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Vortex solutions in a Witten-type model

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Abstract. Straight line vortex solutions in a Witten's superconducting string model are studied. The model has many parameters and this is the main reason of the complexity. We argue the precise conditions of the parameters for finding the solutions of the model. We obtain the rotationally symmetric solutions for the winding numbers $m = 1-4$ with/without the gauge field. For the higher winding numbers, an energy minimization algorithm is used to investigate non-rotational solutions.

1. Introduction

Superconducting strings in the local $U(1) \otimes U(1)$ symmetry was introduced by Witten in 80s' [1]. There are many studies based on the model for the straight line vortex solutions, and also the closed type vortex called vortons. As has been claimed that the solutions may have rich variety of physical applications for the condensed matter physics and especially for the cosmology [2, 3, 4].

This model contains two Abelian gauge fields $R_\mu^{(a)} (a = 1, 2)$ interacting with two complex scalar fields ϕ and σ with the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}\sum_{a=1,2} F_{\mu\nu}^{(a)} F^{(a)\mu\nu} + D_\mu\phi^* D^\mu\phi + D_\mu\sigma^* D^\mu\sigma - U. \tag{1}$$

Here $F_{\mu\nu}^{(a)} = \partial_\mu R_\nu^{(a)} - \partial_\nu R_\mu^{(a)}$ are the Abelian field strengths, the gauge covariant derivatives of scalars are $D_\mu = \partial_\mu - ig_a R_\mu^{(a)}$ where g_a are the gauge coupling constants. The scalar field potential is defined as

$$U = \frac{\lambda_\phi}{4}(|\phi|^2 - \eta_\phi^2)^2 + \frac{\lambda_\sigma}{4}|\sigma|^2(|\sigma|^2 - 2\eta_\sigma^2) + \beta|\phi|^2|\sigma|^2, \tag{2}$$

where $\lambda_\phi, \lambda_\sigma, \eta_\phi, \eta_\sigma$ and β are positive constants. This theory is invariant under the local $U(1) \otimes U(1)$ transformation.

The theory admits the stationary solutions of straight line vortex. If the parameters in the potential are properly chosen, the field ϕ vanishes in the vortex core and approaches the finite value at the infinity, while σ develops a non-zero condensates value in the core and vanishes at the infinity. The fields ϕ, σ are sometimes referred to as a vortex field and a condensate field, respectively.

Several attempts have been done for finding rotationally symmetric solutions of the model [5, 6, 7]. All of them were only of the winding number $m = 1$ solutions of the global limit, i.e., the gauge coupling was ignored. They were quite confusing because they each suggested different conditions for the model parameters. We have tried to confirm the existence of such solutions by employing the conditions, but often failed to find the solutions. We have to say that existence of such vortex solutions in the model is thus still an open problem.

In this paper, first we propose the correct conditions for the model parameters. Within the full gauged model, finding the solutions is still absent thus we try to find the solutions with the gauge field. In [5, 6, 7], only the vortex with unit winding number $m = 1$ was discussed, thus we try to get solutions of higher winding number solutions. For such solutions, breaking of the rotational symmetry might happen. The numerical simulations are done by an energy minimization algorithm based on a simulated annealing [8]. We shall see the winding number m solutions comprise m constituents.

2. The model parameters

We employ a rotationally symmetric ansatz

$$\phi = \phi_0(r)e^{im\theta}, \sigma = \sigma_0(r)e^{i(\omega t+kz)}, \tag{3}$$

where $m \in \mathbb{Z}$ is the winding number and $\omega, k \in \mathbb{R}$ describe the traveling wave along z axis. The boundary conditions are given as

$$\phi_0(0) = 0, \quad \phi_0(\infty) = \eta_\phi, \quad \sigma_0'(0) = 0, \quad \sigma_0(\infty) = 0. \tag{4}$$

For the moment we consider the global limit of this model thus $g_a = 0$. The gauge fields decouple and the Lagrangian reduces to

