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FACULTEIT DER NATUURWETENSCHAPPEN, WISKUNDE EN INFORMATICA

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   Holography for Einstein-Maxwell-dilaton theories
   from generalized dimensional reduction,
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3. J. Smolic and M. Taylor,
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for my family

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Contents

Overview

This thesis represents research conducted during my time as a PhD student at the University of Amsterdam. As I hope will be clear from the ensuing pages, I have been fortunate to work on a number of different aspects of high energy physics. These topics range from the subtleties inherent in studying the dynamics of far from equilibrium systems and the underlying framework of non-equilibrium quantum field theory (QFT), to the intricacies of black hole solutions in higher derivative gravity. Though these fields seem vastly different in character, there is an underlying idea that links them. This is the concept of holography. Taking its name from the way a hologram stores the data of a three-dimensional object on a two-dimensional photographic plate, this ground-breaking concept states the equivalence of a gravitational theory living in a (d+1)-dimensional spacetime background (the bulk) and the non-gravitational QFT living on its boundary, in d-dimensions. It is also known as the gauge/gravity duality since the boundary QFT is often a gauge theory.

The concept of holography was inspired by studies of black holes in the 1970s by Jacob Bekenstein and Stephen Hawking. These early studies revealed that black holes obey laws very similar to the laws of ordinary thermodynamics. Indeed, quantities analogous to entropy and temperature could be defined for them. Analysis of these concepts led Bekenstein to conjecture that the entropy of a Black hole is proportional to the area of the event horizon, rather than the volume. Indeed, he argued that black holes are maximum entropy objects, having more entropy than any other object occupying the same volume. Subsequent studies by Hawking revealed that black holes emit thermal radiation and allowed him to calculate the precise constant of proportionality linking the black hole entropy to the event horizon area. This allowed black holes to become firmly established as thermodynamic objects.

Building on this early work, Gerard 't Hooft postulated in 1993 that it must be possible to describe all phenomena within a region of space of volume V by a set of degrees of freedom which reside on the boundary of V. In 1995 Leonard Susskind gave 't Hooft's ideas a precise string theory interpretation. Indeed, earlier work by Susskind, Klebanov and, independently, Charles Thorn, had shown that string theory in the light front gauge has the form of a (2+1)-dimensional theory with no explicit mention of a longitudinal direction.

The first explicit manifestation of the holographic principle was provided by Maldacena in 1997 in the form of the AdS/CFT correspondence. One of his postulates is the duality of type IIB string theory on the $AdS_5 \times S^5$ background and the $\mathcal{N} = 4$ super-Yang Mills (gauge) theory living on the boundary of the AdS_5. However, as is the case with many dualities in string theory, the AdS/CFT correspondence is a strong-weak coupling duality. Hence, the strong-coupling regime of the gauge theory maps to the weak-coupling regime in the string theory (*i.e.* supergravity), and vice versa. While the strongly-coupled regimes of both theories are difficult to access, this correspondence allows us to, for instance, get a handle on the strongly-coupled gauge theory by doing computations in the dual supergravity theory, a comparatively simple task. It is thus a remarkably useful tool, and indeed much use has been made of it since its discovery. One of the major areas of research which has risen to prominence in recent years is applying holography to study strongly-coupled condensed matter systems. A similarly fruitful research direction has been in holographic modeling of aspects of relativistic heavy-ion collisions, such as those studied at RHIC and the LHC. These theoretical endeavors have given us fundamental insights into the nature of the quark-gluon plasma (QGP) produced in these experiments. The goal of this thesis is to add a tiny contribution to these endeavors and the vast general holographic literature.

This thesis is organized as follows. In Part I, we work solely on the gauge theory side and study the $\mathcal{N} = 4$ SYM theory in a non-equilibrium setting. Non-equilibrium QFT is a notoriously difficult subject, however, advances have been made in this field in recent years which make analytical computations of far-from equilibrium dynamics tractable. These are the *n Particle Irreducible* (nPI) effective actions.

In Chapter 1 we discuss the derivation of the nPI effective action and we introduce the framework of non-equilibrium QFT, discussing the difficulties inherent in studying far from equilibrium dynamics. In Chapter 2 we present the computation of the 2PI effective action, to two loops, of the $\mathcal{N} = 4$ SYM theory. We also compute the evolution equations for the two-point correlators in this theory. Our analysis is in the weak-coupling regime of the gauge theory and thus, via holography, could potentially give us information about the strongly-coupled gravity theory.

In Part II we proceed to tune the coupling all the way up to the strong-coupling regime in the gauge theory, and also study small tunings of the coupling back towards the weak-coupling regime by the inclusion of higher-derivative corrections.

Chapters 3 and 4 contain introductory material. In Chapter 3 we present the basic idea of holography and the AdS/CFT correspondence. We write down the holographic dictionary between a bulk gravitational theory and a boundary gauge theory, and explain the procedure of holographic renormalization. In Chapter 4 we deal with various aspects of holography. We discuss how holography has been used in the study of strongly-coupled condensed matter systems. Specifically, we deal with a special type of bulk theory, the Einstein-Maxwell-Dilaton (EMD) theory, which has proved extremely useful in building holographic models of condensed matter systems. We then move slightly away from the strong-coupling regime by adding higher-derivative terms to the gravity action. We discuss the holographic effect of such higher derivative corrections on, for instance, the KSS bound. Finally we give a brief introduction to black hole thermodynamics, discussing the laws and the well-known expressions for the temperature and Bekenstein-Hawking entropy. We deal also with Wald's formalism for computing these quantities in the presence of higher derivative terms.

In Chapter 5 we derive the holographic dictionary for a class of Einstein-Maxwell-Dilaton theories in the strong-coupling regime. This is challenging since many of the relevant solutions do not have AdS asymptotics. We deal with this problem by using the tool of generalized dimensional reduction, which allows us to set up the dictionary using a theory whose dictionary is already known. We also study the hydrodynamics in this theory.

Finally, in Chapter 6 we tune the coupling slightly towards the weak-coupling regime and look for black hole solutions in various theories of higher derivative gravity. Specifically, we look at higher-derivative effects in 4-dimensional AdS black holes. We also work out the thermodynamics of these corrected black hole solutions.

Part I

Non-equilibrium QFT and nPI effective actions

Preface

Our goal in this part of the thesis is to elaborate on progress that has been made in recent years in the study of the non-equilibrium dynamics of quantum fields. This is a notoriously difficult subject and for many years physicists had to content themselves with close-to equilibrium studies which are a far-cry from understanding the full non-equilibrium physics.

These new developments are based on the method of so-called *n particle irreducible* (nPI) effective actions, which allow us to use powerful non-perturbative approximation schemes to study non-equilibrium dynamics. These techniques have been applied to a number of theories thus far, for instance ϕ^4, QED and QCD. See for example [1], [2], [3], [4].

Here we want to apply them to $\mathcal{N} = 4$ super Yang-Mills theory. This is a very special theory in the realm of high energy physics and string theory, and has been called by many the *simplest* QFT, [5]. It has been widely studied in the context of the AdS/CFT correspondence, where, in loose terms, it is conjectured to be equivalent to a gravitational theory living in the $AdS_5 \times S^5$ spacetime [6], [7]. This is the best-known example of holography, a ground-breaking concept which relates the gravitational (string) theory living in the bulk of a spacetime to the QFT living on its boundary. As we will expound on in greater detail in subsequent chapters, this is a weak-strong coupling duality, which means that the weak-coupling regime on one side is mapped to the strong-coupling regime on the other side. Hence, previously intractable problems in one theory can be mapped to tractable problems in the other theory.

One of the main areas of interest is thermalization in $\mathcal{N} = 4$ SYM. Studying this theory in the non-equilibrium nPI setting would give us insight into thermalization at weak coupling, a problem which is not well understood. Indeed, the question of thermalization is an important one from the holography point of view: horizon formation on the gravity side (in the bulk) is expected to be mapped to thermalization in the boundary QFT. A good understanding of thermalization in the QFT would thus give us insight into black hole formation and evaporation in the bulk.[1]

A further interest is in understanding relativistic heavy-ion collisions, such as the ones performed at RHIC, [8] and, more recently, at the LHC, [9]. These experiments aim at providing a better understanding of QCD through study of the quark-gluon plasma. Modeling the initial stages of these collisions requires considering extreme non-equilibrium dynamics and understanding how the system thermalizes at late times is theoretically challenging. The nPI formalism may give us an analytic way to study these collisions. Indeed, holographic studies in the low energy dual gravitational theory indicate that some aspects of the quark-gluon plasma seem to be captured by the strongly-coupled $\mathcal{N} = 4$ theory, see [10], [11], [12], [13] for recent reviews. The reason for this remains a mystery. However, studying thermalization in the weakly-coupled $\mathcal{N} = 4$ SYM theory and comparing to the gravity result, would allow one to see what features are common to strong and weak coupling and what features are different. That might give insight into why the gravity results reproduce features of the quark gluon plasma.

The powerful techniques of nPI effective actions are our best tool to date to get a handle on the non-equilibrium dynamics of this remarkable theory, which, via holography can then potentially give us new insights into the corresponding gravitational theory.

Our goal in this thesis is a modest one: we wish to write down the two-loop 2PI effective action and evolution equations for $\mathcal{N} = 4$ SYM theory. We present this analysis in Chapter 2.

In the following Chapter we begin with a more detailed discussion of the intricacies of non-equilibrium QFT and nPI effective actions.

[1]This statement comes with the usual caveat inherent in holographic calculations: black hole formation occurs in the bulk theory at large 't Hooft coupling, which makes the corresponding boundary QFT strongly coupled. The nPI calculations we consider, however, are valid at weak coupling in the boundary theory.

Chapter 1

Non-equilibrium QFT

1.1 Introduction

Many interesting physical processes, vital to our understanding of nature, occur out of equilibrium. Such processes are found in a wide range of physical domains, from particle physics and cosmology to astrophysics and condensed matter systems. Some of the interesting physical processes occurring far away from equilibrium include: the early stages of heavy ion collisions performed for instance at RHIC, the generation of density fluctuations during inflation and the explosive particle production at the end of inflation, phase transitions in condensed matter systems, glassy systems.

However, although myriad experimental data exist on non-equilibrium processes, getting a quantitative description of the non-equilibrium time evolution of quantum fields is a highly non-trivial task. This is quite different to systems in thermal equilibrium, where well-understood statistical mechanics forms the underlying framework, and powerful numerical techniques exist, or, indeed, systems weakly perturbed away from equilibrium which obey linear response theory. These are realms where the particle number is large enough for statistical fluctuations to dominate over quantum fluctuations.

In studying non-equilibrium processes, however, both quantum and statistical fluctuations play an important role. The difficulty in studying non-equilibrium dynamics is due to its intrinsically non-perturbative nature. Hence, standard perturbative approaches to quantifying non-equilibrium time evolution fail. Indeed, attempting to write a naive perturbative expansion for a non-equilibrium process results in the appearance of infinitely many spurious time-dependent quantities, so-called *secular* *terms*, which invalidate the expansion even for a weak coupling.

In addition to understanding how quantum fields evolve in a non equilibrium setting, we would also like to understand how these non-equilibrium systems approach thermal equilibrium (thermalize) at late times. This is a formidable problem, not least because of the requirement of *universality*: thermal equilibrium is universal, i.e. it is characterized by a relatively small number of conserved charges, and has no memory of the non-equilibrium initial conditions or the time history of the system. Hence the fundamental issue is that thermal equilibrium describes macroscopic time-irreversible behavior, whereas the evolution of quantum fields is time-reversal invariant.

In recent years much progress has been made in making non-equilibrium QFT calculations more tractable. The main advancement has been the development of so-called *n particle irreducible (nPI) effective actions*, see for instance [14], [15], [16], [17], [18]. These are efficient functional integral techniques which allow us to use non-perturbative approximations to get a handle on non-equilibrium dynamics. Ultimately, they allow us to rearrange the perturbation expansion in a more efficient way so that the problem of secularity is avoided and the requirement of universality is satisfied. For additional information about non-equilibrium quantum fields and thermalization see the textbook [19] and the research publications [20], [21], [22], [23], [24], [25], [26], [27], [28], [29]. For an excellent in-depth review of these topics and the tools of nPI effective actions we refer the reader to [30].

To illustrate the effectiveness of the nPI approximation scheme, as opposed to standard perturbation theory, we relate a toy example from classical mechanics: the anharmonic oscillator, described in [30].

Consider the following differential equation for the oscillation amplitude, $y(t)$, with small parameter ϵ,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + y = -\epsilon \frac{\mathrm{d}y}{\mathrm{d}t} - (\epsilon y)^3 - (\epsilon y)^5 - (\epsilon y)^7 + \cdots, \quad \epsilon \ll 1 \tag{1.1}$$

The arbitrarily high-order of anharmonic terms is similar to the case in QFT where quantum fluctuations can induce self-interaction to high powers in the field, [30]. It is easy to solve this numerically with given initial conditions. However we will now show how, depending on the approximation scheme we use, we can get vastly different results. First we look at a standard perturbative expansion, namely we expand the amplitude $y(t)$ in powers of ϵ,

$$y(t) = y_0(t) + \epsilon y_1(t) + \frac{\epsilon^2}{2} y_2(t) + \cdots \tag{1.2}$$

This leads to the usual system of equations

$$\frac{\mathrm{d}^2 y_0}{\mathrm{d}t^2} + y_0 = 0, \quad \frac{\mathrm{d}^2 y_1}{\mathrm{d}t^2} + y_1 = -\frac{\mathrm{d}y_0}{\mathrm{d}t}, \quad \cdots \tag{1.3}$$

This is where we run into a problem. We may solve the above eqations iteratively for y_0, y_1, $e$tc. To second order in ϵ this yields, [30],

$$y(t) = \frac{1}{2} e^{it} \left(1 - \frac{\epsilon}{2} t + \frac{\epsilon^2}{8} (t^2 - it) \right) + \text{c.c.} \tag{1.4}$$

Hence, we can see that any solution higher than lowest order contains secular terms: terms which grow with time and make the perturbative expansion invalid unless $\epsilon t \ll 1$, i.e. it is invalid for late times.

Now we consider a different expansion, where we don't expand y, but where we truncate (1.1) according to powers of the small expansion parameter ϵ. Namely,

$$\begin{aligned} &\frac{\mathrm{d}^2 y^{(0)}_{nPI}}{\mathrm{d}t^2} + y^{(0)}_{nPI} = 0, \quad \frac{\mathrm{d}^2 y^{(2)}_{nPI}}{\mathrm{d}t^2} + y^{(2)}_{nPI} = -\epsilon \frac{\mathrm{d}y^{(2)}_{nPI}}{\mathrm{d}t}, \\ &\frac{\mathrm{d}^2 y^{(3)}_{nPI}}{\mathrm{d}t^2} + y^{(3)}_{nPI} = -\epsilon \frac{\mathrm{d}y^{(3)}_{nPI}}{\mathrm{d}t} - (\epsilon y^{(3)}_{nPI})^3, \quad \cdots \end{aligned} \tag{1.5}$$

This is a so-called nPI expansion scheme, and it turns out to be non-secular in time. Indeed, solving the above gives,

$$\begin{aligned} y^{(0)}_{nPI}(t) &= \frac{1}{2} e^{it} + \text{c.c.}, \\ y^{(2)}_{nPI}(t) &= \frac{1}{2} e^{it\sqrt{1-\epsilon^2/4} - \epsilon t/2} + \text{c.c.}, \quad \cdots \end{aligned} \tag{1.6}$$

It is clear that, for all times, neither $y^{(0)}_{nPI}(t)$ nor $y^{(2)}_{nPI}(t)$ exceed $\mathcal{O}(\epsilon)$. This is an example of a self-consistent expansion since there is no mixing of different orders in $y^{(m)}_{nPI}$. The drawback of this procedure is that one has to solve non-linear equations at each order. However, upon a deeper examination it can be shown that it is precisely this non-linearity which is required for universality.

This toy example is in the spirit of the nPI effective action technique which will prove useful in studying non-equilibrium dynamics of quantum fields and which we will expand on in this Chapter. As we will elaborate on later, there exists a useful equivalence hierarchy among nPI effective actions. This basically links the order in n required to the degree of computational accuracy we desire. This means that one can avoid having to calculate the full nPI effective action to arbitrarily high order in n, since, for instance, for a q-loop approximation all nPI descriptions with $n \geq q$ are equivalent, so only the qPI effective action is required.

1.2 The nPI Effective Action

1.2.1 The 1PI Effective Action

Before we go into the details of nPI effective actions, and specifically the 2PI effective action, we review the construction of the standard 1PI effective action. For more details we refer the reader to the excellent textbook [31].

We begin with a theory of a d-dimensional scalar field φ governed by the classical action $S[\varphi]$. The generating functional of connected correlation functions, $W[J]$, in the presence of a single source term $J(x)$, is given by,

$$Z(J) = e^{iW[J]} = \int \mathscr{D}\varphi \; e^{i\left(S[\varphi]+\int_x J(x)\varphi(x)\right)}. \tag{1.7}$$

Here, following [30], we use a shorthand notation where $\int_x \equiv \int_{\mathscr{C}} dt \int d^{d-1}\vec{x}$. The time integral is over some contour $\mathscr{C}$, which will be specified later. As we will show, we can extend our results from the vacuum to non-equilibrium scenarios by choosing an appropriate contour.

We can obtain the n-point correlation functions by taking functional derivatives of $W[J]$ with respect to the source. We write below the first two, namely,

$$\left.\frac{\delta W[J]}{\delta J(x)}\right|_{J=0} = \langle \Phi(x) \rangle_{conn.} \equiv \phi(x), \qquad \left.\frac{\delta^2 W[J]}{\delta J(x)\delta J(y)}\right|_{J=0} = \langle T_{\mathscr{C}} \Phi(x)\Phi(y) \rangle_{conn.} \equiv iG(x,y), \tag{1.8}$$

where ϕ and G are the connected one- and two-point functions and $\Phi(x)$ is the Heisenberg field operator. The time ordering is written once again with respect to the contour $\mathscr{C}$ which will be defined in due course and which allows us to make use of these expressions in a non-equilibrium setting.

To formally write down the 1PI effective action we Legendre-transform $W[J]$ with respect to the source $J(x)$,

$$\Gamma[\phi] = W[J] - \int_x \frac{\delta W[J]}{\delta J(x)} J(x) = W[J] - \int_x \phi(x) J(x). \tag{1.9}$$

By functionally differentiating the above effective action with respect to the one-point function, or field expectation value, we find the so-called *stationarity* condition,

$$\frac{\delta \Gamma}{\delta \phi(x)} = -J(x), \tag{1.10}$$

which, upon setting the external source to zero, gives us the equation of motion for ϕ.

As has been demonstrated in [31], we may directly compute the 1PI effective action by perturbatively expanding the generating functional (1.7). This involves, among other things, shifting the field $\varphi(x)$ around the field expectation value $\phi(x)$, namely, $\varphi \to \phi + \varphi$, and evaluating the shifted action $S[\phi+\varphi]$, where we may drop terms linear in φ. This new action will contain both terms quadratic in φ, which will contribute to the propagator, and also terms cubic and higher in φ which contribute to the interaction part $S_{int}(\varphi, \phi)$. Both the propagator, and the vertices coming from S_{int}, will depend on ϕ. This procedure eventually leads us to the following expression for the 1PI effective action, to one loop order,

$$\Gamma[\phi] = S[\phi] + \frac{i}{2} \mathrm{Tr} \ln(G_0^{-1}(\phi)). \tag{1.11}$$

Here $G_0^{-1}(\phi)$ is the classical inverse propagator obtained from the shifted action as

$$iG_0^{-1}(x, y; \phi) = \frac{\delta^2 S[\phi]}{\delta \phi(x) \delta \phi(y)} \tag{1.12}$$

The full inverse propagator is given by the second functional derivative of the effective action with respect to ϕ,

$$\frac{\delta^2 \Gamma[\phi]}{\delta \phi(x) \delta \phi(y)} = iG^{-1}(x, y) = iG_0^{-1}(x, y) - i\Sigma(x, y). \tag{1.13}$$

Here $\Sigma(x, y)$ is the proper self energy which consists of only 1PI diagrams, i.e. diagrams which cannot be disconnected by cutting one propagator line. The lines in the diagrams represent the free propagator G_0. Indeed, the 1PI effective action is the generating functional of 1PI correlation functions, hence the name. That is, for $n \geq 3$,

$$\frac{\delta^n \Gamma[\phi]}{\delta\phi(x_1)\cdots\delta\phi(x_n)} = i\langle\varphi(x_1)\cdots\varphi(x_n)\rangle_{1PI}. \tag{1.14}$$

This ends our digression into the 1PI effective action, and we extend the above framework to the so-called 2PI effective action in the next Section.

1.2.2 The 2PI Effective Action

As we have demonstrated above, the standard prescription for writing down a 1PI effective action involves introducing only one source term when setting up the generating functional. In order to extend our analysis to 2PI effective actions, we need to introduce a second source term. In addition to the Legendre transform of the generating functional necessary to obtain the 1PI effective action, we can then perform a second Legendre transform to get the required 2PI effective action. Once again, we will show explicitly how this is done for the case of a real scalar field φ, with classical action $S[\varphi]$. The extension to multi-component and fermionic fields is straight-forward.

In the presence of two sources, $J(x)$ and $R(x,y)$, the generating functional for connected Green's functions, $W[J,R]$, is given by

$$\begin{aligned} Z[J,R] &= \int \mathscr{D}\varphi \, \exp\left(iS[\varphi] + \int_x J(x)\varphi(x) + \int_{xy} R(x,y)\varphi(x)\varphi(y)\right) \\ &= e^{iW[J,R]}. \end{aligned} \tag{1.15}$$

As in (1.8), we can define the connected one- and two-point functions, ϕ and G, from the above generating functional as follows,

$$\begin{aligned} \phi(x) &= \frac{\delta W[J,R]}{\delta J(x)} \\ \frac{1}{2}\left(\phi(x)\phi(y) + G(x,y)\right) &= \frac{\delta W[J,R]}{\delta R(x,y)}, \end{aligned}$$

where we can see from the second expression above that varying $W[J,R]$ with respect to the sources can now produce disconnected diagrams. Analogously to the 1PI effective action, the 2PI effective action is obtained by a Legendre transform of $W[J,R]$, but now simultaneously with respect to both source terms:

$$\begin{aligned}
\Gamma[\phi,G] &= W[J,R]-\int_x \frac{\delta W[J,R]}{\delta J(x)}J(x)-\int_{xy}\frac{\delta W[J,R]}{\delta R(x,y)}R(x,y) \\
&= W[J,R]-\int_x \phi(x)J(x)-\frac{1}{2}\int_{xy}R(x,y)\phi(x)\phi(y) \\
&\quad -\frac{1}{2}TrGR. \qquad (1.16)
\end{aligned}$$

Once again, in order to obtain the equations of motion for ϕ and G from the above, we simply take functional derivatives as follows,

$$\begin{aligned}
\frac{\Gamma[\phi,G]}{\delta\phi(x)} &= -J(x)-\int_y R(x,y)\phi(y) \\
\frac{\Gamma[\phi,G]}{\delta G(x,y)} &= -\frac{1}{2}R(x,y), \qquad (1.17)
\end{aligned}$$

and set the sources to zero. As in (1.10), the above are called *stationarity conditions*.

We can write the 2PI effective action in diagrammatic form, following [16] and [30],

$$\Gamma[\phi,G]=S[\phi]+\frac{i}{2}TrlnG^{-1}+\frac{i}{2}TrG_0^{-1}(\phi)G+\Gamma_2[\phi,G]+\text{const}, \qquad (1.18)$$

where we have split the one-loop result from the rest, so that $\Gamma_2[\phi,G]$ corresponds to two-loop and higher corrections to $\Gamma[\phi,G]$. Only closed-loop diagrams are included. The constant is arbitrary and is absorbed into the normalization. This derivation involved the same shift of the field φ by ϕ and subsequent evaluation of the shifted action $S[\varphi+\phi]$, as discussed below (1.10).

By varying (1.18) with respect to G and using the stationarity condition (1.17) we find, analogously to (1.13),

$$\begin{aligned}
G^{-1}(x,y) &= G_0^{-1}(x,y;\phi)-iR(x,y)-2i\frac{\delta\Gamma_2[\phi,G]}{\delta G(x,y)} \\
&\equiv G_0^{-1}(x,y;\phi)-iR(x,y)-\Sigma(x,y;\phi,G). \qquad (1.19)
\end{aligned}$$

Again, $\Sigma(x,y;\phi,G)$ is the proper self-energy, and as such consists of 1PI Feynman diagrams. In this case, however, the lines in the diagrams represent the full propagator G, with the vertices being the classical ones. It is precisely this relation which tells us that only 2PI diagrams contribute to $\Gamma_2[\phi,G]$, and allows us to call $\Gamma[\phi,G]$ a “2PI" effective action. Since the proper self-energy only has contributions from 1P diagrams, it is not possible for $\Gamma_2[\phi, G]$ to have contributions from diagrams with parts “joined" by two propagators GG, since $\frac{\delta\Gamma_2}{\delta G}$ would yield a diagram with parts “joined" by one propagator G, which is one particle *reducible*.

Hence, $\Gamma_2[\phi, G]$ is computed by writing down all the 2PI vacuum graphs with the vertices determined by the interaction part of the shifted action, $S_{int}(\phi;\varphi)$, and the propagators given by $G(x, y)$.

When evaluating the corresponding expression of the 2PI effective action for *fermionic* fields, the only difference in the derivation of the one-loop result arises due to the anticommuting nature of the fermions. Thus, instead of obtaining an exact replica of the scalar result $\frac{i}{2}TrlnG_0^{-1}$, in a corresponding calculation for fermionic fields we get $-iTrln\triangle_0^{-1}$, where $\triangle_{0,ij}^{-1}$ is the classical inverse fermion propagator and i, j are flavor indices. Thus, for fermions

$$\Gamma[\psi,\triangle] = S[\psi] - iTrln\triangle^{-1} - iTr\triangle_0^{-1}(\psi)\triangle + \Gamma_2[\psi,\triangle] + \text{const}, \tag{1.20}$$

where ψ is the one point function of the fermionic field. As for the scalar case, there is a relation between the functional derivative of $\Gamma_2[\psi,\triangle]$ and the proper fermion self-energy, namely,

$$\Sigma^{(f)}(x, y;\triangle) = -i\frac{\delta\Gamma_2[\triangle]}{\delta\triangle(y, x)}. \tag{1.21}$$

The equation of motion for the fermion propagator is the expected stationarity condition,

$$\frac{\delta\Gamma[\psi,\triangle]}{\delta\triangle_{ij}(x, y)} = 0. \tag{1.22}$$

The gauge field 2PI effective action will mirror the form of the scalar result, and ghosts will follow the form of the fermionic fields.

Φ-derivable approximations

The 2PI effective action, being a generating functional of correlation functions, in principle contains the full information about the QFT under investigation. In practice, however, one does not wish to calculate all the correlation functions. For computational efficacy, one normally introduces an approximation scheme and truncates the effective action at some value of a suitable expansion parameter. This could for instance be loop order or order in N for the case of a large N expansion. In the case of the 2PI effective action, one would truncate the series of diagrams making up $\Gamma_2[\phi, G]$. This truncated effective action would yield, through the stationarity conditions (1.17) with vanishing sources, the equations of motion for ϕ and G. For the approximate two-point function G, for instance, the equation of motion (1.19) in the absence of sources would read,

$$\begin{aligned} G^{-1}(x,y) &= G_0^{-1}(x,y;\phi) - 2i\frac{\delta\Gamma_2^{trunc}[\phi,G]}{\delta G(x,y)} \\ &= G_0^{-1}(x,y;\phi) - \Sigma(x,y;\phi,G). \end{aligned} \tag{1.23}$$

These Φ (Functional)-derivable approximations were first discussed in [15].

We can see from this discussion that to calculate the self energy we can either directly perform the functional differentiation in (1.23), or by working out the corrections to the propagator in the usual way: by cutting a propagator line on each of the 2PI diagrams at our disposal. For two-loop 2PI diagrams we naturally get the one-loop self energy.

One of the useful properties of the Φ-derivable approximations is that they lead, via the stationarity conditions, to equations of motion that respect the *global* symmetries of the theory. Specifically, they ensure that charge and energy are conserved. Also, these approximations preserve thermodynamic consistency.

An obvious question is whether Φ-derivable approximations respect also local gauge symmetries. It was shown in [32] that Φ-derivable approximations do not preserve Ward identities and thus gauge invariance is lost, i.e. dependence on the gauge is retained in physical quantities. However, it was shown, in the context of the 2PI effective action, in [17], [33], (and also by another method in [34]), that the gauge dependence is controlled, i.e. the gauge-dependent terms occur at higher order than the truncation order in a self-consistent approximation scheme. Thus these approximation schemes may still be applicable in gauge theories.

In the case of the 2PI effective action it is somewhat intuitive to understand why Ward identities are not satisfied. Generally, for the Ward identities to be satisfied there need to be cancellations between self-energy corrections and vertex corrections. Since the truncated 2PI effective action deals only with corrected one- and two-point function, and uses classical vertices, the Ward identities are not expected to be satisfied, necessarily leading to gauge dependent results for most physical quantities.

However, the method of *resummed* effective actions fixes this problem. The *resummed* nPI effective action is defined with respect to the self-consistent solutions of the n-point functions. Resummations of this type were first investigated in [15] and [35].

Explicitly, the self-consistent solution for the two-point function, $\tilde{G}[\phi]$ is obtained from the stationarity condition as

$$\frac{\delta\Gamma^{trunc}[\phi,G]}{\delta G}\Big|_{\tilde{G}[\phi]} = 0. \tag{1.24}$$

We substitute this self-consistent solution into the effective action to obtain the resummed effective action, namely,

$$\tilde{\Gamma}^{trunc}[\phi] = \Gamma^{trunc}[\phi,\tilde{G}[\phi]]. \tag{1.25}$$

We explicitly show that we are working with a truncated effective action. It was shown in [16] that the above resummed effective action is equivalent to (1.18) to the given order of approximation.

This resummed effective action respects all the symmetries of the theory and the n-point functions obtained from it satisfy the Ward identities. It was used in [3] to compute the QED electrical conductivity in the context of the three-loop 2PI effective action.

Now that we have introduced the idea of the 2PI effective action it is straight forward to extend this idea to nPI effective actions. We do this in the next section.

1.2.3 The nPI effective action

It is straightforward to extend the formalism given in the two previous sections to construct the so called nPI effective action, for $n > 2$. We simply include up to nth order sources, $J,R,R_3,...,R_n$, in the definition of the generating functional (1.7).

The nPI effective action, $\Gamma[\phi,G,V_3,...,V_n]$, is then defined as the Legendre transform of $W[J,R,R_3,...,R_n]$ with respect to all n sources. It is a functional of the connected 1-point to n-point functions $\phi,G,V_3,...,V_n$. That is, it depends not only on the full one and two-point functions, but also on the full 3-vertex, 4-vertex, and so on.

Equivalently to the 1PI and 2PI effective actions, the equations of motion for the n-point functions are obtained by functional differentiation of the nPI effective action, in the absence of external sources. Namely,

$$\frac{\delta\Gamma[\phi,G,V_3,\ldots]}{\delta\phi}=0,\quad \frac{\delta\Gamma[\phi,G,V_3,\ldots]}{\delta G}=0,\quad \frac{\delta\Gamma[\phi,G,V_3,\ldots]}{\delta V_3}=0,\ \ldots \tag{1.26}$$

As we discussed in Section 1.18, we do not wish to calculate the entire effective action, but normally supply an approximation scheme. Again, this may involve truncating the effective action at some order in, for example, a loop expansion. There is a further simplification in this case. An equivalence hierarchy exists between nPI effective actions, and means that one can avoid having to calculate $\Gamma[\phi,G,V_3,\ldots,V_n]$ for arbitrarily large n. For a q-loop approximation, this hierarchy states that all nPI descriptions with $n \leq q$ are equivalent, so only the qPI effective action is required. An additional simplification which may reduce the required order in n can be achieved by, for instance, setting the one point function ϕ to zero. In practice, computational limitations mean that one rarely goes beyond the accuracy afforded by a 4PI effective action.

1.3 Non-equilibrium QFT

1.3.1 Non-equilibrium Generating Functional

The starting point in doing non-equilibrium quantum field theory calculations is to specify an initial state of the system. The pioneering work on this topic was done by Schwinger in the early 1960s in [36], which explained how to use the action principle to solve initial value problems. Before him it was known only how to study transition matrix elements from an earlier time to a later time, and not one where the times coincided, [37]. The formalism is known as the CTP (closed time path) formalism, and we elaborate on it further below.

One way in which we can specify the initial state of the system is with a density matrix, $\rho_D(t_0=0)$. Such a density matrix represents a statistical ensemble of several quantum states. We are familiar with the structure of the density matrix in the case of a canonical thermal ensemble, where $\rho_D(0)\sim e^{-\beta H}$ and β is the inverse temperature. Away from thermal equilibrium, however, we need to ensure that $\rho_D(0)\neq e^{-\beta H}$.

The partition function for such a system is given by

$$Z = \mathrm{Tr}[\rho(0)] = \sum_i \langle i|\rho(0)|i\rangle, \tag{1.27}$$

where $|i\rangle$ are a complete set of Heisenberg states. We may compute expectation values of operators $\mathcal{O}$ in the presence of the density matrix as

$$\langle \mathcal{O}(x)\rangle \equiv \frac{\mathrm{Tr}[\rho(0)\,\mathcal{O}(x)]}{\mathrm{Tr}[\rho(0)]} \tag{1.28}$$

We can see from the above that the trace is making explicit the fact that we are dealing with diagonal matrix elements in the sense that we are at initial time. Now we would like to write an expression for the generating functional of this correlator, in the sense of

$$\langle T\Phi(x)\rangle = \frac{1}{Z_0}\frac{\delta}{\delta iJ(x)}\,Z[J]\bigg|_{J=0} \tag{1.29}$$

where (1.27) plays the role of Z_0. This is accomplished by the above mentioned CTP formalism.

The basic idea of the CTP formalism uses the fact that by taking a diagonal matrix element of a system at a given time, $t=0$, and inserting a complete set of states at a later time t', we can express it as a product of transition matrix elements from $t=0$ to $t=t'$, and the time-reverse (complex conjugate) matrix element from $t=t'$ to 0. Since each of these term is then a transition matrix element of the usual or time reversed kind (non-diagonal), we may express each as a path integral in the usual way. However, to get the generating functional we seek, (which normally involves coupling to an external source, J) we have to introduce *two different sources*, $J_\pm$. We then require that the forward time evolution takes place in the presence of the source J_+, and that the reversed time evolution takes place in the presence of J_-. This is necessary since if the same source were used then the two matrix elements we have are complex conjugates of each other and all dependence on the source is lost.

The generating functional, in the presence of the initial density matrix, which we seek may then be written as,

$$Z[J_+,J_-,\rho(0)] = \mathrm{Tr}\left[\rho(0)\left(T^* e^{-i\int_0^{t'}dt\int d^3x J_-(x)\Phi(x)}\right)\left(Te^{i\int_0^{t'}dt\int d^3x J_+(x)\Phi(x)}\right)\right] \tag{1.30}$$

where T means forward time-ordering and T^* backward. $\Phi(x)$ is the Heisenberg field operator. To facilitate writing this in terms of path integrals we introducing a complete set of states, eigenstates of the Heisenberg field operator at initial time,

$$\begin{aligned} Z[J_+,J_-,\rho(0)] &= \int d\varphi^+(\vec{x})d\varphi^-(\vec{x})\langle\varphi^+|\rho(0)|\varphi^-\rangle \\ &\quad \langle\varphi^-|\left(T^* e^{-i\int_0^{t'} dt\int d^3x J_-(x)\Phi(x)}\right)\left(T e^{i\int_0^{t'} dt\int d^3x J_+(x)\Phi(x)}\right)|\varphi^+\rangle \end{aligned}$$

where $|\varphi^\pm\rangle$ are the eigenstates of Φ at initial time: $\Phi(0,\vec{x})|\varphi^\pm\rangle = \varphi^\pm(\vec{x})|\varphi^\pm\rangle$.

Writing the second matrix element above in path integral representation we have,

$$\begin{aligned} Z[J_+,J_-,\rho(0)] &= \int d\varphi^+(\vec{x})d\varphi^-(\vec{x})\langle\varphi^+|\rho(0)|\varphi^-\rangle \\ &\quad \int_{\varphi^+}^{\varphi^-} \mathscr{D}\varphi e^{i\int_0^{t'} dt\int d^3x\left(L[\varphi^+]+J_+\varphi^+-L[\varphi^-]-J_-\varphi^-\right)} \end{aligned} \tag{1.31}$$

This is the point at which we may introduce the closed time contour, from which the CTP formalism gets its name. We now notice that (1.31) may be written more succinctly as

$$\begin{aligned} Z[J,\rho(0)] &= \int d\varphi^+(\vec{x})d\varphi^-(\vec{x})\langle\varphi^+|\rho(0)|\varphi^-\rangle \\ &\quad \int_{\varphi^+}^{\varphi^-} \mathscr{D}\varphi e^{i\int_{\mathscr{C}} dt\int d^3x\left(L[\varphi]+J(x)\varphi(x)\right)} \\ &= \int d\varphi^+(\vec{x})d\varphi^-(\vec{x})\langle\varphi^+|\rho(0)|\varphi^-\rangle \times \\ &\quad \times \int_{\varphi^+}^{\varphi^-} \mathscr{D}\varphi e^{i\left(S[\varphi]+\int_{\mathscr{C}} dt\int d^3x J(x)\varphi(x)\right)} \end{aligned} \tag{1.32}$$

where the contour $\mathscr{C}$, known as the *Schwinger-Keldysh* contour, is given in Figure 1.1 below.

On the forward part of the contour, $\mathscr{C}_+$, we use normal time-ordering and label fields living here with "+", while on the backward piece, $\mathscr{C}_-$, we use reverse time-ordering and label the corresponding fields with "-". For instance, J_+ and J_-. The time integral over the contour, as evidenced by (1.31), is given by,

$$\int_{\mathscr{C}} dx^0 = \int_{0,\mathscr{C}^+}^{t} dx^0 + \int_{t}^{0,\mathscr{C}^-} dx^0 = \int_{0,\mathscr{C}^+}^{t} dx^0 - \int_{0,\mathscr{C}^-}^{t} dx^0. \tag{1.33}$$

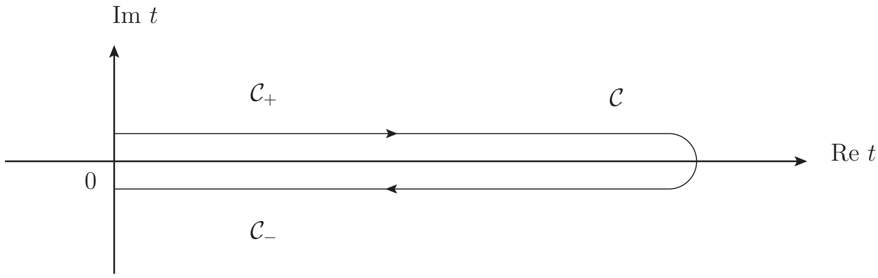

Figure 1.1: Schwinger-Keldysh contour.

We may see from the form of (1.32) that the form of the non-equilibrium generating functional looks very similar to the vacuum one. We simply weigh over the initial conditions and substitute the normal time integration by time integration along the closed time contour. Variation of (1.32) with respect to the source J, in the usual way, will give Green's functions in the state specified by the initial density matrix. In other words, the initial density matrix will determine the initial correlation functions, whose evolution will in turn be determined by the nonequilibrium effective action. We may see how this works explicitly by a further manipulation.

We note that the matrix elements of the density matrix can always be written as the exponential of a polynomial in the fields, [37],

$$\langle\varphi^+|\rho(0)|\varphi^-\rangle = \mathcal{N} e^{i\alpha_{\mathscr{C}}[\varphi]} \tag{1.34}$$

where $\mathcal{N}$ is a normalization factor, and we can expand $\alpha_{\mathscr{C}}[\varphi]$ as,

$$\begin{aligned} \alpha_{\mathscr{C}}[\varphi] &= \alpha_0 + \int_x \alpha_1(x)\varphi(x) + \frac{1}{2}\int_{xy} \alpha_2(x,y)\varphi(x)\varphi(y) \\ &\quad + \frac{1}{3!}\int_{xyz} \alpha_3(x,y,z)\varphi(x)\varphi(y)\varphi(z) + \ldots \end{aligned}$$

The time integrals above are again done over the closed time contour, using the prescription in (1.33). Since $\rho(0)$ is only defined at time $t = 0$, the time integrals in $\alpha_{\mathscr{C}}[\varphi]$ will only pick up contributions when $t = 0$ on the contour. This happens once on $\mathscr{C}^+$ and once on $\mathscr{C}^-$. Thus, when we expand out the first two terms above we have,

$$\int_x \alpha_1(x)\varphi(x) \equiv \int_{\vec{x}} \left\{\alpha_1^+(\vec{x})\varphi^+(\vec{x}) + \alpha_1^-(\vec{x})\varphi^-(\vec{x})\right\},$$

$$\int_{xy} \alpha_2(x,y)\varphi(x)\varphi(y) \equiv \int_{\vec{x}\vec{y}} \{\alpha_2^{++}(\vec{x},\vec{y})\varphi^+(\vec{x})\varphi^+(\vec{y}) - \alpha_2^{+-}(\vec{x},\vec{y})\varphi^+(\vec{x})\varphi^-(\vec{y})$$
$$-\alpha_2^{-+}(\vec{x},\vec{y})\varphi^-(\vec{x})\varphi^+(\vec{y}) + \alpha_2^{--}(\vec{x},\vec{y})\varphi^-(\vec{x})\varphi^-(\vec{y})\}.$$

By using the form of the density matrix elements in (1.34) we easily see that we can write the functions α_n as initial time nonlocal sources in the generating functional (1.31). This yields,

$$Z[J,\rho] = \int \mathscr{D}\varphi \exp\left[i\left(S[\varphi] + \int_x J(x)\varphi(x) + \frac{1}{2}\int_{xy} \alpha_2(x,y)\varphi(x)\varphi(y) + \frac{1}{3!}\int_{xyz} \alpha_3(x,y,z)\varphi(x)\varphi(y)\varphi(z) + \ldots\right)\right] \quad (1.35)$$

where we have absorbed α_0 in the normalization and α_1 in J. As we have seen above, these sources have support only at the endpoints of the time integration. Hence the density matrix can only influence the boundary conditions on the path integral here. The equations of motion at $t \neq 0$ are not influenced by the presence of these terms at $t = 0$. We interpret the coefficient function α_n as providing initial conditions for the n-point function.

This generating functional is valid for general initial conditions, or general initial density matrices. However, for practical purposes, it is often accurate enough to provide only the first few lowest initial correlation functions. Indeed, providing the one- and two-point functions only, means having a Gaussian initial density matrix, for which $\alpha_3 = \alpha_4 = \ldots = 0$.

We would now like to apply the formalism above to write down the non-equilibrium 2PI effective action. Indeed it is straightforward to do this. In the generating functional (1.32), as in the vacuum case, we simply add an additional source term, $R(x,y)$, which is bi-local in the fields,

$$Z[J,R,\rho] = \int d\varphi^+(\vec{x})d\varphi^-(\vec{x})\langle\varphi^+|\rho(0)|\varphi^-\rangle \int_{\varphi^+}^{\varphi^-} \mathscr{D}\varphi e^{i\left(S[\varphi]+\int_x J(x)\varphi(x)+\int_{xy} R(x,y)\varphi(x)\varphi(y)\right)} \quad (1.36)$$

where we again use the closed time contour. Then (1.35) becomes

$$Z[J,R,\rho] = \int \mathscr{D}\varphi \exp\left[i\left(S[\varphi] + \int_x J(x)\varphi(x) + \frac{1}{2}\int_{xy} R(x,y)\varphi(x)\varphi(y)\right.\right.$$

$$+\frac{1}{3!}\int_{xyz}\alpha_3(x,y,z)\varphi(x)\varphi(y)\varphi(z)+\ldots\Big)\Big] \tag{1.37}$$

where we have now also absorbed α_2 into R. Specializing to the case of Gaussian initial density matrices gives us,

$$Z[J,R,\rho^{(\text{gauss})}] = \int \mathscr{D}\varphi e^{i\left(S[\varphi]+\int_x J(x)\varphi(x)+\frac{1}{2}\int_{xy}R(x,y)\varphi(x)\varphi(y)\right)} \equiv Z[J,R] \tag{1.38}$$

where $Z[J,R]$ is the generating functional we discussed previously, (1.15), defined for the closed time contour $\mathscr{C}$.

For the case of more general, non-Gaussian initial density matrices, where we need to include more initial-time sources, the analysis above clearly shows that this is most naturally done through a higher-PI effective action.

In summary, our manipulations have lead us to a form for $Z[J,\rho]$ which is indicative of the nature of nonequilibrium quantum field theory. Indeed, we see that both statistical fluctuations (via the initial conditions) and quantum fluctuations (via the functional integral with action S) are made manifest. When dealing with non-equilibrium systems all the usual results of vacuum field theory follow through, with the additional requirements that we: integrate over the closed-time contour following (1.33), satisfy the initial conditions dictated by ρ and, as we shall explain below, use contour-ordered propagators instead of the usual Feynman propagators when evaluating Feynman diagrams.

1.3.2 CTP propagators and self-energies

The introduction of the closed time contour, $\mathscr{C}$, in the previous section leads us to distinguish between objects living on the upper and lower branches of the contour. For instance, we recall that the source J becomes instead J_+ on the upper branch and J_- on the lower branch, and similarly for the field φ. This leads us to use a different form for the propagators and self-energies when dealing with non-equilibrium systems.

Propagators

Recalling the definition of the connected 2-point function (1.8), we can make the time-ordering along the closed time contour more explicit by using the $\{+,-\}$ components and writing,

$$\frac{\delta^2 W[J,\rho]}{\delta J_a(x)\delta J_b(y)}\Big|_{J_+,J_-=0} \equiv iG_{ab}(x,y). \tag{1.39}$$

Here $a, b = \{+, -\}$ and it is clear that we have a 2×2 matrix of propagators. We write the components out below:

$$\begin{aligned}
iG^{-+}(x,y) &= \langle \Phi(x)\Phi(y)\rangle_{conn.} \equiv G^{>}(x,y) \\
iG^{+-}(x,y) &= \langle \Phi(y)\Phi(x)\rangle_{conn.} \equiv G^{<}(x,y) \\
iG^{++}(x,y) &= \langle T\Phi(x)\Phi(y)\rangle_{conn.} \\
&= \theta(x^0 - y^0)G^{>}(x,y) + \theta(y^0 - x^0)G^{<}(x,y) \\
iG^{--}(x,y) &= \langle T^{*}\Phi(x)\Phi(y)\rangle_{conn.} \\
&= \theta(x^0 - y^0)G^{<}(x,y) + \theta(y^0 - x^0)G^{>}(x,y),
\end{aligned} \tag{1.40}$$

where T represents forward time-ordering and T^* reverse. We see from the above that the contour propagators have two independent components, $G^>$ and $G^<$, in terms of which all the components can be expressed.

A further useful definition is that of the spectral function, $\rho(x,y)$ and statistical propagator, $F(x,y)$,

$$\begin{aligned}
F(x,y) \equiv \frac{1}{2}\langle\{\Phi(x),\Phi(y)\}\rangle &= \frac{1}{2}(G^{>}(x,y) + G^{<}(x,y)) \\
&= \frac{1}{2}[G^{-+}(x,y) + G^{+-}(x,y)],
\end{aligned} \tag{1.41}$$

$$\begin{aligned}
\rho(x,y) \equiv i\langle[\Phi(x),\Phi(y)]\rangle &= i(G^{>}(x,y) - G^{<}(x,y)) \\
&= i[G^{-+}(x,y) - G^{+-}(x,y)].
\end{aligned} \tag{1.42}$$

We may invert these easily to give,

$$\begin{aligned}
G^{-+}(x,y) &= F(x,y) - \frac{i}{2}\rho(x,y) \\
G^{+-}(x,y) &= F(x,y) + \frac{i}{2}\rho(x,y).
\end{aligned} \tag{1.43}$$

The advantage of using $F(x,y)$ and $\rho(x,y)$ is that they are both real, which makes their evolution equations intrinsically more manageable, and, more importantly, they have handy physical interpretations. The spectral function involves the spectrum of the theory while the statistical propagator deals with occupation numbers. In fact, for a real scalar field, we can show that,

$$F(x,y) = F(y,x) \text{ and } \rho(x,y) = -\rho(y,x). \tag{1.44}$$

In addition, the spectral function also comprises the scalar field equal-time commutation relations in,

$$\rho(x,y)|_{x^0=y^0} = 0, \qquad \partial_{x^0}\rho(x,y)|_{x^0=y^0} = \delta(\vec{x}-\vec{y}). \tag{1.45}$$

We may write the above matrix expressions in condensed notation by using the familiar expression for the connected two point function and simply replacing the usual theta function with the CTP contour ordered theta function. That is,

$$G(x,y) = \theta_{\mathscr{C}}(x^0-y^0)\langle\Phi(x)\Phi(y)\rangle_{conn.} + \theta_{\mathscr{C}}(y^0-x^0)\langle\Phi(y)\Phi(x)\rangle_{conn.} \tag{1.46}$$

where,

$$\theta_{\mathscr{C}}(x_0-y_0) \equiv \begin{cases} \theta(x_0-y_0) & \text{for } x_0, y_0 \text{ both on } \mathscr{C}_+ \\ \theta(y_0-x_0) & \text{for } x_0, y_0 \text{ both on } \mathscr{C}_- \\ 1 & \text{for } x_0 \text{ on } \mathscr{C}_-,\ y_0 \text{ on } \mathscr{C}_+ \\ 0 & \text{for } x_0 \text{ on } \mathscr{C}_+,\ y_0 \text{ on } \mathscr{C}_- \end{cases} \tag{1.47}$$

The above allows us to express the two point function compactly in terms of the spectral and statistical functions,

$$G(x,y) = F(x,y) - \frac{i}{2}\rho(x,y)\mathrm{sign}_{\mathscr{C}}(x_0-y_0). \tag{1.48}$$

Here the *sign* function is defined in the usual way in terms of the theta function but in this case the contour ordered theta function is used. Hence,

$$\mathrm{sign}_{\mathscr{C}}(x_0-y_0) = \theta_{\mathscr{C}}(x_0-y_0) - \theta_{\mathscr{C}}(y_0-x_0). \tag{1.49}$$

The above discussion applies for bosonic degrees of freedom, but we can do a very similar one for fermionic degrees of freedom.

In condensed contour-ordered notation, the fermion propagator is given by (we exclude the spinor indices for simplicity),

$$\begin{aligned} \triangle_{\mathscr{C}}(x,y) &= \theta_{\mathscr{C}}(x^0-y^0)\langle\Psi(x)\bar{\Psi}(y)\rangle_{conn.} - \theta_{\mathscr{C}}(y^0-x^0)\langle\bar{\Psi}(y)\Psi(x)\rangle_{conn.} \\ &\equiv \theta_{\mathscr{C}}(x^0-y^0)\triangle^{>}(x,y) - \theta_{\mathscr{C}}(y^0-x^0)\triangle^{<}(x,y), \end{aligned} \tag{1.50}$$

where $\Psi(x)$ is the fermion field Heisenberg operator. Hence the components of the contour-ordered propagator matrix are,

$$\triangle^{++}(x,y) = \theta(x^0-y^0)\langle\Psi(x)\bar{\Psi}(y)\rangle_{conn.} - \theta(y^0-x^0)\langle\bar{\Psi}(y)\Psi(x)\rangle_{conn.}$$

$$
\begin{aligned}
&\equiv \theta(x^0-y^0)\triangle^>(x,y)-\theta(y^0-x^0)\triangle^<(x,y)\\
\triangle^{--}(x,y) &= \theta(y^0-x^0)\langle\Psi(x)\bar{\Psi}(y)\rangle_{conn.}-\theta(x^0-y^0)\langle\bar{\Psi}(y)\Psi(x)\rangle_{conn.}\\
&\equiv \theta(y^0-x^0)\triangle^>(x,y)-\theta(x^0-y^0)\triangle^<(x,y)\\
\triangle^{+-}(x,y) &= -\theta(y^0-x^0)\langle\bar{\Psi}(y)\Psi(x)\rangle_{conn.}=-\langle\bar{\Psi}(y)\Psi(x)\rangle_{conn.}\equiv-\triangle^<(x,y)\\
\triangle^{-+}(x,y) &= \theta(x^0-y^0)\langle\Psi(x)\bar{\Psi}(y)\rangle_{conn.}=\langle\Psi(x)\bar{\Psi}(y)\rangle_{conn.}\equiv\triangle^>(x,y).
\end{aligned}
$$

In the case of the spectral and statistical components, we can do a very similar decomposition for fermions, the only difference being that the field commutator will now correspond to the statistical propagator, and the anti-commutator to the spectral function:

$$
\begin{aligned}
\rho^{(f)}(x,y) \equiv i\langle\{\Psi(x),\bar{\Psi}(y)\}\rangle &= i\left(\triangle^>(x,y)+\triangle^<(x,y)\right)\\
&= i\left(\triangle^{-+}(x,y)-\triangle^{+-}(x,y)\right),
\end{aligned}
\tag{1.51}
$$

$$
\begin{aligned}
F^{(f)}(x,y) \equiv \frac{1}{2}\langle[\Psi(x),\bar{\Psi}(y)]\rangle &= \frac{1}{2}\left(\triangle^>(x,y)-\triangle^<(x,y)\right)\\
&= \frac{1}{2}\left(\triangle^{-+}(x,y)+\triangle^{+-}(x,y)\right).
\end{aligned}
\tag{1.52}
$$

We may invert these easily to give,

$$
\begin{aligned}
\triangle^{-+}(x,y) &= F^{(f)}(x,y)-\frac{i}{2}\rho^{(f)}(x,y)\\
\triangle^{+-}(x,y) &= F^{(f)}(x,y)+\frac{i}{2}\rho^{(f)}(x,y).
\end{aligned}
$$

Once again, the equal-time anti-commutation relations are encoded in $\rho^{(f)}(x,y)$. For a Dirac fermion we have

$$\gamma^0\rho^{(f)}(x,y)|_{x^0=y^0}=i\delta(\vec{x}-\vec{y}). \tag{1.53}$$

Similarly, in condensed contour ordered notation we can write the fermion two point function in the same form as the boson one,

$$\triangle(x,y)=F^{(f)}(x,y)-\frac{i}{2}\rho^{(f)}(x,y)\mathrm{sign}_{\mathscr{C}}(x_0-y_0). \tag{1.54}$$

Self energy

We may do a similar analysis to the above for the self energy. We first separate the self energy into a local and non-local part as,

$$\Sigma(x,y) = -i\delta_{\mathscr{C}}(x,y)\Sigma^{(0)}(x) + \bar{\Sigma}(x,y). \tag{1.55}$$

We absorb the local part into a generalized "mass" term, and we drop the bar on the non-local part in what follows. The above is naturally defined on the time ordered contour, and we interpret the time-ordered delta function as the ordinary delta function for x,y on $\mathscr{C}_+$ and the negative one for x,y on $\mathscr{C}_-$.

Similarly to the propagator above, we separate the self energy in $\{+,-\}$ components as,

$$\begin{aligned}
\Sigma^{++}(x,y) &= -i\Sigma^{(0)}(x)\delta(x,y) + \theta(x^0-y^0)\Sigma^{>}(x,y) + \theta(y^0-x^0)\Sigma^{<}(x,y) \\
\Sigma^{--}(x,y) &= i\Sigma^{(0)}(x)\delta(x,y) + \theta(x^0-y^0)\Sigma^{<}(x,y) + \theta(y^0-x^0)\Sigma^{>}(x,y) \\
\Sigma^{-+}(x,y) &= -\Sigma^{>}(x,y) \\
\Sigma^{+-}(x,y) &= -\Sigma^{<}(x,y).
\end{aligned} \tag{1.56}$$

The minus signs in the definitions of $\Sigma^{-+}(x,y)$ and $\Sigma^{+-}(x,y)$ come from integration along the contour. We may show this more explicitly using the formal definition of the self-energy, namely,

$$G = G_0 + G_0\cdot\Sigma\cdot G_0 + \cdots \tag{1.57}$$

Here we define the convolution by $A\cdot B = \int d^4z A(x,z)B(z,y)$. The propagator and self-energy in (1.57) are defined on the CTP contour and hence we may expand out the second term above as,

$$\begin{aligned}
G_0\cdot\Sigma\cdot G_0 &= \int_{zw} G_0(x,z)\Sigma(z,w)G_0(w,y) \\
&= \int_{\vec{z}\vec{w}} \int_{\mathscr{C}} dz^0 \int_{\mathscr{C}} dw^0\; G_0(x,z)\Sigma(z,w)G_0(w,y) \\
&= \int_{\vec{z}\vec{w}} \left(\int_{0,\mathscr{C}^+}^{t} dz^0 - \int_{0,\mathscr{C}^-}^{t} dz^0\right)\left(\int_{0,\mathscr{C}^+}^{t'} dw^0 - \int_{0,\mathscr{C}^-}^{t'} dw^0\right)\times \\
&\quad \times G_0(x,z)\Sigma(z,w)G_0(w,y) \\
&= \int_{\vec{z}\vec{w}} \left(\int_{0,\mathscr{C}^+}^{t} dz^0 \int_{0,\mathscr{C}^+}^{t'} dw^0\; G_0(x,z)\Sigma(z,w)G_0(w,y)\right. \\
&\quad - \int_{0,\mathscr{C}^+}^{t} dz^0 \int_{0,\mathscr{C}^-}^{t'} dw^0\; G_0(x,z)\Sigma(z,w)G_0(w,y)
\end{aligned}$$

$$
-\int_{0,\mathscr{C}^-}^{t} dz^0 \int_{0,\mathscr{C}^+}^{t'} dw^0\, G_0(x,z)\Sigma(z,w)G_0(w,y)
$$
$$
+\int_{0,\mathscr{C}^-}^{t} dz^0 \int_{0,\mathscr{C}^-}^{t'} dw^0\, G_0(x,z)\Sigma(z,w)G_0(w,y)\Bigg). \tag{1.58}
$$

Studying the two middle terms in the above expressions it is clear that the minus signs are coming from integration over the $\mathscr{C}^-$ part of the CTP contour. The second term above contains z^0 on $\mathscr{C}^+$ and w^0 on $\mathscr{C}^-$, while the opposite is true of the third term above. Hence, the second term corresponds to the contribution from Σ^{+-} while the third corresponds to the contribution from Σ^{-+}.

We may also write the statistical and spectral self-energies in the same way as for the propagator,

$$
\begin{aligned}
\Sigma_F(x,y) &= \frac{1}{2}[\Sigma^>(x,y)+\Sigma^<(x,y)] = \frac{1}{2}[-\Sigma^{-+}(x,y)-\Sigma^{+-}(x,y)] \\
\Sigma_\rho(x,y) &= i[\Sigma^>(x,y)-\Sigma^<(x,y)] = i[-\Sigma^{-+}(x,y)+\Sigma^{+-}(x,y)].
\end{aligned} \tag{1.59}
$$

These satisfy similar relations to (1.44), and we emphasize that they are non-local.

Analogously to the propagator expression (1.48), we may write the self energy in compact notation in terms of the statistical and spectral self energies as,

$$
\Sigma(x,y) = \Sigma_F(x,y) - \frac{i}{2}\Sigma_\rho(x,y)\mathrm{sign}_{\mathscr{C}}(x_0-y_0). \tag{1.60}
$$

An expression of the same form applies for the fermion self energy.

Now that we have the expressions above for the spectral and statistical components of the self-energy, (1.59), we will elaborate a bit more on precisely how to compute them for a given theory and approximation scheme.

We start with a vacuum expression for $\Sigma(x,y)$, which we obtain following the discussion on Φ-derivable approximations. We then use this to obtain the various $\{+,-\}$ components of $\Sigma(x,y)$ in the nonequilibrium case. We do this by noting that

$$
\begin{aligned}
\Sigma^{+-}(x,y) &= -\Sigma^{(vac)}\Big|_{Prop^{+-}(x,y)} \\
\Sigma^{-+}(x,y) &= -\Sigma^{(vac)}\Big|_{Prop^{-+}(x,y)}.
\end{aligned} \tag{1.61}
$$

Here, $Prop^{+-}(x,y)$ means that whenever a propagator (*of any type*) appears in the self-energy, replace it by its corresponding $+-$ component if the argument of that propagator is (x,y), and by its $-+$ component if the argument is (y,x). Similarly for $Prop^{-+}(x,y)$. Once again, the minus sign above is due to integration along the closed time contour.

As an example assume that, for some theory, a one loop contribution to the vacuum scalar self-energy is given by

$$\Sigma(x,y) = G(x,y)G(y,x). \tag{1.62}$$

Then, according to the prescription above,

$$\begin{aligned}
\Sigma^{+-}(x,y) &= -\left(G^{+-}(x,y)G^{-+}(y,x)\right) \\
\Sigma^{-+}(x,y) &= -\left(G^{-+}(x,y)G^{+-}(y,x)\right).
\end{aligned}$$

Note that in the scalar case $G(x,y) = G(y,x)$, so this is a trivial example. However, the principle holds for theories with fermions where we have to distinguish between $\triangle(x,y)$ and $\triangle(y,x)$. The final step is to express the self-energy in terms of $F(x,y)$ and $\rho(x,y)$ by using (1.59).

For completeness, we now present the analysis for the fermion self-energy. It is very similar to the scalar case. Once again, we begin by separating the self energy into a local and non-local part as,

$$\Sigma^{(f)}(x,y) = i\delta_{\mathscr{C}}(x,y)\Sigma^{(f),(0)}(x) + \bar{\Sigma}^{(f)}(x,y), \tag{1.63}$$

where we absorb the local part into a generalized "mass" term and we drop the bar on the non-local part in what follows.

We separate the self-energy into $\{+,-\}$ components as,

$$\begin{aligned}
\Sigma^{(f)++}(x,y) &= i\delta(x,y)\Sigma^{(f),(0)}(x) + \theta(x^0-y^0)\Sigma^{(f)>}(x,y) - \theta(y^0-x^0)\Sigma^{(f)<}(x,y) \\
\Sigma^{(f)--}(x,y) &= -i\delta(x,y)\Sigma^{(f),(0)}(x) + \theta(y^0-x^0)\Sigma^{(f)>}(x,y) - \theta(x^0-y^0)\Sigma^{(f)<}(x,y) \\
\Sigma^{(f)+-}(x,y) &= -\left(-\theta(y^0-x^0)\Sigma^{(f)<}(x,y)\right) = -\left(-\Sigma^{(f)<}(x,y)\right) = \Sigma^{(f)<}(x,y) \\
\Sigma^{(f)-+}(x,y) &= -\left(\theta(x^0-y^0)\Sigma^{(f)>}(x,y)\right) = -\Sigma^{(f)>}(x,y),
\end{aligned} \tag{1.64}$$

As explained in the scalar case, the minus signs in the definitions of $\Sigma^{(f)+-}(x,y)$ and $\Sigma^{(f)-+}(x,y)$ above, come from integration along the contour.

