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CAPTURE EFFICIENCY OF A CONSTANT GRADIENT ELECTRON SYNCHROTRON

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Contents

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Abstract

The problem of high-energy multi-turn injection into a constant gradient electron synchrotron from a linear accelerator has been studied. The injected beam is characterized by a constant current with a well defined duration, a uniform energy distribution, and unspecified distributions in vertical and radial phase space. Calculations yielding the optimum synchrotron parameters and surfaces of capture efficiency over the radial phase space have been made assuming an idealized synchrotron with the design parameters of the Caltech machine. On the basis of these calculations and the parameters of a commercially available linac, an upper limit of 10^{12} captured electrons for 10 Mev injection and half that at 5 Mev can be set.

I Introduction

The number of particles ultimately accelerated to high energies in a synchrotron is determined by the characteristics of two complicated systems: the injector and its associated deflector, and the synchrotron itself. The injector and deflector characteristics determine the distribution in phase space of the injected particles while the synchrotron characteristics define a region of phase space in which a particle's parameters must lie in order that it be accelerated. The basic problem of injection is then the maximizing of the overlap of these two regions. This report is devoted to the determination of the acceptance region of an idealized synchrotron, i.e., one described completely by the linearized synchrotron equations.

II General Statement of the Problem and Basic Assumptions

It will be assumed that all particles are injected into the machine at a certain azimuth and that they are highly relativistic. Then an injected electron is characterized by the instant τ it passes through a vertical plane at the injection azimuth, its energy E, its horizontal and vertical coordinates x, y in the injection plane, and its angles x', y' with respect to a tangent to the equilibrium orbit at this azimuth. Let the number of particles dn in a volume $d^{(6)}V$ of this six dimensional space be

$$dn = f(E, \tau, x, y, x', y')d^{(6)}V$$

where

$$d^{(6)}V = dE\, d\tau\, dx\, dy\, dx'\, dy'$$

Let $g(E, \tau, x, y, x', y')$ be a bivalued function with value one if a particle with these coordinates is ultimately accelerated to high energies and zero otherwise. Thus the total number of particles accelerated is

$$N = \int f\, g\, d^{(6)}V \qquad (1)$$

where the integral is taken over the entire space. In an ideal machine, the function g is determined by the aperture geometry and the detailed manner in which the r.f. accelerating voltage is turned on. It will be assumed that the r.f. is turned on in a very short time to an amplitude V, so that g will be parametrically dependent upon t_1, the r.f. turn-on time, and V, its amplitude (see Appendix I for discussion of this point).

For fixed values of E, x, y, x', y', the function g changes from one to zero in each r.f. period. Thus, if f does not change appreciably on this time scale, the bivalued function $g(\tau)$ may be replaced with little error by its average over an r.f. period at time t. This new continuous function will be designated $\rho(t)$ where the argument t is defined only to within one r.f. period. Thus, we have

$$N = \int f(E,t,x,y,x',y') \; \rho(E,t,x,y,x',y';V,t_1) d^{(6)}V \qquad (2)$$

In an ideal machine, there exists no coupling between the vertical motion of a particle and any of its other coordinates so that the conditions on the vertical acceptance of the synchrotron are independent of the other coordinates. Thus we can write

$$\rho(E,t,x,y,x',y') = \rho_v(y,y') \; \rho_h(E,t,x,x') \qquad (3)$$

Also, although it is by no means necessary, we shall assume that the variables of the injected beam are also independent to some extent. In particular, we assume that

$$f(E,t,x,y,x',y') = I(t) \; f_E(E) \; f_x(x,x') \; f_y(y,y') \qquad (4)$$

where
$$\int f_E(E)dE = 1$$
$$\int f_x(x,x') \, dxdx' = 1$$
$$\int f_y(y,y') \, dydy' = 1$$

and I(t) is the total injected current. Thus, one stringent condition implied by this assumption is an achromatic deflection system. Under these conditions, then, the multidimensional integral for N may be decomposed into the product of two integrals, so that

$$N = \alpha_v N_h$$

where

$$\alpha_v = \int \rho_v(y,y')\ f_y(y,y')\ dy\ dy' \tag{6}$$

$$N_h = \int I(t)\ f_E(E)\ f_x(x,x')\ \rho_h(x,x',E,t)\ dx\ dx'\ dE\ dt$$

III The Conditions Which Determine ρ for an Ideal Machine

(a) The vertical acceptance ρ_v. If s is path length along an electron's equilibrium orbit in the synchrotron (s = 0 at the injection azimuth), and y is the vertical deviation from this orbit, then in an ideal machine[1)]

$$\frac{d^2y}{ds^2} + h_v(s)y = 0 \tag{7}$$

where $h_v(s)$ is a periodic function determined by the magnetic field in the machine. The motion described by this equation is the vertical betatron oscillation. If $y_{max}(y_o, y_o')$ is the largest value of y attained by a particle with initial conditions y_o, y_o' (y must be bounded), we clearly have

$$\rho_v(y_o, y_o') = \begin{cases} 1 \text{ if } y_{max}(y_o, y_o') < b \\ \\ 0 \text{ if } y_{max}(y_o, y_o') \geq b \end{cases} \tag{8}$$

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1) For the theory of the electron synchrotron, see, for example, R. R. Wilson, Handbuch der Physik, <u>44</u>, 170, (1959).

where b is the smallest distance from the equilibrium orbit to the vertical defining aperture in the synchrotron. It can be shown that the condition

$$y_{max}(y_o, y_o') = b$$

defines an ellipse in the (y_o, y_o') plane. Thus α_v is that fraction of the beam which falls inside this ellipse. Fig. 1 gives this vertical acceptance ellipse for injection into the Caltech synchrotron at the beginning of a magnet section.

(b) The radial acceptance ρ_h. The determination of the function $\rho_h(x,x',E,t;\ V,t_1)$ is not so trivial. For, in addition to the radial betatron oscillations which particles execute about their instantaneous equilibrium orbits, the equilibrium orbits themselves execute radial oscillations after the r.f. is turned on.

To every energy E, there corresponds a closed instantaneous equilibrium orbit in the synchrotron, defined by the condition that particles with energy E, launched on it, will remain on it. In general, however, particles of this energy will follow paths which deviate by an amount q(s) in radius from it, where

$$\frac{d^2q}{ds^2} + h_h(s)q = 0 \qquad (7')$$

and $h_h(s)$ is a periodic function determined by the magnetic field. The amplitude of these radial betatron oscillations is determined by q_o and q_o', the initial radial displacement from and angle to the equilibrium orbit. Since the radius of the equilibrium orbit for a particle depends upon its energy, q_o depends not only on x_o (the position of the particle in the injected beam) but also upon E. Furthermore, before the r.f. is turned on, the equilibrium orbit associated with a given energy migrates toward the center of the machine due to the rise in the magnetic field with time. Thus, q_o also depends upon t, the time at which the particle is injected.

After the r.f. is turned on, the energy which particles have is no longer conserved so that their equilibrium orbits stop shrinking. Rather, the orbits oscillate about a synchronous orbit defined by the condition that a particle on it traverses the machine in an integral number of r.f. periods. The energy associated with the synchronous orbit will be called the synchronous energy, E_s. Also, the radial distance from the lip of the deflector to the synchronous orbit will be taken as the radial half aperture, a. Fig. 2 illustrates these distances in the horizontal plane at the injection azimuth at the instant a particle is injected.

Since the magnetic field increases an amount ΔB during one turn, a relativistic particle will remain on the synchronous orbit only if, in one turn, it gains an amount of energy

$$u = eR_s \ \Delta B$$

where R_s is the radius of the synchronous orbit in a magnetic field section. Hence its phase with respect to the r.f. accelerating voltage must be ϕ_s where

$$u = V \sin \phi_s \tag{10}$$

However, a particle whose equilibrium orbit does not coincide with the synchronous orbit or whose phase is not ϕ_s when the r.f. is turned on will execute phase oscillations and concomitant energy and equilibrium orbit oscillations. The equations governing these are

$$\begin{aligned} \ddot{\phi} &= M(1 - \frac{V}{u} \sin \phi) \\ X &= N \dot{\phi} \end{aligned} \tag{11}$$

where ϕ is phase with respect to the r.f., X is radial deviation of the equilibrium orbit from the synchronous orbit, and M and N are (adiabatic) constants depending upon the synchrotron parameters. A first integral yields

$$X = C \sqrt{\phi + \frac{\cos \phi}{\sin \phi_s} - K} \qquad (12)$$

where $C = N \sqrt{M}$ and K is a constant of integration determined by initial conditions. From (12) we find that

$$X_{max} = C \sqrt{\phi_s + \frac{\cos \phi_s}{\sin \phi_s} - K} \qquad (12')$$

is the maximum deviation from the synchronous orbit during the course of the acceleration. Thus, if q_{max} is the maximum amplitude of a particle's radial betatron oscillations, its initial conditions must be such that the relation

$$X_{max} + q_{max} < a \qquad (13)$$

be satisfied if it is to be accelerated to high energies.