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi + \partial_\mu\sigma^*\partial^\mu\sigma - U \tag{5}$$

so that the internal $U(1) \otimes U(1)$ symmetry becomes global. The ungauged model was already studied in [6, 7]. In [6], following three conditions were proposed for the existence of the solutions

$$\lambda_\phi\eta_\phi^4 > \lambda_\sigma\left(\eta_\sigma^2 + \frac{2(\omega^2-k^2)}{\lambda_\sigma}\right)^2, \quad \beta\eta_\phi^2 - \frac{1}{2}\lambda_\sigma\eta_\sigma^2 - \omega^2 + k^2 > 0,$$
$$k^2 - \omega^2 < \frac{1}{2}\lambda_\sigma\eta_\sigma^2 - \sqrt{\beta\lambda_\phi}\eta_\phi^2 \tag{6}$$

while in [7], a simple estimate was given

$$\beta\eta_\phi^2 - \frac{1}{2}\lambda_\sigma\eta_\sigma^2 < \frac{1}{2}\lambda_\phi\eta_\phi^2. \tag{7}$$

Both conditions are, however, not precise one. In fact, we could not find solutions with the parameters based on these conditions (6) or (7). Thus we start with a study of the proper parameter conditions of the model.

The Euler-Lagrange equations in the theory (5) with the ansatz (3) read

$$r\phi_0'' + \phi_0' - \frac{m^2}{r}\phi_0 - \frac{r}{2}\lambda_\phi(\phi_0^2 - \eta_\phi^2)\phi_0 - r\beta\phi_0\sigma_0^2 = 0, \tag{8}$$

$$r\sigma_0'' + \sigma_0' + r(\omega^2 - k^2)\sigma_0 - \frac{r}{2}\lambda_\sigma(\sigma_0^2 - \eta_\sigma^2)\sigma_0 - r\beta\phi_0^2\sigma_0 = 0. \tag{9}$$

We examine their analytical properties by expanding the fields at the origin and also infinity. The critical one is the expansion of the σ field at the infinity and then we describe it in detail. We introduce a new coordinate $x \equiv \frac{r}{1+r}$ which runs from 0 to 1. At the vicinity of infinity, σ_0 can be expanded as $\sigma_0 = \sum_{i=1} \sigma^{(i)}(1-x)^i$ with the coefficients $\sigma^{(i)}, i = 1, 2, 3, \cdots$. Substituting this expansion for (9), we get a following equation

$$\left(k^2 - \omega^2 + \beta\eta_\phi^2 - \frac{1}{2}\eta_\sigma^2\lambda_\sigma\right)\sigma^{(1)} + \left[\left(k^2 - \omega^2 + \beta\eta_\phi^2 - \frac{1}{2}\eta_\sigma^2\lambda_\sigma\right)(\sigma^{(1)} + \sigma^{(2)})\right](1-x)$$
$$+ \left[\left({\sigma^{(1)}}^2\lambda_\sigma - \frac{2\beta({\sigma^{(1)}}^2\beta + m^2)}{\lambda_\phi} - 1\right)\sigma^{(1)} + \left(k^2 - \omega^2 + \beta\eta_\phi^2 - \frac{1}{2}\eta_\sigma^2\lambda_\sigma\right)(\sigma^{(2)} + \sigma^{(3)})\right](1-x)^2$$
$$+ O((1-x)^3) = 0. \tag{10}$$

Each coefficient of $(1-x)^i$ should be zero, then for the zeroth order we have

$$\lambda_\sigma = \frac{2(k^2 - \omega^2 + \beta\eta_\phi^2)}{\eta_\sigma^2}. \tag{11}$$

From this the first order becomes zero, too. Also for the second order, the $\sigma^{(1)}$ should satisfy

$$\sigma^{(1)} = -\sqrt{\frac{2m^2\beta + \lambda_\phi}{\frac{k^2-\omega^2+\beta\eta_\phi^2}{\eta_\sigma^2}\lambda_\phi - 2\beta^2}}.$$

Since the σ_0 is defined as a real, the $\sigma^{(1)}$ is also real. From this, we obtain the additional condition,

$$\frac{k^2 - \omega^2 + \beta\eta_\phi^2}{\eta_\sigma^2}\lambda_\phi > 2\beta^2. \tag{12}$$

All coefficients of the higher orders are automatically guaranteed once the conditions (11) and (12) are employed. Here we get correct form of the conditions (11) and (12) for the parameters. In the following we perform the numerical analysis with the parameters based on these conditions.

