ ,\\ &= -\partial_{x_j}\big(\mathcal{X}(\alpha_jx_j)V(B_2;x_j)\big)\ .\end{aligned} \tag{4.96}$$

Using the above identities, we get

$$C_{n+3}=-\int_1^\infty \mathrm{d}x_1\cdots\mathrm{d}x_n\sum_{j=1}^n\partial_{x_j}E_j\ , \tag{4.97}$$

with

$$\begin{aligned}E_1 &= \langle V(\zeta_{J'};x_{-2})\ V(\lambda_S^{n-};x_{-1})\ (\xi(x_0)-\xi(x_{-1}))\ V(I;x_0)\times\\ &\qquad\times V(B_2;x_1)\mathcal{Y}(\alpha_2,x_2)V(B_2;x_2)\ \cdots\ \mathcal{Y}(\alpha_n,x_n)V(B_2;x_n)\rangle\ ,\\ E_j &= \langle V(\zeta_{J'};x_{-2})\ V(\lambda_S^{n-};x_{-1})\ (\xi(x_0)-\xi(x_{-1}))\ V(I;x_0)\ (-b_{x_1})V(B_2;x_1)\times\\ &\qquad\times\mathcal{Y}(\alpha_2,x_2)V(B_2;x_2)\ \cdots\ \cancel{\mathcal{Y}(\alpha_j,x_j)}\mathcal{X}(x_j)V(B_2;x_j)\ \cdots\ \mathcal{Y}(\alpha_n,x_n)V(B_2;x_n)\rangle\ .\end{aligned} \tag{4.98}$$

We use $\mathcal{Y}(1,x_j)V(B_2;x_j)=\mathrm{i}\partial Z_2\,\mathrm{e}^{2\mathrm{i}k_j\cdot X}(x_j)+\cdots$. The correlator in E_1 is then zero because the $\xi(x_0)-\xi(x_{-1})$ factor cannot be saturated. In addition, the weights in the worldsheet fermion correlator do not add up to $2e_6$. To evaluate E_j, we need an expression for $\mathcal{X}(x_j)V(B_2;x_j)$. Since we have to saturate the ξ-dependent factor, only the η-dependent terms can give a non-zero contribution. Of these, only the $\eta\partial(\mathrm{e}^{2\varphi}b)$ term gives a simple pole:

$$\mathcal{X}(x_j)V(B_2;x_j)=-\frac{1}{4}\eta\,\mathrm{e}^{\lambda(B_2)\cdot H}\mathrm{e}^{2\mathrm{i}k_j\cdot X}(x_j)+\text{other terms which give zero contribution}\ . \tag{4.99}$$

However, the worldsheet fermion correlator now evaluates to zero since the weights λ_i do not add up to $2e_6$. We then find that $E_j = 0$ as well. This gives

$$A_{n+3} = B_{n+3} = \frac{n!(\alpha')^n}{(\vartheta_2)^{n/2}(2\mathrm{i})^{n-1}} \mathrm{B}(-v_2 + \tfrac{1}{2}, v_2 + \tfrac{1}{2})^n \ \mathrm{Tr}(B_2^n I \lambda_S^{n-} \zeta_{J'}) \ . \tag{4.100}$$

We are interested in the non-commutative point particle limit. That is, the limit $\alpha' \to 0$ such that, in addition, the open string metric $G^{2\bar{2}}$ and Poisson bivector $\Theta^{2\bar{2}}$ are constant. These quantities are the right-hand sides of the zero mode commutation relations:

$$[z_2\,, \bar{z}_2] = \Theta^{2\bar{2}} = \frac{2\pi\alpha' b_2}{1+b_2^2} \ , \quad [z_2\,, p_2] = \mathrm{i}G^{2\bar{2}} = \frac{\mathrm{i}}{1+b_2^2} \ . \tag{4.101}$$

Following [SW1], we achieve this by introducing a small parameter $\varepsilon \to 0$ and introducing the following ε dependence for the various objects:

$$\alpha' = \varepsilon^{1/2}, \quad b_2 = \varepsilon^{-1/2}\hat{b}_2, \quad 2\pi\alpha' B_{2\bar{2}} = \mathrm{i}\varepsilon b_2, \quad g_{2\bar{2}} = \varepsilon \ . \tag{4.102}$$

for finite $\hat{b}_2$. The result of the scaling on $B_{2\bar{2}}$ and $g_{2\bar{2}}$ is that the right-hand sides of the commutators pick up an ε^{-1}. In the limit $\varepsilon \to 0$, we get

$$\Theta^{2\bar{2}} = \vartheta_2 = 2\pi(\hat{b}_2)^{-1} \ , \quad G^{2\bar{2}} = \mathrm{i}(\hat{b}_2)^{-2} \ . \tag{4.103}$$

Since $b_2 \to \infty$ in this limit, we have $v_2 \to \frac{1}{2}$. The amplitude hits a pole in this limit:

$$\mathrm{B}(\tfrac{1}{2} - v_2, \tfrac{1}{2} + v_2) = \Gamma(\tfrac{1}{2} - v_2)\Gamma(\tfrac{1}{2} + v_2) \to \frac{1}{\frac{1}{2} - v_2} = \frac{\pi \hat{b}_2}{\varepsilon^{1/2}} \ . \tag{4.104}$$

The amplitude is then finite in this limit and is given by

$$A_{n+3} = \left(\frac{\pi}{2}\right)^{n/2} \frac{n!(\hat{b}_2)^{3n/2}}{(2\mathrm{i})^{n-1}} \ \mathrm{Tr}(B_2^n I \lambda_S^{n-} \zeta_{J'}) \ . \tag{4.105}$$

The corresponding J-term can be read off (up to normalisation) as

$$J_{\lambda_S^{n-}} = J'(B_2)^n I \ . \tag{4.106}$$

The amplitude for an E-term for λ_S^{n-} would have to involve $\overline{\lambda}_S^{n-}$ which would have an

$SO(2)_{34}$ quantum number of $+n$. But there are no holomorphic fields from the various Dp-Dp' sectors which can saturate $+n$. So the E-term for such a fermi multiplet is zero.

Repeating the above analysis for λ_T^{n-} which is in the $\bar{\boldsymbol{n}} \times \boldsymbol{n'}$ of $U(n) \times U(n')$ would give the J-term

$$J_{\lambda_T^{n-}} = J(B_2)^n I' \ . \qquad (4.107)$$

This completes the derivation of the equations for the folded branes. We now have generated the spiked instanton equations (1.33) - (1.37) that we wrote down in the Introduction.

4.5 Additional equations from D5-D5 strings

The states of the $D5_{(12)}$-$D5_{(12)}$ on the two dimensional intersection have the same covariant weights as the D1-D1 strings, summarised in Table 4.2. However, there is an additional doubly infinite tower of massless states corresponding to each additional complex dimension $\mathbf{C}_{a=1}$ and $\mathbf{C}_{a=2}$. In particular, there are fermi multiplets of the form $\lambda_j^{n_1\pm,n_2\pm}$, $j = 2,3,4$ and $n_1, n_2 \in \mathbf{Z}_{\geq 0}$, with vertex operators

$$\begin{aligned} V(\lambda_j^{n_1-,n_2-};x) &= (\vartheta_1)^{-n_1/2}(\vartheta_2)^{-n_2/2}\, c\, (\overline{Z}_1)^{n_1}(\overline{Z}_2)^{n_2}\ \mathrm{e}^{\lambda(\lambda_j)\cdot H}\ \mathrm{e}^{2\mathrm{i}k\cdot X}(x)\ , \\ V(\lambda_j^{n_1-,n_2+};x) &= (\vartheta_1)^{-n_1/2}(\vartheta_2)^{-n_2/2}\, c\, (\overline{Z}_1)^{n_1}(Z_2)^{n_2}\ \mathrm{e}^{\lambda(\lambda_j)\cdot H}\ \mathrm{e}^{2\mathrm{i}k\cdot X}(x) \quad \text{etc.} \end{aligned} \qquad (4.108)$$

There is an E-term for $\lambda_2^{n_1+,n_2+}$ which arises from the following disk amplitude:

$$\begin{aligned} &V(\zeta_J;x_{-2})\ ,\ V(\overline{\lambda}_2^{n_1+,n_2+};x_{-1})\ ,\ V(I;x_0)\ ,\ V(B_1;x_1)\ ,\ \dots \\ &\qquad \dots\ ,\ V(B_1;x_{n_1})\ ,\ V(B_2;x_{n_1+1})\ ,\ \dots\ ,\ V(B_2;x_{n_1+n_2})\ . \end{aligned} \qquad (4.109)$$

Again, the contraction of Chan-Paton factors is in the reverse order. For the region $0 < 1 < x_1 < x_2 < \cdots < x_{n+m} < \infty$, the trace over the Chan-Paton factors is

$$\mathrm{Tr}[\zeta_J(B_2)^{n_2}(B_1)^{n_1} I\ \overline{\lambda}_2^{n_1+,n_2+}]\ . \qquad (4.110)$$

Let us look at the example of $n_1 = 1$, $n_2 = 1$. We will be able to generalise the answer to arbitrary n_1 and n_2. Going through the same procedure as earlier by attaching picture changing operators appropriately, we get the following expression for the amplitude:

$$I(x_1, x_2) = \frac{(\alpha')^2}{2\mathrm{i}(\vartheta_1\vartheta_2)^{1/2}} \times \frac{(x_1-1)^{v_1-\frac{1}{2}}(x_2-1)^{v_2-\frac{1}{2}}}{x_1 x_2} \ . \tag{4.111}$$

