# RECENT DEVELOPMENTS IN NON LINEAR REPRESENTATIONS AND EVOLUTION EQUATIONS

Daniel STERNHEIMER
CNRS, PARIS
and
Physique-Mathématique, Faculté des Sciences Mirande
BP 138 - 21004 - DIJON Cedex FRANCE

**ABSTRACT**

After a short review of the "FPS (Flato, Pinczon, Simon) approach" to the study of nonlinear group representations and of their earlier applications to nonlinear evolution equations, we pass to the most recent applications to equations in 1+1 and 3+1 space-time dimensions. In particular the proof of existence of global solutions to the coupled Maxwell-Dirac equations is briefly outlined.

## I - THE FPS APPROACH.

### 1. Historical Background.

The problem of linearization of group actions or vector-fields goes back almost a hundred years ago, to the works of Sophus Lie (local actions on $\mathbb{R}^n$) and Henri Poincaré (the famous non-resonance condition for the eigenvalues of the linear part to linearize one analytic vector field in a finite-dimensional space around a point where it vanishes). Other authors studied the linearizability of local group actions around a fixed point in finite-dimensional spaces in the past 40 years during which time the theory of linear Lie group representations, and even that of analytic functions, in infinite-dimensional spaces made considerable progress. But it was only ten years ago that M. Flato, G. Pinczon and J. Simon[1] combined in some sense both theories, developing what is now called the "FPS approach" to nonlinear representations and evolution equations.

### 2. The framework.

Let E be a Banach or Fréchet space (Fréchet space are needed since the space of differentiable vectors of a continuous linear Lie group representation on a Banach space has a natural Fréchet topology). We shall denote by $\mathcal{L}_n(E)$ the space of continuous symmetric multilinear maps from $E^n$ to E, which have a natural extension to continuous linear maps from $\hat{\otimes}^n_\pi E$ (the $n^{th}$-order symmetric tensor power of E, endowed and completed with the projective topology $\pi$) to E. To any $f^n \in \mathcal{L}_n(E)$ is associated the monomial $\hat{f}^n$ defined by $\hat{f}^n(\varphi) = f^n(\varphi,\dots,\varphi) \in E$ for $\varphi \in E$. The space F(E) of formal series on E is then defined as the space of $f = \Sigma_{n=1}^{\infty} \hat{f}^n$ , with $f^n \in \mathcal{L}_n(E)$, and endowed with a product by the composition of maps (having the origin as fixed point) from E to E. Now let $\tilde{E} = \bigcup_n E_n$ , $E_n = \bigoplus_{p=1}^{n} \hat{\otimes}^p E$ be the (vacuum-less) "Fock space" built on E. Then we can define the one-to-one map $\Lambda$ from F(E) to $\mathcal{L}(\tilde{E})$ by ($\otimes$ denoting the symmetrized tensor product) :

$$\Lambda(f)^n = \Sigma_{1\leqslant p\leqslant n}\ \Sigma_{i_1+\dots+i_p=n}\ (f^{i_1}\otimes\dots\otimes f^{i_p})$$

which maps the composition f o h of two power series f and h into the product of the linear operators $\Lambda(f)$ and $\Lambda(h)$ in $\mathcal{L}(\tilde{E})$. Its differential $d\Lambda$ maps the bracket $[f,h] : f\times h - h\times f$ of $f,h \in F(E)$, where

$$(f\times h)^n = \Sigma_{1\leqslant p\leqslant n}\ f^p \circ (\Sigma_{0\leqslant q\leqslant p-1}\ I_q \otimes h^{n-p+1} \otimes I_{p-q-1})$$

onto the commutator of $d\Lambda(f)$ and $d\Lambda(h)$, and is thus a Lie algebra morphism for the bracket so defined on F(E).