The fermion statistical and spectral self-energies are then calculated to be,

$$\begin{aligned}
\Sigma_\rho^{(f)}(x,y) &= i\left(\Sigma^{(f)>}(x,y)+\Sigma^{(f)<}(x,y)\right) &= -i\left(\Sigma^{(f)-+}(x,y)-\Sigma^{(f)+-}(x,y)\right) \\
\Sigma_F^{(f)}(x,y) &= \frac{1}{2}\left(\Sigma^{(f)>}(x,y)-\Sigma^{(f)<}(x,y)\right) &= -\frac{1}{2}\left(\Sigma^{(f)-+}(x,y)+\Sigma^{(f)+-}(x,y)\right).
\end{aligned} \tag{1.65}$$

Equivalently to the scalar case in (1.60), we may write the fermion self-energy in compact notation as,

$$\Sigma^{(f)}(x,y) = \Sigma_F^{(f)}(x,y) - \frac{i}{2}\Sigma_\rho^{(f)}(x,y)\mathrm{sign}_{\mathscr{C}}(x_0-y_0). \tag{1.66}$$

Once again, we have,

$$\begin{aligned}
\Sigma^{(f)+-}(x,y) &= -\Sigma^{(\mathrm{vac})(f)}(x,y)\Big|_{\mathrm{Prop}^{+-}(x,y)}, \\
\Sigma^{(f)-+}(x,y) &= -\Sigma^{(\mathrm{vac})(f)}(x,y)\Big|_{\mathrm{Prop}^{-+}(x,y)}.
\end{aligned}$$

We can do an equivalent analysis for ghosts as for fermions explained above, and gauge bosons are treated in the same way as the scalar case.

In the following Chapter we demonstrate in more detail how this is achieved, within the context of $\mathscr{N}=4$ SYM.

1.3.3 Evolution Equations

In this section we will discuss the evolution equations for non-equilibrium systems in the light of the 2PI effective action. For example, in the case of a simple scalar field theory we are interested in the evolution equations for the connected one-point and two-point functions, ϕ and G. In what follows, for the sake of calculational simplicity, we will limit ourselves to the case of a vanishing field expectation value, $\phi=0$. We begin with a discussion of scalar fields followed by a similar analysis for fermionic fields.

As discussed before, we obtain the equations of motion for the fields by using the stationarity conditions (1.17). This leads in turn to (1.19), or, in the case of Φ-derivable approximations with vanishing source $R(x,y)$ to (1.23).

For the case of non-equilibrium evolution we are dealing with *initial value* problems, so we rewrite the equation of motion (1.19) in a more suitable way by convoluting with G, namely,

$$\int_z G_0^{-1}(x,z)G(z,y) - \int_z [\Sigma(x,z) + iR(x,y)]G(z,y) = \delta_{\mathscr{C}}(x-y), \tag{1.67}$$

where $\int_z G^{-1}(x,z)G(z,y) = \delta_{\mathscr{C}}(x-y)$. The theory-dependent value of the classical inverse propagator, G_0^{-1}, is now substituted into this expression. We demonstrate this first for a scalar field theory with classical propagator defined by $iG_0^{-1}(x-y) = (\partial^2 + m^2)\delta(x-y)$. For the sake of simplicity, we will also set the source $R(x,y)$ to zero. The evolution equation above then becomes,

$$(\partial^2 + m^2 - \Sigma^{(0)}(x))G(x,y) - i\int_z \Sigma(x,z)G(z,y) = i\delta_{\mathscr{C}}(x-y), \tag{1.68}$$

where we can see that the local self-energy naturally combines with the mass.

In the case of fermion fields a very similar line of argument leads us to the following evolution equation,

$$\int_z \triangle_0^{-1}(x,z)\triangle(z,y) - \int_z [\Sigma^{(f)}(x,z) + iR(x,y)]\triangle(z,y) = \delta_{\mathscr{C}}(x-y). \tag{1.69}$$

Keeping in mind our ultimate goal, which is to study $\mathscr{N} = 4$ SYM, we now choose the classical inverse massless fermion propagator to be $i\triangle^{-1}_{0,\dot{\alpha}\beta}(x,y) = i\bar{\sigma}^{\mu}_{\dot{\alpha}\beta}\partial^x_\mu\delta(x-y)$. Here the sigma matrices $\sigma^{\mu}_{\alpha\dot{\beta}}$ are the chiral projections of the gamma matrices in four dimensions, with spinor indices $\alpha\,\dot{\beta}$ taking values $1,2$. We again split the self-energy into a local and non-local part,

$$\Sigma^{(f)}_{\dot{\alpha}\beta}(x,y) = i\Sigma^{(0)(f)}_{\dot{\alpha}\beta}(x)\delta(x-y) + \bar{\Sigma}^{(f)}_{\dot{\alpha}\beta}(x,y), \tag{1.70}$$

and drop the bar on the non-local self-energy in what follows. Upon setting the source R to zero, the fermion evolution equation becomes,

$$\left(\bar{\sigma}^{\mu}_{\dot{\alpha}\beta}\partial^x_\mu - i\Sigma^{(0)(f)}_{\dot{\alpha}\beta}(x)\right)\triangle^{\beta\dot{\gamma}}(x,y) - \int_z \Sigma^{(f)}_{\dot{\alpha}\beta}(x,z)\triangle^{\beta\dot{\gamma}}(z,y) = \delta^{\dot{\gamma}}_{\dot{\alpha}}\delta_{\mathscr{C}}(x-y). \tag{1.71}$$

We emphasize that the above equations are on the closed time contour. In order to get a more tangible physical interpretation, we express them in terms of the statistical and spectral components of the propagator and self energy. An efficient way to do this is to make use of the compact expressions (1.48) and (1.60) (and the corresponding expressions for fermions) and to perform a careful integration along $\mathscr{C}$. For a detailed derivation we refer the reader to the excellent review [30].

We give below the evolution equations for the spectral and statistical components of the two point functions for scalars and fermions.

Scalar evolution equations:

$$\begin{aligned}
(\partial^2 - \Sigma^{(0)}(x;G))\rho(x,y) &= \int_{y^0}^{x^0} dz^0 \int d^3z \Sigma_\rho(x,z)\rho(z,y) \\
(\partial^2 - \Sigma^{(0)}(x;G))F(x,y) &= \int_{0}^{x^0} dz^0 \int d^3z \Sigma_\rho(x,z)F(z,y) \\
&\quad - \int_{0}^{y^0} dz^0 \int d^3z \Sigma_F(x,z)\rho(z,y).
\end{aligned} \tag{1.72}$$

Fermion evolution equations:

$$\begin{aligned}
\left(i\bar{\sigma}^\mu_{\dot{\alpha}\beta}\partial^x_\mu + \Sigma^{(f)(0)}_{\dot{\alpha}\beta}(x;\triangle)\right)\rho^{(f)\beta\dot{\gamma}}(x,y) &= \int_{y^0}^{x^0} dz^0 \int d^3z \Sigma^{(f)}_{\rho,\dot{\alpha}\beta}(x,z)\rho^{(f)\beta\dot{\gamma}}(z,y) \\
\left(i\bar{\sigma}^\mu_{\dot{\alpha}\beta}\partial^x_\mu + \Sigma^{(f)(0)}_{\dot{\alpha}\beta}(x;\triangle)\right)F^{(f)\beta\dot{\gamma}}(x,y) &= \int_{0}^{x^0} dz^0 \int d^3z \Sigma^{(f)}_{\rho,\dot{\alpha}\beta}(x,z)F^{(f)\beta\dot{\gamma}}(z,y) \\
&\quad - \int_{0}^{y^0} dz^0 \int d^3z \Sigma^{(f)}_{F,\dot{\alpha}\beta}(x,z)\rho^{(f)\beta\dot{\gamma}}(z,y).
\end{aligned} \tag{1.73}$$

These evolution equations are clearly *causal* since only previous times contribute to the spacetime integration. They are thus known as *memory* integrals.

We have now set up the basics. In the following Chapter we will apply them to a special theory, namely $\mathcal{N} = 4$ super Yang-Mills. All of the comments regarding scalars in the previous sections can be extended to gluons, and all those involving fermions can be extended naturally to ghosts. Our ultimate goal in this part of the thesis is to write down the evolution equations for this theory.

Chapter 2

2PI effective action of $\mathcal{N} = 4$ SYM

2.1 Introduction

In this Chapter we will apply the 2PI effective action formalism derived in the previous Chapter to a special theory, $\mathcal{N} = 4$ SYM. This theory, which has been called the *simplest* quantum field theory, [5], has been widely-studied in the past 15 years in the context of the AdS/CFT correspondence, [6]. For comprehensive reviews see [7], [38]. This correspondence is a concrete realization of holography. This is a groundbreaking concept of high-energy physics which relates the gravitational theory in the bulk of spacetime to a non-gravitational QFT in one less dimension, living on its boundary. We will discuss it in more detail in the coming Chapters.

Holography is a very useful tool since, for instance, it allows us to study long-standing gravitational problems in a new way. One can map them holographically to QFT problems (which can potentially be solved using known methods) and then map the solution back to the gravitational sector. There is a caveat however: this is a weak-strong coupling duality, meaning that the weakly coupled regime of one theory is mapped to the strongly-coupled regime of the other.

We are very interested in understanding thermalization in $\mathcal{N} = 4$ SYM. From a holographic point of view this would potentially give us insight into black hole formation and evaporation, since thermalization in the boundary QFT is expected to be mapped to horizon formation in the bulk, [39]. Applying the methodology of nPI effective actions to study the non-equilibrium dynamics and thermalization of $\mathcal{N} = 4$ SYM would add a powerful new tool to the holographic arsenal. It would also allow us to study holography in the weakly-coupled regime in the gauge theory, while up till now it has mostly been done in the strongly-coupled regime.

$\mathcal{N} = 4$ SYM is a special theory from other perspectives also. For one thing, in addition to being conformal, it has the maximal amount of supersymmetry. Hence, many things which are difficult to compute in conventional theories, may be simpler to handle in this theory.

Comparing with thermalization in QCD may shed light on why the RHIC and LHC data seem to be well-approximated by the strongly coupled $\mathcal{N} = 4$ theory. Indeed, standard theoretical predictions for QCD thermalization times vary dramatically from the experimental observations: the theory seems to thermalize much sooner than expected. See [10], [11], [12], [13] for recent reviews.

Our goal is to write down the 2PI effective action for $\mathcal{N} = 4$ SYM, to two-loop order. For simplicity, we do so in the symmetric phase, meaning that the one-point functions are set to zero. We also set the external sources to zero. We then proceed to write down the evolution equations for the two-point functions in this theory, the necessary starting point towards eventually understanding thermalization.

2.2 $\mathcal{N}$ = 4 super Yang-Mills basics

$\mathcal{N} = 4$ SYM theory is a maximally supersymmetric gauge theory in 4 dimensions. One lovely feature of supersymmetric field theories is that many of the usual divergences present in ordinary QFTs are absent when supersymmetry is introduced, due to non-renormalization theorems in superspace, [40], [41]. This results in our theory preserving conformal invariance at the quantum level. Conformal symmetry is a very constraining property in field theory and is usually anomalous. This is because quantizing the theory inadvertently requires the introduction of a mass scale to remove divergences. This breaks scale invariance and hence conformal invariance and is usually characterized by a non-zero beta function. This function describes the dependence of the coupling constant on the mass scale. For $\mathcal{N} = 4$ SYM, the beta function is believed to vanish to all orders in perturbation theory and nonperturbatively, meaning that superconformal symmetry is preserved at the quantum level, [42]. In our case, these considerations mean that we will not renormalize when computing the 2PI effective action of $\mathcal{N} = 4$ SYM[1]. For a comprehensive review of $\mathcal{N} = 4$ SYM see [43],

[1]There is a caveat when dealing with the 2PI effective action, however, as new infinities may appear in this formalism. This is because there are new possible counterterms at each finite order, which is ultimately due to the fact that the theory is parameterised in terms of both the one- and two-point functions. These counterterms have no analogue in perturbation theory. Counterterms are restricted

[48], [49].

As mentioned above, one of the main claims-to-fame of $\mathcal{N} = 4$ SYM is the well-researched AdS/CFT correspondence. Also known as the Maldacena conjecture, it poses an equivalence between type IIB superstring theory on the background $AdS_5 \times S^5$ and four-dimensional $\mathcal{N} = 4$ SYM living on the boundary of AdS_5. It is a duality between gauge theory and string theory, which was first glimpsed by 't Hooft in 1974 [50]. For a generic $U(N)$ gauge theory, he made the conceptual leap to consider $1/N$ as a coupling constant in the large N limit. He then noticed that there is a connection between the order in N of a Feynman diagram and it's topological structure. Specifically, in the large N limit, for each Feynman diagram there is a corresponding two-dimensional surface, with the N-dependence given by the genus of the surface. He thus made the connection between the perturbative expansion of the field theory in $1/N$ and a perturbative expansion of string theory, since in the string theory the coupling constant is related to the genus of the worldsheet.

The field content of $\mathcal{N} = 4$ super Yang-Mills is made up of four spinors λ_i, $i = 1,\ldots,4$, transforming under the global $SU(4)$ symmetry, and six scalars M^m, $m = 1,\ldots,6$, transforming under $SO(6)$, which is isomorphic to $SU(4)$. The number of bosonic and fermionic on-shell degrees of freedom are equal, in keeping with the supersymmetric nature of the theory. As mentioned above, the theory is conformal and supersymmetric, giving it the $PSU(2,2|4)$ global symmetry. This is the *maximal superconformal group* in 4 dimensions.

It has an $SU(N)$ color gauge symmetry and corresponding gauge field A_μ, from which can be constructed the covariant derivative,

$$\nabla_\mu = \partial_\mu - igA_\mu. \tag{2.1}$$

Here g is the dimensionless coupling constant. All the fields transform in the adjoint representation of the gauge group so they are traceless and hermitian $N \times N$ matrices. We will call a generic such field $\mathcal{W}$, [43]. Under a gauge transformation $U(x) \in SU(N)$ the fields transform as,

$$\mathcal{W} \to U\mathcal{W}U^{-1} \tag{2.2}$$

by the symmetries of the theory and in this case they must be consistent with the 2PI Ward identities. These may allow for more counterterms than in usual perturbation theory. See for instance [44], [45] for 2PI renormalization of theories with scalar fields and [46] for theories with fermions. For gauge theories, see for instance [47] for the case of 2PI renormailzation of QED.

and

$$A_\mu \to U A_\mu U^{-1} + i g^{-1} \partial_\mu U U^{-1}. \tag{2.3}$$

The covariant derivative acts on a field in the adjoint representation as

$$\nabla_\mu \mathscr{W} := [\nabla_\mu, \mathscr{W}] = \partial_\mu \mathscr{W} + i g [A_\mu, \mathscr{W}] \tag{2.4}$$

and we can build the field strength as

$$F_{\mu\nu} = -i g^{-1} [\nabla_\mu, \nabla_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + i g [A_\mu, A_\nu]. \tag{2.5}$$

The $\mathcal{N}$ = 4 super Yang-Mills action is given by

$$\begin{aligned} S_{SYM} &= \int_x Tr\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\nabla_\mu M_m \nabla^\mu M_m + \frac{1}{4}[M_m, M_n]^2 \\ &+ \frac{1}{2} i \lambda_i \sigma^\mu \nabla_\mu \bar{\lambda}^i + \frac{1}{2} i \lambda_i [\lambda_j, (\tilde{\sigma}_m^{-1})^{ij} M_m] - \frac{1}{2} i \bar{\lambda}^i [\bar{\lambda}^j, (\tilde{\sigma}_m)_{ij} M_m]\Big), \end{aligned}$$

where the trace is over the adjoint indices and we use the shorthand notation $\int_x \equiv \int d^4x$. The sigma matrices $\tilde{\sigma}^m_{ij}$ and $\sigma^\mu_{\alpha\dot{\beta}}$ are the chiral projections of the gamma matrices in 6 and 4 dimensions respectively, where the spinor index $i = 1, \cdots, 4$ and the spinor indices $\alpha, \dot{\beta}$ take values 1,2. Here we use the conventions in [51] to manipulate spinors.

In order to quantize this theory properly we need to gauge-fix and include ghosts. The gauge-fixing term which we add is,

$$-\frac{1}{2} Tr(\partial_\mu A^\mu)^2 = \frac{1}{2} Tr\left(A^\mu \partial_\mu \partial_\nu A^\nu\right). \tag{2.6}$$

The ghost Lagrangian is given by

$$Tr\left(-\bar{\eta}\partial^\mu\left(\nabla_\mu \eta\right)\right) = Tr\left(\bar{\eta}(-\partial^2)\eta + i g\left(\partial^\mu \bar{\eta} A_\mu \eta - \partial^\mu \bar{\eta}\eta A_\mu\right)\right), \tag{2.7}$$

where we performed an intergation by parts in going from LHS to RHS in (2.7).

We can thus write the complete action, split into the free and interacting parts, as,

$$S_{SYM} = S^0_{SYM} + S^{int}_{SYM}, \tag{2.8}$$

where

$$
\begin{aligned}
S^0_{SYM} \;=\; & \int_x Tr\bigg(\frac{1}{2}(A^\mu iD^{-1}_{0,\mu\nu}A^\nu)+\frac{1}{2}(M_m iS^{-1}_{0,mn}M_n) \\
& +\bar{\lambda}^{\dot{\alpha}i}i\triangle^{-1}_{0,\dot{\alpha}\beta,ij}\lambda^{\beta j}+\bar{\eta}iG_0^{-1}\eta\bigg),
\end{aligned}
\tag{2.9}
$$

and

$$
\begin{aligned}
& S^{int}_{SYM} = \\
& \int_x Tr\Big(-ig\left(\partial_\mu A_\nu A^\mu A^\nu-\partial_\mu A_\nu A^\nu A^\mu\right)+\frac{1}{2}g^2\left(A_\mu A_\nu A^\mu A^\nu-A_\mu A_\nu A^\nu A^\mu\right) \\
& -ig\left(\partial_\mu M_m A^\mu M^m-\partial_\mu M_m M^m A^\mu\right)+g^2\left(A_\mu M_m A^\mu M^m-M_m A_\mu A^\mu M^m\right) \\
& +\frac{1}{2}\left(M_m M_n M^m M^n-M_m M_n M^n M^m\right)-g\left(\lambda^\alpha_i\sigma^\mu_{\alpha\dot{\beta}}A_\mu\bar{\lambda}^{\dot{\beta}i}-\lambda^\alpha_i\sigma^\mu_{\alpha\dot{\beta}}\bar{\lambda}^{\dot{\beta}i}A_\mu\right) \\
& +\frac{1}{2}i\Big(\lambda^\alpha_i\lambda_{\alpha j}(\tilde{\sigma}^{-1}_m)^{ij}M_m-\lambda^\alpha_i(\tilde{\sigma}^{-1}_m)^{ij}M_m\lambda_{\alpha j}-\bar{\lambda}^i_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}j}(\tilde{\sigma}_m)_{ij}M_m \\
& +\bar{\lambda}^i_{\dot{\alpha}}(\tilde{\sigma}_m)_{ij}M_m\bar{\lambda}^{\dot{\alpha}j}\Big)+ig\left(\partial^\mu\bar{\eta}A_\mu\eta-\partial^\mu\bar{\eta}\eta A_\mu\right)\Big).
\end{aligned}
\tag{2.10}
$$

The free inverse propagators in (2.9) are given by,

$$
\begin{aligned}
iD^{-1}_{0,\mu\nu} &= g_{\mu\nu}\partial^2 \\
iS^{-1}_{0,mn} &= \partial^2\delta_{mn} \\
i\triangle^{-1}_{0,\dot{\alpha}\beta,ij} &= i\delta_{ij}\bar{\sigma}^\mu_{\dot{\alpha}\beta}\partial_\mu \\
iG_0^{-1} &= -\partial^2.
\end{aligned}
$$

Below we make a few extra comments about the propagators. The indices l,k,r,s are $SU(N)$ indices taking values from 1 to N.

The gluon free inverse propagator is $iD^{-1}_{0,\mu\nu}\equiv g_{\mu\nu}iD_0^{-1}=g_{\mu\nu}\partial^2$, where $D^{-1}_{0,\mu\nu}D_0^{\nu\kappa}(x,y)=\delta(x-y)\delta^\kappa_\mu$ holds. Inverting gives $D_0(x,y)=\frac{i}{4\pi^2}\frac{1}{(x-y)^2}$, where $\langle(A^\mu(x))^l_k(A^\nu)^s_r(y)\rangle=D_0^{\mu\nu}(x,y)\delta^l_r\delta^s_k=g^{\mu\nu}D_0(x,y)\delta^l_r\delta^s_k$.

The free scalar inverse propagator is $iS^{-1}_{0,mn}\equiv\delta_{mn}iS_0^{-1}=\delta_{mn}\partial^2$, where $S^{-1}_{0,mn}S_0^{np}(x,y)=\delta(x-y)\delta^p_m$ holds. Inverting gives $S_0(x,y)=\frac{i}{4\pi^2}\frac{1}{(x-y)^2}$, where $\langle(M_m(x))^l_k(M_n)^s_r(y)\rangle=S_0^{mn}(x,y)\delta^l_r\delta^s_k=\delta^{mn}S_0(x,y)\delta^l_r\delta^s_k$.

The free ghost inverse propagator is $iG_0^{-1}=-\partial^2$, where $G_0^{-1}G_0(x,y)=\delta(x-y)$ holds. Inverting gives $G_0(x,y)=-\frac{i}{4\pi^2}\frac{1}{(x-y)^2}$, where $\langle(\eta(x))^l_k(\bar{\eta}(y))^s_r\rangle=G_0(x,y)\delta^l_r\delta^s_k$.

Finally the free fermion inverse propagator is $i\triangle^{-1}_{0,\dot{\alpha}\beta,ij} \equiv \delta_{ij} i\triangle^{-1}_{0,\dot{\alpha}\beta} = i\delta_{ij}\bar{\sigma}^{\mu}_{\dot{\alpha}\beta}\partial_{\mu}$, where $\triangle^{-1}_{0,\dot{\alpha}\beta}\triangle^{\beta\dot{\gamma}}_{0}(x-y) = \delta^{\dot{\gamma}}_{\dot{\alpha}}\delta(x-y)$ holds. Inverting gives $\triangle_{0,\alpha\dot{\beta}}(x-y) = \frac{1}{2\pi^2}\sigma^{\mu}_{\alpha\dot{\beta}}\frac{(x-y)_{\mu}}{(x-y)^4}$, where $\langle(\lambda^{\alpha i}(x))^{l}_{k}(\bar{\lambda}^{\dot{\beta}j}(y))^{s}_{r}\rangle = \delta^{ij}\triangle^{\alpha\dot{\beta}}_{0}(x-y)\delta^{l}_{r}\delta^{s}_{k}$.

Notice that we can write all the free propagators in terms of the gluon free propagator:

$$
\begin{aligned}
S_0(x,y) &= D_0(x,y) \\
G_0(x,y) &= -D_0(x,y) \\
\triangle_{0,\alpha\dot{\beta}}(x-y) &= i\sigma^{\mu}_{\alpha\dot{\beta}}\partial^{x}_{\mu}D_0(x,y).
\end{aligned}
\tag{2.11}
$$

This completes our introduction to $\mathcal{N} = 4$ SYM. We now want to compute the 2PI effective action for this theory, and we do so in the next section.

2.3 2PI Effective Action of $\mathcal{N} = 4$ SYM

In this section we will show how to compute the two-loop 2PI effective action of $\mathcal{N} = 4$ SYM.

In Section 1.2.2 we wrote down the 2PI effective action for the case of a single scalar field. It is straightforward to extend this result to the present case, that of multiple fields. Mirroring the previous discussion, we now include, for each field $\mathcal{W}$, sources $J_{\mathcal{W}}(x)$ and $R_{\mathcal{W}}(x,y)$, and Legendre transform the generating functional with respect to these sources. Analogously to (1.18), we obtain the following expression for the $\mathcal{N} = 4$ SYM 2PI effective action,

$$
\begin{aligned}
\Gamma[\tilde{M},\tilde{A},\tilde{\lambda},\tilde{\eta},S,D,\triangle,G] &= S_{SYM}[\tilde{M},\tilde{A},\tilde{\lambda},\tilde{\eta}] \\
&+ \frac{i}{2}Tr ln S^{-1} + \frac{i}{2}Tr S_0^{-1}S + \frac{i}{2}Tr ln D^{-1} + \frac{i}{2}Tr D_0^{-1}D \\
&- iTr ln\triangle^{-1} - iTr\triangle_0^{-1}\triangle - iTr ln G^{-1} - iTr G_0^{-1}G \\
&+ \Gamma_2[\tilde{M},\tilde{A},\tilde{\lambda},\tilde{\eta},S,D,\triangle,G].
\end{aligned}
$$

Here S, D, $\triangle$ and G are the full scalar, gluon, fermion and ghost propagators respectively, and $\tilde{M}$, $\tilde{A}$, $\tilde{\lambda}$, $\tilde{\eta}$ are the respective one-point functions.

From our previous discussion of the scalar field 2PI effective action, we know that the first nine terms above represent the 2PI effective action to one-loop order, while

Γ_2 is the higher-loop contribution. It is obtained by shifting each of the fields in S_{SYM}, given in (2.8) to (2.10), by their respective one-point functions, and using the vertices obtained from this shifted action to build the higher loop 2PI diagrams. More precisely, we first evaluate,

$$\begin{aligned}&S_{SYM}[M+\tilde{M},A+\tilde{A},\lambda+\tilde{\lambda},\eta+\tilde{\eta}] = \\ &S_{SYM}[\tilde{M},\tilde{A},\tilde{\lambda},\tilde{\eta}] + S^0_{SYM}[\tilde{M},\tilde{A},\tilde{\lambda},\tilde{\eta};M,A,\lambda,\eta] + S^{int}_{SYM}[\tilde{M},\tilde{A},\tilde{\lambda},\tilde{\eta};M,A,\lambda,\eta].\end{aligned}$$

Above, all old and new terms quadratic in the fields contribute to S^0_{SYM}, while S^{int}_{SYM} contains all cubic and quartic vertex terms. In fact, upon doing the shift above, the only terms yielding vertices will be the original quartic terms in (2.10). They will give rise to cubic vertices when shifted. After some trivial manipulations we see that

$$\begin{aligned}
&S^{int}_{SYM}[\tilde{M},\tilde{A},\tilde{\lambda},\tilde{\eta};M,A,\lambda,\eta] = \\
&\int_x Tr\Big(-ig\left(\partial_\mu A_\nu A^\mu A^\nu - \partial_\mu A_\nu A^\nu A^\mu\right) + \frac{1}{2}g^2\left(A_\mu A_\nu A^\mu A^\nu - A_\mu A_\nu A^\nu A^\mu\right) \\
&-ig\left(\partial_\mu M_m A^\mu M^m - \partial_\mu M_m M^m A^\mu\right) + g^2\left(A_\mu M_m A^\mu M^m - M_m A_\mu A^\mu M^m\right) \\
&+\frac{1}{2}\left(M_m M_n M^m M^n - M_m M_n M^n M^m\right) - g\left(\lambda^\alpha_i \sigma^\mu_{\alpha\dot{\beta}} A_\mu \bar{\lambda}^{\dot{\beta}i} - \lambda^\alpha_i \sigma^\mu_{\alpha\dot{\beta}} \bar{\lambda}^{\dot{\beta}i} A_\mu\right) \\
&+\frac{1}{2}i\Big(\lambda^\alpha_i \lambda_{\alpha j}(\tilde{\sigma}^{-1}_m)^{ij} M_m - \lambda^\alpha_i (\tilde{\sigma}^{-1}_m)^{ij} M_m \lambda_{\alpha j} - \bar{\lambda}^i_{\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}j}(\tilde{\sigma}_m)_{ij} M_m \\
&\qquad + \bar{\lambda}^i_{\dot{\alpha}}(\tilde{\sigma}_m)_{ij} M_m \bar{\lambda}^{\dot{\alpha}j}\Big) \\
&+\frac{1}{2}g^2\left(4A_\mu A_\nu A^\mu \tilde{A}^\nu - 2A_\nu A_\mu A^\mu \tilde{A}^\nu - 2A_\mu A^\mu A_\nu \tilde{A}^\nu\right) \\
&+\frac{1}{2}g^2\left(4M_m M_n M^m \tilde{M}^n - 2M_n M_m M^m \tilde{M}^n - 2M_m M^m M_n \tilde{M}^n\right) \\
&+g^2\Big(2M_m A_\mu M^m \tilde{A}^\mu + 2A_\mu M_m A^\mu \tilde{M}^m - A_\mu M^m M_m \tilde{A}^\mu - M^m M_m A_\mu \tilde{A}^\mu \\
&\qquad - A_\mu A^\mu M_m \tilde{M}^m - M_m A_\mu A^\mu \tilde{M}^m\Big) + ig\left(\partial^\mu \bar{\eta} A_\mu \eta - \partial^\mu \bar{\eta}\eta A_\mu\right)\Big) \qquad (2.12)
\end{aligned}$$

Notice that the last five lines above contain the cubic vertices with the shift fields (*i.e.* the one-point functions). We include a diagram of all the vertices in Figure 2.1. Then Γ_2 consists of all 2PI connected vacuum diagrams obtained by using the vertices in Figure 2.1.

We want the 2PI effective action to two loop order only and at two loop order the fourteen diagrams in Figure 2.2 contribute.

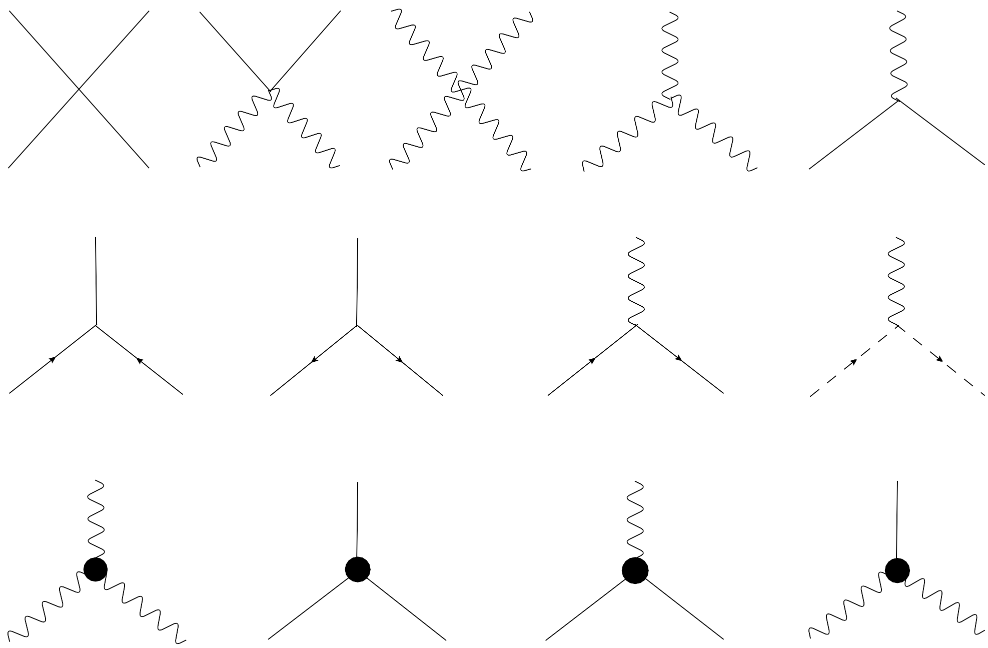

Figure 2.1: Vertices arising from $S^{int}_{SYM}[\tilde{M},\tilde{A},\tilde{\lambda},\tilde{\eta};M,A,\lambda,\eta]$. Here gluons are represented with wavy lines, scalars with straight lines, arrowed lines correspond to fermions and dashed arrowed lines correspond to ghosts. The three-vertices containing the various one-point functions are simply represented with a solid black dot.

At this point we simplify the calculation considerably by working in the so-called *symmetric phase* where we set all the one-point functions to zero. This allows us to remove 6 of the diagrams in Figure 2.2, leaving us with the eight diagrams in Figure 2.3. We will label these eight diagrams according to the propagators they contain. We emphasize that all the propagators in these diagrams are *full*. Thus, starting from the top row in Figure 2.3 and going from left to right, we have the diagrams S^2, DS, D^2, D^3, S^2D, $\triangle\bar{\triangle}D$, $\triangle^2 S$, and $G\bar{G}D$.

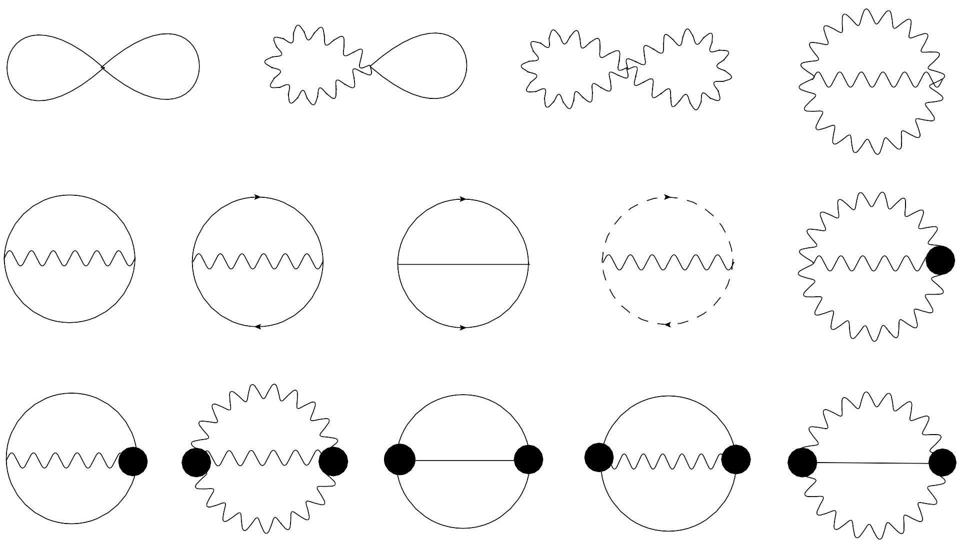

Figure 2.2: 2PI 2-loop diagrams contributing to $\Gamma_2[\tilde{M},\tilde{A},\tilde{\lambda},\tilde{\eta},S,D,\triangle,G]$.

We may then write,

$$\Gamma_2[S,D,\triangle,G]_{SYM} = \Gamma_{\triangle\bar{\triangle}D} + \Gamma_{\triangle^2 S} + \Gamma_{D^3} + \Gamma_{S^2 D} + \Gamma_{D^2} + \Gamma_{S^2} + \Gamma_{DS} + \Gamma_{G\bar{G}D}.$$

We write each of these terms explicitly below.

$$\Gamma_{\triangle\bar{\triangle}D} = \quad -4ig^2(N^3 - N)\sigma^{\mu}_{\alpha\dot{\beta}}\sigma^{\nu}_{\kappa\dot{\rho}} \int_{xy} \triangle^{\alpha\dot{\rho}}(x,y)\triangle^{\kappa\dot{\beta}}(y,x)D_{\mu\nu}(x,y)$$

$$\Gamma_{\triangle^2 S} = \quad -24ig^2(N^3 - N)\varepsilon_{\alpha\beta}\varepsilon_{\dot{\beta}\dot{\alpha}} \int_{xy} \triangle^{\alpha\dot{\alpha}}(x,y)\triangle^{\beta\dot{\beta}}(x,y)S(x,y)$$

$$
\begin{aligned}
\Gamma_{D^3} = \quad & -ig^2(N^3-N)\int_{xy}\Big(\partial^x_\mu\partial^y_\rho D_{\nu\kappa}(x,y)\big(D^{\mu\kappa}(x,y)D^{\nu\rho}(x,y)-D^{\mu\rho}(x,y)D^{\nu\kappa}(x,y)\big)\\
& +\partial^x_\mu D^\rho_\nu(x,y)\big(\partial^y_\rho D^\mu_\kappa(x,y)D^{\nu\kappa}(x,y)-\partial^y_\rho D^\nu_\kappa(x,y)D^{\mu\kappa}(x,y)\big)\\
& +\partial^x_\mu D^\kappa_\nu(x,y)\big(\partial^y_\rho D^\nu_\kappa(x,y)D^{\mu\rho}(x,y)-\partial^y_\rho D^\mu_\kappa(x,y)D^{\nu\rho}(x,y)\big)\Big)\\
\Gamma_{S^2D} = \quad & -6ig^2(N^3-N)\int_{xy} D^{\mu\rho}(x,y)\Big(\partial^x_\mu S(x,y)\partial^y_\rho S(x,y)-\partial^x_\mu\partial^y_\rho S(x,y)S(x,y)\Big)\\
\Gamma_{D^2} = \quad & \frac{1}{2}g^2(N^3-N)\int_x\Big(D^\nu_\mu(x,x)D^\mu_\nu(x,x)-D^\mu_\mu(x,x)D^\nu_\nu(x,x)\Big)\\
\Gamma_{S^2} = \quad & -15g^2(N^3-N)\int_x S^2(x,x)\\
\Gamma_{DS} = \quad & -6g^2(N^3-N)\int_x D^\mu_\mu(x,x)S(x,x)\\
\Gamma_{G\bar{G}D} = \quad & ig^2(N^3-N)\int_{xy}\partial^\mu_x G(y,x)\partial^\nu_y G(x,y)D_{\mu\nu}(x,y).
\end{aligned}
$$

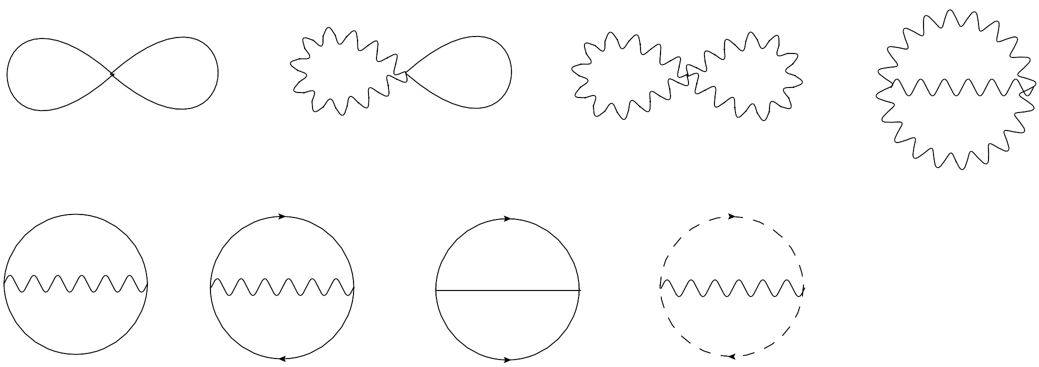

Figure 2.3: 2PI 2-loop diagrams contributing to $\Gamma_2[\tilde{M},\tilde{A},\tilde{\lambda},\tilde{\eta},S,D,\triangle,G]$ in the symmetric phase. Starting at the top row and going from left to right, we label them as S^2, DS, D^2, D^3, S^2D, $\triangle\bar{\triangle}D$, $\triangle^2S$, and $G\bar{G}D$.

As a first check of our results we can evaluate the two-loop 2PI effective action of $\mathcal{N} = 4$ SYM to $\mathcal{O}(g^2)$ by using the *free* propagators in each of the contributing diagrams. Due to the conformal and supersymmetric nature of this theory, we expect the effective action to vanish, and it does. This is a first check that our effective action is indeed the correct one. In Appendix 2.A.1 we demonstrate in more detail how diagrams of this kind are evaluated.

2.3.1 One-loop self-energies

In the previous Chapter we saw that in order to write down the non-equilibrium evolution equations for a theory, we need to know the spectral and statistical components of the various self-energies. The starting point to do this is to evaluate the vacuum self-energy, which begins with finding the corrections to the propagators for each field type, and then truncating external legs. These are obtained by cutting the appropriate propagator line in the two-loop diagrams in Figure 2.3. We sketch these results below and in Appendix 2.A.1 we include a more detailed sample calculation to elaborate our method.

We look at the diagrams contributing to the propagator correction for each field type, and we again label these by the propagators they contain, but now we put the loop propagators in brackets. (For notational simplicity, we leave off all the indices on the diagrams, since it is straightforward to see what they should be from the answers below.) Below, p,q are the scalar field flavor indices, r,s are the fermion field flavor indices and e,f,g,h are the adjoint indices.

We begin with the corrections to the scalar propagator. The contributing diagrams are given in Figure 2.4.

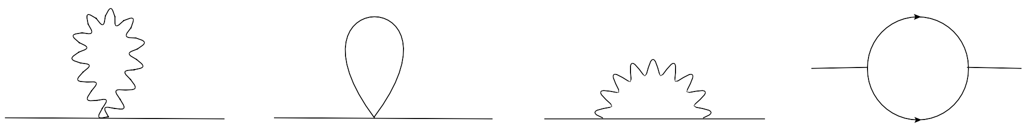

Figure 2.4: Corrections to the scalar propagator.

These diagrams evaluate to,

$$S(D)S = -2ig^2\delta_p^q(N\delta_e^h\delta_g^f - \delta_g^h\delta_e^f)\int_z D_\mu^\mu(z,z)S(x,z)S(y,z)$$

$$S(S)S = -10i\delta_p^q(N\delta_e^h\delta_g^f - \delta_g^h\delta_e^f)\int_z S(x,z)S(y,z)S(z,z)$$

$$S(DS)S = 2g^2\delta_p^q(N\delta_e^h\delta_g^f - \delta_g^h\delta_e^f)\int_{zw} D^{\mu\nu}(z,w)\times$$

$$\times\Big(\partial^z_\mu S(x,z)(\partial^w_\nu S(z,w)S(w,y) - S(z,w)\partial^w_\nu S(w,y)) + S(x,z)(\partial^z_\mu S(z,w)\partial^w_\nu S(w,y) - \partial^z_\mu\partial^w_\nu S(z,w)S(w,y))\Big)$$

$$S(\triangle^2)S = 2\delta^q_p(N\delta^h_e\delta^f_g - \delta^h_g\delta^f_e)\varepsilon_{\alpha\beta}\varepsilon_{\dot\beta\dot\alpha}\int_{zw}(S(x,z)S(w,y) + S(x,w)S(z,y))\times \times\Big(\triangle^{\alpha\dot\alpha}(z,w)\triangle^{\beta\dot\beta}(z,w) + \triangle^{\alpha\dot\alpha}(w,z)\triangle^{\beta\dot\beta}(w,z)\Big)$$

Similarly, the corrections to the gluon propagator are given by the diagrams in Figure 2.5.

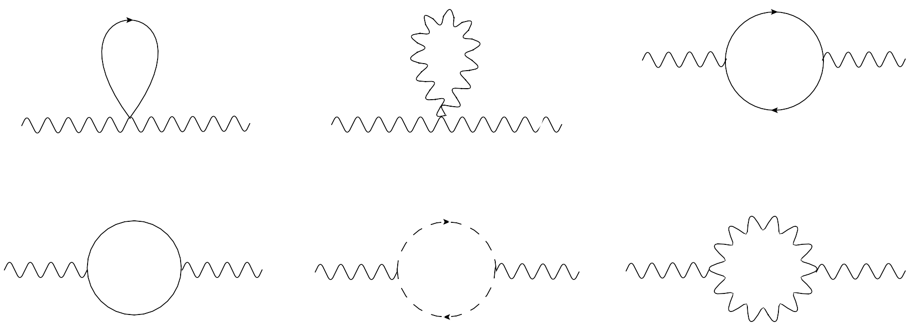

Figure 2.5: Corrections to the gluon propagator.

These diagrams evaluate to,

$$D(S)D = -6ig^2(N\delta^h_e\delta^f_g - \delta^h_g\delta^f_e)\int_z S(z,z)\big(D_{\tau\mu}(x,z)D^\mu_\gamma(y,z) + D^\mu_\tau(x,z)D_{\gamma\mu}(y,z)\big)$$

$$D(\triangle\bar\triangle)D = 4g^2(N\delta^h_e\delta^f_g - \delta^h_g\delta^f_e)\sigma^\mu_{\alpha\dot\beta}s^\nu_{\kappa\dot\rho}\int_{zw}\triangle^{\alpha\dot\rho}(z,w)\triangle^{\kappa\dot\beta}(w,z) \times\big(D_{\tau\mu}(x,z)D_{\gamma\nu}(y,w) + D_{\tau\nu}(x,w)D_{\gamma\mu}(y,z)\big)$$

$$D(S^2)D = 6g^2(N\delta^h_e\delta^f_g - \delta^h_g\delta^f_e)\int_{zw}\big(D^\rho_\tau(x,w)D^\mu_\gamma(y,z) + D^\mu_\tau(x,z)D^\rho_\gamma(y,w)\big) \times\Big(\partial^z_\mu S(z,w)\partial^w_\rho S(z,w) - \partial^z_\mu\partial^w_\rho S(z,w)S(z,w)\Big)$$

$$D(G\bar G)D = -g^2(N\delta^h_e\delta^f_g - \delta^h_g\delta^f_e)\int_{zw}\partial^\mu_z G(w,z)\partial^\nu_w G(z,w)\big(D_{\tau\nu}(x,w)D_{\gamma\mu}(y,z) + D_{\tau\mu}(x,z)D_{\gamma\nu}(y,w)\big)$$

$$D(D)D = ig^2(N\delta^h_e\delta^f_g - \delta^h_g\delta^f_e)\int_z \Big[D^\nu_\mu(z,z)\big(D_{\tau\nu}(x,z)D^\mu_\gamma(y,z)+D^\mu_\tau(x,z)D_{\gamma\nu}(y,z)\big) - D^\nu_\nu(z,z)\big(D_{\tau\mu}(x,z)D^\mu_\gamma(y,z)+D^\mu_\tau(x,z)D_{\gamma\mu}(y,z)\big)\Big]$$

$$\begin{aligned} D(D^2)D = g^2(N\delta^h_e\delta^f_g - \delta^h_g\delta^f_e)\int_{zw} \Big[&4g_{\tau\gamma}\partial^z_\mu\partial^\mu_z D(z,w)D(x,z)D(z,w)D(y,w) \\ &+ 8\partial^z_\gamma\partial^z_\tau D(z,w)D(x,z)D(z,w)D(y,w) \\ &-4\partial^z_\gamma D(z,w)D(x,z)D(z,w)\partial^w_\tau D(y,w) \\ &- 12\partial^z_\tau D(z,w)D(x,z)\partial^z_\gamma D(z,w)D(y,w) \\ &+ 4g_{\tau\gamma}\partial^z_\mu D(z,w)D(x,z)D(z,w)\partial^\mu_w D(y,w) \\ &+ 4\partial^z_\tau D(z,w)\partial^z_\gamma D(x,z)D(z,w)D(y,w) \\ &- 4g_{\tau\gamma}D(z,w)\partial^z_\mu D(x,z)\partial^\mu_z D(z,w)D(y,w) \\ &- 4g_{\tau\gamma}D(z,w)\partial^z_\mu D(x,z)D(z,w)\partial^\mu_w D(y,w) \\ &+ 4D(z,w)\partial^z_\gamma D(x,z)D(z,w)\partial^w_\tau D(y,w)\Big]. \end{aligned}$$

In the same way, the corrections to the fermion propagator are given by the diagrams in Figure 2.6.

Figure 2.6: Corrections to the fermion propagator.

These diagrams evaluate to,

$$\begin{aligned} \triangle(D\triangle)\bar{\triangle} &= -g^2\sigma^\mu_{\alpha\dot{\beta}}\sigma^\nu_{\kappa\dot{\rho}}\delta^{rs}(N\delta^h_e\delta^f_g - \delta^h_g\delta^f_e)\times \\ &\quad \times\int_{zw} D_{\mu\nu}(z,w)\Big[\triangle^{\zeta\dot{\beta}}(x,z)\triangle^{\kappa\dot{\varphi}}(w,y)\triangle^{\alpha\dot{\rho}}(z,w) \\ &\quad\quad +\triangle^{\zeta\dot{\rho}}(x,w)\triangle^{\alpha\dot{\varphi}}(z,y)\triangle^{\kappa\dot{\beta}}(w,z)\Big] \\ \triangle(S\bar{\triangle})\bar{\triangle} &= 6\delta^{rs}(N\delta^h_e\delta^f_g - \delta^h_g\delta^f_e)\int_{zw}\Big[S(z,w)\Big(\triangle^{\zeta\dot{\beta}}(x,w)\triangle^{\alpha\dot{\varphi}}(z,y)\triangle_{\alpha\dot{\beta}}(z,w) \\ &\quad\quad +\triangle^{\zeta\dot{\beta}}(x,z)\triangle^{\alpha\dot{\varphi}}(w,y)\triangle_{\alpha\dot{\beta}}(w,z)\Big)\Big]. \end{aligned}$$

Finally, the ghost propagator correction is given by the single diagram in Figure 2.7. This diagram evaluates to,

$$G(DG)\bar{G} \;=\; g^2(N\delta^h_e\delta^f_g-\delta^h_g\delta^f_e)\int_{zw}\left[D_{\mu\nu}(z,w)\left(\partial^\mu_z G(x,z)G(w,y)\partial^\nu_w G(z,w)\right.\right.$$
$$\left.\left.+\partial^\nu_w G(x,w)G(z,y)\partial^\mu_z G(w,z)\right)\right]. \qquad (2.13)$$

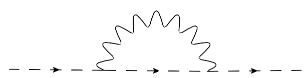

Figure 2.7: Correction to the ghost propagator.

As is well-known, the vacuum self-energy for each field type is given by the truncated forms of these corrections (i.e. without the external propagator lines). We thus truncate each of the expressions above by pre- and post-multiplying them by the relevant propagator inverse. Below, we present the contributions to the various self-energies from each of the diagrams, where we use the notation $\Sigma,\Pi,\Sigma^{(f)},\Sigma^{(gh)}$ for the scalar, gluon, fermion and ghost self-energies respectively.

Scalar self-energy contributions are given by the sum of the truncated diagrams below,

$$\begin{aligned}\Sigma^{pq}_{S(D)S}(w,u)^{fh}_{ge} &\equiv \mathrm{trunc}(S(D)S)\\ &= -2ig^2\delta^{pq}\left(N\delta^f_g\delta^h_e-\delta^f_e\delta^h_g\right)D^\mu_\mu(w,w)\delta(w-u)\end{aligned}$$

$$\begin{aligned}\Sigma^{pq}_{S(S)S}(w,u)^{fh}_{ge} &\equiv \mathrm{trunc}(S(S)S)\\ &= -10ig^2\delta^{pq}\left(N\delta^f_g\delta^h_e-\delta^f_e\delta^h_g\right)\delta(w-u)S(w,w)\end{aligned}$$

$$\begin{aligned}\Sigma^{pq}_{S(DS)S}(w,u)^{fh}_{ge} &\equiv \mathrm{trunc}(S(DS)S)\\ &= -2g^2\delta^{pq}\left(N\delta^f_g\delta^h_e-\delta^f_e\delta^h_g\right)\left[4\partial^u_\nu\partial^w_\mu S(w,u)D^{\mu\nu}(w,u)\right.\\ &\qquad +2\partial^w_\mu S(w,u)\partial^u_\nu D^{\mu\nu}(w,u)\end{aligned}$$

$$
\begin{aligned}
&+2\partial^u_\nu S(w,u)\partial^w_\mu D^\mu(w,u) \\
&+S(w,u)\partial^u_\nu\partial^w_\mu D^{\mu\nu}(w,u)\Big]
\end{aligned}
$$

$$
\begin{aligned}
\Sigma^{pq}_{S(\triangle^2)S}(w,u)^{fh}_{ge} &\equiv \operatorname{trunc}\left(S(\triangle^2)S\right) \\
&= 4\delta^{pq}\varepsilon_{\alpha\beta}\varepsilon_{\dot\beta\dot\alpha}\left(N\delta^f_g\delta^h_e-\delta^f_e\delta^h_g\right)\times \\
&\quad\times\left(\triangle^{\alpha\dot\alpha}(w,u)\triangle^{\beta\dot\beta}(w,u)+\triangle^{\alpha\dot\alpha}(u,w)\triangle^{\beta\dot\beta}(u,w)\right) \qquad (2.14)
\end{aligned}
$$

Gluon self-energy contributions are given by the sum of the truncated diagrams below,

$$
\begin{aligned}
\Pi^{\rho\xi}_{D(S)D}(w,u)^{fh}_{ge} &\equiv \operatorname{trunc}(D(S)D) \\
&= -12ig^2\left(N\delta^f_g\delta^h_e-\delta^f_e\delta^h_g\right)g^{\rho\xi}S(w,w)\delta(w-u)
\end{aligned}
$$

$$
\begin{aligned}
\Pi^{\rho\xi}_{D(\triangle\bar\triangle)D}(w,u)^{fh}_{ge} &\equiv \operatorname{trunc}\left(D(\triangle\bar\triangle)D\right) \\
&= 8g^2\left(N\delta^f_g\delta^h_e-\delta^f_e\delta^h_g\right)\sigma^\rho_{\alpha\dot\beta}\sigma^\xi_{\kappa\dot\rho}\triangle^{\alpha\dot\rho}(w,u)\triangle^{\kappa\dot\beta}(u,w)
\end{aligned}
$$

$$
\begin{aligned}
\Pi^{\rho\xi}_{D(S^2)D}(w,u)^{fh}_{ge} &\equiv \operatorname{trunc}\left(D(S^2)D\right) \\
&= 6g^2\left(N\delta^f_g\delta^h_e-\delta^f_e\delta^h_g\right)\times \\
&\quad\times\left(\partial^\xi_u S(u,w)\partial^\rho_w S(u,w)-\partial^\xi_u\partial^\rho_w S(u,w)S(u,w)\right. \\
&\quad\left.+\partial^\rho_w S(w,u)\partial^\xi_u S(w,u)-\partial^\rho_w\partial^\xi_u S(w,u)S(w,u)\right)
\end{aligned}
$$

$$
\begin{aligned}
\Pi^{\rho\xi}_{D(G\bar G)D}(w,u)^{fh}_{ge} &\equiv \operatorname{trunc}\left(D(G\bar G)D\right) \\
&= -2g^2\left(N\delta^f_g\delta^h_e-\delta^f_e\delta^h_g\right)\partial^\xi_u G(w,u)\partial^\rho_w G(u,w)
\end{aligned}
$$

$$
\begin{aligned}
\Pi^{\rho\xi}_{D(D)D}(w,u)^{fh}_{ge} &\equiv \operatorname{trunc}(D(D)D) \\
&= 2ig^2\left(N\delta^f_g\delta^h_e-\delta^f_e\delta^h_g\right)\times \\
&\quad\times\left(D^{\rho\xi}(w,w)-g^{\rho\xi}D^\nu_\nu(w,w)\right)\delta(w-u)
\end{aligned}
$$

$$
\begin{aligned}
\Pi^{\rho\xi}_{D(D^2)D}(w,u)^{fh}_{ge} &\equiv \operatorname{trunc}\left(D(D^2)D\right) \\
&= 2g^2\left(N\delta^f_g\delta^h_e-\delta^f_e\delta^h_g\right)\times \\
&\quad\times\Big[-8\partial^w_\mu\partial^u_\tau D^{\rho\xi}(w,u)D^{\mu\tau}(w,u)+4\partial^u_\tau\partial^w_\mu D^{\rho\tau}(w,u)D^{\mu\xi}(w,u)
\end{aligned}
$$

$$
\begin{aligned}
&+4\partial^w_\mu \partial^u_\tau D^{\mu\xi}(w,u) D^{\rho\tau}(w,u) - 2\partial^u_\tau \partial^w_\mu D^{\mu\tau}(w,u) D^{\rho\xi}(w,u) \\
&+4\partial^\mu_w \partial^\xi_u D^{\rho\kappa}(w,u) D_{\mu\kappa}(w,u) + 4\partial^\rho_w \partial^\tau_u D^{\nu\xi}(w,u) D_{\nu\tau}(w,u) \\
&-2\partial^\tau_u \partial^\rho_w D_{\nu\tau}(w,u) D^{\nu\xi}(w,u) - 2\partial^\xi_u \partial^\rho_w D_{\nu\kappa}(w,u) D^{\nu\kappa}(w,u) \\
&-2\partial^\mu_w \partial^\xi_u D_{\mu\kappa}(w,u) D^{\rho\kappa}(w,u) \\
&+8\partial^w_\mu D^{\rho\tau}(w,u) \partial^u_\tau D^{\mu\xi}(w,u) - 4\partial^\rho_w D^\tau_\nu(w,u) \partial^u_\tau D^{\nu\xi}(w,u) \\
&-4\partial^\mu_w D^{\rho\kappa}(w,u) \partial^\xi_u D_{\mu\kappa}(w,u) + 2\partial^\rho_w D^{\nu\kappa}(w,u) \partial^\xi_u D_{\nu\kappa}(w,u) \\
&-4\partial^w_\mu D^{\rho\xi}(w,u) \partial^u_\tau D^{\mu\tau}(w,u) + 2\partial^\rho_w D^{\nu\xi}(w,u) \partial^\tau_u D_{\nu\tau}(w,u) \\
&-4\partial^w_\mu D^{\mu\tau}(w,u) \partial^u_\tau D^{\rho\xi}(w,u) + 2\partial^\mu_w D_{\mu\kappa}(w,u) \partial^\xi_u D^{\rho\kappa}(w,u) \\
&+2\partial^w_\mu D^{\mu\xi}(w,u) \partial^u_\tau D^{\rho\tau}(w,u) \Big]
\end{aligned}
\tag{2.15}
$$

Fermion self-energy contributions are given by the sum of the truncated diagrams below,

$$
\begin{aligned}
\Sigma^{(f)}_{\triangle(D\triangle)\bar\triangle,rs,\dot\lambda\tau}(w,u)^{fh}_{ge} &\equiv \operatorname{trunc}\left(\triangle(D\triangle)\bar\triangle\right) \\
&= -g^2\delta_{rs}\left(N\delta^f_g\delta^h_e - \delta^f_e\delta^h_g\right)\times \\
&\quad \times\left(\sigma^\mu_{\alpha\dot\lambda}\sigma^\nu_{\tau\dot\rho}(D_{\mu\nu}(w,u) + D_{\nu\mu}(u,w))\triangle^{\alpha\dot\rho}(w,u)\right) \\
\Sigma^{(f)}_{\triangle(S\bar\triangle)\bar\triangle,rs,\dot\lambda\tau}(w,u)^{fh}_{ge} &\equiv \operatorname{trunc}\left(\triangle(S\bar\triangle)\bar\triangle\right) \\
&= 6g^2\delta_{rs}\left(N\delta^f_g\delta^h_e - \delta^f_e\delta^h_g\right)\times \\
&\quad \times(S(u,w) + S(w,u))\triangle_{\tau\dot\lambda}(u,w)
\end{aligned}
\tag{2.16}
$$

Ghost self-energy contributions are given by the truncated diagram below,

$$
\begin{aligned}
\Sigma^{(gh)}_{G(DG)G}(w,u)^{fh}_{ge} &\equiv \operatorname{trunc}(G(DG)G) \\
&= -g^2\left(N\delta^f_g\delta^h_e - \delta^f_e\delta^h_g\right) \\
&\quad \times\left[\left(\partial^\mu_w D_{\mu\nu}(w,u) + \partial^\mu_w D_{\nu\mu}(u,w)\right)\partial^\nu_u G(w,u)\right. \\
&\quad \left. + \left(D_{\mu\nu}(w,u) + D_{\nu\mu}(u,w)\right)\partial^\mu_w\partial^\nu_u G(w,u)\right].
\end{aligned}
\tag{2.17}
$$

Now that we have evaluated the vacuum self-energies, we are ready to compute the non-equilibrium evolution equations for each of our particle types. We do this in the following Section.

2.4 Evolution equations of $\mathcal{N} = 4$ SYM

In this Section we give the main result of this Part of the thesis: the evolution equations of the various fields in $\mathcal{N} = 4$ SYM. In Section 1.3.3, in equations (1.72) and (1.73) respectively, we wrote down the evolution equations for a generic scalar and fermion field in terms of the spectral and statistical components of the propagator and self-energy. We may extend these results straightforwardly to the $\mathcal{N} = 4$ SYM fields. We begin by computing the various self-energy components and in the next Section we write down the evolution equations for our fields.

2.4.1 Spectral and statistical self-energies

Here we calculate the $\Sigma^{(0)}$ and the spectral and statistical components of the self-energies for the fields in $\mathcal{N} = 4$ SYM. The first of these is given by the local part of the vacuum self-energy calculated in Section 2.3.1. To calculate Σ_F and Σ_ρ we use the non-local parts of the vacuum self-energies and the relations (1.59) and (1.61). We begin with the scalar self-energy.

