Aspects of AdS/CFT Correspondence: Symmetries, Integrability and Solitons

by

Inês Varela Aniceto

Licenciada in Physics, Instituto Superior Técnico, Universidade Técnica de Lisboa

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Date: ______________ ______________________________

Antal Jevicki, Advisor

Recommended to the Graduate Council

Date: ______________ ______________________________

Anastasia Volovich, Reader

Date: ______________ ______________________________

Chung-I Tan, Reader

Approved by the Graduate Council

Date: ______________ ______________________________

Sheila Bonde, Dean of the Graduate School

Curriculum Vitae

1979 Born in Setúbal, Portugal.

1997 High School Degree, Escola Sebastião da Gama, Setúbal, Portugal.

2002 *Licenciatura* in Physics Engineering, Instituto Superior Técnico, Universidade Técnica de Lisboa, Portugal.

2006 M.Sc. (Physics) Brown University.

Teaching

2000-2003 Mathematical Analysis I, IV, Linear Algebra, Instituto Superior Técnico, Department of Mathematics.

2007-08 Classical Mechanics, Introduction to Modern Physics, Brown University, Physics Department.

Publications

Listed in the preface, page v.

Preface

The work present in this dissertation is based on the following published works, presented in a more pedagogical manner with respective introductions to each subject:

[1] Michael C. Abbott and Inês Aniceto
Vibrating giant spikes and the large-winding sector
Journal of High Energy Physics **06** (2008) 088, arXiv:0803.4222

[2] Inês Aniceto
Matrix Reduction and the $\mathfrak{su}(2|2)$ *Superalgebra in AdS/CFT*
Physical Review **D79** 086014 (2009), arXiv:0809.2645

[3] Inês Aniceto and Antal Jevicki
N-body Dynamics of Giant Magnons in $\mathbb{R}\times S^2$
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[4] Michael C. Abbott and Inês Aniceto
Giant Magnons in $AdS_4\times CP^3$*: Embeddings, Charges, and a Hamiltonian*
Journal of High Energy Physics **04** (2009) 136, arXiv:0811.2423

[5] Michael C. Abbott, Inês Aniceto and Olof Ohlsson Sax
Dyonic Giant Magnons in CP^3*: Strings and Curves at Finite-J*
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The following papers published during the same time are not included in this thesis:

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Dedication

À minha avó Maria de Lurdes,

para que ela entenda pelo menos uma frase desta dissertação

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Chapter 1

Introduction

> *"There is a theory which states that if ever anyone discovers exactly what the Universe is for and why it is here, it will instantly disappear and be replaced by something even more bizarre and inexplicable. There is another which states that this has already happened."*
>
> Douglas Adams, *Hitchhiker's Guide to the Galaxy*

Having started as an attempt to describe the physics of strong interactions before the introduction of QCD, string theory has become one of the best candidates for a quantum theory of gravity. In this theory, the oscillations of the strings that can be interpreted as particles. In a low energy limit, the oscillations of these strings will look like localized excitations - different oscillation modes correspond to the different kinds of particles, including gauge fields. From this perspective, gauge fields appear as non-fundamental objects which are excitations of the fundamental strings. Gravity itself is derived from the interaction of these fundamental objects (through splitting and joining). But such a perspective is too narrow. In fact, a better understanding of non-perturbative string theory and D-branes has shed light into another interpretation of gauge fields: it has been seen that string theory in certain space-time backgrounds has a dual description as a gauge field theory, thus putting gauge fields and strings as fundamental objects in the respective theories. This gauge/string duality, also known as Anti-de-Sitter/Conformal Field Theory (AdS/CFT) correspondence, was first proposed by Maldacena [1], and identified string theory on an $AdS_{d+1} \times X$ background with a conformal field theory living on the boundary of this AdS_{d+1}space (d-dimensional).

String/Gauge Dualities

Gauge/string dualities relate two seemingly different quantum physical descriptions, one being a gauge field theory in a number of space-time dimensions, and the other a string theory on a two-dimensional conformal worldsheet. The most studied of these dualities is the AdS_5/CFT_4 correspondence [1], which relates type IIB superstring theory on the $AdS_5 \times S^5$ background and $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) gauge theory in four dimensions.

Both theories in the duality have a coupling constant λ and a genus-counting parameter g_s (string theory) and g_{YM} (on the gauge side). The latter g_s appears naturally on the string side, and related to the coupling of an $U(N_c)$ gauge theory through $g_{YM}^2 = 4\pi g_s$. At N_c large, there is a natural choice for the effective coupling λ in each theory, given by the 't Hooft coupling $\lambda = g_{YM}^2 N_c$ for the gauge theory [2], and is related to the string tension $\lambda = R^4/\alpha^2$ on the string side (R is the radius of the $AdS_5 \times S^5$ background, and α is the the inverse of the string tension). In a path integral formulation of these theories, one can easily see that only the combinations $\hbar g_{YM}^2$ and $R^2/\hbar\alpha$ appear in the gauge and string theories, respectively. Then, obtaining an expansion in $\hbar \ll 1$ (quantum corrections) is equivalent to expanding in $g_{YM}^2 \ll 1$ and $\alpha/R^2 \ll 1$. A direct comparison between these two theories is then non-trivial because of the strong/weak property of this duality. In the perturbative regime of gauge theory ('t Hooft coupling small), string theory is strongly coupled (α large), and vice versa. Thus, this property makes checking this conjectured duality a very hard problem to solve, as one cannot access both perturbative regimes at the same time. On the other hand, if we choose to believe in this correspondence, then it gives us access to previously inaccessible regimes in both sides of the duality.

Even though a direct comparison of most quantities in both sides of the duality can not be done, there are a few cases where such a comparison can be performed [3,4]. One example consists in calculating quantities which are protected from receiving quantum corrections (by supersymmetry). Other examples include certain limits which can be taken, such as the plane-wave limit [5], where the existence of some large charge allows us to define an effective coupling, and thus reach both sides of the duality.

Recently, a new example of an AdS/CFT duality has been conjectured [6], which relates the three dimensional $\mathcal{N}=6$ superconformal Chern-Simons theory (with a gauge group $SU(N)\times SU(N)$) and a theory of $M2$-branes in the background $AdS_4\times S^7/Z_k$ (11-dimensional). The parameter k in this AdS_4/CFT_3 duality is the Chern-Simons level. If we consider the limit $k,N\to\infty$ keeping $\lambda=2\pi N/k$ fixed (scaling limit), then the eleven dimensional M-theory becomes the type IIA superstring theory on the $AdS_4\times CP^3$ background.

One of the main focuses of this dissertation is the study of the classical string and gauge theories of these two examples of the AdS/CFT correspondence, and their semi-classical behaviour. This will be done with using tools such as the symmetries underlying each of these dualities, and their integrability properties.

Symmetries

The symmetries of a physical system are one of our best hopes of truly understanding it. The algebra of symmetries $\mathfrak{psu}(2,2|4)$ is central in $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence, as both the gauge theory and its string theory dual have the same underlying supersymmetry algebra $\mathfrak{psu}(2,2|4)$. The two dimensional sigma-model which gives us the perturbative string theory in $AdS_5\times S^5$ [7] has a manifest global symmetry under $\mathrm{PSU}(2,2|4)$ [8,9], the isometry group of the target space. This is the same group of internal and space-time (superconformal) symmetries of the $\mathcal{N}=4$ SYM (see [10] and references therein). Through AdS/CFT correspondence, the spectrum of energies of string states should coincide with the scaling dimensions of conformal gauge operators. This can be seen at the algebra level as a correspondence between the eigenvalues of the string light-cone Hamiltonian and the eigenvalues of the Dilatation operator in gauge theory, and was checked in the so called thermodynamic [11,12] and BMN [5] limits [13, 14, 15, 16, 17]. In the case of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence the same approach can be taken, but in this case the group of symmetries shared by string and gauge theories is the $OSp(2,2|6)$ [18].

The sigma-models for these dualities have been shown to be classically integrable [19,18], thus having an infinite number of (non)local integrals of motion. It is of great interest to study the algebra of symmetries of these dualities, which together with the integrability of the theories can give a lot of information on several properties of the physical system, such as scattering.

Integrability

Because of the integrable structures found to exist in planar (and perturbative) $\mathcal{N}=4$ SYM and in classical IIB superstring theory on $AdS_5\times S^5$, these theories are thought to have a Yangian symmetry, that is, an infinite number of conserved charges. Together with the symmetry algebra already known, these charges can provide a much more thorough map between gauge and string theories. But we are far from having a full set of explicit solutions with which to compare the two sides of the duality. Nevertheless, even if we don't know explicit form of the solutions, we can try to use the integrability properties of the theories to construct and compare the respective Bethe ansatz equations, allowing us to explore the full energy spectrum of classical strings and of eigenstates of the (planar) dilatation operator in gauge theory.