We see that when $v_1 = v_2$, the integrand is symmetric in x_1 and x_2. Recall that this is one the constraints required to consistently freeze the gauge degrees of freedom on the D5-branes. Now, the full amplitude is given by

$$\begin{aligned} A_{1,1} &= \mathrm{Tr}\, I\ \overline{\lambda}_2^{1+,1+}\zeta_J B_2 B_1 \int_1^\infty \mathrm{d}x_1 \int_{x_1}^\infty \mathrm{d}x_2\ I(x_1,x_2)\ + \\ &\quad + \mathrm{Tr}\, I\ \overline{\lambda}_2^{1+,1+}\zeta_J B_1 B_2 \int_1^\infty \mathrm{d}x_1 \int_1^{x_1} \mathrm{d}x_2\ I(x_1,x_2)\ , \\ &= \mathrm{Tr}\, I\ \overline{\lambda}_2^{1+,1+}\zeta_J (B_2B_1 + B_1B_2) \int_1^\infty \mathrm{d}x_1 \int_{x_1}^\infty \mathrm{d}x_2\ I(x_1,x_2)\ , \\ &= \tfrac{1}{2}\mathrm{Tr}\, I\ \overline{\lambda}_2^{1+,1+}\zeta_J (B_2B_1 + B_1B_2) \int_1^\infty \mathrm{d}x_1 \int_1^\infty \mathrm{d}x_2\ I(x_1,x_2)\ , \\ &= \frac{(\alpha')^2}{2\mathrm{i}(\vartheta_1\vartheta_2)^{1/2}} \mathrm{B}(\tfrac{1}{2}-v_2, \tfrac{1}{2}+v_2)^2\ \tfrac{1}{2}\,\mathrm{Tr}\,(I\ \overline{\lambda}_2^{1+,1+}\zeta_J(B_2B_1+B_1B_2))\ . \end{aligned} \tag{4.112}$$

In the Seiberg-Witten noncommutative point-particle limit, we get

$$A_{1,1} = \frac{\pi \hat{b}^3}{4\mathrm{i}}\ \frac{1}{2}\,\mathrm{Tr}\,(I\ \overline{\lambda}_2^{1+,1+}\zeta_J(B_2B_1+B_1B_2))\ . \tag{4.113}$$

giving the E-term

$$E_{\lambda_2^{1+,1+}} = \tfrac{1}{2}J(B_2B_1+B_1B_2)I\ . \tag{4.114}$$

The same steps apply to general n_1, n_2 provided we constrain $v_1 = v_2$. Then, we get the following E-terms for $\lambda_2^{n_1+,n_2+}$:

$$E_{\lambda_2^{n_1+,n_2+}} = \frac{n_1!n_2!}{(n_1+n_2)!} J s_{n_1,n_2}(B_1,B_2) I\ , \tag{4.115}$$

where $s_{n_1,n_2}(B_1,B_2)$ is the totally symmetrized version of $B_1^{n_1}B_2^{n_2}$ with weight 1. For example

$$s_{2,1}(B_1, B_2) = B_1^2 B_2 + B_1 B_2 B_1 + B_2 B_1^2 \ . \tag{4.116}$$

The other fields λ^{n_1+,n_2-}, λ^{n_1-,n_2+} and λ^{n_1-,n_2-} cannot receive E-terms because there are no holomorphic fields which can soak up the quantum numbers of these fermis. $\square$

Chapter 5

Equivariant elliptic genus of spiked instanton moduli space

In this chapter, we compute an important observable of the spiked instanton gauged linear sigma model: the equivariant elliptic genus a.k.a the flavoured elliptic genus [BEOT1, BEOT2, GG, GGP1, GGP2], more familiarly known to physicists as the twisted index.

The study of elliptic genera for moduli spaces of gauge theories in diverse dimensions was initiated in [N7]. We obtain the elliptic version of the spiked instanton partition functions described in [N1, N3, N4, N6]. We also briefly study the structure of the index, leaving further details to the original works above. A more detailed version of the computation can be found in the forthcoming paper [P].

One can put the $\mathcal{N} = (0, 2)$ theory on a cylinder $\mathbf{S}^1 \times \mathbf{R}$ by imposing periodic boundary conditions for the fermions around $\mathbf{S}^1$. Consider the following twisted index, also known as the *equivariant elliptic genus*:

$$\mathcal{Z}(\tau, \xi) = \mathrm{Tr}_{\mathcal{H}}\ (-1)^{F_L+F_R}\, \mathrm{e}^{2\pi \mathrm{i} J_\xi} q^{H_R} \bar{q}^{H_L}\ , \tag{5.1}$$

Here, $\mathcal{H}$ is the state space of the theory on the cylinder, $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$ and $H_{L,R}$ are the Hamiltonians and $F_{L,R}$ are the fermion numbers in the left-moving and right-moving sectors respectively. $\mathrm{e}^{2\pi\mathrm{i}J_\xi} := \mathrm{e}^{2\pi\mathrm{i}\xi\cdot J}$ is an element in a torus (i.e. compact abelian) subgroup T of the group of rigid symmetries which commute with the $\mathcal{N} = (0, 2)$ superalgebra. One usually considers the maximal such torus subgroup. Due to supersymmetry in the left-moving sector, $\{\mathrm{Q}_+, \overline{\mathrm{Q}}_+\} = 2H_L$, only $H_L = 0$ states contribute to the above index rendering it independent of $\bar{\tau}$.

The expression (5.1) can be rewritten as a (euclidean) path integral of the theory on a torus with complex structure τ. We have

$$\mathcal{Z}(\tau,\xi) = \sum_{\varphi\in\mathcal{H}} \left\langle \varphi \left| (-1)^{F_L+F_R}\, e^{2\pi i \xi\cdot J}\, e^{2\pi i \tau_1 P}\, e^{-2\pi\tau_2 H} \right| \varphi \right\rangle \ . \tag{5.2}$$

where $\tau = \tau_1 + i\tau_2$, $H = H_L + H_R$ is the total Hamiltonian and $P = H_R - H_L$ is the generator of translations in the compact direction x. We recognise the (euclidean) time translation operator $e^{-2\pi\tau_2 H}$ on the right hand side. Choose coordinates (x,t) on the cylinder with $x \sim x + 2\pi$. Let the euclidean time be $\eta = it$ and $z = \frac{1}{2}(x + i\eta)$. The coordinates $x^{\pm\pm} = \frac{1}{2}(t \pm x)$ and derivatives $D_{\pm\pm}$ become

$$x^{++} \to \bar{z}\ , \quad x^{--} \to -z\ , \quad D_{++} \to D_{\bar{z}}\ , \quad D_{--} \to -D_z\ . \tag{5.3}$$

The field configurations on the cylinder, collectively denoted by $\varphi(\eta,x)$, are periodic in x:

$$\varphi(\eta, x+2\pi) = \varphi(\eta,x)\ . \tag{5.4}$$

The trace instructs us to sum over those field configurations which also satisfy twisted-periodic boundary conditions along η:

$$\varphi(\eta + 2\pi\tau_2, x) = e^{-2\pi i J_\xi} e^{-2\pi i \tau_1 P} \varphi(\eta,x) = e^{-2\pi i J_\xi} \varphi(\eta, x - 2\pi\tau_1)\ . \tag{5.5}$$

This corresponds to choosing spacetime to be a cylinder of length $2\pi\tau_2$ and with its ends identified after rotating one end by $2\pi\tau_1$, in other words a torus with complex structure τ. We can undo the twisted-periodic boundary conditions by the following trick. Choose coordinates (θ_1,θ_2) with periodicity 2π such that $x + i\eta = \theta_1 + \tau\theta_2$. The 1-cycles $\theta_2 = \text{const.}$ and $\theta_1 = \text{const.}$ shall be called the a and b cycles respectively. The two boundary conditions in (5.4), (5.5) correspond to

$$\begin{aligned} a:&\quad \varphi(\theta_1 + 2\pi, \theta_2) = \varphi(\theta_1,\theta_2)\ ,\\ b:&\quad \varphi(\theta_1, \theta_2 + 2\pi) = e^{-2\pi i J_\xi}\varphi(\theta_1,\theta_2)\ . \end{aligned} \tag{5.6}$$

The twisting along the b cycle can be undone by first weakly gauging the rigid symmetry

T and then performing the large T-gauge transformation $g(\theta_1,\theta_2)=\mathrm{e}^{\mathrm{i}\theta_2 J_\xi}$:

$$\varphi(\theta_1,\theta_2)\to {}^g\varphi(\theta_1,\theta_2)=\mathrm{e}^{\mathrm{i}\theta_2 J_\xi}\varphi(\theta_1,\theta_2)\ , \tag{5.7}$$

so that ${}^g\varphi$ satisfies periodic boundary conditions along both cycles:

$${}^g\varphi(\theta_1+2\pi,\theta_2)={}^g\varphi(\theta_1,\theta_2+2\pi)={}^g\varphi(\theta_1,\theta_2)\ . \tag{5.8}$$

The large gauge transformation also results in a constant background gauge field

$$v_{\theta_1}=0\ ,\quad v_{\theta_2}=g^{-1}\partial_{\theta_2}g=\mathrm{i}J_\xi\ , \tag{5.9}$$

which adds extra constant pieces to the covariant derivatives D_z and $\mathrm{D}_{\bar z}$:

$$\mathrm{D}_z\longrightarrow \mathrm{D}_z+\tfrac{\mathrm{i}}{2\tau_2}J_\xi\ ,\quad \mathrm{D}_{\bar z}\longrightarrow \mathrm{D}_{\bar z}-\tfrac{\mathrm{i}}{2\tau_2}J_\xi\ . \tag{5.10}$$

Thus, the path integral that calculates $\mathcal{Z}(\tau,\xi)$ in (5.1) is the partition function of the supersymmetric theory on a torus with complex structure τ with the background gauge field in (5.9). The path integral is given by

$$\mathcal{Z}(\tau,\xi)=\int[\mathrm{d}\varphi]\,e^{-\mathcal{S}[\varphi]}\ , \tag{5.11}$$

where φ collectively denotes all the fields which arise from the $\mathcal{N}=(0,2)$ chiral, fermi and gauge multiplets. $\mathcal{S}$ is the sum of (Wick-rotated) $\mathcal{N}=(0,2)$ actions for the various superfields:

$$\mathcal{S}=\mathcal{S}_{\text{gauge}}+\mathcal{S}_{\text{chiral}}+\mathcal{S}_{\text{fermi}}\ . \tag{5.12}$$

The index receives contribution only from states which satisfy $H_L=0$. Using $\{\mathrm{Q}_+\,,\overline{\mathrm{Q}}_+\}=2H_L$, it is easy to see that such states are precisely in the cohomology of $\overline{\mathrm{Q}}_+$ i.e. those states which are annihilated by $\overline{\mathrm{Q}}_+$ but cannot be written as $\overline{\mathrm{Q}}_+$ on another state.