Scalar

The vacuum scalar self-energy is given by the various diagrams evaluated in (2.14). The local part of the scalar self-energy is given by the diagrams $S(D)S$ and $S(S)S$, and thus we can write,

$$-i\delta(w,u)\Sigma^{(0)pq}(w;S)^{fh}_{ge} \equiv \Sigma^{pq}_{S(D)S}(w,u)^{fh}_{ge} + \Sigma^{pq}_{S(S)S}(w,u)^{fh}_{ge}. \tag{2.18}$$

The non-local vacuum self-energy is made of the diagrams $S(DS)S$ and $S(\triangle^2)S$. Thus,

$$\Sigma^{(\text{vac})pq}(w,u)^{fh}_{ge} = \Sigma^{pq}_{S(DS)S}(w,u)^{fh}_{ge} + \Sigma^{pq}_{S(\triangle^2)S}(w,u)^{fh}_{ge}. \tag{2.19}$$

Now, from (1.61) we may write the $\{+,-\}$ components of the self-energy, namely,

$$\begin{aligned}
\Sigma^{+-pq}(w,u)^{fh}_{ge} &= -\Sigma^{(\text{vac})pq}(w,u)^{fh}_{ge}\Big|_{\text{Prop}^{+-}(w,u)} \\
&= -\Sigma^{pq}_{S(DS)S}(w,u)^{fh}_{ge}\Big|_{\text{Prop}^{+-}(w,u)} - \Sigma^{pq}_{S(\triangle^2)S}(w,u)^{fh}_{ge}\Big|_{\text{Prop}^{+-}(w,u)}
\end{aligned}$$

$$\begin{aligned}\Sigma^{-+pq}(w,u)^{fh}_{ge} &= -\Sigma^{(\mathrm{vac})pq}(w,u)^{fh}_{ge}\Big|_{\mathrm{Prop}^{-+}(w,u)} \\ &= -\Sigma^{pq}_{S(DS)S}(w,u)^{fh}_{ge}\Big|_{\mathrm{Prop}^{-+}(w,u)} - \Sigma^{pq}_{S(\triangle^2)S}(w,u)^{fh}_{ge}\Big|_{\mathrm{Prop}^{-+}(w,u)}.\end{aligned}$$

We now express these quantities in terms of ρ and F using the relations (1.43). We show this in a bit more detail for the Σ^{+-} expression above. The Σ^{-+} follows analogously. Hence,

$$\begin{aligned}&-\Sigma^{pq}_{S(DS)S}(w,u)^{fh}_{ge}\Big|_{\mathrm{Prop}^{+-}(w,u)} \\ &= 2g^2\delta^{pq}\left(N\delta^f_g\delta^h_e - \delta^f_e\delta^h_g\right) \\ &\quad\times\Big[4\partial^u_\nu\partial^w_\mu S^{+-}(w,u)D^{\mu\nu+-}(w,u) + 2\partial^w_\mu S^{+-}(w,u)\partial^u_\nu D^{\mu\nu+-}(w,u) \\ &\quad+2\partial^u_\nu S^{+-}(w,u)\partial^w_\mu D^{\mu+-}(w,u) + S^{+-}(w,u)\partial^u_\nu\partial^w_\mu D^{\mu\nu+-}(w,u)\Big] \\ &= 2g^2\delta^{pq}\left(N\delta^f_g\delta^h_e - \delta^f_e\delta^h_g\right) \\ &\quad\times\Bigg[4\Big(\partial^u_\nu\partial^w_\mu F(w,u)F^{\mu\nu}_D(w,u) + \frac{i}{2}\partial^u_\nu\partial^w_\mu F(w,u)\rho^{\mu\nu}_D(w,u) \\ &\qquad\quad+\frac{i}{2}\partial^u_\nu\partial^w_\mu\rho(w,u)F^{\mu\nu}_D(w,u) - \frac{1}{4}\partial^u_\nu\partial^w_\mu\rho(w,u)\rho^{\mu\nu}_D(w,u)\Big) \\ &\qquad+2\Big(\partial^w_\mu F(w,u)\partial^u_\nu F^{\mu\nu}_D(w,u) + \frac{i}{2}\partial^w_\mu F(w,u)\partial^u_\nu\rho^{\mu\nu}_D(w,u) \\ &\qquad\quad+\frac{i}{2}\partial^w_\mu\rho(w,u)\partial^u_\nu F^{\mu\nu}_D(w,u) - \frac{1}{4}\partial^w_\mu\rho(w,u)\partial^u_\nu\rho^{\mu\nu}_D(w,u)\Big) \\ &\qquad+2\Big(\partial^u_\nu F(w,u)\partial^w_\mu F^{\mu\nu}_D(w,u) + \frac{i}{2}\partial^u_\nu F(w,u)\partial^w_\mu\rho^{\mu\nu}_D(w,u) \\ &\qquad+\frac{i}{2}\partial^u_\nu\rho(w,u)\partial^w_\mu F^{\mu\nu}_D(w,u) - \frac{1}{4}\partial^u_\nu\rho(w,u)\partial^w_\mu\rho^{\mu\nu}_D(w,u)\Big) \\ &\qquad+F(w,u)\partial^u_\nu\partial^w_\mu F^{\mu\nu}_D(w,u) + \frac{i}{2}F(w,u)\partial^u_\nu\partial^w_\mu\rho^{\mu\nu}_D(w,u) \\ &\qquad+\frac{i}{2}\rho(w,u)\partial^u_\nu\partial^w_\mu F^{\mu\nu}_D - \frac{1}{4}\rho(w,u)\partial^u_\nu\partial^w_\mu\rho^{\mu\nu}_D\Bigg]\end{aligned}$$

$$\begin{aligned}&-\Sigma^{pq}_{S(\triangle^2)S}(w,u)^{fh}_{ge}\Big|_{\mathrm{Prop}^{+-}(w,u)} \\ &= -4\delta^{pq}\varepsilon_{\alpha\beta}\varepsilon_{\dot\beta\dot\alpha}\left(N\delta^f_g\delta^h_e - \delta^f_e\delta^h_g\right)\end{aligned}$$

$$\times\left(\triangle^{\alpha\dot{\alpha}+-}(w,u)\triangle^{\beta\dot{\beta}+-}(w,u)+\triangle^{\alpha\dot{\alpha}-+}(u,w)\triangle^{\beta\dot{\beta}-+}(u,w)\right)$$

$$\begin{aligned}
&= -4\delta^{pq}\varepsilon_{\alpha\beta}\varepsilon_{\dot{\beta}\dot{\alpha}}\left(N\delta^f_g\delta^h_e-\delta^f_e\delta^h_g\right)\\
&\times\Big[F^{(f)\alpha\dot{\alpha}}(w,u)F^{(f)\beta\dot{\beta}}(w,u)+\frac{i}{2}F^{(f)\alpha\dot{\alpha}}(w,u)\rho^{(f)\beta\dot{\beta}}(w,u)\\
&+\frac{i}{2}\rho^{(f)\alpha\dot{\alpha}}(w,u)F^{(f)\beta\dot{\beta}}(w,u)-\frac{1}{4}\rho^{(f)\alpha\dot{\alpha}}(w,u)\rho^{(f)\beta\dot{\beta}}(w,u)\\
&+F^{(f)\alpha\dot{\alpha}}(u,w)F^{(f)\beta\dot{\beta}}(u,w)-\frac{i}{2}F^{(f)\alpha\dot{\alpha}}(u,w)\rho^{(f)\beta\dot{\beta}}(u,w)\\
&-\frac{i}{2}\rho^{(f)\alpha\dot{\alpha}}(u,w)F^{(f)\beta\dot{\beta}}(u,w)-\frac{1}{4}\rho^{(f)\alpha\dot{\alpha}}(u,w)\rho^{(f)\beta\dot{\beta}}(u,w)\Big]
\end{aligned}$$

We may now write the expressions for Σ_F and Σ_ρ, using the above expressions and (1.59), namely,

$$\begin{aligned}
\Sigma_F^{pq}(w,u)^{fh}_{ge} &= -\frac{1}{2}\left[\Sigma^{+-pq}(w,u)+\Sigma^{-+pq}(w,u)\right]^{fh}_{ge}\\
\Sigma_\rho^{pq}(w,u)^{fh}_{ge} &= i\left[\Sigma^{+-pq}(w,u)-\Sigma^{-+pq}(w,u)\right]^{fh}_{ge}.
\end{aligned}$$

This yields,

$$\begin{aligned}
\Sigma_F^{pq}(w,u)^{fh}_{ge} &= -g^2\delta^{pq}\left(N\delta^f_g\delta^h_e-\delta^f_e\delta^h_g\right)\\
&\times \Big[8\partial^u_\nu\partial^w_\mu F(w,u)F^{\mu\nu}_D-2\partial^u_\nu\partial^w_\mu\rho(w,u)\rho^{\mu\nu}_D\\
&+ 4\partial^w_\mu F(w,u)\partial^u_\nu F^{\mu\nu}_D(w,u)-\partial^w_\mu\rho(w,u)\partial^u_\nu\rho^{\mu\nu}_D(w,u)\\
&+ 4\partial^u_\nu F(w,u)\partial^w_\mu F^{\mu\nu}_D(w,u)-\partial^u_\nu\rho(w,u)\partial^w_\mu\rho^{\mu\nu}_D(w,u)\\
&+ 2F(w,u)\partial^u_\nu\partial^w_\mu F^{\mu\nu}_D(w,u)-\frac{1}{2}\rho(w,u)\partial^u_\nu\partial^w_\mu\rho^{\mu\nu}_D(w,u)\\
&- \varepsilon_{\alpha\beta}\varepsilon_{\dot{\beta}\dot{\alpha}}\Big(4F^{(f)\alpha\dot{\alpha}}(w,u)F^{(f)\beta\dot{\beta}}(w,u)-\rho^{(f)\alpha\dot{\alpha}}(w,u)\rho^{(f)\beta\dot{\beta}}(w,u)\\
&+ 4F^{(f)\alpha\dot{\alpha}}(u,w)F^{(f)\beta\dot{\beta}}(u,w)-\rho^{(f)\alpha\dot{\alpha}}(u,w)\rho^{(f)\beta\dot{\beta}}(u,w)\Big)\Big],
\end{aligned}$$

and

$$\begin{aligned}
\Sigma_\rho^{pq}(w,u)^{fh}_{ge} &= -g^2\delta^{pq}\left(N\delta^f_g\delta^h_e-\delta^f_e\delta^h_g\right)\\
&\times \Big[8\left(\partial^u_\nu\partial^w_\mu F(w,u)\rho^{\mu\nu}_D-2\partial^u_\nu\partial^w_\mu\rho(w,u)F^{\mu\nu}_D\right)\\
&+ 4\left(\partial^w_\mu F(w,u)\partial^u_\nu\rho^{\mu\nu}_D(w,u)-\partial^w_\mu\rho(w,u)\partial^u_\nu F^{\mu\nu}_D(w,u)\right)
\end{aligned}$$

$$
\begin{aligned}
+ & \quad \left(4\partial_\nu^u F(w,u)\partial_\mu^w \rho_D^{\mu\nu}(w,u) - \partial_\nu^u \rho(w,u)\partial_\mu^w F_D^{\mu\nu}(w,u)\right) \\
+ & \quad 2F(w,u)\partial_\nu^u \partial_\mu^w \rho_D^{\mu\nu}(w,u) + 2\rho(w,u)\partial_\nu^u \partial_\mu^w F_D^{\mu\nu}(w,u) \\
- & \quad 4\varepsilon_{\alpha\beta}\varepsilon_{\dot\beta\dot\alpha}\left(F^{(f)\alpha\dot\alpha}(w,u)\rho^{(f)\beta\dot\beta}(w,u) + \rho^{(f)\alpha\dot\alpha}(w,u)F^{(f)\beta\dot\beta}(w,u)\right. \\
- & \quad \left.\left. F^{(f)\alpha\dot\alpha}(u,w)\rho^{(f)\beta\dot\beta}(u,w) - \rho^{(f)\alpha\dot\alpha}(u,w)F^{(f)\beta\dot\beta}(u,w)\right)\right].
\end{aligned}
$$

Gluon

The vacuum gluon self-energy is given by the various diagrams evaluated in (2.15). The local part of the gluon vacuum self-energy is given by the diagrams $D(S)D$ and $D(D)D$, and thus we can write,

$$
-i\delta(w,u)\Pi^{(0)\kappa}_{\ \nu}(w;D)^{fh}_{ge} = \Pi^{\kappa}_{\nu,D(S)D}(w,u)^{fh}_{ge} + \Pi^{\kappa}_{\nu,D(D)D}(w,u)^{fh}_{ge}. \tag{2.20}
$$

There are 4 non-local diagrams which contribute in the vacuum case, namely $D(\triangle\bar\triangle)D$, $D(S^2)D$, $D(G\bar G)D$ and $D(D^2)D$. Thus,

$$
\begin{aligned}
\Pi^{(\mathrm{vac})\rho\xi}(w,u)^{fh}_{ge} \quad = \quad & \Pi^{\rho\xi}_{D(\triangle\bar\triangle)D}(w,u)^{fh}_{ge} + \Pi^{\rho\xi}_{D(S^2)D}(w,u)^{fh}_{ge} + \Pi^{\rho\xi}_{D(G\bar G)D}(w,u)^{fh}_{ge} \\
& + \Pi^{\rho\xi}_{D(D^2)D}(w,u)^{fh}_{ge}.
\end{aligned}
$$

Now, as for the scalar case, we may use (1.61) to write,

$$
\begin{aligned}
& \Pi^{\rho\xi+-}(w,u)^{fh}_{ge} \\
& = -\Pi^{(\mathrm{vac})\rho\xi}(w,u)^{fh}_{ge}\,|_{Prop^{+-}(w,u)} \\
& = -\Pi^{\rho\xi}_{D(\triangle\bar\triangle)D}(w,u)^{fh}_{ge}\,|_{Prop^{+-}(w,u)} - \Pi^{\rho\xi}_{D(S^2)D}(w,u)^{fh}_{ge}\,|_{Prop^{+-}(w,u)} \\
& \quad -\Pi^{\rho\xi}_{D(G\bar G)D}(w,u)^{fh}_{ge}\,|_{Prop^{+-}(w,u)} - \Pi^{\rho\xi}_{D(D^2)D}(w,u)^{fh}_{ge}\,|_{Prop^{+-}(w,u)}\,,
\end{aligned} \tag{2.21}
$$

$$
\begin{aligned}
& \Pi^{\rho\xi-+}(w,u)^{fh}_{ge} \\
& = -\Pi^{(\mathrm{vac})\rho\xi}(w,u)^{fh}_{ge}\,|_{Prop^{-+}(w,u)} \\
& = -\Pi^{\rho\xi}_{D(\triangle\bar\triangle)D}(w,u)^{fh}_{ge}\,|_{Prop^{-+}(w,u)} - \Pi^{\rho\xi}_{D(S^2)D}(w,u)^{fh}_{ge}\,|_{Prop^{-+}(w,u)} \\
& \quad -\Pi^{\rho\xi}_{D(G\bar G)D}(w,u)^{fh}_{ge}\,|_{Prop^{-+}(w,u)} - \Pi^{\rho\xi}_{D(D^2)D}(w,u)^{fh}_{ge}\,|_{Prop^{-+}(w,u)}\,.
\end{aligned} \tag{2.22}
$$

Again we wish to write everything in terms of F and ρ and we show the substitution for the $\Pi^{\rho\xi+-}$ case below,

$$
\begin{aligned}
&-\Pi^{\rho\xi}_{D(\triangle\bar{\triangle})D}(w,u)^{fh}_{ge}\,|_{Prop^{+-}(w,u)}\\
&= -8g^2\left(N\delta^f_g\delta^h_e - \delta^f_e\delta^h_g\right)\sigma^{\rho}_{\alpha\dot{\beta}}\sigma^{\xi}_{\kappa\dot{\rho}}\triangle^{\alpha\dot{\rho}+-}(w,u)\triangle^{\kappa\dot{\beta}-+}(u,w)\\
&= -8g^2\left(N\delta^f_g\delta^h_e - \delta^f_e\delta^h_g\right)\sigma^{\rho}_{\alpha\dot{\beta}}\sigma^{\xi}_{\kappa\dot{\rho}}\times\\
&\quad\times\Big(F^{(f)\alpha\dot{\rho}}(w,u)F^{(f)\kappa\dot{\beta}}(u,w) - \frac{i}{2}F^{(f)\alpha\dot{\rho}}(w,u)\rho^{(f)\kappa\dot{\beta}}(u,w)\\
&\qquad + \frac{i}{2}\rho^{(f)\alpha\dot{\rho}}(w,u)F^{(f)\kappa\dot{\beta}}(u,w) + \frac{1}{4}\rho^{(f)\alpha\dot{\rho}}(w,u)\rho^{(f)\kappa\dot{\beta}}(u,w)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&-\Pi^{\rho\xi}_{D(S^2)D}(w,u)^{fh}_{ge}\,|_{Prop^{+-}(w,u)}\\
&= -6g^2\left(N\delta^f_g\delta^h_e - \delta^f_e\delta^h_g\right)\\
&\quad\times\Big(\partial^{\xi}_u S^{-+}(u,w)\partial^{\rho}_w S^{-+}(u,w) - \partial^{\xi}_u\partial^{\rho}_w S^{-+}(u,w)S^{-+}(u,w)\\
&\qquad + \partial^{\rho}_w S^{+-}(w,u)\partial^{\xi}_u S^{+-}(w,u) - \partial^{\rho}_w\partial^{\xi}_u S^{+-}(w,u)S^{+-}(w,u)\Big)\\
&= -12g^2\left(N\delta^f_g\delta^h_e - \delta^f_e\delta^h_g\right)\\
&\quad\times\Big\{\partial^{\rho}_w F(w,u)\partial^{\xi}_u F(w,u) + \frac{i}{2}\partial^{\rho}_w F(w,u)\partial^{\xi}_u\rho(w,u)\\
&\qquad + \frac{i}{2}\partial^{\rho}_w\rho(w,u)\partial^{\xi}_u F(w,u) - \frac{1}{4}\partial^{\rho}_w\rho(w,u)\partial^{\xi}_u\rho(w,u)\\
&\qquad - \partial^{\rho}_w\partial^{\xi}_u F(w,u)F(w,u) - \frac{i}{2}\partial^{\rho}_w\partial^{\xi}_u F(w,u)\rho(w,u)\\
&\qquad - \frac{i}{2}\partial^{\rho}_w\partial^{\xi}_u\rho(w,u)F(w,u) + \frac{1}{4}\partial^{\rho}_w\partial^{\xi}_u\rho(w,u)\rho(w,u)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&-\Pi^{\rho\xi}_{D(G\bar{G})D}(w,u)^{fh}_{ge}\,|_{Prop^{+-}(w,u)}\\
&= 2g^2\left(N\delta^f_g\delta^h_e - \delta^f_e\delta^h_g\right)\partial^{\xi}_u G^{+-}(w,u)\partial^{\rho}_w G^{-+}(u,w)\\
&= 2g^2\left(N\delta^f_g\delta^h_e - \delta^f_e\delta^h_g\right)\\
&\quad\times\Big\{\partial^{\xi}_u F^{(gh)}(w,u)\partial^{\rho}_w F^{(gh)}(u,w) - \frac{i}{2}\partial^{\xi}_u F^{(gh)}(w,u)\partial^{\rho}_w\rho^{(gh)}(u,w)\\
&\qquad + \frac{i}{2}\partial^{\xi}_u\rho^{(gh)}(w,u)\partial^{\rho}_w F^{(gh)}(u,w) + \frac{1}{4}\partial^{\xi}_u\rho^{(gh)}(w,u)\partial^{\rho}_w\rho^{(gh)}(u,w)\Big\}.
\end{aligned}
\tag{2.23}
$$

In exactly the same way we compute the contribution to the $(+-)$ gluon self energy from the $D(D^2)D$ diagram in terms of F and ρ. It is rather lengthy so we will not quote it here. We will simply include it in the final expression for the spectral and statistical gluon self-energies in (2.24) and (2.25) below.

Now we may write the spectral and statistical gluon self-energies using (1.59),

$$\begin{aligned}\Pi_F^{\rho\xi}(w,u)_{ge}^{fh} &= -\frac{1}{2}\left[\Pi^{\rho\xi+-}(w,u)+\Pi^{\rho\xi-+}(w,u)\right]_{ge}^{fh}\\ \Pi_\rho^{\rho\xi}(w,u)_{ge}^{fh} &= i\left[\Pi^{\rho\xi+-}(w,u)-\Pi^{\rho\xi-+}(w,u)\right]_{ge}^{fh}.\end{aligned}$$

Thus, upon substitution, we finally find,

$$\begin{aligned}
&\Pi_F^{\rho\xi}(w,u)_{ge}^{fh}\\
&= g^2\left(N\delta_g^f\delta_e^h-\delta_e^f\delta_g^h\right)\\
&\quad\times\Big\{\sigma^\rho_{\alpha\dot\beta}\sigma^\xi_{\kappa\dot\rho}\left(8F^{(f)\alpha\dot\rho}(w,u)F^{(f)\kappa\dot\beta}(u,w)+2\rho^{(f)\alpha\dot\rho}(w,u)\rho^{(f)\kappa\dot\beta}(u,w)\right)\\
&\qquad+12\partial_w^\rho F(w,u)\partial_u^\xi F(w,u)-3\partial_w^\rho\rho(w,u)\partial_u^\xi\rho(w,u)\\
&\qquad-12\partial_w^\rho\partial_u^\xi F(w,u)F(w,u)+3\partial_w^\rho\partial_u^\xi\rho(w,u)\rho(w,u)\\
&\qquad-2\partial_u^\xi F^{(gh)}(w,u)\partial_w^\rho F^{(gh)}(u,w)-\frac{1}{2}\partial_u^\xi\rho^{(gh)}(w,u)\partial_w^\rho\rho^{(gh)}(u,w)\\
&\qquad-8\left(2\partial_\mu^w\partial_\beta^u F_D^{\rho\xi}(w,u)F_D^{\mu\beta}(w,u)-\frac{1}{2}\partial_\mu^w\partial_\beta^u\rho_D^{\rho\xi}(w,u)\rho_D^{\mu\beta}(w,u)\right)\\
&\qquad+4\left(2\partial_\mu^w\partial_\beta^u F_D^{\rho\beta}(w,u)F_D^{\mu\xi}(w,u)-\frac{1}{2}\partial_\mu^w\partial_\beta^u\rho_D^{\rho\beta}(w,u)\rho_D^{\mu\xi}(w,u)\right)\\
&\qquad+4\left(2\partial_\mu^w\partial_\beta^u F_D^{\mu\xi}(w,u)F_D^{\rho\beta}(w,u)-\frac{1}{2}\partial_\mu^w\partial_\beta^u\rho_D^{\mu\xi}(w,u)\rho_D^{\rho\beta}(w,u)\right)\\
&\qquad-2\left(2\partial_\mu^w\partial_\beta^u F_D^{\mu\beta}(w,u)F_D^{\rho\xi}(w,u)-\frac{1}{2}\partial_\mu^w\partial_\beta^u\rho_D^{\mu\beta}(w,u)\rho_D^{\rho\xi}(w,u)\right)\\
&\qquad+4\left(2\partial_w^\mu\partial_u^\xi F_D^{\rho\kappa}(w,u)F_{D,\mu\kappa}(w,u)-\frac{1}{2}\partial_w^\mu\partial_u^\xi\rho_D^{\rho\kappa}(w,u)\rho_{D,\mu\kappa}(w,u)\right)\\
&\qquad+4\left(2\partial_w^\rho\partial_u^\beta F_D^{\nu\xi}(w,u)F_{D,\nu\beta}(w,u)-\frac{1}{2}\partial_w^\rho\partial_u^\beta\rho_D^{\nu\xi}(w,u)\rho_{D,\nu\rho}(w,u)\right)\\
&\qquad-2\left(2\partial_w^\rho\partial_u^\beta F_{D,\nu\beta}(w,u)F_D^{\nu\xi}(w,u)-\frac{1}{2}\partial_w^\rho\partial_u^\beta\rho_{D,\nu\beta}(w,u)\rho_D^{\nu\xi}(w,u)\right)\\
&\qquad-2\left(2\partial_w^\rho\partial_u^\xi F_{D,\nu\kappa}(w,u)F_D^{\nu\kappa}(w,u)-\frac{1}{2}\partial_w^\rho\partial_u^\xi\rho_{D,\nu\kappa}(w,u)\rho_D^{\nu\kappa}(w,u)\right)\\
&\qquad-2\left(2\partial_w^\mu\partial_u^\xi F_{D,\mu\kappa}(w,u)F_D^{\rho\kappa}(w,u)-\frac{1}{2}\partial_w^\mu\partial_u^\xi\rho_{D,\mu\kappa}(w,u)\rho_D^{\rho\kappa}(w,u)\right)
\end{aligned}$$

$$
\begin{aligned}
&+8\left(2\partial^w_\mu F^{\rho\beta}_D(w,u)\partial^u_\rho F^{\mu\xi}_D(w,u)-\frac{1}{2}\partial^w_\mu\rho^{\rho\beta}_D(w,u)\partial^u_\rho\rho^{\mu\xi}_D(w,u)\right)\\
&-4\left(2\partial^\rho_w F_{D,\nu\beta}(w,u)\partial^\beta_u F^{\nu\xi}_D(w,u)-\frac{1}{2}\partial^\rho_w\rho_{D,\nu\beta}(w,u)\partial^\beta_u\rho^{\nu\xi}_D(w,u)\right)\\
&-4\left(2\partial^\mu_w F^{\rho\kappa}_D(w,u)\partial^\xi_u F_{D,\mu\kappa}(w,u)-\frac{1}{2}\partial^\mu_w\rho^{\rho\kappa}_D(w,u)\partial^\xi_u\rho_{D,\mu\kappa}(w,u)\right)\\
&+2\left(2\partial^\rho_w F^{\nu\kappa}_D(w,u)\partial^\xi_u F_{D,\nu\kappa}(w,u)-\frac{1}{2}\partial^\rho_w\rho^{\nu\kappa}_D(w,u)\partial^\xi_u\rho_{D,\nu\kappa}(w,u)\right)\\
&-4\left(2\partial^w_\mu F^{\rho\xi}_D(w,u)\partial^u_\beta F^{\mu\beta}_D(w,u)-\frac{1}{2}\partial^w_\mu\rho^{\rho\xi}_D(w,u)\partial^u_\beta\rho^{\mu\beta}_D(w,u)\right)\\
&+2\left(2\partial^\rho_w F^{\nu\xi}_D(w,u)\partial^\beta_u F_{D,\nu\beta}(w,u)-\frac{1}{2}\partial^\rho_w\rho^{\nu\xi}_D(w,u)\partial^\beta_u\rho_{D,\nu\beta}(w,u)\right)\\
&-4\left(2\partial^w_\mu F^{\mu\beta}_D(w,u)\partial^u_\beta F^{\rho\xi}_D(w,u)-\frac{1}{2}\partial^w_\mu\rho^{\mu\beta}_D(w,u)\partial^u_\beta\rho^{\rho\xi}_D(w,u)\right)\\
&+2\left(2\partial^\mu_w F_{D,\mu\kappa}(w,u)\partial^\xi_u F^{\rho\kappa}_D(w,u)-\frac{1}{2}\partial^\mu_w\rho_{D,\mu\kappa}(w,u)\partial^\xi_u\rho^{\rho\kappa}_D(w,u)\right)\\
&+2\left(2\partial^w_\mu F^{\mu\xi}_D(w,u)\partial^u_\beta F^{\rho\beta}_D(w,u)-\frac{1}{2}\partial^w_\mu\rho^{\mu\xi}_D(w,u)\partial^u_\beta\rho^{\rho\beta}_D(w,u)\right)\Bigg\}. \qquad (2.24)
\end{aligned}
$$

$$
\begin{aligned}
\Pi^{\rho\xi}_\rho(w,u)^{fh}_{ge} \;=\;& g^2\left(N\delta^f_g\delta^h_e-\delta^f_e\delta^h_g\right)\\
&\times\Big\{\sigma^\rho_{\alpha\dot\beta}\sigma^\xi_{\kappa\dot\rho}\left(-8F^{(f)\alpha\dot\rho}(w,u)\rho^{(f)\kappa\dot\beta}(u,w)+8\rho^{(f)\alpha\dot\rho}(w,u)F^{(f)\kappa\dot\beta}(u,w)\right)\\
&+12\partial^\rho_w F(w,u)\partial^\xi_u\rho(w,u)+12\partial^\rho_w\rho(w,u)\partial^\xi_u F(w,u)\\
&-12\partial^\rho_w\partial^\xi_u F(w,u)\rho(w,u)-12\partial^\rho_w\partial^\xi_u\rho(w,u)F(w,u)\\
&+2\partial^\xi_u F^{(gh)}(w,u)\partial^\rho_w\rho^{(gh)}(u,w)-2\partial^\xi_u\rho^{(gh)}(w,u)\partial^\rho_w F^{(gh)}(u,w)\\
&-16\left(\partial^w_\mu\partial^u_\beta F^{\rho\xi}_D(w,u)\rho^{\mu\beta}_D(w,u)+\partial^w_\mu\partial^u_\beta\rho^{\rho\xi}_D(w,u)F^{\mu\beta}_D(w,u)\right)\\
&+8\left(\partial^w_\mu\partial^u_\beta F^{\rho\beta}_D(w,u)\rho^{\mu\xi}_D(w,u)+\partial^w_\mu\partial^u_\beta\rho^{\rho\beta}_D(w,u)F^{\mu\xi}_D(w,u)\right)\\
&+8\left(\partial^w_\mu\partial^u_\beta F^{\mu\xi}_D(w,u)\rho^{\rho\beta}_D(w,u)+\partial^w_\mu\partial^u_\beta\rho^{\mu\xi}_D(w,u)F^{\rho\beta}_D(w,u)\right)\\
&-4\left(\partial^w_\mu\partial^u_\beta F^{\mu\beta}_D(w,u)\rho^{\rho\xi}_D(w,u)+\partial^w_\mu\partial^u_\beta\rho^{\mu\beta}_D(w,u)F^{\rho\xi}_D(w,u)\right)\\
&+8\left(\partial^\mu_w\partial^\xi_u F^{\rho\kappa}_D(w,u)\rho_{D,\mu\kappa}(w,u)+\partial^\mu_w\partial^\xi_u\rho^{\rho\kappa}_D(w,u)F_{D,\mu\kappa}(w,u)\right)\\
&+8\left(\partial^\rho_w\partial^\beta_u F^{\nu\xi}_D(w,u)\rho_{D,\nu\beta}(w,u)+\partial^\rho_w\partial^\beta_u\rho^{\nu\xi}_D(w,u)F_{D,\nu\rho}(w,u)\right)\\
&-4\left(\partial^\rho_w\partial^\beta_u F_{D,\nu\beta}(w,u)\rho^{\nu\xi}_D(w,u)+\partial^\rho_w\partial^\beta_u\rho_{D,\nu\beta}(w,u)F^{\nu\xi}_D(w,u)\right)\\
&-4\left(\partial^\rho_w\partial^\xi_u F_{D,\nu\kappa}(w,u)\rho^{\nu\kappa}_D(w,u)+\partial^\rho_w\partial^\xi_u\rho_{D,\nu\kappa}(w,u)F^{\nu\kappa}_D(w,u)\right)
\end{aligned}
$$

$$
\begin{aligned}
&-4\left(\partial^{\mu}_{w}\partial^{\xi}_{u}F_{D,\mu\kappa}(w,u)\rho^{\rho\kappa}_{D}(w,u)+\partial^{\mu}_{w}\partial^{\xi}_{u}\rho_{D,\mu\kappa}(w,u)F^{\rho\kappa}_{D}(w,u)\right)\\
&+16\left(\partial^{w}_{\mu}F^{\rho\beta}_{D}(w,u)\partial^{u}_{\rho}\rho^{\mu\xi}_{D}(w,u)+\partial^{w}_{\mu}\rho^{\rho\beta}_{D}(w,u)\partial^{u}_{\rho}F^{\mu\xi}_{D}(w,u)\right)\\
&-8\left(\partial^{\rho}_{w}F_{D,\nu\beta}(w,u)\partial^{\beta}_{u}\rho^{\nu\xi}_{D}(w,u)+\partial^{\rho}_{w}\rho_{D,\nu\beta}(w,u)\partial^{\beta}_{u}F^{\nu\xi}_{D}(w,u)\right)\\
&-8\left(\partial^{\mu}_{w}F^{\rho\kappa}_{D}(w,u)\partial^{\xi}_{u}\rho_{D,\mu\kappa}(w,u)+\partial^{\mu}_{w}\rho^{\rho\kappa}_{D}(w,u)\partial^{\xi}_{u}F_{D,\mu\kappa}(w,u)\right)\\
&+4\left(\partial^{\rho}_{w}F^{\nu\kappa}_{D}(w,u)\partial^{\xi}_{u}\rho_{D,\nu\kappa}(w,u)+\partial^{\rho}_{w}\rho^{\nu\kappa}_{D}(w,u)\partial^{\xi}_{u}F_{D,\nu\kappa}(w,u)\right)\\
&-8\left(\partial^{w}_{\mu}F^{\rho\xi}_{D}(w,u)\partial^{u}_{\beta}\rho^{\mu\beta}_{D}(w,u)+\partial^{w}_{\mu}\rho^{\rho\xi}_{D}(w,u)\partial^{u}_{\beta}F^{\mu\beta}_{D}(w,u)\right)\\
&+4\left(\partial^{\rho}_{w}F^{\nu\xi}_{D}(w,u)\partial^{\beta}_{u}\rho_{D,\nu\beta}(w,u)+\partial^{\rho}_{w}\rho^{\nu\xi}_{D}(w,u)\partial^{\beta}_{u}F_{D,\nu\beta}(w,u)\right)\\
&-8\left(\partial^{w}_{\mu}F^{\mu\beta}_{D}(w,u)\partial^{u}_{\beta}\rho^{\rho\xi}_{D}(w,u)+\partial^{w}_{\mu}\rho^{\mu\beta}_{D}(w,u)\partial^{u}_{\beta}F^{\rho\xi}_{D}(w,u)\right)\\
&+4\left(\partial^{\mu}_{w}F_{D,\mu\kappa}(w,u)\partial^{\xi}_{u}\rho^{\rho\kappa}_{D}(w,u)+\partial^{\mu}_{w}\rho_{D,\mu\kappa}(w,u)\partial^{\xi}_{u}F^{\rho\kappa}_{D}(w,u)\right)\\
&+4\left(\partial^{w}_{\mu}F^{\mu\xi}_{D}(w,u)\partial^{u}_{\beta}\rho^{\rho\beta}_{D}(w,u)+\partial^{w}_{\mu}\rho^{\mu\xi}_{D}(w,u)\partial^{u}_{\beta}F^{\rho\beta}_{D}(w,u)\right)\Big\}. \qquad (2.25)
\end{aligned}
$$

Ghost

The vacuum ghost self-energy is given by the diagram $G(DG)G$, evaluated in (2.17). It is a single non-local diagram, hence,

$$
\Sigma^{(gh)(0)}(x;G)=0. \qquad (2.26)
$$

Thus, for the vacuum case,

$$
\Sigma^{(gh)(\mathrm{vac})}(w,u)^{fh}_{ge} = \Sigma^{(gh)}_{G(DG)G}(w,u)^{fh}_{ge}. \qquad (2.27)
$$

We can again use the vacuum relation (1.61) to write,

$$
\begin{aligned}
\Sigma^{(gh)+-}(w,u)^{fh}_{ge} &= -\Sigma^{(gh)(\mathrm{vac})}(w,u)^{fh}_{ge}\big|_{D^{+-}(w,u),G^{+-}(w,u)}\\
&= g^{2}\left(N\delta^{f}_{g}\delta^{h}_{e}-\delta^{f}_{e}\delta^{h}_{g}\right)\\
&\quad\times\Big[\left(\partial^{\mu}_{w}D^{+-}_{\mu\nu}(w,u)+\partial^{\mu}_{w}D^{-+}_{\nu\mu}(u,w)\right)\partial^{\nu}_{u}G^{+-}(w,u)\\
&\quad+\left(D^{+-}_{\mu\nu}(w,u)+D^{-+}_{\nu\mu}(u,w)\right)\partial^{\mu}_{w}\partial^{\nu}_{u}G^{+-}(w,u)\Big]
\end{aligned}
$$

$$
\begin{aligned}
\Sigma^{(gh)-+}(w,u)^{fh}_{ge} &= -\Sigma^{(gh)(\mathrm{vac})}(w,u)^{fh}_{ge}\big|_{D^{-+}(w,u),G^{-+}(w,u)}\\
&= g^{2}\left(N\delta^{f}_{g}\delta^{h}_{e}-\delta^{f}_{e}\delta^{h}_{g}\right)\\
&\quad\times\Big[\left(\partial^{\mu}_{w}D^{-+}_{\mu\nu}(w,u)+\partial^{\mu}_{w}D^{+-}_{\nu\mu}(u,w)\right)\partial^{\nu}_{u}G^{-+}(w,u)
\end{aligned}
$$

$$+\left(D^{-+}_{\mu\nu}(w,u)+D^{+-}_{\nu\mu}(u,w)\right)\partial^{\mu}_{w}\partial^{\nu}_{u}G^{-+}(w,u)\Big].$$

Again, we may use (1.59) to write the spectral and statistical components of the ghost self-energy using the expressions above, namely,

$$\begin{aligned}\Sigma^{(gh)}_{F}(w,u)^{fh}_{ge} &= -\frac{1}{2}\left[\Sigma^{(gh)+-}(w,u)+\Sigma^{(gh)-+}(w,u)\right]^{fh}_{ge}\\ \Sigma^{(gh)}_{\rho}(w,u)^{fh}_{ge} &= i\left[\Sigma^{(gh)+-}(w,u)-\Sigma^{(gh)-+}(w,u)\right]^{fh}_{ge}.\end{aligned}$$

Hence we have,

$$\begin{aligned}\Sigma^{(gh)}_{F}(w,u)^{fh}_{ge} &= -\frac{1}{2}g^{2}\left(N\delta^{f}_{g}\delta^{h}_{e}-\delta^{f}_{e}\delta^{h}_{g}\right)\\ &\times\Big[\left(\partial^{\mu}_{w}D^{+-}_{\mu\nu}(w,u)+\partial^{\mu}_{w}D^{-+}_{\nu\mu}(u,w)\right)\partial^{\nu}_{u}G^{+-}(w,u)\\ &+\left(D^{+-}_{\mu\nu}(w,u)+D^{-+}_{\nu\mu}(u,w)\right)\partial^{\mu}_{w}\partial^{\nu}_{u}G^{+-}(w,u)\\ &+\left(\partial^{\mu}_{w}D^{-+}_{\mu\nu}(w,u)+\partial^{\mu}_{w}D^{+-}_{\nu\mu}(u,w)\right)\partial^{\nu}_{u}G^{-+}(w,u)\\ &+\left(D^{-+}_{\mu\nu}(w,u)+D^{+-}_{\nu\mu}(u,w)\right)\partial^{\mu}_{w}\partial^{\nu}_{u}G^{-+}(w,u)\Big]\end{aligned}$$

$$\begin{aligned}\Sigma^{(gh)}_{\rho}(w,u)^{fh}_{ge} &= ig^{2}\left(N\delta^{f}_{g}\delta^{h}_{e}-\delta^{f}_{e}\delta^{h}_{g}\right)\\ &\times\Big[\left(\partial^{\mu}_{w}D^{+-}_{\mu\nu}(w,u)+\partial^{\mu}_{w}D^{-+}_{\nu\mu}(u,w)\right)\partial^{\nu}_{u}G^{+-}(w,u)\\ &+\left(D^{+-}_{\mu\nu}(w,u)+D^{-+}_{\nu\mu}(u,w)\right)\partial^{\mu}_{w}\partial^{\nu}_{u}G^{+-}(w,u)\\ &-\left(\partial^{\mu}_{w}D^{-+}_{\mu\nu}(w,u)+\partial^{\mu}_{w}D^{+-}_{\nu\mu}(u,w)\right)\partial^{\nu}_{u}G^{-+}(w,u)\\ &-\left(D^{-+}_{\mu\nu}(w,u)+D^{+-}_{\nu\mu}(u,w)\right)\partial^{\mu}_{w}\partial^{\nu}_{u}G^{-+}(w,u)\Big].\end{aligned}$$

Notice that,

$$D^{-+\nu\mu}(y,x)=\left\langle A^{\nu}(y)A^{\mu}(x)\right\rangle=D^{+-\mu\nu}(x,y). \tag{2.28}$$

This gives,

$$\begin{aligned}\Sigma^{(gh)}_{F}(w,u)^{fh}_{ge} &= -g^{2}\left(N\delta^{f}_{g}\delta^{h}_{e}-\delta^{f}_{e}\delta^{h}_{g}\right)\\ &\times\Big[\partial^{\mu}_{w}D^{+-}_{\mu\nu}(w,u)\partial^{\nu}_{u}G^{+-}(w,u)+D^{+-}_{\mu\nu}\partial^{\mu}_{w}\partial^{\nu}_{u}G^{+-}(w,u)(w,u)\\ &+\partial^{\mu}_{w}D^{-+}_{\mu\nu}(w,u)\partial^{\nu}_{u}G^{-+}(w,u)+D^{-+}_{\mu\nu}(w,u)\partial^{\mu}_{w}\partial^{\nu}_{u}G^{-+}(w,u)\Big]\end{aligned}$$

$$\Sigma^{(gh)}_{\rho}(w,u)^{fh}_{ge} = 2ig^{2}\left(N\delta^{f}_{g}\delta^{h}_{e}-\delta^{f}_{e}\delta^{h}_{g}\right)$$

$$\times\left[\partial^{\mu}_{w}D^{+-}_{\mu\nu}(w,u)\partial^{\nu}_{u}G^{+-}(w,u)+D^{+-}_{\mu\nu}\partial^{\mu}_{w}\partial^{\nu}_{u}G^{+-}(w,u)(w,u)\right.$$
$$\left.-\partial^{\mu}_{w}D^{-+}_{\mu\nu}(w,u)\partial^{\nu}_{u}G^{-+}(w,u)-D^{-+}_{\mu\nu}(w,u)\partial^{\mu}_{w}\partial^{\nu}_{u}G^{-+}(w,u)\right].$$

Finally, we write these expressions in terms of F and ρ using (1.43),

$$\begin{aligned}\Sigma^{(gh)}_{F}(w,u)^{fh}_{ge} &= -g^{2}\left(N\delta^{f}_{g}\delta^{h}_{e}-\delta^{f}_{e}\delta^{h}_{g}\right)\\ &\times\left[2\partial^{\mu}_{w}F_{D,\mu\nu}(w,u)\partial^{\nu}_{u}F^{(gh)}(w,u)-\frac{1}{2}\partial^{\mu}_{w}\rho_{D,\mu\nu}(w,u)\partial^{\nu}_{u}\rho^{(gh)}(w,u)\right.\\ &\left.+2F_{D,\mu\nu}(w,u)\partial^{\mu}_{w}\partial^{\nu}_{u}F^{(gh)}(w,u)-\frac{1}{2}\rho_{D,\mu\nu}(w,u)\partial^{\mu}_{w}\partial^{\nu}_{u}\rho^{(gh)}(w,u)\right]\end{aligned}$$

$$\begin{aligned}\Sigma^{(gh)}_{\rho}(w,u)^{fh}_{ge} &= -2g^{2}\left(N\delta^{f}_{g}\delta^{h}_{e}-\delta^{f}_{e}\delta^{h}_{g}\right)\\ &\times\left[\partial^{\mu}_{w}F_{D,\mu\nu}(w,u)\partial^{\nu}_{u}\rho^{(gh)}(w,u)+\partial^{\mu}_{w}\rho_{D,\mu\nu}\partial^{\nu}_{u}F^{(gh)}(w,u)\right.\\ &\left.+F_{D,\mu\nu}(w,u)\partial^{\mu}_{w}\partial^{\nu}_{u}\rho^{(gh)}(w,u)+\rho_{D,\mu\nu}(w,u)\partial^{\mu}_{w}\partial^{\nu}_{u}F^{(gh)}(w,u)\right].\end{aligned}$$

Fermion

The vacuum fermion self-energy is given by the various diagrams evaluated in (2.16). As for the ghost case, we have no local fermion self-energy diagrams. Hence,

$$\Sigma^{(f)(0)}_{\dot{\alpha}\beta}(x;\triangle)=0. \tag{2.29}$$

However, there are two non-local diagrams, $\triangle(D\triangle)\bar{\triangle}$ and $\triangle(S\bar{\triangle})\bar{\triangle}$, hence the vacuum non-local fermion self-energy is,

$$\Sigma^{(f)(\mathrm{vac})}_{rs,\dot{\lambda}\tau}(w,u)^{fh}_{ge}=\Sigma^{(f)}_{\triangle(D\triangle)\bar{\triangle},rs,\dot{\lambda}\tau}(w,u)^{fh}_{ge}+\Sigma^{(f)}_{\triangle(S\bar{\triangle})\bar{\triangle},rs,\dot{\lambda}\tau}(w,u)^{fh}_{ge}. \tag{2.30}$$

We can again write Σ^{+-} and Σ^{-+} using the vacuum relation:

$$\begin{aligned}\Sigma^{(f)+-}_{rs,\dot{\lambda}\tau}(w,u)^{fh}_{ge} &= -\Sigma^{(f)(\mathrm{vac})}_{rs,\dot{\lambda}\tau}(w,u)^{fh}_{ge}\,|_{\triangle^{+-}(w,u),D^{+-}(w,u),S^{+-}(w,u)}\\ &= -\Sigma^{(f)}_{\triangle(D\triangle)\bar{\triangle},rs,\dot{\lambda}\tau}(w,u)^{fh}_{ge}\,|_{\triangle^{+-}(w,u),D^{+-}(w,u),S^{+-}(w,u)}\\ &\quad-\Sigma^{(f)}_{\triangle(S\bar{\triangle})\bar{\triangle},rs,\dot{\lambda}\tau}(w,u)^{fh}_{ge}\,|_{\triangle^{+-}(w,u),D^{+-}(w,u),S^{+-}(w,u)}\end{aligned}$$

$$\begin{aligned}\Sigma^{(f)-+}_{rs,\dot{\lambda}\tau}(w,u)^{fh}_{ge} &= -\Sigma^{(f)(\mathrm{vac})}_{rs,\dot{\lambda}\tau}(w,u)^{fh}_{ge}\,|_{\triangle^{-+}(w,u),D^{-+}(w,u),S^{-+}(w,u)}\\ &= -\Sigma^{(f)}_{\triangle(D\triangle)\bar{\triangle},rs,\dot{\lambda}\tau}(w,u)^{fh}_{ge}\,|_{\triangle^{-+}(w,u),D^{-+}(w,u),S^{-+}(w,u)}\end{aligned}$$

$$-\Sigma^{(f)}_{\triangle(S\bar{\triangle})\bar{\triangle},rs,\dot{\lambda}\tau}(w,u)^{fh}_{ge}\,\big|_{\triangle^{-+}(w,u),D^{-+}(w,u),S^{-+}(w,u)}. \tag{2.31}$$

Thus,

$$\begin{aligned}
\Sigma^{(f)+-}_{rs,\dot{\lambda}\tau}(w,u)^{fh}_{ge} &= -g^2\delta_{rs}\left(N\delta^f_g\delta^h_e-\delta^f_e\delta^h_g\right)\\
&\quad\times\Big(-\sigma^{\mu}_{\alpha\dot{\lambda}}\sigma^{\nu}_{\tau\dot{\rho}}\triangle^{\alpha\dot{\rho}+-}(w,u)\left[D^{+-}_{\mu\nu}(w,u)+D^{-+}_{\nu\mu}\right]\\
&\quad+6\triangle^{-+}_{\tau\dot{\lambda}}(u,w)\left[S^{-+}(u,w)+S^{+-}(w,u)\right]\Big)
\end{aligned}$$

$$\begin{aligned}
\Sigma^{(f)-+}_{rs,\dot{\lambda}\tau}(w,u)^{fh}_{ge} &= -g^2\delta_{rs}\left(N\delta^f_g\delta^h_e-\delta^f_e\delta^h_g\right)\\
&\quad\times\Big(-\sigma^{\mu}_{\alpha\dot{\lambda}}\sigma^{\nu}_{\tau\dot{\rho}}\triangle^{\alpha\dot{\rho}+-}(w,u)\left[D^{-+}_{\mu\nu}(w,u)+D^{+-}_{\nu\mu}\right]\\
&\quad+6\triangle^{+-}_{\tau\dot{\lambda}}(u,w)\left[S^{+-}(u,w)+S^{-+}(w,u)\right]\Big).
\end{aligned}$$

We know from (1.59) that,

$$\begin{aligned}
\Sigma^{(f)}_{F,rs,\dot{\lambda}\tau}(w,u)^{fh}_{ge} &= -\frac{1}{2}\left[\Sigma^{(f)+-}_{rs,\dot{\lambda}\tau}(w,u)^{fh}_{ge}+\Sigma^{(f)-+}_{rs,\dot{\lambda}\tau}(w,u)^{fh}_{ge}\right]\\
\Sigma^{(f)}_{\rho,rs,\dot{\lambda}\tau}(w,u)^{fh}_{ge} &= i\left[\Sigma^{(f)+-}_{rs,\dot{\lambda}\tau}(w,u)^{fh}_{ge}-\Sigma^{(f)-+}_{rs,\dot{\lambda}\tau}(w,u)^{fh}_{ge}\right].
\end{aligned} \tag{2.32}$$

Hence, using (1.43) we can finally write the spectral and statistical fermion self-energies as,

$$\begin{aligned}
\Sigma^{(f)}_{F,rs,\dot{\lambda}\tau}(w,u)^{fh}_{ge} &= -2g^2\delta_{rs}\left(N\delta^f_g\delta^h_e-\delta^f_e\delta^h_g\right)\\
&\quad\times\Big[\sigma^{\mu}_{\alpha\dot{\lambda}}\sigma^{\nu}_{\tau\dot{\rho}}\left(F^{(f)\alpha\dot{\rho}}(w,u)F_{D,\mu\nu}(w,u)-\frac{1}{4}\rho^{(f)\alpha\dot{\rho}}(w,u)\rho_{D,\mu\nu}\right)\\
&\quad-6\left(F^{(f)}_{\tau\dot{\lambda}}(u,w)F(w,u)+\frac{1}{4}\rho^{(f)}_{\tau\dot{\lambda}}(u,w)\rho(w,u)\right)\Big]
\end{aligned}$$

$$\begin{aligned}
\Sigma^{(f)}_{\rho,rs,\dot{\lambda}\tau}(w,u)^{fh}_{ge} &= -2g^2\delta_{rs}\left(N\delta^f_g\delta^h_e-\delta^f_e\delta^h_g\right)\\
&\quad\times\Big[\sigma^{\mu}_{\alpha\dot{\lambda}}\sigma^{\nu}_{\tau\dot{\rho}}\left(\rho^{(f)\alpha\dot{\rho}}(w,u)F_{D,\mu\nu}(w,u)+F^{(f)\alpha\dot{\rho}}(w,u)\rho_{D,\mu\nu}\right)\\
&\quad+6\left(\rho^{(f)}_{\tau\dot{\lambda}}(u,w)F(w,u)-F^{(f)}_{\tau\dot{\lambda}}(u,w)\rho(w,u)\right)\Big].
\end{aligned}$$

The above expressions for the spectral and statistical self-energies appear in the evolution equation for each field. We write these evolution equations in the following Section.

2.4.2 The evolution equations

Here we extend the results of Section 1.3.3, where we gave the evolution equations for the spectral and statistical components of the two-point function of a generic scalar and fermion field in equations (1.72) and (1.73) respectively. The extension to $\mathcal{N} = 4$ SYM is the same as that used to derive these equations from the evolution equations for the two-point functions on the closed time contour, (1.68) and (1.71) respectively. The gluon derives naturally from the scalar analysis and the ghost from the fermion analysis. In Appendix 2.A.2 we include some expressions needed for this derivation for each particle type, but for the details we again refer the reader to the excellent review [30].

For the scalar and gluon fields the evolution equations have the same form. They are,

$$\begin{aligned}\left(\partial_x^2 - \Sigma^{(0)}(x;S)\right) F(x,y) &= \int_0^{x^0} dz^0 \int d^3z \Sigma_\rho(x,z) F(z,y) \\ &\quad - \int_0^{y^0} dz^0 \int d^3z \Sigma_F(x,z)\rho(z,y) \\ \left(\partial_x^2 - \Sigma^{(0)}(x;S)\right) \rho(x,y) &= \int_{y^0}^{x^0} dz^0 \int d^3z \Sigma_\rho(x,z)\rho(z,y),\end{aligned} \tag{2.33}$$

and

$$\begin{aligned}\left(g^\kappa_\nu \partial_x^2 - \Pi^{(0)\kappa}{}_\nu(x;D)\right) F_D^{\nu\gamma}(x,y) &= \int_0^{x^0} dz^0 \int d^3z \Pi_{(\rho)\nu}{}^{\kappa}(x,z) F_D^{\nu\gamma}(z,y) \\ &\quad - \int_0^{y^0} dz^0 \int d^3z \Pi_{(F)\nu}{}^{\kappa}(x,z)\rho_D^{\nu\gamma}(z,y) \\ \left(g^\kappa_\nu \partial_x^2 - \Pi^{(0)\kappa}{}_\nu(x;S)\right) \rho_D^{\nu\gamma}(x,y) &= \int_{y^0}^{x^0} dz^0 \int d^3z \Pi_{(\rho)\nu}{}^{\kappa}(x,z)\rho_D^{\nu\gamma}(z,y)\end{aligned} \tag{2.34}$$

respectively. The fermion evolution equation are,

$$\begin{aligned}
\left(i\bar{\sigma}^{\mu}_{\dot{\alpha}\beta}\partial^{x}_{\mu} + \Sigma^{(f)(0)}_{\dot{\alpha}\beta}(x;\triangle)\right)F^{(f)\beta\dot{\gamma}}(x,y) &= \int_0^{x^0} dz^0 \int d^3z \Sigma^{(f)}_{\rho,\dot{\alpha}\beta}(x,z)F^{(f)\beta\dot{\gamma}}(z,y) \\
&\quad - \int_0^{y^0} dz^0 \int d^3z \Sigma^{(f)}_{F,\dot{\alpha}\beta}(x,z)\rho^{(f)\beta\dot{\gamma}}(z,y) \\
\left(i\bar{\sigma}^{\mu}_{\dot{\alpha}\beta}\partial^{x}_{\mu} + \Sigma^{(f)(0)}_{\dot{\alpha}\beta}(x;\triangle)\right)\rho^{(f)\beta\dot{\gamma}}(x,y) &= \int_{y^0}^{x^0} dz^0 \int d^3z \Sigma^{(f)}_{\rho,\dot{\alpha}\beta}(x,z)\rho^{(f)\beta\dot{\gamma}}(z,y).
\end{aligned} \tag{2.35}$$

Finally, the ghost evolution equations are,

$$\begin{aligned}
\left(\partial^2_x - \Sigma^{(gh)(0)}(x;G)\right)F^{(gh)}(x,y) &= \int_0^{y^0} dz^0 \int d^3z \Sigma^{(gh)}_{F}(x,z)\rho^{(gh)}(z,y) \\
&\quad - \int_0^{x^0} dz^0 \int d^3z \Sigma^{(gh)}_{\rho}(x,z)F^{(gh)}(z,y) \\
\left(\partial^2_x - \Sigma^{(gh)(0)}(x;G)\right)\rho^{(gh)}(x,y) &= -\int_{y^0}^{x^0} dz^0 \int d^3z \Sigma^{(gh)}_{\rho}(x,z)\rho^{(gh)}(z,y).
\end{aligned} \tag{2.36}$$

2.5 Discussion

The next step, now that we have obtained the evolution equations of $\mathcal{N} = 4$ SYM, would be to solve them numerically given certain initial conditions. This would conceptually allow us to study thermalization in this theory. Indeed, from the highly non-linear and coupled nature of the evolution equations it is clear that numerics would have to be used. As we have discussed already, the natural initial conditions for the 2PI effective action are gaussian. Afterwards one could move on to more physical initial conditions, for instance, one would like to use supersymmetric initial conditions to exploit the high degree of supersymmetry in the theory.

On the way to observing thermalization in $\mathcal{N} = 4$ SYM the natural first step is to determine transport coefficients. These quantities tell us about the linear response of the system to small perturbations away from equilibrium. They can be obtained from the two-point correlators using the Kubo formulae and they are usually calculated using kinetic theory. Common transport coefficients include the shear and bulk viscosity and the electrical conductivity of the system. The shear viscosity of $\mathcal{N} = 4$ SYM has been computed in [52] using kinetic theory. Transport coefficients characterize the behavior of the system close to equilibrium and therefore one would like to have a good handle on computing them before proceeding to more advanced non-equilibrium calculations.

However our computation of the $\mathcal{N} = 4$ SYM two-loop 2PI effective action is just the beginning in a calculation of transport coefficients in this theory in the nPI framework. Indeed, recent work in calculating transport coefficients in gauge theories such as QED and QCD has shown that the three-loop 3PI effective action is needed in order to compare with kinetic theory, [2], [3], [4], [53].

The main issue is the appearance of singularities which complicate the power counting. In other words, it is not trivial which diagrams one may drop in a calculation to a given order. In gauge theories in particular, the power counting is subtle, since Ward identities need to be satisfied to produce a gauge-invariant result. Indeed, as we have already discussed, the truncated 2PI (and indeed nPI) effective action in general has a gauge dependence, [17], [34], [54]. This appears at higher order than the truncation but nonetheless care needs to be taken that the final result is gauge invariant for a specific calculation. As we have discussed in the section on Φ-derivable approximations, the use of a *resummed* effective action would ensure that gauge invariance is automatically satisfied.

On the one hand there are pinch singularities which essentially arise from the form of the Kubo formula, which involves taking a low-energy limit. This limit results in pairs of propagators carrying the same momenta, leading to the resulting integral having a "pinch" singularity. There are infinitely many of these ladder diagrams which contribute at the same order and which need to be resummed. They essentially relate to a certain set of all the $2 \to 2$ particle processes. In fact, this resummation has naturally been done in the framework of the resummed 2PI formalism, to two-loops, in the calculation of the QED electrical conductivity in [55]. This gave the result to leading-log order. In [3] this computation has been done using the 2PI effective action to three-loop order. This naturally resums the contribution from all the $2 \to 2$ particle processes but still fails to produce the leading-order result for the transport coefficient.

The issue is the fact that QED is a gauge theory and as such contains collinear singularities also. This means that we in addition have to take into account all the $1 \to 2$ particle scattering processes, which means resumming another infinite set of diagrams. This is a manifestation of the so-called *Landau-Pomeranchuk-Migdal* (LPM) effect, [56], [57], [58], [59], [60]. This effect is due to collinearly-enhanced, inelastic processes such as bremsstrahlung. This resummation was naturally performed for the QED electrical conductivity in the framework of the resummed 3PI effective action, to three-loops, in [4]. Indeed the three-loop 3PI effective action also has been used to compute the full leading-order result for the QCD shear viscosity and electrical conductivity in [2], [3], [4], without the need for power counting. This result agrees with the result for the QED electrical conductivity in [60], [61], [62], calculated using kinetic theory. This seems to suggest, [53], that this theory provides the natural framework for the leading order computation of gauge theory transport coefficients. Indeed, further results of [63] for a scalar theory with cubic and quartic interactions seem to indicate that the four-loop 4PI effective action is the natural framework to do the next-to-leading order computation.

Hence the first step in making our results suitable for the calculation of transport coefficients is to push the analysis to 3PI and 3-loop order. Also, using the resummed effective action approach would ensure that Ward identities are satisfied and that physical quantities are gauge-invariant. The transport coefficients which could then be calculated could be checked against those obtained via kinetic theory in [52].

Hence, our current result of the two-loop 2PI effective action of $\mathcal{N} = 4$ SYM may not allow us to see thermalization. However, it is clear that the evolution equations presented here are fairly complicated even at two-loop order. Hence, we present here the first nontrivial step on the way to a 3-loop computation.

2.A Appendix

2.A.1 Propagator corrections

Here we give the details of the evaluation of a few of the propagator correction diagrams. In an attempt to give the most general treatment we show the evaluation of 3 different diagrams: a gluon propagator correction with a scalar loop, $D(S)D$, one with a fermion loop, $D(\triangle\bar{\triangle})D$, and a fermion propagator correction with a gluon loop, $\triangle(D\triangle)\bar{\triangle}$.