The $\mathcal{N}=4$ SYM theory was first seen to have integrable structures by Minahan and Zarembo in [20], where they found that in the invariant $\mathfrak{so}(6)$ sector (sector containing scalars) and at one-loop order in the 't Hooft coupling λ, the dilatation operator is isomorphic to the Hamiltonian of a $\mathfrak{so}(6)$ integrable quantum spin chain. There it was shown that determining the anomalous dimensions of operators (eigenvalues of the dilatation operator) in the $\mathfrak{so}(6)$ sector of the gauge theory is equivalent to solving the corresponding Bethe equations of this sector. This description of gauge theory operators as spin-chains can be easily pictured in the large N_c limit. In this limit one can restrict local operators to be single-trace operators, and we interpret the traces as cyclic spin-chains, and the fields inside the traces as spin sites. A specific example is the $\mathfrak{su}(2)$ sector, where the operators are traces of products of two scalar fields, and corresponds to the spin-1/2 Heisenberg model (only nearest neighbour interactions). The $\mathfrak{so}(6)$ sector will correspond to a spin-chain where at each site the spin can take six different values. This equivalence between conformal dimensions of operators and integrable spin chains has been generalized to the full $\mathfrak{psu}(2,2|4)$ algebra [21], and the gauge theory was also seen to be integrable to higher loops [21, 22, 23, 10, 24].

On the string side, the BMN limit allowed one to study the string action in a near plane-wave geometry, and compare results to near BPS gauge operators. On the other hand, the thermodynamic limit of string theory was seen to correspond to a semi-classical expansion about solitonic string solutions in the $AdS_5 \times S^5$ background [11]. This led to the study of multi-spin string configurations rotating on $AdS_5 \times S^5$ [12, 25] which were then compared to the $\mathfrak{so}(6)$ sector of the gauge theory. This work was later generalized to other classical string solutions with very large angular momentum in some directions of S^5, called giant magnons [26, 27], or of AdS^5 [28] in a limit which differs somewhat from the BMN limit, allowing us to take the large spin and semi-classical limits separately.

One can also study the comparison of integrable structures of string and gauge theories through the algebraic curve formalism. This was first studied in [29], where the $\mathfrak{su}(2)$ conserved currents of the classical string sigma-model were used to construct a Lax pair and re-write the equations of motion as a Riemann-Hilbert problem in terms of hyperelliptic curves. It was also shown that the same can be obtained from the gauge theory side up to two loops. These results have been generalized to other sectors of the theory [30,31,32,33]. The full spectrum of $AdS_5 \times S^5$ superstrings has been studied in [34] (at the classical level) and in [35] (quantum generalization). A Bethe ansatz for quantum strings was first proposed in [36], and further studied in [37, 38, 39, 40, 41, 42], including comparisons to the Bethe ansatz equations from the gauge side.

The integrable structures of the new gauge/string duality, the $\mathrm{AdS}_4/\mathrm{CFT}_3$, have also been studied: the sigma model has been seen to be classically integrable [18], and semi-classical expansions of rotating string configurations have also been studied [43,44,45]. These results were compared to the ones obtained through the algebraic curve formalism [46, 47].

Solitons

As with any physical system, it is of great interest to determine any solitonic solutions of the theory, and consider them as the fundamental excitations which one can then use to build the other states of the theory. In the limit of long spinning strings in $AdS_5 \times S^5$ such solitonic solutions were seen to exist [26] and were called giant magnons. Related to these by a generalized T-dual transformation another kind of solutions were also found [48], called the giant single spikes. The solitonic properties of these solutions were discussed through a reduction to the sine-Gordon field theory using the so called Pohlmeyer map [49]. Through this, the giant magnons were mapped to the sine-Gordon kinks (and the single spikes to an unstable kink) and some of their classical and semiclassical properties were determined, such as the time delay of scattering of magnons and their phase shift. This phase shift was found to be different for scattering of giant magnons and of sine-Gordon kinks, a consequence of a different symplectic structure associated to each of the problems [50]. Some attempts have been made to reconcile this issue [51], where the conjectured existence of a bi-Hamiltonian structure on the relativistic N-body model related to both sine-Gordon solitons and giant magnons was seen to lead to such a difference on the phase shifts.

Overview

The main text of this dissertation is divided into six parts. The first three chapters focus of the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence, introducing the algebra of symmetries and studying dynamics of giant magnon soliton solutions and the related giant spikes at a semi-classical level. The fourth chapter gives an introduction to the algebraic curve formalism from the string sigma model in $AdS_5 \times S^5$. The last two chapters focus on the AdS_4/CFT_3 correspondence, determining the spectrum of magnon solutions in the $\mathbb{R} \times CP^3$ space in the algebraic curve, and comparing it to several magnon solutions in this space.

Chapter 2

The Algebra of $\mathrm{AdS}_5/\mathrm{CFT}_4$

The gauge/string duality has been the object of study for more than a decade by means of the AdS/CFT correspondence [1,52,53], between IIB superstrings on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ $U(N_c)$ super Yang-Mills theory in four dimensions. But while the results calculated from the gauge theory are perturbative in 't Hooft coupling $\lambda = g_{YM}^2 N_c$, the calculations on the string side are valid for strong coupling λ.

This strong/weak property of the duality limited its study to operators/states in sectors protected by supersymmetry, as these would receive no quantum corrections. But a heuristic comparison of the algebraic structures in the weak/strong coupling limits was possible by taking the plane-wave limit, or BMN limit [5]. On the gauge theory side, the BMN limit is taken by considering single trace operators, i.e. N_c very large, with large R-charge of $\mathfrak{so}(6)$ $J \sim \sqrt{N_c}$ and conformal dimension Δ , keeping $\Delta - J$ finite. These operators consist of a chiral primary (trace of a large number of a complex field) with some impurities (other complex fields, bosonic or fermionic). Even though the 't Hooft coupling $\lambda = g_{YM}^2 N_c$ is very large, one can use perturbation theory provided some effective coupling $\lambda' = g_{YM}^2 N_c / J^2 \sim g_{YM}^2$ is kept fixed and small.

On the string side we start from the Green-Schwarz action on the $AdS_5 \times S^5$ [7], with $J \sim \sqrt{N_c}$ now being the angular momentum in one of the directions of S^5. We also take the energy E (generator of time translations in AdS_5) to be large, obeying $E - J$ finite, thus originating point-like closed strings with large angular momentum in S^5. In light-cone gauge, the quantity $E - J$ is just the light-cone Hamiltonian, and the light-cone momentum

$P_+ = E + J$ is very large. In this case, there is an effective coupling just $\tilde{\lambda} = 4\lambda/P_+^2$, which is equivalent to λ' in the limit $J \to \infty$ $(P_+/2 \to J)$.This limit allowed a direct comparison of the dilatation operator in SYM (anomalous dimensions of operators in the conformal field theory) to the energies $(E-J)$ of point like semiclassical string oscillations in the plane-wave geometry.

The algebra of symmetries $\mathfrak{psu}(2,2|4)$ is central in $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence, as both the gauge theory and its string theory dual have the same underlying supersymmetry algebra. The two dimensional sigma-model, which gives us the perturbative string theory in $AdS_5 \times S^5$ [7], has a manifest global symmetry under $\mathrm{PSU}(2,2|4)$ [8,9], which is the same group of internal and space-time symmetries of the $\mathcal{N} = 4$ SYM (see [10] and references therein).

In particular, one can use the algebra to compare the scattering of particles in the duality. For large 't Hooft coupling the scattering is best described by string theory, but for small 't Hooft coupling the spin-chain description is more adequate. It was shown by Beisert that in this limit the non-perturbative S-matrix is almost completely determined by the centrally extended $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ algebra [54,55,56], up to an overall dressing phase (determined by a crossing symmetry restriction [57,58,59,60]). In fact, the invariance of the S-matrix under this algebra (that is, its trivial commutation relations with every generator of the algebra) led to discovery that it satisfies the Yang-Baxter equation, requirement for a factorized scattering, which is a property of integrability [55]. The related Yangian symmetry was also found in [61].

Each of these centrally extended algebras $\mathfrak{su}(2|2)$ has the following structure: bosonic (kinematical) generators $\mathbf{R}^a_b, \mathbf{L}^\alpha_\beta$, corresponding to the rotation generators of the bosonic subalgebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$; fermionic (dynamical) supersymmetry generators $\mathbf{Q}^\alpha_a$, $\mathbf{Q}^{\dagger b}_\beta$; and three central charges $\mathbf{H}$, $\mathbf{C}$, $\mathbf{C}^\dagger$ (Hamiltonian, generator of space translations and of boosts).[1]

[1]Note that $(\mathbf{Q}^\alpha_a)^\dagger = \mathbf{Q}^{\dagger a}_\alpha$ and the same relation holds to the central elements $\mathbf{C}$ and $\mathbf{C}^\dagger$.

