Gravitational Anyon

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Abstract

We show that, in the (2+1)-dimensional Einstein's gravity without the Chern-Simons term, the topological interaction of a point source with or without spin necessarily make the composite state of the source and a neutral test particle a gravitational anyon. In particular for a massless spinning source with spin σ, the angular momentum l of the gravitational anyon is given by $l = n - \sigma E/2\pi$, where E is the energy and n is an integer.

In a fundamental paper Wilczek[1] has shown that in (2+1)-dimensional physics an anyonic state is possible which can carry a fractional angular momentum. The anyon can best be visualized as a charge-flux composite state in (2+1)-dimensional electrodynamics: In the presence of a (non-electromagnetic) binding force a particle with charge e orbiting around a point source of magnetic flux Φ carries a fractional angular momentum $l = n - e\Phi/2\pi$, where n is an integer. The purpose of this paper is to demonstrate that another anyonic state is possible in (2+1)-dimensional Einstein's gravity (without the Chern- Simons interaction), and to establish a remarkable similarity between an energy-spin composite state in gravity and a charge-flux composite state in electrodynamics. The existence of an anyonic state in (2+1)-dimensional gravity with the Chern-Simons interaction has recently been argued by Deser[2]. Our result not only supports this claim but also shows that a gravitational anyon is possible even without the Chern-Simons interaction.

To establish the existence of a gravitational anyon we first provide a simple geometric argument why the charge-flux composite can have a fractional angular momentum, in such a way that is extremely useful and enlightening for later purpose. For this we identify the (2+1)-dimensional electrodynamics coupled with gravitation as a (2+1)-dimensional effective theory of the (3+1)-dimensional Kaluza-Klein unification[3] of the Einstein-Maxwell theory. In this picture the "fourth" coordinate describes the internal space of the $U(1)$ gauge group. Now consider the following unified metric

$$\begin{aligned} ds^2 &= g_{ij}dx^i dx^j + (d\theta + A_i dx^i)^2 \\ &= g_{ij}dx^i dx^j + (d\theta + \frac{\Phi}{2\pi}(\partial_i\varphi)dx^i)^2, \quad (1) \end{aligned}$$

where g_{ij} $(i,j = 0,1,2)$ is the (2+1)-dimensional metric, A_i is the electromagnetic potential, φ is the azimuthal angle, and θ is the internal coordinate. Clearly, when g_{ij} is flat, the above metric describes a point source of magnetic flux Φ. But notice that with the (singular) coordinate transformation

$$\theta \longrightarrow \theta' = \theta + \frac{\Phi}{2\pi}\varphi, \quad (2)$$

the metric (1) becomes flat (except at the origin),

$$ds^2 = g_{ij}dx^i dx^j + d\theta'^2.$$

In this flat coordinates the motion of a particle with mass m orbiting around the magnetic source may be described by the "free" Klein-Gordon equation[4]. Assuming that the wave function of the particle Ψ in the flat coordinates is an eigenstate of charge e, energy E, and angular momentum

l, one must have (in the polar coordinates t, ρ, φ)

$$\begin{aligned} \Psi &= e^{-iEt+il\varphi+ie\theta'}R(\rho) \\ &= e^{-iEt+i(l+\frac{e\Phi}{2\pi})\varphi+ie\theta}R(\rho). \end{aligned} \tag{3}$$

But since Ψ must be single-valued in the original coordinates, one must have $l = n - e\Phi/2\pi$, where n is an integer. So the wave function must satisfy the following Klein-Gordon equation

$$\begin{aligned} [-\frac{\partial^2}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{l^2}{\rho^2} + V(\rho)]R(\rho) \\ = (E^2 - m^2 - e^2)R(\rho), \end{aligned} \tag{4}$$

where we have put in an external binding potential $V(\rho)$. With $V = 0$ one obtains the "scattering" solution

$$\Psi = e^{-iEt+ie\theta}\sum_{n=-\infty}^{+\infty} a_n e^{in\varphi}J_{|l|}(\kappa\rho), \tag{5}$$

where $J_{|l|}$ is the Bessel function of the first kind of order l and $\kappa = (E^2 - m^2 - e^2)^{1/2}$. This shows that the angular momentum felt by the orbiting particle is l, not an integer n, even though the azimuthal angular dependence of Ψ is "normal". This confirms the well-known anyonic nature of the charge-flux composite state.