3. The rotationally symmetric solutions

3.1. The ungauged model

First we present the solutions with the rotational symmetry. In [6, 7] the authors presented only solutions with unit winding number, then we investigate the higher winding numbers as well. Unlike the standard Abelian-Higgs model, the model possesses the solutions only with the scalar fields. We solve the Euler-Lagrange equations (8) and (9) in terms of a relaxation technique. In Figure 1 we plot typical profiles of the scalar fields and also each value of energy. One can easily see that for the higher winding numbers the behavior at the origin becomes shallower and also the speed of approaching the vacuum gets slower.

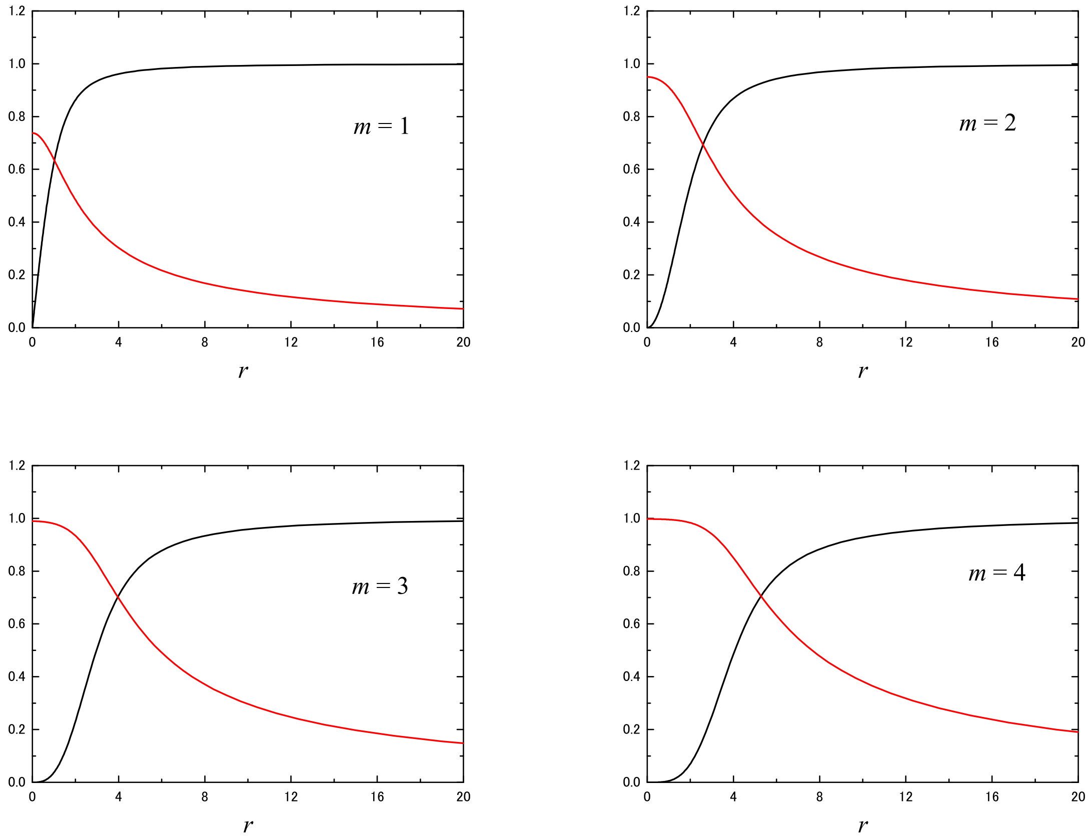

Figure 1. The profiles of $\phi_0(r)$(black line) and $\sigma_0(r)$(red line) of the ungauged model for each winding number for the parameters $\lambda_\phi = 5.5, \lambda_\sigma = 3.0, \eta_\phi = 1.0, \eta_\sigma = 1.0, \beta = 1.5$. The energies are $E_{m=1} = 39.84, E_{m=2} = 132.63, E_{m=3} = 269.83, E_{m=4} = 446.61$, respectively.

3.2. The gauged model

The model (1) originally was proposed as the gauged model, thus we have to introduce the gauge field in our numerical analysis. Physically it is important because the gauge field describes the magnetic field penetrating a superconductor. The gauged model was already studied in [5] but only for the unit winding number.