However, even if this relation is satisfied, it will not be accelerated if it does not survive from the instant it is injected till the r.f. is turned on. If the magnetic field did not rise in this time interval, the particle would surely be lost since it would strike the deflector lip on a subsequent return to the injection azimuth. However, due to the shrinking of the equilibrium orbit, this does not necessarily happen. If we define P_n as the radial distance measured toward the machine center from the deflector lip to the particle's instantaneous position at the injection azimuth on its n^{th} return to it, it is clearly necessary that

$$P_n > 0 \quad , \quad n = 1, 2, 3, \ldots\ldots$$

if the particle is to miss the deflector before the r.f. is turned on. This distance is made up of two parts: one due to the displacement of the particle from its equilibrium orbit and the other due to the migration of the equilibrium orbit itself. Since the betatron oscillation equation is linear, the radial deviation from the equilibrium orbit must be a linear function of the initial deviations, q_o and q_o', but with coefficients which are dependent upon turn number, n. In the linear approximation, the equilibrium orbit corresponding to any energy shrinks a constant amount σ on each turn. Thus we have

$$P_n = n\sigma + (A_n q_o + B_n q_o') \tag{14}$$

IV Calculation of the Radial Capture Efficiency in Various Approximations.

(a) Persico's approximation. If the injected particles are assumed to be monoenergetic and launched tangent to the equilibrium orbit from a point, we have the situation studied in detail by E. Persico[2). It is assumed that the duration of injection is comparable to the time required for the equilibrium orbit to migrate across the half aperture. The small effects on the betatron oscillations due to the straight sections will be ignored. Thus, no particles injected after the equilibrium orbit has passed the injection point are lost due to striking the deflector lip. If the time origin be taken as the instant the equilibrium orbit coincides with the injection point, we have

$$q_{max} = |q_o| = \gamma\,|t|$$

where γ is the rate of shrinkage of the equilibrium orbit.

Hence, all particles whose initial phase with respect to the r.f. is such that the conditions

$$X_{max} < (a - \gamma t) \equiv p \tag{15}$$

and

$$t > 0$$

2) E. Persico, A Theory of the Capture in a High Energy Injected Synchrotron, Nuovo Cimento Suppl. (Ser. 10) 2, 459 (1955). This section follows closely Persico's treatment.

are satisfied will be accelerated[3]; the quantity p will be called the maximum available aperture for synchrotron oscillations. Since $X = 0$ when the r.f. is turned on, the constant K in equation (12) is

$$\left(\phi_o + \frac{\cos \phi_o}{\sin \phi_s}\right)$$

where ϕ_o is a particle's initial phase. Thus, if $\phi_{o_{max}}(\alpha;\ \phi_s)$ and $\phi_{o_{min}}(\alpha;\ \phi_s)$ are the two solutions to the equation

$$\alpha = \frac{p}{C} = \sqrt{\left(\phi_s + \frac{\cos \phi_s}{\sin \phi_s}\right) - \left(\phi_o + \frac{\cos \phi_o}{\sin \phi_s}\right)}\ , \tag{16}$$

particles for which

$$\phi_{o_{min}} < \phi_o < \phi_{o_{max}}$$

will be accelerated. Thus, in keeping with the approximation which led from $g(\tau)$ to $\rho(t)$, we have

$$\rho_h(E{=}E_s,\ t,\ x{=}0,\ x'{=}0;\ \phi_s) = \frac{\phi_{o_{max}} - \phi_{o_{min}}}{2\pi} \tag{17}$$

This function is displayed versus α for several values of ϕ_s in Fig. 3.

To proceed further, it is necessary to know the time dependence of the current $I(t)$. We shall assume that the current is turned on sharply at $t = t_i$ and held at a constant value $\underline{I}$ until it is turned off sharply at $t = t_f$. Then

$$N_h\ (\phi_s,\ t_i,\ t_f) = I\ \Theta_{eff} \tag{18}$$

where

$$\Theta_{eff} = \int_{t_i}^{t_f} \rho(t;\ \phi_s)\ dt$$

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3) It is, of course, assumed that the particle energy is E_s.

and is the effective duration of injection.

By changing variables from time to α, the normalized aperture available for synchrotron oscillations, we can express Θ_{eff} as

$$\frac{C}{\gamma} \int_{\alpha_f}^{\alpha_i} \rho(\alpha; \phi_s)\, d\alpha \tag{19}$$

This integral is independent of machine parameters except for ϕ_s; thus it is useful to define the universal function

$$G(\alpha; \phi_s) = \int_0^{\alpha} \rho(\alpha; \phi_s)\, d\alpha \tag{20}$$

This function has been computed by Persico[2)] for $\alpha < 1.2$ and ϕ_s from 30^o to 80^o. Its extension[4)] to $\alpha = 4$ and for angles down to 4^o is shown in Fig. 4. It should be noted that the parameter α has a maximum value for a given machine. In particular

$$\alpha_{max} = \frac{a}{C}$$

corresponding to those particles injected just when the equilibrium orbit passes the injection point. Furthermore, for relativistic particles, C is proportional to

$$\frac{1}{\sqrt{E_s}}$$

so that α_{max} increases as the root of the injection energy in a given synchrotron.

Since the characteristics of the injector will be fully specified by one more parameter, the duration of injection Θ, this will be taken as an independent variable. We then wish to find the values of t_i and ϕ_s

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4) This integral as well as all others given in this report were evaluated numerically on a Burroughs 220 digital computer. A resume of calculational procedures is given in Appendix III.

which maximize Θ_{eff} for constant Θ. It is clear that t_i should be taken as the instant that the equilibrium orbit passes the injection point since this leads to the smallest average betatron oscillation amplitude[5]. The optimum value of ϕ_s for a given E_s, injection duration $\Theta = (t_f - t_i)$, and machine can be determined by using the curves in Fig. 4 and the appropriate values of C and γ. The pertinent machine constants for the Caltech synchrotron are shown in Tables I and II. Figs. 5, 6 show Θ_{eff} as a function of ϕ_s for various Θ's at different energies.

(b) Arbitrary energy distribution. In this section, the function

$$\rho_h(E, t, x = 0, x' = 0)$$

will be considered in detail. For convenience, we shall consider ρ as a function of q_o, the initial displacement of the injected particle from its equilibrium orbit, rather than time.

In the linear approximation, the radial distance between two equilibrium orbits is proportional to the difference of their corresponding energies and is independent of their radii. Thus, if R_s is the radius corresponding to energy E_s and R to energy E, we can write

$$X = (R - R_s) = \kappa\eta \tag{21}$$

where κ is the proportionality constant and

$$\eta = \frac{E - E_s}{E_s} \tag{22}$$

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5) Strictly speaking, Θ_{eff} will be larger if t_i is taken earlier by one revolution period than the indicated instant. For particles injected at this time also miss the deflector and the average betatron amplitude will be in fact slightly smaller. This is a negligible effect, however, in the assumed multiturn injection.

TABLE I.

Energy Independent Parameters for Caltech Synchrotron.

Synchronous orbit radius, R_s	376 cm
Energy gain per turn, u	720 ev
Rate of rise of magnetic field	64 kg/sec
Field index, n	0.6
R.F. harmonic number, k	4
Number of straight sections	4
Ratio of peripheral length of synchronous orbit to that of a circle of radius R_s, Λ	1.264
Radial half aperture, a	13 cm
Vertical half aperture, b	3 cm

TABLE II.

Energy Dependent Parameters.

Synchronous energy, E_s	10	5 Mev
γ	0.64	1.16 cm/μsec
C	1.60	2.16 cm
Orbit shrinkage per turn, σ	.064	.116 cm
α_{max} (corresponding to $t_i = 0$)	4.00	2.95

If once again we take the time origin as the instant that the equilibrium orbit corresponding to the synchronous energy coincides with the point of injection, we have

$$q_o = \gamma t - \kappa\eta \tag{23}$$

and since $q_{max} = |q_o|$, we have for the aperture available for synchrotron oscillations

$$p = a - |\gamma t - \kappa\eta| \tag{24}$$

If a particle is injected before its equilibrium orbit has migrated past the injection point, it will strike the deflector on its first traversal of the machine. Thus, we have

$$\rho = 0$$

when $q_o < 0$.

When the accelerating voltage is turned on, there will be particles in the machine with equilibrium orbits differing from the synchronous orbit. Thus, phase oscillations are excited by initial radial deviations, X, from the synchronous orbit, as well as initial phases ϕ_o differing from ϕ_s. Since $X_{max}(X, \phi_o) \geq |X|$ in general, the capture efficiency will be zero if $|X| > p$. Also for a given equilibrium phase angle ϕ_s, there is a maximum amplitude of synchrotron radial oscillation for captured particles. If this maximum is Ξ_{max}, then the capture efficiency is zero if $|X| \geq \Xi_{max}$. Equality in this condition gives the maximum possible energy spread that the synchrotron can accept. Furthermore, since a particle can utilize only Ξ_{max} of its aperture available for synchrotron oscillations, ρ is independent of p for $p > \Xi_{max}$. These features of the capture efficiency are illustrated in Fig. 7.

The function $\rho(p,\eta)$ may once again be computed using equation (12') with

$$K = \phi_o + \frac{\cos \phi_o}{\sin \phi_s} - \left(\frac{X}{C}\right)^2 \tag{25}$$

As before

$$\rho(q_o,\eta) = \frac{\phi_{o_{max}} - \phi_{o_{min}}}{2\pi} \tag{26}$$

and

$$N_h = \int_t \int_E I(t)\ f_E(E)\ \rho(q_o,\eta)\ dE\ dt \tag{27}$$

Making the same assumptions about the current as before, we can perform the time integration and obtain the average capture efficiency as a function of energy, parametrically dependent upon t_i, Θ, and ϕ_s. With $f_E(E)$ given, the effective duration of injection can be determined. Fig. 8 gives a typical capture efficiency as a function of η, but averaged over time.