We begin with the calculation of the $D(S)D$ diagram.

$$\begin{aligned} D(S)D \quad = \quad & \Big\langle (A_\tau)^b_a(x)(A_\gamma)^f_e(y) i g^2 \int_z \Big((A_\mu)^j_i (M_m)^k_j (A^\mu)^l_k (M^m)^i_l(z) \\ & \qquad - (M_m)^j_i (A_\mu)^k_j (A^\mu)^l_k (M^m)^i_l(z) \Big) \Big\rangle . \end{aligned}$$

We now consider the Wick contraction of the first term above. The second may be obtained directly from the first by a reshuffling of adjoint indices. We calculate,

$$\begin{aligned} & \Big\langle (A_\tau)^b_a(x)(A_\gamma)^f_e(y)(A_\mu)^j_i(M_m)^k_j(A^\mu)^l_k(M^m)^i_l(z) \Big\rangle \\ & = (A_\tau)^b_a(x)(A_\gamma)^f_e(y)(A_\mu)^j_i(M_m)^k_j(A^\mu)^l_k(M^m)^i_l(z) \\ & \quad + (A_\tau)^b_a(x)(A_\gamma)^f_e(y)(A_\mu)^j_i(M_m)^k_j(A^\mu)^l_k(M^m)^i_l(z) \end{aligned}$$

$$\begin{aligned} & = \Big\langle (M_m)^k_j(z)(M^m)^i_l(z) \Big\rangle \\ & \quad \times \Big(\Big\langle (A_\tau)^b_a(x)(A_\mu)^j_i \Big\rangle \Big\langle (A_\gamma)^f_e(y)(A^\mu)^l_k(z) \Big\rangle \\ & \qquad + \Big\langle (A_\tau)^b_a(x)(A^\mu)^l_k(z) \Big\rangle \Big\langle (A_\gamma)^f_e(y)(A_\mu)^j_i(z) \Big\rangle \Big) \end{aligned}$$

$$\begin{aligned} & = \delta^m_m S(z,z) \delta^k_l \delta^i_j \Big(D_{\tau\mu}(x,z) \delta^b_i \delta^j_a D^\mu_\gamma(y,z) \delta^f_k \delta^l_e \\ & \qquad\qquad + D^\mu_\tau(x,z) \delta^b_k \delta^l_a D_{\gamma\mu}(y,z) \delta^f_i \delta^j_e \Big) \\ & = 6S(z,z) \Big(D_{\tau\mu}(x,z) D^\mu_\gamma(y,z) \delta^k_l \delta^i_j \delta^b_i \delta^j_a \delta^f_k \delta^l_e \\ & \qquad\qquad + D^\mu_\tau(x,z) D_{\gamma\mu}(y,z) \delta^k_l \delta^i_j \delta^b_k \delta^l_a \delta^f_i \delta^j_e \Big) . \end{aligned}$$

Thus,

$$
\begin{aligned}
D(S)D &= 6ig^2\int_z S(z,z)\Big(D_{\tau\mu}(x,z)D^{\mu}_{\gamma}(y,z)(\delta^k_l\delta^i_j\delta^b_i\delta^j_a\delta^f_k\delta^l_e-\delta^j_l\delta^i_i\delta^b_j\delta^k_a\delta^f_k\delta^l_e)\\
&\qquad +D^{\mu}_{\tau}(x,z)D_{\gamma\mu}(y,z)(\delta^k_l\delta^i_j\delta^b_k\delta^l_a\delta^f_i\delta^j_e-\delta^j_l\delta^i_i\delta^b_k\delta^l_a\delta^f_j\delta^k_e)\Big)\\
&= -6ig^2(N\delta^f_a\delta^b_e-\delta^b_a\delta^f_e)\int_z S(z,z)\big(D_{\tau\mu}(x,z)D^{\mu}_{\gamma}(y,z)\\
&\qquad +D^{\mu}_{\tau}(x,z)D_{\gamma\mu}(y,z)\big).
\end{aligned}
$$

Next, we evaluate $D(\triangle\bar{\triangle})D$.

$$
\begin{aligned}
D(\triangle\bar{\triangle})D &= \Big\langle (A_\tau)^b_a(x)(A_\gamma)^f_e(y)\frac{i^2}{2!}(-g)^2\int_{zw}\Big(\lambda^{\alpha}_i\sigma^{\mu}_{\alpha\dot{\beta}}A_\mu\bar{\lambda}^{\dot{\beta}i}-\lambda^{\alpha}_i\sigma^{\mu}_{\alpha\dot{\beta}}\bar{\lambda}^{\dot{\beta}i}A_\mu\Big)^k_k(z)\\
&\qquad\times\Big(\lambda^{\kappa}_j\sigma^{\nu}_{\kappa\dot{\rho}}A_\nu\bar{\lambda}^{\dot{\rho}j}-\lambda^{\kappa}_j\sigma^{\nu}_{\kappa\dot{\rho}}\bar{\lambda}^{\dot{\rho}j}A_\nu\Big)^l_l(w)\Big\rangle\\
&= -\frac{g^2}{2}\sigma^{\mu}_{\alpha\dot{\beta}}\sigma^{\nu}_{\kappa\dot{\rho}}\int_{zw}\Big\langle (A_\tau)^b_a(x)(A_\gamma)^f_e(y)\times\\
&\qquad\times\Big((\lambda^{\alpha}_i)^m_k(A_\mu)^n_m(\bar{\lambda}^{\dot{\beta}i})^k_n(z)(\lambda^{\kappa}_j)^p_l(A_\nu)^q_p(\bar{\lambda}^{\dot{\rho}j})^l_q(w)\\
&\qquad-(\lambda^{\alpha}_i)^m_k(A_\mu)^n_m(\bar{\lambda}^{\dot{\beta}i})^k_n(z)(\lambda^{\kappa}_j)^p_l(\bar{\lambda}^{\dot{\rho}j})^q_p(A_\nu)^l_q(w)\\
&\qquad-(\lambda^{\alpha}_i)^m_k(\bar{\lambda}^{\dot{\beta}i})^n_m(A_\mu)^k_n(z)(\lambda^{\kappa}_j)^p_l(A_\nu)^q_p(\bar{\lambda}^{\dot{\rho}j})^l_q(w)\\
&\qquad+(\lambda^{\alpha}_i)^m_k(\bar{\lambda}^{\dot{\beta}i})^n_m(A_\mu)^k_n(z)(\lambda^{\kappa}_j)^p_l(\bar{\lambda}^{\dot{\rho}j})^q_p(A_\nu)^l_q(w)\Big)\Big\rangle.
\end{aligned}
$$

Again we only look at the first term in the expression above, and obtain the other three by a suitable reshuffling of adjoint indices.

$$
\begin{aligned}
&\Big\langle (A_\tau)^b_a(x)(A_\gamma)^f_e(\lambda^{\alpha}_i)^m_k(A_\mu)^n_m(\bar{\lambda}^{\dot{\beta}i})^k_n(z)(\lambda^{\kappa}_j)^p_l(A_\nu)^q_p(\bar{\lambda}^{\dot{\rho}j})^l_q(w)\Big\rangle\\
&= \Big\langle (A_\tau)^b_a(x)(A_\gamma)^f_e(\lambda^{\alpha}_i)^m_k(A_\mu)^n_m(\bar{\lambda}^{\dot{\beta}i})^k_n(z)(\lambda^{\kappa}_j)^p_l(A_\nu)^q_p(\bar{\lambda}^{\dot{\rho}j})^l_q(w)\Big\rangle\\
&\quad+\Big\langle (A_\tau)^b_a(x)(A_\gamma)^f_e(\lambda^{\alpha}_i)^m_k(A_\mu)^n_m(\bar{\lambda}^{\dot{\beta}i})^k_n(z)(\lambda^{\kappa}_j)^p_l(A_\nu)^q_p(\bar{\lambda}^{\dot{\rho}j})^l_q(w)\Big\rangle\\
&= \Big(D_{\tau\mu}(x,z)D_{\gamma\nu}(y,w)\delta^b_m\delta^n_a\delta^f_p\delta^q_e+D_{\tau\nu}(x,w)D_{\gamma\mu}(y,z)\delta^b_p\delta^q_a\delta^f_m\delta^n_e\Big)
\end{aligned}
$$

$$\times\left[-\left\langle(\lambda^\alpha_i)^m_k(z)(\bar\lambda^{\dot\rho j})^l_q(w)\right\rangle\left\langle(\lambda^\kappa_j)^p_l(w)(\bar\lambda^{\dot\beta i})^k_n(z)\right\rangle\right]$$

$$\begin{aligned}
=&\quad\left(D_{\tau\mu}(x,z)D_{\gamma\nu}(y,w)\delta^b_m\delta^n_a\delta^f_p\delta^q_e+D_{\tau\nu}(x,w)D_{\gamma\mu}(y,z)\delta^b_p\delta^q_a\delta^f_m\delta^n_e\right)\\
&\quad\times\left[-\left(\delta^j_i\triangle^{\alpha\dot\rho}(z,w)\delta^m_q\delta^l_k\right)\left(\delta^i_j\triangle^{\kappa\dot\beta}(w,z)\delta^p_n\delta^k_l\right)\right]\\
=&\quad-4\triangle^{\alpha\dot\rho}(z,w)\triangle^{\kappa\dot\beta}(w,z)\Big(D_{\tau\mu}(x,z)D_{\gamma\nu}(y,w)\delta^b_m\delta^n_a\delta^f_p\delta^q_e\delta^m_q\delta^l_k\delta^p_n\delta^k_l\\
&\qquad+D_{\tau\nu}(x,w)D_{\gamma\mu}(y,z)\delta^b_p\delta^q_a\delta^f_m\delta^n_e\delta^m_q\delta^l_k\delta^p_n\delta^k_l\Big).
\end{aligned}$$

Using the above and re-shuffling indices, we get overall,

$$\begin{aligned}
&D(\triangle\bar\triangle)D\\
&=-\frac{g^2}{2}\sigma^\mu_{\alpha\dot\beta}\sigma^\nu_{\kappa\dot\rho}\int_{zw}(-4)\triangle^{\alpha\dot\rho}(z,w)\triangle^{\kappa\dot\beta}(w,z)\\
&\quad\times\Big(D_{\tau\mu}(x,z)D_{\gamma\nu}(y,w)\Big(\delta^b_m\delta^n_a\delta^f_p\delta^q_e\delta^m_q\delta^l_k\delta^p_n\delta^k_l-\delta^b_m\delta^n_a\delta^f_q\delta^l_e\delta^m_p\delta^q_k\delta^p_n\delta^k_l\\
&\qquad-\delta^b_n\delta^k_a\delta^f_p\delta^q_e\delta^m_q\delta^l_k\delta^p_m\delta^n_l+\delta^b_n\delta^k_a\delta^f_q\delta^l_e\delta^m_p\delta^q_k\delta^p_m\delta^n_l\Big)\\
&\quad+D_{\tau\nu}(x,w)D_{\gamma\mu}(y,z)\Big(\delta^b_p\delta^q_a\delta^f_m\delta^n_e\delta^m_q\delta^l_k\delta^p_n\delta^k_l-\delta^b_q\delta^l_a\delta^f_m\delta^n_e\delta^m_p\delta^q_k\delta^p_n\delta^k_l\\
&\qquad-\delta^b_p\delta^q_a\delta^f_n\delta^k_e\delta^m_q\delta^l_k\delta^p_m\delta^n_l+\delta^b_q\delta^l_a\delta^f_n\delta^k_e\delta^m_p\delta^q_k\delta^p_m\delta^n_l\Big)\Big)\\
&=4g^2(N\delta^b_e\delta^f_a-\delta^b_a\delta^f_e)\sigma^\mu_{\alpha\dot\beta}\sigma^\nu_{\kappa\dot\rho}\int_{zw}\triangle^{\alpha\dot\rho}(z,w)\triangle^{\kappa\dot\beta}(w,z)\big(D_{\tau\mu}(x,z)D_{\gamma\nu}(y,w)\\
&\qquad+D_{\tau\nu}(x,w)D_{\gamma\mu}(y,z)\big).
\end{aligned}$$

Finally, we evaluate $\triangle(D\triangle)\bar\triangle$.

$$\begin{aligned}
&\triangle(D\triangle)\bar\triangle\\
&=\Big\langle(\lambda^{\zeta r})^b_a(x)(\bar\lambda^{\dot\varphi s})^e_d(y)\frac{i^2}{2!}(-g)^2\int_{zw}\left(\lambda^\alpha_i\sigma^\mu_{\alpha\dot\beta}A_\mu\bar\lambda^{\dot\beta i}-\lambda^\alpha_i\sigma^\mu_{\alpha\dot\beta}\bar\lambda^{\dot\beta i}A_\mu\right)^k_k(z)\\
&\qquad\left(\lambda^\kappa_j\sigma^\nu_{\kappa\dot\rho}A_\nu\bar\lambda^{\dot\rho j}-\lambda^\kappa_j\sigma^\nu_{\kappa\dot\rho}\bar\lambda^{\dot\rho j}A_\nu\right)^l_l(w)\Big\rangle\\
&=-\frac{g^2}{2}\sigma^\mu_{\alpha\dot\beta}\sigma^\nu_{\kappa\dot\rho}\int_{zw}\Big\langle(\lambda^{\zeta r})^b_a(x)(\bar\lambda^{\dot\varphi s})^e_d(y)\times\\
&\qquad\times\Big((\lambda^\alpha_i)^m_k(A_\mu)^n_m(\bar\lambda^{\dot\beta i})^k_n(z)(\lambda^\kappa_j)^p_l(A_\nu)^q_p(\bar\lambda^{\dot\rho j})^l_q(w)\\
&\qquad-(\lambda^\alpha_i)^m_k(A_\mu)^n_m(\bar\lambda^{\dot\beta i})^k_n(z)(\lambda^\kappa_j)^p_l(\bar\lambda^{\dot\rho j})^q_p(A_\nu)^l_q(w)
\end{aligned}$$

$$-(\lambda_i^\alpha)_k^m(\bar\lambda^{\dot\beta i})_m^n(A_\mu)_n^k(z)(\lambda_j^\kappa)_l^p(A_\nu)_p^q(\bar\lambda^{\dot\rho j})_q^l(w)$$
$$+(\lambda_i^\alpha)_k^m(\bar\lambda^{\dot\beta i})_m^n(A_\mu)_n^k(z)(\lambda_j^\kappa)_l^p(\bar\lambda^{\dot\rho j})_p^q(A_\nu)_q^l(w)\Big)\Big\rangle .$$

We again consider the Wick contraction of the first term above,

$$\left\langle(\lambda^{\zeta r})_a^b(x)(\bar\lambda^{\dot\varphi s})_d^e(y)(\lambda_i^\alpha)_k^m(A_\mu)_m^n(\bar\lambda^{\dot\beta i})_n^k(z)(\lambda_j^\kappa)_l^p(A_\nu)_p^q(\bar\lambda^{\dot\rho j})_q^l(w)\right\rangle$$

$$= \left\langle(\lambda^{\zeta r})_a^b(x)(\bar\lambda^{\dot\varphi s})_d^e(y)(\lambda_i^\alpha)_k^m(A_\mu)_m^n(\bar\lambda^{\dot\beta i})_n^k(z)(\lambda_j^\kappa)_l^p(A_\nu)_p^q(\bar\lambda^{\dot\rho j})_q^l(w)\right\rangle$$

$$+ \left\langle(\lambda^{\zeta r})_a^b(x)(\bar\lambda^{\dot\varphi s})_d^e(y)(\lambda_i^\alpha)_k^m(A_\mu)_m^n(\bar\lambda^{\dot\beta i})_n^k(z)(\lambda_j^\kappa)_l^p(A_\nu)_p^q(\bar\lambda^{\dot\rho j})_q^l(w)\right\rangle$$

$$\begin{aligned}
&= \left\langle(\lambda^{\zeta r})_a^b(x)(\bar\lambda^{\dot\beta i})_n^k(z)\right\rangle\left\langle(\lambda_j^\kappa)_l^p(w)(\bar\lambda^{\dot\varphi s})_d^e(y)\right\rangle\left\langle(\lambda_i^\alpha)_k^m(z)(\bar\lambda^{\dot\rho j})_q^l(w)\right\rangle \times\\
&\times\left\langle(A_\mu)_m^n(A_\nu)_p^q(w)\right\rangle\\
&+(-1)^2\left\langle(\lambda^{\zeta r})_a^b(x)(\bar\lambda^{\dot\rho j})_q^l(w)\right\rangle\left\langle(\lambda_i^\alpha)_k^m(z)(\bar\lambda^{\dot\varphi s})_d^e(y)\right\rangle\left\langle(A_\mu)_m^n(z)(A_\nu)_p^q(w)\right\rangle\times\\
&\times\left\langle(\lambda_j^\kappa)_l^p(w)(\bar\lambda^{\dot\beta i})_n^k(z)\right\rangle
\end{aligned}$$

$$\begin{aligned}
&= (\delta^{ri}\delta_j^s\delta_i^j)D_{\mu\nu}(z,w)\triangle^{\zeta\dot\beta}(x,z)\triangle^{\kappa\dot\varphi}(w,y)\triangle^{\alpha\dot\rho}(z,w)\delta_n^b\delta_a^k\delta_d^p\delta_l^e\delta_q^m\delta_k^l\delta_p^n\delta_m^q\\
&+(\delta^{rj}\delta_i^s\delta_j^i)D_{\mu\nu}(z,w)\triangle^{\zeta\dot\rho}(x,w)\triangle^{\alpha\dot\varphi}(z,y)\triangle^{\kappa\dot\beta}(w,z)\delta_q^b\delta_a^l\delta_d^m\delta_k^e\delta_p^n\delta_m^q\delta_n^p\delta_l^k .
\end{aligned} \tag{2.37}$$

Thus, our diagram is given by,

$$\begin{aligned}
&\triangle(D\triangle)\bar\triangle\\
&= -\frac{g^2}{2}\sigma^\mu_{\alpha\dot\beta}\sigma^\nu_{\kappa\dot\rho}\int_{zw}\delta^{rs}D_{\mu\nu}(z,w)\Big[\triangle^{\zeta\dot\beta}(x,z)\triangle^{\kappa\dot\varphi}(w,y)\triangle^{\alpha\dot\rho}(z,w)\times\\
&\qquad\times\Big(\delta_n^b\delta_a^k\delta_d^p\delta_l^e\delta_q^m\delta_k^l\delta_p^n\delta_m^q-\delta_n^b\delta_a^k\delta_d^p\delta_l^e\delta_p^m\delta_k^q\delta_q^n\delta_m^l\\
&\qquad-\delta_m^b\delta_a^n\delta_d^p\delta_l^e\delta_q^m\delta_k^l\delta_p^k\delta_n^q+\delta_m^b\delta_a^n\delta_d^p\delta_l^e\delta_p^m\delta_k^q\delta_q^k\delta_n^l\Big)\\
&\qquad+\triangle^{\zeta\dot\rho}(x,w)\triangle^{\alpha\dot\psi}(z,y)\triangle^{\kappa\dot\beta}(w,z)\times\\
&\qquad\times\Big(\delta_q^b\delta_a^l\delta_d^m\delta_k^e\delta_p^n\delta_m^q\delta_n^p\delta_l^k-\delta_p^b\delta_a^q\delta_d^m\delta_k^e\delta_q^n\delta_m^l\delta_n^p\delta_l^k\\
&\qquad-\delta_q^b\delta_a^l\delta_d^m\delta_k^e\delta_p^k\delta_n^q\delta_m^p\delta_l^n+\delta_p^b\delta_a^q\delta_d^m\delta_k^e\delta_q^k\delta_n^l\delta_m^p\delta_l^n\Big)\Big]
\end{aligned}$$

$$= -g^2 \sigma^{\mu}_{\alpha\dot{\beta}} \sigma^{\nu}_{\kappa\dot{\rho}} \delta^{rs} (N \delta^b_d \delta^e_a - \delta^b_a \delta^e_d) \int_{zw} D_{\mu\nu}(z,w) \times$$
$$\times \left[\triangle^{\zeta\dot{\beta}}(x,z) \triangle^{\kappa\dot{\varphi}}(w,y) \triangle^{\alpha\dot{\rho}}(z,w) + \triangle^{\zeta\dot{\beta}}(x,w) \triangle^{\alpha\dot{\varphi}}(z,y) \triangle^{\kappa\dot{\beta}}(w,z) \right].$$

The propagator correction diagrams are evaluated in the same way as those demonstrated above.

2.A.2 Evolution equations

We give here some of the details needed for the derivation of the evolution equations of the $\mathcal{N}=4$ SYM fields. We begin with the scalar field.

Scalar Evolution Equations

We begin with the expression (1.19), obtained from the stationarity condition, where we have set R to zero. In our case we have matrix indices, so this expression is,

$$S^{-1}_{mn}(x,y) = S_0^{-1}(x,y) - \Sigma_{mn}(x,y;S), \tag{2.38}$$

where $S_0^{-1}{}_{mn}(x,z) = -i\Box_x \delta_{mn} \delta_{\mathscr{C}}(x-z)$. Upon convoluting we get an equation suitable for initial value problems, namely,

$$\delta_{\mathscr{C}}(x-y)\delta_{mp} = \int_z S_0^{-1}{}_{mn}(x,z) S_{np}(z,y) - \int_z \Sigma_{mn}(x,z;S) S_{np}(z,y).$$

Hence,

$$\Box_x S(x,y) - i \int_z \Sigma(x,z;S) S(z,y) = i\delta_{\mathscr{C}}(x-y) \tag{2.39}$$

Writing the self energy as a local and non-local part, that is,

$$\Sigma(x,y;S) = -i\Sigma^{(0)}(x,S)\delta(x-y) + \bar{\Sigma}(x,y;S), \tag{2.40}$$

gives,

$$\left(\Box_x - \Sigma^0(x,S)\right) S(x,y) - i \int_z \bar{\Sigma}(x,z;S) S(z,y) = i\delta_{\mathscr{C}}(x-y). \tag{2.41}$$

We may write the above in terms of F, ρ, Σ_F and Σ_ρ, where,

$$\begin{aligned} S(x,y) &= F(x,y) - \frac{i}{2}\rho(x,y)\,\mathrm{sign}_{\mathscr{C}}(x^0 - y^0) \\ \bar{\Sigma}(x,y) &= \Sigma_F(x,y) - \frac{i}{2}\Sigma_\rho(x,y)\,\mathrm{sign}_{\mathscr{C}}(x^0 - y^0). \end{aligned}$$

We also have $\rho(x,y)|_{x^0=y^0}=0$ and $\partial_{x^0}\rho|_{x^0=y^0}=\delta(\vec{x}-\vec{y})$, and we have dropped flavour indices for simplicity.

Gluon Evolution Equations

The stationarity condition for gluons gives,

$$D^{-1}_{\mu\nu}=D^{-1}_{0\ \mu\nu}-\Pi_{\mu\nu}, \tag{2.42}$$

where $iD^{-1}_{0\ \mu\nu}(x,z)=g_{\mu\nu}\Box_x\delta_{\mathscr{C}}(x-z)$. Upon convoluting we get,

$$\int D^{-1}_{\mu\nu}(x,z)D^{\nu\gamma}(z,y)=\int D^{-1}_{0\ \mu\nu}(x,z)D^{\nu\gamma}(z,y)-\int \Pi_{\mu\nu}(x,z)D^{\nu\gamma}(z,y). \tag{2.43}$$

Hence,

$$ig^{\kappa\gamma}\delta_{\mathscr{C}}(x-y)=g^{\kappa}_{\nu}\Box_x D^{\nu\gamma}(x,y)-i\int_z \Pi^{\kappa}_{\nu}(x,z)D^{\nu\gamma}(z,y) \tag{2.44}$$

This has the same form as the scalar equation, except for factors of $g^{\mu\nu}$. Again writing the self-energy in a local and non-local part,

$$\Pi^{\kappa}_{\nu}(x,y;D)=-i\Pi^{(0)\ \kappa}_{\nu}(x;D)\delta_{\mathscr{C}}(x-y)+\bar{\Pi}^{\kappa}_{\nu}(x,y;D) \tag{2.45}$$

gives,

$$ig^{\kappa\gamma}\delta_{\mathscr{C}}(x-y)=\Big(g^{\kappa}_{\nu}\Box_x-\Pi^{(0)\ \kappa}_{\nu}(x;D)\Big)D^{\nu\gamma}(x,y)-i\int_z \Pi^{\kappa}_{\nu}(x,z)D^{\nu\gamma}(z,y) \tag{2.46}$$

We may again write the above in terms of F, ρ and Π_F, Π_ρ, where,

$$\begin{aligned} D^{\nu\gamma}(x,y)&=F^{\nu\gamma}(x,y)-\frac{i}{2}\rho^{\nu\gamma}(x,y)\ \mathrm{sign}_{\mathscr{C}}(x^0-y^0)\\ \bar{\Pi}^{\kappa}_{\nu}(x,y;D)&=\Pi^{\kappa}_{F\ \nu}(x,y)-\frac{i}{2}\Pi^{\kappa}_{\rho\ \nu}(x,y)\ \mathrm{sign}_{\mathscr{C}}(x^0-y^0). \end{aligned} \tag{2.47}$$

We also have $\rho^{\mu\kappa}(x,y)|_{x^0=y^0}=0$ and $\partial_{x^0}\rho^{\mu\kappa}(x,y)|_{x^0=y^0}=g^{\mu\kappa}\delta(\vec{x}-\vec{y})$.

Fermion Evolution Equations

We begin with,

$$\triangle^{-1}_{\dot\alpha\beta}(x,y)=\triangle^{-1}_{0\ \dot\alpha\beta}-\Sigma^{(f)}_{\dot\alpha\beta}(x,y;\triangle), \tag{2.48}$$

where $\triangle_{0\,\dot\alpha\beta}^{-1}(x,y)=\bar\sigma^{\mu}_{\dot\alpha\beta}\,\partial^x_\mu\,\delta_{\mathscr{C}}(x-y)$. We now convolute the above equation to give,

$$\int_z \triangle^{-1}_{\dot\alpha\beta}(x,z)\ \triangle^{\beta\dot\gamma}(z,y)=\int_z \triangle_{0\,\dot\alpha\beta}^{-1}(x,z)\triangle^{\beta\dot\gamma}(z,y)-\int_z \Sigma^{(f)}_{\dot\alpha\beta}(x,z)\triangle^{\beta\dot\gamma}(z,y). \tag{2.49}$$

Hence,

$$\bar\sigma^{\mu}_{\dot\alpha\beta}\ \partial^x_\mu\ \triangle^{\beta\dot\gamma}(x,y)-\int_z \Sigma^{(f)}_{\dot\alpha\beta}(x,z)\triangle^{\beta\dot\gamma}(z,y)=\delta^{\dot\gamma}_{\dot\alpha}\delta_{\mathscr{C}}(x-y). \tag{2.50}$$

We write the self-energy in a local and non-local part as,

$$\Sigma^{(f)}_{\dot\alpha\beta}(x,y)=i\Sigma^{(0)}_{\dot\alpha\beta}(x;\triangle)\delta(x-y)+\bar\Sigma^{(f)}_{\dot\alpha\beta}(x,y;\triangle) \tag{2.51}$$

which gives

$$\left(\bar\sigma^{\mu}_{\dot\alpha\beta}\ \partial^x_\mu-i\Sigma^{(0)}_{\dot\alpha\beta}(x;\triangle)\right)\triangle^{\beta\dot\gamma}(x,y)-\int_z \Sigma^{(f)}_{\dot\alpha\beta}(x,z)\triangle^{\beta\dot\gamma}(z,y)=\delta^{\dot\gamma}_{\dot\alpha}\delta_{\mathscr{C}}(x-y). \tag{2.52}$$

We may rewrite the above expression in terms of $F^{(f)}$, $\rho^{(f)}$ and $\Sigma^{(f)}_F$, $\Sigma^{(f)}_\rho$, where,

$$\begin{aligned}\triangle^{\alpha\dot\beta}(x,y)&=F^{(f)\alpha\dot\beta}(x,y)-\frac{i}{2}\rho^{(f)\alpha\dot\beta}(x,y)\ \mathrm{sign}_{\mathscr{C}}(x^0-y^0)\\ \bar\Sigma^{(f)}_{\dot\alpha\beta}(x,y)&=\Sigma^{(f)}_{F\ \dot\alpha\beta}(x,y)-\frac{i}{2}\Sigma^{(f)}_{\rho\ \dot\alpha\beta}(x,y)\ \mathrm{sign}_{\mathscr{C}}(x^0-y^0).\end{aligned} \tag{2.53}$$

The equal-time canonical anticommutation relation reads,

$$\rho^{(f)\ \alpha\dot\beta}(x,y)^j_i|_{x^0=y^0}\ \bar\sigma^0_{\dot\beta\beta}=i\delta(\vec x-\vec y)\delta^\alpha_\beta\delta^j_i, \tag{2.54}$$

which implies

$$-i\bar\sigma^0_{\dot\alpha\beta}\rho^{(f)\ \beta\dot\gamma}(x,y)\delta_{\mathscr{C}}(x^0-y^0)=\delta_{\mathscr{C}}(x-y)\delta^{\dot\gamma}_{\dot\alpha}. \tag{2.55}$$

Ghost Evolution Equations

The stationarity condition for the ghost field gives,

$$G^{-1}(x,y)=G_0^{-1}(x,y)-\Sigma^{(g)}(x,y;G), \tag{2.56}$$

where $G_0^{-1}(x,y)=i\Box_x\delta(x-y)$. Convoluting the above equation gives,

$$\int_z G^{-1}(x,z)G(z,y)=\int_z G_0^{-1}(x,z)G(z,y)-\int_z \Sigma^{(g)}(x,z;G)G(z,y), \tag{2.57}$$

which implies,

$$\delta_{\mathscr{C}}(x-y) = i\Box_x G(x,y) - \int_z \Sigma^{(g)}(x,z;G)G(z,y). \tag{2.58}$$

Writing the self-energy in a local and non-local part as,

$$\Sigma^{(g)}(x,y) = i\Sigma^{(g)(0)}(x,G)\delta(x-y) + \bar{\Sigma}^{(g)}(x,y;G), \tag{2.59}$$

gives,

$$\delta_{\mathscr{C}}(x-y) = i\left(\Box_x - \Sigma^{(g)(0)}(x,G)\right)G(x,y) - \int_z \Sigma^{(g)}(x,z;G)G(z,y). \tag{2.60}$$

We may rewrite this in terms of $F^{(g)}$, $\rho^{(g)}$, $\Sigma_F^{(g)}$ and $\Sigma_\rho^{(g)}$, where,

$$\begin{aligned} G(x,y) &= F^{(g)}(x,y) - \frac{i}{2}\rho^{(g)}(x,y)\ \mathrm{sign}_{\mathscr{C}}(x^0-y^0) \\ \bar{\Sigma}^{(g)}(x,y) &= \Sigma_F^{(g)}(x,y) - \frac{i}{2}\Sigma_\rho^{(g)}(x,y)\ \mathrm{sign}_{\mathscr{C}}(x^0-y^0). \end{aligned} \tag{2.61}$$

We also have $\partial_{x^0}\rho^{(g)}(x,y)|_{x^0=y^0} = -\delta(\vec{x}-\vec{y})$.

Part II

Holography: from weak to strong coupling

Preface

In the light of the holographic principle, Part I of this thesis was devoted solely to the boundary QFT in the weak coupling regime. In Part II we shift our attention to the bulk gravitational side of the correspondence. Specifically, we will perform computations in the bulk in order to obtain insight into the dual boundary gauge theory. Since holography is a strong-weak coupling duality this means that we have tuned the gauge theory coupling from the weak all the way up to the strong coupling regime. We will consider various aspects of holography, in an attempt to demonstrate its wide relevance. These will split naturally into two topics: the application of holography in the study of strongly coupled condensed matter systems and higher derivative corrections in holography.

To facilitate these analyses, we begin in Chapter 3 with a general overview of holography. We present the holographic dictionary between the bulk gravitational theory and the boundary gauge theory and deal with such issues as holographic renormalization. For simplicity, we deal with the scalar field in AdS spacetime and then extrapolate to general bulk fields.

The impact of holography in the study of strongly-coupled condensed matter systems is touched on in part of Chapter 4. Indeed, this is one of the biggest applications of holography in recent years, and a wide range of systems exhibiting behavior ranging from that resembling high T_c superconductors to non-Fermi liquid behavior have been modeled holographically.

To successfully model strongly coupled condensed matter systems at finite temperature and charge density, we necessarily need to consider bulk actions which admit suitable black hole solutions and also contain a bulk gauge field. One of the simplest such theories is Einstein Maxwell gravity with a cosmological constant and its corresponding extremal Reissner-Nordström (RN) black hole solution. We discuss some of the characteristics of such models in Section 4.1. Another extremely useful class of theories are the Einstein-Maxwell-Dilaton (EMD) gravity theories, which include scalar fields in the bulk. We discuss these models in more detail in Section 4.1.1.

Continuing in this direction, we go on to compute the holographic dictionary for a class of EMD theories in Chapter 5. Many of the relevant solutions in this case do not have AdS asymptotics, making it *a priori* unclear how to set up holography. We overcome this problem by making use of the method of *generalized dimensional reduction*. This is a consistent dimensional reduction where we allow the dimension of the compactification manifold to be a *real* parameter and where the reduced theory depends smoothly on this parameter. If our theory is related to a higher-dimensional theory with a known holographic dictionary *via* such a generalized reduction, then we can obtain its holographic dictionary simply by reducing the dictionary of the higher-dimensional theory. Indeed, the method has universal and far-reaching applications. In our case, we start with higher dimensional Einstein gravity and reduce it to an EMD theory. We also study the hydrodynamics in the reduced theory and compute transport coefficients, from the hydrodynamics of higher-dimensional AdS.

The final aspect we consider is higher derivative corrections in holography. This essentially involves going beyond the leading order and including higher derivative terms in the classical gravity action, representing corrections from stringy or quantum effects. From the point of view of the dual field theory, we are tuning the coupling slightly away from the strong-coupling regime towards weak coupling. In recent years there has been increasing interest in going beyond the leading-order. Indeed, as we discuss in Section 4.2, such higher derivative effects could resolve a number of puzzles. For instance, the seeming violation of the Coleman-Mermin-Wagner theorem in holographic superconductors, or the question of what is the true ground state in models using the extremal RN black hole. As we will show in Section 4.2.2, another area of interest is the effect of higher derivative corrections on the KSS bound of the ratio of shear viscosity to entropy density, η/s.

We will deal with such higher derivative effects in more detail in Chapter 6, where we will attempt to find corrected 4-dimensional AdS black hole solutions in various theories of higher derivative gravity. We will compute also the thermodynamics of these black holes, using holography to find some of the thermodynamic variables and show that, indeed, the appropriate thermodynamics relations are satisfied. As a precursor to this we present a small introduction to black hole thermodynamics in the final Section of Chapter 4. We deal with the four laws of black hole thermodynamics as well as Unruh and Hawking radiation and give the well-known formula for the Bekenstein-Hawking entropy. These expressions are valid for classical Einstein gravity. When higher derivative effects are taken into account, the basic thermodynamic expressions change. They are calculated using the more general formalism developed by Wald in Section 4.3.6, and we will make use of this formalism in Chapter 6.

Chapter 3

Holography and the AdS/CFT correspondence

3.1 Introduction

One of the most ground-breaking discoveries in the last 15 years in string theory is so-called gauge/gravity duality. The basic postulate says that a string (gravitational) theory living in a $(d+1)$-dimensional bulk spacetime with a d-dimensional boundary, is dual to a QFT (gauge theory), without gravity, living in d-dimensions. It is also known as the *Holographic Principle*, by analogy with the way an ordinary optical hologram stores a three-dimensional image on a two-dimensional photographic plate. In the same way, we are describing a $(d+1)$-dimensional gravity theory in terms of a d-dimensional theory without gravity. This novel idea was pioneered by 't Hooft, [64] and Susskind, [65]. They were inspired by the earlier work of Bekenstein [66] and Hawking [67], [68], who showed that the entropy of a black hole scales, contrary to expectations, as the area of the event horizon and not as the volume.

The most researched example of this duality is the *AdS/CFT Correspondence*, which states the equivalence of string theory or supergravity on an asymptotically anti-de Sitter (AdS) spacetime (times a compact manifold) and a QFT living on the boundary of this spacetime. The first examples were provided by Maldacena in 1997, [6]. He conjectured, among other things, the equivalence between type IIB string theory on $AdS_5 \times S^5$ and four-dimensional $\mathcal{N} = 4$ SYM (CFT) theory living on the boundary of AdS_5.

Since "knowing" a QFT means knowing all correlation functions, this duality means that we can compute all the QFT correlation functions by doing computations on the gravity side. However, the correspondence is of the strong-weak type. This means that the strong coupling regime of one theory is necessarily mapped to the weak-coupling regime of the other.

The $\mathcal{N} = 4$ SYM theory is an $SU(N)$ gauge theory, and it contains two dimensionless coupling constants: N, the rank of the gauge group and g_{YM}, the usual coupling constant of the theory (it also has a theta angle which is mapped to the 10-dimensional axion, but we will not review this here). For large N the effective coupling in the field theory is the 't Hooft coupling, given by the combination $\lambda \equiv g_{YM}^2 N$ and the field theory simplifies in this limit as only planar Feynman diagrams contribute. The type IIB string theory on the $AdS_5 \times S^5$ background also has two dimensionless parameters: the string coupling, g_s, which determines the strength of string interactions and the radius of the S^5 expressed in units of the string length, $R/\sqrt{\alpha'}$, where $\alpha' = l_s^2$. The correspondence tells us to identify the parameters in the two theories as follows,

$$g_s = g_{YM}^2, \qquad \frac{R^4}{\alpha'^2} = g_{YM}^2 N \equiv \lambda. \tag{3.1}$$

On the string theory side, to make computations easier, instead of working with the full string theory on AdS spacetime, one rather looks at the low energy limit of this theory, which is supergravity. In this limit we require that the string coupling g_s is small and that the radius of curvature of the S^5 is large. Having g_s small means that g_{YM} is small on the field theory side and having R^4/α'^2 large means that λ is large. These two things are satisfied if N is large. However, for large N the effective coupling constant in the dual field theory is λ and therefore for the gauge theory to be tractable we require that λ be small. Hence we can see that the AdS/CFT duality is a strong-weak coupling duality in λ, [69].

We can therefore see that the correspondence allows us to, for instance, use the classical gravity solutions in the bulk (i.e. we may neglect stringy corrections) to compute the correlation functions in the boundary QFT at strong coupling.

3.1.1 Anti de Sitter spacetime

Anti de Sitter spacetime was first discovered as a maximally symmetric vacuum solution of Einstein's equations with negative cosmological constant. It is the Lorentzian analogue of hyperbolic spacetime. The metric of $(d+1)$-dimensional AdS spacetime is given by

$$ds^2 = -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2 d\Omega_{d-1} \tag{3.2}$$

with boundary topology $R \times S^{d-1}$ and the AdS radius set to unity. We will define the notion of asymptotically AdS spaces in the following section.

The Euclidean AdS metric in Poicare coordinates is given by,

$$ds^2 = r^2\left(dx_0^2 + dx_1^2 + \cdots + dx_{d-1}^2\right) + \frac{dr^2}{r^2}, \tag{3.3}$$

where $r \in (0,\infty)$ and $(x_0, \cdots, x_d)$ are the boundary coordinates. By changing coordinates as $z = \frac{1}{r}$ this becomes.

$$ds^2 = \frac{1}{z^2}\left(dz^2 + dx_0^2 + dx_1^2 + \cdots + dx_{d-1}^2\right), \tag{3.4}$$

where $z \in (0,\infty)$, with the boundary at zero and the origin of coordinates at infinity. Each radial slice is a d-dimensional Euclidean space. Now, the AdS metric has the scale symmetry

$$x^\mu \to \lambda x^\mu. \tag{3.5}$$

Hence, as we go close to the boundary, $z \to 0$, we scale the boundary directions as $x^i \to 0$ and hence we are in the UV regime in the boundary QFT. Similarly, as $z \to \infty$ we scale $x^i \to \infty$ and thus we are in the IR regime in the boundary QFT. However, $z \to 0$ implies $r \to \infty$, and hence close to the boundary in a gravitational theory means we are in the IR. Similarly, $z \to \infty$ means we are in the UV regime of the bulk theory. This UV/IR connection is a general phenomenon of gauge/gravity duality. Now, as is commonly known, QFT correlation functions suffer from UV divergences. Hence we expect that the bulk theory will have IR divergences, i.e. divergences near the boundary. Just as we have to renormalize the QFT to remove the UV divergences, so one also needs to renormalize the bulk theory near the boundary to remove the IR divergences. The subject of holographic renormalization deals with how this is accomplished, and we will discuss it in more detail in the following section.

3.1.2 The holographic dictionary

The statement of the equality is at the level of the equivalence of the partition functions on either side, [70], [71]. For a field ϕ living in the bulk, with corresponding boundary conditions $\phi_{(0)}$, related to the single-trace operator $\mathcal{O}$ in the QFT, we have,

$$Z_{Gravity}[\phi_{(0)}(x)] = Z_{QFT}[\phi_{(0)}(x)] \equiv e^{W_{QFT}[\phi_{(0)}]} = \langle e^{\int d^d x\ \phi_{(0)}(x)\mathcal{O}(x)} \rangle \tag{3.6}$$

where we recognize W_{QFT} as the usual generating functional of connected correlators in the boundary QFT. The gravity theory is a string theory and to evaluate the partition function on the gravity side in full is extremely difficult. Hence, the partition function is often evaluated in the low-energy limit in which the string theory reduces to a supergravity theory. This limit is equivalent to doing a saddle point approximation. Hence, the leading term is given by the on-shell supergravity action, evaluated on a solution satisfying the given boundary conditions. Namely,

$$Z_{Gravity}[\phi_{(0)}(x)] = e^{-S_{on-shell}[\phi_{(0)}]}. \tag{3.7}$$

Hence,

$$S_{on-shell}[\phi_{(0)}] = -W_{QFT}[\phi_{(0)}]. \tag{3.8}$$

We may compute connected correlation functions of operators using the standard prescription we know from QFT, namely,

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\cdots\mathcal{O}(x_n)\rangle_c = (-1)^n \frac{\delta}{\delta\phi_{(0)}(x_1)}\frac{\delta}{\delta\phi_{(0)}(x_2)}\cdots\frac{\delta}{\delta\phi_{(0)}(x_n)} W_{QFT}[\phi_{(0)}(x)]\Big|_{\phi_{(0)}=0}. \tag{3.9}$$

To reiterate, for each operator in the QFT we have a corresponding bulk field, satisfying certain boundary conditions $\phi_{(0)}$. In the case of a scalar bulk field ϕ, the source of $\mathcal{O}$ (the dual scalar operator) is $\phi_{(0)}$. For the metric field $g_{\mu\nu}$, the background metric $g_{(0)ij}$ sources the boundary stress-energy tensor T_{ij}. For the case of a gauge bulk field A_μ, the boundary gauge field $A_{(0)i}$ sources a conserved current on the boundary, J^i. Hence, a gauge symmetry in the bulk corresponds to a global symmetry on the boundary. Using the holographic dictionary we may write the one-point functions of the these operators, in the presence of sources, as,

$$\begin{aligned}
\langle \mathcal{O}(x)\rangle_s &= -\frac{1}{\sqrt{g_{(0)}}}\frac{\delta W}{\delta\phi_{(0)}(x)} \\
\langle J^i(x)\rangle_s &= -\frac{1}{\sqrt{g_{(0)}}}\frac{\delta W}{\delta A_{(0)i}(x)} \\
\langle T_{ij}(x)\rangle_s &= -\frac{2}{\sqrt{g_{(0)}}}\frac{\delta W}{\delta g^{ij}_{(0)}(x)}.
\end{aligned} \tag{3.10}$$

We will see how these expressions change in the framework of holographic renormalization.

Hence, to calculate the boundary generating functional, we need to determine the solution of the equations of motion arising from the classical supergravity action, with the given boundary conditions $\phi_{(0)}$, and evaluate the action on this solution.

To reiterate, the holographic dictionary (3.6) allows us to calculate QFT correlation functions from the bulk on-shell action. The relation as it stands is divergent. Hence, to make the correspondence consistent we have to remove these divergences. On the QFT side this is achieved through the usual renormalization procedure. On the gravity side we need to do holographic renormalization. This essentially involves removing the divergences in the on-shell supergravity action by adding suitable boundary counterterms. We explain the procedure in more detail in the next section.

3.2 Holographic renormalization

In order to perform holographic renormalization, one first determines the most general solution of the field equations with arbitrary Dirichlet boundary conditions. One then calculates the on-shell action with a regulator ϵ and calculates all infinities. This gives the regulated on-shell action, $S_{reg}[f_{(0)}]$, where we have an arbitrary bulk field $\mathscr{F}$ taking value $f_{(0)}$ on the boundary. These infinities are then removed by adding a counterterm action, where

$$S_{ct}[\mathscr{F}(x,\epsilon)] = -S_{reg}[f_{(0)}[\mathscr{F}(x,\epsilon)], \tag{3.11}$$

where we have made explicit the fact that the counterterm action needs to be defined in terms of the full bulk field $\mathscr{F}(x,\epsilon)$ in order to transform properly under bulk diffeomorphisms. To get the renormalized on-shell action we define the subtracted action as

$$S_{sub}[\mathscr{F}(x,\epsilon);\epsilon] = S_{on-shell}[f_{(0)}] + S_{ct}[\mathscr{F}(x,\epsilon);\epsilon], \tag{3.12}$$

and then remove the regulator in the counterterm action, namely,

$$S_{ren}[f_{(0)}] = \lim_{\epsilon\to 0} S_{sub}[\mathscr{F};\epsilon]. \tag{3.13}$$

This procedure was first carried out in [72].

We will first look at the case where the only bulk field is the metric and the action is the classical Einstein-Hilbert action which takes the form,

$$S[G] = \frac{1}{\kappa^2}\left(\int_{\mathcal{M}} d^{d+1}x\sqrt{G}(R-2\Lambda) - \int_{\partial\mathcal{M}} d^d x\sqrt{\gamma}\mathcal{K}\right) \tag{3.14}$$

where γ is the induced metric on the boundary and $\mathcal{K}$ is the trace of the extrinsic curvature on the boundary. This second term is the Gibbons-Hawking-York boundary term, [73], and is introduced to make the variational problem well-defined in the case of a spacetime with boundary. The resulting equations of motion for the metric field are Einstein's equations,

$$R_{\mu\nu} - \frac{1}{2}G_{\mu\nu}R = \Lambda G_{\mu\nu}. \tag{3.15}$$

We mentioned at the beginning of this section that the starting point in the procedure is to find the most general solution of the equations of motion with arbitrary Dirichlet boundary conditions. Hence, in this case, we wish to find a metric G satisfying Einstein's equations and where $g_{(0)}$ is the background metric for the dual field theory. In order to proceed with this question we now need to define precisely the notion of asymptotically AdS spacetimes.

3.2.1 Asymptotically AdS spacetimes

As mentioned previously, AdS space is defined as the maximally symmetric solution of Einstein's equations with negative cosmological constant. Examples of the metric in different coordinate systems are given by (3.2) and (3.3). However, we have seen that the gauge/gravity duality focuses our attention of the boundary of spacetime. Hence a natural question to ask is what boundary metric does the AdS metric induce? Analyzing this question in detail one comes to the conclusion that the induced metric is not unique, but depends on the choice of a so-called *defining function* (positive in the interior of the spacetime and containing a simple zero at the boundary) introduced essentially to make the boundary metric non-singular. (In a convenient coordinate system it can be shown that the AdS metric has a second order pole on the boundary.) The resulting boundary metric is well-defined, up to conformal transformations, and we say that the AdS metric induces a *conformal structure* at the boundary.

We may now define the notion of an asymptotically locally AdS spacetime: it is a conformally compact Einstein metric. This means that it satisfies the Einstein equations and in addition is a conformally compact metric. This is a metric G defined on a manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$, which has a second-order pole on the boundary, for which exists a defining function f such that f^2G smoothly extends to the boundary.

Returning now to the discussion of holographic renormalization, for the case of the metric field, the natural question now is whether, if given some conformal structure, on can obtain an asymptotically AdS spacetime with this conformal structure on the boundary.

This question was studied by the mathematics community. Fefferman and Graham in [74], managed to find the most general solution to Einstein's equations given a representative of a conformal structure. This asymptotic solution takes the form[1],

$$ds^2 = G_{\mu\nu}dx^\mu dx^\nu = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}g_{ij}(x,\rho)dx^i dx^j,$$
$$g(x,\rho) = g_{(0)} + g_{(2)}\rho \cdots + \rho^{\frac{d}{2}} g_{(d)} + \rho^{\frac{d}{2}} \log\rho \; h_{(d)} + \cdots . \tag{3.16}$$

Here the boundary is at $\rho = 0$, and any asymptotically AdS space looks like this near the boundary. For odd dimension d the coefficient of the logarithmic term, $h_{(d)}$, is zero. Hence, for the sake of clarity, we may write the above expansion more explicitly for odd and even d. For even d it is,

$$g(x,\rho) = g_{(0)} + g_{(2)}\rho \cdots + \rho^{\frac{d}{2}} \log\rho \; h_{(d)} + \rho^{\frac{d}{2}} g_{(d)} + \cdots , \tag{3.17}$$

while for odd d we have instead,

$$g(x,\rho) = g_{(0)} + g_{(2)}\rho \cdots + \rho^{\frac{d-1}{2}} g_{(d-1)} + \rho^{\frac{d}{2}} g_{(d)} + \cdots . \tag{3.18}$$

Given $g_{(0)}$ one can now determine the coefficients $g_{(2)}, \cdots, g_{(d-2)}$ and $h_{(d)}$ in terms of $g_{(0)}$. One does this by solving Einstein's equations order by order in ρ. Only the trace and covariant divergence of $g_{(d)}$ can be determined in this way. As we will discuss in due course, $h_{(d)}$ is equal to the metric variation of the coefficient of the logarithmic term in the regulated on-shell action, $a_{(d)}$ in (3.19) below, and it is proportional to the holographic conformal anomaly. $g_{(d)}$ is related to the one-point function of the dual stress energy tensor. These coefficients were found in [75].

As mentioned above, the next step is to evaluate the on-shell action by inserting the above solution in (3.14). We may then write down the regularized on-shell action, by cutting off the ρ integral at $\rho = \epsilon$ and collecting the divergent terms as $\epsilon \to 0$. The IR

[1]We change the radial coordinate from z to $\rho = z^2$. This is convenient when explicitly solving the equations of motion.

divergences depend only on the coefficients $g_{(2)}$ to $g_{(d-2)}$. These are uniquely determined in terms of $g_{(0)}$. The regularized on-shell action is then,

$$S_{reg} = \frac{1}{\kappa^2}\int_{\epsilon}\sqrt{g_{(0)}}\left(\epsilon^{-\frac{d}{2}}a_{(0)} + \epsilon^{-\frac{d}{2}+1}a_{(2)} + \cdots + \epsilon^{-\frac{1}{2}}a_{(d-1)} - \log\epsilon\ a_{(d)}\right) \tag{3.19}$$

where the $a_{(n)}$ are local covariant terms depending on the boundary metric $g_{(0)}$ and its curvature tensor. The explicit expressions can be found in [75]. We may now write down the counterterm action by means of (3.11). We emphasize that in order to get an expression with the right transformation properties with respect to bulk diffeomorphisms, we need to express it in terms of the bulk field G, defined on the regulating hypersurface $\rho = \epsilon$. In this case this means inverting (3.16) up to the required order. The answer is,

$$\begin{aligned} S_{ct} = \frac{1}{2\kappa^2}\int_{\rho=\epsilon}\sqrt{\gamma}\Bigg[&2(1-d) + \frac{1}{d-2}R - \\ &- \frac{1}{(d-2)(d-4)^2}\left(R_{ij}R^{ij}\frac{d}{4(d-1)}R^2\right) - \log\epsilon\ a_{(d)}\Bigg] \end{aligned} \tag{3.20}$$

where $\gamma_{ij} = \frac{g_{(0)ij}}{\epsilon}$ is the metric induced on the regulating hypersurface. The counterterm action contains only divergent counterterms. Looking at (3.17) and (3.18), we can say that in even dimensions, $d = 2p$, the logarithmic counterterm is non-zero, so we include only it and the first p counterterms. In odd dimensions, $d = p+1$, we include only the first $k+1$ counterterms.

From the counterterm action we may define the subtracted action, (3.12), and finally the renormalized on-shell action using (3.13). The renormalized on-shell action depends on the representative of the conformal structure, $g_{(0)}$, we have chosen. Choosing a different representative, namely, $e^{2\sigma}g_{(0)}$ we get,

$$S_{ren}[e^{2\sigma}g_{(0)}] = S_{ren}[g_{(0)}] + \mathscr{A}[g_{(0)}, \sigma]. \tag{3.21}$$

This has been computed in [72], for infinitesimal σ, to give,

$$\mathscr{A} = \frac{1}{16\pi G_N^{d+1}}\left(-2a_{(d)}\right). \tag{3.22}$$

This anomaly in the bulk corresponds to the Weyl anomaly in the dual QFT and hence is referred to as the holographic Weyl anomaly.

We may write the one point-function of the stress-energy tensor of the dual theory from this renormalized on-shell action. The quasi-local stress-energy tensor was defined by Brown and York in [76] to be the functional derivative of the on-shell action with respect to the boundary metric. In the context of holographic renormalization we can thus write, [77],

$$\langle T_{ij}(x)\rangle_s = \frac{2}{\sqrt{g_{(0)}}}\frac{\delta S_{ren}}{\delta g^{ij}_{(0)}(x)}. \tag{3.23}$$

This can be calculated from the subtracted action by taking the $\epsilon \to 0$ limit,

$$\lim_{\epsilon\to 0}\frac{2}{\sqrt{g(x,\epsilon)}}\frac{\delta S_{sub}}{\delta g^{ij}(x,\epsilon)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon^{\frac{d}{2}-1}}T_{ij}[\gamma] \sim g_{(d)ij}(x) + C(g_{(0)ij}(x)), \tag{3.24}$$

where $T_{ij}[\gamma]$ is the stress-energy tensor defined in terms of the induced metric on the regulating hypersurface and the $C(g_{(0)ij}(x))$ are contact terms which are renormalization scheme-dependent and reflect the conformal anomalies of the boundary theory. The exact result for (3.24) can be found in [75]. For odd boundary dimensions the contact terms are zero.

Once we have the holographic one-point functions we can determine the Ward identities. For instance, for the case of pure Einstein gravity, the conformal Ward identity is

$$\langle T^i_i\rangle = \mathscr{A}, \tag{3.25}$$

while for a four-dimensional scalar field, of dimension Δ, coupled to gravity, we get the following diffeomorphism and Weyl Ward identities, [78],

$$\begin{aligned}\nabla^i\langle T_{ij}\rangle_s &= -\langle\mathscr{O}\rangle_s\nabla_j\phi_{(0)},\\ \langle T^i_i\rangle_s &= (\Delta-4)\phi_{(0)}\langle\mathscr{O}\rangle_s + \mathscr{A}.\end{aligned} \tag{3.26}$$

Here $\mathscr{A}$ denotes the appropriate holographic conformal anomaly.

Higher correlation functions can be computed by carrying out further functional differentiations, with respect to the sources, of the one-point function result. However, in order to do this we need to solve the Dirichlet problem in full and not simply asymptotically, and we may then expand the solution to look at the near-boundary behavior. This is an intractable problem at this stage since the bulk field equations are coupled and non-linear. The procedure instead is to linearize the bulk field equations and solve them for the fluctuations. We may then obtain the two-point functions. For the higher n-point functions one then solves the bulk field equations perturbatively.

3.2.2 General bulk field

We may write the analogous analysis for an arbitrary bulk field $\mathscr{F}$ with Dirichelt boundary condition $f_{(0)}$. In this case we write the asymptotic solution in a Fefferman-Graham expansion as,

$$\mathscr{F}(x,\rho) = \rho^m \left[f_{(0)}(x) + f_{(2)}(x)\rho + \cdots + \rho^n \left(f_{(2n)}(x) + \tilde{f}_{(2n)}(x)\log\rho \right) + \cdots \right]. \tag{3.27}$$

We interpret the coefficient $f_{(0)}$ as the source of the dual operator $\mathscr{O}_{\mathscr{F}}$. Since the other bulk fields are related to the metric via the field equations, assuming the asymptotic metric satisfies (3.16) will in turn restrict the boundary behavior of the other fields and also the value of m in (3.27). For a scalar bulk field, for instance, $m = \frac{(d-\triangle)}{2}$ and $n = \triangle - \frac{d}{2}$. Similarly as we have seen for the metric bulk field, if n is not an integer, the logarithmic term above would be zero. As we showed for the case of the metric bulk field, we can solve for the coefficients $f_{(2k)}$, $k \ll n$, in terms of $f_{(0)}$ and its derivatives, but this is generally not the case for $f_{(2n)}$. This coefficient is instead related to the one-point function of the dual operator. The logarithmic term is once again related to the conformal anomaly and may also be solved in terms of $f_{(0)}$. For instance, in the case of a scalar field, it is related to the matter conformal anomaly.

The regularized on-shell action is then computed as

$$S_{reg}[f_{(0)};\epsilon] = \int_{\rho=\epsilon} d^d x \sqrt{g_{(0)}} \left(\epsilon^{-\nu} a_{(0)} + \epsilon^{-(\nu+1)} a_{(2)} + \cdots + \cdots - \log\epsilon \; a_{(2\nu)} \right) \tag{3.28}$$

where ν is positive and depends on the scaling dimension of $\mathscr{O}_{\mathscr{F}}$. For instance, for a scalar field $\nu = \triangle - \frac{d}{2}$. The $a_{(n)}$ are local terms depending on the boundary field $f_{(0)}$. From the regularized action we define the counterterm and subtracted actions as in (3.11) and (3.12). As we have already mentioned, this involves an inversion of the expansion (3.27), and the action depends also on the induced metric on the $\rho = \epsilon$ hypersurface, γ_{ij}. Finally, the renormalized on-shell action is given by taking the $\epsilon \to 0$ limit in the subtracted action (3.13).

We define the one-point function of $\mathscr{O}_{\mathscr{F}}$, in the presence sources, as,

$$\langle \mathscr{O}_{\mathscr{F}} \rangle_s = \frac{1}{\sqrt{g_{(0)}}} \frac{\delta S_{ren}}{\delta f_{(0)}} = \lim_{\epsilon\to 0} \frac{1}{\epsilon^{\frac{d}{2}-m}} \frac{1}{\sqrt{\gamma}} \frac{\delta_{sub}}{\delta \mathscr{F}(x;\epsilon)}. \tag{3.29}$$

Explicit evaluation yields,

$$\langle \mathscr{O}_{\mathscr{F}} \rangle_s \sim f_{(2d)} + C(f_{(0)}) \tag{3.30}$$

where $C(f_{(0)})$ are contact terms.

We have already seen the analogue of this expression for the metric bulk field in (3.24), and below we give the results for the scalar field ϕ in a fixed gravitational background and a gauge bulk field A_μ,

$$\begin{aligned} \langle \mathcal{O}(x) \rangle_s &= \frac{1}{\sqrt{g_{(0)}}} \frac{\delta S_{ren}}{\delta \phi_{(0)}(x)} \sim -(2\triangle - d)\phi_{(2\triangle-d)}(x), \\ \langle J^i(x) \rangle_s &= \frac{1}{\sqrt{g_{(0)}}} \frac{\delta S_{ren}}{\delta A_{(0)i}(x)} \sim A_{(m)i}(x). \end{aligned} \tag{3.31}$$

This ends our introduction to the intricacies of holography. In the remaining Chapters we will attempt to demonstrate what an incredibly useful tool it is in the study of a large number of different physical systems. In the next Chapter we will touch on the new and fast-growing field of holographic condensed matter theory, followed in Chapter 5 by an explicit calculation. Also in Chapter 4 we will look at higher derivative effects in holography and do a brief introduction to black hole thermodynamics. Finally, in the last Chapter we will use holography as a tool to work out the thermodynamics of certain corrected black hole solutions.

Chapter 4

Aspects of holography

4.1 Holography in condensed matter theory (AdS/CMT)

One of the biggest applications of the gauge/gravity duality in recent years has been in the realm of condensed matter physics. Some excellent reviews are [79], [80], [81], [82]. Many interesting condensed matter systems, for instance high-T_c superconductors, appear to be strongly interacting and not well described by simple perturbative models. In the context of holography this makes the dual bulk theory necessarily weakly coupled and thus accessible for investigation and computation. In order to study strongly coupled condensed matter systems at finite temperature and charge density, we need to consider weakly-coupled bulk actions which admit suitable black hole solutions on the one hand and also contain a bulk gauge field. This is due to the fact that the finite charge density implies having a conserved global charge, and hence a conserved global current on the field theory side.[1] One may then infer various properties of the dual condensed matter system, such as conductivities, superconductor-like behavior, non-Fermi liquid behavior and so on, from the corresponding black hole. Various kinds of black hole solutions have been considered, one of the simplest and most explicit being the extremal Reissner-Nordström (RN) black hole of Einstein-Maxwell gravity with a cosmological constant. This proved to be one of the simplest

[1]This requires some clarification. Condensed matter systems often possess a U(1) symmetry, which is commonly electromagnetism. This is of course a gauge symmetry and corresponds to a local current, not a global one. However, in AdS/CFT we can currently only describe global currents in the boundary theory using the bulk, not gauge currents. Hence, although we would ideally like to work with gauge currents in the boundary theory, we can't do this at present. However, many condensed matter processes have an effective field theory description of the dynamics involving effective degrees of freedom and containing charged fields but no gauge bosons for the U(1) symmetry. This amounts to neglecting the virtual photons and allows the electromagnetic symmetry to be treated as a global symmetry, [79].

setups in which to study phase transitions at finite density [83], [84]. Indeed this system has yielded a rich holographic phenomenological structure, including non-Fermi liquid behavior and the emergence of a scaling symmetry at zero temperature, as well as superfluidity and superconductivity. However this setup has an important unphysical feature: the extremal RN black hole has non-zero entropy at zero temperature

One property of condensed matter systems is that they satisfy Nerst's theorem, which states that a generic physical system at zero temperature should have zero entropy because it occupies a unique ground state. It is basically a statement of the third law of thermodynamics. However, the black hole solutions dual to condensed matter systems, including the extremal RN black hole, often exhibit the unwanted property of having non-zero entropy at zero temperature. In fact, the entropy scales with the appropriate power of the chemical potential and increases when it increases. From the point of view of holography, this means that we should look at finite temperature black holes which have a degenerate horizon as the temperature goes to zero.

The presence of the large entropy at $T = 0$ suggests that the ground state is unstable due to the large degeneracy (or approximate degeneracy) in the large N limit. There is still ongoing debate as to whether any of these states is the true ground state, [85], [86], [87], [88]. Indeed, it is believed that the apparent degeneracy is an artifact of the large N limit and, since supersymmetry is absent, should go away once finite N corrections are included. Hence, beyond the gravity approximation, it is likely that these states are energetically closely-spaced, rather than exactly degenerate and that the extremal black hole at $T = 0$ reflects the average behavior of a large number of states rather than that of a single ground state, [86]. Indeed, $1/N$ contributions seem to have substantial effects [89], [90]. However, it is subtle to work out what happens when $1/N$ effects are comparable in size to sub-leading effects, because then there is no perturbation expansion. In the final Chapter we will look at a warm up to this problem, when we analyze small $1/N$ corrections to finite temperature black holes. As a precursor to this, we discuss higher derivative corrections in a little more detail in Section 4.2 below.

4.1.1 EMD theories and AdS/CMT

An extension of this very simple model is to include scalar fields in the bulk which are covariantly (minimally) coupled to the gauge potential. These kinds of models often possess a critical temperature below which there is a phase transition and a charged condensate forms in the dual field theory. This is reminiscent of systems which exhibit superconducting or superfluid behavior.

One may extend this analysis by considering non-minimal coupling of the scalar field. These theories are known as Einstein-Maxwell-Dilaton (EMD) gravity theories, and we may write their generic action as,

$$S = \int d^4x\sqrt{-g}\left(R - \frac{f(\phi)}{4}F^2 - \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi)\right), \tag{4.1}$$

where ϕ is the dilaton and F is the usual field strength for the Maxwell field. The function $f(\phi)$ parameterizes the coupling between the scalar and Maxwell fields and $V(\phi)$ gives the scalar self-interaction.

There are a number of advantages to considering these theories. The non-minimal couplings are very common in supergravity and in the low energy effective action of string theory models. The charged black brane solutions seem to exhibit interesting thermodynamics, for instance allowing for the formation of a condensate in the dual field theory. The phenomenology of holographic models that arise form this theory seems to be rather rich.

Charged black hole solutions of low energy string theory have been known for a long time. One fairly interesting class of these are the charged dilatonic black holes. They emerge as solutions of these EMD gravity theories, and a number were found in the nineties [91], [92], [93]. Charged dilatonic black holes with AdS asymptotics are known in only a few cases, [94].

Charged dilatonic black holes were studied more recently in [95], as solutions of Einstein-Maxwell theories with a scalar dilaton gauge coupling, $e^{2\alpha\psi}F^2$ in asymptotically flat spacetime. Extremal magnetically and electrically charged black hole solutions carrying non-trivial scalar hair were found.

These planar solutions have the attractive quality of having zero entropy at extremality. Their asymptotic behavior is unconventional. Indeed, for the magnetically charged black holes, the dilaton blows up at the horizon and hence quantum loop corrections should be taken into account there. For the electrically charged black holes on the other hand, the dilaton is zero at the horizon, but the curvature in the string frame blows up, hence higher derivative corrections are needed to study the near-horizon behavior. This is a common feature of dilatonic black holes and black branes, and makes it difficult to calculate properties which depend on the near-horizon geometry[2]. In

[2]However, many interesting quantities are not sensitive to the regions in which the dilaton is large.

the context of the AdS/CFT correspondence, these black holes may allow us to study strongly coupled toy models in condensed matter. Indeed, in flat space, these extremal dilaton black holes exhibit several unique features and may thus be used to study novel holographic phases of matter, in addition to having the desirable quality of zero entropy at extremality.

In recent years EMD theories have been used fairly extensively to model strongly-interacting condensed matter systems holographically, [96], [97], [98], [99], [100], [101], [102], [103], [104], [105], [106], [107], [108], [109].

An important area of interest involving EMD theories is in studying non-relativistic field theories exhibiting Lifshitz symmetry. This particular study has emerged form more general attempts to explore holography with reduced symmetries and to find generalizations of AdS/CFT to non-relativistic systems, [110].

Theories with Lifshitz symmetry exhibit Lifshitz scaling: an anisotropic scale-invariant behavior of the form,

$$x \to \lambda x, \qquad t \to \lambda^z t. \tag{4.2}$$

The Lifshitz symmetry group contains the above scale transformations, and also space translations, space rotations and time translations. Indeed, if one includes also a special conformal transformation, Galilean boosts and a mass operator, one arrives at the Schrödinger symmetry group. This group can be viewed as the non-relativistic version of conformal symmetry. Restricting to the Lifshitz symmetry in fact means that particle number is not conserved.

Systems with Lifshitz scaling have been widely studied in condensed matter physics [111] and such scaling arises naturally at quantum critical points. For instance, $z = 1$ refers to spin systems, $z = 2$ to anti-ferromagnetism in heavy fermion metals and systems obeying the Schrödinger equation and $z = 3$ to ferromagnetism in heavy fermion metals.

Gravitational theories realizing the Lifshitz symmetry should asymptotically approach the Lifshitz-metric [112], and such a solution may be obtained in Einstein gravity only with matter fields present. One of the simplest contexts in which Lifshitz metrics emerge is in EMD theories. Indeed, one of the first papers to find the gravity dual of a strongly coupled system exhibiting Lifshitz scaling (but with a running coupling, which breaks the Lifshitz symmetry) was [96], which dealt with, among others, a matter Lagrangian containing a massless scalar coupled appropriately to a gauge field:

an EMD theory. Indeed asymptotically Lifshitz black hole solutions exist in EMD theories and can be used to model systems exhibiting Lifshitz symmetry at finite temperature, see for instance [113].