The reason why we present the above argument is that the argument can easily be generalized to demonstrate the existence of a gravitational anyon. To show this we simply replace the above metric with the one which describes an arbitrary gravitational point source[5]

$$\begin{aligned} ds^2 &= g_{ij}dx^i dx^j + (d\theta + \frac{\Phi}{2\pi}d\varphi)^2 \\ &= -(dt + \frac{\sigma}{2\pi}d\varphi)^2 + d\rho^2 \\ &\quad +(1-\mu)^2\rho^2 d\varphi^2 + (d\theta + \frac{\Phi}{2\pi}d\varphi)^2, \end{aligned} \tag{6}$$

where σ is the spin and μ is the mass (in the unit of the Newton's constant) of the point source. Notice that with the following (singular) coordinate transformation

$$\begin{aligned} t &\longrightarrow t' = t + \frac{\sigma}{2\pi}\varphi \\ \varphi &\longrightarrow \varphi' = (1-\mu)\varphi \\ \theta &\longrightarrow \theta' = \theta + \frac{\Phi}{2\pi}\varphi, \end{aligned} \tag{7}$$

one can again reduce the metric (6) into a flat form (except at the origin)

$$ds^2 = -dt'^2 + d\rho^2 + \rho^2 d\varphi'^2 + d\theta'^2.$$

But this time the wave function of a charged particle orbiting around the point source in this flat coordinates should have the following form

$$\begin{aligned} \Psi &= e^{-iEt'+il\varphi'+ie\theta'}R(\rho) \\ &= e^{-iEt+in\varphi+ie\theta}R(\rho). \end{aligned} \tag{8}$$

This means that the composite state now must have the angular momentum given by

$$l = \frac{n - \frac{e\Phi}{2\pi} + \frac{\sigma E}{2\pi}}{1-\mu}. \tag{9}$$

Indeed one can easily convince oneself that under the coordinate transformation (7) the angular momentum operator l_{op} should transform as

$$\begin{aligned} l_{op} &= -i\frac{\partial}{\partial\varphi} \longrightarrow -i\frac{\partial}{\partial\varphi'} \\ &= -\frac{i}{1-\mu}(\frac{\partial}{\partial\varphi} - \frac{\Phi}{2\pi}\frac{\partial}{\partial\theta} - \frac{\sigma}{2\pi}\frac{\partial}{\partial t}), \end{aligned} \tag{10}$$

which confirms the above result.

To obtain the fractional statistics, consider two identical anyons. Assuming the anyon is made of bosons, one can easily obtain the condition that the wave function $\hat{\Psi}$ of two anyons must satisfy under the exchange of the identical anyons[1,6]. Since the exchange operation is nothing but the parallel transport of one anyon around the other, it corresponds to the parallel transport of the anyon wave function (8) around the origin under the influence of the metric (6). The parallel transport is best visualized in the flat coordinates, where the wave function acquires a phase $e^{i2\pi l}$ under the parallel transport. This means that under the exchange we must have

$$\begin{aligned} \hat{\Psi} \longrightarrow \hat{\Psi}' &= e^{i2\pi l}\hat{\Psi} \\ &= e^{i\frac{2\pi n - e\Phi + \sigma E}{1-\mu}}\hat{\Psi}. \end{aligned} \tag{11}$$