What is the corresponding operator in superspace? We have the algebra of the gauge-covariant supercovariant derivatives:

$$\nabla_+^2 = 0\ ,\quad \overline{\nabla}_+^2 = 0\ ,\quad \{\nabla_+\,,\overline{\nabla}_+\} = 2\mathrm{i}\nabla_{\bar z}\ , \tag{5.13}$$

It turns out that we need to consider the cohomology of the operator $\overline{\nabla}_+$ rather than that of the superspace counterpart $\overline{\mathcal{Q}}_+$ of $\overline{\mathrm{Q}}_+$. A proof of this statement can be found in the thesis [De].

In euclidean space, the field strength becomes $v_{01} \to \mathrm{i}v_{01} =: F_{z\bar z}$ and the auxiliary field D also gets an extra i. Recall the following transformations:

$$\begin{aligned}
&\overline{\nabla}_+\lambda_- = 0\ , && \overline{\nabla}_+\phi_i = 0\ , && \overline{\nabla}_+\psi_{a-} = \sqrt{2}E_a\ ,\\
&\overline{\nabla}_+\bar\lambda_- = -\mathrm{i}D - F_{z\bar z}\ , && \overline{\nabla}_+\bar\phi^i = -\sqrt{2}\,\bar\zeta_+^i\ , && \overline{\nabla}_+\bar\psi_-^a = \sqrt{2}\bar G^a\ .
\end{aligned} \tag{5.14}$$

The field configurations in the cohomology of $\overline{\nabla}_+$ then satisfy

$$E_a(\phi_i) = 0\ ,\quad D = 0 = F_{z\bar z}\ ,\quad \bar\zeta_+^i = 0\ ,\quad \bar G^a = 0\ . \tag{5.15}$$

Further, by using the action of ∇_+, we also get $\mathrm{D}_{\bar z}\phi_i = 0$, $\mathrm{D}_{\bar z}\psi_{a-} = 0$ and $\mathrm{D}_{\bar z}\lambda_- = 0$ where $\mathrm{D}_{\bar z}$ is the ordinary space gauge-covariant derivative. The potential energy and the field equations for the auxiliary fields D and G_a are

$$\begin{aligned}
&V(\phi) = \sum_a (|J^a|^2 + |E_a|^2) + \frac{1}{g^2}\mathrm{Tr}\,D^2\ ,\\
&\bar G^a = -J^a\ ,\quad \frac{2}{g^2}D = \sum_i \phi_i\bar\phi^i - r\cdot\mathbb{1} =: \mu^{\mathbf{R}} - r\cdot\mathbb{1}\ .
\end{aligned}$$

The moduli space of classical vacua $\mathcal{M}_c$ is given by:

$$\mathcal{M}_c = \left\{(\phi_i,\bar\phi^i)\ \middle|\ E_a = 0\ ,\ J^a = 0\ ,\ \mu^{\mathbf{R}} = r\cdot\mathbb{1}\right\}\Big/\ G\ , \tag{5.16}$$

where G is the gauge group. Plugging the auxiliary field equations into (5.15), we see that the cohomology of $\overline{\nabla}_+$ consists of configurations that satisfy

$$\begin{gathered}\zeta_{i+}=0\ ,\quad \mathrm{D}_{\bar z}\psi_{a-}=0\ ,\quad \mathrm{D}_{\bar z}\lambda_-=0\ ,\quad F_{z\bar z}=0\ ,\ \mathrm{D}_{\bar z}\phi_i=0\ ,\\ E_a=0\ ,\quad J^a=0\ ,\quad \mu^{\mathbf{R}}=r\cdot\mathbb{1}\ .\end{gathered} \tag{5.17}$$

The last line in fact consists of the equations defining the classical moduli space of vacua of the theory. Thus, the index receives contributions only from (a subset of) configurations in the classical vacuum moduli space. Let us study the equations defining $\overline{\nabla}_+$-cohomology next.

5.1 $\overline{\nabla}_+$ Cohomology

First, we look at $\mathrm{D}_{\bar z}\phi_i=0$. We suppress the i index in the following. Write

$$\mathrm{D}_{\bar z}\phi=(\partial_{\bar z}+\mathrm{i}\mathcal{C})\phi=0\quad\text{with}\quad \mathcal{C}:=v_{\bar z}-\tfrac{1}{2\tau_2}J_\xi\ , \tag{5.18}$$

where $v_{\bar z}$ is a flat G-connection (since $F_{z\bar z}=0$) in the G-representation that ϕ belongs to. This has solution

$$\phi(z,\bar z)=\mathrm{e}^{-\mathrm{i}\mathcal{C}\bar z}\phi^0\quad\text{for some constant }\phi^0\ . \tag{5.19}$$

Imposing periodicity under $\bar z\to\bar z+1$, $\bar z\to\bar z+\bar\tau$, we get the conditions

$$\mathrm{e}^{-\mathrm{i}\mathcal{C}}\phi^0=\mathrm{e}^{-\mathrm{i}\mathcal{C}\bar\tau}\phi^0=\phi^0\ . \tag{5.20}$$

When $\tau_2\neq 0$, we have non-trivial solutions only when the matrix $\mathcal{C}=v_{\bar z}-\frac{1}{2\tau_2}J_\xi$ has zero eigenvalues and the solutions are $\phi(z,\bar z)=\phi^0\in\ker\mathcal{C}$. In the degenerate limit $\tau_2\to 0$, in order to retain the equivariant parameters ξ, we need to introduce a τ_2 dependence for ξ such that $\xi(\tau_2)\sim 2\tau_2$ in the limit $\tau_2\to 0$. In that case, when τ_1 is a rational number h/k with g.c.d$(h,k)=1$, we have non-trivial solutions when the above matrix has eigenvalues $2\pi nk$ for some $n\in\mathbf{Z}$. Thus, $\overline{\nabla}_+$-cohomology consists of the configurations

$$v_{\bar z}\ ,\quad (\phi_i)^0\ ,\quad (\psi_{a-})^0\ ,\quad (\lambda_-)^0\ , \tag{5.21}$$

which satisfy

$$F_{z\bar{z}} = 0 \ , \quad \mathcal{C}(\phi_i)^0 = 0 \ , \quad \mathcal{C}(\psi_{a-})^0 = 0 \ , \quad \mathcal{C}(\lambda_-)^0 = 0 \ , \tag{5.22}$$

where the matrix $\mathcal{C}$ is in the appropriate representation of G and T for each of the fields above.

5.1.1 Primer: ADHM equations

Let us study the $\overline{\nabla}_+$-cohomology of the $\mathcal{N} = (0, 2)$ sigma model which describes ordinary instanton moduli space $\mathcal{M}_{n,k}$. We studied this in the Introduction. The gauge group is $\mathrm{U}(k)$ and we have a rigid symmetry group $\mathrm{U}(n)$. The multiplets are

$$\text{Chirals: } B_1 \ , \ B_2 \ , \ I \ , \ J \ ; \quad \text{Fermis: } \lambda_- \ , \ \psi_2 \ , \ \psi_I \ , \ \psi_J \ , \tag{5.23}$$

with B_1, B_2, λ_- and ψ_2 in the adjoint of $\mathrm{U}(k)$, I, ψ_I in the $\boldsymbol{k} \times \overline{\boldsymbol{n}}$, and J, ψ_J in the $\overline{\boldsymbol{k}} \times \boldsymbol{n}$ of $\mathrm{U}(k) \times \mathrm{U}(n)$. The equations are

$$E_2 = [B_1, B_2] + IJ = 0 \ , \quad \mu^{\mathbf{R}} = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = r \cdot \mathbb{1}_k \ . \tag{5.24}$$

We saw in the D1-D5 analysis that r is proportional to the self-dual part of the B-field $v_1 + v_2$ with $v_1 v_2 > 0$. We can flip the sign of r by flipping the signs of v_1 and v_2 simultaneously. Recall that the fields I and $J^\dagger$ correspond to the open string states in the $\boldsymbol{k} \times \overline{\boldsymbol{n}}$ with energies

$$I : \ -\tfrac{1}{2}(v_1 + v_2) \ , \quad J : \ \tfrac{1}{2}(v_1 + v_2) \ . \tag{5.25}$$

When we take $r \to -r$, the fields I and $J^\dagger$ are exchanged. Hence, we can choose $r > 0$ without loss of generality. The rigid symmetries we consider are

1. **Framing rotations**: Let $g = \mathrm{e}^{-\mathrm{i} a_\alpha T_\alpha} \in \mathrm{U}(1)^n$, the maximal torus of $\mathrm{U}(n)$. Then,

$$I \to I g^{-1} \ , \quad J \to g J \ , \quad B_a \to B_a \ . \tag{5.26}$$

2. **Rotational invariance**: Let $(\mathrm{e}^{\mathrm{i}\epsilon_1 J_1}, \mathrm{e}^{\mathrm{i}\epsilon_2 J_2}) \in \mathrm{U}(1)^2$, arising from mutually commuting spatial rotations of the four dimensional support of the instanton. Then,

$$I \to \mathrm{e}^{\frac{\mathrm{i}}{2}(\epsilon_1+\epsilon_2)} I \ , \quad J \to \mathrm{e}^{\frac{\mathrm{i}}{2}(\epsilon_1+\epsilon_2)} J \ , \quad B_a \to \mathrm{e}^{\mathrm{i}\epsilon_a} B_a \ . \tag{5.27}$$

The derivative $\overline{\nabla}_+$ also transforms as

$$\overline{\nabla}_+ \to \mathrm{e}^{\frac{\mathrm{i}}{2}(\epsilon_1+\epsilon_2)}\, \overline{\nabla}_+ \ . \tag{5.28}$$

3. **R-symmetry**: The R-charge of $\overline{\nabla}_+$ is $+1$. The charges of the various superfields are as follows:

$$\begin{aligned} &[R\,, B_1] = 0 \ , \quad [R\,, B_2] = B_2 \ , \quad [R\,, I] = \tfrac{1}{2} I \ , \quad [R\,, J] = \tfrac{1}{2} J \ , \\ &[R\,, \lambda_-] = \lambda_- \ , \quad [R\,, \psi_2] = 0 \ , \quad [R\,, \psi_I] = \tfrac{1}{2}\psi_I \ , \quad [R\,, \psi_J] = \tfrac{1}{2}\psi_J \ . \end{aligned} \tag{5.29}$$

We have to choose the rigid symmetries such that they commute with the $\mathcal{N} = (0,2)$ superalgebra. This requires us to include an R-transformation with parameter $-\frac{1}{2}(\epsilon_1+\epsilon_2)$. Thus,

$$\xi = \{-a_1, \ldots, -a_n, \epsilon_1, \epsilon_2, -\tfrac{1}{2}(\epsilon_1+\epsilon_2)\} \ .$$

The compensating R-transformation vanishes when $\epsilon_1+\epsilon_2 = 0$. We proceed with this case.