In addition to theories with Lifshitz scaling, an extensive investigation has been conducted in [98] into extremal and non-extremal charged dilaton black branes in AdS_4. The authors find that the entropy vanishes in all cases and that the non-extremal solutions have positive specific heat. They may provide novel holographic duals of insulators. Indeed, this is thought to be dual to $(3+1)$-dimensional layered systems. It has also been shown for this system, [99], that below a certain critical temperature there is a phase transition from the RN black hole to a charged dilatonic black hole. Transport coefficients have been calculated in the dual field theory and the optical conductivity seems to show a "Drude peak" at low frequency, while the resistivity shows a Kondo-like effect. The results of [98] were extended to general dimensions in [100]. These authors studied the thermodynamics of the near-extremal black holes and calculated the AC conductivity. They also looked at higher derivative corrections, namely, how Gauss-Bonnet corrections affect the $\frac{\eta}{s}$ ratio in five-dimensions at finite temperature. We will discuss this ration in a little more detail in Section 4.2.2. In [101], a general framework to study the holographic dynamics of EMD theories, in the context of effective holographic theory was investigated. The authors consider EMD theories parameterized by two real numbers, appearing in the scalar potential and gauge coupling function respectively. These parameters control the IR dynamics. They find regions where the finite-density theory at zero temperature may be mapped to a Mott-like insulator. Anisotropic background solutions of EMD theories with a Liouville background and their holography duals have also been studied, in [102], [103]. The AC and DC conductivities in these theories were calculated and strange metal-like behavior for some region of parameter space was found.

Holographic superconductors have been constructed from four-dimensional EMD gravity with a charged scalar and two adjustable couplings, [105]. This is an example of a generalized holographic superconductor model. For certain values of the couplings it was found that there is a critical temperature at which a second-order phase transition occurs form a hairy black hole to the RN AdS black hole. This is identified with the transition to the superconducting phase in the dual field theory. For a small ratio of the couplings, the conductivity is similar to that found in so-called minimal models of holographic superconductors. However, for a large enough ratio of the couplings it exhibits very different characteristics from the minimal model, for instance the novel feature of a Drude peak.

Mostly the studies of EMD gravity have been restricted to the case of an exponential scalar-Maxwell coupling function $f(\phi)$, since these arise naturally in low energy effective string theory. Also one may solve the field equations exactly in the extremal limit for the case of the exponential coupling functions. This was broadened in [107] where finite temperature black brane solutions of four-dimensional AdS EMD gravity at zero and finite temperature were studied in the presence of a power-law-type coupling. Electric, magnetic and dyonic charge configurations were studied and it was found that at finite temperature, the dual field theory resembles electron motion in metals, and exhibits such features as phase transitions and the Hall effect, while at zero temperature it resembles charged plasmas.

Charged fermions have also been added to the extremal black brane background solutions of EMD gravity, [108]. The dual field theory exhibits Fermi-liquid like behavior for a certain region of parameter space, and there is also evidence for a non-Fermi liquid in other regions. Fermions in strongly coupled systems with a chemical potential have proved difficult to study using conventional techniques.

The following Chapter deals with holography for a specific EMD theory. We will compute the holographic dictionary and the hydrodynamics in the dual field theory. In the final Chapter we will consider higher derivative corrections to Einstein gravity and compute black hole solutions in these theories. We give a little motivation for this in the section below.

4.2 Higher derivative effects in holography

Much of the early work on gauge/gravity dualities has been performed at leading-order, for the obvious reasons of computational simplicity. Indeed, recall from Section 3.1 that the supergravity approximation in the bulk is valid for small g_s and small α'. This translates into large 't Hooft coupling λ and large N in the dual field theory and means that the dual field theory is strongly coupled with only planar Feynman diagrams contributing. However, in recent years, motivated by research into the quark-gluon plasma[3] and with the increasingly popular trend of applying holography

[3] The quark-gluon plasma was initially thought to be weakly-coupled, [114]. However, there is a large body of evidence, provided by the heavy ion collision experiments at RHIC and the LHC, as well as holographic computations, that the quark-gluon plasma is in fact strongly coupled (sQGP), see for example [115] and the more recent reviews [116], [117], [13]. However, it is not entirely clear whether the QGP is strongly coupled as opposed to moderately strongly coupled. That is, it does not seem to have an almost infinite 't Hooft coupling, as in the gravity dual, but nor it is entirely in the perturbative regime.

to study condensed matter systems, there has been more and more research into going beyond the leading-order. This essentially means including higher derivative terms in the classical Einstein-Hilbert action, representing corrections from stringy or quantum effects. In other words, this means including finite λ and finite N effects. These extra terms may take many forms, for instance, the addition of terms with higher powers of the curvature. On the dual field theory side this translates to including $1/\lambda$ and $1/N$ corrections in the analysis.

One place where higher derivative effects could play an important role is in the study of holographic superconductors. This is a subject that has seen extensively researched in recent years, [79], [118], [119], [120], [121], and these studies indicate a seeming violation of the Coleman-Mermin-Wagner theorem. This well-known theorem of condensed matter physics forbids continuous symmetry breaking in systems at finite temperature in 2 and 3 spacetime dimensions. This is because long-range fluctuations are thermodynamically favored. Hence, one of the key requirements for superconductivity, the formation of a superconducting condensate, is forbidden in principle. However, studies involving classical gravity in the bulk in $(3+1)$-dimensions, namely the AdS-Schwarzschild black hole solution, have resulted in superconducting behavior on the dual field theory side. Indeed, holographic superconductors were mostly studied in $(3+1)$ bulk dimensions. A conjectured reason for this obvious violation of the Coleman-Mermin-Wagner theorem is that classical gravity corresponds to taking the large N limit in the field theory, and could very well be the reason that the fluctuations are suppressed, [79], [120], [121], [122].

The natural next step seems to be to deviate away from classical bulk gravity and include higher derivative corrections in an attempt to suppress condensation. This has been done [123] for $(3+1)$-dimensional holographic superconductors in Einstein-Gauss-Bonnet gravity, where it was found that addition of the higher curvature terms does indeed make condensation more difficult. If one would like to do an analogous study of $(2+1)$-dimensional superconductors, the obvious place to start is to consider gravity theories in $(3+1)$-dimensions with different higher derivative corrections and look for modifications to the AdS-Schwarzschild black hole solution.

4.2.1 Clues from quantum gravity

Quite aside from any motivations from holography, higher derivative corrections to classical $(3+1)$-dimensional gravity have been studied for a long time, mostly in the search for a toy model of quantum gravity. These studies have faced a number of challenges, not the least of which is that adding higher derivative corrections means adding higher-order time derivatives to the theory, and consequently ghosts. See [124], [125] for some of the original work, and more recently [126], [127]. Things seem to be somewhat easier in 3-dimensions, and indeed this is where subsequent research was done in the context of so-called topologically massive gravity [128], [129].

A well known fact about pure Einstein gravity in 3-dimensions is that it has no propagating degrees of freedom and is in this sense trivial. However, it was shown in [128], [129] that, upon adding a topological Chern-Simons term to the usual Einstein action (without cosmological constant), the theory acquires new massive propagating spin-2 modes. Since the addition of the Chern-Simons term introduces higher-order time derivatives, normally such a massive mode is tachyonic.

If the same topological term is added to Einstein gravity with a cosmological constant, namely,

$$S = \int d^3x \left[(R - 2\Lambda) + \frac{1}{2\mu}(\Gamma d\Gamma + \Gamma^3) \right], \tag{4.3}$$

where Γ is the one-form Christoffel symbol, then the theory admits asymptotically AdS solutions, for instance the BTZ black hole. However, it once again suffers from the massive spin-2 modes. For some time it was thought that the solution to this problem was the advent of so-called *critical* gravity theory. It was shown in [130] that there is a critical value ($\mu = 1$) of the coefficient of the Chern-Simons term for which the massive modes seemed to disappear and the BTZ black hole has positive energy. This became known as the chiral point as the theory was believed to be dual to a 2-dimensional chiral CFT. However, using a detailed holographic analysis, the authors of [131] demonstrated that the theory is dual to logarithmic CFT at the chiral point and is therefore non-unitary.

Similar attempts were made in recent years to find a consistent theory of 4-dimensional gravity [126], [127]. Driven by requirements of renormalizability, the natural starting point in this case was to consider curvature-squared modifications, as studied, for instance, in the earlier work of [124], [125]. The most general action of this form can be written as,

$$I = \int d^4x \, \sqrt{-g}(R - 2\Lambda + \alpha R^{\mu\nu} R_{\mu\nu} + \beta R^2 + \gamma R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}). \tag{4.4}$$

However, it is well known that the Gauss-Bonnet invariant,

$$E_4 = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 4R^{\mu\nu} R_{\mu\nu} + R^2, \tag{4.5}$$

does not contribute to the equations of motion in 4-dimensions, but, since it is a total derivative, yields only a surface term. The last term in the action above can thus be eliminated, which makes the most general action one needs to consider (modulo the E_4 term),

$$I = \int d^4x \sqrt{-g}(R - 2\Lambda + \alpha R^{\mu\nu} R_{\mu\nu} + \beta R^2). \tag{4.6}$$

The equations of motion arising from the above action show that solutions of the $\alpha = \beta = 0$ theory are also solutions of the full theory, making the AdS - Schwarzschild black hole a solution of this higher curvature theory. The theory, upon linearization about the AdS_4 background, was found to describe a massless spin-2 graviton, a massive scalar and a massive spin-2 field which is again ghostlike. Once again there seemed to be a critical point at which ghost-like modes appeared to be absent. By tuning the coefficients so that $\alpha = -3\beta$ the massive scalar mode could be eliminated. Then, as in the case of 3-dimensional topologically massive gravity, the coefficient β could be tuned in an attempt to get rid of the massive ghostlike modes also. However, also in this four-dimensional case, a detailed holographic analysis shows that the theory is still non-unitary, even at the so-called critical point.

In the final Chapter we will revisit such gravity theories containing higher derivative corrections from a different perspective, and not one where they are quantized. Indeed, we will assert that, in the context of string theory and holography, the higher curvature terms should always be viewed as small corrections to the leading order effective action. Hence, they will appear in the action multiplied by a *small* coefficient. Since this action will not be quantized, there is no issue with ghost-like modes. More specifically, we will be interested in the modified AdS-Schwarzschild black hole solutions in these theories.

4.2.2 The KSS bound

Another area where higher derivative corrections could prove interesting is in studying the ratio of shear viscosity to entropy density, η/s. In classical fluid dynamics, this ratio is a measure of how close a fluid is to being perfect.

In the next section we will see that black holes possess thermodynamic properties. For instance, they have a temperature and an entropy associated with them. In higher dimensions, other "black" objects have been found to exist. One such object is a *black brane*, which is essentially a black hole with a horizon that is translationally invariant. The concept of black hole thermodynamics can be extended to hydrodynamics for these black branes and they are found to possess also fluid properties, for instance viscosity. Through holography, the hydrodynamics of the black brane horizon corresponds to the hydrodynamics of the dual field theory. The next Chapter concerns a holographic calculation of the hydrodynamics related to such a black brane.

One of the finest achievements of holography was the calculation of the ratio η/s, in the large-N limit, for a large class of 4-dimensional strongly-interacting quantum field theories dual to Einstein gravity in the bulk, [132]. It was found to be,

$$\frac{\eta}{s} = \frac{1}{4\pi}. \tag{4.7}$$

This value is far lower than the value calculated for most well-known liquids, with water, for instance, having a ratio 380 times larger [132]. So far, all known liquids satisfy it. Indeed, preliminary estimates of η/s for the quark-gluon plasma at RHIC seem to indicate a value very close to $1/4\pi$, which would make it the most perfect fluid known to date [115], [133]. The authors of [132] in fact conjectured that there is a universal bound, the KSS bound, satisfied by the ratio η/s for a large class of thermal quantum field theories,

$$\frac{\eta}{s} \geq \frac{1}{4\pi}. \tag{4.8}$$

This conjecture makes sense from the point of view that (4.7) is valid for strongly coupled gauge theories dual to classical Einstein gravity, possibly coupled to matter also. However, Einstein gravity is taken in this context as the low energy effective theory descending from string theory or quantum gravity. Hence one can go beyond Einstein gravity by the addition of higher derivative terms related to stringy or quantum corrections. As we have already mentioned, these corrections correspond to $1/\lambda$ and $1/N$ corrections in the dual field theory. Hence, it is to be expected that inclusion of such higher-derivative terms should modify the ratio (4.7).

Indeed, this was done for the primary AdS/CFT pair: the type IIB string theory on $AdS_5 \times S^5$ dual to $\mathcal{N}$ = SYM, in [134]. These authors included some of the leading-order $1/\lambda$ correction, arising from stringy correction (of form $\alpha'^3 R^4$), and found a positive correction to (4.7), showing that the KSS bound (4.8) is satisfied.

However, this begs the question [132], [135], [136]: do all fluids, including non-relativistic ones, satisfy this bound? Indeed, we need not limit ourselves to the known string theory corrections because, for one thing, we don't know very much about string theory higher derivative corrections in general and, secondly, we can expect generic corrections to occur since the string landscape is vast [137].

Indeed, it was found that when such generic curvature corrections were included, the KSS bound was violated. This was done for instance in [137], [138], [139], [140], [141], where curvature-squared corrections were added to the bulk gravitational theory. These correspond to finite N corrections. See also [142] for KSS bound violation in 5-dimensional $\mathcal{N}=2$ gauged supergravity with up to fourth derivative corrections.

More specifically, the authors of [137] and [140] considered the case of five-dimensional Gauss-Bonnet gravity. This theory exhibits certain special properties that allowed the authors to compute the viscosity nonperturbatively in the Gauss-Bonnet coupling. They then found that, by tuning this coupling, the ratio η/s could be adjusted to any positive value, thus violating the KSS bound. A similar analysis for four-dimensional gravity theories has not yet been carried out. This is mainly due to the lack of suitable toy black hole solutions to higher derivative theories in four dimensions. Indeed, four-dimensional Gauss-Bonnet gravity is trivial and η/s is unchanged. Similarly, the cases in which one just switches on the square of the Ricci tensor or the square of the Ricci scalar are also trivial and η/s will be unchanged. It will be obvious to infer this from the analysis of our last Chapter, where we will show that both η and s are effectively just rescaled in the same way in the deformed theory. It would be interesting to see what further investigation of η/s in four bulk dimensions would yield.

The deep reason behind this violation is not yet understood, although certain authors have suggested that it may related to the consistency of the bulk theory as an effective theory [137].

4.2.3 3-d CFTs

A further motivation for going beyond the leading order is given by the recent advances in three-dimensional conformal field theories. This originated in the search for the worldvolume theory of multiple M2-branes. The pioneering Lagrangians, comprising Chern-Simons theories coupled to matter, were written down for the case of 2 branes with $\mathcal{N}=8$ supersymmetry by Gustavsson, Bagger and Lambert in [143]. Their analysis was extended to N M2-branes by means of the ABJM construction (Aharony, Bergman, Jafferis and Maldacena) in [144]. The 3-dimensional $\mathcal{N}=6$ supersymmetric Chern Simons-matter theory they discovered is conjectured to be dual to the low energy CFT living on N coincident M2-branes probing a $\mathbb{C}^4/\mathbb{Z}_k$ singularity. In the large N limit, the gravity dual of this theory corresponds to M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$. For $N \gg k^5$ this description is weakly curved. The 't Hooft coupling in this theory is given by $\lambda = N/k$, and in the 't Hooft limit ($K, N \to \infty$ with λ fixed) the appropriate description is in terms of type IIA string theory on $AdS_4 \times \mathbb{CP}^3$. Since we again have $R^4/\alpha'^2 = \lambda$, this description is weakly curved for $k \ll N$, [144]. This investigation into 3-dimensional CFTs is interesting from the condensed matter point of view as, for one thing, Chern-Simons theories often arise in interesting condensed matter systems, and thus it would be useful to have a dual gravitational description also.

Significant progress was made by the authors of [145], who succeeded in deriving the matrix model of ABJM theory. This model captures all observables of the full theory (which preserve certain supercharges) in a finite dimensional integral. This matrix model was solved exactly in the large N limit in [146], where the exact expression for the vev of a 1/6 BPS Wilson loop in the ABJM theory was computed, as a function of λ, and in the planar limit. This expression gives an exact interpolating function between the weak and the strong coupling regimes and, indeed, the strong coupling behavior agrees precisely with the prediction of the AdS string dual. Similarly, the free energy in the ABJM theory was computed by means of the matrix model in [147], as a function of λ and in the planar limit. Once again, in the strong coupling limit, there is precise agreement with the supergravity result.

A possible next step would be to extend these results for the free energy to the first sub-leading orders, which would conceptually allow one to compare with the higher derivative effects in the bulk. This would be quite interesting as it may shed light on the precise coefficients of the higher derivative terms in the bulk, which is a difficult calculation from first principles.

In this Section we have given some motivation for why one would want to go beyond Einstein gravity and add higher derivative corrections. We will deal with this in more detail in the final Chapter where we consider corrections to 4-dimensional AdS black hole solutions.

In the next Section we briefly introduce a very deep subject which will prove to be an invaluable tool in our study of black holes in the final Chapter: black hole thermodynamics. This concept, while being highly interesting in itself, also vastly broadens the scope of holographically accessible systems. Indeed, equilibrium thermodynamic properties of black holes are mapped holographically to field theories in thermal equilibrium.

4.3 Black hole thermodynamics

The study of black hole thermodynamics began some 40 years ago with Hawking's discovery that black holes emit thermal radiation [67], [68]. This is an intrinsically quantum effect since classical black holes are perfect absorbers whose temperature is absolute zero. However, in the years preceding Hawking's discovery, it had already been observed that stationary black holes, at the classical level, obey laws uncannily similar to the well-known laws of thermodynamics. Indeed, quantities analogous to temperature and entropy could be defined for them. The discovery of Hawking radiation put these laws on a firm footing as equal to the laws of thermodynamics for systems in which a black hole is present. Indeed, this opened the door to a deep and beautiful connection between thermodynamics, gravity and quantum theory.

Ordinary thermodynamics has a firm base in statistical mechanics, and so the thermodynamic behavior of black hole systems suggests that there must exist a statistical mechanics formulation, in terms of microstates, at the level of quantum gravity. Since at this stage we do not know how to quantize gravity, this may give us a handle on this long-standing problem.

The discovery of black hole thermodynamics has led also to fundamental advances in string theory and quantum theory. The fact that the entropy scales with the horizon area and not the volume is what ultimately led to the discovery of the holographic principle, followed by the AdS/CFT correspondence [7], [64], [65], [148]. Similarly, advances were made in the theory of quantum fields in curved spacetime, [149].

We will start our discussion with the early classical studies that hinted at the deeper connection between black holes and thermodynamics. We will look at Einstein gravity and set all fundamental constants to unity in this discussion. We follow closely the excellent works [150], [151] and the relevant sections in [152], and refer the reader to them for more detailed results.

4.3.1 The Schwarzschild black hole

We begin with the oldest-known and simplest black hole solution: the eternal Schwarzschild black hole. It is a vacuum solution of classical 4-dimensional gravity, with metric

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{4.9}$$

This metric has a coordinate singularity at $r = 2M$ and a curvature singularity ar $r = 0$. This coordinate system does not cover the whole of spacetime, so a better set of coordinates, in which we can remove the coordinate singularity, are the maximally-extended Kruskal coordinates, defined by,

$$u' = \left(\frac{r}{2M} - 1\right)^{1/2} e^{(r+t)/4M}, \tag{4.10}$$

$$v' = \left(\frac{r}{2M} - 1\right)^{1/2} e^{(r-t)/4M}, \tag{4.11}$$

where the metric is given by,

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M} du'dv' + r^2 d\Omega_2, \tag{4.12}$$

and $r(u', v')$ can be calculated from (4.10) and (4.11). The time-translation symmetry in these coordinates is given by,

$$\partial_t = \frac{1}{4M}(u'\partial_{u'} - v'\partial_{v'}). \tag{4.13}$$

4.3.2 The AdS-Schwarzschild black hole

In the context of holography we are interested in AdS black holes, hence we will add a negative cosmological constant to the theory and consider the d-dimensional static AdS-Schwarzschild black hole solution, namely,

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_{d-2}, \tag{4.14}$$

with $f(r)$ given by,

$$f(r) = \left(1 - \frac{\omega_d M}{r^{d-3}} + \frac{r^2}{l^2}\right), \tag{4.15}$$

and

$$\omega_d = \frac{16\pi G}{(d-2)\mathrm{Vol}(S^{d-2})}. \tag{4.16}$$

The cosmological constant is given by $\Lambda = -(d-1)(d-2)/2l^2$ and the above metric has a coordinate singularity at the event horizon, $r = r_+$, such that $f(r_+) = 0$. In this case, the coordinate singularity-removing Kruskal coordinates are defined as

$$u' = e^{\kappa u}, \qquad v' = -e^{-\kappa v} \tag{4.17}$$

where,

$$\kappa = \frac{1}{2} f'(r_+). \tag{4.18}$$

In this case, the light-cone coordinates u, v are defined as

$$u, v = t \pm r_* \tag{4.19}$$

where r_* is defined such that $\frac{dr_*}{dr} = f(r)^{-1}$. The time-translation Killing vector in these coordinates is given by

$$\partial_t = \kappa(u'\partial_{u'} - v'\partial_{v'}). \tag{4.20}$$

For the case of this black hole, the presence of the negative cosmological constant prevents the radiation emitted by the black hole from escaping to infinity and if we include quantum corrections this black hole can reach an equilibrium approximately described by the solution above.

The event horizon is in this case also a *Killing horizon*, a null hypersurface on which there is a null Killing vector field, ξ. The Killing horizon here is bifurcate with respect to the time-translation Killing vector $\xi = \partial_t$ and consists of the two branches $u' = 0$ and $v' = 0$, that is, two intersecting null curves. In (4.18), κ is actually the surface gravity, the parameter that determines the acceleration of a near-horizon observer. It is defined on a general Killing horizon as,

$$\xi^\mu \nabla_\mu \xi^\nu = \kappa \xi^\nu. \tag{4.21}$$

A bifurcate Killing horizon is typical of stationary black hole spacetimes.

4.3.3 The four laws of black hole mechanics

One of the first discoveries ultimately allowing us to relate certain aspects of the dynamics of black holes with that of thermodynamics was provided by Hawking in the form of the area theorem, [153], which essentially states that the area of an event horizon won't decrease provided that spacetime is regular and the stress tensor satisfies the weak energy condition. Bekenstein took this one step further and proposed that black holes have an entropy which is proportional to the area of the event horizon, [66].

This was followed, upon further investigation and analogously to the four laws of thermodynamics, by the four laws of black hole mechanices, [154].

The zeroth and first laws essentially deal with equilibrium and quasi-equilibrium processes, that is, nearby stationary black holes or adiabatic changes between them. The laws are as follows.

The zeroth law
This states that a stationary black hole has constant surface gravity κ over its event horizon. Making a comparison to the zeroth law of thermodynamics, which states that a system in thermal equilibrium has constant temperature, this in essence suggests that the surface gravity is analogous to temperature.

The first law
For an adiabatic transition between two nearby stationary black hole solutions, there is a relationship between the change of the mass M and the changes of the horizon area A, angular momentum J and electric charge Q. The transition is for constant surface gravity κ, angular velocity Ω and electric potential Φ. This expression, satisfied by a rotating charged black hole, is given by,

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ. \tag{4.22}$$

One may similarly interpret this from the point of view that it demonstrates the change in horizon area when a small amount of mass, angular momentum or charge is thrown into the black hole. The first law of black hole mechanics, like the first law of thermodynamics, is essentially a statement about energy conservation. The latter has the term dE on the LHS and hence this seems to indicate the the mass M plays the role of energy in the black hole system. The first law of thermodynamics similarly has TdS on its right hand side and, since the zeroth law of black hole mechanics points to an identification of the surface gravity with temperature, this again points to the black hole entropy being identified with the area of the event horizon.

The second law
This is essentially the statement of Hawking's area theorem: the area of the horizon does not decrease, subject to the weak energy condition,

$$\frac{dA}{dt} \geq 0. \tag{4.23}$$

In analogy to the second law of thermodynamics, which insists that the entropy of an isolated system cannot decrease, this again points to the identification of the horizon area with entropy. However, this is too simplistic. Indeed, matter, which should satisfy the second law of thermodynamics, loses its entropy upon falling into the black hole. Hence the second law was modified to the *Generalized second law*, where the entropy is defined as the total entropy of the black hole and the external matter.

The third law

This states that the surface gravity κ of a black hole cannot reach zero. Once again, this is to be compared to the third law of thermodynamics, which sets the entropy of a system at absolute zero to a well-defined constant. Extremal black holes have $\kappa = 0$.

From the point of view of classical black holes, the laws above are simply analogies. This is because, classically, black holes only absorb, and thus they appear to have zero temperature. Also, by the no hair theorem, they have zero entropy. Hence we are led to the conclusion that quantum effects are important and that it is necessary to consider the black hole not simply as classical solutions of GR but as the background spacetime of a quantum field theory.

The first major step in this direction was undertaken by Hawking in [67], [68]. He demonstration that black holes radiate once quantum effects are taken into account. This ultimately allowed the laws of black hole mechanics to become firmly established and, indeed, the direct relations between temperature and surface gravity, and entropy and horizon area to be determined.

4.3.4 Horizons and Unruh radiation

The laws of black hole mechanics as related to the laws of thermodynamics do not, in fact, exclusively concern black holes. The fundamental relation is between a spacetime with a *Killing horizon* and thermodynamics. Indeed, this horizon radiates upon carrying out a semi-classical analysis using the formalism of quantum field theory in curved spacetime.

The simplest example of this is provided by the analysis of the Rindler horizon in flat Minkowski spacetime. This horizon appears in the spacetime seen by an accelerated observer in Minkowski spacetime. The Rindler horizon is the simplest example of a bifurcate Killing horizon and, indeed, is the near-horizon limit of any bifurcate Killing horizon. By looking at the horizon of the Schwarzschild-AdS solution (4.14), in the neighborhood of a null-generator, one ends up with flat spacetime, but in Rindler coordinates instead of the usual Cartesian coordinates. This coordinate transformation is given by,

$$X = x\cosh\kappa t \qquad Y = x\sinh\kappa t. \tag{4.24}$$

The Killing horizon for Killing vector $\xi = \partial_t$ is at $x = 0$ in Rindler coordinates. This horizon occurs because this observer cannot, in fact, see the whole of spacetime as the Rindler coordinates only cover a portion of it. It can be shown that the Minkowski vacuum radiates in a semi-classical treatment. Indeed, by considering a free quantum field in this background spacetime there exists a unique globally nonsingular state of this field which is invariant under the spacetime isometries: the inertial vacuum state. In other words, when the field is in this inertial vacuum state, a uniformly accelerating observer would describe the field as being in a thermal equilibrium state at a fixed temperature. The technical reasoning behind this is that the two-point function in the Minkowski vacuum is periodic of period $\frac{2\pi}{\kappa}$ in imaginary time and as such satisfies the KMS condition as required by a thermal state. It has the corresponding temperature

$$T_R = \frac{\kappa}{2\pi} = \frac{a_R}{2\pi} \tag{4.25}$$

associated with it. Here a_R is the acceleration of the observers and it is equal to the surface gravity of the Killing horizon in this case. This phenomenon is called Unruh radiation, [155].

The above analysis was extended to include cosmological horizons also. Indeed, Gibbons and Hawking showed, [156], that the horizon of de Sitter space also radiates thermally.

The above procedure yielding Unruh radiation from flat Minkowski spacetime may also be applied to a stationary black hole spacetime and its Killing horizon. The simplest example is again the Schwarzschild black hole solution (4.9). The first step is to express the metric in Euclidean space by complexifying the time coordinate as $t \to i\tau$. The horizon at $r = 2M$ is now an origin in the r,τ plane in the Euclidean Schwarzschild spacetime. This is a conical singularity, and smoothness at this point, that is, *regularity at the horizon*, requires that τ be periodic with period $\beta = 2\pi/\kappa$. We can calculate the surface gravity from (4.18) to be $\kappa = 1/4M$. If we now consider the Euclidean Green's function and continue back to the Lorentzian spacetime, the resulting two-point function will satisfy the KMS condition. It is thus a thermal state at temperature

$$T = 1/\beta = \kappa/2\pi. \tag{4.26}$$

However, there is once again an obvious caveat to considering this black hole solution, since one finds that the black hole plus radiation system is unstable to small fluctuations. Indeed, the Schwarzschild black hole's temperature is $T = 1/8\pi M$, resulting in a a negative specific heat. Hence small decreases in mass result in an increase in temperature, and hence the black hole will radiate more than it absorbs.

Once again, the situation is somewhat improved by considering instead the stationary AdS-Schwarzschild black hole solution (4.14). In this case, for large horizon radius, the specific heat becomes positive and a stable equilibrium exists for the black hole plus radiation system.

4.3.5 Hawking radiation

The above two examples were of stationary black holes. Physical black holes are not stationary but are formed by the gravitational collapse of a massive body. Hawking was the first to prove that physical black holes radiate at late times [67], [68], the phenomenon being called *Hawking radiation* in his honor. His analysis in fact preceded that of Unruh. He started off by looking at the situation where there is a classical spacetime collapsing to a Schwarzschild black hole, and a free quantum field, initially in its vacuum state, propagating in this background spacetime. By computing the late-time particle content of this field at infinity, he found that it corresponds to emission from a perfect black body at Hawking temperature,

$$T_H = \frac{\kappa}{2\pi}. \tag{4.27}$$

The basic idea is that once the matter collapsing to form the black hole falls behind the horizon, the geometry on and just outside the horizon approaches an almost vacuum. Hence the stationary observers outside the horizon are almost the Rindler observers since any neighborhood of the horizon is almost flat. If one now assumes regularity of the quantum state on the horizon, this results in these almost Rindler observers measuring perfect blackbody radiation emanating from the horizon at the Hawking temperature, T_H.

This discovery by Hawking of the equilibrium between a black hole and a quantum field, where the field must be in a thermal state at Hawking temperature (4.27), is what finally allowed the entropy of the black hole to be calculated. Indeed, it can be obtained directly from the first law, by writing it in the form,

$$\begin{aligned} dM &= \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ \\ &= T dS + \Omega dJ + \Phi dQ \end{aligned} \tag{4.28}$$

where we now know that T is the Hawking temperature. This resulted in the expression for the entropy,

$$S = \frac{A}{4}, \tag{4.29}$$

where we remind the reader that we have set all physical constants to unity in this discussion. Hence, the laws of black hole mechanics, which were merely an analogy up until this point, have been put on a firm footing and are shown to be equivalent to the laws of thermodynamics.

4.3.6 Wald entropy

If we wish to go beyond the semi-classical approximation we need to add quantum corrections to the classical Einstein-Hilbert action. This represents the so-called higher derivative corrections as discussed in Section 4.2 and can take the form of, for instance, terms with higher powers of the curvature. Indeed, this is a subject we are interested in here and it forms the focus of much of the final Chapter of this work. Once such higher curvature terms are included in the theory, the expressions for many of the thermodynamic quantities defined above, like (4.29) for instance, no longer hold. They need to be derived using a more general formalism. This formalism was developed by Wald [157]. There, a general relationship between black hole entropy and Noether charge was found.

Those authors considered a general n-dimensional diffeomorphism-invariant Lagrangian, which we may write as an arbitrary functional of the Riemann tensor and it's covariant derivatives, $L(R_{\mu\nu\rho\sigma}, \nabla_\lambda R_{\mu\nu\rho\sigma}, ...)$. Then one may associate a local symmetry to each spacetime vector field ξ^a and, hence, a Noether current $\mathbf{j}$ and a Noether charge Q (for solutions of the field equations). This current is an $(n-1)$-form, while the charge is an $(n-2)$-form and satisfies $\mathbf{j} = d\mathbf{Q}$. Both of these quantities are local functionals of the Lagrangian fields and ξ^a.

They then considered a stationary black hole solution in this theory, possessing a bifurcate Killing horizon with bifurcation surface Σ and well-defined asymptotic mass and angular momentum. In this framework, they were able to show that the first law of black hole mechanics always holds. The expression for the entropy was now proportional to the integral of the $\mathbf{Q}$ charge associated with $\tilde{\xi}$, the Killing field of the horizon (i.e. the field vanishing on Σ), and with the surface gravity normalized to unity. Specifically,

$$\delta\mathscr{E} = \frac{\kappa}{2\pi}\delta S + \text{work terms}, \tag{4.30}$$

where $\mathscr{E}$ is the *canonical energy* of the theory and is associated with an asymptotic time translation. Indeed, in the absence of matter fields, it is the *ADM* mass of General Relativity, [158]. We see from the above that the expression for the black hole temperature remains unchanged in this general setting, while the entropy S is given by,

$$\begin{aligned} S &= 2\pi \int_{\Sigma} \mathbf{Q}[\tilde{\xi}] \\ &= 2\pi \int_{\Sigma} \mathbf{X}^{cd} \epsilon_{cd}, \end{aligned} \tag{4.31}$$

where

$$\mathbf{X}_{ab} = -\frac{\delta L}{\delta \mathbf{R}_{ab}}, \tag{4.32}$$

and $\mathbf{R}_{ab} = R_{ab\mu\nu} = \epsilon_a^\rho \epsilon_b^\sigma R_{\rho\sigma\mu\nu}$. Hence we can see that the black hole entropy always is given by a local geometrical expression on the horizon. Indeed, for the Einstein-Hilbert action this reduces to the familiar expression (4.29).

What is the state of affairs between this generalized entropy and the second law of black hole mechanics? The author of [157] shows that, for stationary black holes evolving to nearby stationary black holes, whether or not the second law is satisfied has to do with whether the total Noether flux is positive or not. It was shown, [159], using the weak-energy condition and the first law that, for this class of black hole processes, the second law does indeed hold. However a proof for general dynamical processes is still at large.

This formalism also fits the Euclidean formalism for calculating the black hole entropy. We will use it extensively in the last chapter where we consider higher derivative corrections to four-dimensional AdS black holes.

4.3.7 The Euclidean formalism

Now that we have calculated the entropy of a black hole, an obvious next question is why it scales with the horizon area as opposed to some form of volume-like quantity. Indeed, the entropy of a system has a statistical interpretation in that it counts the number of states accessible to the system for given values of the state variables. One would of course like to know what are these dynamical quantum degrees of freedom of a black hole.

In order to investigate this question further, it is necessary to go beyond the semi-classical approach described above and to use instead the full framework of quantum gravity to directly calculate the entropy. One possible approach to formulating quantum gravity is the Euclidean path integral approach. This provides a very general link between thermodynamic potentials and geometry. Indeed, Gibbons and Hawking performed the first such direct quantum gravity computation of the black hole entropy in [73].

The Euclidean path integral approach uses a formal functional integral expression for the canonical ensemble partition function in quantum gravity. It consists of a sum over all smooth Riemannian geometries, with the (imaginary) time periodically identified with period $T = \beta^{-1}$. We write this partition function as,

$$Z[\beta] = \int \mathscr{D}[g] e^{-I[g]}, \tag{4.33}$$

where g represents collectively all possible metric fields satisfying some user-defined boundary condition but allowed to fluctuate freely in the interior. To simplify our analysis we have excluded any matter dependence and we have set Planck's constant to unity.

It is well-known that there are fundamental problems in writing down a consistent finite theory of quantum gravity. It suffers from non-renormalizable UV divergences, among other problems and the on-shell gravitational action normally diverges. We circumvent these problems by looking at the setup in the semi-classical regime, for energies well below the Planck length, where the classical solution is weakly curved. Here we perform the saddle-point approximation where the classical solution should dominate the path integral. In other words, we may approximate the path integral as

$$\ln Z[\beta] \sim -I_s \tag{4.34}$$

where I_s is the classical on-shell action evaluated on a solution of the equations of motion which satisfies the appropriate boundary conditions.

The Euclidean action I is defined from the usual Lorentzian action S by Wick-rotating the time coordinate, so that $I = iS$. This Wick-rotation to Euclidean space and making the resulting coordinate periodic is what provides the link to statistical mechanics, where the phase space has positive signature. We then have that $Z[\beta]$ is the canonical statistical mechanics partition function and we may write

$$Z[\beta] = e^{-\beta F} = e^{-\beta\langle E\rangle + S}, \tag{4.35}$$

where F is the free energy, S is the entropy and $\langle E\rangle$ is the expectation value of the energy of the system. Then we may read off the above

$$\beta F = I_s, \tag{4.36}$$

and we may use the standard statistical mechanics expressions to evaluate the remaining quantities. Namely,

$$\langle E\rangle = -\frac{\partial}{\partial\beta}\ln Z \sim \frac{\partial}{\partial\beta}I_s, \tag{4.37}$$

$$S = \beta\langle E\rangle + \ln Z = -\frac{\partial}{\partial\beta}\ln Z + \ln Z \sim \frac{\partial}{\partial\beta}I_s - I_s. \tag{4.38}$$

In order to evaluate all of these quantities we of course first need to evaluate the gravitational on-shell action. This is written as

$$I = -\frac{1}{16\pi}\int_{\mathscr{M}}\sqrt{g}(R-2\Lambda) - \frac{1}{8\pi}\int_{\partial\mathscr{M}}\sqrt{h}\mathscr{K}. \tag{4.39}$$

Here $\mathscr{M}$ is the background spacetime with boundary $\partial\mathscr{M}$ and $(h,\mathscr{K})$ are the induced metric and the trace of the intrinsic curvature on the boundary respectively. That is, $\mathscr{K} = h^{\mu\nu}\nabla_\mu n_\nu$, where n^μ is the boundary normal.

The action needs to be supplemented by the boundary term above, called the Gibbons-Hawking-York term [160], to ensure that the variational problem is well-defined for spacetimes where the boundary metric is held fixed. We will deal with the challenge of writing down such terms for more general theories of gravity, containing higher curvature terms, in the final Chapter.

Next comes the challenge of evaluating this action on-shell, which, as we have mentioned above, diverges for the most part. In the context of GR there are a number of techniques that can be used to evaluate it. For the case of certain spacetimes a possible approach is background subtraction. Here the on-shell action is regularized by subtracting from it an appropriately chosen background piece. In many cases this can simply be the asymptotic metric, but there are many choices that could suffice. This ambiguity is rather undesirable. Another method makes use of the Hamiltonian formulation which uses the full boundary conditions supplied and removes the ambiguities inherent in the background subtraction approach. Such methods do, in fact, reproduce the value of the entropy (4.29).

However, a more modern approach and one which has had numerous successes, is to once again make use of the gauge/gravity duality for situations with AdS asymptotics. As we have already seen, this allows us to compute a set of covariant counterterms which render the on-shell action finite. This correspondence is useful for cases where the bulk spacetime is weakly curved, and, as we have already mentioned, this is essentially the scenario we are considering here.

We recall from (3.6) that the bulk gravitational partition function is equivalent to the generating functional of correlation functions of the boundary field theory. In this case we have that the generating functional of thermal correlation functions in the field theory is equivalent to the string theory version of the quantum gravity partition function discussed above.

Using the gauge/gravity duality we may calculate the free energy F of the system from (4.36) by using the renormalized on-shell action. Similarly, we may calculate the mass M of black hole solutions in the bulk holographically by calculating the expectation value of the $(0,0)$ component of the boundary stress-energy tensor. As we will show in the final chapter, calculating certain thermodynamic quantities holographically does result in the appropriate thermodynamic relations being satisfied, for instance $F = M - TS$. The thermodynamic relations for general asymptotically AdS black holes were proved in [161].

This ends our brief review of black hole thermodynamics. It has been hugely successful in the past 40 years in deepening our understanding of some of the fundamental aspects of gravity and quantum theory. Yet many questions fundamental to our understanding still remain open. For one thing, despite many attempts, the question of what precisely are the black hole microstates responsible for the entropy remains unsolved. Similarly, ordinary matter thrown into a black hole results in the black hole emitting thermal radiation. This radiation is specified by one number, the temperature, meaning that most of the information contained in the matter is lost. This is known as the black hole information loss paradox and as yet there is no resolution of this paradox. As we have mentioned in the first Part of this thesis, deeper insight into this problem could come from holography giving us a better understanding of the precise process of black hole formation. This requires a good handle on the non-equilibrium dynamics in the dual quantum field theory, and a step in this direction may be afforded by the nPI effective actions discussed in Part I.

Chapter 5

Holography from generalized dimensional reduction

5.1 Introduction

In Section 4.1.1 we introduced the subject of Einstein-Maxwell-Dilaton (EMD) theories and their applications in the holographic modeling of various strongly-coupled condensed matter systems. Indeed these applications are rather rich, as evidenced by the fact that much work has been done in recent years in this direction, [96], [97], [98], [99], [100], [101], [102], [103], [104], [105], [106], [107], [108], [109], [162].

However, there are a number of challenges inherent in studying these theories holographically. Indeed, it is normally not clear off hand how to set up holography for these theories since many of the relevant solutions do not have AdS assymptotics. In this Chapter we will circumvent this problem for a class of such theories and compute the holographic dictionary.

As we have explained in Chapter 3, there are a number of steps which need to be taken in order to set up holography for a given theory. One must first understand the asymptotic structure of the field equations and identify where the source of the dual operator is located. One then evaluates the on-shell action and, since this is normally divergent, one needs to holographically renormalize by determining a set of (local covariant) counterterms. Finally, one may compute the renormalized one-point functions in the presence of sources (see [78], [163] for reviews). This procedure has been carried out early on for AdS gravity coupled to matter fields, [75] (see also [164],

[165]). More recently it has also been applied to a gravity-dilaton system with gravity coupled to a scalar field with exponential potential, [166], [167]. In fact, this case is associated with the near-horizon limit of the non-conformal branes studied in [168], [169]. It was realized later in [170] that the early case involving AdS gravity and the gravity-dilaton system above are, in fact, closely related. Indeed one is able to obtain the results in [167] from those in the earlier paper [75] *via* a *generalized dimensional reduction*. Simply put, this is a reduction where the dimension of the compactification manifold is allowed to be a *real* parameter, σ, and where the reduced theory depends smoothly on σ.

In this specific case, the authors of [167] showed that, by starting with $\mathrm{AdS}_{2\sigma+1}$ gravity and reducing it diagonally over the $T^{2\sigma-d}$ torus, one obtained the $(d+1)$-dimensional gravity-dilaton system, with an exponential potential $V(\phi) \sim \exp(-\delta\phi)$. This is, in fact, a consistent reduction in that the solutions of the equations of motion of the reduced theory are also solutions of the higher-dimensional theory. This is a beautiful result from the point of view of holography: due to the consistency, one may deduce not only the structure of the solutions of the equations of motion of the reduced theory from that of AdS gravity, but also all other results needed in order to set up holography.

The results readily show a smooth dependence on σ for $\sigma > d/2$, which is also apparent from the requirement that the torus dimension, $(2\sigma - d)$, be positive[1]. Hence, for $\sigma > d/2$, we may set up the holographic dictionary for this theory by means of the generalized dimensional reduction[2] by using the known holographic dictionary of AdS. The authors of [167] also performed a direct analysis of the field equations, etc. and these results are identical with those from the generalized reduction. This method was subsequently used in [171], where, starting from the higher-dimensional holographic dictionary in [172] and applying the generalized reduction, non-conforml branes managed to be probed.

This method has universal and far-reaching applications. Indeed, simply starting with a theory whose holographic dictionary we know, we can now set up holography for any other theory related to it *via* such a generalized consistent reduction. Thus we are allowed to start with the simplest example and take the higher-dimensional theory to be AdS gravity coupled to general matter (scalar fields, fermions, gauge fields, form fields).

[1] this translates into a constraint on the slope of the potential, $\delta^2 < 2/(d-1)$.

[2] The non-conformal branes correspond to specific rational values of σ.

We now carry out in detail the dimensional reduction procedure described above by starting with higher dimensional Einstein gravity and reducing it to an EMD theory. We show how to construct the holographic dictionary for the reduced theory from the higher-dimensional one. The lower-dimensional theory, thus, will contain Maxwell fields, and there are, in fact, a number of ways to achieve this. For one thing, we could begin with a Maxwell field in the upstairs theory. For another, we may replace the diagonal torus reduction by a general non-diagonal reduction. We may also do away with the torus altogether and reduce instead over an Einstein manifold,[3] which will result in a reduced theory with a potential containing two exponential terms. Indeed, another possibility is to have form fields in higher dimensions, [109].

In order to be as widely applicable as possible, we would like our reduced theory to have black hole solutions but with non-trivial Maxwell and scalar fields. It turns out that the theories obtained *via* a generalized dimensional reduction from a higher-dimensional AdS gravity (possibly coupled to a Maxwell field) are the same as the theories where non-extremal black hole solutions are explicitly known. For instance, they may fall in the class of EMD theories. As we have discussed in Section 4.1.1, these solutions often have unconventional asymptotics as the scalar field blows up at infinity. This is a common feature in dilatonic black holes and black branes and, hence, the computation of conserved charges in these theories is difficult. However, this is no longer the case from the point of view of the generalized dimensional reduction. Indeed, we can compute the reduced theory conserved charges from the holographic stress energy tensor and conserved current and these will, in principle, satisfy all expected thermodynamic identities since they originate from the higher-dimensional AdS gravity conserved charges, which are well-behaved, [161]. Thus, although the solutions in the reduced theory may look complicated, they actually originate from simpler higher-dimensional solutions.

We may also study the hydrodynamics in the reduced theory and compute transport coefficients, from the hydrodynamics of higher-dimensional AdS. In this way, the well-known AdS hydrodynamics may give insight into the new hydrodynamics. For instance, it was seen in [170] that the bulk to shear viscosity ratio has a special value for all backgrounds which asymptote to the non-conformal brane background. This was found to be ultimately due to the conformal symmetry of the higher-dimensional theory. In [173], this method was applied to a holographic model of QCD in order to compute transport coefficients in the quark-gluon plasma.

[3] This is consistent as long as we only keep the *breathing mode*: the mode parametrizing the overall size of the compact manifold

We will carry out this generalized reduction procedure below and the reduction we consider contains one gauge field and two Kaluza-Klein scalars. We will first derive the generalized Kaluza-Klein reduction map, followed by the holographic dictionary. Finally we will calculate the universal holographic hydrodynamics for this model.

5.1.1 Generalized dimensional reduction

Einstein gravity with a negative cosmological constant in $(2\sigma+1)$ dimensions is given by the action,

$$S_{(2\sigma+1)} = L_{AdS} \int \mathrm{d}^{2\sigma+1}x \sqrt{-g_{(2\sigma+1)}}\,[R + 2\sigma(2\sigma-1)], \tag{5.1}$$

where $L_{AdS} = \ell^{2\sigma-1}_{(2\sigma+1)}/(16\pi G_{2\sigma+1})$ and $\ell_{(2\sigma+1)}$ is the AdS radius.

We Kaluza-Klein reduce the above theory on the torus $\mathbf{T}^{(2\sigma-d)}$ with the following reduction ansatz,

$$\mathrm{d}s^2_{(2\sigma+1)} = \mathrm{d}s^2_{(d+1)}(\rho,z) + e^{2\phi_1(\rho,z)}\left(\mathrm{d}y - A_M \mathrm{d}x^M\right)^2 + e^{\frac{2\phi_2(\rho,z)}{(2\sigma-d-1)}}\mathrm{d}y^a \mathrm{d}y^a, \tag{5.2}$$

where we have included a Kaluza-Klein gauge field. We have $a = 1,\ldots,(2\sigma-d-1)$ and the coordinates (y, y^a) are periodic, with period $2\pi R$. We also identify $x^M = (\rho, z^i)$, where $M = 0,\ldots,d$ and $i = 1,...,d$. The action of the resulting lower-dimensional theory is

$$\begin{aligned} S_{(d+1)} \;\; = \;\; & L \int \mathrm{d}^{d+1}x \sqrt{-g_{(d+1)}} e^{\phi_1+\phi_2} \bigg[R + 2\partial\phi_1\partial\phi_2 + \frac{2\sigma-d-2}{2\sigma-d-1}(\partial\phi_2)^2 \\ & - \frac{1}{4} e^{2\phi_1} F_{MN}F^{MN} + 2\sigma(2\sigma-1) \bigg]. \end{aligned} \tag{5.3}$$

where $L = L_{AdS}(2\pi R)^{2\sigma-d}$. By working out the field equations of the lower-dimensional theory and seeing that they are equivalent to those of the original theory, we see that the reduction we have implemented is consistent.

To facilitate the derivation of the lower-dimensional action above, we in fact performed two successive reductions. The first was on the $(2\sigma-d-1)$-dimensional torus, which resulted in the action

$$S_{(d+2)} = L_{AdS}(2\pi R)^{2\sigma-d-1} \int \mathrm{d}^{d+2}x \sqrt{-g_{(d+2)}} e^{\phi_2} \left[R_{(d+2)} + \frac{2\sigma-d-2}{2\sigma-d-1}(\partial\phi_2)^2 + 2\sigma(2\sigma-1) \right], \tag{5.4}$$

where we used the fact that

$$R_{(2\sigma+1)} = R_{(d+2)} - 2\nabla^2\phi_2 - \frac{2\sigma-d}{2\sigma-d-1}(\partial\phi_2)^2. \tag{5.5}$$

The second reduction was on the y direction including a Kaluza-Klein gauge field. Note that,

$$\mathrm{d}s^2_{(d+2)} = \mathrm{d}s^2_{(d+1)} + e^{2\phi_1}\left(\mathrm{d}y - A_M \mathrm{d}x^M\right)^2, \tag{5.6}$$

and thus

$$R_{(d+2)} = R_{(d+1)} - 2\nabla^2\phi_1 - 2(\partial\phi_1)^2 - \frac{1}{4}e^{2\phi_1}F_{MN}F^{MN}. \tag{5.7}$$

Substituting the above into (5.4) then gives us (5.3). We may now make contact with the action for non-conformal branes derived in [167], [169], namely,

$$S = L\int \mathrm{d}^{d+1}x\sqrt{-g_{d+1}}e^{\phi}\left(R + \frac{2\sigma-d-1}{2\sigma-d}(\partial\phi)^2 + 2\sigma(2\sigma-1)\right). \tag{5.8}$$

We simply set $F_{MN} = 0$, and rescale the scalars as

$$\phi_1 = \frac{\phi}{(2\sigma-d)}, \qquad \phi_2 = \frac{\phi(2\sigma-d-1)}{(2\sigma-d)}, \tag{5.9}$$

with $\phi = (\phi_1 + \phi_2)$.

To get a more natural expression for the reduced action (5.3) we recombine the two scalars ϕ_1 and ϕ_2 as,

$$\psi = \phi_1 + \phi_2, \quad \zeta = (2\sigma-d-1)\phi_1 - \phi_2. \tag{5.10}$$

This combination gives a simple expression for the determinant of the torus metric, namely $\sqrt{g_{T^{2\sigma-d}}} = e^{\psi}$. In terms of these new scalars the reduction of the metric is,

$$\mathrm{d}s^2_{(2\sigma+1)} = \mathrm{d}s^2_{(d+1)} + e^{2\frac{(\psi+\zeta)}{(2\sigma-d)}}(\mathrm{d}y - A_M\mathrm{d}x^M)^2 + e^{\frac{2\psi}{(2\sigma-d)} - \frac{2\zeta}{(2\sigma-d)(2\sigma-d-1)}}\mathrm{d}y^a\mathrm{d}y_a, \tag{5.11}$$

and the reduced action becomes

$$\begin{aligned} S_{(d+1)} \quad = \quad & L\int \mathrm{d}^{d+1}x\sqrt{-g_{(d+1)}}e^{\psi}\left[R - \frac{1}{(2\sigma-d)(2\sigma-d-1)}(\partial\zeta)^2 \right. \\ & \left. + \frac{2\sigma-d-1}{2\sigma-d}(\partial\psi)^2 - \frac{1}{4}e^{\frac{2(\zeta+\psi)}{(2\sigma-d)}}F_{MN}F^{MN} + 2\sigma(2\sigma-1)\right]. \end{aligned} \tag{5.12}$$

As an aside we now note that we may always consistently set $\zeta = 0$ when $F_{MN} = 0$. Indeed, this is obvious from the equation of motion for ζ,

$$\nabla[e^{\psi}\partial\zeta] = \frac{1}{4}(2\sigma - d - 1)e^{\psi}e^{\frac{2(\zeta+\psi)}{(2\sigma-d)}}F_{MN}F^{MN}. \tag{5.13}$$

The action with both ζ and F set to zero is precisely that given above, with the identification $\psi = \phi$.

We now give the equations of motion for the remaining fields.

The equation of motion for ψ is,

$$\begin{aligned}\nabla[e^{\psi}\partial\psi] \;\; &= \;\; \frac{2\sigma-d}{2(2\sigma-d-1)}e^{\psi}\left[R - \frac{1}{(2\sigma-d)(2\sigma-d-1)}(\partial\zeta)^2 + \frac{2\sigma-d-1}{2\sigma-d}(\partial\psi)^2\right. \\ &\quad \left. -\frac{1}{4}\frac{(2\sigma-d+2)}{(2\sigma-d)}e^{\frac{2(\zeta+\psi)}{(2\sigma-d)}}F_{MN}F^{MN} + 2\sigma(2\sigma-1)\right].\end{aligned} \tag{5.14}$$

The equation of motion for the metric is,

$$\begin{aligned}& R_{MN} - \frac{1}{2}g_{MN}R - \frac{1}{(2\sigma-d)(2\sigma-d-1)}\left(-\frac{1}{2}g_{MN}(\partial\zeta)^2 + \partial_M\zeta\partial_N\zeta\right) \\ + \;\; & \frac{1}{2\sigma-d}\left(\frac{1}{2}(2\sigma-d+1)g_{MN}(\partial\psi)^2 - \partial_M\psi\partial_N\psi\right) \\ - \;\; & \frac{1}{4}e^{\frac{2(\zeta+\psi)}{2\sigma-d}}\left(-\frac{1}{2}g_{MN}F_{PQ}F^{PQ} + 2F^{Q}_{M}F_{NQ}\right) - \sigma(2\sigma-1)g_{MN} \\ - \;\; & \nabla_N\nabla_M\psi + g_{MN}\Box\psi = 0.\end{aligned} \tag{5.15}$$

Finally, the equation of motion for the gauge field is,

$$\nabla_M\left(e^{\frac{2(\zeta+\psi)}{(2\sigma-d)}}F^{MN}\right) = 0. \tag{5.16}$$

The above equations of motion point to certain special values of σ. For $2\sigma = (d+1)$ we see from (5.11) that there is no additional scalar field ζ and the reduction is simply along a circle. For $2\sigma = d$, there is no reduction at all and we are left with only the metric in the original conformal case. We may also set the gauge field to zero while keeping both the scalars.

Next, we write the action in the Einstein frame, by conformally rescaling the metric as follows,

$$g_{MN} = e^{-2\psi/(d-1)}\bar{g}_{MN}. \tag{5.17}$$

This then yields the action,

$$\begin{aligned}S_{(d+1)} &= L\int \mathrm{d}^{d+1}x\sqrt{-\bar{g}_{(d+1)}}\Big[\bar{R}-\frac{1}{(2\sigma-d)(2\sigma-d-1)}(\partial\zeta)^2+\frac{1-2\sigma}{(2\sigma-d)(d-1)}(\partial\psi)^2 \\ &\quad -\frac{1}{4}e^{\frac{2(\zeta+\psi)}{(2\sigma-d)}+\frac{2\psi}{d-1}}F_{MN}F^{MN}+e^{-\frac{2\psi}{d-1}}2\sigma(2\sigma-1)\Big], \end{aligned} \tag{5.18}$$

where we can see that the potential is independent of ζ. In order to obtain canonically normalized scalar kinetic terms we in addition rescale as follows,

$$\psi=\sqrt{\frac{(2\sigma-d)(d-1)}{2(2\sigma-1)}}\bar{\psi}, \qquad \zeta=\sqrt{\frac{(2\sigma-d)(2\sigma-d-1)}{2}}\bar{\zeta}. \tag{5.19}$$

This rescaling implicitly assumes that $2\sigma>(d+1)$, since ζ has a negative kinetic term whenever $2\sigma<(d+1)$ and, therefore, cannot be canonically normalized. For such values of σ, one would not expect that the scalar ζ is part of a physical compactification, as we discuss next[4].

We may now finally write the action in the Einstein frame,

$$\begin{aligned}S_{(d+1)} &= L\int \mathrm{d}^{d+1}x\sqrt{-\bar{g}_{(d+1)}}\Big[\bar{R}-\frac{1}{2}(\partial\bar{\psi})^2-\frac{1}{2}(\partial\bar{\zeta})^2+2\sigma(2\sigma-1)e^{-\bar{\psi}\sqrt{\frac{2(2\sigma-d)}{(d-1)(2\sigma-1)}}} \\ &\quad -\frac{1}{4}e^{\sqrt{\frac{2(2\sigma-1)}{(d-1)(2\sigma-d)}}\bar{\psi}+\sqrt{\frac{2(2\sigma-d-1)}{2\sigma-d}}\bar{\zeta}}F_{MN}F^{MN}\Big]. \end{aligned} \tag{5.20}$$

Brane interpretation

We will now analyze whether our $(d+1)$-dimensional reduced action (5.20) has a brane interpretation. More specifically, we would like to know whether (5.20) originates from sphere reductions of decoupled Dp-brane, M-brane and string solutions and, specifically, consistent truncations of these solutions.

The simpler case, containing only one scalar ψ and the metric and no gauge field, was already analyzed in [167], [169]. We in turn truncate our action (5.20) to include just these fields, and fix the dimension as

$$2\sigma=d+\frac{(p-3)^2}{(5-p)}, \tag{5.21}$$

[4]It is interesting to note that a similar action was recently discussed in [174], in the context of p-branes with curved worldvolumes. However, the scalar potentials in this case are different, and one cannot interpret the action given here in terms of branes with curved worldvolumes.

where $d = p+1$ and $p \neq 5$. The action we obtain thus does have a brane interpretation. Indeed it originates from the sphere reduction of the corresponding decoupled p-brane background. We may then interpret the metric and scalar field ψ to be dual to the world volume theory energy momentum tensor and running coupling respectively.

The general setup we have introduced above in fact contains naturally a number of cases. Namely, the three conformal cases of D3-branes, M2-branes and M5-branes and also fundamental strings, D0-branes, D1-branes, D2-branes and D4-branes. The conformal M-branes arise from setting $2\sigma = d$, while the fundamental strings correspond to $p = 1$. This setup excludes, however, 5-branes and Dp-branes with $p \geq 6$.

The fundamental strings, D1-branes and D4-branes are non-conformal, and require us to set

$$(2\sigma - d) = 1. \tag{5.22}$$

Hence, in these cases, we can view our reduced action (5.20) as an S^1 reduction of a conformal theory. Also, since we are simply doing this one standard KK reduction over a circle, there is no second scalar field ζ and the gauge field is simply the Kaluza-Klein gauge field of the reduction. It corresponds to the dual field theory conserved current.

For the D2-branes we have

$$(2\sigma - d) = 1/3 < 1, \tag{5.23}$$

and we now have the extra scalar field ζ. In addition, it will have a negative kinetic term, in the Einstein frame, in the action (5.12). Or, viewed differently, we will find that the action (5.20) will be complex due to the fact that the coefficient of F^2 becomes complex.

From the analysis of [169], it seems that decoupled D2-branes, reduced on an S^6, can be consistently truncated to the $ISO(7)$ gauged supergravity theory [175], [176]. The dual field theory operators of these gauged supergravity fields belong to the same supermultiplet as the stress energy tensor. However, since this theory doesn't have any scalars with negative kinetic terms, ζ has no support in this framework and, hence, it does not apply in our case. Indeed, ζ seems to have no support in the dual supersymmetric gauge theory either.

Finally, for the case of D0-branes, we have,

$$(2\sigma - d) = \frac{9}{5}, \tag{5.24}$$

and this time ζ has a positive kinetic term. These branes occur in ten-dimensional type IIA string theory and if we reduce it on an S^8 there seems to be nothing to forbid us from interpreting ζ as one of the scalars which arises. However, this is not a sufficient condition for actually being able to identify ζ, or the gauge field, with fields in the reduced theory.

There is another natural way to proceed in the search for a brane interpretation for our reduced $d+1$-dimensional theory. One can always reduce, on a circle, the action for a non-conformal p-brane in $d'+1 \equiv (p+2)$ dimensions, such that the reduced theory, in one less dimension, contains an additional scalar and gauge field. The dual theory in this case would be the KK reduction of the non-conformal p-brane theory. This would require that σ and p satisfy:

$$2\sigma = p+2+\frac{(p-3)^2}{(5-p)}, \tag{5.25}$$

with $p \neq 5$.

5.1.2 Holographic dictionary

We would now like to study the dimensionally reduced theory (5.20) holographically. In order to do this we must write down the appropriate holographic dictionary, which, following the discussion in Section 3.2, means we need to analyze the asymptotic structure of the field equations. In general this is rather difficult to achieve. However, since we obtained the reduced theory from a generalized consistent reduction of higher-dimensional Einstein gravity, all solutions of the reduced theory descend from solutions of (5.1).

We have already written down the most general asymptotic solution of this theory in (3.16), namely, the Fefferman-Graham expansion,

$$\begin{aligned} \mathrm{d}s^2_{(2\sigma+1)} &= \frac{\mathrm{d}\rho^2}{4\rho^2}+\frac{1}{\rho}g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu, \\ g_{\mu\nu} &= g_{(0)\mu\nu}+\rho g_{(2)\mu\nu}+\cdots+\rho^\sigma\left(g_{(2\sigma)\mu\nu}+h_{(2\sigma)\mu\nu}\log\rho\right)+\cdots. \end{aligned} \tag{5.26}$$

Here, as we have already seen, $g_{(0)\mu\nu}$ is the source in terms of which all other coefficients, except for $g_{(2\sigma)\mu\nu}$, are completely determined. Only the trace and divergence of $g_{(2\sigma)\mu\nu}$ are determined locally in terms of the source. The logarithmic terms $h_{(2\sigma)}$ are non-zero only for integer values of σ.

If we now wish to use this expansion to obtain the general asymptotic solution of the reduced theory (5.12), we need to look at the subset of asymptotic solutions that fits the specific form of the reduction we have implemented, (5.11).

We expand the $(d+1)$-dimensional metric in the Fefferman-Graham form, as above. The scalar fields and the gauge field can be expanded as

$$\begin{aligned} e^{\frac{2\psi}{(2\sigma-d)}} &= \frac{1}{\rho} e^{\frac{2\kappa}{(2\sigma-d)}}, \\ \kappa &= \kappa_{(0)} + \rho\kappa_{(2)} + \cdots + \rho^{\sigma}\kappa_{(2\sigma)}, \\ \zeta &= \zeta_{(0)} + \rho\zeta_{(2)} + \cdots + \rho^{\sigma}\zeta_{(2\sigma)}, \\ A_i(\rho,z) &= A_{i(0)}(z) + \rho A_{i(2)}(z) + \ldots + \rho^{\sigma} A_{i(2\sigma)}(z) + \ldots . \end{aligned} \tag{5.27}$$

Once again, the log terms are present when σ is an integer, but, since we are primarily interested in the case where σ is non-integral, these are suppressed. Here $\kappa_{(0)}, \zeta_{(0)}$ are the sources of the dual scalar operators $\mathcal{O}_\psi$ and $\mathcal{O}_\zeta$. $A_{(0)}$ is the source of the boundary conserved current, J^i. Once again, the sub-leading coefficients depend locally on the sources, up to the term which is related to the vev of the dual operator. We know the precise way in which the sub-leading coefficients in the metric expansion, (5.26), depend locally on the source $g_{(0)}$, (see appendix A of [75]). From this we can work out the dependence on the sources of the sub-leading terms in the expansions for the other fields, but we will omit this as we will not need these relations here.

We have now obtained the asymptotic solution. As we have discusses in Section 3.2, the on-shell action is intrinsically divergent. Hence, in order to remove the divergences we need the renormalized on-shell action and to do this we need to compute the local boundary counterterms. In principle this is a difficult problem, however, in this case we may do it easily using the method of generalized dimensional reduction, [170].