This is a straightforward generalization of the results obtained by Wilczek[1] and 't Hooft[6]. Indeed with $e\Phi = \sigma E = 0$ our result becomes precisely the "looping boundary condition" of 't Hooft, which must be satisfied by the wave function of two identical spinless gravitational point particles[6]. Of course when the anyon is made of fermions, $\hat{\Psi}'$ will pick up an extra phase factor according to the Fermi-Dirac statistics. From this we conclude that, with $e\Phi = 0$, the composite state of a gravitational point source and a neutral test particle orbiting around it should behave as a gravitational anyon. In particular, with $\mu = 0$, we obtain a striking similarity between a charge-flux composite in electrodynamics and an energy-spin composite in gravitation, with the role of charge and magnetic flux of the electromagnetic anyon replaced by energy and spin of the gravitational one. This correspondence between gravitation and electrodynamics has already been noticed in the gravitational scattering of cosmic string and the Bohm-Aharonov scattering of electron[7]. Our result tells that the correspondence can be extended between the gravitational and the electromagnetic anyons. Of course with $\mu \neq 0$ we still obtain a gravitational anyon, as first implied by Wilczek, and others[5,6]. But this is due to the conic structure of the 2-dimensional space, and has no electromagnetic analogy.

To repel any remaining doubt about the gravitational anyon we now provide another argument based on a (2+1)-dimensional point of view, why the energy-spin composite must form a gravitational anyon. For this purpose we consider the classical motion of a neutral test particle orbiting around a massless spinning source in a (2+1)-dimensional space-time described by the following metric

$$ds^2 = -(dt + \frac{\sigma}{2\pi} d\varphi)^2 + d\rho^2 + \rho^2 d\varphi^2, \qquad (12)$$

but this time we allow σ to become time-dependent. Now the motion of a neutral test particle orbiting around the source should be described by the geodesic equation

$$p^i \nabla_i p^k = a^k, \qquad (13)$$

where p^i is the velocity vector of the particle. Notice that here we have introduced a (non-gravitational) acceleration a^k which provides a binding force. Assuming that the binding force is central, one can easily integrate (13). When σ is independent of time we have

$$\begin{aligned} \frac{dE}{ds} &= 0 \\ \frac{dl}{ds} &= 0, \end{aligned} \qquad (14)$$

where s is the affine parameter of the trajectory, and E and l are the energy and the angular momentum given by

$$\begin{aligned} E &= \frac{dt}{ds} + \frac{\sigma}{2\pi}\frac{d\varphi}{ds} \\ l &= \rho^2 \frac{d\varphi}{ds}. \end{aligned} \qquad (15)$$

However, when σ becomes time-dependent, we obtain

$$\begin{aligned} \frac{dE}{ds} &= \frac{\dot{\sigma}}{2\pi}\frac{l}{\rho^2} E \\ \frac{d}{ds}(l &- \frac{\sigma E}{2\pi}) = 0. \end{aligned} \qquad (16)$$

The result shows that, when we increase σ from zero to a finite value slowly, the angular momentum gets shifted by an amount

$$\Delta l = \Delta(\frac{\sigma E}{2\pi}). \qquad (17)$$

This confirms our previous result.

In conclusion we have shown that a gravitational anyon is possible in (2+1)-dimensional Einstein's theory. In this paper we have presented a (2+1)-dimensional point of view as well as a (3+1)-dimensional unified point of view to obtain the same conclusion. The advantage of the unified view is that it can treat the electromagnetic and the gravitational anyons on the same footing. When the 2-dimensional space has a conic structure, the existence of a gravitational anyon has been expected by many authors[5,6]. But our energy-spin composite state whose existence does not depend on the conic structure clearly demonstrates the existence of another non-electromagnetic anyon, which nevertheless is strikingly similar to the well-known electromagnetic one. This correspondence between the Einstein's theory and the Maxwell's theory can easily be extended to (3+1)-dimension, to show the existence of a gravitational monopole[8] as the counter-part of the Dirac's electromagnetic monopole. A more detailed discussion on the subject will be published elsewhere[9].

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