Let the eigenvalues of the flat connection $v_{\bar{z}}$ in the $\boldsymbol{k}$ of $\mathrm{U}(k)$ be $\frac{1}{2\tau_2}\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$. Let us analyse the kernel of the matrix $\mathcal{C}$ for each of the fields above.

1. B_a. For the matrix element $(B_a)_i{}^j$, we have

$$i \neq j : \ \sigma_i - \sigma_j = \epsilon_a \ , \quad i = j : \ 0 = \epsilon_a \ , \tag{5.30}$$

Let $\epsilon_a \neq 0$. Then the diagonal components of B_a are zero. Since $(B_a)_i{}^j$ and $(B_a)_j{}^i$ correspond to $\sigma_i - \sigma_j = \pm\epsilon_a$ respectively, only one of them can be non-zero.

2. I and J. For the matrix elements $I_i{}^\alpha$ and $J_\alpha{}^i$ with $i \in \boldsymbol{k}$, $\alpha \in \boldsymbol{n}$, we have

$$\begin{aligned} I_i{}^\alpha : &\quad \sigma_i = a_\alpha + \tfrac{1}{2}\epsilon_1 + \tfrac{1}{2}\epsilon_2 = a_\alpha \ , \\ J_\alpha{}^i : &\ -\sigma_i = -a_\alpha + \tfrac{1}{2}\epsilon_1 + \tfrac{1}{2}\epsilon_2 = -a_\alpha \ . \end{aligned} \tag{5.31}$$

It can be shown that the condition $r > 0$ implies that $J = 0$ on the moduli space. By taking the trace of the real moment map, we see that $\mathrm{Tr}\, II^\dagger = kr$ indicating that atleast one component of I must be non-zero. This forces us to pick at least one equation in the I row in (5.31). It is straightforward to check that, to get a valid solution, the remaining σ_i have to be solved for using (5.30). This frees up the appropriate number of components of B_1 and B_2 such that the moment map equations are satisfied. The set of values that σ_i can take is precisely encoded in n-coloured partitions of k. That is, an n-tuple of partitions $\{\lambda_\alpha\}_{\alpha=1}^n$ with $|\lambda_\alpha| = k_\alpha$ and $\sum_\alpha k_\alpha = k$. Write a partition λ_α of k_α of length ℓ as

$$k_\alpha = \lambda_{\alpha 1} + \lambda_{\alpha 2} + \cdots + \lambda_{\alpha \ell} \quad \text{with} \quad \lambda_{\alpha 1} \geq \lambda_{\alpha 1} \geq \cdots \geq \lambda_{\alpha \ell} \geq 0 \ . \tag{5.32}$$

Then, for each string of k_α's and a partition λ_α of k_α, the values of σ_i are in the set

$$\bigcup_{\{k_\alpha\}} \bigcup_{\lambda_\alpha} \{a_\alpha + (m-1)\epsilon_1 + (n-1)\epsilon_2 \mid 1 \leq m \leq \ell(\lambda_\alpha) \ , 1 \leq n \leq \lambda_{\alpha m}\} \tag{5.33}$$

The locations of the above values can be written as an n-tuple of Ferrers diagrams by giving each of a_α, ϵ_1 and ϵ_2 small positive imaginary parts. This is to avoid coincident values of σ_i when ϵ_1 and ϵ_2 are not independent. An example is given in Figure 5.1 for our choice $\mathrm{Re}(\epsilon_1 + \epsilon_2) = 0$.

In this section, we have seen that the cohomology of $\overline{\nabla}_+$ is (a subspace of) the vacuum moduli space of the gauged linear sigma model. More precisely, the fixed points of the $\overline{\nabla}_+$ action coincides with the fixed points of the action of the torus group T in the vacuum moduli space (from $\mathrm{D}_{\bar{z}}\phi = 0$). The holonomies of the $\mathrm{U}(k)$ gauge field which belong to the $\overline{\nabla}_+$ cohomology contains information about these fixed points. Finally, as we discussed earlier, the twisted index localises on to (the T-invariant subspace of) the vacuum moduli space. Thus, the twisted index encodes geometric information about the vacuum moduli

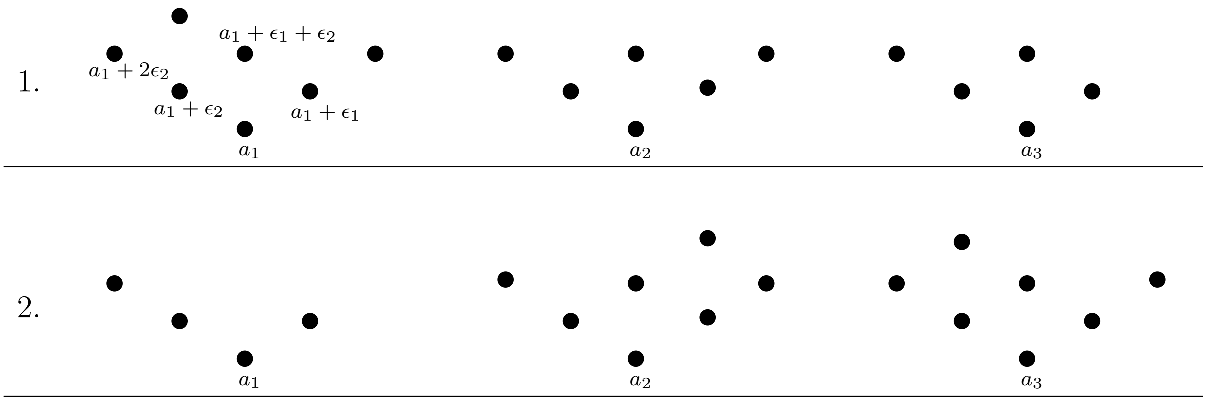

Figure 5.1: Two examples for the value set of $\{\sigma_i\}$ for $k = 18$, $n = 3$. Here, $\mathrm{Re}\,\epsilon_2 = -\mathrm{Re}\,\epsilon_1$.

space, including the action of various symmetries.

It turns out that the opposite is also true: starting from the moduli space and its symmetries, one can write down a path integral that computes precisely the twisted index above! In fact, this path integral coincides with the path integral of the $\mathcal{N} = (0, 2)$ gauge theory that we described above. This is the machinery of Cohomological Field Theory (CohFT) introduced by Witten [W5]. The canonical lift of a CohFT in dimension d which localises to the moduli space $\mathcal{M}$ of solutions to some PDE to the corresponding theory in dimension $d + 2$ which computes the elliptic genus of $\mathcal{M}$ is described in the paper [BLN].

A familiar example is the Donaldson-Witten CohFT in $d = 4$ which localises onto the moduli space of framed instantons and lifts to gauge theory in $d = 6$. The finite dimensional version of the $d = 4$ partition function is described in terms of the matrix model in $d = 0$ with ADHM moduli space as target -- this described the collective dynamics of instanton moduli. It lifts to the $d = 2$ theory in the fashion described in [BLN] and computes the equivariant elliptic genus of ADHM moduli space. We consider generalised versions of this where we are interested in the moduli space of spiked instantons.

The path integral of the gauged linear sigma model involves integrating over the σ_i since we must be integrate over the $\mathrm{U}(k)$ gauge field. It turns out that the integrand has poles in the σ_i which are located precisely at the fixed points described above. The advantage of this approach is that one can add $\overline{\nabla}_+$-exact terms to the action which do not change the answer but simplify the evaluation of the path integral.

5.2 Cohomological Field Theory

To make contact with the CohFT paradigm, we need to reduce the manifest supersymmetry to $\mathcal{N} = (0,1)$. Define the derivatives $\mathrm{D}_+, \mathrm{Q}_+$

$$\mathrm{D}_+ = \frac{\nabla_+ + \overline{\nabla}_+}{\sqrt{2}} \ , \ \mathrm{Q}_+ = \frac{\nabla_+ - \overline{\nabla}_+}{\sqrt{2}\mathrm{i}} \quad \text{with} \quad \mathrm{Q}_+^2 = \mathrm{D}_+^2 = \mathrm{i}\nabla_{\bar{z}} \ , \quad \{\mathrm{D}_+, \mathrm{Q}_+\} = 0 \ . \quad (5.34)$$

D_+ is the real $\mathcal{N} = (0,1)$ gauge-covariant supercovariant derivative and Q_+ is the generator of the extra (non-manifest) supersymmetry. The $(0,2)$ chiral and fermi multiplets (and their antichiral counterparts) become complex $(0,1)$ scalar and fermi multiplets with components

$$\begin{aligned} \text{Chiral}: \quad & \phi_i \ , \quad \mathrm{D}_+\phi_i = \zeta_{i+} \ , & \text{Fermi}: \quad & \psi_{a-} \ , \quad \mathrm{D}_+\psi_{a-} = G_a + E_a =: F_a \ , \\ \text{Antichiral}: \quad & \bar{\phi}^i \ , \quad \mathrm{D}_+\bar{\phi}^i = -\bar{\zeta}^i_+ \ , & \text{Antifermi}: \quad & \bar{\psi}^a_- \ , \quad \mathrm{D}_+\bar{\psi}^a_- = \overline{G}_a + \overline{E}_a =: \overline{F}_a \ . \end{aligned}$$

The $(0,2)$ field strength fermi multiplet splits up into two hermitian $(0,1)$ fermi multiplets λ^D_- and λ^F_-, one containing the auxiliary field D and the other containing the field strength:

$$\lambda^F_- = -\tfrac{1}{\sqrt{2}}(\mathcal{F}_- + \overline{\mathcal{F}}_-) \ , \quad \mathrm{D}_+\lambda^F_- = F_{z\bar{z}} \ ; \qquad \lambda^D_- = \tfrac{1}{\sqrt{2}\mathrm{i}}(\mathcal{F}_- - \overline{\mathcal{F}}_-) \ , \quad \mathrm{D}_+\lambda^D_- = D \ . \quad (5.35)$$