Their commutation relations are:

$$
\begin{aligned}
\left[\mathbf{L}^{\alpha}_{\beta},\mathbf{J}^{\gamma}\right] &= \delta^{\gamma}_{\beta}\mathbf{J}^{\alpha}-\frac{1}{2}\delta^{\alpha}_{\beta}\mathbf{J}^{\gamma}, \qquad [\mathbf{R}^{a}_{b},\mathbf{J}^{c}] = \delta^{c}_{b}\mathbf{J}^{a}-\frac{1}{2}\delta^{a}_{b}\mathbf{J}^{c}\,,\\
\left[\mathbf{L}^{\alpha}_{\beta},\mathbf{J}_{\gamma}\right] &= -\delta^{\alpha}_{\gamma}\mathbf{J}_{\beta}+\frac{1}{2}\delta^{\alpha}_{\beta}\mathbf{J}_{\gamma}, \qquad [\mathbf{R}^{a}_{b},\mathbf{J}_{c}] = -\delta^{a}_{c}\mathbf{J}_{b}+\frac{1}{2}\delta^{a}_{b}\mathbf{J}_{c}\,,\\
\left\{\mathbf{Q}^{\alpha}_{a},\mathbf{Q}^{\dagger b}_{\beta}\right\} &= \delta^{b}_{a}\mathbf{L}^{\alpha}_{\beta}+\delta^{\alpha}_{\beta}\mathbf{R}^{b}_{a}+\frac{1}{2}\delta^{b}_{a}\delta^{\alpha}_{\beta}\mathbf{H},\\
\left\{\mathbf{Q}^{\alpha}_{a},\mathbf{Q}^{\beta}_{b}\right\} &= \varepsilon^{\alpha\beta}\varepsilon_{ab}\mathbf{C}\,, \qquad \left\{\mathbf{Q}^{\dagger a}_{\alpha},\mathbf{Q}^{\dagger b}_{\beta}\right\} = \varepsilon^{ab}\varepsilon_{\alpha\beta}\mathbf{C}^{\dagger}\,.
\end{aligned}
\tag{2.1}
$$

In the above expressions, $\mathbf{J}^{M}$ (where $M \in \{a,\alpha\}$, a being bosonic indices and α being the fermionic ones), is any element of the Lie algebra. From the elements of the algebra, the dilatation operator (or central charge Hamiltonian) has been studied in detail (see [10] and references therein).

Much has been done on the study of sectors of this superconformal algebra on the string side [62,63,64,65,66]. On the gauge side, Beisert has perturbatively studied and determined the action of the generators of the superalgebra $\mathfrak{su}(2|2)$ up to two loops, by first restricting to the subalgebra $\mathfrak{su}(2|3)$ whose fundamental representation consists of three complex scalars and two complex fermions [67], and finally considering an infinite chain of one of the scalar operators [55]. Using Bethe Ansatz techniques it was later conjectured an all loop result in this sector for the action of the algebra generators [56].

In this chapter, we present the SUSY algebra in terms of a matrix model reduction of Yang-Mills Theory in the large N limit. The matrix model has played a very useful role in large N theories. In fact, the $\frac{1}{2}$ BPS sector of $N=4$ SYM is completely described in terms of a complex matrix model [68,69,70,71,72], and the $\frac{1}{4}$ BPS generalization is also of great interest (work in progress). Presently, the interest is in the detailed construction and comparison of supercharges and their commutation relations both on the Yang-Mills and on the string side. We will demonstrate that the algebra given by Beisert in [56] (at least at one loop) is correctly reproduced from the reduced Matrix model point of view.

In [73] it was seen that the plane-wave matrix theory [5,74] arises when compactifying $\mathcal{N}=4$ SYM in $\mathbb{R}\times S^3$ followed by a consistent truncation in order to keep only the lowest Kaluza-Klein modes (see also [75,76]). These modes have masses proportional to a mass parameter, given by $\left(\frac{m}{3}\right)^3 = \frac{32\pi^2}{g_{YM}^2}$. This theory was shown (in some sectors) to still be integrable up to four-loops [77, 78]. Study of this model is simpler than the full $\mathcal{N} = 4$ SYM, and can be found in Section 2.4. In this section we present a detailed study of the supercharges Q and S, following the approach of [73]. In Section 2.5 we restrict the action of the generators of the algebra to a subsector $\mathfrak{su}(2|2)$. The results presented in this paper are one-loop, and we compare our results with the non-local generators presented in [56], evaluating some of the parameters defining these generators.

Some methods have been employed in the gauge theory side that allowed a comparison of the Hamiltonian to string theory equivalent algebra generator. Such methods include the use of coherent states [13, 17, 79], collective field theory and string field theory [80, 81]. In this framework one can compare a discrete (first quantized) version of the supercharges on the string side with the oscillator expansion of the charges in SYM, in the BMN limit.

2.1 The algebra of strings in a $AdS_5 \times S^5$ background

We start from the Green-Schwarz string theory on the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background, following [66,64]. The string sigma-model has a global symmetry given by the supergroup PSU$(2,2|4)$. Because it is a string theory, it also has local worldsheet diffeomorphism symmetries and the fermionic κ-symmetry. So one is interested in fixing these symmetries and keeping only the physical degrees of freedom. In order to do so, the uniform light-cone gauge will be used.[2]

Our main interest is in studying the generators of the superisometry algebra $\mathfrak{psu}(2,2|4)$, which can be divided into two groups, according to whether they (Poisson) commute with the Hamiltonian or not. The first case corresponds to the subalgebra $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ which is the algebra we will be focusing on. We can also separate the generators into dynamical and kinematical generators, depending on whether they depend or not on some unphysical field

[2] One thing to remember is that the manifold space is given by the full group of isometries divided by the little group (that is, the group of isometries, or rigid rotations, given one fixed point). In the case of S^5 we have $S^5 \equiv \frac{\mathrm{SO}(6)}{\mathrm{SO}(5)}$, and also for $\mathrm{AdS}_5 \equiv \frac{\mathrm{SO}(2,4)}{\mathrm{SO}(1,4)}$. The full group of isometries of $\mathrm{AdS}_5 \times \mathrm{S}^5$ is $\mathrm{PSU}(2,2|4)$, and so the space manifold is equivalent to

$$\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(1,4) \times \mathrm{SO}(5)}.$$

x_- (one of the light-cone co-ordinates). The zero mode of this unphysical field x_- will be seen to be the conjugate variable to the total light-cone momentum P_+, and consequently, the dynamical variables change the light-cone momentum. The large J limit mentioned before corresponds to the limit of infinite P_+ in light-cone variables, and in this limit one can see that the zero mode x_-^0 can be dropped.

The derivative x_-' can also be seen to be a density of the world-sheet momentum (related to rigid symmetries of the $\sigma-$direction of the string action, after gauge fixing), and for closed strings the periodicity of the fields requires the total worldsheet momentum to vanish. This imposes a constraint in x_-, called the level-matching condition. It can be shown in the string theory frame that relaxing this level-matching condition and considering the limit of infinite light-cone momentum P_+ leads to a central extended $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ algebra. This extension consists of an extra common central element, proportional to the level-matching condition, with the Hamiltonian remaining central (as in the on-shell algebra, where the level-matching condition is obeyed).

The same thing should be obtainable from the gauge theory side. In $\mathcal{N}=4$ gauge theory, the level-matching condition corresponds to working with traces of products of fields (gauge-invariant operators), and the relaxing of such constraint corresponds to consider infinitely long operators. From [55] one has that the opening of the traces and considering infinitely long operators will add two central charges (to the one already existing, the Hamiltonian) to the algebra $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$, functions of the momentum carried by the one-particle excitations.