In the higher-dimensional theory, given σ, we choose any half-integer $\tilde{\sigma}$ which is bigger than σ. We then determine the $[\sigma]+1$ most singular $\mathrm{AdS}_{(2\tilde{\sigma}+1)}$-counterterms as a function of $\tilde{\sigma}$, where $[\sigma]$ denotes the largest integer less than or equal to σ. When σ is an integer one of these counterterms is logarithmic. Reducing these $\mathrm{AdS}_{(2\tilde{\sigma}+1)}$-counterterms and replacing $\tilde{\sigma}$ by σ, yields the counterterms appropriate for the reduced theory, (5.12).

As an example let us consider the counterterm action for $1 < \sigma < 2$, for which we only need two counterterms. From the general expression (3.20), we see that the two most singular counterterms in $\mathrm{AdS}_{2\tilde{\sigma}+1}$ defined on a regulating hypersurface $\rho = \epsilon$ are given by (see appendix B of [75])[5]

$$S^{ct}_{(2\tilde{\sigma})} = L_{AdS} \int_{\rho=\epsilon} \mathrm{d}^{2\tilde{\sigma}} x \sqrt{-\gamma_{2\tilde{\sigma}}} \left[2(2\tilde{\sigma}-1) + \frac{1}{2\tilde{\sigma}-2} \hat{R}[\gamma_{2\tilde{\sigma}}] \right], \tag{5.28}$$

where $\gamma_{2\tilde{\sigma}ij}$ is the induced metric on the $(2\tilde{\sigma})$-dimensional hypersurface and $\hat{R}[\gamma_{2\tilde{\sigma}}]$ the corresponding curvature. The counterterm action to (5.12), for $1 < \sigma < 2$, is then given by reducing (5.28) to d dimensions and replacing $\tilde{\sigma}$ with σ,

$$\begin{aligned} S^{ct}_{(d)} \;\; = \;\; & L \int_{\rho=\epsilon} \mathrm{d}^d x \sqrt{-\gamma_d}\, e^{\psi} \left[2(2\sigma-1) + \frac{1}{2\sigma-2} \left(\hat{R}_d + \frac{2\sigma-d-1}{2\sigma-d} (\partial\psi)^2 \right.\right. \\ & \left.\left. - \frac{1}{(2\sigma-d-1)(2\sigma-d)} (\partial\zeta)^2 - \frac{1}{4} e^{\frac{2(\zeta+\psi)}{2\sigma-d}} F_{ij} F^{ij} \right) \right]. \end{aligned} \tag{5.29}$$

This result matches early results for $d = 3$, [177]. When $\sigma > 2$ more gravitational counterterms need to be included and the reduction is then carried out in the same way.

Now that we have the counterterms we can compute the renormalized on-shell action for the reduced theory, and, as explained in the previous Chapter, from it the correlation functions of the dual operators. For instance, we compute the holographic one-point functions by functionally differentiating the renormalized on-shell action, S_{ren}, as in (3.29). However, in this case we may equivalently use the generalized dimensional reduction, which turns out to be considerably easier. We simply need to reduce the formula for the higher-dimensional 1-point function, (3.24), given exactly in [75] as,

$$\langle T_{\mu\nu} \rangle_{2\sigma} = \frac{2}{\sqrt{-g_{(0),2\sigma}}} \frac{\delta S_{ren}}{\delta g^{\mu\nu}_{(0)}} = 2\sigma L_{AdS} g_{(2\sigma)\mu\nu} + \ldots, \tag{5.30}$$

where the ellipses denote the contact terms which are only present for σ integer. They do not play an important role in the discussion here and hence we will suppress them.

Now, to find the expectation values of operators in the lower-dimensional field theory, we simply reduce the higher-dimensional stress energy tensor expectation value above. To facilitate this we write the higher-dimensional expression in terms of components longitudinal and transverse to the reduction torus, also taking into account the additional prefactor $(2\pi R)^{2\sigma-d}$ of the lower-dimensional action in (5.3). This is

[5]Note that convention for the curvature tensor used in [75] has the opposite sign.

essentially as a result of the integration over the torus and because the determinant of the metric in the definition of the vev changes: $\sqrt{g_{(0),d}} = e^{-\kappa_{(0)}}\sqrt{g_{(0),2\sigma}}$. As a result we define,

$$\langle t_{\mu\nu}\rangle_d \equiv e^{\kappa_{(0)}}(2\pi R)^{2\sigma-d}\langle T_{\mu\nu}\rangle_{2\sigma}\,, \tag{5.31}$$

where T is the higher-dimensional stress energy tensor and t the lower-dimensional one. Then one obtains the following for the various components of the stress-energy tensor transverse and longitudinal to the reduction torus,

$$\begin{aligned}
\langle t_{ij}\rangle_d &= 2\sigma L e^{\kappa_{(0)}} g_{(2\sigma)ij} + \cdots \\
&= 2\sigma L\left[e^{\kappa_{(0)}} g_{(2\sigma)ij} + 2e^{\frac{(2\sigma-d+2)\kappa_{(0)}+2\zeta_{(0)}}{2\sigma-d}}\left(A_{(i(0)}A_{j)(2\sigma)} + \frac{A_{i(0)}A_{j(0)}}{2\sigma-d}\left(\kappa_{(2\sigma)}+\zeta_{(2\sigma)}\right)\right)\right], \\
\langle t_{iy}\rangle_d &= 2\sigma L e^{\kappa_{(0)}} g_{(2\sigma)iy} + \cdots \\
&= -2\sigma L e^{\frac{(2\sigma-d+2)\kappa_{(0)}+2\zeta_{(0)}}{2\sigma-d}}\left(A_{i(2\sigma)} + \frac{2}{2\sigma-d}\left(\kappa_{(2\sigma)}+\zeta_{(2\sigma)}\right)A_{i(0)}\right), \\
\langle t_{yy}\rangle_d &= 2\sigma L e^{\kappa_{(0)}} g_{(2\sigma)yy} + \cdots \\
&= \frac{4\sigma L}{(2\sigma-d)} e^{\frac{(2\sigma-d+2)\kappa_{(0)}+2\zeta_{(0)}}{2\sigma-d}}\left(\kappa_{(2\sigma)}+\zeta_{(2\sigma)}\right) + \ldots \\
&\equiv -e^{\frac{2}{(2\sigma-d)}\left(\kappa_{(0)}+\zeta_{(0)}\right)}\langle \mathcal{O}_1\rangle_d, \\
\langle t_{ab}\rangle_d &= 2\sigma L e^{\kappa_{(0)}} g_{(2\sigma)ab} + \cdots \\
&= \frac{4\sigma L}{2\sigma-d} e^{\frac{1}{2\sigma-d}\left((2\sigma-d+2)\kappa_{(0)}-\frac{2}{(2\sigma-d-1)}\zeta_{(0)}\right)}\left(\kappa_{(2\sigma)} - \frac{1}{(2\sigma-d-1)}\zeta_{(2\sigma)}\right)\delta_{ab} + \ldots \\
&\equiv -e^{\frac{2}{2\sigma-d}\left(\kappa_{(0)}-\frac{1}{(2\sigma-d-1)}\zeta_{(0)}\right)}\langle \mathcal{O}_2\rangle_d \delta_{ab},
\end{aligned} \tag{5.32}$$

where the ellipses again denote contact terms which depend on curvatures of the boundary metric $g_{(0)ij}$ and derivatives of $\kappa_{(0)}$ and $\zeta_{(0)}$.

From the expressions above we may read off,

$$\begin{aligned}
\langle \mathcal{O}_1\rangle_d &= -\frac{4\sigma L}{2\sigma-d} e^{\kappa_{(0)}}\left(\kappa_{(2\sigma)}+\zeta_{(2\sigma)}\right) + \ldots, \\
\langle \mathcal{O}_2\rangle_d &= -\frac{4\sigma L}{2\sigma-d} e^{\kappa_{(0)}}\left(\kappa_{(2\sigma)} - \frac{1}{(2\sigma-d-1)}\zeta_{(2\sigma)}\right) + \ldots\,.
\end{aligned} \tag{5.33}$$

As we can see, the reduction has resulted in a symmetric tensor operator t_{ij}, a vector operator t_{iy} and two scalar operators, $\mathcal{O}_1$ and $\mathcal{O}_2$. This fits what we expected from the field content of the reduced theory.

The dual theory stress-energy tensor, conserved current and scalar operators are given by linear combinations of these operators. Hence, we need not worry about their normalization at this point. In order to determine which linear combinations form the d-dimensional theory operators, we may follow the usual route of varying the renormalized on-shell action with respect to the appropriate d-dimensional sources. However, a simpler way to do this is to reduce the higher-dimensional Ward identities. With the benefit of hindsight, let us introduce the following linear combinations of the scalar operators,

$$\begin{aligned}
\langle \mathcal{O}_\psi \rangle_d &= \frac{1}{(2\sigma - d)}[(2\sigma - d - 1)\langle \mathcal{O}_2 \rangle_d + \langle \mathcal{O}_1 \rangle_d], \\
\langle \mathcal{O}_\zeta \rangle_d &= \frac{1}{(2\sigma - d)}[\langle \mathcal{O}_1 \rangle_d - \langle \mathcal{O}_2 \rangle_d],
\end{aligned} \tag{5.34}$$

where we are anticipating that the field ψ will in fact act as a source for $\mathcal{O}_\psi$ whilst the field ζ will source $\mathcal{O}_\zeta$. Indeed, in the $\zeta = 0$ limit we have $\langle \mathcal{O}_1 \rangle_d = \langle \mathcal{O}_2 \rangle_d$, with the operator $\langle \mathcal{O}_\phi \rangle_d$ of [167] having the expectation value

$$\langle \mathcal{O}_\phi \rangle_d = \langle \mathcal{O}_1 \rangle_d = \langle \mathcal{O}_2 \rangle_d. \tag{5.35}$$

The conformal Ward identity of the higher-dimensional theory, given in (3.25), is,

$$\langle T^\mu_\mu \rangle_{2\sigma} = \mathcal{A}_{2\sigma}. \tag{5.36}$$

This can be reduced to

$$\begin{aligned}
&\langle t^i_i \rangle_d - 2A^i_{(0)} \langle t_{iy} \rangle_d - (2\sigma - d - 1)\langle \mathcal{O}_2 \rangle_d - \left(1 + e^{\frac{2(\kappa_{(0)} + \zeta_{(0)})}{2\sigma - d}} A_{(0)i} A^i_{(0)}\right) \langle \mathcal{O}_1 \rangle_d \\
&= e^{\kappa_{(0)}} (2\pi R)^{2\sigma - d} \mathcal{A}_{2\sigma} \equiv \mathcal{A}_d\,.
\end{aligned} \tag{5.37}$$

If we now write,

$$\begin{aligned}
&\langle J_i \rangle_d = \langle t_{iy} \rangle_d + A_{(0)i} \langle t_{yy} \rangle_d\,, \\
&\langle T_{ij} \rangle_d = \langle t_{ij} \rangle_d + \left(A_{(0)i} \langle J_j \rangle + A_{(0)j} \langle J_i \rangle\right) + A_{(0)i} A_{(0)j} e^{\frac{2(\kappa_{(0)} + \zeta_{(0)})}{2\sigma - d}} \langle \mathcal{O}_1 \rangle_d\,,
\end{aligned} \tag{5.38}$$

so that

$$\langle J_i\rangle_d = -2\sigma L e^{\frac{1}{2\sigma-d}\left((2\sigma-d+2)\kappa_{(0)}+2\zeta_{(0)}\right)} A_{(2\sigma)i} + \cdots, \qquad (5.39)$$
$$\langle T_{ij}\rangle_d = 2\sigma L e^{\kappa_{(0)}} g_{(2\sigma)ij} + \cdots,$$

the dilatation Ward identity becomes simply,

$$\langle T^i_i\rangle_d - (2\sigma-d)\langle\mathscr{O}_\psi\rangle_d = \mathscr{A}_d, \qquad (5.40)$$

and we can see that $\mathscr{O}_\zeta$ does not contribute.

Similarly, the conservation equation for the higher-dimensional stress energy tensor reduces to,

$$\nabla^i\langle T_{ij}\rangle_d + \partial_j\kappa_{(0)}\langle\mathscr{O}_\psi\rangle_d + \partial_j\zeta_{(0)}\langle\mathscr{O}_\zeta\rangle_d - F^i_{(0)j}\langle J_i\rangle_d = 0, \qquad (5.41)$$

and the divergence equation for a current,

$$\nabla^i\langle J_i\rangle_d = 0. \qquad (5.42)$$

This is the linear combination of operators which we were looking for.

The first divergence equation is simply the standard diffeomorphism Ward identity for a theory whose stress energy tensor is T_{ij} and where the other operators are defined in the usual way terms of the generating functional W:

$$\langle J^i\rangle_d = -\frac{1}{\sqrt{g_{(0)}}}\frac{\delta W}{\delta A_{(0)i}}; \qquad \langle\mathscr{O}_\psi\rangle_d = -\frac{1}{\sqrt{g_{(0)}}}\frac{\delta W}{\delta\kappa_{(0)}}; \qquad \langle\mathscr{O}_\zeta\rangle_d = -\frac{1}{\sqrt{g_{(0)}}}\frac{\delta W}{\delta\zeta_{(0)}}. \qquad (5.43)$$

We can thus clearly see that the non-normalizable modes of ψ and ζ do indeed source $\mathscr{O}_\psi$ and $\mathscr{O}_\zeta$ and that $A_{(0)i}$ sources the conserved current J^i. Indeed we would get the same relations by following the standard procedure, namely, varying the renormalized bulk on-shell action.

5.1.3 Black branes

The setup discussed in the previous Section can be realized by means of black branes, as we will elaborate on now. This is in view of the fact that we are ultimately interested in studying the hydrodynamics of the boundary theory dual to the reduced theory we have obtained. Indeed, the authors of [178] studied the boosted black D3 brane geometry and succeeded in deriving conformal hydrodynamics on the boundary by studying the long wavelength fluctuations of this geometry. In a similar way the authors of [170] studied the non-conformal boosted black Dp branes and derived the universal hydrodynamics on the boundary by studying the long wavelength fluctuations of this geometry. This was done by carrying out the generalized dimensional reduction on the conformal black brane solution in $(2\sigma+1)$ dimensions:

$$\begin{aligned} \mathrm{d}s^2_{(2\sigma+1)} &= \frac{\mathrm{d}\rho^2}{4\rho^2 f(\rho)} + \frac{1}{\rho}\left[-f(\rho)\mathrm{d}t'^2 + \mathrm{d}y'^2 + \mathrm{d}z_r \mathrm{d}z^r + \mathrm{d}y_a \mathrm{d}y^a\right], \\ f(\rho) &= 1 - m^{2\sigma}\rho^\sigma, \end{aligned} \tag{5.44}$$

where (y, y^a, z^r) encompass all the transverse coordinates ($a = d+1, \ldots, 2\sigma - 1$). For 2σ an integer the above is an Einstein metric with negative curvature and, as such, it has an event horizon (situated at $\rho = m^{-2}$) as well as a Hawking temperature T and Bekenstein-Hawking entropy density s. These are given by

$$T = \frac{m\sigma}{2\pi}, \qquad s = 4\pi L_{AdS} m^{2\sigma-1}. \tag{5.45}$$

We may introduce a wave on the above background by Lorentz transforming as follows: $t = \cosh\omega\ t' - \sinh\omega\ y'$, $y = \cosh\omega\ y' - \sinh\omega\ t'$. This results in,

$$\begin{aligned} \mathrm{d}s^2_{(2\sigma+1)} &= \frac{\mathrm{d}\rho^2}{4\rho^2 f(\rho)} - \rho^{-1}K(\rho)^{-1} f(\rho)\mathrm{d}t^2 + \frac{K(\rho)}{\rho}\left[\mathrm{d}y - \left((K'(\rho))^{-1} - 1\right)\mathrm{d}t\right]^2 \\ &\quad + \rho^{-1}\mathrm{d}z_r \mathrm{d}z^r + \rho^{-1}\mathrm{d}y_a \mathrm{d}y^a, \\ f(\rho) &= 1 - m^{2\sigma}\rho^\sigma, \qquad K(\rho) = \left(1 + Q\rho^\sigma\right), \\ \left(K'(\rho)\right)^{-1} &= \left(1 - \bar{Q}\rho^\sigma K(\rho)^{-1}\right), \end{aligned} \tag{5.46}$$

where

$$Q = m^{2\sigma}\sinh^2\omega, \qquad \bar{Q} = m^{2\sigma}\sinh\omega\cosh\omega. \tag{5.47}$$

We may remove the wave by setting $\omega = 0$, while for extremality we simply take the limit $m \to 0$ with $\omega \to \infty$ and Q finite.

By performing a standard non-extremal intersection of one of the conformal branes (D3, M2, M5) with a wave (see for example [179], [180]), taking a decoupling limit (which focuses the geometry near the brane) and then reducing over the transverse sphere, we obtain the solution above. This is valid for $(2\sigma+1)$ an integer. We will discuss the physical meaning of non-integer $(2\sigma+1)$ in the next Section.

We are interested in obtaining the boundary universal hydrodynamics dual to our reduced theory, (5.20). With this in mind, we begin by dimensionally reducing the above metric along the lines specified in subsection 5.1.1. We start by periodically identifying the coordinates (y, y^a) with period $2\pi R$ and rewrite the geometry as,

$$\begin{aligned} \mathrm{d}s^2_{(2\sigma+1)} &= \frac{\mathrm{d}\rho^2}{4\rho^2 f(\rho)} + \frac{1}{\rho}\left(\mathrm{d}z^r \mathrm{d}z_r - \mathrm{d}t^2\right) + \frac{1}{\rho}\left(1 - K(\rho)^{-1} f(\rho)\right)\mathrm{d}t^2 \\ &\quad + \frac{1}{\rho}\mathrm{d}y^a \mathrm{d}y_a + \frac{K(\rho)}{\rho}\left[\mathrm{d}y - \left((K'(\rho))^{-1} - 1\right)\mathrm{d}t\right]^2. \end{aligned} \tag{5.48}$$

We now boost this geometry along the non-compact boundary dimensions, $z^i = (t, z^r)$, with boost parameter $\hat{u}_i$. This results in

$$\begin{aligned} \mathrm{d}s^2_{(2\sigma+1)} &= \frac{\mathrm{d}\rho^2}{4\rho^2 f(\rho)} + \frac{1}{\rho}\left(\mathrm{d}z^i \mathrm{d}z_i\right) + \frac{1}{\rho}\left(1 - K(\rho)^{-1} f(\rho)\right)\hat{u}_i \hat{u}_j \mathrm{d}z^i \mathrm{d}z^j \\ &\quad + \frac{1}{\rho}\mathrm{d}y^a \mathrm{d}y_a + \frac{K(\rho)}{\rho}\left[\mathrm{d}y - \left((K'(\rho))^{-1} - 1\right)\hat{u}_i \mathrm{d}z^i\right]^2. \end{aligned} \tag{5.49}$$

Note that the fluid velocity $\hat{u}^i$ squares to -1 with η_{ij}, i.e. $\eta_{ij}\hat{u}^i\hat{u}^j = -1$, and *not* with the 2σ-dimensional boundary metric. We will also include an external, uniform gauge field, $A_{(0)i}\mathrm{d}z^i$, in the setup. This is achieved by performing a coordinate transformation on y in the metric above: $\mathrm{d}y \to \mathrm{d}y + A_{(0)i}\mathrm{d}z^i$. If we allow the temperature, charge, fluid velocity and external gauge field to become position-dependent the metric no longer *a. priori* satisfies the field equations, but must be corrected at each order.

Upon reducing (5.49) we get,

$$\mathrm{d}s^2_{(d+1)} = \frac{\mathrm{d}\rho^2}{4\rho^2 f(\rho)} + \frac{1}{\rho}\left(\mathrm{d}z^i \mathrm{d}z_i\right) + \frac{1}{\rho}\left(1 - K(\rho)^{-1} f(\rho)\right)\hat{u}_i \hat{u}_j \mathrm{d}z^i \mathrm{d}z^j, \tag{5.50}$$

where we may read off the scalar fields and gauge field as,

$$e^{\frac{2\phi_2}{(2\sigma-d-1)}} = \frac{1}{\rho}, \qquad e^{2\phi_1} = \frac{K(\rho)}{\rho}, \tag{5.51}$$

and

$$A = \left[A_{(0)i} + \left((K'(\rho))^{-1} - 1\right)\hat{u}_i\right]\mathrm{d}z^i, \tag{5.52}$$

respectively.

Rewriting the scalar fields in terms of ψ and ζ we obtain,

$$e^{\psi} = \frac{1}{\rho^{\sigma - d/2}} K(\rho)^{1/2}, \qquad e^{\zeta} = K(\rho)^{\frac{1}{2}(2\sigma - d - 1)}. \tag{5.53}$$

We now rewrite quantities in Fefferman-Graham coordinates (see [170] for the derivation of the coordinate transformation). The reduced metric in these coordinates is,

$$\begin{aligned} ds^2_{(d+1)} &= \frac{d\tilde{\rho}^2}{4\tilde{\rho}^2} + \frac{1}{\tilde{\rho}}\left(1+\frac{m^{2\sigma}\tilde{\rho}^{\sigma}}{4}\right)^{\frac{2}{\sigma}} dz_i dz^i \\ &\quad + \frac{1}{\tilde{\rho}}\left(1+\frac{m^{2\sigma}\tilde{\rho}^{\sigma}}{4}\right)^{\frac{2}{\sigma}} \left[1-K(\rho(\tilde{\rho}))^{-1}f(\rho(\tilde{\rho}))\right]\hat{u}_i\hat{u}_j dz^i dz^j, \end{aligned} \tag{5.54}$$

with the scalar fields and gauge field given by

$$e^{\frac{2\phi_2}{(2\sigma-d-1)}} = \frac{1}{\tilde{\rho}}\left(1+\frac{m^{2\sigma}\tilde{\rho}^{\sigma}}{4}\right)^{\frac{2}{\sigma}}, \qquad e^{2\phi_1} = \frac{K(\rho(\tilde{\rho}))}{\tilde{\rho}}\left(1+\frac{m^{2\sigma}\tilde{\rho}^{\sigma}}{4}\right)^{\frac{2}{\sigma}}, \tag{5.55}$$

and

$$A = A_{(0)i}dz^i + \left[\left(K^{'}(\rho(\tilde{\rho}))\right)^{-1} - 1\right]\hat{u}_i dz^i, \tag{5.56}$$

respectively. Note that the term in square brackets in (5.56) above goes to zero as $\tilde{\rho}\to 0$ and, hence, $A_{i(0)}$ is the source for the gauge field.

Rewriting the scalar fields in terms of ψ and ζ gives us

$$e^{\psi} = \frac{K(\rho(\tilde{\rho}))^{\frac{1}{2}}}{\tilde{\rho}^{\frac{2\sigma-d}{2}}}\left(1+\frac{m^{2\sigma}\tilde{\rho}^{\sigma}}{4}\right)^{\frac{2\sigma-d}{\sigma}}, \qquad e^{\zeta} = K(\rho(\tilde{\rho}))^{\frac{2\sigma-d-1}{2}}. \tag{5.57}$$

Now, since,

$$e^{\frac{2\psi}{2\sigma-d}} = \frac{1}{\tilde{\rho}}e^{\frac{2\kappa}{2\sigma-d}},$$

we get that,

$$e^{\kappa} = K(\rho(\tilde{\rho}))^{\frac{1}{2}}\left(1+\frac{m^{2\sigma}\tilde{\rho}^{\sigma}}{4}\right)^{\frac{2\sigma-d}{\sigma}}. \tag{5.58}$$

Expanding the above results in $\tilde{\rho}$ we get,

$$\begin{aligned} \kappa_{(0)} = 0 \quad &, \quad \kappa_{(2\sigma)} = \frac{1}{2}Q + \frac{2\sigma-d}{\sigma}\frac{m^{2\sigma}}{4}, \\ \zeta_{(0)} = 0 \quad &, \quad \zeta_{(2\sigma)} = \frac{2\sigma-d-1}{2}Q, \\ & \quad A_{i(2\sigma)} = \hat{u}_i\bar{Q}. \end{aligned} \tag{5.59}$$

Now, using (5.32) and the expressions worked out above, we may write down the expectation values of the dual operators. This results in the following,

$$\begin{aligned}
\langle T_{ij}\rangle_d &= Lm^{2\sigma}\eta_{ij} + 2\sigma L(Q+m^{2\sigma})\hat{u}_i\hat{u}_j \qquad (5.60)\\
&= Lm^{2\sigma}\left(\eta_{ij} + 2\sigma\cosh^2\omega\hat{u}_i\hat{u}_j\right);\\
\langle J_i\rangle_d &= 2\sigma L\bar{Q}\hat{u}_i\\
&= 2\sigma Lm^{2\sigma}\sinh\omega\cosh\omega\hat{u}_i;\\
\langle \mathcal{O}_1\rangle_d &= -m^{2\sigma}L - 2\sigma LQ\\
&= -Lm^{2\sigma}\left(1+2\sigma\sinh^2\omega\right)\\
\langle \mathcal{O}_2\rangle_d &= -Lm^{2\sigma}.
\end{aligned}$$

We may directly verify that this indeed satisfies the dilatation Ward identity (5.40). We may also write down the energy density $\hat{\epsilon}$, the charge density $\hat{q}$ and the pressure $\hat{P}$ of the reduced spacetime (5.54). These are given by,

$$\hat{\epsilon} = Lm^{2\sigma}(2\sigma\cosh^2\omega - 1), \qquad \hat{q} \equiv \bar{Q} = 2\sigma Lm^{2\sigma}\sinh\omega\cosh\omega, \qquad \hat{P} = Lm^{2\sigma}. \qquad (5.61)$$

One may obtain the equation of state, $\hat{P} = \hat{P}(\hat{\epsilon},\hat{q})$, from the above. This is done by inverting the first two expressions to obtain m and ω in terms of $\hat{\epsilon}$ and $\hat{q}$ and then, in turn, using these in the last relation. The equation of state thus obtained is,

$$\hat{P}(\hat{\epsilon},\hat{q}) = \frac{1}{2\sigma-1}\left(\sqrt{\hat{\epsilon}^2(\sigma-1)^2 + (\hat{\epsilon}^2-\hat{q}^2)(2\sigma-1)} - \hat{\epsilon}(\sigma-1)\right). \qquad (5.62)$$

(Since $\hat{P} = Lm^{2\sigma}$, this relation also expresses m in terms of $\hat{\epsilon}$ and $\hat{q}$, while $\sinh 2\omega = \hat{q}/(\sigma\hat{P})$ gives ω in terms of $\hat{\epsilon}$ and $\hat{q}$). The equation of state for the non-conformal branes is obtained by taking the $\hat{q}\to 0$ limit. The extremal limit is obtained by sending $\hat{\epsilon}\to|\hat{q}|$ and we see from the equation of state that the pressure is zero, as expected. Away from extremality, namely for $\hat{\epsilon} > |\hat{q}|$, we see that the argument of the square root is manifestly positive.

The temperature and entropy of the reduced theory are,

$$\hat{T} = \frac{m\sigma}{2\pi\cosh\omega}, \qquad \hat{s} = 4\pi L\cosh\omega m^{2\sigma-1}. \qquad (5.63)$$

The equation of state (5.62) also gives us the adiabatic speed of sound[6], subject to

[6] See [181], Chapter XV, equation (134.14) and (134.7).

keeping the ratio $\hat{s}/\hat{q}$ fixed:

$$\hat{c}_s^2 = \left.\frac{\partial \hat{P}}{\partial \hat{\epsilon}}\right|_{\hat{s}/\hat{q}}. \tag{5.64}$$

Then (5.63) and (5.61) allow us to calculate,

$$\mathrm{d}\left(\frac{\hat{s}}{\hat{q}}\right) = 0 \quad \Rightarrow \quad \mathrm{d}\omega = -\tanh\omega \frac{\mathrm{d}m}{m}, \tag{5.65}$$

so that

$$\hat{c}_s^2 = \frac{1}{2(\sigma-1)\cosh^2\omega + 1}. \tag{5.66}$$

This reduces to the result for the neutral black branes derived in [170].

We may also obtain the chemical potential of the reduced theory. This is done using (5.52) and yields,

$$\hat{\mu} = -\left(\hat{u}^i A_i\Big|_{\rho=0} - \hat{u}^i A_i\Big|_{\rho=m^{-2}}\right) = \tanh\omega. \tag{5.67}$$

One may also verify that the usual thermodynamic identities hold,

$$\hat{P} + \hat{\epsilon} = \hat{T}\hat{s} + \hat{q}\hat{\mu}, \qquad \mathrm{d}\hat{P} = \hat{s}\mathrm{d}\hat{T} + \hat{q}\mathrm{d}\hat{\mu}. \tag{5.68}$$

Note that we fix the external gauge field in terms of the chemical potential by requiring regularity at the horizon, namely $\hat{u}^i A_i|_{\rho=m^{-2}} = 0$. However, in the next Section we would like to have a general external gauge field and so we will relax this regularity condition. We would like to state, though, that all of the main results (transport coefficients etc.) can also be obtained without turning on an additional external field beyond that required by the presence of the chemical potential.

We may also express the expectation values of the scalar operators $\langle \mathscr{O}_\psi \rangle_d$ and $\langle \mathscr{O}_\zeta \rangle_d$ in terms of the energy density and pressure. This yields,

$$\begin{aligned}
\langle \mathscr{O}_\psi \rangle_d &= \frac{1}{(2\sigma-d)}\langle T^i_i\rangle_d = \frac{1}{(2\sigma-d)}\left[(d-1)\hat{P} - \hat{\epsilon}\right], \\
\langle \mathscr{O}_\zeta \rangle_d &= \frac{1}{(2\sigma-d)}\left[(2\sigma-1)\hat{P} - \hat{\epsilon}\right].
\end{aligned} \tag{5.69}$$

This tells us that, as expected, the expectation value of the scalar operator $\langle \mathscr{O}_\psi \rangle_d$ characterizes the deviation of the equation of state from conformality. On the other hand, for zero charge, the expectation value of the second operator $\langle \mathscr{O}_\zeta \rangle_d$ is zero, and the equation of state indeed reduces to that of the non-conformal branes, $\hat{P} = \hat{\epsilon}/(2\sigma-1)$.

5.1.4 Universal Hydrodynamics

We would now like to obtain the universal hydrodynamics corresponding to the charged dilatonic solutions. We do this by means of the generalized dimensional reduction.

We begin by reminding ourselves of the expression for the hydrodynamic energy-momentum tensor of a (2σ)-dimensional conformal fluid. At first-derivative order and on a curved manifold with metric $g_{(0)\mu\nu}$, it is given by,

$$\begin{aligned}
\langle T_{\mu\nu}\rangle_{2\sigma} &= \langle T^{\text{eq}}_{\mu\nu}\rangle_{2\sigma} + \langle T^{\text{diss}}_{\mu\nu}\rangle_{2\sigma}, \\
\langle T^{\text{eq}}_{\mu\nu}\rangle_{2\sigma} &= P(g_{(0)\mu\nu} + 2\sigma u_\mu u_\nu), \qquad \langle T^{\text{diss}}_{\mu\nu}\rangle_{2\sigma} = -2\eta_{2\sigma}\sigma_{\mu\nu}, \\
\sigma_{\mu\nu} &= P^\kappa_\mu P^\lambda_\nu \nabla_{(\kappa} u_{\lambda)} - \frac{1}{2\sigma-1} P_{\mu\nu}(\nabla\cdot u), \qquad P_{\mu\nu} = g_{(0)\mu\nu} + u_\mu u_\nu,
\end{aligned} \tag{5.70}$$

where T is the temperature, u_μ the velocity and $\eta_{2\sigma}$ the shear viscosity of the fluid. We denote by ∇_μ the covariant derivative corresponding to the metric $g_{(0)\mu\nu}$. Note that we are working in Landau-Lifshitz frame,

$$u^\mu \langle T^{\text{diss}}_{\mu\nu}\rangle_{2\sigma} = 0\,. \tag{5.71}$$

The conservation of the energy-momentum tensor,

$$\nabla^\mu \langle T_{\mu\nu}\rangle_{2\sigma} = 0, \tag{5.72}$$

governs the fluid evolution.

The pressure and shear viscosity of the AdS black brane can be determined from (5.45) to be,

$$P = L_{AdS} m^{2\sigma}, \qquad \eta_{2\sigma} = \frac{s}{4\pi} = L_{AdS} m^{2\sigma-1}. \tag{5.73}$$

We will now compute the various quantities for the reduced theory, starting with the reduced fluid velocity.

From the reduction ansatz (5.11), we may read off the boundary metric using the expansions of the fields (5.26) and (5.27). Taking a cue from the AdS black brane (5.59), we will set $\kappa_{(0)} = \zeta_{(0)} = 0$. Then,

$$g_{(0)ij} = \eta_{ij} + A_{(0)i}A_{(0)j}, \qquad g_{(0)iy} = -A_{(0)i}, \qquad g_{(0)yy} = 1, \tag{5.74}$$

with the inverse metric given by

$$g^{ij}_{(0)} = \eta^{ij}, \qquad g^{iy}_{(0)} = A^{i}_{(0)}, \qquad g^{yy}_{(0)} = 1 + \eta^{ij} A_{(0)i} A_{(0)j}\,. \tag{5.75}$$

Note that the reduced boundary metric is simply the Minkowski metric η_{ij}[7]. In order to obtain the reduced fluid velocity, $\hat{u}^i$, we now impose the following:

$$u^\mu u_\mu = -1, \qquad u^\mu = g^{\mu\nu}_{(0)} u_\mu, \qquad \hat{u}^i \hat{u}_i = -1, \qquad \hat{u}^i = \eta^{ij} \hat{u}_j. \tag{5.76}$$

For convenience we now choose

$$u_y = \sinh\omega. \tag{5.77}$$

This also allows us to make a link with the wave generating coordinate transformation of the previous Section. Setting $u_a = 0$ along y^a, the remaining compact dimensions, yields the following,

$$u_i = \cosh\omega \hat{u}_i - \sinh\omega A_{(0)i}\,, \quad u_y = \sinh\omega\,, \quad u^i = \cosh\omega \hat{u}^i\,, \quad u^y = \sinh\omega + \cosh\omega \hat{u}\cdot\partial A_{(0)}\,. \tag{5.78}$$

We now have the required expressions to write down the equilibrium hydrodynamics, from the expression for $\langle T^{\rm eq}_{\mu\nu}\rangle_{2\sigma}$ in (5.70). Recalling the correct linear combinations of (5.38) we obtain,

$$\begin{aligned}
\langle T^{\rm eq}_{ij}\rangle_d &= \hat{P}\left[\eta_{ij} + 2\sigma\left(u_i + u_y A_{(0)i}\right)\left(u_j + u_y A_{(0)j}\right)\right],\\
\langle J^{\rm eq}_{i}\rangle_d &= 2\sigma \hat{P} u_y \left(u_i + u_y A_{(0)i}\right),\\
\langle \mathcal{O}^{\rm eq}_{1}\rangle_d &= -\hat{P}\left(1 + 2\sigma u_y^2\right),\\
\langle \mathcal{O}^{\rm eq}_{2}\rangle_d \mathrm{d}_{ab} &= -\hat{P}\left(\mathrm{d}_{ab} + 2\sigma u_a u_b\right).
\end{aligned}$$

Using (5.78), these become

$$\langle T^{\rm eq}_{ij}\rangle_d = \hat{P}\left(\eta_{ij} + 2\sigma\cosh^2\omega \hat{u}_i \hat{u}_j\right), \tag{5.79}$$

$$\langle J^{\rm eq}_{i}\rangle_d = 2\sigma \sinh\omega\cosh\omega \hat{P}\hat{u}_i\,, \tag{5.80}$$

$$\langle \mathcal{O}^{\rm eq}_{1}\rangle_d = -\hat{P}\left(1 + 2\sigma\sinh^2\omega\right), \tag{5.81}$$

$$\langle \mathcal{O}^{\rm eq}_{2}\rangle_d = -\hat{P}\,, \tag{5.82}$$

$$\langle \mathcal{O}^{\rm eq}_{\psi}\rangle_d = -\frac{2\sinh^2\omega\hat{P}}{2\sigma - d}\,, \tag{5.83}$$

[7]The hydrodynamics at first-order is independent of the curvature of the reduced boundary metric, so our results will still hold at first-order for a curved boundary in d dimensions.

$$\langle \mathcal{O}_{\zeta}^{\mathrm{eq}} \rangle_d \quad = \quad -\frac{\hat{P}}{2\sigma - d}\left(2\sigma \cosh^2 \omega - d\right). \tag{5.84}$$

Hence, we calculate the equilibrium quantities to be,

$$\hat{P} = \frac{L}{L_{AdS}} P, \qquad \hat{\epsilon} = \left(2\sigma \cosh^2 \omega - 1\right) \hat{P}, \qquad \hat{q} = 2\sigma \sinh \omega \cosh \omega \hat{P}. \tag{5.85}$$

Inserting into the above the value of the AdS black brane pressure, (5.73), allows us to recover the correct pressure, energy and charge density of the reduced theory, (5.61), as well as the dual operators (5.60) and (5.69).

This was the equilibrium part of the hydrodynamics. We will now turn to the dissipative part.

In order to compute this we start with the expression for $\langle T_{\mu\nu}^{\mathrm{diss}} \rangle_{2\sigma}$ given in (5.70). We insert $u^{\mu} = (u^i, 0, u^y)$ and reduce the resulting expression to d dimensions. It is important to note that, in the reduced theory, the Landau-Lifshitz frame condition (5.71) becomes,

$$\begin{aligned} \hat{u}^i \langle J_i^{\mathrm{diss}} \rangle_d \quad &= \quad \tanh \omega \langle \mathcal{O}_1^{\mathrm{diss}} \rangle_d, \\ \hat{u}^i \langle T_{ij}^{\mathrm{diss}} \rangle_d \quad &= \quad -\tanh \omega \langle J_j^{\mathrm{diss}} \rangle_d. \end{aligned} \tag{5.86}$$

Hence, the reduced frame is *not* in the Landau frame and, indeed, it shows that upon reduction one does not end naturally in the Landau or Eckhart frame. As a result we will need to be careful in computing the transport coefficients of the reduced theory. To this end, we will make use of the frame independent formulation discussed in [182]. These authors determine the most general, first-order form of the equations of relativistic superfluid hydrodynamics consistent with Lorentz invariance and the second law of thermodynamics, without assuming any choice of frame. They rely only on invariance and symmetry considerations. The method relies on ensuring that the divergence of the entropy current is positive semi-definite.

The Landau-frame entropy current, in $2\sigma + 1$ dimensions, is

$$\langle J_s^{\mu} \rangle_{2\sigma} = s u^{\mu}. \tag{5.87}$$

This obeys the following divergence equation,

$$\nabla_{\mu} \langle J_s^{\mu} \rangle_{2\sigma} = -\nabla_{\mu} \left(\frac{u_{\nu}}{T}\right) \langle T_{diss}^{\mu\nu} \rangle_{2\sigma} = -\frac{1}{T} \sigma_{\mu\nu} \langle T_{diss}^{\mu\nu} \rangle_{2\sigma}, \tag{5.88}$$

(see for example [182]). In order for this to be positive semi-definite, we require that the shear viscosity, $\eta_{2\sigma}$, given by,

$$P^{\kappa}_{\mu}P^{\lambda}_{\nu}\langle T^{\rm diss}_{\kappa\lambda}\rangle_{2\sigma} - \frac{1}{2\sigma-1}P_{\mu\nu}P^{\kappa\lambda}\langle T^{\rm diss}_{\kappa\lambda}\rangle_{2\sigma} = -2\eta_{2\sigma}\sigma_{\mu\nu}, \tag{5.89}$$

be non-negative.

The entropy current for charged fluids is given in [182] as [8],

$$\langle J^i_s\rangle_d = \hat{s}\hat{u}^i - \frac{\hat{u}_j}{\hat{T}}\langle T^{ij}_{diss}\rangle_d - \frac{\hat{\mu}}{\hat{T}}\langle J^i_{diss}\rangle_d\,. \tag{5.90}$$

We find that this coincides with the reduction of the entropy current (5.87), upon imposition of the reduced Landau frame conditions (5.86). Namely, we obtain,

$$\langle J^i_s\rangle_d = \hat{s}\hat{u}^i, \qquad \hat{s} = \frac{L\cosh\omega}{L_{AdS}}s\,. \tag{5.91}$$

Similarly, reducing the divergence equation (5.88) yields

$$\partial_i\langle J^i_s\rangle_d = -\partial_i\left(\frac{\hat{u}_j}{\hat{T}}\right)\langle T^{ij}_{diss}\rangle_d - \left[\partial_i\left(\frac{\hat{\mu}}{\hat{T}}\right) - \frac{\hat{u}_k}{\hat{T}}F^k_{(0)i}\right]\langle J^i_{diss}\rangle_d\,, \tag{5.92}$$

which coincides with equation (2.19) of [182].

We may extract the reduced transport coefficients from the formulae. The reduced shear viscosity $\hat{\eta}$, the heat conductivity $\hat{\kappa}$ and the bulk viscosity $\hat{\zeta}_s$, are thus calculated to be:

$$\hat{P}^i_k\hat{P}^j_l\langle T^{\rm diss}_{ij}\rangle_d - \frac{1}{d-1}\hat{P}_{kl}\hat{P}^{ij}\langle T^{\rm diss}_{ij}\rangle_d \;=\; -2\hat{\eta}\hat{\sigma}_{kl}\,, \tag{5.93}$$

$$\hat{P}^j_i\left(\langle J^{\rm diss}_j\rangle_d + \frac{\hat{q}}{\hat{\epsilon}+\hat{P}}\hat{u}^i\langle T^{\rm diss}_{ij}\rangle_d\right) \;=\; -\hat{\kappa}\left(\hat{P}_{ij}\partial^j\frac{\hat{\mu}}{\hat{T}} + \frac{F_{(0)ij}\hat{u}^j}{\hat{T}}\right), \tag{5.94}$$

$$\frac{\hat{P}^{ij}\langle T^{\rm diss}_{ij}\rangle_d}{d-1} - \frac{\partial\hat{P}}{\partial\hat{\epsilon}}\hat{u}^i\hat{u}^j\langle T^{\rm diss}_{ij}\rangle_d + \frac{\partial\hat{P}}{\partial\hat{q}}\hat{u}^i\langle J^{\rm diss}_i\rangle_d \;=\; -\hat{\zeta}_s\partial_i\hat{u}^i\,. \tag{5.95}$$

[8] Note that our conventions relate to those of [182] by changing $\hat{A}_{(0)i} \to -\hat{A}_{(0)i}$ and consequently $F^{ij}_{(0)} \to -F^{ij}_{(0)}$. This has no impact on (5.93) or (5.95), but changes the relative signs in (5.94) as well as in the conservation equation for the reduced boundary stress-energy tensor (5.41).

Using the reduced Landau-frame condition (5.86), the last two expressions above become,

$$\hat{P}^{ij}\langle J_j^{\rm diss}\rangle_d\left(1-\frac{\hat{q}}{\hat{\epsilon}+\hat{P}}\tanh\omega\right) = -\hat{\kappa}\left(\hat{P}^{ij}\partial_j\frac{\hat{\mu}}{\hat{T}}+\frac{F_{(0)}^{ij}\hat{u}_j}{\hat{T}}\right), \tag{5.96}$$

$$\frac{\hat{P}^{ij}\langle T_{ij}^{\rm diss}\rangle_d}{d-1}+\left(\frac{\partial\hat{P}}{\partial\hat{\epsilon}}\tanh^2\omega+\frac{\partial\hat{P}}{\partial\hat{q}}\tanh\omega\right)\langle\mathscr{O}_1^{\rm diss}\rangle_d = -\hat{\zeta}_s\partial_i\hat{u}^i\,. \tag{5.97}$$

The fluid conservation equations,

$$\partial_i\langle T^{ij}\rangle_d = F_{(0)}^{ij}\langle J_i\rangle_d\,, \tag{5.98}$$

$$\partial^i\langle J_i\rangle_d = 0\,, \tag{5.99}$$

yield the following constraints,

$$\partial_j\log m = \frac{\cosh\omega}{\sinh\omega}\,\hat{u}\cdot\partial\omega\;\hat{u}_j-\cosh^2\omega\;\hat{u}\cdot\partial\hat{u}_j+\sinh\omega\cosh\omega\;\hat{u}^iF_{(0)ij}\,, \tag{5.100}$$

$$\hat{u}\cdot\partial\omega = \frac{\sinh\omega\,\cosh\omega}{2(\sigma-1)\cosh^2\omega+1}\partial\cdot\hat{u}\,. \tag{5.101}$$

We also calculate the following quantities,

$$\begin{aligned}\langle T_{ij}^{diss}\rangle_d &= -2\eta_d\Bigg[\cosh\omega\hat{\sigma}_{ij}+\sinh\omega\hat{u}_{(i}\left(\partial_{j)}\omega+\frac{1}{2}\sinh 2\omega\hat{u}\cdot\partial\hat{u}_{j)}-\cosh^2\omega\hat{u}_kF^k_{(0)j)}\right)+\\ &\quad+\cosh\omega\frac{\hat{P}_{ij}}{d-1}\left(1-\frac{(d-1)\cosh^2\omega}{2(\sigma-1)\cosh^2\omega+1}\right)\partial\cdot\hat{u}\Bigg]\,,\end{aligned} \tag{5.102}$$

$$\langle J_j^{\rm diss}\rangle_d = \eta_d\cosh\omega\left[\hat{u}_j\hat{u}\cdot\partial\omega-\partial_j\omega-\sinh\omega\cosh\omega\hat{u}\cdot\partial\hat{u}_j-\cosh^2\omega\hat{u}_iF^i_{(0)j}\right], \tag{5.103}$$

$$\langle\mathscr{O}_1^{\rm diss}\rangle_d = \langle\mathscr{O}_2^{\rm diss}\rangle_d=\langle\mathscr{O}_\psi^{\rm diss}\rangle_d=2\eta_d\frac{\cosh^2\omega}{\sinh\omega}\hat{u}\cdot\partial\omega\,, \tag{5.104}$$

$$\langle\mathscr{O}_\zeta^{\rm diss}\rangle_d = 0\,. \tag{5.105}$$

This allows us to finally write down the transport coefficients of the reduced theory as:

$$\hat{\eta} = \eta_d\cosh\omega = Lm^{2\sigma-1}\cosh\omega\,, \tag{5.106}$$

$$\hat{\kappa} = \frac{\eta_d\hat{T}}{\cosh\omega}=\frac{\sigma Lm^{2\sigma}}{2\pi\cosh^2\omega}\,, \tag{5.107}$$

$$\hat{\zeta}_s \quad = \quad \frac{2\eta_d \cosh\omega}{2\sigma-1}\left[\frac{2\sigma-d}{d-1}-\frac{2\sinh^2\omega\left((\sigma-1)\cosh^2\omega+\sigma\right)}{\left(2(\sigma-1)\cosh^2\omega+1\right)^2}\right]. \tag{5.108}$$

Here η_d is the shear viscosity of the (reduced) neutral case,

$$\eta_d = \frac{L}{L_{AdS}}\eta_{2\sigma} = Lm^{2\sigma-1}. \tag{5.109}$$

We obtain the first equality above from the reduction and the second simply by substituting the universal value of $\eta_{2\sigma}$ for conformal AdS black branes given in (5.73).

The transport coefficients (5.106)-(5.107)-(5.108) are *universal*. In other words, they are valid for any solution possessing the same assymptotics as the black brane solution of the previous Section.

From (5.63) and (5.106), we find that the KSS bound [132] is saturated for charged branes, namely $\hat{\eta}/\hat{s} = 1/4\pi$. This is a direct consequence of the fact that it is saturated for the conformal branes, similar to the neutral case discussed in [170]. Indeed, the value of η/s is fixed by the requirements of regularity in the interior, while singular solutions can have η/s different from $1/4\pi$.

Analyzing next the ratios $\hat{\zeta}_s/\hat{\eta}$ and $\hat{\kappa}/\hat{\eta}$, we find that each is, in fact, individually kinematically fixed by the reduction. Indeed, any solution with the same assymptotics, irrespective of whether it is regular or singular, will yield the same individual result for each of the ratios. This stems from the fact that the factor $\eta_{2\sigma}$ cancels out upon taking the ratios, as is discussed in [170].

We now turn our attention to a recent formula in [183],

$$\frac{\hat{\zeta}_s}{\hat{\eta}} = \sum_i \left(\hat{s}\frac{\mathrm{d}\phi_i^h}{\mathrm{d}\hat{s}} + \hat{q}_a\frac{\mathrm{d}\phi_i^h}{\mathrm{d}\hat{q}_a}\right)^2, \tag{5.110}$$

and investigate how our result for $\hat{\zeta}_s/\hat{\eta}$ compares. In the above expression, $\hat{q}_a$ represent conserved charge densities while the ϕ_i^h are a collection of scalar fields, evaluated at the event horizon. This expression is valid in the Einstein frame where the entropy density s is given by the usual expression $s = \frac{A}{4}$ where A is the area of the horizon. Since this formula reproduces all known results it constitutes an important check of our result (5.108).

In our case, the expressions for the entropy, (5.63), and charge density, (5.61), still hold if we go to the Einstein frame. We may thus implement them to derive the expressions below,

$$\begin{aligned}
\mathrm{d}(\log\hat{s})|_{\hat{q}} &= -\frac{2(\sigma-1)\cosh^2\omega+1}{2\sigma\cosh\omega\;\sinh\omega}\,\mathrm{d}\omega,\\
\mathrm{d}(\log\hat{q})|_{\hat{s}} &= \frac{2(\sigma-1)\cosh^2\omega+1}{(2\sigma-1)\cosh\omega\;\sinh\omega}\,\mathrm{d}\omega,\\
\mathrm{d}(\psi_h)|_{\hat{s}} &= \sqrt{\frac{2(d-1)}{(2\sigma-1)(2\sigma-d)}}\tanh\omega\,\mathrm{d}\omega,\\
\mathrm{d}(\psi_h)|_{\hat{q}} &= -\sqrt{\frac{2(2\sigma-1)}{(d-1)(2\sigma-d)}}\frac{2(\sigma-d)\cosh^2\omega\;+d}{2\sigma\cosh\omega\;\sinh\omega}\,\mathrm{d}\omega,\\
\mathrm{d}(\zeta_h)|_{\hat{s}} &= d(\zeta_h)|_{\hat{q}} = \sqrt{\frac{2(2\sigma-d-1)}{(2\sigma-d)}}\tanh\omega\,\mathrm{d}\omega.
\end{aligned} \tag{5.111}$$

Now, combining everything together in (5.110), we do indeed recover (5.108). This constitutes an important and indeed very non-trivial check of our result, since the two methods are completely different.

We now analyze how our result stands up to the bound proposed in [184]:

$$\frac{\hat{\zeta}_s}{\hat{\eta}} \geq 2\left(\frac{1}{d-1}-\hat{c}_s^2\right). \tag{5.112}$$

Indeed, our result yields,

$$\frac{\hat{\zeta}_s}{\hat{\eta}} = 2\left(\frac{1}{d-1}-\hat{c}_s^2\right) - \frac{4\sinh^2\omega\left((\sigma-1)\cosh^2\omega+1\right)}{\left(2(\sigma-1)\cosh^2\omega+1\right)^2}. \tag{5.113}$$

This means that the abovementioned bound is always violated[9], except if

$$\sigma < \hat{\mu}^2. \tag{5.114}$$

Now, since $\hat{\mu}^2 = \tanh^2\omega \leq 1$, this is possible only if $\sigma < 1$. However, $\sigma > 1$ for all values in (5.25). The bound is saturated either for the neutral case $\hat{\mu} = 0$ or for $\hat{\mu}^2 = \sigma$. Now, we have already mentioned that the ratio $\hat{\zeta}_s/\hat{\eta}$ is fixed kinematically for given asymptotics and this leads us to conclude that there is no reason to expect that a general system would satisfy such an inequality. The asymptotic behavior of the charged case is different from the neutral case because we have turned on a non-normalizable mode

[9]See [185] for recent work containing other such examples.

for the gauge field. This is due to the presence of a chemical potential and regularity at the horizon, (5.67).

However, by replacing the adiabatic speed of sound, $\hat{c}_s^2$, in (5.112) with $\hat{c}_q^2$, where

$$\hat{c}_q^2 \equiv \left.\frac{\partial \hat{P}}{\partial \hat{\epsilon}}\right|_{\hat{q}} = \frac{\cosh 2\omega}{(2\sigma-2)\cosh^2\omega+1}, \tag{5.115}$$

we find that the resulting inequality is indeed saturated by the neutral branes and satisfied by the charged ones. When $\omega = 0$, $\hat{c}_q^2$ reduces to the speed of sound of the conformal branes. Furthermore,

$$\frac{\hat{\zeta}_s}{\hat{\eta}} - 2\left(\frac{1}{d-1} - \hat{c}_q^2\right) = \frac{(\sigma-1)\sinh^2(2\omega)}{(2(\sigma-1)\cosh^2\omega+1)^2}. \tag{5.116}$$

For $\sigma > 1$, *i.e.* for all values in (5.25), the right hand side above is manifestly positive. It would be interesting to check whether there are any counterexamples to this inequality.

We may also calculate the DC conductivity for our results. This can be deduced using the Franz-Wiedemann law:

$$\hat{\sigma}_{DC} = \frac{\hat{\kappa}}{\hat{T}} = \frac{\eta_d}{\cosh\omega} = \frac{Lm^{2\sigma-1}}{\cosh\omega}. \tag{5.117}$$

At this point, in order to compare more directly with other results, we express the transport coefficients for the reduced AdS black brane in terms of the temperature and chemical potential. This yields the following expressions for the transport coefficients and DC conductivity:

$$\hat{\eta} = L\left(\frac{2\pi\hat{T}}{\sigma}\right)^{2\sigma-1}\left(1-\hat{\mu}^2\right)^{-\sigma}, \tag{5.118}$$

$$\hat{\kappa} = \frac{\sigma L}{2\pi}\left(\frac{2\pi\hat{T}}{\sigma}\right)^{2\sigma}\left(1-\hat{\mu}^2\right)^{1-\sigma}, \tag{5.119}$$

$$\hat{\zeta}_s = \frac{2(2\sigma-d)\hat{\eta}}{(d-1)(2\sigma-1)}\left[1-\frac{2(d-1)\hat{\mu}^2\left(2\sigma-1-\sigma\hat{\mu}^2\right)}{(2\sigma-d)\left(2\sigma-1-\hat{\mu}^2\right)^2}\right], \tag{5.120}$$

$$\hat{\sigma}_{DC} = L\left(\frac{2\pi\hat{T}}{\sigma}\right)^{2\sigma-1}\left(1-\hat{\mu}^2\right)^{1-\sigma}. \tag{5.121}$$

In the neutral limit $\hat{\mu} \to 0$ the DC conductivity is finite. This is to be expected since we have in fact computed the microscopic fluctuations around the background, which is irrespective of whether it is neutral or charged.

In the zero density limit, we may compare our result for the DC conductivity with results from the flavour branes approach, [101], [109], [186], [187]. In this approach the gauge field lives on the brane in a neutral background and does not backreact, while in our case we obtain (5.121) by working out the fluctuations of the metric and gauge field around a charged black hole.

5.2 Discussion and conclusions

In this Chapter we began with higher-dimensional AdS gravity theory and reduced it to a class of EMD theories by means of a generalized dimensional reduction over compact Einstein manifolds. This allowed us to set up holography for non-asymptotically AdS solutions: the solutions related to the EMD theories. Here we have extended the discussion of generalized consistent reductions pioneered by [170] by including an additional scalar and gauge fields into the analysis.

The higher-dimensional AdS theory is fundamental in explaining the behavior of the reduced theory. It controls both the UV and IR physics of the strongly-coupled field theory dual to the reduced gravity theory. Specifically, since the solution uplifts to an asymptotically AdS solution, there exists a fixed point at $d+\epsilon$ dimensions, for ϵ the dimension of the compact space. This controls the UV behavior in the dual qft, and translates into specific running of coupling constants.

The hydrodynamics of the higher-dimensional theory also controls the IR behavior of the EMD theories in the hydrodynamic regime. Specifically, transport coefficients in the reduced theory come directly from the transport coefficients of AdS and the reduced theory possesses an entropy current with non-negative divergence precisely because there is one in the higher-dimensional theory, [170]. Hence the transport coefficients satisfy certain kinematical relations, such as the ratios of the bulk to shear viscosity and conductivity to shear viscosity being fixed to specific values, irrespective of whether the bulk solution is regular or singular in the interior. In addition, the putative bound on the bulk to shear viscosity proposed in [184] is violated in the presence of a chemical potential.

One important thing to note of is that the duality described here is not in general valid at all energy scales. Take for instance the prototype example of the holographic duality for non-conformal branes, [167]. Here, one assumes that the effective 't Hooft coupling $g_{eff}^2 N$ is fixed while N^2 is taken to infinity. However, since the effective coupling constant depends on the energy scale in these theories, there is always a regime where $g_{eff}^2 N$ grows faster than N^2. This implies that the dilaton blows up and hence a new description is needed. For the case of Dp branes this is typically an M-brane description. We expect the same argument to hold here and that our holographic description is only valid below some energy scale.

The driving force behind these types of investigations is to attempt to model condensed matter systems holographically, specifically by looking at interesting IR fixed points. Indeed, there has been considerable work on models that interpolate between the IR behavior described here and an AdS region in the UV, [98], [101], [104], [106], [109], [188], [189]. However, as our discussion shows and also on general grounds, one would expect to be able to model the IR region without a reference to such UV completion.

There are many directions which this work could take. For instance, two gauge fields could be investigated since these systems may provide a holographic description of imbalanced superconductors, [190]. Also, the higher-dimensional theory that is considered need not be Einstein gravity. For instance, the authors of [191] are considering the Einstein-Gauss-Bonnet gravity instead. Indeed it would be interesting to have a complete set of cases where one could set up holography using such a generalized consistent reduction.

In the final Chapter we return to the question of higher derivative effects in holography and look for black hole solutions in a number of theories of higher derivative gravity. Specifically, we are looking for higher derivative effects in 4-dimensional AdS black holes. We will also work out the thermodynamics of these corrected black hole solutions using holography and the formalism presented in Section 4.3.6.

Chapter 6

Higher-derivative effects for 4d AdS gravity

6.1 Introduction

In this Chapter we will explore higher derivative corrections to gravity theories in $(3+1)$-dimensions with negative cosmological constant. Our main motivation for looking at higher derivative corrections to four dimensional AdS black holes is in the context of holography and, in particular, applied holography, AdS/CMT, where many of the systems of interest are modelled by four dimensional bulk spacetimes. The addition of higher derivative terms allows us to probe the dual physics as one moves away from infinite N and infinite 't Hooft coupling.

Finite N effects can change the physics qualitatively. For example, let us consider holographic superconductors, a subject which has been extensively studied in recent years, initiated by [118], [119] and [121]. Working with classical gravity there is an apparent violation of the Coleman-Mermin-Wagner theorem. This well-known theorem states that, for system in two spatial dimensions, we cannot have continuous symmetry breaking in systems at finite temperature and hence the formation of a symmetric breaking condensate is forbidden. However, holographic superfluids have been found in $(3+1)$ bulk dimensions, in which a symmetry breaking operator in the dual $(2+1)$ dimensional CFT acquires an expectation value. As explored in [122], this is an infinite N effect and at finite N quantum effects in the bulk indeed ensure that the symmetry breaking operator does not have a well defined expectation value, in accordance with the expected field theory behaviour [192].

One does not see a qualitative finite N effect such as the restoration of the Coleman-Mermin Wagner theorem by evaluating higher derivative corrections on the leading order gravity solution but rather by exploring quantum effects in the bulk. Evaluating higher derivative corrections rather shifts the saddle point and allows one to compute corrections to thermodynamic quantities, transport coefficients and so on. In the context of five bulk dimensions, a considerable effort has been put into investigating higher derivative corrections and exploring the effects on the ratio of the shear viscosity η to the entropy density s, see for example [193], [194], [139], [137].

In particular, [137] used Gauss-Bonnet curvature corrections and initiated a bottom up exploration of the constraints on the higher derivative corrections imposed by unitarity of the dual CFT. Working with the Gauss Bonnet term is particularly convenient because the corrections to AdS planar black holes are known analytically for any value of the Gauss Bonnet coupling constant, see [195] and also [196], [197], [198]. Note that an effect of the Gauss-Bonnet term relevant to the superfluids mentioned above was discussed in [123], where it was found that addition of the higher curvature terms makes condensation to a superfluid phase more difficult.

The Gauss-Bonnet terms, and corresponding corrected AdS black holes, are a useful way to go beyond classical gravity in bulk dimensions higher than five. However, such terms are trivial in four bulk dimensions, in the sense that an Einstein metric is uncorrected and therefore one needs to include higher order curvature invariants to obtain non trivial corrections[1]. An alternative possibility is to couple Einstein gravity to a dilaton in four dimensions because Gauss-Bonnet type corrections to diatonic black holes are then non trivial, see for example [141], but this does not address the question of how AdS black holes with no dilaton are corrected.

Apart from AdS/CMT motivations mentioned above, for which dilatonic AdS black holes may indeed already capture many relevant features [191], there are a number of other important motivations in exploring higher derivative corrections to Einstein gravity with a negative cosmological constant. The first is in understanding the AdS/CFT correspondence when the dual theory is on an S^3. In recent years there has been considerable progress in understanding dual (supersymmetric) 3d CFTs, following the works of BLG [143] and ABJM [144], and localisation techniques have been used to compute free energies of the dual theories placed on an S^3. Taking the limit of large N and large 't Hooft coupling, the free energies have been matched to the

[1]A review of higher order gravity theories and their black hole solutions may be found in [199].

onshell renormalised action of AdS_4 with an S^3 boundary in Einstein gravity [147], [200]. Localisation techniques also allow us to access the subleading terms in the free energy which should be compared to the effects of higher derivative terms evaluated on the bulk AdS_4. Comparing these subleading terms with the gravity results we develop here can be used to test the correspondence and indeed restrict which higher derivative terms can arise in the four dimensional bulk action.

The second motivation in exploring higher derivative terms in four dimensions is in the context of understanding the holographic dictionary. One of the main points of this paper is that the addition of higher derivative terms generically involves additional data being required for the variational problem to be well-defined. In the context of holography, the additional data corresponds to a new operator in the dual CFT, in addition to the stress energy tensor which is dual to the bulk metric. For generic higher derivative terms added to the action the dual operator has complex dimension and/or negative norm, reflecting the fact that the corrected added violates unitarity. This analysis provides a very direct probe of the unitarity properties arising from the higher derivative terms.

Historically the main context in which higher curvature corrections to four-dimensional gravity has been studied is as a toy model for a quantum theory of gravity. In this context the key problem is that adding higher curvature corrections adds higher-order time derivatives to the theory and consequently ghosts. Recently there has been considerable interest in so-called critical gravity theories, in which ghostlike modes appear to be absent, in both three [130] and four [126], [127] bulk dimensions. The four dimensional story that we develop here is the exact analogue of the discussions in [131], [201] for topologically massive gravity in three dimensions [128], [129]: the higher derivative terms in TMG were shown to be associated with a new operator in the dual two dimensional CFT. In TMG, regardless of the value of the coupling of the higher derivative term a violation of unitarity was found in the dual field theory, either by a complex operator dimension or by an operator whose two point function was non-positive. Note that this violation of unitary persisted even at the so-called critical point, where the new operator together with the stress energy tensor were non-diagonalizable. In this paper we will show that analogous problems are found in the four-dimensional higher derivative theories.