We assume the rotational symmetry of the fields, and perform the asymptotic expansion of both the scalar fields and the gauge fields. The main difficulty of the problem is that the gauge field coupled with σ has no asymptotic solution at the infinity, while the gauge field coupled with ϕ can exist. This indicates that there are no solutions of the full gauged model (1) at least within the rotational symmetry.

Here we consider the following Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) + (\partial_\mu\sigma)^*(\partial^\mu\sigma) - U. \tag{13}$$

Here $F_{\mu\nu} = \partial_\mu R_\nu - \partial_\nu R_\mu$ is the Abelian field strength, the gauge covariant derivative of scalar field ϕ is $D_\mu\phi = (\partial_\mu - igR_\mu)\phi$, where g is the gauge coupling constant.

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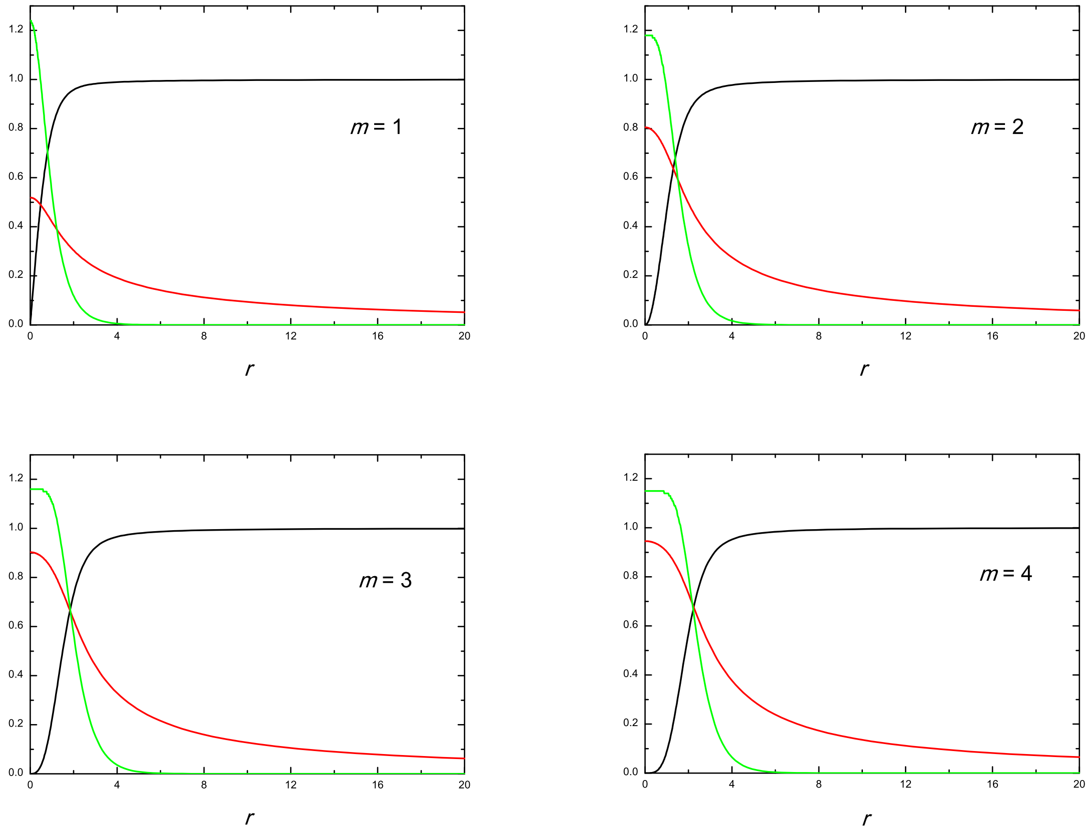

Figure 2. The profiles of $\phi_0(r)$(black line) , $\sigma_0(r)$(red line) and the magnetic field calculated from the gauge field coupled with ϕ(green line) for each winding number for the parameters $\lambda_\phi = 5.5, \lambda_\sigma = 3.0, \eta_\phi = 1.0, \eta_\sigma = 1.0, \beta = 1.5, g = 1.0$. The energies are $E_{m=1} = 7.57, E_{m=2} = 15.14, E_{m=3} = 22.47, E_{m=4} = 29.71$, respectively.