We have $\nabla_+\overline{\nabla}_+ = -\mathrm{i}\mathrm{D}_+\mathrm{Q}_+ + \mathrm{i}\nabla_{\bar{z}}$. We can discard the second term since it gives rise to a total derivative term. Using that Q_+ acts as $-\mathrm{i}\mathrm{D}_+$ on superfields satisfying $\overline{\nabla}_+ = 0$, we can write the $(0,2)$ actions in $(0,1)$ superspace:

$$\begin{aligned} \mathcal{S}_{\text{chiral}} &= \frac{\mathrm{i}}{2}\int \mathrm{d}^2x\, \mathrm{D}_+ \left(\bar{\zeta}^i_+ \nabla_z\phi_i + \bar{\phi}^i\nabla_z\zeta_{i+} + 2\mathrm{i}\bar{\phi}^i\lambda^D_-\phi_i\right) \ , \\ \mathcal{S}_{\text{fermi}} &= \int \mathrm{d}^2x\, \mathrm{D}_+ \left(\bar{\psi}^a_- \left(\tfrac{1}{2}F_a - \mu_a\right) + \left(\tfrac{1}{2}\overline{F}^a - \bar{\mu}^a\right)\psi_{a-}\right) \ , \\ \mathcal{S}_{\text{gauge}} &= \frac{1}{g^2}\int \mathrm{d}^2x\, \mathrm{D}_+ \,\mathrm{Tr}\left(\lambda^D_-(D + g^2 r) + \lambda^F_-\left(F_{z\bar{z}} + \tfrac{g^2}{2\pi}\theta\right)\right) \ , \end{aligned}$$

where $\mu_a = E_a + \bar{J}_a$. The moduli space of vacua is then

$$\mathcal{M}_c = \left\{ \phi_i \,, \bar{\phi}^i \;\middle|\; \mu_a = 0 \;,\; \mu^{\mathbf{R}} - r \cdot \mathbb{1} = 0 \right\} \; / \; G \;. \tag{5.36}$$

The CohFT formalism computes observables associated to a cohomology theory of the moduli space $\mathcal{M}$ defined by the triple of $\{\textsc{Fields}\,, \textsc{Equations}\,, \textsc{Symmetries}\}$:

$$\mathcal{M} = \left\{ \textsc{Fields} \;\middle|\; \textsc{Equations} = 0 \right\} \Big/ \textsc{Symmetries} \;. \tag{5.37}$$

In our context, the equivariant elliptic genus probes the elliptic cohomology of the moduli space of classical vacua defined in (5.36). The triple is described as follows:

1. The set FIELDS consists of the scalars ϕ_i, $\bar{\phi}^i$.
2. EQUATIONS is given by the equations in (5.36) defining the moduli space of classical vacua:
$$\textsc{Equations} = \{\mu_a, \quad \mu^{\mathbf{R}} - r \cdot \mathbb{1}\} \;.$$
3. SYMMETRIES correspond to the gauge group G with Lie algebra valued parameter $v_{\bar{z}}$. The notation for the parameter will become clear in a moment.
4. RIGID SYMMETRIES: There are also rigid symmetries in the theory with which we can work equivariantly.
   (a) A torus subgroup of the group of internal rigid symmetries acting on the fields ϕ_i with constant parameters ξ and generators J:
$$\delta\phi_j = (\mathrm{i}\xi \cdot J)\phi_j \;. \tag{5.38}$$
   (b) There are also the compact $\bar{z}$-translations $\bar{z} \to \bar{z} + \bar{c}$ with $\bar{c} \in \mathbf{C}/(\mathbf{Z} + \tau\mathbf{Z})$ which act as
$$\delta_c\phi_i = \bar{c}\,\partial_{\bar{z}}\phi_i \;. \tag{5.39}$$

To complete the description of the cohomological field theory, we introduce additional fields and a fermionic symmetry δ_s such that $\delta_s^2 =$ gauge + rigid symmetries.

1. We define the action of δ_s on the parameters $v_{\bar{z}}$ and $\xi \cdot J$ to be

$$\delta_s v_{\bar{z}} = 0 \ , \quad \delta_s(\xi \cdot J) = 0 \ . \tag{5.40}$$

2. For each field ϕ_j in Fields, introduce a fermionic field ζ_j with the same quantum numbers as ϕ_j such that

$$\begin{aligned} &\delta_s\phi_i = \zeta_i \ , \quad \delta_s\zeta_i = \mathrm{i}(\partial_{\bar{z}} + \mathrm{i}v_{\bar{z}} - \tfrac{\mathrm{i}}{2\tau_2}\xi \cdot J)\phi_i =: \mathrm{i}\mathrm{D}_{\bar{z}}\phi_i \ , \\ &\delta_s\overline{\phi}^i = -\overline{\zeta}^i \ , \quad \delta_s(-\overline{\zeta}^i) = \mathrm{i}(\partial_{\bar{z}} - \mathrm{i}v_{\bar{z}} + \tfrac{\mathrm{i}}{2\tau_2}\xi \cdot J)\overline{\phi}^i =: \mathrm{i}\mathrm{D}_{\bar{z}}\overline{\phi}^i \ . \end{aligned} \tag{5.41}$$

3. For each equation μ_a in Equations, introduce a doublet (ψ_a, F_a) with ψ_a fermionic and F_a bosonic, and having the same quantum numbers as μ such that

$$\delta_s\psi_a = F_a \ , \quad \delta_s F_a = \mathrm{i}\mathrm{D}_{\bar{z}}\psi_a \ , \quad \delta_s\overline{\psi}^a = \overline{F}^a \ , \quad \delta_s\overline{F}^a = \mathrm{i}\mathrm{D}_{\bar{z}}\overline{\psi}^a \ . \tag{5.42}$$

   We include the real moment map $\mu^{\mathbf{R}} = \mu_0$ in the above discussion with $F_0 = D$, $\psi_0 = \lambda^D$ such that these are real.

4. From the transformation rules above, we see that $v_{\bar{z}}$ plays the role of a connection. In the theory with Minkowski signature, $v_{\bar{z}}$ becomes the right-moving part of the gauge field. It is necessary to include a left-handed component v_z as well. We introduce a new fermion λ^F and define

$$\delta v_z = \lambda^F \ , \quad \delta_s\lambda^F = -\mathrm{D}_z v_{\bar{z}} + \partial_{\bar{z}} v_z = -F_{z\bar{z}} \ . \tag{5.43}$$

   The first term is the gauge transformation of v_z with parameter $v_{\bar{z}}$ and the second comes from $\bar{z}$-translations. As we can see, these combine to give the field strength $F_{z\bar{z}}$ of the gauge field. Further, we have

$$\delta_s F_{z\bar{z}} = -\delta_s^2\lambda^F = -\mathrm{i}(\partial_{\bar{z}}\lambda^F + \mathrm{i}[v_{\bar{z}}, \lambda^F]) = -\mathrm{i}\mathrm{D}_{\bar{z}}\lambda^F \ . \tag{5.44}$$

For the action of the cohomological field theory we choose an expression of the form $\delta_s\Psi$ (in addition to θ-terms) such that the kinetic energy terms for all the fields are non-degenerate.

$$\mathcal{S}_{\text{CohFT}} = \int \mathrm{d}^2z\ \delta_s(\Psi_{\text{symm.}} + \Psi_{\text{eqnn.}}) + \frac{\mathrm{i}\theta}{2\pi}\int \mathrm{d}^2z\,\mathrm{Tr}\,F_{z\bar{z}}\ , \tag{5.45}$$

where

$$\Psi_{\text{symm.}} = \frac{\mathrm{i}}{2}\sum_j \left[\bar{\phi}^j \mathrm{D}_z\zeta_j + \bar{\zeta}^j\mathrm{D}_z\phi_j\right] + \frac{1}{g^2}\mathrm{Tr}\big(\lambda^F F_{z\bar{z}}\big)\ , \tag{5.46}$$

$$\Psi_{\text{eqnn.}} = \frac{1}{g^2}\mathrm{Tr}\big(\lambda^D(D - g^2\mu^{\mathbf{R}} + g^2 r)\big) + \sum_a \left[\bar{\psi}^a\big(\tfrac{1}{2}F_a - \mu_a\big) + \big(\tfrac{1}{2}\bar{F}^a - \bar{\mu}^a\big)\psi_a\right]\ , \tag{5.47}$$

are the gauge fermions corresponding to SYMMETRIES and EQUATIONS. After Wick rotating back to Minkowski space, we see that the above action matches exactly with the $(0,1)$ actions above and the fermionic symmetry δ_s is in fact identical to the $\mathcal{N} = (0,1)$ supercharge D_+. The condition $\delta_s(\xi \cdot J) = 0$ corresponds to choosing the torus subgroup T to be one which commutes with the supercharges ∇_+ and $\overline{\nabla}_+$.

Localisation

As shown by Witten, the path integral in a cohomological field theory localises onto the δ_s-invariant field configurations. These configurations satisfy

$$\begin{aligned} &\zeta_i = 0\ , \quad \mathrm{D}_{\bar{z}}\phi_i = 0\ , \quad F_a = 0\ , \quad \mathrm{D}_{\bar{z}}\psi_a = 0\ , \\ &D = 0\ , \quad \mathrm{D}_{\bar{z}}\lambda^D = 0\ , \quad \mathrm{D}_{\bar{z}}\lambda^F = 0\ , \quad F_{z\bar{z}} = 0\ . \end{aligned} \tag{5.48}$$

These equations are identical to the ones from $\overline{\nabla}_+(\cdot) = 0$ in (5.17) after solving for the auxiliary field equations and also by using the identity $E_aJ^a = 0$.

5.3 Computing the path integral

We start with the CohFT version of the $\mathcal{N} = (0,2)$ theory. Starting here, we fix the gauge group to be $\mathrm{U}(k)$ for simplicity. The same analysis can be applied to other groups.