We can therefore define[1] a formal (resp. analytic) nonlinear representation of a Lie group G as a mapping $G \ni g \to S_g \in F(E)$ such that, for all $\varphi \in E$, $g \to S_g^n(\varphi)$ is measurable from G to E (resp. also analytic near 0 in E for g in a neighbourhood of the identity in G), with the abovementioned product law on the invertible elements of F(E). Similarly one defines a formal nonlinear representation of a Lie algebra $\mathfrak{g}$ as a Lie algebra homomorphism from $\mathfrak{g}$ into F(E) endowed with the abovedefined bracket. When E is a Banach space, an analytic function[2] on E belongs to a Banach space

$$H_r(E) = \{f \in F(E)\ ;\ \|f\|_r = \Sigma_{r=1}^{\infty}\ r^n\ \|f^n\| < \infty\}$$

for some $r > 0$ ; for a Fréchet space one has similar notions involving sets of seminorms.

As for linear representations, we shall say[1] that two formal (resp. analytic) nonlinear group representations (S,E) and (S',E' = E) are equivalent if there is an invertible formal (resp. analytic) series $A \in F(E)$ which intertwines S and S', i.e. $S'_g = A\, S_g\, A^{-1}$ . In particular S will be said linearizable if there is an equivalent S' which is linear.

3. Relations with 1-cohomology and first applications.

Let G be a Lie group, (S,E) a nonlinear representation. Then, for $g, g' \in G$ :

$$S^2(gg') = S^1(g) \circ S^2(g') + S^2(g) \circ (S^1(g') \otimes S^1(g'))$$

which shows that when G acts on $\mathcal{L}_2(E)$ by

$$\mathcal{L}_2(E) \ni A \to S^1(g) \circ A \circ (S^1(g^{-1}) \otimes S^1(g^{-1})),$$

then $R^2(g) = S^2(g) \circ (\otimes^2 S^1(g^{-1}))$ is a one-cocycle of G valued in $\mathcal{L}_2(E)$, and therefore $\Lambda(S^2)$ is obtained as an extension[3] of $S^1$ by $S^1 \otimes S^1$. Similarly, one sees that $\Lambda(S)$ is obtained by successive extensions of $S^1$ by the $(S^1)^{\otimes n}$ , with cocycles $R^n(g) = S^n(g) \circ (\otimes^n S^1(g^{-1}))$. Now let us suppose that $R^2$ is a 2-coboundary, i.e. that there exists $B^2 \in \mathcal{L}_2(E)$ such that $S^2(g) = S^1(g) \circ B^2 - B^2 \circ (\otimes^2 S^1(g))$. Then

$$S_2(g) = (I-B^2)^{-1} S(g)(I-B_2) = S^1(g) + \Sigma_{n \geqslant 3} S_2^n(g).$$

Similarly one shows that if all cocyles $R^2, \ldots, R^m$ are coboundaries then $P_m = \prod_{k=2}^{m} (I-B^k)$, where the $B^k$ are built successively as $B^2$, intertwines S with a representation $S_m$ that has no nonlinear terms of order $\leqslant m$. Therefore[4], if (S,E) is a formal nonlinear representation of a Lie group G in the Fréchet space of differentiable vectors for the linear part $S^1$, and if the (differentiable) 1-cohomology spaces $H^1_\infty(G, \mathcal{L}_n(E)) = 0$ for all $n \geqslant 2$, S is linearizable by $A = \lim P_m$ .

One of the main features of the FPS approach is therefore first to build such an A, and then to show that it is analytic (and not only a formal series).

A related result is the "smoothening" of analytic representations[1] : an analytic representation S in a Banach space E is said smooth if the function $g \to R(g) = S^1(g^{-1}) \circ S(g)$ is $C^\infty$ from some neighbourhood of the identity in G into some space $H_r(E)$ (for some $r>0$), i.e. if all the "obstructions to differentiability" lie in the linear part. One can show that if (S,E) is analytic there always exists an equivalent representation which is smooth (and linear on any compact subgroup of G). The basic reason is that one can always substract coboundaries in the successive extensions process so as to get differentiable 1-cocycles. A smooth nonlinear group representation can always[1] be differentiated to a nonlinear Lie algebra representation on the space of differentiable vectors for the linear part ; conversely[1] a formal representation T of a Lie algebra $\mathfrak{g}$, the linear part $T^1$ of which is the differential of a linear Lie group representation and such that all $T^n(x)$, $x \in \mathfrak{g}$ , are continuous n-linear maps, is the differential of a unique formal group representation.