For a given energy distribution f_E and duration of injection Θ, one can maximize Θ_{eff} by appropriate choice of t_i and ϕ_s. This has been done for a uniform energy distribution for the Caltech synchrotron at two injection energies, 5 and 10 Mev. In this case, we take

$$f_E(E) = \frac{1}{\Delta E}$$

so that I is to be interpreted as the current contained in an energy interval ΔE. Thus, we can write

$$N_h = I \frac{1}{\left(\frac{\Delta E}{E_s}\right)} \int_t \int_\eta \rho(q_o,\eta)\ d\eta\ dt$$

If we take

$$\frac{\Delta E}{E_s} = 1 \text{ per cent}$$

so that $I_{1\%}$ is the current in a 1 per cent energy spread about E_s, then

$$N_h = I_{1\%}\ \Theta_{eff}$$

where $$\Theta_{eff} = \int_t \int_\eta \rho(q_o,\eta)\, d(100\ \eta)\, dt$$

Fig. 9 shows the variation of Θ_{eff} with t_i for a typical value of ϕ_s; Fig. 10 gives its variation with ϕ_s for a constant t_i. Finally, Figs. 11 and 12 give the maximum value of Θ_{eff} and the corresponding optimum phase ϕ_s at the two energies as functions of injection duration. These curves also give the percentage energy range over which particles are accepted at the optimum values of t_i and ϕ_s.

(c) Injection from a distributed source with an angular distribution. The major modification to the preceding considerations is the fact that the consequences of the condition

$$P_n > 0$$

are no longer trivial. Of much less importance is the fact that the amplitude of the radial betatron oscillation q_{max} is no longer simply $|q_o|$; instead q_{max}^2 is a quadratic in q_o and q'_o. Also, it is necessary that on its <u>first</u> traversal of the machine a particle stay within the aperture. This condition has not been explicitly incorporated in the calculations; rather, it has been assumed that the good field aperture extends to a sufficiently large radius for it to be always satisfied. In practice, this implies that the deflector lie about 8σ within the good field region.

It is again convenient to take q_o as an independent variable since the condition that particles not strike the deflector on any subsequent turn can be conveniently expressed using it and because the aperture available for synchrotron oscillations can be readily expressed in terms of it.

By using Eq. (14), we can write the condition for missing the deflector in the form

$$q_o > q_o(q'_o;\ x_o) \qquad (28)$$

where $q_o(q'_o;\ x_o)$ is that value of q_o such that $P = 0$ on some turn.

Fig. 13 gives the family of these functions for the Caltech synchrotron. Thus, for given values of x_o and $x'_o = q'_o$, the same considerations as in the previous section can be used to find ρ_h, its average over time, and the effective injection duration for a uniform energy distribution. It is only necessary to set

$$\rho = 0$$

when $q_o > q_o(q'_o; x_o)$ and change the expression for q_{max} to take the initial angle into account. The results of such calculations for various values of x_o and x'_o yield a surface over the (x_o, x'_o) plane

$$\theta_{eff} (x_o, x'_o; t_i, \theta, \phi_s) .$$

Such surfaces have been computed for typical values of θ using the value of ϕ_s which maximizes

$$\theta_{eff} (x_o = 0, x'_o = 0)$$

and values of t_i near its optimum. These surfaces are shown in Figs. 14 to 18 for the Caltech synchrotron as contour maps. The four diagrams for 10 Mev show the sensitivity of the shape of the capture efficiency surfaces to changes in the injection parameters.

V Application to the Caltech Synchrotron

(a) The present injection system. The principal components of the present injector are a 1 Mev pulse transformer and a 90^o electrostatic deflector. A 60 ma. pulse of electrons at the exit from the deflector is maintained for several revolution periods, corresponding to about 10^{11} electrons. The quality of the injected beam is about 1/5 cm - milliradians[6)] at each energy, but, due to the chromatic nature of the deflector, the position of this "spot" in radial phase space is a function of

6) This assumes a cathode 1/2 cm in diameter at a temperature of 1500^oC.

the particles' precise energy. However, at a given instant, the injected beam has an entirely negligible energy spread (thermal) and provisions are taken to maintain the energy constant to within terms cubic in time. With these injection characteristics, one would expect beam intensities in the synchrotron of a few times 10^{10} electrons. In fact, typical intensities of about 2×10^9 and, rarely, 4×10^9 electrons are observed.

This discrepancy of an order of magnitude is believed to be due to the low injection energy. Since no one has yet succeeded in calculating the observed intensities, it cannot be said that the reason for the discrepancy is entirely understood. However, the injection energy is suspect for the following reasons:

1) The injection field is only 13 gauss, well below even the iron's remnant magnetization. This necessitates elaborate corrective measures to properly shape the injection guide field. Measurements made by R. Macek of the vertical betatron oscillation frequency near injection indicate that the effective vertical field index varies in time and dips down to very near 0.5. Thus, the machine is near a dangerous half-integral resonance for some time near injection.

2) An upper limit on the number of electrons which can be accelerated is set by the space charge forces generated by the beam. The defocussing force due to the electrons themselves tends to reduce the vertical and horizontal betatron oscillation frequencies while the focussing force due to positive ions created by the beam tends to increase these frequencies. For purposes of estimating these perturbations, we ignore the effects of the conducting walls and assume the beam is long and uniformly distributed within a circular cross section. The increase in the vertical and horizontal restoring forces is easily shown to be

$$2\pi\, r_e\, mc^2 \left[n_+ - n_- \left(\frac{m}{E}\right)^2 \right] \xi$$

where $r_e = 2.82 \times 10^{-13}$ cm, the classical electron radius

m = mass of electron

E = energy of electron in beam

n_+ = number of quasi-stationary positive charges minus the number of quasi-stationary negative charges per unit volume

n_- = number of high energy electrons per unit volume

ξ = q or y as the case may be.

Letting ν_ξ be the ratio of the betatron frequency to the circulation frequency ω and $\delta\nu_\xi$ its space charge perturbation, we have

$$\delta\nu_\xi = \frac{\pi r_e c^2}{\omega^2} \frac{1}{\nu_\xi} \left(\frac{1 + \Lambda}{2}\right)^2 \left[n_+ \left(\frac{m}{E}\right) - n_- \left(\frac{m}{E}\right)^3\right]$$

At operating pressures in the machine, the average time required for an electron to produce an ion is of the order 100 μsec. Since it is not expected that the beam will exhibit a net positive charge, we shall assume that the ratio of the number of positive ions to the number of electrons increases from zero to about one in the first hundred μsec after injection, and then remains about one.

The most serious resonances near the operating point are at $\nu_q = 2/3$ and $\nu_q = 3/4$ which are within about $\pm.05$ of the ideal value of $\nu_q = .71$. The number of electrons in a beam 3" in diameter filling about half the machine perimeter which will produce this large a $\delta\nu_q$ at some time after injection is 8.5×10^{10}. The limitation in this case is the over-compensation of the positive charge density; in the other extreme of entirely neglecting the ions, one finds that the corresponding number is 2.5×10^{11}.

However, it has been pointed out by H. Snyder that the synchrotron phase oscillations produce a peripheral compaction of the beam. In particular, injected particles within an energy spread ΔE and initial phases near the equilibrium phase are compressed into a short bunch twice per phase oscillation. The ratio of minimum to maximum longitudinal length of a given number of such electrons is

$$\sqrt{\frac{E_s}{u}\ \frac{\tan \phi_s}{2\pi\Lambda(1-n)k}} \qquad \left(\frac{\Delta E}{E_s}\right)$$

where E_s = synchronous energy

u = energy gain per turn

ϕ_s = equilibrium phase

Λ = ratio of peripheral length of machine to an equivalent circular machine

k = r.f. harmonic number

Near injection, this constant has a value of .007 per kev of energy spread.

Thus, for a 1 kev energy spread in the injected beam, the negative space charge density will take on values about 100 times larger than that used above. For this large an n_-, the positive ions can be neglected and a few times 10^9 electrons will depress the operating point down to the $\nu_q = 2/3$ resonance twice per synchrotron oscillation. A smaller energy spread in the injected beam will lead to a proportionally smaller number of electrons.

Clearly, an order of magnitude calculation of this nature can only indicate what physical effects are of importance. No one to our knowledge has as yet undertaken a detailed consideration of the coupling of synchrotron to betatron oscillations through space charge effects. However, the above crude calculation suggests that the Caltech synchrotron may be space charge limited.

(b) A 5 to 10 Mev linac injector. It is known that the errors in the synchrotron guide field tend to stay constant in time while the average field increases linearly; thus, at higher energies, the uncorrected field shape is more nearly ideal. Furthermore, any space charge effects are depressed at higher energies. However, the wide energy distribution in the output of the linac and the slow spiraling rate of high energy orbits in the synchrotron must be taken into account. It was for this purpose that the present study was undertaken. Since an ideal machine has been assumed throughout, which the Caltech synchrotron is decidedly not, the calculated beam intensities can only be taken as upper limits.

The pertinent parameters for the electron linac manufactured by the Applied Radiation Corporation are as follows:

A. Current and Energy Resolution -

In the output energy range of 5 to 10 Mev, 100 ma or more is available in a 1 per cent energy spread. The total current is about 1/2 amp.

B. Beam Quality -

Spot sizes of < 0.3 cm have been observed, but this is a function of beam tuning. A spot size of 0.6 cm can, however, be guaranteed. The angles are less than 1 milliradian.