2.2 Superstrings on $AdS_5 \times S^5$: action and symmetry algebra generators

The full superstring action is a sum of the non-linear σ-model and a topological Wess-Zumino term (uniquely fixed by requiring $\mathrm{PSU}(2,2|4)$ and κ-symmetry invariance). Its target space is given by the coset manifold [7]

$$\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)}.$$

We want to study the algebra of isometries of this action, the $\mathfrak{psu}(2,2|4)$ superalgebra. This analysis follows closely [64]. To study this coset space, consider 8×8 matrices of the form

$$M=\begin{pmatrix} A & X \\ Y & D \end{pmatrix},$$

where the 4×4 matrices A and D are Grassmann even, while the matrices X and Y are Grassmann odd. The superalgebra $\mathfrak{su}(2,2|4)$ can be described by requiring that M has vanishing supertrace, $\mathrm{str}M=\mathrm{tr}A-\mathrm{tr}D=0$, and satisfies $HM+M^{\dagger}H=0$. In this last condition, the matrix H is a hermitian matrix, chosen to be

$$H=\begin{pmatrix} \Sigma & 0 \\ 0 & \mathbb{I} \end{pmatrix}, \text{ with } \quad \Sigma=\begin{pmatrix} \mathbb{I}_{2\times 2} & 0 \\ 0 & -\mathbb{I}_{2\times 2} \end{pmatrix}.$$

This choice of H allows us to easily see that $A\in\mathfrak{u}(2,2)$ and $D\in\mathfrak{u}(4)$, and also that $Y=-X^{\dagger}\Sigma$ (conjugated to each other). There is a $\mathfrak{u}(1)$ generator from each of the $\mathfrak{u}(2,2)$ and $\mathfrak{u}(4)$, but only the supertraceless combination of the two belongs to $\mathfrak{su}(2,2|4)$. So, the bosonic subalgebra of $\mathfrak{su}(2,2|4)$ is

$$\mathfrak{su}(2,2)\oplus\mathfrak{su}(4)\oplus\mathfrak{u}(1).$$

Finally the superalgebra $\mathfrak{psu}(2,2|4)$ corresponds to the quotient algebra of $\mathfrak{su}(2,2|4)$ over this last $\mathfrak{u}(1)$, and cannot be represented in 8×8 matrices. Nevertheless, this last result will be enough for the present discussion, and we now turn to building the superstring action.

To construct the superstring action, one uses a $\mathbb{Z}_4$ grading of the superalgebra $\mathfrak{su}(2,2|4)$.

This grading is defined by the automorphisms $M \to \Omega(M)$, where

$$\Omega(M) = \begin{pmatrix} KA^tK & -KY^tK \\ KX^tK & KD^tK \end{pmatrix}, \text{ with } \quad K = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Given this automorphism, any matrix $M \in \mathfrak{su}(2,2|4)$ can be decomposed into 4 term

$$M = M^{(0)} + M^{(2)} + M^{(1)} + M^{(3)},$$

where each term will obey $\Omega\left(M^{(p)}\right) = i^p M^{(p)}$. Note that $M^{(0,2)}$ are Grassmann even, while $M^{(1,3)}$ are Grassmann odd. It can be seen that the matrices $M^{(0)}$ span the $\mathfrak{so}(4,1) \times \mathfrak{so}(5)$ subalgebra, to be mod out in the coset space.

One now considers a group element g of $\mathrm{PSU}(2,2|4)$, and constructs a flat current ($\mathbb{Z}_4$ decomposition)

$$\mathbf{A} = -g^{-1}\mathrm{d}g = \mathbf{A}^{(0)} + \mathbf{A}^{(1)} + \mathbf{A}^{(2)} + \mathbf{A}^{(3)},$$

from which the superstring action can be written as [7]:

$$S = -\frac{\sqrt{\lambda}}{4\pi}\int_{-\pi}^{\pi} \mathrm{d}\sigma \mathrm{d}\tau \left(\gamma^{\alpha\beta}\mathrm{str}\left(\mathbf{A}_\alpha^{(2)}\mathbf{A}_\beta^{(2)}\right) + \kappa\varepsilon^{\alpha\beta}\mathrm{str}\left(\mathbf{A}_\alpha^{(1)}\mathbf{A}_\beta^{(3)}\right)\right).$$

In the above expression, λ is the string effective tension, and from κ-symmetry we have $\kappa = \pm 1$. Also $\gamma^{\alpha\beta} = \sqrt{-h}h^{\alpha\beta}$ where $h^{\alpha\beta}$ is the worldsheet metric.

Parametrization of coset group elements

Before we start, it is important to parametrize the bosonic part of the orthogonal complement of $\mathfrak{so}(4,1) \times \mathfrak{so}(5)$ (the latter spanned by the matrices $M^{(0)}$), namely the even matrices $M^{(2)}$. The total bosonic algebra corresponds to $\mathfrak{su}(2,2) \oplus \mathfrak{su}(4)$, and we will be treating the algebras $\mathfrak{so}(4,1)$ and $\mathfrak{so}(5)$ and their orthogonal complements separately. The algebra $\mathfrak{su}(4)$ has 15 generators, out of which 10 span the algebra $\mathfrak{so}(5)$, and the other 5 span its orthogonal complement.

From the eigenstates of the matrices $M^{(p)}$ under the map Ω, one can easily check that if

$$M^{(0,2)} = \begin{bmatrix} A_{0,2} & 0 \\ 0 & D_{0,2} \end{bmatrix},$$

then we have the relations

$$\begin{cases} KA_0^t K = A_0 \\ KD_0^t K = D_0 \end{cases} \quad \text{and} \quad \begin{cases} KA_2^t K = -A_2 \\ KD_2^t K = -D_2 \end{cases}.$$

In these, $D_{0,2} \in \mathfrak{su}(4) \equiv \mathfrak{so}(6)$, while $A_{0,2} \in \mathfrak{su}(2,2) \equiv \mathfrak{so}(4,2)$. So we can conclude that the D_0 will span $\mathfrak{so}(5)$ while D_2 spans its orthogonal complement (and the analogous relation can be found between $A_0 \in \mathfrak{so}(4,1)$, and $A_2 \in \mathfrak{so}(4,1)^{\perp}$).

In order to parametrize $M^{(2)}$, one introduces the 5 (hermitian) Dirac matrices for $\mathrm{SO}(5)$ $\{\gamma_s,\, s = 1, ..., 4,\, \gamma_5 = \Sigma\}$, such that

$$K\gamma_s^t K = -\gamma_s \,, \qquad K\Sigma^t K = -\Sigma.$$

These 5 generators span the orthogonal complement to the algebra $\mathfrak{so}(5)$, understood as a subalgebra of $\mathfrak{su}(4)$, while the other 10 generators of $\mathfrak{su}(4)$ will obey the same relations with a plus sign on the r.h.s..

To parametrize the orthogonal algebra to $\mathfrak{so}(4,1)$, one can take $\{\gamma_a, a = 1, ..., 4, i\Sigma\}$, where the γ_a are the Dirac matrices pointed above.

We can finally write a matrix $M^{(2)}$ as

$$M^{(2)} = x_M \Sigma_M + \begin{pmatrix} it\Sigma & 0 \\ 0 & i\phi\Sigma \end{pmatrix} + im_0 \mathbb{I},$$

where we have real parameters $x_M = \{z_a, y_s\}, t, \phi, m_0$, and the 8×8 generators:[3]

$$\Sigma_M = \left\{ \begin{pmatrix} \gamma_a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & i\gamma_s \end{pmatrix} \right\}, \quad \Sigma_\pm = \begin{pmatrix} \pm\Sigma & 0 \\ 0 & \Sigma \end{pmatrix}. \tag{2.2}$$

The real parameters are such that $\{t, z_a\}$ parametrize AdS_5 space, and $\{\phi, y_s\}$ parametrize S^5.

We are now ready to parametrize a general element of the coset group. For that choose $g(\chi,x,t,\phi) = \Lambda(t,\phi)\, g(\chi)\, g(x)$. Matrices $\Lambda(t,\phi), g(x)$ are even, and give us an embedding of the target space into $\mathrm{SU}(2,2)\times\mathrm{SU}(4)$, and the matrix $g(\chi)$ is odd, and includes all the 32 fermionic degrees of freedom. We have

$$g(x) = \begin{pmatrix} g_a(z) & 0 \\ 0 & g_s(y) \end{pmatrix}, \quad \Lambda(t,\phi) = \exp\left\{ \frac{i}{2}\left(x_+\Sigma_+ + x_-\Sigma_-\right) \right\}, \tag{2.3}$$

where $x_\pm$ are the light-cone co-ordinates, given in the next section, and

$$g_a(z) = \left(4 - z_a^2\right)^{-\frac{1}{2}} \left(2 + z_a\gamma_a\right), \quad g_s(y) = \left(4 + y^2\right)^{-\frac{1}{2}} \left(2 + iy_s\gamma_s\right).$$

Using this parametrization, the metric of the target space is:

$$ds^2 = -G_{tt}(z)\, dt^2 + G_{\phi\phi} d\phi^2 + \frac{16}{\left(4 - z^2\right)^2} dz_a dz_b + \frac{16}{\left(4 + y^2\right)^2} dy_s dy_r, \tag{2.4}$$

with $G_{tt} = \left(\frac{4+z^2}{4-z^2}\right)^2$ and $G_{\phi\phi} = \left(\frac{4-y^2}{4+y^2}\right)^2$.

Finally we need to parametrize the fermionic degrees of freedom. They are given by

$$g(\chi) = \chi + \sqrt{1+\chi^2}, \quad \chi = \begin{pmatrix} 0 & \Theta \\ -\Theta^\dagger\Sigma & 0 \end{pmatrix}.$$

The next step is to fix κ-symmetry and use the uniform light-cone gauge.