In [5], The conditions for the parameters were suggested

$$\lambda_\phi \eta_\phi^4 > \lambda_\sigma \eta_\sigma^4, \quad \beta\eta_\phi^2 - \frac{1}{2}\lambda_\sigma \eta_\sigma^2 > 0, \quad \sqrt{\lambda_\phi}\eta_\phi < \sqrt{\lambda_\sigma}\eta_\sigma. \tag{14}$$

These conditions are again not precise. In fact, the conditions which ensure the existence of the solutions hold the same as (11),(12) because the gauge coupling has no effect for analytical problems.

For the gauge field, we consider the following rotationally symmetric ansatz,

$$R_\theta = R(r), \tag{15}$$

and the boundary conditions

$$R(0) = R(\infty) = 0. \tag{16}$$

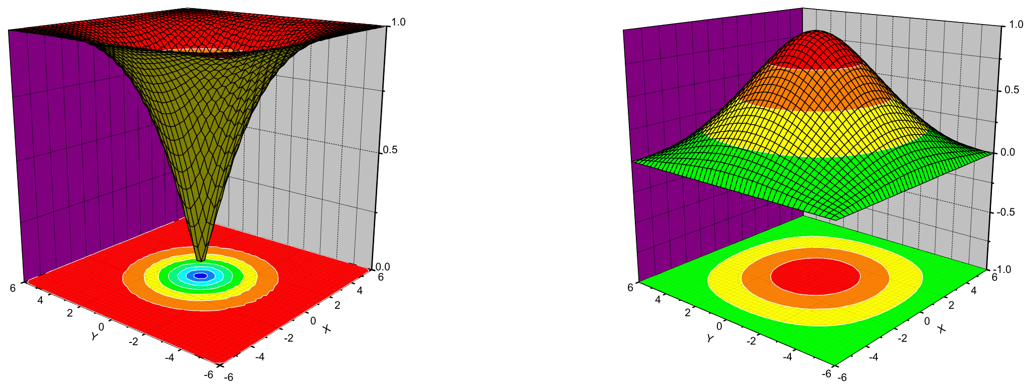

Figure 3. The profiles of ϕ(left) , σ(right) in the case of $m = 1$ for the parameters $\lambda_\phi = 10.0, \lambda_\sigma = 9.0, \eta_\phi = 1.0, \eta_\sigma = 1.0, \beta = 4.5$.

The equations of these fields are given as following forms,

$$r\phi_0'' + \phi_0' - r\left(\frac{m}{r} - gR\right)^2 \phi_0 - \frac{r}{2}\lambda_\phi(\phi_0^2 - \eta_\phi^2)\phi_0 - r\beta\phi_0\sigma_0^2 = 0, \tag{17}$$

$$r\sigma_0'' + \sigma_0' + r(\omega^2 - k^2)\sigma_0 - \frac{r}{2}\lambda_\sigma(\sigma_0^2 - \eta_\sigma^2)\sigma_0 - r\beta\phi_0^2\sigma_0 = 0. \tag{18}$$

$$rR'' + R' - \frac{R}{r} + 2rg\left(\frac{m}{r} - gR\right)\phi_0^2 = 0, \tag{19}$$

Results of the gauged model are shown in Figure 2. The behavior of scalar fields is similar to that of the ungauged model, and the width of the magnetic field gets broader for the higher winding numbers.

4. The solutions without the rotational symmetry

So far, we have discussed the rotationally symmetric solutions by solving the Euler-Lagrange equations. It seems reasonable at least for the unit winding number. For the higher winding numbers, however, the premise may be wrong in the case of the ungauged model. In fact, the solutions of the ungauged model always satisfy

$$E_m > mE_{m=1} \tag{20}$$

which indicates that the winding number m ($m > 1$) solutions have to break into the number m of unit winding number ones. One possibility to avoid such collapse is to relax the assumption of the rotational symmetry. It is known that non-rotationally symmetric solutions with higher winding numbers (or with the higher topological charges) often appear in several field theory models [9, 10]. Here we compute the solutions without the rotational symmetry in terms of an energy minimization scheme. We employ the method called the simulated annealing algorithm [8] in the two dimensional Cartesian grid. As in the case of the symmetric solutions, we suppose that ϕ carries the winding number and σ exhibits the traveling wave along the z axis. The