A consequence of this seemingly inocuous result is a trivial proof[1] of a theorem (due 20 years ago to Guillemin and Sternberg and suggested 40 years ago by Palais) that any analytic action, with fixed point, of a semi-simple Lie group in a finite- dimensional space is linearizable (one just uses Weyl's "unitary trick") ; formal linearizability is even more evident (Whitehead's lemma).

Conversely, non-trivial 1-cocycles can give rise to non-linearizable representations -though the nonlinearity of these representations, as the triviality of the cocycles, is sensitive to the choice of space. Examples of such phenomena can be found in Ref.5, with actions of $G = \mathbb{R}$, $SL(2, \mathbb{R})$ or $SL(2,\mathbb{C})$, built using non-trivial 1-cocycles of G valued in the linear representation space E. Other cute examples include[6] truly nonlinear representations of inhomogeneous classical groups (such as Poincaré) in the (finite-dimensional) translation subgroup space (where two types of nonlinear representations are found, coupled by a non-dimensional coupling constant taking discrete values).

4. An ordinary differential evolution equation.

To end this part, and as a transition to the next one, let us mention a simple result[1] and its consequences. If (S,E) is a nonlinear analytic representation of a nilpotent Lie group G in a Hilbert space H, the linear part $S^1$ of which is unitary and nontrivial, then S is analytically linearizable. Now taking $H = \mathbb{C}$ and $G = \mathbb{R}$ represented linearly by $t \to e^{it}$ , we see that all evolution equations of the type $\frac{dx}{dt} = ix + F(x)$, where F is an analytic series with no constant or linear term, expressing the generator of a nonlinear representation of G in H, are linearizable. Therefore if $|x(0)|$ is small enough, this property being preserved by the unitary linear part, all those equations will have global solutions in time (no singularity appears as $t \to \infty$ ). This would of course be true only for $t>0$ or $t<0$ if the factor i was replaced by a real number.

## II - APPLICATIONS TO GROUPS AND EVOLUTION EQUATIONS IN 3+1 AND 1+1 SPACE-TIME DIMENSIONS.

5. Poincaré group and Euclidean groups.

We shall here only mention some results, the proofs of which are often quite involved and require fine estimates.

a) Let G be the Euclidean group of 2-space $E(2) = SO(2).\mathbb{R}^2$ or the same with dilatations $(SO(2) \times \mathbb{R}_+).\mathbb{R}^2$. Then[7] an analytic nonlinear representation (S,E) of G such that $S^1$ is unitary with no 1-dimensional subrepresentation is (analytically) linearizable if and only if the cocycle $R^2$ defined by $S^2$ is trivial (in other words, obstructions to linearizability lie in the quadratic term).

b) In 3-space, a nonlinear representation (S,E) of $SO(3).\mathbb{R}^3$ with linear part $S^1$ coming from a finite direct sum of unitary irreducible representations (UIR) of the (3+1) Poincaré group with mass $m^2 \geq 0$ and energy $P_0 > 0$ is linearizable.

c) If G is the Poincaré group $SL(2,\mathbb{C}).\mathbb{R}^4$ and (S,E) a nonlinear representation of G on the space E of differentiable vectors for $S^1$, then [4,8,9,10]:

i) If $S^1$ is a finite sum of UIR with $m^2>0$, it is formally linearizable, and analytically linearizable if all energy signs are the same and $m_1+m_2 > m_3$ for any combination of masses of the UIR.

ii) If $S^1$ is a finite sum of massless UIR with arbitrary energy sign, then $P_0$ (and the Euclidean group) are linearizable.

d) For the Poincaré group in 1+1 dimensions $SO(1,1).\mathbb{R}^2$ there exist[10] truly nonlinear (on the translations) representations (with linear part massless). The same holds[11] in 3+1 dimensions, with linear part massless with helicities $\pm 1$ or $\pm 1/2$, but there the nonlinearity occurs on the Lorentz boosts.