Recall that the Lagrangian can be written as $\delta_s\Psi$ upto θ-angle terms. The gauge fermion Ψ is given by $\Psi = \Psi_{\text{symm.}} + \Psi_{\text{eqnn.}}$ with

$$\Psi_{\text{symm.}} = \frac{\mathrm{i}}{2}\sum_j \left[\bar{\phi}^j \mathrm{D}_z\zeta_j + \bar{\zeta}^j \mathrm{D}_z\phi_j\right] + \frac{1}{g^2}\mathrm{Tr}\big(\lambda^F F_{z\bar{z}}\big) \ , \tag{5.49}$$

$$\Psi_{\text{eqnn.}} = \mathrm{Tr}\big(\lambda^D(\tfrac{1}{g^2}D - \mu^{\mathbf{R}} + r)\big) + \sum_a \left[\bar{\psi}^a\big(\tfrac{1}{2}F_a - \mu_a\big) + \big(\tfrac{1}{2}\bar{F}^a - \bar{\mu}^a\big)\psi_a\right] \ . \tag{5.50}$$

We next add the following terms to the gauge fermion:

$$\Delta\Psi = g_1 \mathrm{Tr}\,(\lambda^D D) + g_2\bar{\psi}^a F_a \ . \tag{5.51}$$

which give the following terms in the Lagrangian:

$$g_1\,\mathrm{Tr}(D^2 - \mathrm{i}\lambda^D \mathrm{D}_{\bar{z}}\lambda^D) + g_2\,(\bar{F}^a F_a - \mathrm{i}\bar{\psi}^a \mathrm{D}_{\bar{z}}\psi_a) \ . \tag{5.52}$$

In the limit $g_1 \to \infty$, the other terms in the action involving D and λ^D are negligible and we can perform the path integral over D to set $D = 0$. The path integral over λ^D becomes gaussian and gives

$$\sqrt{\mathrm{Det}'\,(\partial_{\bar{z}} + \mathrm{i}[v_{\bar{z}}\,,\cdot])} \ . \tag{5.53}$$

The factors of g_1 cancel between the integration of D and λ^D since one is bosonic and the other is fermionic. There are zero modes for the fermion λ_D which do not transform under any of the rigid symmetries. This will render the determinant equal to zero. The prime on Det' is to indicate that we have removed the zero modes. Since the coupling constant g^2 appears inside a δ_s-exact term, we may evaluate the integral by taking $g^2 \to 0$. In this limit, the integral over gauge fields localises on to the space of flat connections. The adjoint fermion zero modes and the $g^2 \to 0$ limit have been treated systematically in [BEOT1, BEOT2].

Flat connections are parametrized by their holonomies around the two cycles of the torus. A convenient gauge choice is $v_z = 0$. By z-dependent gauge transformations, one can set $v_{\bar{z}} =$ constant and rotate it into the Cartan subalgebra of $\mathrm{U}(k)$. The holonomies are then parametrised by the constant eigenvalues of $v_{\bar{z}}$ in the fundamental representation of

$\mathrm{U}(k)$. There is still freedom due to Weyl transformations which permutes the eigenvalues. Thus, one has to divide the answer by the order of the Weyl group.

Similarly, in the $g_2 \to \infty$ limit, the auxiliary field F_a can be set to zero we get the following determinant from the F, ψ integration:

$$\prod_a \mathrm{Det}(\partial_{\bar{z}} + \mathrm{i}v_{\bar{z}}^{(a)} - \tfrac{\mathrm{i}}{2\tau_2} J_\xi^{(a)}) \ . \tag{5.54}$$

Here, $v_{\bar{z}}^{(a)}$ and $J_\xi^{(a)}$ are taken to be in the representations of $\mathrm{U}(k)$ and T that ψ_a belongs to. We notice that the terms involving the moduli space equations have completely dropped out of the action! The remaining terms for the fluctuating fields are:

$$\mathcal{S} = \int \mathrm{d}^2x \left[-\mathrm{i}\bar{\zeta}^j \mathrm{D}_z \zeta_j + \bar{\phi}^j \mathrm{D}_z \mathrm{D}_{\bar{z}} \phi_j - \tfrac{1}{g^2} \mathrm{Tr}\,(\mathrm{i}\lambda^F \mathrm{D}_{\bar{z}} \lambda^F) \right] \ . \tag{5.55}$$

Each of the above are quadratic actions for the various fields. The path integral then gives

$$\sqrt{\mathrm{Det}'(\partial_{\bar{z}} + \mathrm{i}[v_{\bar{z}}\,,\cdot])}\ \prod_i \frac{\mathrm{Det}(\partial_z + \frac{\mathrm{i}}{2\tau_2} J_\xi^{(i)})}{\mathrm{Det}(\partial_z + \frac{\mathrm{i}}{2\tau_2} J_\xi^{(i)}) \mathrm{Det}(\partial_{\bar{z}} + \mathrm{i}v_{\bar{z}}^{(i)} - \frac{\mathrm{i}}{2\tau_2} J_\xi^{(i)})} \ . \tag{5.56}$$

For generic values of the equivariant parameters ξ, the determinant for $\partial_z + \frac{\mathrm{i}}{2\tau_2} J_\xi^{(i)}$ cancels between the numerator and denominator. This is a consequence of supersymmetry in the left-moving sector.

The final integrand for the integration over the holonomies of the gauge field is

$$\mathrm{Det}'(\partial_{\bar{z}} + \mathrm{i}[v_{\bar{z}}\,,\cdot])\ \frac{\prod_a \mathrm{Det}(\partial_{\bar{z}} + \mathrm{i}v_{\bar{z}}^{(a)} - \frac{\mathrm{i}}{2\tau_2} J_\xi^{(a)})}{\prod_i \mathrm{Det}(\partial_{\bar{z}} + \mathrm{i}v_{\bar{z}}^{(i)} - \frac{\mathrm{i}}{2\tau_2} J_\xi^{(i)})} \tag{5.57}$$

Each of the determinants can be calculated either in the path integral by a suitable regularisation or in the Hamiltonian formalism. We choose the latter. The determinant with periodic boundary conditions around the two cycle of the torus in the presence of the flat connection $v_{\bar{z}} - \frac{1}{2\tau_2} J_\xi$ is nothing but the twisted index (5.1) evaluated over the appropriate right-moving Fock spaces! The expression for the index is

$$\mathrm{Tr}_{\mathcal{H}}\ (-1)^{F_R}\,\mathrm{e}^{2\pi\mathrm{i}(-2\tau_2 v_{\bar{z}}+J_\xi)}q^{H_R}\ . \tag{5.58}$$

We need the following expression for a particular Jacobi θ function:

$$\theta_1(\tau|z)=\mathrm{i}\eta(\tau)\mathrm{e}^{\pi\mathrm{i}z}q^{1/12}\prod_{n=1}^{\infty}(1-q^n\mathrm{e}^{2\pi\mathrm{i}z})(1-q^{n-1}\mathrm{e}^{-2\pi\mathrm{i}z})\ , \tag{5.59}$$

and also for the ratio

$$\frac{\theta_1(\tau|z)}{\mathrm{i}\eta(\tau)}=:\Theta(z)\ . \tag{5.60}$$

Note that $\theta_1(\tau|z)$, and consequently $\Theta(z)$, has a simple zero at $z=\mathbf{Z}+\tau\mathbf{Z}$.

Let us calculate the first determinant in (5.57). Recall that it arises from the path integral over the adjoint fermions λ^D and λ^F after excluding their zero modes. Let the eigenvalues of $v_{\bar{z}}$ taken in the fundamental representation of $\mathrm{U}(k)$ be $\frac{1}{2\tau_2}\{\sigma_1,\dots,\sigma_k\}$ and let

$$y_i=\exp(-2\pi\mathrm{i}\sigma_i)\ .$$

Let us consider the complex combination $\lambda=\mathrm{i}\lambda^D-\lambda^F$. Then, the component $\lambda_i{}^j$ transforms with gauge parameter $\sigma_i-\sigma_j$ due to the commutator $[v_{\bar{z}}\,,\lambda]$ and it receives a contribution $y_i y_j^{-1}$ in the above trace. The trace for each diagonal component $\lambda_i{}^i$ (with zero modes removed) is

$$q^{2/24}(1-q)^2(1-q^2)^2\cdots=\eta(\tau)^2\ . \tag{5.61}$$

The prefactor $q^{2/24}$ arises from the zero-point energy for a single complex fermion that is periodic along the spatial direction of the torus. This arose in Chapter 2 (equation (2.81)) where we considered more general boundary conditions along the spatial direction. Thus, the total determinant for the diagonal components of λ is $\eta(\tau)^{2k}$. For an off-diagonal component $\lambda_i{}^j$, $i \neq j$, we get

$$q^{2/24}(y_i^{1/2}y_j^{-1/2} - y_i^{-1/2}y_j^{1/2})\prod_{n=1}^{\infty}(1 - y_i y_j^{-1}q^n)(1 - y_i^{-1}y_j q^n) = \frac{\theta_1(\tau|\sigma_j - \sigma_i)}{\mathrm{i}\eta(\tau)} = \Theta(\sigma_j - \sigma_i)\ . \tag{5.62}$$

The first factor is the zero point energy for a single complex fermion. The second factor comes from the zero mode of the complex fermion $\lambda_i{}^j$ which gives rise to a two-dimensional ground state. One state is bosonic and one is fermionic and they pick up factors $(y_i y_j^{-1})^{\pm 1/2}$ respectively. The rest are contributions from non-zero modes of $\lambda_i{}^j$ and their complex conjugates. Thus, the full contribution from the complex fermion λ (with zero modes removed from the diagonal components) is

$$\mathrm{Det}'(\partial_{\bar{z}} + \mathrm{i}[v_{\bar{z}}\,,\cdot]) = \eta(\tau)^{2k}\prod_{i\neq j}\Theta(\sigma_j - \sigma_i) \tag{5.63}$$

In fact, this is the full contribution of the $\mathcal{N} = (0,2)$ vector multiplet (cf. [BEOT2] and references therein).

The rest of the determinants can be derived in a similar fashion once we specify the representations of the various matter fields and equations of the theory. We specialise to the case of spiked instantons from now on.