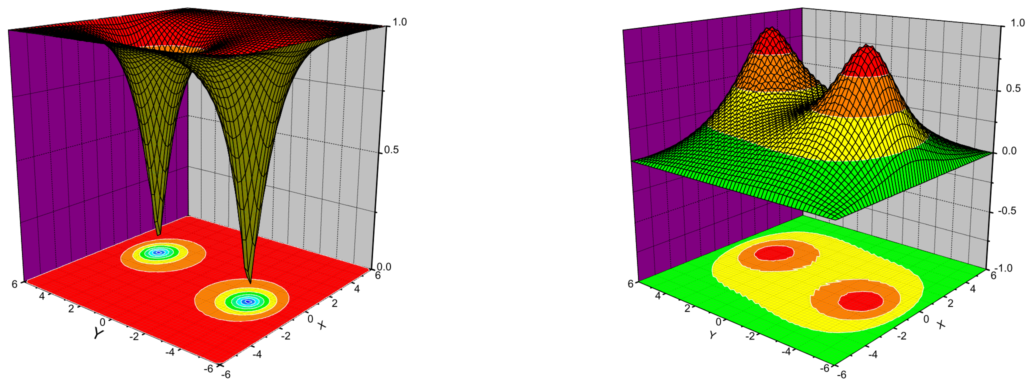

Figure 4. The profiles of ϕ(left) , σ(right) in the case of $m = 2$ for the parameters $\lambda_\phi = 10.0, \lambda_\sigma = 9.0, \eta_\phi = 1.0, \eta_\sigma = 1.0, \beta = 4.5$.

Hamiltonian is thus

$$\mathcal{H} = (\partial_x\phi)^2 + (\partial_y\phi)^2 + (\partial_x\sigma)^2 + (\partial_y\sigma)^2 + (\omega^2 + k^2)|\sigma|^2 + U. \tag{21}$$

Since the fields ϕ and σ are complex scalar fields, in the actual simulation we separate them into the real part and the imaginary part, and the Hamiltonian is minimized about the components ϕ_R, ϕ_I, σ_R and σ_I (where the subscript R, I mean the real and imaginary part, respectively). Here, we consider the bare case (ω=k=0). The profiles of the scalar fields are shown with the contour plot in Figure 3 of $m = 1$ and Figure 4 of $m = 2$. In both result, x and y are from -6 to 6. As was expected that the solutions with $m = 1$ has rotational symmetry. For the higher winding numbers, the solutions split into constituents. Thus we conclude the solutions do not have the rotational symmetry but are composed of two centers. The solutions, however, might be affected by the edge of the area. Of course if we consider the system with finite area such as a superconductor of a small material, the solutions can be stable. More thorough, large-scale numerical analysis is needed to reach the final conclusion. In addition, the choice of the parameters in the non-rotational system is unknown again and should be examined.

In the case of the gauged model, the solutions with the higher winding number may be stable because the energies exhibit $E_m \leqq mE_{m=1}$ (see caption of Figure 2). It means the gauge field might help the stability of higher winding number solutions. Since they are quite subtle, we may have ground states with breaking the rotational symmetry. The analysis is currently studied and the result will be reported elsewhere.

5. Conclusion

In this paper, we obtained several straight line vortex solutions in a Witten-type model. First we found the precise conditions of the model parameters for the existence of the solutions. For the gauged model, the solutions exist when only a gauge field couple with ϕ within the rotational symmetry. We thus obtained the solutions of such semi-local model. In the previous attempts, only the solutions with unit winding number was discussed, and then we investigated several higher winding number solutions as well. For the higher winding numbers, we suggested the possibility that the rotational symmetry should be broken and as a result, the winding number m solutions always split into m constituents.

For the rotational symmetry, only one gauge field is possible to implement. If the symmetry is broken, however, it might not be true. It is certainly interesting that analysis of energy minimization for the gauged model. The result will be reported in near future.

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