From Figs. 11 and 12, a real time duration of injection beyond which it is not especially profitable to operate the linac can be set; these times are indicated as Θ_{opt} in the figures. We shall assume that the linac can maintain a beam for at least this length of time. In Table III we give the calculated synchrotron parameters which result in optimum performance for injection from a point source with a uniform energy distribution. Figs. 14 through 18 indicate that so long as the beam does not have too large a radial size at its entrance into the synchrotron and a quality of .6 milliradian-cm, one can expect the distributed source to be about 70 per cent as efficient as a point source. Thus, taking Θ_{opt} as the actual duration of injection, we find an effective duration of injection of 1.5 μsec at 10 Mev and half that at 5 Mev. Since this number is normalized to the current in a 1 per cent energy interval, the number of accelerated electrons is about 10^{12} for 10 Mev injection and half that for 5 Mev injection. The parameters of Table III may be used to make an estimate of the space charge limited number of electrons as was done for the present injector. Making the same assumption concerning the positive ions, one finds that 1.2×10^{12} is the limiting number of electrons for 10 Mev injection and 7.0×10^{11} for 5 Mev

TABLE III.

SUMMARY OF RESULTS FOR TANGENTIAL INJECTION FROM A POINT WITH A UNIFORM ENERGY DISTRIBUTION

Synchronous Energy, E_s	5	10 Mev
Optimum Injection Duration, Θ_{opt}	6.0	11.2 μsec
Effective Duration at Θ_{opt}	1.1	2.2 μsec
Accepted Energy Spread at Θ_{opt}	0.9	0.9 %
Optimum Synchronous Phase, ϕ_s	15^o	10^o

injection. Thus, the actual intensities may very well be space charge limited. However, if the ions are ignored, the limiting intensities are about an order of magnitude larger. This suggests that an ion sweeping arrangement in the machine may be useful in attaining the calculated beam intensity with high energy linac injection.

Appendix I The Effect of Turning On the Accelerating Voltage in a Finite Time[7)]

In practice, the voltage on the cavity is turned on according to some law

$$V = V(t)$$

characterized by a turn-on time T. Including this function in the analysis complicates it in two ways. First, a particle's phase oscillation initial conditions (ϕ_o, X_o) at time t_o when the turn-on is started will be transformed into different values (ϕ_o', X_o') at the time $t_1 = (t_o + T)$ when the voltage has essentially reached its final value V_f. From time t_1 on, the phase oscillations are described as in the body of the report. Secondly, during the interval t_o to t_1, equilibrium orbits no longer shrink linearly with time, but the precise manner in which they move depends upon ϕ_o and X_o. Thus q_o is no longer a simple linear function of time and energy. The importance of the first effect depends upon the relative sizes of T and the circular period of the synchrotron oscillations

$$\left(\frac{T_{sync.\ osc.}}{2\pi} \right) ;$$

that of the second, upon the relative sizes of T and the time required for an equilibrium orbit to migrate across the half-aperture a. These times are given in Table IV for the Caltech synchrotron at 5 and 10 Mev, where equilibrium phases ϕ_s indicated by the fast turn-on computations have been used.

---

7) In the approximation of monoenergetic injection tangent to the machine from a point for a time equal to that required for the equilibrium orbit to shrink from the injection point to the synchronous orbit, Persico and Bernardini have calculated the effect of power law voltage rises, i.e., $V(t) = V_f(\frac{t}{T})^n$ for $t < T$ and constant thereafter. See Teoria della Cattura Semirapida e Lenta, Rapporto n.12, I.N.F.N. - Sezione Acceleratore.

TABLE IV.

Synchronous Energy	5	10 Mev
Time for equilibrium orbit to migrate 13 cm	11.2	20.3 μsec
Circular period of synchrotron oscillation for ϕ_s as in Table III	.62	.70 μsec
Final peak voltage to obtain ϕ_s	2.8	4.1 kev

Appendix II The Coupling of Horizontal to Vertical Betatron Oscillations

It has been a tacit assumption in all the calculations made that it is feasible to allow horizontal betatron oscillations with amplitudes of the order of the horizontal aperture. However, this will not be the case if there is any appreciable feeding of energy from horizontal oscillations to vertical since the vertical aperture is relatively small. Hence, it is pertinent to examine the coupling between these two oscillations.

Coupling between the modes arises from several sources:

1) The effect of construction errors on the linear terms of the restoring forces:
   a) torsional misalignment of the guide field
   b) azimuthal dependence of the guide field giving rise to an azimuthal component of the magnet field.
2) Non-linear terms in the restoring forces in a perfect machine.
3) The effect of constructional errors on the non-linear terms.

If the construction errors are taken to be of first order in smallness as are the deviations from the equilibrium orbit, then the indicated couplings are of the following orders:

1) 1st and higher
2) 2nd and higher
3) 3rd and higher

We shall be interested here only in terms up to second order, so that (3) will be ignored. We shall first consider only the longitudinal field and the non-linear terms, deferring torsional misalignments till later. Let the vertical component of the magnetic field in the plane of symmetry ($y = 0$) be

$$B_y(q,o,z) = B_o \left[1 - n\frac{q}{R} + \frac{1}{2}\left(n^2 - R\frac{\partial n}{\partial x}\right)\frac{q^2}{R^2} + \ldots\ldots \right]$$

where q is radial displacement from an equilibrium orbit of radius R and z is distance along it. In the ensuing discussion, we shall neglect the effects of the straight sections so that B_o and n depend upon z only because of constructional errors. Then it can be shown that up to terms bilinear in x, y and their derivatives[8], that the equation describing the vertical deviations from the equilibrium orbit is

$$y'' + n\,y = \frac{q}{R}\,y'' + \frac{q'}{R}\,y' - \frac{B_o'}{B_o}\,\frac{q'}{R}\,y + \left[n^2 - n - R\,\frac{\partial n}{\partial x}\right]\frac{q}{R}\,y$$

where primes denote derivatives with respect to $\Phi = \frac{z}{R}$. Since constructional errors are considered to be of first order in smallness, the terms in B_o' and $\frac{\partial n}{\partial x}$ are of higher order than is here being considered. Thus we have

$$y'' + k_y^2\,y = d\,(\Phi)$$

where

$$d(\Phi) = \frac{q}{R}\,y'' + \frac{q'}{R}\,y' - k_y^2\,k_x^2\,\frac{q}{R}\,y$$

$$k_y^2 = n$$

$$k_x^2 = 1 - n$$

If we take for y and q in the perturbation term their unperturbed values, we can obtain an approximate solution to this equation. Since the driving term is than a product of two sinusoids, a particular solution can be written by inspection. In fact, if

$$q_{unpert.} = \mathrm{Re}_i\; a\; e^{ik_x\Phi} \quad , \quad a = A\,e^{i\alpha}$$

$$y_{unpert.} = \mathrm{Re}_j\; b\; e^{jk_y\Phi} \quad , \quad b = B\,e^{j\beta}$$

where A, B, α, β are real, then (suppressing the two real value signs) we have

---

8) See, for example, P. A. Sturrock, Nonlinear Effects in Alternating - Gradient Synchrotrons, Ann. Phys. 3, 113 (1958), page 119.

$$d(\Phi) \cong \frac{ab}{R} \left[- k_y^2 (1 + k_x^2) + i\, j\, k_x\, k_y \right] e^{(i\, k_x + j\, k_y)\Phi}$$

and a particular solution to the equation is

$$\Delta_1 y(\Phi) = \frac{ab}{R} \; \frac{-k_y^2 (1 + k_x^2) + i\, j\, k_x\, k_y}{-k_x^2 + 2\, i\, j\, k_x\, k_y} \; e^{(i\, k_x + j\, k_y)\Phi}$$

Hence, if y_o and y_o'/R are the initial vertical displacement from and angle with respect to the equilibrium orbit, the non-linear coupling effectively modifies then by

$$\Delta_1 y_o = - \mathrm{Re}_i \, \mathrm{Re}_j \, \frac{ab}{R} \; \frac{-k_y^2 (1 + k_x^2) + i\, j\, k_x\, k_y}{-k_x^2 + 2\, i\, j\, k_x\, k_y}$$

$$\Delta_1 \left(\frac{y_o'}{R}\right) = - \mathrm{Re}_i \, \mathrm{Re}_j \, (i\, k_x + j\, k_y) \, \frac{ab}{R} \; \frac{-k_y^2 (1 + k_x^2) + i\, j\, k_x\, k_y}{-k_x^2 + 2\, i\, j\, k_x\, k_y}$$

Then we can write

$$y = (y_o + \Delta_1 y_o) \cos k_y \Phi + \frac{1}{k_y} (y_o' + \Delta_1 y_o') \sin k_y \Phi + \Delta_1 y$$

Using the parameters for the Caltech synchrotron, we can easily find upper limits for these perturbations to be

$$\Delta_1 y_o \text{ and } \Delta_1 y < .06 \text{ inches}$$

$$\Delta_1 \left(\frac{y_o'}{R}\right) < .3 \text{ milliradians}$$

We now consider the effect of torsional misalignments. If the guide field at each point along the orbit is ideal about a local plane of symmetry but this symmetry plane is rotated through an angle $\tau(\Phi) \ll 1$ about the equilibrium orbit, then with reference to coordinates taken in and perpendicular to the ideal symmetry plane, the magnetic field is

$$\vec{B} = \vec{B}_{ideal} - \Delta\vec{B}$$

where $$\Delta\vec{B} = -2\,n\,B_o\,\tau\,\frac{y}{R}\,\vec{1}_y - B_o\,\tau\,(1 - 2n\,\frac{q}{R})\,\vec{1}_q$$

and $\vec{1}_q$ and $\vec{1}_y$ are unit vectors in the radial and vertical directions. This field increment gives an additional force on the electron so that the vertical oscillation equation becomes

$$y'' + k_y^2\,y = 2\,n\,q\,\tau - R\,\tau$$

Since τ is periodic in Φ with period 2π, we can write

$$\tau = \sum_m \alpha_m\,e^{jm\Phi}$$

Then the particular solution which yields the first order driving term $-R\,\tau$ is

$$Y(\Phi) = R\sum_m \frac{\alpha_m}{m^2 - k_y^2}\,e^{jm\Phi}$$

In the absence of any further perturbations, an electron starting on the orbit defined by the periodic function $Y(\Phi)$ will stay on it. Thus, the first order perturbation arising from the torsional errors results in a shift of the equilibrium orbit from its ideal position. Letting y now designate deviations from the orbit defined by $Y(\Phi)$, we have

$$y'' + k_y^2\,y = 2\,n\,q\,\tau$$

Exactly as before, we can write the particular solution to this equation using the unperturbed value for q

$$\Delta_2 y = 2\,n\,a\,e^{ik_x\Phi}\sum_m \frac{\alpha_m\,e^{jm\Phi}}{k_y^2 + (i\,k_x + j\,m)^2}$$