6. Nonlinear evolution equations.

An evolution equation $(d/dt)\varphi_t = P_0(\varphi_t)$ is said covariant under a Lie algebra $\mathfrak{g}$ if there exists a representation $(T,H_\infty)$ of $\mathfrak{g}$ on the space $H_\infty$ of differentiable vectors for a linear representation U of the corresponding Lie group G in a Banach space H, such that $T^1 = dU$, $T^n(x) \in \mathcal{L}_n(H)$ for all $n \geq 2$ and $x \in \mathfrak{g}$, $\Sigma_{n\geq 2} T^n(x)$ being analytic near the origin in H for all $x \in \mathfrak{g}$, and such that $P_0 = T(x_0)$ for some $x_0 \in \mathfrak{g}$.

In particular, one sees that if the cohomology spaces

$H^1_\infty(G,\mathcal{L}_n(H))=0$ for all $n\geq 2$, and (S,H) denotes the corresponding representation of G, then the solution $\varphi_t = S(\exp tx_0)\varphi$ of the evolution equation with initial value $\varphi \in H_\infty$ can be formally expressed by

$$\varphi_t = A^{-1}\, U(\exp tx_0)\, A\, \varphi$$

where $A \in F(E)$ is the formal linearization operator built in section 3.

One of the main ideas in the applications of the FPS approach to nonlinear equations is thus to build such an A and show that it converges in a suitably chosen space of initial data, for small enough initial data, which ensures global existence (in time) of solutions when U is isometric, and then (this works especially well in 1+1 dimensions) to extend analytically to more general initial conditions.

7. A short survey of the 1+1 dimensional case.

(For more complete reviews and references, see Refs 12 and 13). As for other methods, the FPS approach gives excellent results for all known completely integrable systems. In fact it has been shown (see Ref.12c) that the inverse scattering transform method, the Hirota $\tau$ formalism and the Kac-Moody constructions of Date et al. all give the same inverse linearization map $A^{-1}$ as the one built following the FPS approach, indicated in the previous section. In the 1+1 dimensional case the group has two generators, the (nonlinear) evolution operator given by the equation and the (linear) generator of space-translations. Published explicit examples of equations treated in this manner include (Ref.13) the Korteweg-De Vries equation, the Burgers equation [for which a Hamiltonian formalism and an infinite sequence of (nonlocal) constants of motion have been found], and the integro-differential Benjamin-Ono equation (both in the nonsoliton and in the soliton sectors).

Theses cases examplify the general result that equations which are linearizable by the FPS approach have Lax pairs. The converse is not true since the Kowalewski top, which has no linear part, does have Lax pairs[14]. However such examples can in some sense be viewed as limiting cases (when the linear part vanishes) of nonlinear equations to which the FPS appproach is applicable, and this fact suggests a further extension of that approach to cases that can be solved by introducing a linear part (given a priori) and by linearizing against that linear part (obviously the linearizing operator becomes singular when the linear part is made to vanish, but some properties such as the existence of Lax pairs may remain).