[3]It includes $\mathbb{I}$ the identity 8×8 matrix, which is a $\mathrm{U}(1)$ generator, included in the algebra $\mathfrak{su}(2,2|4)$, but divided in the $\mathfrak{psu}(2,2|4)$.

The Uniform Light-Cone Gauge

To impose the light cone gauge on a string propagating on some target space, we will use the time co-ordinate of the manifold, called t, and we assume there is an angle variable ϕ with respect to which the manifold has a $U(1)$ isometry under shifts of ϕ [62]. Also the string σ-model action has to be invariant under shifts of both t and ϕ, as well as all of the bosonic and fermionic fields (the string action does not depend explicitly on t or ϕ and depends only on the derivatives of the fields). In particular we will be interested in studying the Green-Schwarz superstring in $\mathrm{AdS}_5 \times \mathrm{S}^5$ [7]. For this case, t is just the global time co-ordinate of AdS_5 and ϕ is an angle of S^5.

Invariance under shifts of t and ϕ result in two conserved currents E^α, J^α, and the corresponding conserved charges

$$E = \int_0^{2\pi} \frac{d\sigma}{2\pi} E^0 \qquad ; \qquad J = \int_0^{2\pi} \frac{d\sigma}{2\pi} J^0 .$$

In the above expression $E^0 = -p_t$ and $J^0 = p_\phi$ are just the momenta conjugate to the co-ordinates t, ϕ, and E and J are the target space energy and the total $U(1)$ charge of the string (or angular momentum), respectively.

We now need to define the light-cone co-ordinates $x_\pm$

$$\left\{ \begin{array}{lcl} t & = & x_+ - x_- \\ \phi & = & x_+ + x_- \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{lcl} x_+ & = & \frac{1}{2}(\phi + t) \\ x_- & = & \frac{1}{2}(\phi - t) \end{array} \right. ,$$

and the respective momentum conjugates

$$\left\{ \begin{array}{lcl} p_t & = & \frac{1}{2}(p_+ + p_-) \\ p_\phi & = & \frac{1}{2}(p_+ - p_-) \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{lcl} p_+ & = & p_\phi + p_t \\ p_- & = & p_t - p_\phi \end{array} \right. .$$

Note that p_+ (p_-) is the momentum conjugate to x_- (x_+).

The uniform light-cone gauge is obtained by fixing

$$x_+ = \tau + \frac{m}{2}\sigma , \quad p_+ = P_+ = E + J = \text{constant}. \tag{2.5}$$

The integer m is just the winding number related to the angle variable ϕ. To fix this gauge we also need to fix the local fermionic κ-symmetry, which will be done next.

In this gauge the worldsheet Hamiltonian corresponds to p_-,

$$H = \int_0^{2\pi} \frac{d\sigma}{2\pi} p_- = E - J.$$

But the Hamiltonian can be also written as a function of P_+, and so we get an equation for the energy $E = J + H(E+J)$, which will give the energy E as a function just of J - the dispersion relation.

One now proceeds to fixing the light-cone gauge, and κ-symmetry [64].[4] If one only considered the bosonic degrees of freedom, then it would be enough to have introduced the momenta canonically conjugate to the light-cone co-ordinates $x_\pm$, But if we keep the fermionic degrees of freedom as well, the expressions for the momenta will be obtained by introducing a auxiliary field π, in the algebra of the $M^{(2)}$:[5]

$$\pi = \frac{i}{4}\pi_+\Sigma_+ + \frac{i}{4}\pi_-\Sigma_- + \frac{1}{2}\pi_M\Sigma_M + \pi_0 i\mathbb{I}. \tag{2.6}$$

Then we write the superstring action as:

$$S = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\sigma\mathrm{d}\tau \mathrm{Str}\left(\pi A_0^{(2)} + \kappa\frac{\sqrt{\lambda}}{2}\varepsilon^{\alpha\beta}\mathbf{A}_\alpha^{(1)}\mathbf{A}_\beta^{(3)} - \frac{1}{2\sqrt{\lambda}\gamma^{00}}\left(\pi^2 + \lambda(\mathbf{A}_1^{(2)})^2\right) + \frac{\gamma^{01}}{\gamma^{00}}(\pi\mathbf{A}_1^{(2)})\right). \tag{2.7}$$

We get the original action by solving the equations of motion for π:

$$\pi = \sqrt{\lambda}\gamma^{\tau\beta}\mathbf{A}_\beta^{(2)}, \tag{2.8}$$

and plugging them back into the action. Note that the momenta conjugate to the light-cone co-ordinates will also be written as functions of the auxiliary field π. The last two

[4] A simpler version of this gauge fixing, only considering the bosonic part of the theory, can be found in [82].

[5] The field π is an even field that lives on the $\mathfrak{su}(2,2|4)$ algebra. It lives on this algebra because it is a momentum field (When the manifold is a group, the momenta, conjugate to the fields that live on the manifold, live on the corresponding algebra). Consequently, we have a $\mathbb{Z}_4$ decomposition, $\pi = \pi^{(0)} + \pi^{(2)}$ and as we are dividing by the subspace of the $M^{(0)}$, we can consider $\pi = \pi^{(2)}$.

terms in the action correspond to the Virasoro constraints, to be imposed after gauge and κ-symmetry fixing.

$$C_1 = \mathrm{Str}\left(\pi^2 + \lambda\left(\mathbf{A}_1^{(2)}\right)^2\right) = 0, \tag{2.9}$$

$$C_2 = \mathrm{Str}\left(\pi \mathbf{A}_1^{(2)}\right) = 0. \tag{2.10}$$

κ-Symmetry

Recalling that the fermionic part of the group element of the coset space, $g(\chi)$, depends solely on a complex 4×4 odd matrix $\mathbf{\Theta}$, we need only to restrict the entries of this matrix to fix κ-symmetry. In fact, κ-symmetry can be fixed by choosing

$$\mathbf{\Theta} = \begin{pmatrix} 0 & \mathbf{\Theta}_1 \\ \mathbf{\Theta}_2 & 0 \end{pmatrix},$$

where $\mathbf{\Theta}_{1,2}$ are odd 2×2 complex matrices. Then the fixed χ obeys the relations:

$$\Sigma_\pm \chi = \mp \chi \Sigma_\pm .$$

These relations can be seen as defining the fixed κ-symmetry. Using these relations, together with $g^{-1}(\chi) = g(-\chi)$, we obtain the following properties for $g(\chi)$:

$$g^{-1}(\chi)\Sigma_+ = \Sigma_+ g(\chi) \quad \text{and} \quad g^{-1}(\chi)\Sigma_- = \Sigma_- g^{-1}(\chi).$$

Note that the bosonic $g(x)$ has similar relations, namely $\Sigma_\pm g^{-1}(x) = g(x)\Sigma_\pm$ and $g^{-1}(x) = g(-x)$.

One can write the current $\mathbf{A} = \mathbf{A}_{even} + \mathbf{A}_{odd}$ fully parametrized:

$$\begin{aligned}
\mathbf{A}_{even} &= -g^{-1}(x)\frac{i}{2}\left\{dx_+\Sigma_+\left(1+2\chi^2\right) + dx_-\Sigma_-\right\}g(x) - \\
&\quad -g^{-1}(x)\left\{\sqrt{1+\chi^2}\,d\sqrt{1+\chi^2} - \chi d\chi + dg(x)g^{-1}(x)\right\}g(x) \\
\mathbf{A}_{odd} &= -g^{-1}(x)\left\{idx_+\Sigma_+\chi\sqrt{1+\chi^2} + \sqrt{1+\chi^2}\,d\chi - \chi d\sqrt{1+\chi^2}\right\}g(x).
\end{aligned}$$

Next, we want to separate the kinetic term as the product of the time derivative of light cone co-ordinates $x_\pm$ with the corresponding momentum operators $p_\mp$ (the symplectic term). The momentum p_+ does not have a contribution from $\mathbf{A}_{odd}$, as only $\mathbf{A}_{even}$ depends on x_-. Once we substitute $\mathbf{A}$ into the action, we easily get that[6]

$$p_+ = -\frac{i}{2}\mathrm{Str}\left(\pi\Sigma_- g\left(x\right)^2\right) = G_+\pi_+ - G_-\pi_-, \tag{2.11}$$

where $G_\pm = \frac{1}{2}\left(G_{tt}^{\frac{1}{2}} \pm G_{\phi\phi}^{\frac{1}{2}}\right)$.