This results in an effective perturbation to the initial conditions

$$\Delta_2 y_o = -2\ n\ a \sum_m \frac{\alpha_m}{k_y^2 + (i\ k_x + j\ m)^2}$$

$$\Delta_2 \left(\frac{y_o'}{R}\right) = -2\ n\ \frac{a}{R} \sum_m \frac{(i\ k_x + j\ m)\ \alpha_m}{k_y^2 + (i\ k_x + j\ m)^2}$$

These expressions could be used to set limits on the coefficients α_m to keep the coupling tolerable. Alternatively, the coefficients could be estimated and the perturbations calculated. We shall attempt the latter.

From the residue of carbonized vacuum pump oil on the outer vacuum chamber wall caused by radiation at high energies ("beam stain"), one can measure the vertical deviations of the equilibrium orbit from the ideal. In addition to torsional misalignments, these deviations are also caused by vertical displacements of the magnetic symmetry planes from the ideal geometric plane. Although we might obtain underestimates for α_m due to possible destructive interference between these effects, we shall assume that all wanderings of the equilibrium orbit are caused by torsional misalignments.

Beam stain measurements[9)] were made in April, 1959, and they are fit very well by the function

$$Y(\Phi) = -.178 - .161 \cos \Phi - .241 \sin \Phi + .036 \cos 2\Phi$$
$$-.006 \sin 2\Phi \qquad \text{(in.)}$$

Using these coefficients, we obtain generous upper limits for the perturbation as follows

---

9) As a result of these measurements, measures to restore the equilibrium orbit to its optimum position were taken. Thus, the present calculations, using these measurements, will presumably yield over-estimates.

$$\Delta_2 \, y_o \; < \; .02 \text{ inch}$$

$$\Delta_2 \left(\frac{y_o'}{R}\right) \; < \; .1 \text{ milliradians}$$

$$\Delta_2 \, y \; < \; .05 \text{ inch}$$

Thus, with reservations concerning the torsional misalignments, it appears that couplings will not make the proposed large radial oscillations unfeasible.

Appendix III Computational Procedures

The calculations leading to the capture efficiency diagrams have been performed on a Burroughs Datatron 220 digital computer. Two primary programs were written. We here give a resume of the formulas and numerical techniques used in these programs.

1) The first program computes the effective duration of injection for tangential injection from a point source with a uniform energy distribution.

$$\Theta_{eff} = \frac{1}{\gamma} \int_{\eta} \int_{\alpha} \rho(p,\eta;\, \phi_s)\, dp\, d\eta$$

where p is the aperture available for synchrotron oscillations and η is the fractional energy deviation from the synchronous energy.

$$\rho(p,\eta;\, \phi_s) = \begin{cases} \dfrac{\phi_M - \phi_m}{2\pi} & p \leq X_{max} \\[2ex] \left.\dfrac{\phi_M - \phi_m}{2\pi}\right|_{p = X_{max}} & p > X_{max} \end{cases}$$

$\phi_M > \phi_m$ are the two roots of

$$\frac{p}{C} = \sqrt{F(\phi_s) - F(\phi) + \left(\frac{X}{C}\right)^2}$$

$$F(\phi) = \frac{\cos \phi}{\sin \phi_s} + \phi$$

$$X = -\frac{R}{1-n}\,\eta$$

$$X_{max} = C \sqrt{2 \cot \phi_s + 2\, \phi_s - \pi}$$

$$C = \frac{R}{1-n} \sqrt{\frac{u\Lambda(1-n)}{\pi k \; E_s}}$$

The machine parameters have been defined previously. The integration limits on p are

$$p_{upper} \; (\eta) = \min \; (a - \gamma T_i - X, \; a)$$

$$p_{lower} \; (\eta) = \max \; (a - \gamma T_f - X, \; X)$$

while those on η are

$$\eta_{upper} = \frac{1-n}{R} \min \; (X_{max}, \; a, \; \gamma T_f)$$

$$\eta_{lower} = \frac{1-n}{R} \max \; (-X_{max}, \; -a, \; \tfrac{1}{2}(\gamma T_i - a))$$

where T_i = injection starting time

T_f = injection stopping time

The function ρ is evaluated by an iterative procedure employing Newton's rule and the integrations are performed by segmenting the range and applying 3-point Gauss-Legendre quadrature to each segment.

This program has three options:

a) Compute and print the function

$$\varepsilon(\eta) = \int_p \rho(p,\eta; \; \phi_s) \; dp$$

b) Compute and print Θ_{eff}

c) Search values of ϕ_s and T_i to maximize Θ_{eff} for a given duration of injection. After ϕ_s and T_i are found, Θ_{eff} and $\varepsilon(\eta)$ are printed.

2) The second program computes the effective duration of injection for particles at an arbitrary point in radial phase space. Let x and x' be the point under consideration. The origin in this phase plane is taken tangent to the equilibrium orbit and at the inner edge of the septum.

$$\Theta_{eff} = \frac{1}{\gamma} \int_{\eta} \int_{q_o} \rho(p(q_o), \eta; \phi_s)\, d\eta\, dq_o$$

where η, ρ, and p are the same as before. However, we now take the initial distance from the injection point to the equilibrium orbit as the independent variable. The available aperture is

$$p = a - \frac{1}{\cos\beta} \sqrt{q_o^2 - 2q_o \lambda' x' \sin\beta + (\lambda' x')^2}$$

where λ' and β are machine constants given later.

Since the integration limits are moderately complicated, it is convenient to define several auxiliary variables. Let

$$q_{max} = \min_{A_n<0} \left(\frac{1}{A_n} \left[B_n x' + x - n\sigma \right] \right) \qquad 1 \leq n \leq 100$$

$$q_{min} = \max_{A_n>0} \left(\frac{1}{A_n} \left[B_n x' + x - n\sigma \right] \right) \qquad 1 \leq n \leq 100$$

$$q_{\pm} = \lambda' x' \sin\beta \pm \cos\beta \sqrt{a^2 - (\lambda' x')^2}$$

$$q_{u1} = \min(q_{max}, q_+)$$

$$q_{\ell 1} = \max(q_{min}, q_-)$$

$$q_{u2} = \lambda' x' \sin\beta + \cos\beta \sqrt{(a - X)^2 - (\lambda' x')^2}$$

$$q_{\ell 2} = \lambda' x' \sin\beta - \cos\beta \sqrt{(a - X)^2 - (\lambda' x')^2}$$

$$q_{u3} = \gamma T_f + x + X$$

$$q_{\ell 3} = \gamma T_i + x + X$$

Then we have for integration limits on q_o,

$$q_{o\ upper} = \min (q_{u1}, q_{u2}, q_{u3})$$

$$q_{o\ lower} = \max (q_{\ell 1}, q_{\ell 2}, q_{\ell 3})$$

Similarly, let

$$\eta_{u1} = \frac{1-n}{R} (a - \lambda' x')$$

$$\eta_{\ell 1} = - \frac{1-n}{R} (a - \lambda' x')$$

$$\eta_{u2} = \frac{1-n}{R} X_{max}$$

$$\eta_{\ell 2} = - \frac{1-n}{R} X_{max}$$

$$\eta_{u3} = \frac{1-n}{R} (\gamma T_f + x - q_{\ell 1})$$

$$\eta_{\ell 3} = \frac{1-n}{R} (\gamma T_i + x - q_{u1})$$

Then the integration limits on η are

$$\eta_{upper} = \min (\eta_{u1}, \eta_{u2}, \eta_{u3})$$

$$\eta_{lower} = \max (\eta_{\ell 1}, \eta_{\ell 2}, \eta_{\ell 3})$$

In the formulas for q_{max} and q_{min}, the constants are

$$A_n = 1 - \cos n \mu + \tan \beta \sin n \mu$$

$$B_n = - \lambda' \frac{\sin n \mu}{\cos \beta}$$

where the quantities λ', β, and μ are machine constants arising in the solution of the radial betatron equation in a machine with straight sections. For radial oscillations in the Caltech synchrotron, they have the values

$$\lambda' = 643.2 \text{ cm}$$
$$\beta = 6.95^{\circ}$$
$$\mu = 256.57^{\circ}$$

The various critical values of q_o and η given above have simple physical interpretations. Reference to Fig. 13 shows that for a given x' ($= q_o'$) and x, there will be in general a minimum and maximum value of q_o consistent with the condition that the injected particles do not strike the deflector on a subsequent turn; these are q_{min} and q_{max}. In the program, the first 100 turns are investigated. Also, for a given x', there is a maximum (generally positive) and minimum (generally negative) value of q_o which lead to betatron oscillations of amplitude a; these are $q_{\pm}$. These four critical values lead to absolute maximal values which q_o can assume, i.e., q_{u1} and $q_{\ell 1}$. However, a particle is surely lost if its aperture available for synchrotron oscillations is smaller than X; equality in this condition leads to q_{u2} and $q_{\ell 2}$. Finally, the beginning and end of the injection pulse yield maximal values of q_o. The formulas for the q_o integration limits then follow immediately.