Given that higher derivative terms generically give rise to new boundary conditions and hence dual CFT operators, whose properties are not consistent with unitary, one may ask how this observation can be consistent with the fact that top down models arising from string theory are necessarily unitary. To understand this point, one should first note that in the context of string theory and holography the higher curvature terms are always viewed as an infinite series of small corrections to the leading order effective action. The action with higher derivative terms is not quantized, which makes the issue of ghostlike modes moot. In other words, the effective action takes the form

$$I = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}(R - 2\Lambda + \alpha_n l_p^n R^n + \cdots) \tag{6.1}$$

where Λ is the cosmological constant; R^n denotes schematically an n-th order invariant[2]; α_n is a dimensionless numerical constant and l_p denotes the effective Planck length. The effective Newton constant in the Einstein theory is $\kappa^2 = 8\pi G$.

One is by assumption working in a regime where l_p is small and therefore the corrections should be treated perturbatively. Suppose $g_{(0)}$ is a solution of the Einstein theory, namely

$$\mathcal{G}_{\mu\nu}(g_{(0)}) = R_{\mu\nu}(g_{(0)}) - \frac{1}{2}R(g_{(0)})g_{(0)\mu\nu} + \Lambda g_{(0)\mu\nu} \tag{6.2}$$

Then the corresponding solution of the corrected theory can be expressed as a perturbative series

$$g = g_{(0)} + l_p^{n+1} g_{(n)} + \cdots \tag{6.3}$$

with

$$\mathcal{G}_{\mu\nu}(g_{(n)}) = -\alpha_n \frac{\delta R^n}{\delta g^{\mu\nu}}(g_{(0)}) \tag{6.4}$$

and so on.

One should emphasize at this point the conceptual difference between evaluating the higher derivative terms on the lowest order solution and treating the higher derivative term non perturbatively. In the former case, the equations for all the metric corrections $g_{(n)}$ are second order inhomogeneous differential equations, rather than higher order differential equations. Since the equations are second order, the only boundary data that needs to be supplied for the variational problem to be well-defined is the metric. When one is considering the higher derivative terms evaluated on the lowest order solution, an analogue of the Gibbons-Hawking-York [73], [202] term in the action can always be defined such that the variational problem is well-defined for a Dirichlet condition on the metric.

By contrast, as we will explore in sections 6.2, 6.4, whenever the higher derivative terms are treated non-perturbatively or we consider the spectrum around a given

[2] Derivatives of the curvature can also arise but will not be considered here.

background, the resulting equations of motion are generically higher order[3]. This means that additional boundary data needs to be supplied. In the context of holography one can understand the additional data as corresponding to additional dual operators in the field theory, beyond the stress energy tensor. The variational problem in such contexts will be well-defined only if one supplies additional information together with the Dirichlet condition on the metric; the actual information which is needed depends on which higher derivative terms are added.

Thus, given a background which solves the supergravity equations at leading order, the variational problem will be well-defined when one computes the corrections to this solution without specifying additional data. However, when one looks at the spectrum around this background, the higher order nature of the field equations manifests itself and additional data, corresponding to a new dual operator, is required.

A top down model arising from string theory must be consistent with unitarity. This is guaranteed if the curvature invariant is such that the resulting equations are actually second order. (Note that since one is treating the corrections perturbatively it is guaranteed that the shift to η/s is small and is consistent with unitarity, in contrast to the discussions of [139], [137] in which the coupling constant of the higher derivative term is allowed to be of order one.)

As we discuss at the end of section 6.4 another case in which the higher derivative invariant is automatically consistent with unitarity is when the linearised field equation around AdS remains second order. This is a weaker condition than requiring that the equation of motion is always second order, but suffices to ensure that there is no non-unitary dual operator induced by adding the higher derivative term. Examples of such curvature invariants are those built out of the Weyl tensor of order three and higher.

Finally it is interesting to note that reducing a curvature invariant of a given order from ten or eleven dimensions to four dimensions on a curved manifold gives rise to curvature invariant in the effective four dimensional action which is of that order and smaller, see for example (6.37). In the context of AdS solutions the reduction required is indeed always on curved manifolds such as spheres. This implies in particular that a curvature invariant such as one quartic in the Riemann tensor never arises without an accompanying term quadratic in the Riemann tensor and a shift of the cosmolog-

[3]The Lovelock theories [203] are a well-known counterexample in which the equations of motion remain second order.

ical constant. Here we show that the term quadratic in the Riemann tensor gives rise to a new boundary condition for the linearised theory around AdS, and hence a dual operator in the CFT, which turns out to be non-unitarity. When one combines all terms arising from the corrections at a given order in the upstairs theory, the resulting four dimensional theory must be unitarity and this may be achieved either by the linearised theory around AdS being second order or by the higher order terms conspiring to give a unitarity dual operator.

The plan of this Chapter is as follows. In section 6.2 we discuss in more detail the variational problem in higher derivative theories and show that it is well-posed with only boundary data for the metric when one treats higher derivative terms perturbatively about a leading order Einstein solution. In section 6.3 we first discuss what curvature invariants are expected to arise in the effective four-dimensional action from a top down perspective and then we explore the effects of various curvature invariants on four dimensional planar AdS black holes. Our goal is to find an analogue of the Gauss-Bonnet corrected black hole in five and higher dimensions, i.e. a representative corrected AdS_4 planar black hole, and we find that the solution in the Weyl corrected theory is the closest analogue. In section 6.4 we look in detail at the spectrum in theories with curvature squared corrections, demonstrating that there are indeed new dual operators associated with the higher derivative terms and these are non-unitary. Noting that the spectrum in the Weyl cubed theory is unchanged again this seems to be the simplest case of a representative correction. In section 6.5 we conclude.

6.2 The variational problem in higher derivative theories

In general one cannot define an analogue of the Gibbons-Hawking-York term [73], [202] such that the variational problem is well-defined with only a Dirichlet condition on the metric - one must impose additional conditions. This observation explains a long standing problem in the literature: for generic higher derivative corrections the analogue of the Gibbons-Hawking-York (GHY) term has never been found.

There is considerable literature discussing the variational problem in higher derivative theories. In the context of corrections arising in string theory, boundary terms were discussed in [204] where the analogue of the GHY term was found for Gauss-Bonnet. This is a very special case, however, as the field equations are second order. For corrections involving powers of the Ricci scalar, the variational problem was discussed in [205]. The generic issues in setting up a variational problem for higher derivative gravity given only a boundary condition on the metric were highlighted in [206]: the boundary terms which arise in varying the bulk action cannot in general be integrated to give an analogue of the GHY term.

Here we argue that the problem in finding a GHY term results from the fact that in general such a term *cannot exist*: one must specify additional data together with the metric. In special cases an analogue of the GHY term was found, for example, for Lovelock theories. However, Lovelock theories are themselves special in that the equations of motion are actually second order and this fact explains why a GHY term could be found.

A useful approach to dealing with higher derivative theories is the auxiliary field method and the variational problem in such a context was discussed in [207]. In this approach the higher order equations are reduced to coupled second order equations for the metric and the auxiliary fields, and one specifies boundary data for both the metric and for the auxiliary field. In the context of perturbatively evaluating higher derivative corrections on leading order Einstein solutions, the boundary condition of the auxiliary field does not involve new data, but rather can be built out of the boundary data for the metric. When one looks at the spectrum, however, one sees that there is indeed generically new data required for the auxiliary field. These points will be illustrated further when we use the auxiliary field method to discuss the spectrum in Section 6.4.

Before moving on to consider specific models for higher derivative corrections in four dimensions, let us discuss the issue with the variational problem. We consider a general action in $(d+1)$ dimensions

$$I = \int_{\mathcal{M}} d^{d+1}x\sqrt{-g}\mathscr{L}, \tag{6.5}$$

where the Lagrangian $\mathscr{L}$ depends only on the metric and the Riemann tensor. The variation of the action with respect to the metric gives

$$\begin{aligned}
\delta I \quad = \quad & \int_{\mathcal{M}} d^{d+1}x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\mathscr{L}+\mathscr{L}^{\mu\nu}\right)\delta g_{\mu\nu} \\
& +\int_{\mathcal{M}} d^{d+1}x\sqrt{-g}\left(\mathscr{L}^{\mu\nu\rho\sigma}R_{\mu\nu\rho}{}^{\lambda}\delta g_{\sigma\lambda}+2\nabla_{\rho}\nabla_{\mu}\mathscr{L}^{\mu\nu\rho\sigma}\delta g_{\nu\sigma}\right) \\
& +2\int_{\partial\mathcal{M}} d\Sigma^{\mu}\mathscr{L}_{\mu\nu\rho\sigma}\nabla^{\rho}\delta g^{\sigma\nu}+\cdots
\end{aligned} \tag{6.6}$$

Here $\partial\mathcal{M}$ is the boundary of the manifold $\mathcal{M}$ and we define

$$\mathcal{L}^{\mu\nu} = \frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} \qquad \mathcal{L}^{\mu\nu\rho\sigma} = \frac{\delta\mathcal{L}}{\delta R^{\mu\nu\rho\sigma}} \tag{6.7}$$

while the ellipses denote boundary terms which vanish with a Dirichlet boundary condition on the metric, $\delta g = 0$.

In the case of Einstein gravity

$$\mathcal{L} = \frac{1}{2\kappa^2}(R - 2\Lambda) \tag{6.8}$$

and thus the boundary term in the variation is

$$\int_{\partial\mathcal{M}} d\Sigma^{\mu}(g^{\nu\sigma}\nabla_{\mu}\delta g_{\nu\sigma} - g_{\mu\sigma}\nabla_{\nu}\delta g^{\nu\sigma}). \tag{6.9}$$

As is well-known, one can set up a well-defined variational problem by noticing that

$$\delta\left(-\frac{1}{\kappa^2}\int_{\partial M} dx\sqrt{-\gamma}K\right) = -\frac{1}{2\kappa^2}\int_{\partial M} d\Sigma^{\mu}(g^{\nu\sigma}\nabla_{\mu}\delta g_{\nu\sigma} + \cdots) \tag{6.10}$$

where the ellipses again denote terms which depend only the on restriction of the metric variation to the boundary (and which hence vanish given the boundary condition). Thus the addition of this term, the Gibbons-Hawking-York term, to the action gives a well-defined variational problem in which the metric on the boundary is held fixed.

For generic Lagrangians involving higher powers of the curvature, the boundary terms involving metric derivatives cannot be canceled by those in the variation of a boundary term. To illustrate this it is useful to look at a specific example,

$$\mathcal{L} = \frac{1}{2\kappa^2}(R - 2\Lambda - \alpha R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}). \tag{6.11}$$

When $\alpha = 0$ this reduces to the Einstein theory. When α is small the higher derivative term can be treated perturbatively. It is thus useful to express the equations of motion in the form

$$\begin{aligned} R_{\mu\nu} &= \bar{T}_{\mu\nu} \equiv \Lambda g_{\mu\nu} + \alpha t_{\mu\nu}; \qquad (6.12)\\ t_{\mu\nu} &= \frac{1}{(d-1)}R^{\rho\sigma\tau\eta}R_{\rho\sigma\tau\eta}g_{\mu\nu} + \frac{4}{(d-1)}\nabla_{\rho}\nabla_{\sigma}R^{\rho\sigma}g_{\mu\nu} \\ &\quad -2R_{\mu\rho\sigma\lambda}R_{\nu}{}^{\rho\sigma\lambda} - 4\nabla^{\rho}\nabla^{\sigma}R_{\rho\mu\nu\sigma}, \end{aligned}$$

with $\bar{T}$ being the effective (trace adjusted) stress energy tensor. In later sections we will be interested in four dimensional models, in which the Riemann squared term can be rewritten in terms of the Ricci tensor and the Ricci scalar, but in this section we will work in general dimension. The reason for consider terms involving the Riemann tensor is that such terms will always arise from top down models, and (unlike Ricci scalar and Ricci terms) they cannot be removed by field redefinitions. We can use the Bianchi identities to simplify the stress tensor as

$$\begin{aligned} t_{\mu\nu} &= \frac{1}{(d-1)} R^{\rho\sigma\tau\eta} R_{\rho\sigma\tau\eta} g_{\mu\nu} - 2 R_{\mu\rho\sigma\lambda} R_{\nu}{}^{\rho\sigma\lambda} \\ &\quad + \frac{2}{(d-1)} \Box R g_{\mu\nu} - 4 \Box R_{\mu\nu} + 4 \nabla^{\rho} \nabla_{\mu} R_{\nu\rho}, \end{aligned} \tag{6.13}$$

where $\Box = \nabla^{\rho}\nabla_{\rho}$.

A perturbative treatment of the field equations means that one looks for a solution such that

$$g_{\mu\nu} = g_{(0)\mu\nu} + \alpha g_{(1)\mu\nu} + \cdots \tag{6.14}$$

where $g_{(0)}$ is Einstein with cosmological constant Λ and $g_{(1)}$ satisfies

$$\begin{aligned} (\mathscr{L}_R - \Lambda) g_{(1)\mu\nu} &= \frac{1}{(d-1)} R^{\rho\sigma\tau\eta}(g_{(0)}) R_{\rho\sigma\tau\eta}(g_{(0)}) g_{(0)\mu\nu} \\ &\quad - 2 R_{\mu\rho\sigma\lambda}(g_{(0)}) R_{\nu}{}^{\rho\sigma\lambda}(g_{(0)}), \end{aligned} \tag{6.15}$$

where $\mathscr{L}_R$ is the linearized Ricci operator and terms on the right hand side are evaluated on the metric $g_{(0)}$ using the connection of that metric. Note that the terms involve derivatives of the Ricci tensor do not contribute since the covariant derivative of the Einstein metric $g_{(0)}$ is zero. As emphasised earlier, this equation is a second order inhomogeneous equation for $g_{(1)}$ and therefore it does not require any new boundary condition. Note that we regard here the boundary conditions for the metric as being given as a power series in α; i.e. the homogenous part of the solution $g_{(1)}$ is determined by this data.

Let us now turn to the question of the variational problem for such a theory. The new (relative to Einstein gravity) boundary term that arises in varying the action is then

$$\frac{2\alpha}{\kappa^2} \int_{\partial\mathcal{M}} d\Sigma^{\mu} R_{\mu\nu\rho\sigma} \nabla^{\rho} \delta g^{\sigma\nu} + \cdots \tag{6.16}$$

where we again suppress terms which vanish for the boundary condition $\delta g = 0$. This term can be manipulated using the Gauss-Codazzi relations as follows. The metric on

$\mathscr{M}$ can be decomposed as

$$ds^2 = (N^2 + N_\mu N^\mu)dr^2 + 2N_\mu dx^\mu dr + \gamma_{\mu\nu}dx^\mu dx^\nu \tag{6.17}$$

in terms of hypersurfaces Σ_r of constant r with the unit normal to each hypersurface being given by n^μ. As the notation suggests, we are most interested in the case where the finite boundary is at spatial infinity, so r is indeed a radial coordinate[4]. Defining the radial flow vector r^μ such that $r^\mu \partial_\mu r = 1$, the components of r^μ tangent and normal to the hypersurfaces define the shift N^μ and the lapse Nn^μ respectively. The extrinsic curvature $K_{\mu\nu}$ of the hypersurface is given by

$$K_{\mu\nu} = \frac{1}{2}\mathscr{L}_n \gamma_{\mu\nu}, \tag{6.18}$$

where $\mathscr{L}$ is the Lie derivative. The Riemann tensor of the $(d+1)$ dimensional manifold can now be expressed entirely in terms of the intrinsic curvature and extrinsic curvature of Σ_r via the Gauss-Codazzi relations

$$\begin{aligned}
\gamma^\alpha_\mu \gamma^\beta_\nu \gamma^\gamma_\rho \gamma^\delta_\sigma R_{\alpha\beta\gamma\delta} &= \hat{R}_{\mu\nu\rho\sigma} + K_{\mu\sigma}K_{\nu\rho} - K_{\mu\rho}K_{\nu\sigma}; \\
\gamma^\rho_\nu n^\sigma R_{\rho\sigma} &= D_\mu K^\mu_\nu - D_\nu K^\mu_\mu; \\
n^\rho n^\sigma R_{\mu\rho\nu\sigma} &= -n^\rho \nabla_\rho K_{\mu\nu} - K_{\mu\rho}K^\rho_\nu,
\end{aligned} \tag{6.19}$$

where D_μ is the covariant derivative of the metric γ and $\hat{R}$ denotes the curvature of this metric. A useful manipulation of these equations gives the following identities

$$\begin{aligned}
K^2 - K^{\mu\nu}K_{\mu\nu} &= \hat{R} + 2G_{\mu\nu}n^\mu n^\nu; \\
\mathscr{L}_n K_{\mu\nu} + KK_{\mu\nu} - 2K^\rho_\mu K_{\rho\nu} &= \hat{R}_{\mu\nu} - \gamma^\rho_\mu \gamma^\sigma_\nu R_{\rho\sigma},
\end{aligned} \tag{6.20}$$

with $G_{\mu\nu}$ the (bulk) Einstein tensor. One can simplify these expressions by fixing the gauge freedom such that $N = 1$ and $N^\mu = 0$. In this gauge

$$\begin{aligned}
ds^2 &= dr^2 + \gamma_{ij}dx^i dx^j; \\
K_{ij} &= \frac{1}{2}\partial_r \gamma_{ij}.
\end{aligned} \tag{6.21}$$

Moreover the Gauss-Codazzi relations which we will need can be written in terms of the trace adjusted stress energy tensor as

$$R_{rirj} = -\partial_r K_{ij} + K^k_i K_{kj}; \tag{6.22}$$

[4]Such a foliation would also be appropriate near timelike infinity, in which case r would be a time coordinate and $g_{rr} < 0$ in Lorentzian signature.

$$
\begin{aligned}
K^2 - K_{ij}K^{ij} &= \hat{R} + \bar{T}_{rr} - \gamma^{ij}\bar{T}_{ij}; \\
\partial_r K_{ij} - 2K_i^l K_{lj} + KK_{ij} &= \hat{R}_{ij} - \bar{T}_{ij}.
\end{aligned}
$$

Returning to (6.16) the terms in the variation which do not vanish given a Dirichlet condition on the metric, $\delta\gamma = 0$, in this gauge take the form

$$
\begin{aligned}
\frac{2\alpha}{\kappa^2}\int_{\partial\mathcal{M}} d^d x\sqrt{-\gamma}R_{rirj}\delta\gamma^{ij} &= \frac{2\alpha}{\kappa^2}\int_{\partial\mathcal{M}} d^d x\sqrt{-\gamma}(-\partial_r K_{ij} + K_i^k K_{kj})\partial_r\delta\gamma^{ij} \qquad (6.23) \\
&= \frac{2\alpha}{\kappa^2}\int_{\partial\mathcal{M}} d^d x\sqrt{-\gamma}(KK_{ij} - K_i^k K_{kj} - \hat{R}_{ij} + \bar{T}_{ij})\partial_r\delta\gamma^{ij},
\end{aligned}
$$

where in the last equality the bulk equation of motion in Gauss-Codazzi form has been used. Next we note that

$$
\begin{aligned}
\delta(KK^{ij}K_{ij}) &= \frac{1}{2}K^{ij}K_{ij}\gamma^{kl}\partial_r\delta\gamma_{kl} + KK^{ij}\partial_r\delta\gamma_{ij} + \cdots \qquad (6.24) \\
\delta(K^3) &= \frac{3}{2}K^2\gamma^{ij}\partial_r\delta\gamma_{ij} + \cdots; \\
\delta(\hat{R}K) &= \frac{1}{2}\hat{R}\gamma^{ij}\partial_r\delta\gamma_{ij} + \cdots; \\
\delta(K_{ij}K^{jk}K_k^i) &= \frac{3}{2}K^{kj}K_j^i\partial_r\delta\gamma_{ij} + \cdots; \\
\delta(\hat{R}^{ij}K_{ij}) &= \frac{1}{2}\hat{R}^{ij}\partial_r\delta\gamma_{ij} + \cdots;
\end{aligned}
$$

where ellipses denote terms which do not depend on the normal derivative of the metric derivation. Then

$$
\begin{aligned}
&\frac{2\alpha}{\kappa^2}\int_{\partial\mathcal{M}} d^d x\sqrt{-\gamma}(KK_{ij} - K_i^k K_{kj} - \hat{R}_{ij} + \bar{T}_{ij})\partial_r\delta\gamma^{ij} \qquad (6.25) \\
&= \frac{2\alpha}{\kappa^2}\int_{\partial\mathcal{M}} d^d x\sqrt{-\gamma}\delta(KK^{ij}K_{ij} - \frac{1}{3}K^3 + \hat{R}K - \frac{2}{3}K_i^k K_{kj}K^{ij} - 2\hat{R}_{ij}K^{ij}) \\
&\quad + \frac{2\alpha}{\kappa^2}\int_{\partial\mathcal{M}} d^d x\sqrt{-\gamma}(\gamma^{ij}(\bar{T}_k^k - \bar{T}_{rr}) + \bar{T}^{ij})\partial_r\delta\gamma_{ij} + \cdots
\end{aligned}
$$

The terms in the second line are written in terms of quantities intrinsic to the boundary and define an analogue of the GHY term but the remaining terms left over in the last line cannot, in general, be expressed in terms of such quantities.

Suppose however that one works perturbatively in α, evaluating the corrections as a perturbative series on the leading order metric, so that

$$
g_{\mu\nu} = g^{(0)}_{\mu\nu} + \alpha g^{(1)}_{\mu\nu} + \cdots \qquad (6.26)
$$

Working to order α in the action one needs to evaluate the terms involving $\bar{T}$ only to zeroth order in α, i.e.

$$(\gamma^{ij}(\bar{T}^k_k - \bar{T}_{rr}) + \bar{T}^{ij}) \to d\Lambda\gamma^{ij} \tag{6.27}$$

so that

$$\int_{\partial\mathcal{M}} d^d x\sqrt{-\gamma}(\gamma^{ij}(\bar{T}^k_k - \bar{T}_{rr}) + \bar{T}^{ij})\partial_r\delta\gamma_{ij} \to \int_{\partial\mathcal{M}} d^d x\sqrt{-\gamma}2d\Lambda\delta K + \cdots \tag{6.28}$$

That is, applying the field equations perturbatively, the problematic term can indeed be reexpressed in terms of quantities which are intrinsic to the boundary.

Putting the terms together, we see that working up to order α the boundary term needed to set up a well-defined Dirichlet variational problem at a finite radial boundary is

$$\begin{aligned} I_{GHY} &= -\frac{1}{\kappa^2}\int_{\partial M} d^d x\sqrt{-\gamma}K \\ &\quad -\frac{2\alpha}{\kappa^2}\int_{\partial\mathcal{M}} d^d x\sqrt{-\gamma}\Big(KK^{ij}K_{ij} - \frac{1}{3}K^3 + \hat{R}K \\ &\quad -\frac{2}{3}K^k_i K_{kj}K^{ij} - 2\hat{R}_{ij}K^{ij} + 2d\Lambda K\Big), \end{aligned} \tag{6.29}$$

where implicitly in the first line one needs the metric to order α whilst in the second line one needs the metric only to zeroth order in α.

It would be interesting to extend this proof to show that the variational problem is well-defined to arbitrary order. To do this one would need to argue that the problematic term

$$\frac{2\alpha}{\kappa^2}\int_{\partial\mathcal{M}} d^d x\sqrt{-\gamma}(\gamma^{ij}(\bar{T}^k_k - \bar{T}_{rr}) + \bar{T}^{ij})\partial_r\delta\gamma_{ij} \tag{6.30}$$

can always be expressed as the variation of a term intrinsic to the boundary, when the bulk equations of motion are used iteratively. Such an all orders proof could be developed using similar inductive techniques to [208], [209].

6.3 Gravity models

In this section we will consider higher derivative corrections to Einstein gravity with a negative cosmological constant in four bulk dimensions. Before we describe the features of various models, let us comment on top down derivations of the effective action. One might think that it would be straightforward to work out the leading order corrections to the action from the reduction of ten or eleven dimensional actions, i.e. one could exploit our knowledge of the M theory action or the type II string actions. Here we point out that there are many subtleties in implementing such as strategy and our knowledge of these actions is not currently adequate to derive the corrections to AdS gravity actions in lower dimensions.

To illustrate this point let us consider the best understood top down possibility to obtain AdS_4, the reduction of M theory on a seven dimensional Sasaki-Einstein SE_7 to four dimensions. At the level of supergravity, it is always consistent to retain just the four-dimensional graviton in the lower dimensional theory, i.e. the eleven dimensional equations are solved by eleven-dimensional fields such that

$$\begin{aligned} ds^2_{11} &= ds^2_4(E_4) + ds^2_7(SE_7); \\ F_4 &= \eta_{E_4}, \end{aligned} \tag{6.31}$$

where E_4 is any Einstein manifold with negative cosmological constant, $\eta_(E_4)$ is the volume form of this manifold and the metric reduction is diagonal over the SE_7. The effective four dimensional action is written only in terms of the metric on E_4, $g_{\mu\nu}$. Note however that not only the eleven-dimensional metric g_{mn} but also the four form F_4 in eleven dimensions are non trivial, and the Riemann tensor of the SE_7 is also non trivial since the manifold has positive curvature.

Let us consider what this implies for the higher derivative corrections to the effective four-dimensional action. Since the four form is non-trivial at leading order, to compute the higher derivative corrections to the leading order solution, one would need to know higher derivative corrections to the eleven-dimensional action involving not just curvatures but also the four form. Building on [210], [211], leading corrections involving the latter in eleven dimensions were worked out in [212]; they have the structure

$$\begin{aligned} I &= \int d^{11}x\sqrt{-g}a\left(t_8 t_8 R^4 + \frac{1}{4!}\epsilon_{11}R^4\right) \\ &\quad + \int d^{11}x\sqrt{-g}b\left(t_8 t_8 R^4 - \frac{1}{4!}\epsilon_{11}R^4 - \frac{1}{6}\epsilon_{11}t_8 A R^4 + [R^3F^2] + [R^2(DF)^2]\right), \end{aligned} \tag{6.32}$$

where a and b are coefficients. It is known by comparison with IIA string calculations that

$$b = \frac{1}{2\kappa^2_{11}}\frac{l_p^6}{2^8 4!}\frac{\pi^2}{3}, \tag{6.33}$$

with $2\kappa_{11}^2 = (2\pi)^8 l_p^9$. Here ϵ_{11} is the eleven-dimensional epsilon, t_8 consists of 4 Kronecker deltas and $t_8 t_8 R^4$ denotes a specific product of such such t_8 tensors and four Riemann tensors; the explicit expressions will not be needed here. A is the three form of which F is the four form field strength. The tensor structure of the terms denoted $[R^3 F^2]$ and $[R^2 (DF)^2]$ is also not important here; all we need is this schematic form, in which D denotes the covariant derivative.

One might think that the knowledge of such terms would suffice to compute the leading corrections to the eleven-dimensional solution of interest, (6.31), and that the these corrections could be rewritten in terms of a corrected equation for the four-dimensional metric $g_{\mu\nu}$, and hence in terms of a corrected four dimensional action. Apart from the complexity of the actual calculation, there would be a number of subtleties in actually carrying this out.

First of all, one cannot assume a priori that the higher order terms do not induce additional four-dimensional fields, as well as the metric, although it seems reasonable that in some cases they do not. For example, consider a four-dimensional massless scalar field ϕ which corresponds to a modulus of the dual conformal field theory. In principle, even though this field is constant at leading order, it could be sourced by a higher derivative correction, i.e. one could have an equation such as

$$\Box\phi \sim R^n \tag{6.34}$$

where R^n denotes schematically a scalar curvature invariant of order n. The latter must be zero when evaluated on AdS itself, as one does not expect the conformal invariance to be broken, but this argument could not exclude invariants of the Weyl tensor occurring.

Even if could argue that a four-dimensional action involving only $g_{\mu\nu}$ exists, there is a second obstacle in actually computing such an action To illustrate this point, consider just one of the tensor structures occurring in the R^4 invariant

$$\frac{1}{l_p^3}(R_{mnpq}R^{mnpq})^2. \tag{6.35}$$

Evaluated on the lowest order metric this picks up contributions

$$\frac{1}{l_p^3}\Big((R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})^2 + (R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})(R_{abcd}R^{abcd}) + (R_{abcd}R^{abcd})^2\Big), \tag{6.36}$$

where R_{abcd} is the Riemann curvature of the Sasaki Einstein. For any given Sasaki Einstein this would then result in a term of the form

$$I \sim \frac{V_{SE_7}}{l_p^3} \int d^4x \sqrt{-g}((R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})^2 + b_2(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) + b_0), \tag{6.37}$$

in the four-dimensional action with V_{SE_7} the volume of the Sasaki-Einstein and (b_0, b_2) computable (dimensionful) parameters. That is, a quartic invariant in eleven dimensions can lead to quadratic and constant terms in four dimensions, with the latter shifting the cosmological constant.

Since the Sasaki-Einstein has a curvature radius of the same order as the four dimensional manifold, none of these terms is subleading. Let L be the scale of the curvature radius for both; then each of the three terms in the action is of order L^3/l_p^3, using the fact that Riemann squared is of order $1/L^4$. Note that the Einstein term in the action would be of order L^9/l_p^9. Terms arising from the reduction of higher order invariants in eleven dimensions would be subleading in a power series in L/l_p.

Recall that in the case of $AdS_4 \times S^7$ the radius L scales according to $L^6 \sim N l_p^6$ where N is the rank of the dual gauge group. Therefore the Einstein term gives the well-known scaling of $N^{3/2}$ [72] whilst the terms given above scale as $N^{1/2}$ and thus are suppressed by a factor of $1/N$ relative to the leading order terms. We will use this scaling later when discussing the spectrum. Similarly for the case of ABJM [144] where the eleven-dimensional geometry is $AdS_4 \times S^7/Z_k$ the curvature radius scales according to $L^6 \sim (kN)l_p^6$ where N is the rank of the dual gauge group. Recalling that the volume of the compact space scales as $1/k$ this gives a scaling of $k^{1/2}N^{3/2}$ for the leading Einstein term. One can rewrite this as $k^2\lambda^{3/2}$ where the 't Hooft coupling is $\lambda = N/k$, and this scaling was reproduced from the ABJM theory in [147], [200]. In the ABJM case the term give above would contribute at order $\lambda^{1/2}$, i.e. it differs by a factor of $1/(k^2\Lambda)$ from the leading term. (Note that validity of the eleven-dimensional description requires $N \gg k^5$.)

In conclusion, identifying the leading order corrections in four dimensions is very subtle. The leading order correction in four dimensions indeed derives from the leading order correction in eleven dimensions, but terms involving higher curvature invariants in four dimensions can actually contribute at the same order! Similarly terms in the higher dimensional action involving R^3F^2 and so on can give rise to corrections in four dimensions involving R^3.

Note that the term picked out above (6.35) shifts the cosmological constant and is non zero even when evaluated on AdS itself. This would mean, in particular, that it would be expected to adjust the value of the free energy for the dual theory (at zero temperature) evaluated on an S^3. If one can argue that there is no such renormalisation, then the four-dimensional contributions from such a term must cancel those arising from the reduction of other eleven-dimensional terms. A series of corrections expressed in terms of the Weyl tensor, which vanishes on a maximally symmetric space, would not induce such a change in the free energy.

From the string theory perspective one might think that one should in any case start from a higher dimensional with curvature corrections involving only the Weyl tensor, since corrections involving Ricci and Ricci scalar can always be absorbed into field redefinitions. Here we will look nonetheless look at terms such as (6.35) as well as Weyl terms. Firstly it is is interesting to look at the effects of different curvature invariant structures but secondly the usual field redefinition argument refers to the bulk field equations but does not take into account boundary conditions and onshell thermodynamic quantities. We will see below that it is possible to have terms which do not contribute to the field equations perturbed around a given leading order solutions but which nonetheless change the action and change the spectrum. In particular, curvature squared corrections in four dimensions do not change the metric, so in the past they would have been viewed as trivial, but here we show that they still introduce additional (non-unitary) operators into the dual CFT spectrum.

In what follows, we will pursue a bottom up perspective, in which we consider case by case the effects of various higher derivative terms in four dimensions. In other words, we discuss the effects of adding particular scalar curvature invariants to the four dimensional action. We will then return to the issue of which scalar invariants are expected to arise in top down models.

6.3.1 Curvature squared corrections

Motivated by requirements of renormalizability of gravity, curvature-squared modifications to Einstein's theory were first discussed in [124], [125] and they have been extensively explored in the literature. The most general action involves curvature squared terms can be written as

$$I = \frac{1}{2\kappa^2}\int d^4x\, \sqrt{-g}(R - 2\Lambda + \alpha R^{\mu\nu}R_{\mu\nu} + \beta R^2 + \gamma R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}) \tag{6.38}$$

However, it is well known that the Gauss-Bonnet invariant,

$$E_4 = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2 \tag{6.39}$$

does not contribute to the equations of motion in four dimensions but yields only a surface term. Hence for analysing the field equations we can eliminate the Riemann squared term in the action above, making the most general action we need to consider, modulo the E_4 term, simply

$$I = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}(R - 2\Lambda + \alpha R^{\mu\nu}R_{\mu\nu} + \beta R^2), \tag{6.40}$$

(where implicitly the coefficients (α, β) have been shifted relative to the above.) The equations of motion following from this action are

$$\mathcal{G}_{\mu\nu} + E_{\mu\nu} = 0 \tag{6.41}$$

where

$$\mathcal{G}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} \tag{6.42}$$

and

$$\begin{aligned} E_{\mu\nu} &= 2\alpha(R_{\mu\rho}R^{\rho}_{\nu} - \frac{1}{4}R^{\rho\sigma}R_{\rho\sigma}g_{\mu\nu}) + 2\beta R(R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}) \\ &\quad + \alpha(\Box R_{\mu\nu} + \frac{1}{2}\Box R g_{\mu\nu} - 2\nabla_{\rho}\nabla_{(\mu}R^{\rho}_{\nu)}) + 2\beta(g_{\mu\nu}\Box R - \nabla_{\mu}\nabla_{\nu}R) \end{aligned} \tag{6.43}$$

By analyzing the above equations of motion it immediately follows that all solutions of the $\alpha = \beta = 0$ theory are also solutions of the full theory as $E_{\mu\nu}$ is zero for any Einstein spacetime. In particular, AdS-Schwarzschild black holes

$$ds^2 = -dt^2(\epsilon - \frac{m}{r} + \frac{|\Lambda|}{3}r^2) + \frac{dr^2}{(-\frac{m}{r} + \frac{|\Lambda|}{3}r^2 + \epsilon)} + r^2 d\Omega_2^2(k) \tag{6.44}$$

are solutions of the higher curvature theory. Here $k = 0$ and $\epsilon = 0$ corresponds to the case in which the horizon is flat, with $k = 1$ and $\epsilon = 1$ corresponding to the case in which the horizon is a two-sphere. Note however that the thermodynamic properties are modified in the deformed theory and depend explicitly on the deformation parameters.

It is straightforward to derive the thermodynamic properties in the deformed theory, exploiting the fact that the metric remains Einstein. (For earlier discussions of thermodynamics in bulk dimensions higher than four see [213].) The free energy of the black holes can be obtained by considering the onshell value of the action. In order for the variational problem to be well-defined, the action must be supplemented by boundary terms. For the Einstein part of the action the appropriate boundary term is the Gibbons-Hawking-York term discussed earlier

$$I_{GHY} = -\frac{1}{\kappa^2}\int d^3x K\sqrt{-\gamma}. \tag{6.45}$$

where K denotes the second fundamental form and γ is the boundary metric. This term is not however sufficient to ensure a well-defined variational problem: in varying the bulk action the following boundary terms arise, analogously to those given in (6.16)

$$\frac{1}{\kappa^2}\int_{\partial\mathcal{M}} d\Sigma^{\mu}\left(2\alpha R_{\nu\sigma} + 2\beta R g_{\nu\sigma}\nabla_{\mu}\delta g^{\sigma\nu} + \cdots\right) \tag{6.46}$$

where we again suppress terms which vanish for the boundary condition $\delta g = 0$. Using the equation of motion $R_{\mu\nu} = \Lambda g_{\mu\nu}$ we note that the variational problem will be well posed if we add the following boundary terms

$$I = -\frac{1}{\kappa^2}\int d^3x\sqrt{-\gamma}K(2\alpha\Lambda + 8\beta\Lambda). \tag{6.47}$$

In this case the fact that the solution remains Einstein implies that this term is sufficient to evaluate the onshell action, to all perturbative orders in α and β. (It does not however suffice for discussing fluctuations around the Einstein solution, as we will discuss in section four.)

Evaluating the complete onshell action gives

$$\begin{aligned} I &= \frac{1}{2\kappa^2}2\Lambda(1 + 2\alpha\Lambda + 8\beta\Lambda)\int d^4x\sqrt{-g} \\ &\quad -\frac{1}{\kappa^2}(1 + 2\alpha\Lambda + 8\beta\Lambda)\int d^3x\sqrt{-\gamma}K. \end{aligned} \tag{6.48}$$

Relative to the case of $\alpha = \beta = 0$, there is just an overall prefactor, which means that we can immediately read off from [72], [214], [75] the required holographic counterterms as

$$I_{ct} = -\frac{1}{2\kappa^2}(1 + 2\alpha\Lambda + 8\beta\Lambda)\int d^3x\sqrt{-\gamma}\left(\frac{4}{l} + lR(\gamma)\right), \tag{6.49}$$

where $l^2 = |3/\Lambda|$. The asymptotic expansion of the metric g is [72], [75]

$$\begin{aligned} ds^2 &= l^2\left(\frac{d\rho^2}{\rho^2} + \frac{1}{\rho^2} g_{ij}(x,\rho) dx^i dx^j\right); \\ g_{ij}(x,\rho) &= g_{(0)ij}(x) + \rho^2 g_{(2)ij}(x) + \rho^3 g_{(3)ij} + \cdots, \\ g_{(2)ij} &= -R_{ij}(g_{(0)}) + \frac{R(g_{(0)})}{4} g_{(0)ij}, \end{aligned} \tag{6.50}$$

with $g_{(3)}$ being traceless and divergenceless but otherwise undetermined by the asymptotic analysis. The renormalized stress energy tensor obtained by varying the action with respect to $g_{(0)}$ is then shifted by an overall prefactor relative to [75]

$$\langle T_{ij}\rangle = \frac{3}{2\kappa^2}(1 + 2\alpha\Lambda + 8\beta\Lambda) g_{(3)ij}. \tag{6.51}$$

We can now immediately evaluate thermodynamic quantities for the black hole solutions (6.44); the free energies and masses are clearly shifted relative to those in Einstein gravity by a proportionality factor:

$$-\beta_T F \equiv I^E_{\text{onshell}} = \frac{\beta_T V_{xy}}{2\kappa^2}(1 + 2\alpha\Lambda + 8\beta\Lambda) m \tag{6.52}$$

$$M \equiv \int d^2x \sqrt{-\gamma}\langle T_{00}\rangle = \frac{V_{xy}}{\kappa^2}(1 + 2\alpha\Lambda + 8\beta\Lambda) m \tag{6.53}$$

with β_T the inverse temperature (not to be confused with the coupling constant β) and the temperature being

$$T = \frac{1}{4\pi}\left(\frac{2|\Lambda|}{3} r_h + \frac{m}{r_h^2}\right), \tag{6.54}$$

and r_h is the horizon position. I^E denotes the Euclidean action, which in this static case is straightforwardly computed by analytic continuation of the time. Note that under such a continuation $iI \to -I^E$. One can also work out the black hole entropy using Wald's method [157]. Define

$$\mathcal{Q}^{\mu\nu} = -2\mathcal{L}^{\mu\nu\rho\sigma}\nabla_\rho\, l_\sigma + \cdots \tag{6.55}$$

where $\mathcal{Q}^{\mu\nu}$ is antisymmetric and the terms denoted by ellipses vanish for stationary horizons. Here

$$\mathcal{L}^{\mu\nu\rho\sigma} \equiv \frac{\delta L}{\delta R_{\mu\nu\rho\sigma}}. \tag{6.56}$$

For a stationary horizon the black hole entropy is then given by

$$S = \frac{1}{T}\int_{\mathcal{H}} \mathcal{Q}^{\mu\nu}\, d\Sigma_{\mu\nu}, \tag{6.57}$$

with T being the horizon temperature, $\mathcal{H}$ denoting the horizon and l^κ being the horizon normal.

For Einstein gravity

$$L = \frac{1}{2\kappa^2}\sqrt{-g}(R - 2\Lambda) \tag{6.58}$$

where $\kappa^2 = 8\pi G$ and G is the Newton constant. Hence

$$\frac{\delta L}{\delta R_{\mu\nu\rho\sigma}} = \frac{\sqrt{-g}}{4\kappa^2}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) \tag{6.59}$$

And so

$$\mathcal{Q}^{\mu\nu} = -\frac{\sqrt{-g}}{2\kappa^2}\left(\nabla^\mu l^\nu - \nabla^\nu l^\mu\right) \tag{6.60}$$

and

$$S = \frac{1}{\kappa^2 T}\int_{\mathcal{H}} \sqrt{-g}\left(\nabla^\nu l^\mu \, d\Sigma_{\mu\nu}\right) \equiv \frac{A_h}{4G} \tag{6.61}$$

with A_h the horizon area, using $l_\nu \nabla^\nu l^\mu = \kappa_h l^\mu$ where $\kappa_h = 2\pi T$ is the surface gravity of the horizon.

For the curvature squared corrections, using

$$\begin{aligned} \frac{\delta R^2}{\delta R_{\mu\nu\rho\sigma}} &= R(g^{\mu\rho}g^{\sigma\nu} - g^{\mu\sigma}g^{\nu\rho}); \\ \frac{\delta(R^{\tau\eta}R^{\tau\eta})}{\delta R^{\mu\nu\rho\sigma}} &= (R^{\mu\rho}g^{\sigma\nu} - R^{\mu\sigma}g^{\nu\rho}), \end{aligned} \tag{6.62}$$

the Wald entropy becomes

$$S = (1 + 2\alpha\Lambda + 8\beta\Lambda)\frac{A_h}{4G}. \tag{6.63}$$

Putting these results together one finds that the thermodynamic relations

$$F = M - TS; \qquad dM = TdS, \tag{6.64}$$

are indeed satisfied.

To summarise, the metric is uncorrected but the thermodynamic properties of the black holes are adjusted: the entropy, the temperature, the mass and the free energy are all changed, albeit by just an overall factor. It is also interesting to note that the action evaluated on AdS_4 with S^3 boundary is also changed. In the latter case the relevant bulk metric is

$$ds^2 = \frac{3}{|\Lambda|}\left(\mathrm{d}\rho^2 + \sinh^2\rho\ \mathrm{d}\Omega_3^2\right), \tag{6.65}$$

where $0 < \rho < \infty$. Using the renormalised action given above, one can compute the onshell Euclidean action to be

$$I^E_{\text{onshell}} = \frac{12\psi^2}{|\Lambda|\kappa^2}(1 + 2\Lambda\alpha + 8\Lambda\beta). \tag{6.66}$$

This is therefore corrected by the curvature squared terms except when $\alpha = -4\beta$, which corresponds to the case in which the correction is Riemann squared minus E_4. Given a holographic dual in which one can compute the free energy on S^3 by localisation techniques, the answer will give a criterion restricting the terms which can arise in the effective four-dimensional action. In particular, the case of ABJM theory, for which the exact expression for the planar free energy was obtained in [147], [200], will be explored in detail elsewhere.

At this point it would seem as if the addition of such terms to the action is rather trivial because the thermodynamic quantities are shifted by an overall factor, which could be reabsorbed into the cosmological constant. However, we will discuss in section 6.4, these terms are highly non-trivial when one looks at the spectrum of the theory. To find the spectrum of the dual CFT linearize the above field equations about the background solution AdS_4. As we discuss in section 6.4, the bulk theory is found to describe a massless spin-2 graviton, a massive scalar and a massive spin-2 field. By tuning the coefficients so that $\alpha = -3\beta$ one may eliminate the massive scalar mode. One can also tune the remaining coefficient β to the so-called critical value [126]

$$\beta = -\frac{1}{2\Lambda} \tag{6.67}$$

where the massive spin two mode becomes logarithmic [215]. Noting that the AdS-Schwarzschild mass when $\alpha = -3\beta$ behaves as

$$M = \frac{m}{\kappa^2}(1 + 2\beta\Lambda) \tag{6.68}$$

we see that in the critical theory the black hole solution has zero mass. One can show furthermore that the Wald entropy vanishes at the critical point.

However we should emphasize that the critical value (6.67) can clearly never be achieved when the Planck length is small compared to the curvature radius of AdS. If one is viewing the higher curvature corrections as arising from a top down string model then $\beta\Lambda$ is necessarily much smaller than one. The critical theory does not therefore provide a good model for corrections to macroscopic AdS_4 black holes.

Let us now make a connection to conformal gravity. With the first parameter choice of $\alpha = -3\beta$ we may rewrite the higher curvature term in terms of the Weyl tensor:

$$-\frac{1}{3}\alpha(R^2 - 3R^{\mu\nu}R_{\mu\nu}) = \frac{1}{2}\alpha(C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} - E_4) \tag{6.69}$$

Hence the Lagrangian is equivalent to

$$I = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}\left(R - 2\Lambda + \frac{1}{2}\alpha(C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} - E_4)\right). \tag{6.70}$$

By taking the limit of $\alpha \to \infty$ one recovers Weyl gravity, see related discussions in [216], but again this would not be reached as a small correction from an Einstein solution.

The variation of the Weyl squared term with respect to the metric is linear in the Weyl tensor. This means that this correction vanishes identically when evaluated on AdS_4 since it has vanishing Weyl tensor. Therefore we can deduce from (6.66) that the onshell Euclidean action evaluated on AdS_4 with S^3 boundary for

$$I^E = -\frac{1}{2\kappa^2}\int d^4x\sqrt{g}\left(R - 2\Lambda - \frac{1}{2}\alpha E_4\right) \tag{6.71}$$

is

$$I^E_{\text{onshell}} = \frac{12\pi^2}{|\Lambda|\kappa^2}(1 - \frac{2\alpha\Lambda}{3}). \tag{6.72}$$

In other words, the topological invariant does of course contribute to the action even though it does not affect the field equations. The renormalised E_4 term captures the Euler invariant of the manifold with S^3 conformal boundary. Tuning to the critical value (6.67) this action is zero.

6.3.2 $f(R)$ Gravity

In our exploration of corrected black hole solutions we will now move on to consider an $f(R)$ theory. The $f(R)$ theory is obtained when we add a generic polynomial in the Ricci scalar R to the usual Einstein action,

$$I = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}(R - 2\Lambda + f(R)) \tag{6.73}$$

where

$$f(R) = \sum_{n\geq 2} \alpha_n R^n, \tag{6.74}$$

with arbitrary coefficients α_n. There is considerable interest in $f(R)$ theories in the context of phenomenology and cosmology, see the reviews of [217], [218], [219], even though such corrections are not well motivated from top down considerations, since they can be removed by field redefinitions.

Indeed it is well known that such a correction will not change the leading order black hole solution non-trivially, although it will change its thermodynamic properties. This follows from the equations of motion

$$\mathcal{G}_{\mu\nu} + F_{\mu\nu} = 0 \tag{6.75}$$

with $\mathcal{G}_{\mu\nu}$ defined in (6.42) and

$$\begin{aligned} F_{\mu\nu} &= \sum_{n\geq 2}\alpha_n n R^{n-1}(R_{\mu\nu} - \frac{1}{2n}Rg_{\mu\nu}) \\ &+ \sum_{n\geq 2}\alpha_n n(g_{\mu\nu}\Box R^{n-1} - \nabla_\mu\nabla_\nu R^{n-1}). \end{aligned} \tag{6.76}$$

Consider an Einstein solution $g_{\mu\nu}$ which satisfies

$$R_{\mu\nu} = \lambda g_{\mu\nu}; \qquad R_{\mu\nu} = \frac{1}{4}Rg_{\mu\nu}, \tag{6.77}$$

Evaluated on such a solution both the second line together with the $n = 2$ term in the first line of $F_{\mu\nu}$ vanish and

$$F_{\mu\nu} = \sum_{n>2}\alpha_n(n-2)(4\lambda)^{n-1}\lambda g_{\mu\nu} \equiv -\delta\Lambda g_{\mu\nu}. \tag{6.78}$$

The field equations (6.75) are then satisfied provided that

$$\Lambda = \lambda + \sum_{n>2} \alpha_n (2-n)(4\lambda)^{n-1}\lambda; \tag{6.79}$$

i.e. the higher derivative term acts to shift the effective cosmological constant. Treating the $f(R)$ term as a small perturbation around the leading order solution by setting all coefficients $\alpha_n \ll 1$, we may express

$$\lambda \approx \Lambda(1 - \alpha_3(4\Lambda)^2 - 2\alpha_4(4\Lambda)^3 + \cdots), \tag{6.80}$$

where in the non-linear terms we use the leading order behavior $\lambda \sim \Lambda$. Therefore any Einstein solution remains an Einstein solution in the corrected theory, but with a shifted cosmological constant.

One should again note that the corrected theory does admit non-Einstein solutions, but any solution which reduces to an Einstein solution in the leading order theory remains Einstein in the corrected theory. In other words, when one treats the higher order terms perturbatively one discards solutions which do not reduce to Einstein solutions on setting α_n to zero. The higher derivative terms with recur, however, when one discusses the spectrum as one will obtain new propagating modes.

As a warm up exercise for the non-trivial corrections discussed in the following sections it is useful to derive the corrections to static solutions as follows, using a similar method to that of [220] and also [221]. Let the metric be parameterized as

$$ds^2 = -a(r)b^2(r)dt^2 + \frac{dr^2}{a(r)} + r^2(dx^2 + dy^2), \tag{6.81}$$

where we now focus on the case of flat horizons. Substituting this metric ansatz, the action reduces to

$$I = \frac{1}{2\kappa^2}\beta_T V_{xy}\hat{I} \tag{6.82}$$

with β_T the periodicity in time; V_{xy} the regulated volume of the $(x-y)$ plane and

$$\begin{aligned} \hat{I} \quad = \quad & -\int dr \left[2rb'(-\frac{\Lambda}{3}r^2 - a) + \sum_n (-1)^n \alpha_n \frac{A^n}{(br^2)^{n-1}}\right] \\ & - \left[\frac{2}{3}\Lambda b r^3 + 2ab'r^2 + a'br^2 + 2abr\right]_{r_h}^{\infty} \end{aligned} \tag{6.83}$$

where we have used the fact that the Ricci scalar is given by

$$R = -\frac{A(r)}{b(r)r^2} \tag{6.84}$$

with $A(r)$ given by

$$A(r) \equiv 3a'b'r^2 + 2ab''r^2 + a''br^2 + 4a'br + 4ab'r + 2ab. \tag{6.85}$$

The second line in (6.83) arises from partial integrations. Varying the bulk term in the action we find the following equations of motion for a and b:

$$0 = 2rb' - \sum_n l^{n-1}(-1)^n \alpha_n \frac{\delta}{\delta a}\left(\frac{A^n}{(br^2)^{n-1}}\right) \tag{6.86}$$

where

$$\begin{aligned}\frac{\delta}{\delta a}\left(\frac{A^n}{(br^2)^{n-1}}\right) &= \frac{nA^{n-1}}{(br^2)^{n-1}}(2b''r^2 + 4b'r + 2b) - \frac{d}{dr}\left(\frac{nA^{n-1}}{(br^2)^{n-1}}(3r^2b' + 4b)\right) \\ &\quad + \frac{d^2}{dr^2}\left(\frac{nA^{n-1}}{(br^2)^{n-1}}(br^2)\right)\end{aligned} \tag{6.87}$$

The other equation of motion is

$$0 = -2\Lambda r^2 - 2a - 2a'r - \sum_n l^{n-1}(-1)^n \alpha_n \frac{\delta}{\delta b}\left(\frac{A^n}{(br^2)^{n-1}}\right) \tag{6.88}$$

where

$$\begin{aligned}\frac{\delta}{\delta b}\left(\frac{A^n}{(br^2)^{n-1}}\right) &= \frac{(1-n)A^n}{r^{2n-2}b^n} + \frac{nA^{n-1}}{(br^2)^{n-1}}(a''r^2 + 4a' + 2a) \\ &\quad - \frac{d}{dr}\left(\frac{nA^{n-1}}{(br^2)^{n-1}}(3r^2a' + 4ar)\right) + \frac{d^2}{dr^2}\left(\frac{nA^{n-1}}{(br^2)^{n-1}}(2ar^2)\right).\end{aligned} \tag{6.89}$$

Solving these equations of motion to linear order in the coupling constants α_i we expand as:

$$\begin{aligned}a(r) &= a_{(0)}(r) + \sum_n \alpha_n a_{(n)}(r); \\ b(r) &= b_{(0)}(r) + \sum_n \alpha_n b_{(n)}(r).\end{aligned} \tag{6.90}$$

To leading order, namely all $\alpha_n = 0$, the equations are solved by

$$b_{(0)}(r) = 1; \qquad a_{(0)}(r) = -\frac{1}{3}\Lambda r^2 - \frac{m}{r} \tag{6.91}$$

To linear order in the perturbations, the general solution to the equations of motion is

$$\begin{aligned} b_{(n)}(r) &= 0; \\ a_{(n)}(r) &= (-)^n (n-2)(4\Lambda)^{n-1}\Lambda r^2 - \frac{m_n}{r}. \end{aligned} \tag{6.92}$$

The latter renormalizes the cosmological constant and in addition allows for a shift in the integration constant which parameterizes the black hole mass: the corrected metric is

$$ds^2 = \frac{dr^2}{(-\frac{1}{3}\lambda r^2 - \frac{m}{r} - \sum_n \frac{\alpha_n m_n}{r})} - dt^2(-\frac{1}{3}\lambda r^2 - \frac{m}{r} - \sum_n \frac{\alpha_n m_n}{r}) + r^2 dx \cdot dx. \tag{6.93}$$

Black hole thermodynamics in $f(R)$ theory

For the $f(R)$ term the analysis is of the variational problem is subtle: varying the bulk term gives rise to a boundary variation

$$\delta I = \frac{1}{\kappa^2}\int d^3x f'(R)\delta(K\sqrt{-\gamma}), \tag{6.94}$$

where $f'(R) = \partial_R f(R)$. The appropriate boundary term for a four-dimensional $f(R)$ theory was argued by Hawking and Luttrell [205] to be

$$I_{HL} = -\frac{1}{\kappa^2}\int d^3x f'(R) K\sqrt{-\gamma} \tag{6.95}$$

However, in general this is not satisfactory since R is not intrinsic to the boundary, i.e. it is the scalar curvature of the bulk metric, rather than the boundary metric. In the case at hand however one can use the fact that the onshell Ricci scalar is constant to write this term in terms of quantities manifestly intrinsic to the boundary. Putting all terms together the complete action is

$$\begin{aligned} I &= \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}(R - 2\Lambda + f(R)) \\ &\quad -\frac{1}{\kappa^2}\int d^3x\sqrt{-\gamma}K\left(1 + f'(R)\right). \end{aligned} \tag{6.96}$$

To evaluate the free energy one needs to holographically renormalize this action. To linear order in the couplings for the higher derivative terms, however, one can immediately carry out this procedure using the known results for asymptotically locally AdS Einstein manifolds. Recall that the deformed solution is Einstein, with a different cosmological constant, the metric can always be expressed as

$$ds^2 = \tilde{l}^2 d\bar{s}^2 \tag{6.97}$$

with

$$\tilde{l}^2 = \frac{3}{|\lambda|}; \qquad \bar{R}_{\mu\nu} = -3\bar{g}_{\mu\nu} \tag{6.98}$$

and $\lambda < 0$. The asymptotic expansion of the metric $\bar{g}$ is known

$$\begin{aligned} ds^2 &= \frac{d\rho^2}{\rho^2} + \frac{1}{\rho^2}\bar{g}_{ij}(x,\rho)dx^i dx^j; \\ \bar{g}_{ij}(x,\rho) &= \bar{g}_{(0)ij}(x) + \rho^2 \bar{g}_{(2)ij}(x) + \rho^3 \bar{g}_{(3)ij} + \cdots, \\ \bar{g}_{(2)ij} &= -\bar{R}_{ij}(\bar{g}_{(0)}) + \frac{\bar{R}(\bar{g}_{(0)})}{4}\bar{g}_{(0)ij}, \end{aligned} \tag{6.99}$$

with $\bar{g}_{(3)}$ being traceless and divergencless but otherwise undetermined by the asymptotic analysis. Given this form for the asymptotic expansion one can now compute the regulated action and hence the counterterms. In doing so one can use the fact that to linear order in the new couplings

$$f(R) \to f(R)|_{R=4\Lambda}. \tag{6.100}$$

Setting $\Lambda = -3$ so that the metric to leading order is normalized to unit curvature radius, the required counterterms are then

$$\begin{aligned} I_{ct} &= -\frac{1}{2\kappa^2}(1+\sum_n 2\alpha_n(-12)^{n-1})\int d^3x\sqrt{-\bar{\gamma}}\left(4+\bar{R}(\bar{\gamma})\right); \\ &= \frac{1}{2\kappa^2}(1+\sum_n(\frac{3n}{2}-1)\alpha_n(-12)^{n-1})\int d^3x\sqrt{-\bar{\gamma}}\left(4+(1+\sum_n(n-2)\alpha_n(-12)^{n-1})\bar{R}(\bar{\gamma})\right), \end{aligned} \tag{6.101}$$

and the renormalized stress energy tensor obtained by varying the action with respect to $\bar{g}_{(0)}$ is

$$\langle T_{ij}\rangle = \frac{3}{2\kappa^2}(1+\sum_n 2\alpha_n(-12)^{n-1})\bar{g}_{(3)ij}. \tag{6.102}$$

One can then compute the mass of the black hole in (6.93) as

$$M = \int d^2x\langle T_{00}\rangle = \frac{V_{xy}}{2\kappa^2}(1+\sum_n 2\alpha_n(-12)^{n-1})(m+\sum_n \alpha_n m_n), \tag{6.103}$$

and evaluating the onshell action gives

$$-\beta_T F = I^E \equiv = \beta_T \frac{V_{xy}}{2\kappa^2}(1+\sum_n 2\alpha_n(-12)^{n-1})(m+\sum_n \alpha_n m_n) \tag{6.104}$$

with F the free energy and the black hole temperature being $1/\beta_T$.

In the $f(R)$ theory using the fact that

$$\frac{\delta R^n}{\delta R_{\mu\nu\rho\sigma}} = \frac{1}{2} n R^{n-1}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}), \tag{6.105}$$

the Wald entropy (6.57) is given by

$$S = (1+\sum_n \alpha_n (4\Lambda)^{n-1})\frac{2\pi A_h}{\kappa^2}. \tag{6.106}$$

Evaluating this one obtains

$$S = \frac{2\pi V_{xy}}{\kappa^2}(1+\sum_n 2\alpha_n(-12)^{n-1})(m+\sum_n \alpha_n m_n)^{2/3}. \tag{6.107}$$

Finally the temperature of the black hole is given by

$$T = \frac{3}{4\pi}(m+\sum_n \alpha_n m_n)^{1/3}. \tag{6.108}$$

Putting these results together one sees that the relation $F = M - TS$ is satisfied together with the first law $dM = TdS$. Moreover, it is clear that by choosing the integration constants m_n such that

$$m_n = -2(-12)^{n-1} m \tag{6.109}$$

the black hole in the $f(R)$ theory has unchanged thermodynamic properties to leading order in the coupling constants α_n.

6.3.3 Einstein + C^3

We now move on to consider the addition to the action of curvature invariants of degree three or higher. At this point it is useful to look at classifications of scalar curvature invariants in our dimensions. One such set of invariants are the Carminati-McLenaghan invariants, [222]. At degree three the possible invariants include both those built of lower degree invariants, for example $RC^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}$, and invariants built by contracting three tensors with each other. At degree three the latter gives the new invariants

$$\begin{array}{lll} S^{\mu\rho}S_{\rho\nu}S^{\nu}_{\mu} & C^{\mu\nu}{}_{\rho\sigma}C^{\rho\sigma}{}_{\tau\eta}C^{\tau\eta}{}_{\mu\nu} & *\,C^{\mu\nu}{}_{\rho\sigma}C^{\rho\sigma}{}_{\tau\eta}C^{\tau\eta}{}_{\mu\nu} \\ S^{\mu\nu}S_{\rho\sigma}C_{\mu\rho\nu\sigma} & S^{\mu\nu}S_{\rho\sigma}*C_{\mu\rho\nu\sigma} & \end{array} \tag{6.110}$$

where $S_{\mu\nu}$ is the traceless Ricci tensor, $C_{\mu\nu\rho\sigma}$ is the Weyl tensor and $*C_{\mu\nu\rho\sigma}$ denotes the dual of the Weyl tensor. Note that these comprise an over complete set of invariants for a planar static spacetime. Curvature invariants built from the Ricci scalar or Ricci tensor will behave qualitatively similarly to those at quadratic order, leaving the metric unchanged but shifting the action. Therefore in this section we will focus on the effect of the cube of the Weyl tensor on planar black hole solutions, which is qualitatively different.

The action we consider is therefore

$$I = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}\left(R-2\Lambda+\alpha C_{\mu\nu}{}^{\rho\sigma}C_{\rho\sigma}{}^{\eta\lambda}C_{\eta\lambda}{}^{\mu\nu}\right) \tag{6.111}$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor.

Since the general field equations are somewhat complicated in this case, the easiest way to obtain the corrections to the planar black holes is as follows. Evaluated on a static ansatz (6.81) the action reduces to and effective one dimensional action

$$\begin{aligned} \hat{I} \;=\; & \int_{r_H}^{\infty} dr\,\left[2rb'(a+\frac{\Lambda}{3}r^2)-\alpha\frac{B^3}{18r^4b^2}\right] \\ & -\left[\frac{2}{3}\Lambda br^3+2ab'r^2+a'br^2+2abr\right]_{r_h}^{\infty} \end{aligned} \tag{6.112}$$

where

$$B \equiv 2ab-2rba'-2rab'+3r^2a'b'+r^2ba''+2r^2ab'' \tag{6.113}$$

Varying this action we get the equations of motion for a and b and these are solved perturbatively in α as in Sections 6.3.2 and 6.3.4. There is in this case a non-trivial correction to the planar black hole solution:

$$a_{(1)}(r) = \frac{a_1}{r}-\frac{8m^2\Lambda}{r^4}-\frac{16m^3}{r^7} \tag{6.114}$$

and

$$b_{(1)}(r) = b_1-6\frac{m^2}{r^6}. \tag{6.115}$$

Here a_1 and b_1 are arbitrary integration constants. The former acts as a redefinition of the mass parameter m at order α and the latter changes the norm of the time Killing vector at infinity at order α. We will discuss the interpretation of these integration constants further below but setting them to zero we obtain

$$\begin{aligned} a(r) &= a_{(0)}(r) + \alpha a_{(1)}(r) \\ &= -\frac{\Lambda}{3}r^2 - \frac{m}{r} + \alpha\left(-\frac{8m^2\Lambda}{r^4} - \frac{16m^3}{r^7}\right), \end{aligned} \tag{6.116}$$

$$\begin{aligned} b(r) &= b_{(0)}(r) + \alpha b_{(1)}(r) \\ &= 1 + \alpha\left(-6\frac{m^2}{r^6}\right). \end{aligned}$$

Note that the AdS solution itself is uncorrected, as one would expect: the contribution to the field equations from the variation of the Weyl cubed term is given below in (6.122) and evaluated on a solution with vanishing Weyl tensor it is zero.