Gauge Fixing

Now we have all the equations needed to write the gauge-fixed Lagrangian. As was done in [64], we first solve equation (2.11) using (2.5), in order to get $\pi_+\left(P_+,\pi_-\right)$. We then solve the Virasoro constraint (2.10) to get the unphysical field x_- as a function of the physical fields. In fact one gets the worldsheet momentum density x'_-. Finally, one solves the constraint (2.9) in order to get π_-. The kinematic term of the Lagrangian thus obtained is not in the right form, as it will lead to a complicated Poisson structure. By performing a field redefinition, such that the final fields satisfy the canonical commutation relations, we get:

$$S = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\sigma\mathrm{d}\tau\left(\mathscr{L}_{kin} - \mathscr{H}\right), \tag{2.12}$$

where

$$\mathscr{L}_{kin} = p_M\dot{x}_M - i\mathrm{Str}\left(\Sigma_+\chi\dot{\chi}\right) = p_M\dot{x}_M + i\eta_a^\dagger\dot{\eta}_a + i\theta_a^\dagger\dot{\theta}_a, \tag{2.13}$$

and x_M being the physical bosonic degrees of freedom (and p_M their conjugate momenta), and η_a, θ_a are the fermionic ones. Before writing down the expression for the Hamiltonian density $\mathscr{H}$, a few comments are in order. We will be interested in the limit $P_+ \to \infty$, with the effective coupling $\tilde{\lambda} = \frac{4\lambda}{P_+^2}$ fixed, called the near plane-wave limit. In this limit we can

[6]The only contribution to p_+ will come from the term $\mathrm{Str}\left(\pi\mathbf{A}_0^{(2)}\right)$, as it will be the only term with $\dot{x}_- = \partial_\tau x_-$, apart from the Virasoro constraints.

expand the Hamiltonian in powers of P_+^{-1},

$$\mathscr{H} = \mathscr{H}_2 + \frac{1}{P_+}\mathscr{H}_4 + ... + \frac{1}{P_+^{n-1}}\mathscr{H}_{2n} + ..., \tag{2.14}$$

where the term $\mathscr{H}_{2n}$ in the Hamiltonian contains a product of $2n$ fields. The first two terms, the quadratic and quartic Hamiltonians, were deduced in [64], and are[7]

$$\begin{aligned}
\mathscr{H}_2 &= \frac{1}{2}P_M^2 + \frac{1}{2}x_M^2 + \frac{\tilde{\lambda}}{2}x_M'^2 + \kappa\frac{\sqrt{\tilde{\lambda}}}{2}\mathrm{Str}\left(\Sigma_+\chi\tilde{\Omega}\chi''\Omega\right) + \frac{1}{2}\mathrm{Str}\chi^2, \\
\mathscr{H}_4 &= \tilde{\lambda}\left(y'^2z^2 - z'^2y^2 + z'^2z^2 - y'^2y^2\right) - \\
&\quad -\frac{\tilde{\lambda}}{2}\mathrm{Str}\left(\frac{1}{2}\chi\chi'\chi\chi' + \chi^2\chi'^2 + \frac{1}{4}\left(\chi\chi' - \chi'\chi\right)\Omega\left(\chi\chi' - \chi'\chi\right)^t\Omega + \chi\tilde{\Omega}\chi''\Omega\chi\tilde{\Omega}\chi''\Omega\right) \\
&\quad +\frac{\tilde{\lambda}}{2}\mathrm{Str}\left(\left(z^2 - y^2\right)\chi'\chi' + \frac{1}{2}x_M'x_N\left[\Sigma_M,\Sigma_N\right]\left(\chi\chi' - \chi'\chi\right) - 2x_Mx_N\Sigma_M\chi'\Sigma_N\chi'\right) \\
&\quad +i\kappa\frac{\sqrt{\tilde{\lambda}}}{8}\left(x_Np_M\right)'\mathrm{Str}\left(\left[\Sigma_N,\Sigma_M\right]\left(\tilde{\Omega}\chi^t\Omega\chi - \chi\tilde{\Omega}\chi^t\Omega\right)\right).
\end{aligned}$$

One last important quantity is the worldsheet momentum density x_-', which integrated over σ will give the total worldsheet momentum:

$$p_{ws} = \int_{-\pi}^{\pi} d\sigma x_-' = -\frac{1}{P_+}\int_{-\pi}^{\pi} d\sigma\left(p_Mx_M' - \frac{i}{2}\mathrm{Str}\left(\Sigma_+\chi\chi'\right)\right).$$

The level-matching condition mentioned before becomes $p_{ws} = 0$.

Conserved Charges

Because of the invariance of the action under the $\mathrm{PSU}(2,2|4)$, one can determine the following conserved currents (in terms of $\mathbf{A}_\alpha$):

$$J^\alpha = \sqrt{\lambda}g(x,\chi)\left(\gamma^{\alpha\beta}\mathbf{A}_\beta^{(2)} - \frac{\kappa}{2}\varepsilon^{\alpha\beta}\left(\mathbf{A}_\beta^{(1)} - \mathbf{A}_\beta^{(3)}\right)\right)g(x,\chi)^{-1},$$

[7] Note that Ω, $\tilde{\Omega}$ are the following matrices

$$\Omega = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} \quad \text{and} \quad \tilde{\Omega} = \begin{pmatrix} K & 0 \\ 0 & -K \end{pmatrix}.$$

and the corresponding conserved charges

$$Q = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\sigma J^{\tau}.$$

By using the equations of motion for π, and also using $F_\alpha = \sqrt{1+\chi^2}\partial_\alpha\chi - \chi\partial_\alpha\sqrt{1+\chi^2}$ (the odd component of $g^{-1}(\chi)\,\partial_\alpha g(\chi)$), we obtain the charges in the form[8]

$$Q = \int_{-\pi}^{\pi}\frac{d\sigma}{2\pi}\Lambda U \Lambda^{-1}, \tag{2.15}$$

where Λ was given above (2.3), and

$$U = g(\chi)\,g(x)\left(\pi + i\frac{\kappa}{2}\sqrt{\lambda}\,g(x)\,\tilde{\Omega}F_\sigma^t\Omega\right)g(x)^{-1}\,g(\chi)^{-1}.$$

Combining the components of Q will give the conserved charges corresponding to rotations, dilatations, supersymmetries, etc. To obtain any particular one, we need only to multiply Q by a particular matrix $\mathscr{M} \in \mathfrak{su}(2,2|4)$ and take the supertrace:

$$Q_{\mathscr{M}} = \mathrm{Str}(Q\mathscr{M}). \tag{2.16}$$

The charges Q live on the algebra $\mathfrak{psu}(2,2|4)$, and we know from the generators of this algebra that the trace of the product of two generators will be either zero (if the generators are different) or proportional to the identity (if it is the same generator). This happens because the generators are either Dirac matrices or $\Sigma_\pm$. So taking the product of Q with a particular matrix $\mathscr{M}$ of the algebra followed by the trace acts as a projection on the direction of that matrix.

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[8] The definition of F_σ allows us to write

$$\mathbf{A}_\sigma^{(1)} - \mathbf{A}_\sigma^{(3)} = -ig(x)\,\tilde{\Omega}F_\sigma^t\Omega g(x)^{-1}.$$

2.3 Superstring on $AdS_5 \times S^5$: Superalgebra generators

This section will be devoted to determining the action of the supercharges from the string action in $AdS_5 \times S^5$ on a lattice string, starting from the definition obtained above for the generators $Q_{\mathcal{M}}$.

An important property of the generators $Q_{\mathcal{M}}$ is that if the generator of a symmetry $Q_{\mathcal{M}}$ is independent of x_+, it Poisson commutes with the Hamiltonian. That happens because for a symmetry generator in the light-cone gauge $x_+ = \tau$ we have:

$$\frac{dQ_{\mathcal{M}}}{d\tau} = \frac{\partial Q_{\mathcal{M}}}{\partial \tau} + \{H, Q_{\mathcal{M}}\} = 0.$$

This means that the generators independent of x_+ form an algebra that contains the Hamiltonian H as a central element.