For a given value of x', the smallest possible betatron oscillation amplitude is $\lambda' x'$; thus, the largest possible available aperture is $(a - \lambda' x')$. This fact leads to the maximal values of energy deviation η_{u1} and η_{u2}. The second pair of critical values occur when

$$X = X_{max}$$

and the final pair lead to the absolute maximal values of q_o.

The function evaluation and integration in this program are carried out exactly as in the previous one.

This program has two options. It will either compute and print the effective injection duration for a specified value of x and x', or it will produce a table of $\Theta_{eff}(x, x')$ with specified steps in the two variables, halting when the effective duration drops below a specified value.

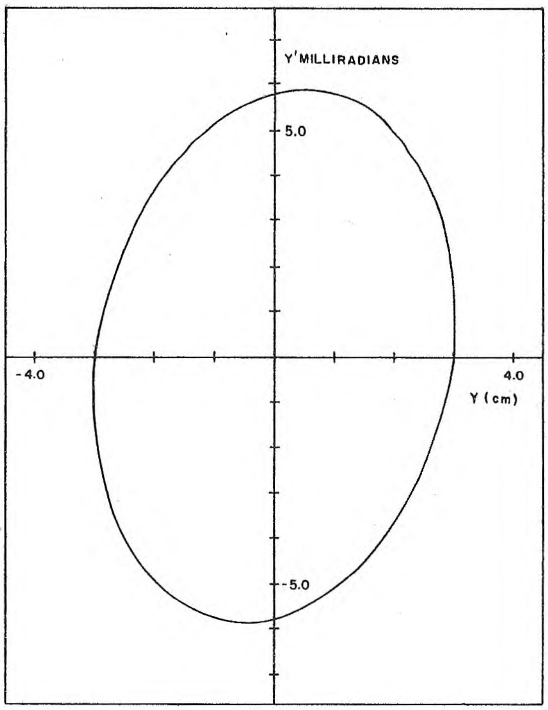

FIG. 1

FIGURE 1 - Vertical acceptance diagram for injection at entrance to magnet section.

It has been assumed that the maximum allowable amplitude of vertical oscillation is 3 cm.

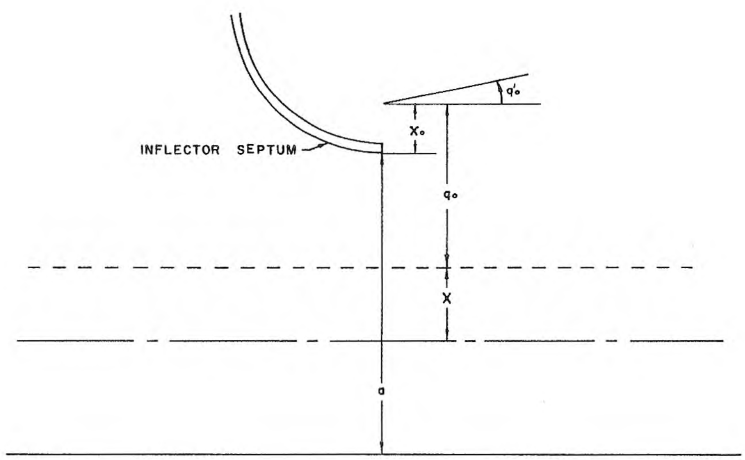

FIG. 2

FIGURE 2 - Definition of Initial Conditions

The dotted line represents the position of the instantaneous equilibrium orbit of a particle with energy deviation η when it is injected. The dot-dash line represents the position of the equilibrium orbit for a particle with synchronous energy, i.e., $\eta = 0$. The solid line is the position of the synchronous orbit; this is taken to be at the center of the aperture.

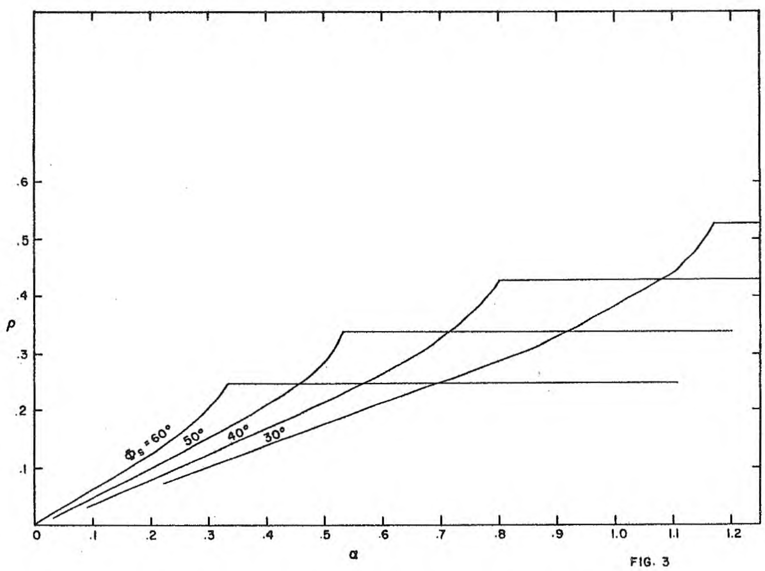

FIGURE 3
Capture efficiency as a function of normalized aperture available for synchrotron oscillations in Persico's approximation.

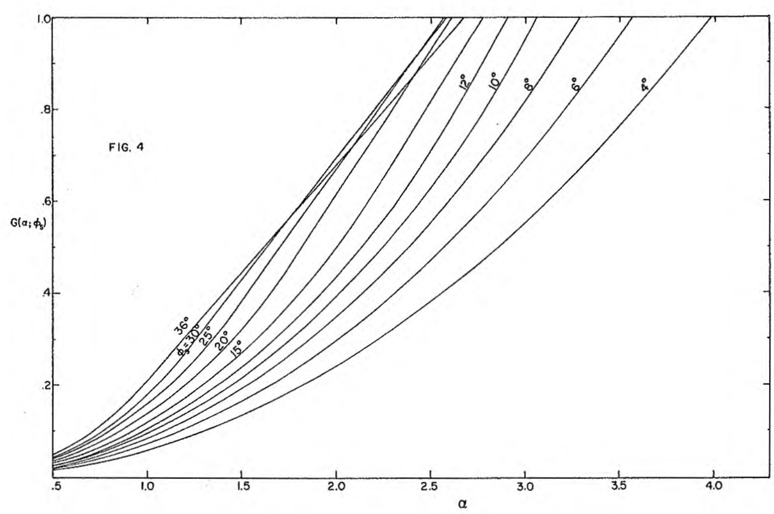

FIGURE 4
Extension of the universal function $G(\alpha;\ \phi_s)$. For smaller values of α and larger values of ϕ_s, see E. Persico, loc.cit., p. 466.

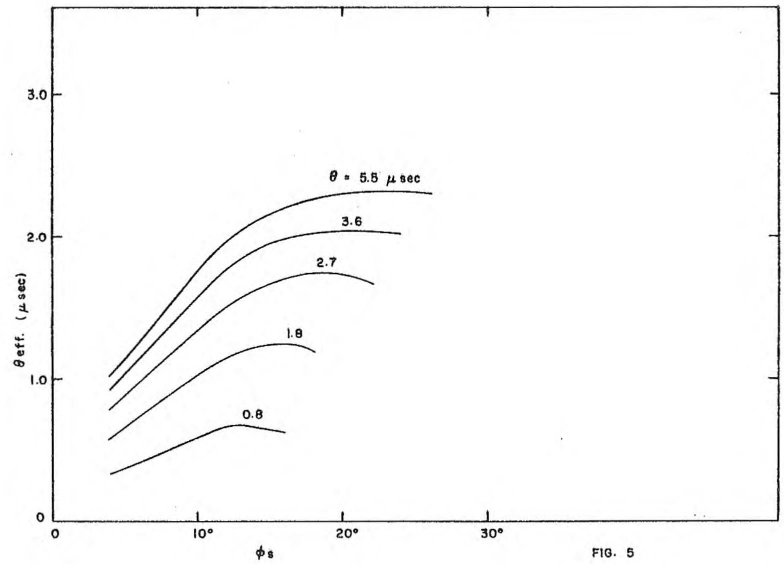

FIGURE 5
Effective duration of injection as a function of synchronous phase in Persico's approximation for the Caltech synchrotron. 5 Mev injection energy.

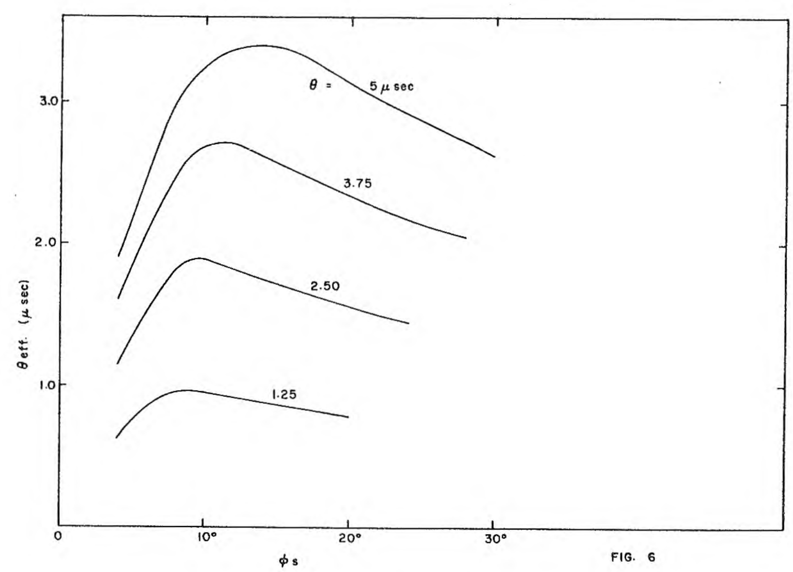

FIGURE 6
Same as Fig. 5, but for 10 Mev injection energy.