To be a little more explicit, let us look at Burgers equation

$$u_t = (u_x+u^2)_x = \partial^2 u + 2u\partial u = T_1(u)$$

with $T_0(u) = \partial u = (\partial u/\partial x)$ and inverse $\partial^{-1}(u)(x) = \int_{-\infty}^{x} u(\xi)d\xi$. Then it is seen that the linearization map A can be written $A(u) = \partial\exp(\partial^{-1}u)$ and maps the Schwartz space $\mathcal{S}(\mathbb{R})$ into an open subset of $\mathcal{S}(\mathbb{R})$. An infinite set of constants of motion (local in $\partial^{-1}u$) can be obtained in transforming by A those of the heat equation. The most instructive example is however probably the Benjamin-Ono equation, for which an explicit linearization map on large initial data (including solitons) is constructed using the cohomological algorithm of the FPS approach. This shows that for more complicated examples the algorithm can be worked out on a computer (using e.g. Reduce or Macsyma), and that approach has indeed been tested for the so-called chiral model ($g^{\mu\nu}\,\partial_\mu a^{-1}\,\partial_\nu a = 0$ ; $\mu,\nu = 0,1$, $g^{\mu\nu}$ hyperbolic metric and $a \in GL(n,\mathbb{C})$).

8. Relativistic evolution equations in 3+1 dimensional space-time.

As a consequence of the results mentioned in section 5c, we see[8] that classical field equations such as $(\Box + m^2)\varphi = F(\varphi)$, with F analytic without constant or linear terms, are analytically linearizable on a neighbourhood of zero in the subspace of smooth positive (or negative) energy initial conditions. Similarly, the results in the massless case show[8] that Yang-Mills equations $\partial_\nu F^{\mu\nu} + [A_\nu,F^{\mu\nu}] = 0$ supplemented by a relativistic gauge condition such as $\partial_\mu A^\mu = 0$ (or $\Box\,\partial_\mu A^\mu = 0$, which is conformally covariant), or Einstein equations (without matter), etc.. are linearizable. Recently results have been obtained, in the massive case, with arbitrary energy signs for the initial data. The method used is to prove the existence of a wave operator W (taking the free solutions at $t = +\infty$ to the initial data at t=0) on a domain of small entire test functions of exponential type by solving a Yang-Feldman equation for W by iteration as an element of F(H) defined in a Banach subspace D of H of functions with Fourier transforms compactly supported (and 14 times continuously differentiable) with values in the Sobolev space H of functions from $\mathbb{R}^3$ to a complex vector space which are square integrable together with their derivatives up to third order ; and then by showing that W is analytic in a neighbourhood of 0 in D, using a stationary phase technique. This method has been applied[15] to nonlinear Klein-Gordon and nonlinear Schrödinger systems of equations ($F_j(0) = F'_j(0) = 0$)

$$(\Box + m_j^2)\varphi_j = F_j(\varphi, \nabla\varphi, \nabla^2\varphi), \quad m^2_j > 0$$

$$(\partial_t \pm i\Delta)\varphi_j = F_j(\varphi, \nabla\varphi)$$

In particular, a quadratic nonlinearity is permitted here. For the case of the classical Maxwell-Dirac equations[16]

$$\Box A_\mu = \bar{\psi}\gamma_\mu\psi \quad , \quad \partial_\mu A^\mu = 0$$

$$(i\gamma^\mu \partial_\mu + m)\psi = A_\mu \gamma^\mu \psi \;, \; m > 0$$

a variation of this method had to be used. The essential difference with the case of the nonlinear "massive" Klein-Gordon equations is due to the presence of the long-range electromagnetic interaction in the Maxwell-Dirac equations, which causes the non-existence of the usual wave operator. However the solution of the Hamilton-Jacobi equation for a classical electron in an exterior potential $A_\mu$ permits to construct an approximate solution to these equations, absorbing the most slowly decaying (in space-time) parts of $A_\mu$ and $\psi$. The remaining part of the solution is then easy to handle and the global existence (for $t \geqslant 0$) of solutions can be proved. The modified wave operator so constructed intertwines the nonlinear and the linear evolutions. Here $A_\mu \in C^\infty(\mathbb{R}^3)$ and $\psi$ belongs to a Sobolev space $W^{\infty,2}(\mathbb{R}^3, \mathbb{C}^4)$ of functions which are square-integrable together with any (finite) number of their derivatives ; the final states are small enough in spaces of entire analytic functions of exponential type. It is the first time that such a result, of great physical importance, could be obtained.

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