The dependence of the charge Q on $x_\pm$ comes from the matrix $\Lambda = e^{\frac{i}{2}x_+\Sigma_+ + \frac{i}{2}x_-\Sigma_-}$, and consequently we can check the dependence of a charge $Q_{\mathcal{M}}$ on the light-cone variables $x_\pm$ b y checking whether $\mathcal{M}$ commutes with $\Sigma_\pm$. Also, using the definitions of kinematical vs dynamical charges (according to their dependence on x_-, $[\Sigma_-, \mathcal{M}_{kin}] = 0$), and the property of odd/even $\mathcal{M}$, we can separate the matrices $\mathcal{M}$ into four categories:

$$\begin{aligned}
\Lambda^{-1}\mathcal{M}^{odd}_{dyn}\Lambda &= e^{-\frac{i}{2}x_-\Sigma_-}\mathcal{M}^{odd}_{dyn}, \\
\Lambda^{-1}\mathcal{M}^{odd}_{kin}\Lambda &= e^{ix_+\Sigma_+}\mathcal{M}^{odd}_{kin}, \\
\Lambda^{-1}\mathcal{M}^{even}_{dyn}\Lambda &= \Lambda^2\mathcal{M}^{even}_{dyn}, \\
\Lambda^{-1}\mathcal{M}^{even}_{kin}\Lambda &= \mathcal{M}^{even}_{kin}.
\end{aligned} \tag{2.17}$$

This information tells us that only $\mathcal{M}^{odd}_{dyn}$ and $\mathcal{M}^{even}_{kin}$ will give rise to charges $Q_{\mathcal{M}}$ independent of x_+.[9] The matrices of these categories are of the form

$$\mathscr{M}^{odd}_{dyn} = \begin{pmatrix} 0 & 0 & d_1 & 0 \\ 0 & 0 & 0 & d_2 \\ d_3 & 0 & 0 & 0 \\ 0 & d_4 & 0 & 0 \end{pmatrix} \quad , \qquad \mathscr{M}^{even}_{kin} = \begin{pmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}, \tag{2.18}$$

where the d_i, k_i are 2×2 matrices. The space generated by these matrices is the subalgebra $\mathscr{J} \in \mathfrak{psu}(2,2|4)$ that leaves the Hamiltonian invariant, i.e., the charge $Q_{\mathscr{M}}$ produced by a matrix $\mathscr{M} \in \mathscr{J}$ will commute with the Hamiltonian.

Note that we can write both the Hamiltonian and the constant P_+ as some of these conserved charges:

$$H = -\frac{i}{2}\mathrm{Str}\left(Q\Sigma_+\right) \ , \quad P_+ = \frac{i}{2}\mathrm{Str}\left(Q\Sigma_-\right). \tag{2.19}$$

From the commutation relations of the subalgebra $\mathscr{J}$, one can determine the Poisson bracket of the corresponding Noether charges $Q_{\mathscr{M}}$:

$$\{Q_{\mathscr{M}}, Q_{\mathscr{N}}\} = (-1)^{\pi(\mathscr{M})\pi(\mathscr{N})}\,\mathrm{Str}\left(Q\left[\mathscr{M},\mathscr{N}\right]\right) + C\left(\mathscr{M},\mathscr{N}\right). \tag{2.20}$$

In the above expression $\mathscr{M}, \mathscr{N} \in \mathscr{J}$, π is the parity of the supermatrices, and $[\mathscr{M},\mathscr{N}]$ is the graded commutator (the anti-commutator if both matrices are odd, and the commutator if one or both of them is even). The first term of (2.20) states that the Poisson bracket of two charges $Q_{\mathscr{M}}$ and $Q_{\mathscr{N}}$ gives a charge corresponding to the commutator $[\mathscr{M},\mathscr{N}]$. The element $C\left(\mathscr{M},\mathscr{N}\right)$ Poisson-commutes with all Noether charges $Q_{\mathscr{M}}$, $\mathscr{M} \in \mathscr{J}$, and will correspond to the central extension.

[9] That is because

$$Q_{\mathscr{M}} = \int \frac{d\sigma}{2\pi}\mathrm{Str}\left(\Lambda U \Lambda^{-1}\mathscr{M}\right) = \int \frac{d\sigma}{2\pi}\mathrm{Str}\left(U\Lambda^{-1}\mathscr{M}\Lambda\right),$$

and if $\Lambda^{-1}\mathscr{M}\Lambda$ does not depend on x_+ then the charge will not depend on it. One should keep in mind that the dependence on $x_\pm$ is only in Λ^{-1} and Λ.

The Central Extension

The relation of this central extension with the level-matching constraint will be discussed at a later time. We will now focus on the properties of the algebra $\mathscr{J}$, in order to get an expression for $C(\mathscr{M},\mathscr{N})$.

We have seen before that the bosonic sector of $\mathfrak{psu}(2,2|4)$ is given by $\mathfrak{su}(2,2)\oplus\mathfrak{su}(4)$. So the bosonic algebra generated by the matrices given in (2.18) will be generated only by $\mathscr{M}_{kin}^{even}$, and will correspond to

$$\mathscr{J}_{even} = \mathfrak{su}(2)\oplus\mathfrak{su}(2)\oplus\mathfrak{su}(2)\oplus\mathfrak{su}(2)\oplus\mathbb{R}^2 \tag{2.21}$$

. The $\mathfrak{su}(2)^4$ comes directly from separating the $\mathfrak{su}(2,2)$ into two $\mathfrak{su}(2)$ and the same to $\mathfrak{su}(4)$. The $\mathbb{R}^2$ corresponds to the fact that we can have a nonzero trace on each of the $\mathfrak{su}(2)$, but require an overall traceless $\mathfrak{su}(2,2)$ and $\mathfrak{su}(4)$. This is accomplished by the use of the generators $\Sigma_\pm$.

Including the fermionic sector we have the decomposition:

$$\mathscr{J} = \mathfrak{psu}(2|2)\oplus\mathfrak{psu}(2|2)\oplus\Sigma_+\oplus\Sigma_-. \tag{2.22}$$

In the limit P_+ infinite, this means that we have one less conserved quantity (the P_+), which implies that the corresponding generator Σ_- won't belong to the invariant subalgebra, and we will have $\mathscr{J} = \mathfrak{psu}(2|2)\oplus\mathfrak{psu}(2|2)\ltimes\mathbb{R}$. This $\mathbb{R}$ direction corresponds to the central element Σ_+ and the corresponding charge, the light-cone Hamiltonian.[10] The central charge $C(\mathscr{M},\mathscr{N})$ will be part of a centrally extended algebra where we include two extra central directions to $\mathscr{J}$, $\mathscr{J}_{ext} = \mathfrak{psu}(2|2)\oplus\mathfrak{psu}(2|2)\ltimes\mathbb{R}^3$.

Finally we can separate the generators of each of the two $\mathfrak{psu}(2|2)$ subalgebras in (2.22). They correspond to keeping k_1, k_3, d_1, d_3 ($\mathfrak{psu}(2|2)_L$) or keeping k_2, k_4, d_2, d_4 ($\mathfrak{psu}(2|2)_R$) from (2.18).

Our goal is to find the expression for the central charge. To do so, some properties have

[10]Note that this subalgebra $\mathscr{J}$ can also be written as $\mathscr{J} = \mathfrak{su}(2|2)\oplus\mathfrak{su}(2|2)$, where the Hamiltonian is a central element to both subalgebras.

to be taken into consideration. An important property of $C(\mathscr{M},\mathscr{N})$ is that it vanishes if $\mathscr{M}$ or $\mathscr{N}$ are bosonic (even). Also it is a bilinear (graded), anti-symmetric form on the algebra $\mathscr{J}$ that Poisson commutes with all other charges $Q_{\mathscr{M}}, \mathscr{M} \in \mathscr{J}$. Together with the Jacobi identity for the Lie brackets, we conclude it is a 2-cocycle.[11] One final property is the invariance under the adjoint action of the group G_{even} corresponding to the algebra J_{even}. The charge $Q_{\mathscr{M}}$ is invariant under the transformation (note that the action of the group preserves the $\mathbb{Z}_2$-grading of $\mathscr{J}$)

$$Q \to gQg^{-1}\,,\; \mathscr{M} \to g\mathscr{M}g^{-1},$$

where $g \in G_{even}$. So the Poisson brackets of two charges also remains unchanged, and consequently

$$C\left(g\mathscr{M}_1 g^{-1}, g\mathscr{M}_2 g^{-1}\right) = C\left(\mathscr{M}_1, \mathscr{M}_2\right).$$

The most general expression for this central element, with such properties, can be found in [66] to be

$$C\left(\mathscr{M}_1, \mathscr{M}_2\right) = \mathrm{Str}\left(\left(\rho\mathscr{M}_1\rho\mathscr{M}_2^t + (-1)^{\pi(\mathscr{M}_1)\pi(\mathscr{M}_2)}\rho\mathscr{M}_2\rho\mathscr{M}_1^t\right)\Delta\right),$$

where

$$\Delta = -\frac{1}{2}\begin{pmatrix} c_3\mathbb{I}_{2\times 2} & 0 & 0 & 0 \\ 0 & c_1\mathbb{I}_{2\times 2} & 0 & 0 \\ 0 & 0 & c_4\mathbb{I}_{2\times 2} & 0 \\ 0 & 0 & 0 & c_2\mathbb{I}_{2\times 2} \end{pmatrix}, \quad \rho = \begin{pmatrix} \sigma & 0 & 0 & 0 \\ 0 & \sigma & 0 & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & \sigma \end{pmatrix},$$

with $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The coefficients $c_i, i = 1,\ldots,4$ can depend on the physical fields and they commute with any charge $Q_{\mathscr{M}}, \mathscr{M} \in \mathscr{J}$. Remember that the algebra $\mathscr{J}$ has two identical subalgebras $\mathfrak{psu}(2|2)$, with the generators given above. Then looking at $\Delta \in \mathscr{J}$,