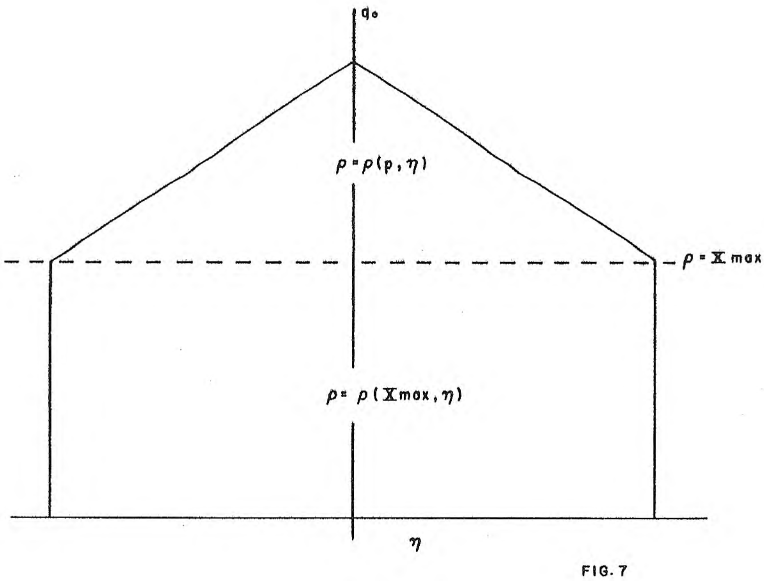

FIGURE 7
The region in (q_o, η) space in which the capture efficiency, ρ, is non-zero for tangential injection from a point. The capture efficiency is zero outside the indicated region.

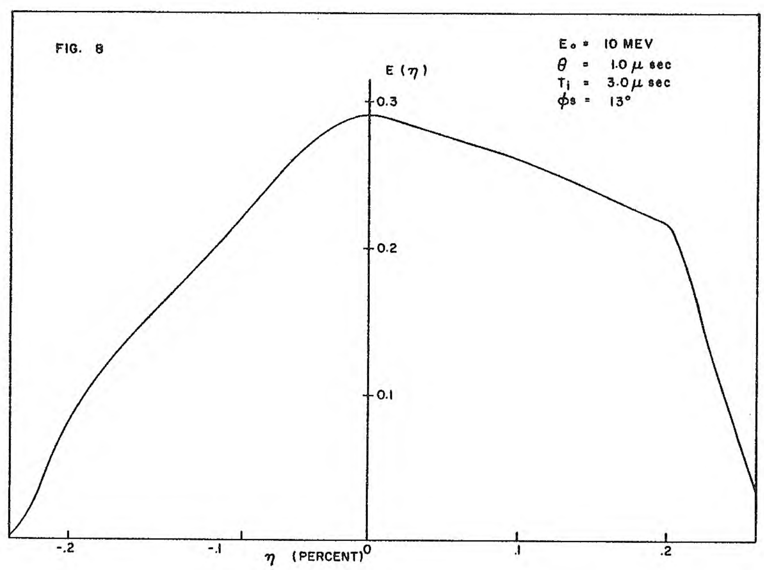

FIGURE 8
A typical capture efficiency averaged over time as a function of energy deviation from synchronous energy for tangential injection from a point.

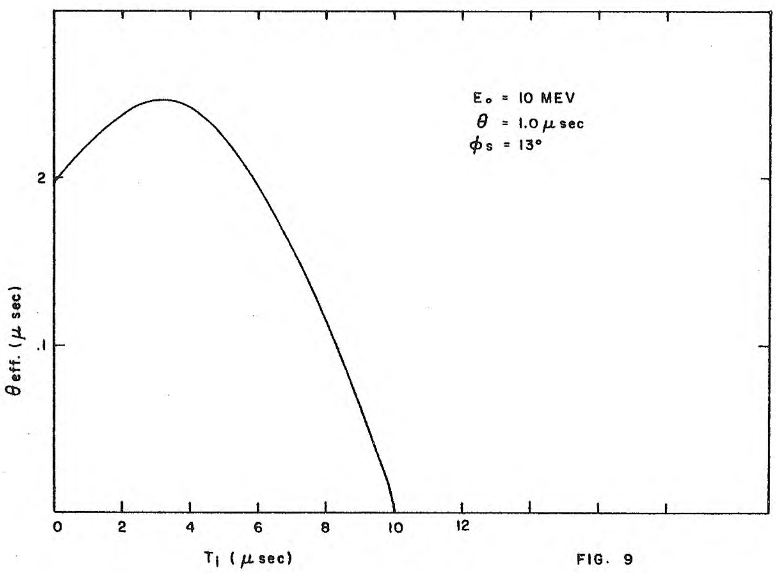

FIGURE 9

Effective duration of injection as a function of injection starting time for a given equilibrium phase for tangential injection from a point. A uniform energy distribution is assumed.

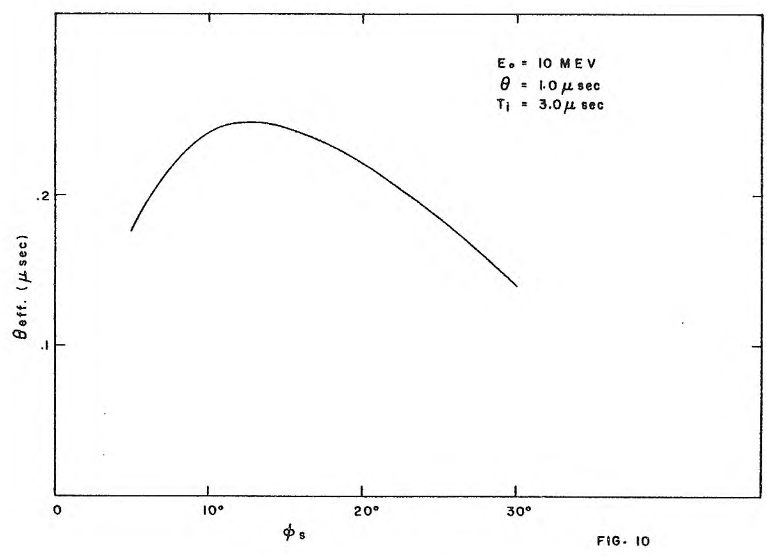

FIGURE 10

Effective duration of injection as a function of synchronous phase for a given injection starting time for tangential injection from a point. A uniform energy distribution is assumed.

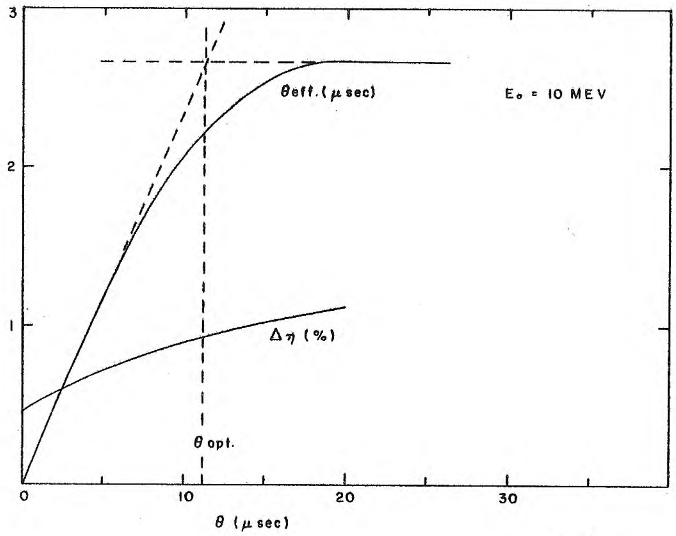

FIG. 11

FIGURE 11
Maximum effective duration of injection and accepted energy spread as a function of Θ for tangential injection from a point. A uniform energy distribution is assumed. The synchronous energy is 10 Mev.

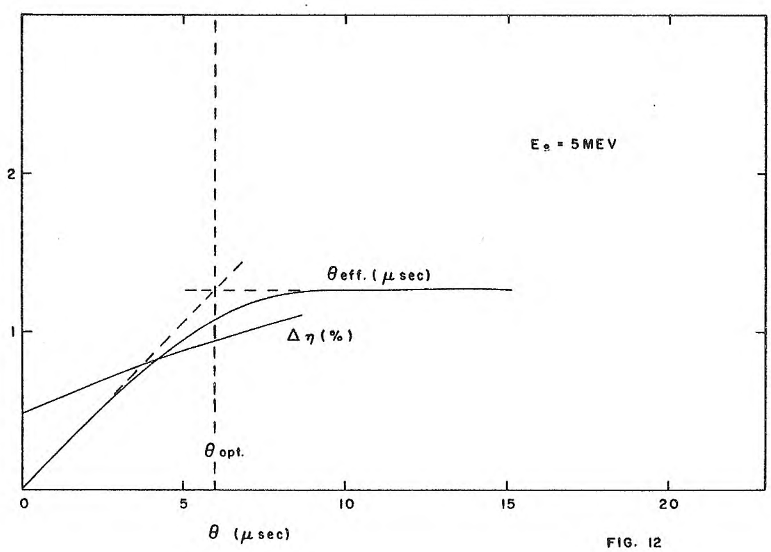

FIG. 12

FIGURE 12
Same as Fig. 11, but for a synchronous energy of 5 Mev.