Thermodynamics of corrected black hole solutions

Let us now work out the thermodynamics of the corrected black hole solution. The horizon is given by r_H such that $a(r_H) = 0$. Since $a(r) = a_{(0)}(r) + \alpha a_{(1)}(r)$, we find r_H also to order α. Let

$$r_H = r_{H(0)} + \alpha r_{H(1)} \tag{6.117}$$

where

$$\begin{aligned} r^3_{H(0)} &= -\frac{3m}{\Lambda}; \\ r_{H(1)} &= r_{H(0)}\left(-\frac{2^3}{3^3}\Lambda^2\right). \end{aligned} \tag{6.118}$$

The temperature of this black hole solution is given by:

$$\begin{aligned} T &= \frac{a'(r_H)b(r_H)}{4\pi} \\ &= \frac{|\Lambda|}{4\pi}r_{H(0)}\left(1 - \alpha\Lambda^2\frac{2}{27}\right). \end{aligned} \tag{6.119}$$

We can also work out the black hole entropy (6.57) giving

$$S = \frac{r^2_{H(0)}}{4G}(1 + \frac{2\alpha\Lambda^2}{27}). \tag{6.120}$$

Note that although both the temperature and the entropy are corrected at order α the combination TS is actually uncorrected at this order. Moreover, imposing the thermodynamic relation

$$dM = TdS, \tag{6.121}$$

and using the fact that the C^3 term evaluated on pure AdS is zero, we can infer that the mass must also be unchanged at order α. (In varying the entropy note that both α and Λ are held fixed.) Using the relation $F = M - TS$ we can also then infer that the onshell action must also be unchanged at order α.

One can also argue that the free energy and mass are unchanged at order α by considering their direct evaluation. Let us consider first the onshell action. The first step is to ensure that the variational problem, including the additional C^3 term, is well-posed at finite radius. To investigate this we vary the bulk action (6.111) with respect to the metric. From the term at order α one obtains the following contribution to the bulk field equation

$$\begin{aligned}
\mathcal{G}^{\mu\nu} \;=\;& \frac{1}{2}g^{\mu\nu}C_{\rho\sigma}{}^{\tau\eta}C_{\tau\eta}{}^{\lambda\kappa}C_{\lambda\kappa}{}^{\rho\sigma} - 6C_{\rho\sigma}{}^{\mu\eta}C^{\nu}{}_{\eta\lambda\kappa}C^{\lambda\kappa\rho\sigma} + 3C^{\tau\eta\lambda\kappa}C_{\lambda\kappa}{}^{\mu\sigma}R^{\nu}{}_{\sigma\tau\eta} \\
&+ 4C^{\tau\nu\lambda\kappa}C_{\lambda\kappa}{}^{\mu\sigma}R_{\tau\sigma} - 2C^{\tau\nu\lambda\kappa}C_{\lambda\kappa}{}^{\mu\sigma}Rg_{\tau\sigma} - C^{\rho\sigma\tau\eta}C_{\rho\sigma\tau\eta}R^{\mu\nu} \\
&+ 6\nabla_\sigma\nabla_\tau(C^{\tau\nu\lambda\kappa}C_{\lambda\kappa}{}^{\mu\sigma}) + 2\nabla_\tau\nabla^\mu(C^{\tau\eta\lambda\kappa}C_{\lambda\kappa}{}^{\rho\nu}g_{\rho\eta}) + 2\nabla_\sigma\nabla^\mu(C^{\nu\eta\lambda\kappa}C_{\lambda\kappa}{}^{\rho\sigma}g_{\rho\eta}) \\
&- 2\Box(C^{\mu\eta\lambda\kappa}C_{\lambda\kappa}{}^{\rho\nu}g_{\rho\eta}) - 2\nabla_\sigma\nabla_\tau(C^{\tau\eta\lambda\kappa}C_{\lambda\kappa}{}^{\rho\sigma}g_{\rho\eta}g^{\mu\nu}) \\
&- \Box(C^{\rho\sigma\tau\eta}C_{\rho\sigma\tau\eta}g^{\mu\nu}) + \nabla^\mu\nabla^\nu(C^{\rho\sigma\tau\eta}C_{\rho\sigma\tau\eta}),
\end{aligned} \tag{6.122}$$

where $\mathcal{G}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}$. The variation results in the following boundary terms at order α involving derivatives of the metric

$$\begin{aligned}
\frac{1}{\kappa^2}\int_{\partial\mathcal{M}} d^3x\,\sqrt{-\gamma}\,\alpha\Big[\, & 3n_\rho C^{\rho\sigma\eta\tau}C_{\eta\tau}{}^{\mu\nu}\nabla_\nu\delta g_{\mu\sigma} + n^\lambda C^{\rho\sigma\eta\tau}C_{\eta\tau}{}^{\mu\nu}g_{\mu\sigma}\nabla_\rho\delta g_{\lambda\nu} \\
&+ n^\lambda C^{\rho\sigma\eta\tau}C_{\eta\tau}{}^{\mu\nu}g_{\mu\sigma}\nabla_\nu\delta g_{\lambda\rho} - n_\zeta C^{\rho\sigma\eta\tau}C_{\eta\tau}{}^{\mu\nu}g_{\mu\sigma}\nabla^\zeta\delta g_{\rho\nu} \\
&- n_\rho C^{\rho\sigma\eta\tau}C_{\eta\tau}{}^{\mu\nu}g_{\mu\sigma}g^{\xi\lambda}\nabla_\nu\delta g_{\xi\lambda} - C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}\,\delta\mathcal{K}\,\Big],
\end{aligned} \tag{6.123}$$

with n the normal to the boundary. (There are additional boundary terms involving the metric which automatically vanish for a Dirichlet boundary condition.) As one would have anticipated, boundary terms involving the normal derivative of the metric arise in this variation and it would therefore seem as if one needs additional Gibbons-Hawking like terms in order for the variational problem to be well-posed. Moreover, working iteratively in α and then using the bulk field equations to simplify the boundary terms looks a very non-trivial calculation in this case. However, it turns out that one only needs to use the fact that the leading order metric is Einstein and is asymptotically locally AdS: in Fefferman-Graham coordinates (6.50), the leading power in the Weyl tensor necessarily behaves as

$$C_{\mu\nu\rho\sigma} \sim \frac{1}{\rho^2}. \tag{6.124}$$

One can use this behaviour to show that the boundary terms needed for the variational problem to be well-posed all go to zero as a positive power of ρ. For example, the term

$$\frac{1}{\kappa^2}\int_{\partial\mathscr{M}} d^3x\sqrt{-\gamma}C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}\mathscr{K} \tag{6.125}$$

evaluated at $\rho = \epsilon \ll 1$ behaves as ϵ, and therefore does not contribute in the limit $\epsilon \to 0$. Therefore, although one could indeed use the explicit expansion of the onshell Weyl tensor to express the boundary terms in terms of quantities intrinsic to the boundary, the resulting boundary action cannot give a finite contribution to the onshell action.

The onshell action is thus given by

$$\begin{aligned} I_{\text{onshell}} &= \frac{1}{\kappa^2}\int d^4x\sqrt{-g}(\Lambda + \alpha C_{\mu\nu\rho\sigma}C^{\rho\sigma\eta\lambda}C_{\eta\lambda}^{\ \ \mu\nu}) \\ &\quad -\frac{1}{\kappa^2}\int d^3x\sqrt{-\gamma}\left(K + 2/l + \frac{l}{2}R(\gamma)\right), \end{aligned} \tag{6.126}$$

where we have used the onshell relation $R = 4\Lambda + C^3$. The terms in the second line denote the Gibbons-Hawking term along with the counterterms. The latter suffice to remove the divergences at leading order, but do not in general suffice to remove additional divergences at order α. However, it again turns out that the Weyl correction falls off sufficiently fast at the boundary that there are no additional terms needed at order α. To show this one needs to use the fact that, in Fefferman-Graham coordinates, C^3 is of order ρ^6 or smaller and the correction to the metric at order α is of order ρ^6 or smaller. Looking at the terms in the onshell action, this means that the contributions at order α are of order ϵ^3 or smaller, and thus vanish in the limit $\epsilon \to 0$. For example, the term

$$\int d^4x\sqrt{-g}C^3 \sim \int_\epsilon d\rho\frac{1}{\rho^4}\cdot\rho^6 \sim \epsilon^3. \tag{6.127}$$

Thus only the above terms are needed in computing the renormalised action. Note that this implies that, as expected, the onshell action for AdS is uncorrected at order

α, regardless of the choice of conformal class of the boundary metric. In particular, if one computes the free energy for the dual theory on an S^3, it is not changed at order α.

It is still non-trivial that the actual value of the free energy for the planar black hole is uncorrected, as the metric is corrected, the horizon position is shifted and the C^3 term in the action all give finite contributions at the horizon. Explicitly evaluating the onshell action using (6.116) together with

$$C^3 = \frac{12m^3}{r^9} \tag{6.128}$$

one obtains

$$\begin{aligned} F = -\beta_T I^E_{\text{onshell}} &= \frac{V_{xy}}{\kappa^2}\int_{r_h}^{r_c} dr r^2 b(r)(\Lambda + \alpha\frac{12m^3}{r^9}) \\ &+ \frac{V_{xy}}{\kappa^2}(\sqrt{a(r)}\partial_r(\sqrt{a(r)}b(r)r^2) + \frac{2}{l}\sqrt{a(r)}b(r)r^2)_{r_c}, \end{aligned} \tag{6.129}$$

where $r_c \gg 1$ is the cutoff radius. Integrating the bulk term and looking at the horizon contribution one obtains

$$\frac{V_{xy}}{\kappa^2}(\frac{2}{3}\Lambda r^3 + \frac{4\alpha m^2\Lambda}{r^3} - \frac{4\alpha m^3}{r^6})_{r_h} = \frac{V_{xy}}{\kappa^2}2m. \tag{6.130}$$

i.e. the terms of order α cancel! Looking at the contribution from the cutoff boundary, as already argued the terms of order α fall off too quickly to contribute and one is left with a contribution

$$-\frac{V_{xy}}{\kappa^2}\frac{5m}{2}, \tag{6.131}$$

with the total free energy being

$$F = -\frac{V_{xy}}{2\kappa^2}m, \tag{6.132}$$

i.e. unchanged at order α. One can similarly argue why the mass $M = V_{xy}m/\kappa^2$ is unchanged at this order: varying the renormalised onshell action with respect to the source for the stress energy tensor, all terms at order α are subleading in the radial expansion and do not contribute.

To summarise: the C^3 term leads to a correction of the metric of the planar black hole. The temperature and entropy are both changed at order α but the mass and the free energy are unchanged.

At this point we return to the physical interpretation of the integration constants in the corrected solution. The first integration constant a_1 corresponds to a shift in the mass parameter,

$$m \to m - \alpha a_1. \tag{6.133}$$

This shift will affect the entropy, temperature, mass and free energy. The second integration constant corresponds to a redefinition of the time coordinate and hence of the temperature. One can see this by looking at the form of the metric

$$\begin{aligned} ds^2 &= -(1 + 2\alpha b_1 + \cdots) a(r) dt^2 + \cdots \\ &= -a(r) d\hat{t}^2 + \cdots \end{aligned}$$

i.e. by redefining the time coordinate one can absorb the integration constant b_1. This in turn corresponds to a shift of the temperature by

$$T \to T(1 + \alpha b_1), \tag{6.134}$$

with the free energy and mass shifted by the same factor.

By an appropriate choice of the integration constant $a_1 = -m\Lambda^2/9$ one can make the entropy be uncorrected at order α. However, this value of a_1 is such that the temperature, mass and free energy are corrected:

$$T \to T(1 - \frac{\alpha \Lambda^2}{9}); \qquad M \to M(1 - \frac{\alpha \Lambda^2}{9}); \quad F \to F(1 - \frac{\alpha \Lambda^2}{9}). \tag{6.135}$$

(These corrections are clearly consistent with the thermodynamic relation.) By fixing the integration constant b_1 appropriately and redefining the time coordinate, one can undo these corrections at order α, leaving all thermodynamic quantities unchanged to order α^2. However, such a redefinition is somewhat unnatural from the perspective of the holographic duality, as it implies that the time coordinate for the field theory is redefined at order α.

Thus the thermodynamics at order α depends on which quantities one has chosen to hold fixed. In the context of supersymmetric black holes one fixes the mass (and charge), with the temperature necessarily being zero and the entropy being corrected by the higher derivative terms. In the context of finite temperature black holes, it would seem natural to fix the mass also, as we did above, with the temperature and the entropy being corrected.

6.3.4 Einstein + R^4

In order to obtain an eleven-dimensional correction which cannot be rendered trivial by field redefinitions we need to add a curvature invariant involving the Riemann tensor, with the first non-trivial term arising at fourth order. As discussed earlier, the reduction of this term will result in terms quartic in the Riemann tensor in the effective four dimensional action. In this section we will work with one representative curvature invariant at this order, the same tensor structure considered earlier, but the generalization of the analysis to other tensor structures would be straightforward. The reason for considering this particular term is because, we discussed earlier, such a term would accompany any Riemann squared term occurring in the effective lower dimensional action.

The action we consider is therefore

$$I = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}\Big(R - 2\Lambda + \alpha(R_{\rho\sigma\tau\lambda}R^{\rho\sigma\tau\lambda})^2\Big) \tag{6.136}$$

The resulting field equation is

$$\begin{aligned}\mathscr{G}^{\mu\nu} \;=\; & \frac{1}{2}(R_{\rho\sigma\tau\lambda}R^{\rho\sigma\tau\lambda})^2 g^{\mu\nu} \;-\; 4R_{\rho\sigma\tau\lambda}R^{\rho\sigma\tau\lambda}R^{\mu\beta\gamma\delta}R^{\nu}{}_{\beta\gamma\delta} \\ & +\; 8\nabla_\rho\nabla_\sigma(R_{\tau\lambda\gamma\delta}R^{\tau\lambda\gamma\delta}R^{\mu\sigma\rho\nu}).\end{aligned} \tag{6.137}$$

Note also that there is a new boundary term involving derivatives of the metric variation obtained when varying the term at order α

$$\frac{4}{\kappa^2}\int_{\partial\mathcal{M}} d^3x\,\sqrt{-\gamma}\;\alpha\;R_{\rho\sigma\tau\lambda}R^{\rho\sigma\tau\lambda}R^{\alpha\beta\gamma\delta}\;n_\gamma\nabla_\beta\delta g_{\alpha\delta}. \tag{6.138}$$

We will discuss this term in the context of the variational problem below.

Now let us turn to the effect of such a correction on the planar black hole metric. Evaluated on the static ansatz (6.81) the action reduces to

$$\hat{I} = \int_0^\infty dr\;\left[2rb'(a-r^2) + \alpha\frac{D^2}{(br^2)^3}\right] \tag{6.139}$$

where to simplify formulae in this section we have imposed $\Lambda = -3$ and

$$\begin{aligned}D \;\equiv\; & 8a^2b'^2r^2 + 8aa'b'br^2 + 4a'^2b^2r^2 + 4a^2b^2 + 9a'^2b'^2r^4 + 12aa'b'b''r^4 \\ & +6a'a''b'br^4 + 4a^2b''^2r^4 + 4aa''b''br^4 + a''^2b^2r^4\end{aligned} \tag{6.140}$$

Varying this action we obtain equations of motion for a and b consisting of the piece coming from the Einstein action and a piece proportional to α coming from the correction. Once again we solve these perturbatively in α as in Section 6.3.2. The planar black hole solution is indeed corrected:

$$\begin{aligned} a(r) &= a_{(0)}(r)+\alpha a_{(1)}(r) \\ &= r^2-\frac{m}{r}+\alpha\left(\left(\frac{a_1}{r}-96r^2\right)-\frac{672m^2}{r^4}-\frac{1200m^3}{r^7}-\frac{536m^4}{r^{10}}\right) \end{aligned} \quad (6.141)$$

and

$$\begin{aligned} b(r) &= b_{(0)}(r)+\alpha b_{(1)}(r) \\ &= 1+\alpha\left(b_1+\frac{336m^2}{r^6}+\frac{224m^3}{r^9}\right). \end{aligned} \quad (6.142)$$

Again there are two arbitrary integration constants, corresponding to redefining the mass parameter and the time coordinate at order α. These constants will be set to zero and their effect will be discussed further below. Note however that the AdS metric itself is corrected by the quartic Riemann term, since the latter does not evaluate to zero, unlike the previous Weyl example.

Thermodynamics of corrected black hole solutions

Let us first calculate the corrected horizon position and temperature of this black hole solution. The horizon is located at

$$r_H=\frac{m^{1/3}}{3}(1+\alpha\frac{104}{3}), \quad (6.143)$$

and the temperature of this black hole solution is given by:

$$T=\frac{1}{4\pi}m^{1/3}(3-208\alpha) \quad (6.144)$$

We also work out the Wald entropy using (6.57) giving

$$S=\frac{A}{4G}-\frac{4\alpha}{\kappa^2}\int_H(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma})R^{vrvr}\sqrt{\gamma}d^2x, \quad (6.145)$$

where we use ingoing coordinates for the horizon, namely

$$ds^2=-a(r)b(r)^2dv^2+2b(r)drdv+r^2(dx^2+dy^2), \quad (6.146)$$

and the integral is over the spatial part of the horizon. The integrand in the second term however vanishes when evaluated on the leading order solution since

$$R^{vrvr} = \frac{1}{2}a'' = 2 - \frac{2m}{r^3} \tag{6.147}$$

which vanishes at the horizon. (The Riemann squared term is non-zero at the horizon.) Hence, in this case, the black hole entropy is given only in terms of the corrected area of the horizon

$$S = \frac{V_{xy}}{4G} m^{2/3} \left(1 + \alpha \frac{208}{3}\right). \tag{6.148}$$

Combining the entropy and the temperature we notice that the combination

$$TS = \frac{V_{xy}}{2\kappa^2} m + \mathcal{O}(\alpha^2) \tag{6.149}$$

is again unchanged at order α. As in the previous section we can now argue that for the relation

$$dM = TdS \tag{6.150}$$

to hold the correction to dM at order α must also vanish.

This argument on its own however does not exclude there being a term in the mass (and free energy) at order α which is independent of the parameter m: recall that, unlike the Weyl example, the AdS metric itself is corrected and the R^4 term in the action evaluated on AdS is non-zero. In other words, the holographically renormalized higher derivative action evaluated on AdS could be non-vanishing. Computation of the renormalised mass and action is somewhat involved as it requires analysing the corrections to asymptotically locally AdS solutions, isolating the divergences, computing the counterterms and so on. Fortunately there is a short cut: when the dual field theory is supersymmetric, the mass of the $m = 0$ solution is necessarily zero, as is the free energy, and therefore there cannot be any contributions to the mass and free energy at order α which are independent of m. (Note that the free energy of the dual theory on a curved space would in general indeed be expected to be corrected at order α.)

Thus, in summary, as for the C^3 case, the temperature and the entropy are corrected whilst the mass and free energy are not. By choosing the integration constants a_1 and b_1 one can adjust which thermodynamic quantities are corrected at order α, but the most natural physical choice from holographic considerations is indeed that where the mass is fixed and the entropy is corrected.

6.3.5 Corrections arising in string theory

In this section we have explored the effect of various curvature invariants added to the four dimensional action. We have shown that the thermodynamic properties of black hole solutions are in general corrected even when the metric is not corrected. From a top down perspective it would be complicated to determine which curvature invariants arise in the four dimensional action, with a given higher dimensional invariant contributing to invariants of different derivative order in four dimensions.

From the dual holographic perspective, one can try to restrict the invariants which arise in four dimensions using the free energy on an S^3. This would not restrict at all Weyl invariants which do not contribute to the free energy. One would also anticipate that other specific combinations of invariants involving Riemann, Ricci and Ricci scalar can be made in which the correction to the free energy also vanishes.

In the following section we will turn to another criterion for higher derivative corrections: the spectrum of fluctuations and the corresponding dual operators. In general, imposing that such corrections lead to CFT operators which are unitary and have positive norm, rules out many curvature invariants.

6.4 Spectrum of curvature squared theories

6.4.1 Linearized equations of motion

In this section we discuss the spectra of the higher derivative theories. For the sake of brevity, we will mostly focus on the case of curvature squared corrections, but the analysis for other higher-derivative theories would be similar and will be discussed at the end.

We consider again the action (6.40), whose equations of motion are given in (6.41) to (6.43). Since our interest here is in the context of holography, we consider the spectrum of excitations around AdS_4, and when we need an explicit form for the metric we will work in the Poincaré patch in which

$$ds^2 = \frac{d\rho^2}{\rho^2} + \frac{1}{\rho^2}(-dt^2 + dx^2 + dy^2). \tag{6.151}$$

We denote by x^i the coordinates on the three dimensional slices of constant ρ. We vary the metric as $g_{\mu\nu} \to \hat{g}_{\mu\nu} + \delta g_{\mu\nu} = \hat{g}_{\mu\nu} + h_{\mu\nu}$, where $\hat{g}_{\mu\nu}$ is the AdS_4 background metric. It was shown in [126] that the linear variations of the various tensors appearing in the equations of motion are

$$\begin{aligned}
\mathcal{G}^L_{\mu\nu} &= R^L_{\mu\nu} - \frac{1}{2}R^L\hat{g}_{\mu\nu} - \Lambda h_{\mu\nu} \\
R^L_{\mu\nu} &= \nabla^\lambda\nabla_{(\mu}h_{\nu)\lambda} - \frac{1}{2}\Box h_{\mu\nu} - \frac{1}{2}\nabla_\mu\nabla_\nu h \\
R^L &= \nabla^\mu\nabla^\nu h_{\mu\nu} - \Box h - \Lambda h
\end{aligned} \tag{6.152}$$

where ∇_μ is the covariant derivative associated with $\hat{g}$. Note that $h = \hat{g}^{\mu\nu}h_{\mu\nu}$. We write explicitly

$$\begin{aligned}
R &= (\hat{g}^{\mu\nu} - h^{\mu\nu})R_{\mu\nu} \\
&= (\hat{g}^{\mu\nu} - h^{\mu\nu})(R^{(0)}_{\mu\nu} + R^L_{\mu\nu}) \\
&\equiv R_{(0)} + R^L,
\end{aligned} \tag{6.153}$$

where $R_{(0)}$ is the Ricci scalar in the background.

To linear order in the variation, the equations of motion then become [126]

$$\begin{aligned}
\delta(\mathcal{G}_{\mu\nu} + E_{\mu\nu}) &= [1 + 2\Lambda(\alpha + 4\beta)]\mathcal{G}^L_{\mu\nu} + \alpha[(\Box - \frac{2\Lambda}{3})\mathcal{G}^L_{\mu\nu} - \frac{2\Lambda}{3}R^L\hat{g}_{\mu\nu}] \\
&\quad + (\alpha + 2\beta)[-\nabla_\mu\nabla_\nu + \hat{g}_{\mu\nu}\Box + \Lambda\hat{g}_{\mu\nu}]R^L.
\end{aligned} \tag{6.154}$$

The most common gauge used in holography is radial axial gauge, $h_{\rho\mu} = 0$ for $\mu = (\rho, t, x, y)$. However, [126] used a covariant gauge

$$\nabla^\mu h_{\mu\nu} = \nabla_\nu h, \tag{6.155}$$

since in this gauge the equations of motion immediately simplify.

Substituting (6.155) into the linearized tensors (6.152) gives

$$\begin{aligned}
\mathcal{G}^L_{\mu\nu} &= \frac{1}{2}\nabla_\mu\nabla_\nu h - \frac{1}{2}\Box h_{\mu\nu} + \frac{\Lambda}{3}h_{\mu\nu} + \frac{1}{6}\Lambda h\hat{g}_{\mu\nu} \\
R^L &= -\Lambda h
\end{aligned} \tag{6.156}$$

which can then be substituted into (6.154), the variation of the equations of motion. Tracing over the result yields

$$0 = \delta(\mathcal{G}_{\mu\nu} + \mathcal{E}_{\mu\nu}) = \Lambda[h - 2(\alpha + 3\beta)\Box h]. \tag{6.157}$$

Imposing the above constraint equation for the trace, we find that the variation of the field equations is,

$$\begin{aligned} 0 = \delta(\mathcal{G}_{\mu\nu}+\mathcal{E}_{\mu\nu}) &= -\frac{\alpha}{2}\Box^2 h_{\mu\nu} - \frac{1}{2}\left(1+\frac{2\Lambda\alpha}{3}+8\Lambda\beta\right)\Box h_{\mu\nu} \\ &\quad +\frac{\Lambda}{3}\left(1+\frac{4\Lambda\alpha}{3}+8\Lambda\beta\right)h_{\mu\nu} \\ &\quad +3(\alpha+2\beta)\left(\frac{1}{4(\alpha+3\beta)}+\Lambda\right)\nabla_\mu\nabla_\nu h \\ &\quad -\frac{\Lambda}{12}\left(\frac{5\alpha+6\beta}{\alpha+3\beta}+\frac{4\Lambda}{3}(\alpha+6\beta)\right)\hat{g}_{\mu\nu}h. \end{aligned} \tag{6.158}$$

In [126] this equation was analysed only in the case of $(\alpha+3\beta)=0$, with a view to critical gravity, but we will not impose this constraint here. From (6.158) we wish to extract the equation of motion for $h_{\langle\mu\nu\rangle}$, the traceless part of $h_{\mu\nu}$, where

$$h_{\mu\nu} = h_{\langle\mu\nu\rangle} + \frac{h}{4}\hat{g}_{\mu\nu}. \tag{6.159}$$

This yields, provided that $\beta \neq 0$,

$$\begin{aligned} 0 &= -\frac{\alpha}{2}\left(\Box-\frac{2\Lambda}{3}\right)\left(\Box+\frac{1}{\alpha}+\frac{4\Lambda}{3}+\frac{8\beta\Lambda}{\alpha}\right)h_{\langle\mu\nu\rangle} \\ &\quad +\frac{3}{4}\frac{(\alpha+2\beta)}{(\alpha+3\beta)}\left(1+4\Lambda(\alpha+3\beta)\right)\nabla_{\langle\mu}\nabla_{\nu\rangle}h. \end{aligned} \tag{6.160}$$

This equation represents an inhomogeneous equation for the traceless part of the metric fluctuation. However one can rewrite the equation as a homogeneous equation by defining a new traceless tensor $\psi_{\langle\mu\nu\rangle}$ as

$$\psi_{\langle\mu\nu\rangle} = h_{\langle\mu\nu\rangle} + \lambda\nabla_{\langle\mu}\nabla_{\nu\rangle}h, \tag{6.161}$$

and choosing λ such that the final term in (6.160) is zero. This value turns out to be

$$\lambda = -\frac{6(\alpha+3\beta)}{3+8\Lambda(\alpha+3\beta)}, \tag{6.162}$$

making the resulting equation of motion for ψ homogeneous

$$0 = \left(\Box-\frac{2\Lambda}{3}\right)\left(\Box-\frac{2\Lambda}{3}-M^2\right)\psi_{\langle\mu\nu\rangle}, \tag{6.163}$$

where

$$M^2 = -2\Lambda - \frac{1}{\alpha} - \frac{8\Lambda\beta}{\alpha}. \tag{6.164}$$

Moreover it is easy to verify that $\psi_{\langle\mu\nu\rangle}$ is transverse.

In the case where $\alpha = 0$, the analogue of (6.160) is

$$(1+8\beta\Lambda)(\Box - \frac{2\Lambda}{3})h_{\langle\mu\nu\rangle} + (1+12\beta\Lambda)\nabla_{\langle\mu}\nabla_{\nu\rangle} = 0, \tag{6.165}$$

which can be rewritten as a homogeneous equation

$$\begin{aligned} (\Box - \frac{2\Lambda}{3})\psi_{\langle\mu\nu\rangle} &= 0; \\ \psi_{\langle\mu\nu\rangle} &= h_{\langle\mu\nu\rangle} - \frac{6\beta}{1+8\beta\Lambda}\nabla_{\langle\mu}\nabla_{\nu\rangle}h. \end{aligned} \tag{6.166}$$

Note that this equation is only second order.

The interpretation of these equations is as follows. In the Einstein theory the only propagating mode is the traceless part of the metric, which couples to the dual stress energy tensor. In the theory with generic values of (α,β) the trace of the metric is a propagating mode dual to a scalar operator $\mathcal{O}_h$ of dimension

$$\Delta_{\mathcal{O}_h} = \frac{3}{2} + \sqrt{\frac{9}{4} + \frac{1}{2\alpha + 6\beta}} \tag{6.167}$$

whilst the equation of motion for the traceless part of the metric fluctuation is fourth order. One can write a basis for solutions of this fourth order equation as

$$\begin{aligned} \psi_{\langle\mu\nu\rangle} &= \psi^{(1)}_{\langle\mu\nu\rangle} + \psi^{(2)}_{\langle\mu\nu\rangle}; \\ (\Box - \frac{2\Lambda}{3})\psi^{(1)}_{\langle\mu\nu\rangle} &= 0; \\ (\Box - \frac{2\Lambda}{3} - M^2)\psi^{(2)}_{\langle\mu\nu\rangle} &= 0, \end{aligned} \tag{6.168}$$

with the propagating massless mode $\psi^{(1)}_{\langle\mu\nu\rangle}$ coupling to the dual stress tensor and the new mode $\psi^{(2)}_{\langle\mu\nu\rangle}$ being associated with a spin two operator X of dimension

$$\Delta_X = \frac{3}{2} + \frac{1}{2}\sqrt{9 + M^2}. \tag{6.169}$$

In this section we will show explicitly how these modes are associated with the dual spin two operator and we will discuss how the variational problem is defined.

Note that the special case in which the action on AdS is uncorrected, with the bulk term in the action reducing to Riemann squared, is obtained by choosing $\alpha = 4\gamma$, $\beta = -\gamma$, in which case

$$\begin{aligned} \Delta_{\mathcal{O}_h} &= \frac{3}{2} + \sqrt{\frac{9}{4} + \frac{1}{2\gamma}}; \\ \Delta_X &= \frac{3}{2} + \frac{1}{2}\sqrt{9 - \frac{1}{\gamma}}. \end{aligned} \tag{6.170}$$

We will discuss later when these operators are unitary, but let us already note here that for $\gamma \to 0$, which is indeed the case when the higher derivative corrections are small, either one or the other operator necessarily has a complex dimension and thus violates unitarity.

For special values of (α, β) one has to look more carefully to obtain the spectrum. At $\alpha = 0$ the higher derivative term consists just of the Ricci scalar squared, and the only new propagating mode in the bulk is the trace of the metric fluctuation, dual to a scalar operator. At $\alpha + 3\beta = 0$, when the bulk term reduces to Weyl squared, the metric trace is zero so there is no dual scalar operator but there is still a propagating spin two mode dual to a spin two operator. Whenever

$$(1 + 2\Lambda\alpha + 8\Lambda\beta) = 0, \tag{6.171}$$

the second spin two mode becomes massless, with the dual operator becoming the logarithmic partner of the stress energy tensor in the dual (L)CFT [215]. Note that this mode can become massless even when the trace is a propagating mode, with $\alpha + 3\beta = 0$ being an additional constraint used to remove the scalar operator.

6.4.2 Derivation of equations of motion in general gauge

In this subsection we derive an elegant form for the linearized equations of motion without imposing a gauge. Taking the trace of the equations of motion (6.41) to (6.43) one obtains

$$(2\alpha + 6\beta)\Box R - 12 - R = 0, \tag{6.172}$$

where we use the explicit value of the cosmological constant together with the Bianchi identity

$$\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R. \tag{6.173}$$

Now letting $r = (R+12)$ one obtains a diagonal equation of motion for r

$$(2\alpha + 6\beta)\Box r - r = 0. \tag{6.174}$$

Note that this equation did not rely on the linearized approximation and is exact. Defining

$$R_{\mu\nu} + 3g_{\mu\nu} = s_{\mu\nu} + \frac{1}{4} r g_{\mu\nu} \tag{6.175}$$

where $s_{\mu\nu}$ is traceless, i.e. $g^{\mu\nu}s_{\mu\nu} = 0$, the traceless part of the linearized equation of motion gives

$$(\alpha\Box + (1 - 4\alpha - 24\beta))s_{\mu\nu} = (\alpha + 2\beta)\nabla_{\langle\mu}\nabla_{\nu\rangle} r, \tag{6.176}$$

with the parentheses denoting the symmetric traceless combination. This equation can be diagonalized by defining

$$\psi^r_{\mu\nu} = s_{\mu\nu} - \gamma\nabla_{\langle\mu}\nabla_{\nu\rangle} r \tag{6.177}$$

with

$$\gamma = \frac{2\alpha + 3\beta}{3(1 - 8\alpha - 24\beta)} \tag{6.178}$$

to give

$$(\alpha\Box + (1 - 4\alpha - 24\beta))\psi^r_{\mu\nu} = 0. \tag{6.179}$$

The equations (6.174) and (6.179) represent second order equations for the linearized curvature tensor and hold in any gauge. In the covariant gauge $\nabla^\mu h_{\mu\nu} = \nabla_\nu h$ used previously

$$r = 3h; \qquad s_{\mu\nu} = -\frac{1}{2}(\Box + 2)h_{\langle\mu\nu\rangle} + \frac{1}{2}\nabla_{\langle\mu}\nabla_{\nu\rangle} h. \tag{6.180}$$

A complete basis for the solutions to (6.174) and (6.179) can be obtained by setting the metric fluctuation $h_{\mu\nu}$ to be

$$h_{\mu\nu} = h^T_{\mu\nu} + h^X_{\mu\nu}, \tag{6.181}$$

with

$$r(h^T) = 0; \qquad s_{\mu\nu}(h^T) = 0, \tag{6.182}$$

and $(r(h^X), \psi^r_{\mu\nu}(h^X))$ are non-zero, satisfying (6.174) and (6.179).

In understanding the holography dictionary it is useful to look at the asymptotic solutions for (6.174) and (6.179). Since r is simply a scalar field, of a specific mass, the general asymptotic solution to (6.174) is as usual

$$r(\rho,x) = \rho^{3-\Delta_{\mathcal{O}_r}}(r_0(x) + \rho^2 r_2(x) + \cdots) + \rho^{\Delta_{\mathcal{O}_r}}(r_{2\Delta-3}(x) + \cdots), \tag{6.183}$$

where $\Delta_{\mathcal{O}_r} = \frac{3}{2} + \sqrt{\frac{9}{4} + \frac{1}{2\alpha+6\beta}}$. Here $r_0(x)$ acts as the source for the dual operator, with $r_{2\Delta-3}(x)$ being the normalisable mode, and all other terms in the expansion being fixed by the field equation.

Equation (6.179) is an equation for a massive spin two field of a given mass. Such fields are considered less frequently in holography (they were first analysed in [223]) but one can analyse the general asymptotic solutions to the field equations as follows. The independent solutions are

$$\begin{aligned}
\psi_{\langle\rho\rho\rangle}(\rho,\vec{k}) &= \rho^{d-\Delta}(f(x)+\cdots) + \rho^{\Delta}(\tilde{f}(x)+\cdots) \\
\psi_{\langle i\rho\rangle}(\rho,\vec{k}) &= \rho^{d-\Delta-1}(B_i(x)+\cdots) + \rho^{\Delta-1}(\tilde{B}_i(x)+\cdots) \\
\psi_{\langle ij\rangle}(\rho,\vec{k}) &= \rho^{d-\Delta-2}(X_{ij}(x)+\cdots) + \rho^{\Delta-2}(\tilde{X}_{ij}(x)+\cdots),
\end{aligned} \tag{6.184}$$

where

$$\Delta = \frac{d}{2} + \frac{1}{2}\sqrt{d^2+4M^2} \tag{6.185}$$

with $d = 3$ in this case and M^2 given in (6.164). The fields without tildes denote the non-normalizable modes and those with tildes are the normalizable modes. Only the transverse traceless part of X_{ij} and $\tilde{X}_{ij}$ are independent data, however, since the field equations imply

$$\begin{aligned}
X^i_i &= \tilde{X}^i_i = 0; \\
B_i &= -\frac{1}{2-\Delta}\partial^j X_{ji}; \\
\tilde{B}_i &= -\frac{1}{\Delta-1}\partial^j \tilde{X}_{ij}; \\
f &= -\frac{1}{3-\Delta}\partial^i B_i; \qquad \tilde{f} = -\frac{1}{\Delta}\partial^i \tilde{B}_i.
\end{aligned} \tag{6.186}$$

Thus the defining data for the spin two field indeed corresponds to a transverse traceless spin two operator in the dual field theory.

The “new" defining boundary data is $r_{(0)}(x)$ in (6.183) and X_{ij} in (6.184), namely the near boundary behaviour of the scalar curvature and the (trace adjusted) Ricci tensor. One can obtain a geometric interpretation of these boundary conditions as follows. As commented earlier, the most natural gauge for holography is the radial axial gauge in which the metric perturbations satisfy $h_{\rho\mu} = 0$ and

$$ds^2 = \frac{d\rho^2}{\rho^2} + \frac{1}{\rho^2}(\eta_{ij} + H_{ij})dx^i dx^j. \tag{6.187}$$

In this gauge the linearized Ricci scalar $r[h]$ defined above is given by

$$r[H] = \rho^2 \hat{R}[H] - \rho^2 \mathrm{tr}(H'') + 3\rho\,\mathrm{tr}(H'), \tag{6.188}$$

with a prime denoting a radial derivative and $\hat{R}_{ij}$ being the linearized curvature of H_{ij}, namely

$$\begin{aligned} \hat{R}_{ij} &= \frac{1}{2}\left(\partial^k \partial_j H_{ik} + \partial^k \partial_i H_{jk} - \partial_i \partial_j \mathrm{tr}(H) - \partial^k \partial_k H_{ij}\right); \\ \hat{R} &= \partial^i \partial_j H_{ij} - \Box \mathrm{tr}(H). \end{aligned} \tag{6.189}$$

From the equation for the linearized Ricci scalar we note that the leading asymptotic behavior of the metric perturbation corresponding to the propagating scalar mode satisfies

$$\mathrm{tr}H = \frac{1}{(3-\Delta)(1+\Delta)}\rho^{3-\Delta_{\mathcal{O}_r}} r_{(0)} + \cdots \tag{6.190}$$

We can also express this condition in a more geometric way, in terms of the extrinsic curvature of the hypersurface with induced metric $\gamma_{\mu\nu}$, as

$$\gamma^{\mu\nu} \mathscr{L}_n K_{\mu\nu} \to -4 r_{(0)x} \rho^{3-\Delta_{\mathcal{O}_r}}. \tag{6.191}$$

Therefore the new data $r_{(0)}(x)$ supplied corresponds to specifying the boundary condition for the trace of the normal derivative of the extrinsic curvature.

6.4.3 Two point functions

To extract the two point functions of the scalar and spin two operators, the field equations alone do not suffice: one needs to compute the onshell renormalized action. This is a non-trivial issue, as even when the bulk field has a mass such that the dual operator would be unitary, the corresponding two point function of that operator is not guaranteed to be positive. In other words, the sickness of the higher derivative theory can manifest itself in negative norms.

A useful trick for obtaining the two point functions is the following, borrowed from the three dimensional discussions in [224]. Let us first rewrite the bulk terms in the action as

$$I = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}\left[R - 2\Lambda + (\beta + \frac{\alpha}{4})R^2 + \alpha S^{\mu\nu}S_{\mu\nu}\right], \tag{6.192}$$

where $S_{\mu\nu}$ is the traceless part of the Ricci tensor. We next introduce a scalar auxiliary field ϕ and a traceless spin two auxiliary field $\phi^{\mu\nu}$ and write the action as

$$\begin{aligned} I &= \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}\left[R - 2\Lambda + (\beta + \frac{\alpha}{4})(2R\phi - \phi^2)\right] \\ &\quad + \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}\left[\alpha(2S^{\mu\nu}\phi_{\mu\nu} - \phi^{\mu\nu}\phi_{\mu\nu})\right]. \end{aligned} \tag{6.193}$$

Eliminating the auxiliary fields using their equations of motion gives the previous action. For this action, the boundary term needed for a well-defined variational problem is

$$I_{GHY} = -\frac{1}{\kappa^2}\int d^3x\,\sqrt{-\gamma}\Big(K\left[1 + (2\beta + \frac{\alpha}{2})\phi\right] + \alpha K_{\mu\nu}\phi^{\mu\nu}\Big). \tag{6.194}$$

Note that the problems in setting up a variational problem have been solved here, by the introduction of the auxiliary fields. A similar approach to dealing with the variational problem in higher derivative theories was discussed in [207]. The action with auxiliary fields admits Einstein manifolds as solutions, in which

$$g_{\mu\nu} = \hat{g}_{\mu\nu}; \qquad \phi = R = 4\Lambda; \qquad \phi_{\mu\nu} = S_{\mu\nu} = 0. \tag{6.195}$$

The action of course also admits other solutions, but in this section we are interested in the spectrum around an Einstein solution. For such solutions, the boundary counterterms given previously in (6.47) renormalise the onshell action. Note that, as previously anticipated, when one looks at the leading order Einstein solutions, the boundary conditions for the auxiliary fields do not involve non-trivial data (i.e. unlike the metric boundary condition, the boundary data for the auxiliary fields is not specified by arbitrary scalars or tensors) and indeed this remains true when evaluating corrections on such solutions. When we compute the spectrum below, however, we find that there is indeed non-trivial data required for the auxiliary fields, which is expressed in terms of arbitrary scalars and tensors.

Let us now consider perturbations around such an Einstein solution $\hat{g}_{\mu\nu}$ of the equations of motion, i.e. we let

$$g_{\mu\nu} = \hat{g}_{\mu\nu} + h_{\mu\nu}; \qquad \phi = 4\Lambda + \delta\phi; \qquad \phi_{\mu\nu} = \delta\phi_{\mu\nu}. \tag{6.196}$$

The boundary data for $\delta\phi$ and $\delta\phi_{\mu\nu}$ specify the defining data for dual scalar and tensor operators, respectively.

Let us begin with the $\alpha = 0$ case. To quadratic order in the fluctuations one obtains the following for the bulk terms in the action

$$\begin{aligned} \delta I &= -\frac{\mu}{2\kappa^2}\int d^4x\,\sqrt{-\hat{g}}\,h^{\mu\nu}\left(\mathcal{G}^L_{\mu\nu}[h]\right) \\ &\quad +\frac{\beta}{\kappa^2}\int d^4x\sqrt{-\hat{g}}h^{\mu\nu}\left(\nabla_\mu\nabla_\nu\delta\phi-\Box\delta\phi\hat{g}_{\mu\nu}\right) \\ &\quad +\frac{\beta}{\kappa^2}\int d^4x\,\sqrt{-\hat{g}}\,\delta\phi\left(R[h]-\Lambda h-\delta\phi\right). \end{aligned} \tag{6.197}$$

where $\mu = (1+8\beta\Lambda)$, $h = \hat{g}^{\mu\nu}h_{\mu\nu}$ and the linearisation of the Einstein equation is

$$\mathcal{G}^L_{\mu\nu}[h] = R_{\mu\nu}[h]-\Lambda h_{\mu\nu}-\frac{1}{2}R[h]\hat{g}_{\mu\nu}+\frac{1}{2}\Lambda h\hat{g}_{\mu\nu}, \tag{6.198}$$

and the linearised Ricci tensor is given by

$$R_{\mu\nu}[h] = \frac{1}{2}\left(\nabla^\rho\nabla_\mu h_{\rho\nu}+\nabla^\rho\nabla_\nu h_{\rho\mu}-\nabla_\mu\nabla_\nu h-\Box h_{\mu\nu}\right), \tag{6.199}$$

with $R[h] = \hat{g}^{\mu\nu}R_{\mu\nu}[h]$ being

$$R[h] = \nabla^\mu\nabla^\nu h_{\mu\nu}-\Box h. \tag{6.200}$$

The action can be diagonalised with the field redefinition

$$h_{\mu\nu} = \bar{h}_{\mu\nu}+\zeta(\delta\phi)\hat{g}_{\mu\nu} \tag{6.201}$$

and letting

$$\zeta = -2\beta/\mu. \tag{6.202}$$

The bulk action at the quadratic level then becomes

$$\begin{aligned} \delta I &= -\frac{\mu}{2\kappa^2}\int\,d^4x\,\sqrt{-\hat{g}}\bar{h}^{\mu\nu}\left(\mathcal{G}^L_{\mu\nu}[\bar{h}]\right) \\ &\quad -\frac{\beta}{\kappa^2}\int d^4x\sqrt{-\hat{g}}\delta\phi\left[-6\frac{\beta}{\mu}\Box\delta\phi+\delta\phi\right]. \end{aligned} \tag{6.203}$$

where we have used

$$R(\delta\phi\hat{g}_{\mu\nu}) = -3\Box\delta\phi. \tag{6.204}$$

The equations of motion resulting from this action describe the graviton together with the scalar field, and agree with those found in the previous sections.

Having obtained the action, it is now straightforward to extract the two point functions of the dual operators. To do this we need to keep careful track of the boundary terms, include those which arise in the field redefinitions. In working out these terms it is convenient to fix a gauge for the metric perturbation $\bar{h}_{\mu\nu}$, the holographic radial axial gauge in which

$$\bar{h}_{\rho\mu} = 0. \tag{6.205}$$

Thus the perturbed metric may be written as

$$ds^2 = \frac{d\rho^2}{\rho^2} + \frac{1}{\rho^2}\left(\eta_{ij} + H_{ij}\right) dx^i dx^j \tag{6.206}$$

where $H_{ij} = \rho^2 \bar{h}_{ij}$. Evaluating all boundary terms involving H_{ij}, including those from the counterterms needed to renormalise the action for the background solution, one obtains

$$I_{\text{onshell}} = -\frac{\mu}{4\kappa^2}\int d^3x \frac{1}{\rho^2}\left(H^{ij}\partial_\rho H_{ij} + 2H\partial_\rho H\right), \tag{6.207}$$

where $H = \eta^{ij} H_{ij}$. These terms can be processed using the Fefferman-Graham expansion, namely

$$H_{ij} = H_{(0)ij} + \rho^3 H_{(3)ij} + \cdots \tag{6.208}$$

with $H_{(3)ij}$ being traceless and transverse. Thus

$$I_{\text{onshell}} = -\frac{3\mu}{4\kappa^2}\int d^3x H^{ij}_{(0)} H_{(3)ij}. \tag{6.209}$$

This action is clearly finite, without the need for additional counterterms, as expected as the counterterms should take of all divergences of Einstein solutions of the field equations. Moreover, recalling that

$$\langle T_{ij}\rangle = -\frac{2}{\sqrt{g_{(0)}}}\frac{\delta I_{\text{onshell}}}{\delta g^{(0)ij}} \tag{6.210}$$

we recover the formula

$$\langle T_{ij}\rangle = \frac{3\mu}{2\kappa^2} H_{(3)ij}, \tag{6.211}$$

which is the linearisation of the renormalised stress tensor given earlier. Relative to Einstein gravity, this formula is shifted by a factor of μ which in turn implies that the two point function for the stress energy tensor will be shifted by factor of μ relative to the Einstein case [75]. In this theory the ratio η/s is unchanged by the higher order correction. This was already apparent on general grounds, since the correction evaluated on an Einstein solution can be removed by field redefinitions. Here the derivation is somewhat non-trivial as both quantities are shifted by the factor of μ: the Wald entropy was computed earlier, and η is obtained from the two point function of the stress energy tensor, which according to the formula above will only be shifted by μ relative to the Einstein case.

What remains is to collect together all of the terms involving the scalar field. These give

$$I_{\text{onshell}} = -\frac{13\beta^2}{\mu\kappa^2}\int d\Sigma^\mu \partial_\mu(\delta\phi)\delta\phi. \tag{6.212}$$

This is the action in Lorentzian signature. The corresponding action in Euclidean signature is

$$\begin{aligned} I^E_{\text{onshell}} &= \frac{13\beta^2}{\mu\kappa^2}\int d\Sigma^\mu \partial_\mu(\delta\phi)\delta\phi; \\ &= \frac{13\beta^2}{\mu\kappa^2}\int d^3x \frac{1}{\rho^2}\delta\phi\partial_\rho\delta\phi. \end{aligned} \tag{6.213}$$

Recalling that the asymptotic expansion of such a scalar field dual to an operator of dimension Δ is

$$\delta\phi = \rho^{d-\Delta}(\delta\phi_{(d-\Delta)} + \ldots) + \rho^{\Delta}(\delta\phi_\Delta + \cdots) \tag{6.214}$$

we see that this part of the onshell action still has divergences as $\rho \to 0$. This was indeed to be expected, as the counterterms computed earlier were for Einstein solutions of the field equations only.

The holographic renormalization required for such a scalar field is already known: if the onshell (non-renormalized) Euclidean action for a free scalar field is

$$I^E_{\text{onshell}} = -\frac{1}{2}\int d\Sigma^\mu \varphi\partial_\mu\varphi \tag{6.215}$$

then the renormalised two point function of the operator of dimension Δ dual to the field φ is [78]

$$\langle \mathcal{O}_\varphi(x)\mathcal{O}_\varphi(0)\rangle = \frac{(2\Delta-d)\Gamma(\Delta)}{\pi^{d/2}\Gamma(\Delta-d/2)}\mathcal{R}\left(\frac{1}{x^{2\Delta}}\right) \equiv c_\Delta \mathcal{R}\left(\frac{1}{x^{2\Delta}}\right), \tag{6.216}$$

where we denote by $\mathscr{R}$ the renormalised quantity. Comparing with our case we obtain the following for the norm of the two point function of the operator dual to $\delta\phi$:

$$\langle \mathscr{O}_\phi(x)\mathscr{O}_\phi(0)\rangle = -\frac{13\beta^2}{\mu\kappa^2} c_\Delta \mathscr{R}\left(\frac{1}{x^{2\Delta}}\right). \tag{6.217}$$

The norm is never positive and recall that the operator also has complex dimension for negative β. It would be interesting to carry out a similar analysis for the spin two operator, setting $\alpha \neq 0$, to find for which values of α the norm of the dual operator is non-positive.

6.4.4 Spectra for other curvature corrections

A similar analysis can be carried out for the spectrum around AdS induced by other curvature corrections. For any given curvature invariant one might anticipate that generically the operator associated with the higher derivative term is either non-unitary or has non-positive norm. There are certain exceptions to this generic case, however.

If a curvature invariant which is built out of the Weyl tensor is added to the action, then the equations of motion linearised around AdS are necessarily unchanged since the Weyl tensor vanishes identically on the background. More precisely, any curvature at least cubic in the Weyl tensor implies that all contributions to the linearised field equations are at least linear in the Weyl tensor of the background, which vanishes for AdS. Therefore the Weyl terms do not change the spectrum of operators in the dual CFT, although they can modify the correlation functions of these operators. This fits with the observation that the Weyl terms on asymptotically locally AdS spacetimes fall off sufficiently fast at infinity that the variational problem is unchanged from the Einstein case. Put differently, one needs no additional new boundary conditions and therefore there are no new associated propagating modes and corresponding dual operators.

If one adds several different curvature invariants to the action, the diagonalization of the linearised field equations becomes more complicated, as we saw in the case of curvature squared corrections. For each new boundary condition there is an associated new dual operator. The dimensions and norms of the dual operators are obtained non-trivially from diagonalising the field equations and manipulating the onshell action.

From a top-down perspective, the leading higher derivative terms in the four-dimensional action must be consistent with unitarity. This implies that they must give rise to a linearised spectrum around AdS which is consistent with dual operators of real conformal dimension and positive norm. We have shown that individual terms such as R^2 are not consistent with unitarity, but we also noted that in reducing a higher dimensional curvature invariant any such curvature squared term always appears with fourth order curvature terms and a shift of the cosmological constant. Moreover, one needs to include all higher dimensional curvature invariants to respect unitarity at the required order, supersymmetry and so on.

Finally let us consider how the dimensions of the dual operators relate to the parameters of the dual CFT. The dimensions of the operators for the case of Riemann squared were given in (6.170). From the discussion around (6.37) we note that when such a term arises from a reduction of eleven dimensions the coupling constant γ would be the ratio of the term descending from R^4 to the leading order Einstein term. In other words, for the case of an S^7 reduction γ would be of order $1/N$ and in ABJM it would be of order $1/(kN) = 1/N'$. The dimensions of the operators scale as $1/\sqrt{\gamma}$, i.e. $N^{1/2}$ and $(N')^{1/2} = k\lambda^{1/2}$ respectively. Riemann squared on its own is not unitary, so one of the operators always has complex dimensions, but the combination with other terms arising from top down would give operators whose dimensions scale similarly. If no such operators exist in the dual CFT, then the net effect of the reduction of the leading top down terms must be trivial at the linearized level.

6.5 Conclusions

In this Chapter we showed that the variational problem is not in general well-posed in higher derivative gravity theories without specifying additional data to the boundary metric. When the higher derivative terms are treated perturbatively around the leading order Einstein solution, the higher derivative equations always become inhomogeneous second order equations, for which the variational problem is well-posed with only a boundary condition for the metric. However, in analyzing the spectrum around the corrected background, the linearized equations of motion are generically higher order and do indeed require additional boundary conditions. In the context of holography these additional boundary conditions correspond to data for operators in the dual conformal field theory. For the curvature invariants we analyzed the operators are non-unitary since their conformal dimensions are generically complex and their norms are non-positive definite. From a top down perspective, the reduction of any given higher dimensional curvature invariant results in a lower dimensional action involving several curvature invariants of different derivative order. When the lower dimensional curvature invariants are combined, the resulting spectrum must be unitary and thus either the dual operators must have real dimensions and positive norms or (perhaps more likely) the resulting lower dimensional linearized field equations remain second order with no new operators arising.

Even when the new operators induced by the higher derivative terms are non-unitary, one might try to look for a unitary subsector of the theory, by switching off these operators. This is indeed the perspective of [130], [216], whose boundary conditions effectively set to zero the sources for the irrelevant operators associated with the higher derivative terms. As explained in detail in [131], [201], however, switching off such sources does not switch off expectation values for such operators. Moreover, even if one restricts to a subsector of the theory in which the irrelevant operators are neither sourced nor acquire expectation values, the theory itself is non-unitary. In particular, the extra fields do contribute in computation of stress energy tensor correlation functions and there is no guarantee that the latter would be unitary, nor is it immediately apparent that the additional operators can always be decoupled from the stress energy tensor.

Many interesting questions deserve further study. By comparison with dual field theory results, one could, at least in the highly supersymmetric examples of ABJM and ABJ models, restrict what curvature invariants can arise in the effective four-dimensional action. It would also be interesting to work out the spectrum for the reduction of a top down curvature invariant, i.e. putting together curvature invariants of different order in four dimensions. In this work we have found that the simplest representative correction would involve the Weyl tensor at orders higher than two: such corrections are in some ways analogous to the Gauss-Bonnet examples in higher dimensions, in that the black holes are corrected but there are no new operators induced in the spectrum. One of the initial motivation for looking at higher derivative terms in AdS_4 was to explore subleading effects in applied holography and the Weyl solution would be a natural candidate for such investigations.

One approach to holographic cosmology exploits the domain wall cosmology correspondence [225] to obtain a field theoretic description of cosmologies in one higher dimension [226]. Since the primary focus is naturally on four dimensional cosmologies, the results obtained here would be relevant in discussing higher derivative effects in holographic cosmology. In particular it would be interesting to understand whether corrections which give rise to non-unitary effects on the AdS side are automatically excluded from being physical on the cosmological side, and whether the irrelevant operators associated with the higher derivative terms could actually be useful on the cosmology side in, for example, exiting from the inflationary era.

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Samenvatting

Dit proefschrift vertegenwoordigt het onderzoek dat ik heb gedaan als promovendus aan de Universiteit van Amsterdam. Ik heb het genoegen gehad om aan een aantal verschillende aspecten van de hoge energie fysica te kunnen werken; met uiteenlopende onderwerpen van kwantumveldentheorie en dynamica ver buiten evenwicht tot zwarte gaten in zwaartekrachtstheorieën met hogere afgeleiden. Deze onderzoeksvelden lijken misschien erg verschillend, maar er blijkt een onderliggend idee dat hen verbindt. Dat idee is holografie. De naam 'holografie' duidt op de manier waarop een hologram de informatie van een drie-dimensionaal (ruimtelijk) object opslaat op een twee-dimensionaal (plat) vlak. Het baanbrekende concept van holografie is een wiskundige equivalentie tussen twee verschillende natuurkundige theorieën. Het vertelt ons dat een zwaartekrachtstheorie gedefiniëerd in $n+1$ dimensies volkomen equivalent is met een kwantumveldenheorie (zonder zwaartekracht) in n dimensies. Het staat ook bekend als ijk/zwaartekracht dualiteit, omdat de zwaartekrachtstheorie vaak equivalent (ook wel 'duaal') blijkt te zijn aan een zogenoemde ijktheorie.

Het idee van holografie begon met het werk uit de jaren zeventig van Jacob Bekenstein en Stephen Hawking. Deze vroege werken toonden aan dat zwarte gaten een stelsel wetten gehoorzamen die verdacht veel lijken op de wetten van de thermodynamica. Op deze manier kon men fysische grootheden zoals temperatuur en entropie (maat van wanorde/informatie) aan zwarte gaten toekennen. Bekenstein analyseerde dit zorgvuldig en vond dat de entropie niet evenredig was met het volume, zoals verwacht, maar evenredig met de oppervlakte van het zwarte gat. Verder beredeneerde hij dat zwarte gaten meer entropie bevatten dan welk ander (gelijkvormig) object dan ook. Hawking liet op zijn beurt zien dat zwarte gaten warmtestraling uitzendt en kon daarmee de evenredigheidsconstante bepalen, waarmee de entropie van een zwarte gat uitgedrukt kan worden in termen van de oppervlakte van haar horizon. Deze ontwikkelingen stelden dus vast dat zwarte gaten werkelijk thermische objecten zijn.

Dit eerdere werk leidde Gerard 't Hooft in 1993 tot het postulaat dat alle gebeurtenissen in een bepaald volume V beschreven konden worden door een verzameling 'vrijheidsgraden' (ook wel informatiedragers) die zich enkel op het omhullende oppervlak rond V (oft. de rand van V) bevinden. Vervolgens was het Leonard Susskind die 't Hoofts ideeën een precieze interpretatie gaf binnen snarentheorie.

In 1997 gaf Juan Maldacena het eerste expliciet uitgewerkte voorbeeld van het holografische principe: de AdS/CFT correspondentie. Een van Maldacena's postulaten is dat Type II snarentheorie gedefiniëerd op een $AdS_5 \times S^5$ achtergrond duaal is aan een $\mathcal{N} = 4$ supersymmetische Yang-Mills ijktheorie gedefiniëerd op de rand van AdS_5. De AdS/CFT correspondentie is een dualiteit tussen sterke en zwakke interactie. Dit betekent in het bijzonder dat een ijktheorie met sterke interactie kan worden vertaald naar een snarentheorie met zwakke interactie (m.a.w. supersymmetrische zwaartekracht) en vice versa. Een theorie met sterke interactie is zeer lastig te beschrijven, terwijl een theorie met zwakke interactie juist relatief veel handzamer is. Deze correspondentie stelt ons dus in staat om een lastige ijktheorie te beschijven in termen van een relatief makkelijke supersymmetrische zwaartekrachtstheorie. Het is dus een ontzettend handig stuk gereedschap en het is veelvuldig gebruikt sinds haar ontdekking.

Een belangrijk recent onderzoeksveld is de toepassing van holografie op de fysica van vaste stoffen. Nog zo'n belangrijk onderzoeksgebied is het gebruik van holografie voor het modelleren van botsingen tussen zware ionen, zoals dat in de praktijk word bestudeerd aan RHIC and de LHC. Deze onderzoeken hebben ons informatie gegeven over de fundamentele eigenschappen van zgn. quark-gluon plasma's. Het doel van dit proefschrift is om een bijdrage te leveren aan deze onderzoeksgebieden en aan de (omvangrijke) literatuur van algemene holografische dualteiten.

In het eerste deel van dit proefschrift zullen we alleen werken aan de ijktheorie kant van de dualiteit. We zullen namelijk de $\mathcal{N} = 4$ supersymmetrische Yang-Mills (SYM) ijktheorie bestuderen, ver buiten evenwicht. Kwantumveldentheorie buiten evenwichten is een lastig onderwerp, maar er is in recente jaren veel ontwikkeling geweest in het uitvoeren van analytische berekeningen van systemen die ver buiten evenwicht zijn. Dit valt onder de noemer n-deeltjes-irreducibele (in het Engels afgekort met nPI, n-particle irreducible) effectieve acties. We zullen de 2PI effectieve actie berekenen tot en met tweede orde in storingstheorie van $\mathcal{N} = 4$ SYM. Bovendien zullen we de evolutievergelijkingen voor de tweepuntsfuncties uitrekenen. Deze berekeningen worden gedaan met zwakke interactie en kan ons dus via de holografische dualiteit informatie geven over de duale snarentheorie met sterke interactie.

In het tweede deel van dit proefschrift bekijken we een ijktheorie met sterke interactie en zullen we bestuderen wat er gebeurt als we de sterkte van de interactie een klein beetje verlagen door het toevoegen van correcties van hogere afgeleiden. We zullen verscheidene aspecten van holografie bestuderen. Zo laten we zien dat hoe holografie wordt gebruikt in de beschrijving van vaste stoffen met sterke interacties. We bekijken in het bijzonder een systeem die gedefiniëerd is aan de hand van een holografische duale zwaartekrachtstheorie die bekend staat als de Einstein-Maxwell-Dilaton (EMD) theorie. Deze theorie bleek zeer handig in het beschrijven van vaste stoffen. In het voorlaatste hoofdstuk leiden we een holografisch 'woordenboek' af voor een klasse theorieën die duaal zijn aan EMD-theorieën. Dit is geen gemakkelijke taak, omdat zoveel belangrijke oplossingen (theorieën) niet de bekende asymptotiek van de AdS-ruimte bezitten. Om dit probleem op te lossen gebruiken we een stuk gereedschap dat bekend staat als 'veralgemeniseerde dimensionele reductie'. Dit stelt ons in staat om een woordenboek op te stellen door terug te grijpen op een theorie wiens woordenboek reeds bekend was. We zullen ook hydrodynamica bestuderen in deze theorieën.

Vervolgens verlagen we de sterkte van de interactie door hogere afgeleiden toe te voegen aan de actie van de duale zwaartekrachtstheorie en we beschrijven het holografische effect ervan. In het laatste hoofdstuk zullen op zoek gaan naar zwarte gaten in de duale zwaartekrachtstheorie inclusief de hogere afgeleiden. We bestuderen hoe de hogere afgeleiden de vierdimensionale AdS zwarte gaten beïnvloeden en we bekijken tot slot de thermodynamische eigenschappen van deze zwarte gaten.