5.4 Elliptic genus for spiked instantons

Let us specify the Fields, Equations and Symmetries for the spiked instanton moduli space that we described both in the Introduction and in Chapter 4. The fields from the $\overline{\mathrm{D1}}$-$\overline{\mathrm{D1}}$ strings are

$$\begin{aligned} &B_1, B_2, B_3, B_4 \ :\ \text{in the adjoint of } \mathrm{U}(k)\ , \\ &I_A,\ J_A\ :\ \text{in the } \boldsymbol{k}\times\overline{\boldsymbol{n}}_{\boldsymbol{A}} \text{ and } \overline{\boldsymbol{k}}\times\boldsymbol{n}_{\boldsymbol{A}} \text{ of } \mathrm{U}(k)\times\mathrm{U}(n_A) \text{ for } A\in\underline{\mathbf{6}}\ . \end{aligned} \tag{5.64}$$

There are additional infinite towers of fields

The equations are

1. The real moment map:

$$\mu_{\mathbf{R}} - r \cdot \mathbb{1}_k := \sum_{a \in \underline{\mathbf{4}}} [B_a, B_a^\dagger] + \sum_{A \in \underline{\mathbf{6}}} (I_A I_A^\dagger - J_A^\dagger J_A) - r \cdot \mathbb{1}_k = 0 \ . \tag{5.65}$$

2. For $A = (ab) \in \underline{\mathbf{6}}$ with $a < b$,

$$\mu_A^{\mathbf{C}} := [B_a, B_b] + I_A J_A = 0 \ . \tag{5.66}$$

3. For $A \in \underline{\mathbf{6}}$, $\bar{A} = \underline{\mathbf{4}} \smallsetminus A$ and $\bar{a} \in \bar{A}$,

$$\sigma_{\bar{a}A}^{\mathbf{C}} := B_{\bar{a}} I_A = 0 \ , \quad \tilde{\sigma}_{\bar{a}A}^{\mathbf{C}} := J_A B_{\bar{a}} = 0 \ . \tag{5.67}$$

4. For $A \in \underline{\mathbf{6}}$, $\bar{A} = \underline{\mathbf{4}} \smallsetminus A$,

$$\Upsilon_A^{\mathbf{C}} := J_{\bar{A}} I_A = 0 \ . \tag{5.68}$$

5. For $A, B \in \underline{\mathbf{6}}$ such that $A \cap B = \{c\} \in \underline{\mathbf{4}}$, and $n = 0,\, 1,\, \ldots$

$$\Upsilon_{A,B,n} := J_A (B_c)^n I_B = 0 \ . \tag{5.69}$$

6. For $A \in \underline{\mathbf{6}}$, $A = (ab)$ with $a < b$, and $m,\, n = 0,\, 1,\, 2,\, \ldots$

$$\Upsilon_{A,m,n} := J_A (B_a)^m (B_b)^n I_A = 0 \ . \tag{5.70}$$

To recast the above equations into the CohFT form, we define $\underline{\mathbf{3}} = \{(12), (13), (14)\}$. Take the following combinations of the complex equations above:

$$\begin{aligned} s_A &:= \mu_A^{\mathbf{C}} + \varepsilon_{A\bar{A}} \left(\mu_{\bar{A}}^{\mathbf{C}}\right)^\dagger = 0 \ , \quad \text{for} \quad A \in \underline{\mathbf{3}} \ , \\ \sigma_{\bar{a}A} &:= \sigma_{\bar{a}A}^{\mathbf{C}} + \varepsilon_{\bar{a}\bar{b}A} \left(\tilde{\sigma}_{\bar{b}A}^{\mathbf{C}}\right)^\dagger = 0 \ , \quad \text{for} \quad A \in \underline{\mathbf{6}} \ , \quad \bar{a} \in \bar{A} \ , \\ \Upsilon_A &:= \Upsilon_A^{\mathbf{C}} - \varepsilon_{A\bar{A}} \left(\Upsilon_{\bar{A}}^{\mathbf{C}}\right)^\dagger = 0 \quad \text{for} \quad A \in \underline{\mathbf{3}} \ . \end{aligned} \tag{5.71}$$

Here, ε is the totally antisymmetric symbol in four indices ε_{abcd}. For example, when $A = (12)$, we have $\varepsilon_{A\bar{A}} = \varepsilon_{1234} = +1$.

There is a $\mathrm{U}(k)$ gauge invariance which acts in the appropriate representations on the fields and equations. The rigid symmetries were listed in Chapter 4 and are given by

$$\mathrm{P}\left(\bigtimes_{A\in\underline{\mathbf{6}}}\mathrm{U}(n_A)\right)\times\mathrm{U}(1)^3\ . \tag{5.72}$$

The $\mathrm{U}(1)^3$ is given by the mutually commuting rotations F_{12}, F_{34}, F_{56}, F_{78} with parameters ϵ_1, ϵ_2, ϵ_3, ϵ_4 satisfying

$$\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4=0\ . \tag{5.73}$$

This constraint is required since the supercharges of the $\mathcal{N}=(0,2)$ algebra transform with the phase $\mathrm{e}^{\frac{\mathrm{i}}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4)}$ and the symmetries one considers are those which commute with the $(0,2)$ algebra.

Thus, the equivariant parameters along with their exponentiated versions are

$$\xi=\left\{\bigcup_A\{-a_{m,A}\}_{m=1}^{n_A}\,,\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\right\}\ ,\qquad \mathrm{e}^{2\pi\mathrm{i}\xi}=\left\{\bigcup_A\{x_{m,A}\}_{m=1}^{n_A}\,,q_1,q_2,q_3,q_4\right\}\ . \tag{5.74}$$

We now calculate the various determinants. First, let us look at the fields. The calculation of the index for the B_a, $a\in\underline{\mathbf{4}}$ proceeds exactly as for the complex adjoint fermion in the previous section. There is an additional factor of q_a in the trace from the $\mathrm{U}(1)^3$ rotations. The result for a diagonal component of B_a is

$$\frac{1}{q^{2/24}(q_a^{1/2}-q_a^{-1/2})(1-q_aq)(1-q_a^{-1}q)(1-q_aq^2)(1-q_a^{-1}q^2)\cdots}=\frac{\mathrm{i}\eta(\tau)}{\theta_1(\tau|\epsilon_a)}=\frac{1}{\Theta(\epsilon_a)}\ . \tag{5.75}$$

The contributions are in the denominator since the Fock space is bosonic. The first factor is the zero-point energy for a complex boson with periodic boundary conditions in the spatial direction. The second factor is due to the zero modes. The remaining are the non-zero modes from B_a and its hermitian conjugate field $B_a^\dagger$. Similarly, an off-diagonal component $(B_a)_i{}^j$, $i\neq j$, gives

$$\frac{\mathrm{i}\eta(\tau)}{\theta_1(\tau|\epsilon_a-\sigma_i+\sigma_j)}=\frac{1}{\Theta(\epsilon_a+\sigma_j-\sigma_i)}\ . \tag{5.76}$$

The total contribution for B_a is

$$B_a : \prod_{i,j} \frac{1}{\Theta(\epsilon_a + \sigma_j - \sigma_i)} . \tag{5.77}$$

Let us look at the bifundamental fields next. The action of $\mathrm{U}(1)^3$ can be read off from the covariant weights in Tables 4.3 and 4.4:

$$I_{(ab)} \to \mathrm{e}^{\mathrm{i}(-v_a+\frac{1}{2})\epsilon_a+\mathrm{i}(-v_b+\frac{1}{2})\epsilon_b} I_{(ab)} , \quad J_{(ab)} \to \mathrm{e}^{\mathrm{i}(v_a+\frac{1}{2})\epsilon_a+\mathrm{i}(v_b+\frac{1}{2})\epsilon_b} J_{(ab)} . \tag{5.78}$$

Let $\theta_a = \frac{1}{2} - v_a$ and $\bar{\theta}_a = \frac{1}{2} + v_a$. Then, the index for the components $(I_A)_i{}^{m_A}$, $(J_A)_{m_A}{}^i$ is given by

$$(I_A)_i{}^m : \frac{1}{\Theta(\theta_a\epsilon_a + \theta_b\epsilon_b + a_{m,A} - \sigma_i)} , \quad (J_A)_m{}^i : \frac{1}{\Theta(\bar{\theta}_a\epsilon_a + \bar{\theta}_b\epsilon_b - a_{m,A} + \sigma_i)} . \tag{5.79}$$

In the Seiberg-Witten point-particle limit, the $v_a \to \frac{1}{2}$ as we saw at the end of Chapter 4. This gives the index

$$I_A : \prod_{m\in[n_A],i} \frac{1}{\Theta(a_{m,A} - \sigma_i)} , \quad J_A : \prod_{m\in[n_A],i} \frac{1}{\Theta(\epsilon_a + \epsilon_b - a_{m,A} + \sigma_i)} . \tag{5.80}$$

Here, $[n_A]$ is the set of labels $\{1, 2, \ldots, n_A\}$.

Next, let us look at the equations. The fermion for the real moment map is λ^D which has been dealt with in the previous section. Under $\mathrm{U}(1)^3$ the various equations transform as follows:

$$\begin{aligned} &s_A \to \mathrm{e}^{\mathrm{i}(\epsilon_a+\epsilon_b)} s_A , \quad \Upsilon_A = \mathrm{e}^{\mathrm{i}(\epsilon_{\bar{a}}+\epsilon_{\bar{b}})} \Upsilon_A , \quad \sigma_{\bar{a}A} \to \mathrm{e}^{\mathrm{i}\epsilon_{\bar{a}}} \sigma_{\bar{a}A} , \\ &\Upsilon_{A,B,n} \to \mathrm{e}^{\mathrm{i}(\epsilon_a+\epsilon_b+n\epsilon_c)} \Upsilon_{A,B,n} , \quad \Upsilon_{A,m,n} \to \mathrm{e}^{\mathrm{i}(m+1)\epsilon_a+\mathrm{i}(n+1)\epsilon_b} \Upsilon_{A,m,n} . \end{aligned} \tag{5.81}$$

The index contributions for the corresponding complex fermions are

$$s_A : \prod_{i,j} \Theta(\epsilon_a + \epsilon_b - \sigma_i + \sigma_j) \ , \qquad \Upsilon_A : \prod_{\overline{m}\in[n_{\overline{A}}], m\in[n_A]} \Theta(\epsilon_{\bar{a}} + \epsilon_{\bar{b}} - a_{\overline{m}} + a_m) \ ,$$
$$\sigma_{\bar{a}A} : \prod_{m\in[n_A],i} \Theta(\epsilon_{\bar{a}} - \sigma_i + a_{m,A}) \ , \qquad \Upsilon_{A,B,n} : \prod_{\substack{m\in[n_A]\\ m'\in[n_B]}} \Theta(\epsilon_a + \epsilon_b + n\epsilon_c - a_{m,A} + a_{m',B}) \ . \tag{5.82}$$