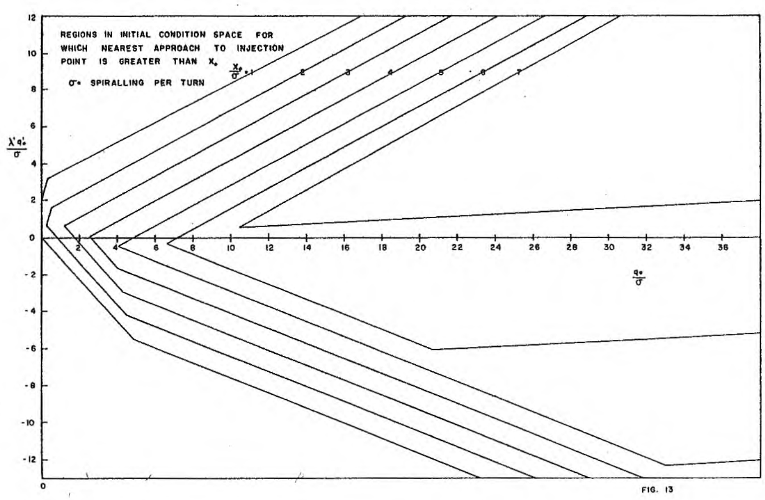

FIGURE 13

Regions in (q_0, q_0') space within which a particle must lie in order to miss the deflector during spiralling for various values of x_0 for the Caltech synchrotron. The acceptance regions are contained within the lines. The orbit shrinkage per turn, σ, is a function of injection energy while the effective wavelength of radial betatron oscillations, λ', is a machine constant; for the Caltech machine, $\lambda' = 643.2$ cm.

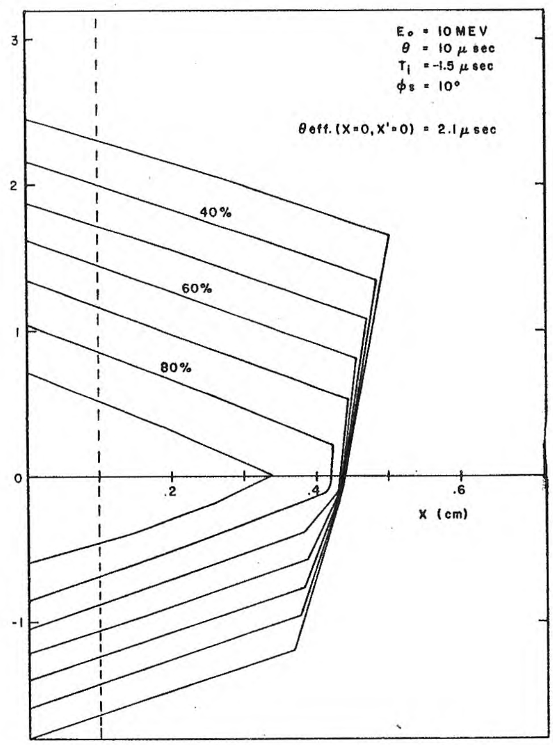

FIG. 14

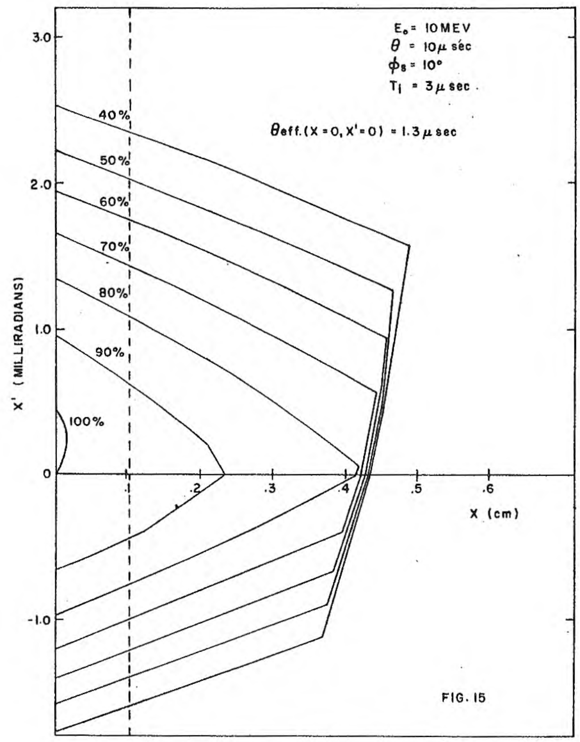

FIG. 15

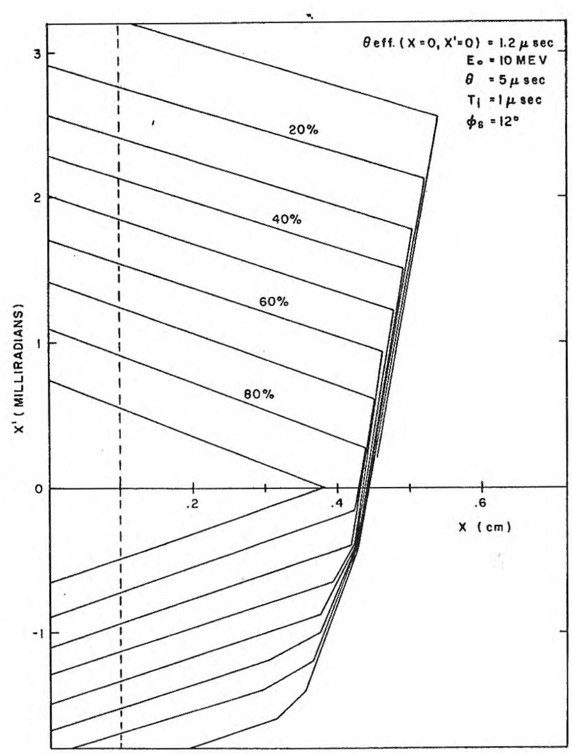

FIG. 16

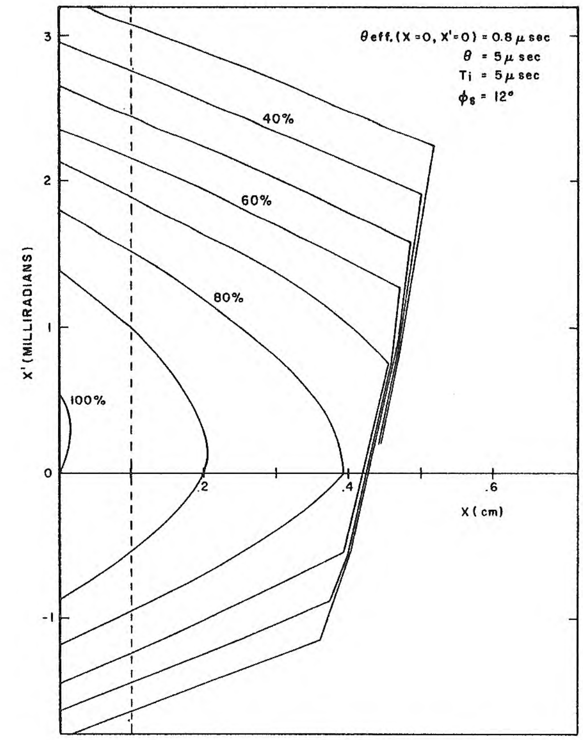

FIG. 17

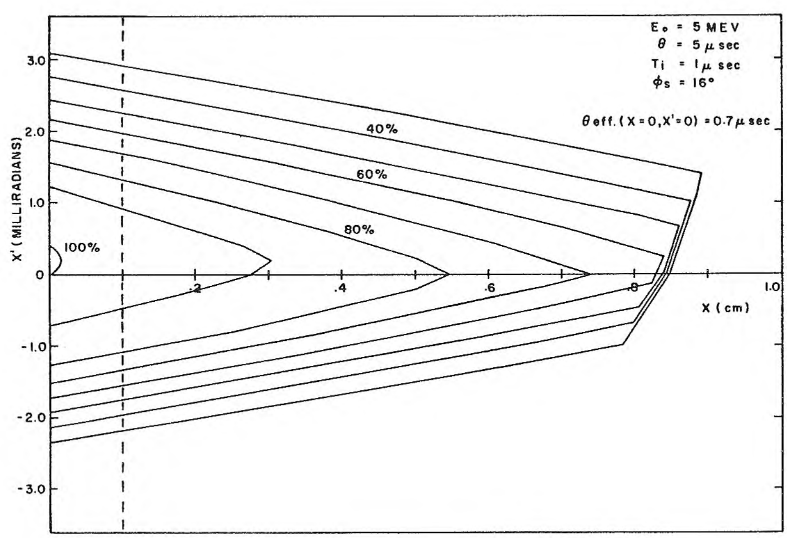

FIG. 18

FIGURES 14 through 18
Contours of constant capture efficiency over radial phase space for a uniform energy distribution. The contour parameter is per cent of $\Theta_{eff}(x = 0, x' = 0)$. It has been assumed that a 1 mm thick septum is feasible so that the region within the dotted line is inaccessible.