For the equations $\Upsilon_{A,m,n}$ we have

$$\Upsilon_{A,m,n} : \prod_{p,p'\in[n_A]} \Theta((m+1)\epsilon_a + (n+1)\epsilon_b - a_{p,A} + a_{p',A}) \ . \tag{5.83}$$

Thus, the integrand of the integration over holonomies y_i is given by

$$\frac{\eta(\tau)^{2k} \prod_{i\neq j} \Theta(\sigma_j - \sigma_i) \ \times \prod_{A\in\underline{\mathbf{3}}} \prod_{i,j} \Theta(\epsilon_a + \epsilon_b - \sigma_i + \sigma_j) \ \times \prod_{\substack{A\in\underline{\mathbf{6}}\\ \bar{a}\in\bar{A}}} \prod_{\substack{m\in[n_A]\\ i}} \Theta(\epsilon_{\bar{a}} - \sigma_i + a_{m,A})}{\prod_{a\in\underline{\mathbf{4}}} \prod_{i,j} \Theta(\epsilon_a + \sigma_j - \sigma_i) \times \prod_{A\in\underline{\mathbf{6}}} \prod_{m\in[n_A],i} \Theta(a_{m,A} - \sigma_i)\Theta(\epsilon_a + \epsilon_b - a_{m,A} + \sigma_i)} \ . \tag{5.84}$$

and the integration measure is given by

$$\frac{1}{k!} \int \frac{\mathrm{d}y_1 \mathrm{d}y_2 \cdots \mathrm{d}y_k}{y_1 y_2 \cdots y_k} \ . \tag{5.85}$$

There is a σ-independent prefactor as well:

$$\prod_{A\in\underline{\mathbf{3}}} \Upsilon_A \ \times \prod_{A,B\,\in\,\underline{\mathbf{6}},\ n} \Upsilon_{A,B,n} \ \times \prod_{A\,\in\,,\underline{\mathbf{6}},\ m,n} \Upsilon_{A,m,n} \ , \tag{5.86}$$

where we have used the symbol of the equation itself for the corresponding expressions above. The infinite products in the last two factors have to be regularised suitably. In [N1, N2, N4] an additional prefactor is considered which is of the form

$$\frac{1}{\prod_A \prod_{p,p'\in[n_A]} \Theta(m\epsilon_a + n\epsilon_b - m_A - a_{p,A} + a_{p',A})} \ , \tag{5.87}$$

where m_A is either $\epsilon_{\bar{a}}$ or $\epsilon_{\bar{b}}$.

Conjecture: These prefactors arise from the scalars in the D5$_A$-D5$_A$ vector multiplet reduced to $\mathbf{R}^{1,1}$ following the procedure in Chapter 4 in the section on Folded instantons.

The discussion from here onwards has been borrowed from [N4]. Since the function $\Theta(z)$ has a simple zeros at $z = \mathbf{Z} + \tau\mathbf{Z}$, we see that the integrand above has poles for values of the holonomies which satisfy

$$y_j y_i^{-1} = q_a \ , \quad y_i = x_{m,A}^{-1} \ , \quad y_i = q_a q_b x_{m,A} \ . \tag{5.88}$$

The integral over the U(k) holonomies should be thought of contour integrals which have poles at the above specified locations. Which poles are picked up depends on the way the contours for the y_j are closed. The contours are specified by first studying the various possible fixed point sets of the action of subgroups of the maximal torus T.

The most general subgroup of T can be specified as follows [N4]: Consider hyperplanes in the space of equivariant parameters of the form

$$L_\alpha(\boldsymbol{a}, \boldsymbol{\epsilon}) = \sum_{A \in \underline{\mathbf{6}}} \sum_{m \in [n_A]} \omega_{\alpha;m,A}\, a_{m,A} + \sum_{a \in \underline{\mathbf{4}}} n_{\alpha;a} \epsilon_a = 0 \ , \tag{5.89}$$

where $\omega_{\alpha;m,A} = +1, 0, -1$ and $n_{\alpha;a} \in \mathbf{Z}$. For certain values of the parameters $\boldsymbol{a}$ and $\boldsymbol{\epsilon}$, the above equations can be inverted to yield the subgroup T_L.

For example, one can find six sets of ordinary ADHM instantons living on each of the six $\mathbf{C}^2$'s by considering the fixed points of the subgroup $\mathsf{T}_\times = \mathrm{U}(1)^5$ with action:

$$(I_A, J_A) \to (\mathrm{e}^{\mathrm{i}\vartheta_A} I_A, \mathrm{e}^{-\mathrm{i}\vartheta_A} J_A) \ , \tag{5.90}$$

with the overall scaling set to 1. The fixed point set is the direct sum of 6-tuples of ordinary instanton moduli spaces:

$$\mathcal{M}_k(\boldsymbol{n})^{\mathsf{T}_\times} = \bigcup_{\sum_A k_A = k} \ \bigtimes_A \ \mathcal{M}_{k_A, n_A} \ . \tag{5.91}$$

Indeed, one can take completely generic parameters for $\boldsymbol{a}$ and $\boldsymbol{\epsilon}$ and look at the poles of the twisted index above. They are found to be precisely at the values of σ_i that was described in the previous section when we studied the cohomology of $\overline{\nabla}_+$ in the ADHM case.

One can now stitch together these six separate sets of moduli spaces by considering fixed point loci of small torus subgroups which interpolate between the various ADHM moduli spaces. These interpolating manifolds are picked up by the contour of the y_j above provided the appropriate congruences hold between $\boldsymbol{a}$ and $\boldsymbol{\epsilon}$. This way one can stitch their way up to obtain the entire spiked instanton moduli space. In particular, this moduli space includes regions which interpolate between ADHM moduli spaces of different instanton number on the same stack of D5-branes.

We conclude by stating the compactness theorem of Nekrasov which places very strong constraints on the non-perturbative behaviour of gauge theory. The statement is that the fixed point loci in $\mathcal{M}_k(\boldsymbol{n})$ of the various torus subgroups T_L are compact. Let

$$x_A = \frac{1}{n_A} \sum_{m \in [n_A]} a_{m,A} \ . \tag{5.92}$$

Then, one of the consequences of the compactness theorem is that the twisted index including the prefactors written above is a polynomial in the x_A. The x_A correspond to the centred of mass of the various stacks of $D5$-branes. The fact that the twisted index is a polynomial in these variables implies that it well-defined for any values of the centres of mass and in particular, suggests that there are no runaway-like transitions in the theory. Somehow, the non-perturbative effects have rendered the theory docile! □

Chapter 6

Conclusions and Outlook

In this thesis, we have studied the low-energy dynamics of D1-branes bound to a maximal set of supersymmetric intersecting D5-branes in Type IIB string theory. A particular low-energy limit enabled us to study the collective dynamics of instantons in four dimensional gauge theory, including processes in which the instantons escape to an auxiliary four dimensional world.

A constant NSNS B-field binds the instantons (D1-branes) to the D5-branes. A peculiar feature was that $\overline{\text{D1}}$-branes bind in a supersymmetric fashion to the D5-branes while D1-branes do not. The equations governing the collective dynamics were derived by studying open string amplitudes in the constant B-field background. This required the calculation of certain simple $(n+3)$-point tree level amplitudes. The equations described the classical moduli space of the theory, also called as the spiked instanton moduli space.

The spiked instanton moduli space is, in a sense, the most conservative way of describing processes that change instanton number. The high amount of symmetry present in the problem allows one to compute various observables that encode these transitions as equivariant integrals over spiked instanton moduli space. In Chapter (5), we computed one such basic observable which is the equivariant elliptic genus.

The intermediate system of crossed instantons which was described in the first part of Chapter (4) is interesting in its own right. The field theory dual of $AdS_3 \times S^3 \times S^3 \times S^1$ has been evading discovery for some time now. The setup of D-branes that comes closest to solving the puzzle seems to the one of crossed instantons. Tong [To] has showed that the central charge for the $\mathcal{N} = (0, 4)$ gauged linear sigma model of crossed instantons agrees with the calculation on the gravity side. The presence of a constant B-field modifies the setup while making it more tractable. It would be interesting to see if any of the calculations in this thesis are applicable to this problem.

Theories with $\mathcal{N} = (0,2)$ supersymmetry display very interesting features like dynamical supersymmetry breaking [GGP1, GGP2], accidental enhancement of symmetries in the infrared [BMP] and so on. The infrared limit of $\mathcal{N} = (0,2)$ theories also furnish possible consistent vacua for heterotic strings. Interesting work has been done on exploring the infrared of $\mathcal{N} = (0,2)$ theories by studying chiral algebras in the spirit of [W6, Ta, De] and others. It would a logical next step to explore the infrared limit of the $\mathcal{N} = (0,4)$ gauge theory of crossed instantons and its $\mathcal{N} = (0,2)$ spiked generalisation along the lines of [SiWi1, SiWi2].

It is well known that the worldvolume theory of D1-brane probes of Calabi-Yau fourfolds preserve $(0,2)$ supersymmetry. Quite a lot of work has been done in studying a version of mirror symmetry for these theories in [FLS1, FLS2, FLSV]. It would be interesting to explore if our spiked instanton system is part of a duality web involving theories of the above type.

On a separate note, the spiked instanton moduli space is an instrumental tool in the overarching program of the BPS/CFT correspondence which relates BPS observables in four dimensions with analogous observables in two dimensional conformal field theories subsumes most such relations. Recently, there has been a spur in uncovering such novel infinite dimensional symmetries in four dimensional quantum field theory. When one considers the grand canonical ensemble of instantons of all windings in the gauge theory, the infinite dimensional symmetry becomes evident. One can (and indeed it has been done) generalise this idea to BPS objects in higher dimensions, say six and eight. This would correspond to studying ensembles of bound states of D0-D6 and D0-D8 branes. The study of the grand canonical ensemble of D0-D6 branes reveals the existence of an SO(10) isometry of the theory and the partition function can essentially be written in terms of free fields in eleven dimensions [NO2]. We would like to explore a similar point of view for D0-D8 bound states.

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