[11] Simple Lie algebras possess no nontrivial central extension. This is not the case of super Lie algebras. In fact, for finite super Lie algebras, the cocycle vanishes if one of the elements is bosonic. See [83], page 101 ff.

we see that the matrix blocks proportional to c_3, c_4 correspond to one of these algebras, while the matrix blocks with c_1, c_2 correspond to the other. As they are identical, we can set $c_1 = c_3 \equiv c$ and $c_2 = c_4$. Also because we are dealing with the algebra $\mathfrak{psu}(2|2)$, the following conjugation property holds:

$$c_1 + c_2^* = 0 \quad \Rightarrow \qquad c_2 = -c^*.$$

Commutation Relations

The easiest way to determine the commutation relations of the Noether charges is to choose an explicit basis for the space of the charges $Q_{\mathscr{M}}, \mathscr{M} \in \mathscr{J}$. As was mentioned $\mathscr{J}$ has two identical $\mathfrak{psu}(2|2)$ subalgebras (Left and Right), whose extended (off-shell) algebra shares the same central elements corresponding to the worldsheet light-cone Hamiltonian (already in the original algebra), plus two other central charges. We will be focusing on just the $\mathfrak{psu}(2|2)_R$, and a possible basis can be found in section 3 of [66].

So for the centrally extended $\mathfrak{su}(2|2)$ algebra, we have the bosonic (kinematical) generators $\mathbf{R}^a_b, \mathbf{L}^\alpha_\beta$, corresponding to the rotation generators of the bosonic subalgebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, the fermionic (dynamical) supersymmetry generators $\mathbf{Q}^\alpha_a$, $\mathbf{Q}^{\dagger b}_\beta$ and 3 central charges $\mathbf{H}$, $\mathbf{C}$, $\mathbf{C}^\dagger$.[12] Their commutation relations can be found by writing the bracket (2.20) in this basis, and are given by (2.1).

The next step is determining the value of the central charges.

The Level-Matching Condition & the Central Charges

The central charges $\mathbf{C}, \mathbf{C}^\dagger$ have zero eigenvalues on physical states, and so the algebra on physical states is effectively $\mathfrak{su}(2|2)$. But how is this related with the level-matching condition?

The level matching constraint comes from requiring that the unphysical field x_- is periodic in σ, or equivalently having the worldsheet momentum vanish, $p_{ws} = 0$. But this cannot be solved in the classical level, in fact it is the worldsheet momentum operator $\mathbf{P}_{ws}$ that van-

[12] Note that $(\mathbf{Q}^\alpha_a)^\dagger = \mathbf{Q}^{\dagger a}_\alpha$ and the same relation holds to the central elements $\mathbf{C}$ and $\mathbf{C}^\dagger$.

ishes on physical states. So we will treat the variable p_{ws} as a non-trivial (non-vanishing) variable, in the off-shell theory. Note that such an operator commutes with the Hamiltonian, as p_{ws} depends on the non-physical field x_- and the Hamiltonian H depends only on physical fields. In the same way, the central charges $\mathbf{C}, \mathbf{C}^\dagger$ exist as quantum operators which vanish when acting on physical states.

In order to determine the central charges, we will need to simplify the problem by using a perturbative expansion. For the time being we will consider the light-cone momentum P_+ to be finite, and set λ to be large. This allows us to have a perturbative expansion in powers of $\zeta = 2\pi\lambda^{-1/2}$.[13] But the central charges are expected to have a non-trivial expansion on ζ, so the results obtained are not exact, and to bypass this problem another expansion is used, in which we only determine the part of the central charges that are independent of the fermionic fields.

Starting with a dynamical generator, one can conclude from (2.16) and (2.17) that it will have the following structure

$$Q_{\mathscr{M}} = \int_{-r}^{r} d\sigma e^{i\alpha x_-} \Omega\left(x, p, \chi; \zeta\right). \tag{2.23}$$

In the expression above Ω is a local function of the transverse bosonic fields and of fermionic fields, and α can take two values $\pm\frac{1}{2}$ (one for Q and the other for its conjugate $Q^\dagger \equiv \bar{Q}$). See the appendix of [66] for explicit formulas. We then expand the function Ω in powers of ζ:

$$\Omega\left(x, p, \chi; \zeta\right) = \Omega_2\left(x, p, \chi\right) + \zeta\Omega_4\left(x, p, \chi\right) + \cdots .$$

As with the Hamiltonian, the subscript in the above expansion denotes the number of fields in the product. The field x_- present in the expression for the generator won't be expanded.

The form of the central charges is completely fixed by their bosonic part, which is the one that has a dependence on the level matching constraint. So to determine the bosonic

[13] Such an expansion is equivalent to the expansion done to the Hamiltonian in (2.14), because in the former case we had considered both P_+ and λ going to infinity, but with $\tilde{\lambda} = \frac{4\lambda}{P_+^2}$ finite. This means that we can change the expansion from powers of P_+^{-1} to powers of $\lambda^{-1/2}$. Also with a redefinition of $\sigma \to \sigma\frac{P_+}{2}$ we get a Hamiltonian that only depends on the variable P_+ on the limits of the integral in σ, $r = \frac{\pi P_+}{2}$.

charges, we only need to keep the terms linear in fermions in $Q_{\mathscr{M}}$ (and keep the bosonic terms up to the desired order in ζ). We then determine the anti-commutators (or Poisson brackets) and keep only the terms independent of fermions. We write the charge as

$$Q_{\mathscr{M}} = \int d\sigma e^{i\alpha x_-} \chi \left(B_1(x,p) + \zeta B_3(x,p) + \cdots\right) + \mathscr{O}\left(\chi^3\right), \tag{2.24}$$

where we only kept the term linear in fermion fields, and kept all the bosonic terms of the expansion ($B_n(x,p)$ is the term with a product of n bosonic fields).

The Poisson brackets of two charges with $\alpha_1 = \alpha_2$ can now be determined. For example

$$\left\{Q_a^\alpha, Q_b^\beta\right\} \sim \varepsilon^{\alpha\beta}\varepsilon_{ab}\int_{-r}^{r} d\sigma e^{-ix_-}\left(x_-' + \frac{d}{d\sigma} f(x,p)\right),$$

where $f(x,p)$ is a local function of the transverse fields (see appendix of [66] for details). The result for $\left\{\bar{Q}_\alpha^a, \bar{Q}_\beta^b\right\}$ can be obtained by conjugation. Integrating this expression, we get

$$\left\{Q_a^\alpha, Q_b^\beta\right\} \sim \varepsilon^{\alpha\beta}\varepsilon_{ab}\int_{-r}^{r} d\sigma \frac{d}{d\sigma} e^{-ix_-} = \varepsilon^{\alpha\beta}\varepsilon_{ab} e^{-ix_-(-r)}\left(e^{-i[x_-(r)-x_-(-r)]} - 1\right).$$

We know that $p_{ws} = x_-(r) - x_-(-r)$. We also impose the boundary condition $x_-(-r) = x_-^0$, which is the zero mode of x_-, conjugate to P_+. Then:

$$\begin{aligned}
\left\{Q_a^\alpha, Q_b^\beta\right\} &\sim \frac{1}{\zeta}\varepsilon^{\alpha\beta}\varepsilon_{ab} e^{-ix_-^0}\left(e^{-ip_{ws}} - 1\right), \\
\left\{\bar{Q}_\alpha^a, \bar{Q}_\beta^b\right\} &\sim \frac{1}{\zeta}\varepsilon^{ab}\varepsilon_{\alpha\beta} e^{ix_-^0}\left(e^{ip_{ws}} - 1\right),
\end{aligned}$$

and consequently, the central charges are c, c^* with:

$$c = \frac{1}{\zeta} e^{-ix_-^0}\left(e^{-ip_{ws}} - 1\right). \tag{2.25}$$

Some Comments

In the case of P_+ infinite, the zero mode x_-^0 vanishes, but the same is not true for finite light-cone momentum. This brings some problems, as for P_+ finite, the transverse fields don't have to vanish at the string points, and the symmetry algebra is thus changed. It can be seen that the Poisson bracket of the Hamiltonian H with a dynamical charge is then non-zero, which means that the extension to P_+ finite (which is effectively the length of the string) does not allow us to keep the $\mathfrak{psu}(2,2|4)$ symmetry algebra.

At the quantum level, both p_{ws} and x_-^0 are promoted to operators $\mathbf{P},\mathbf{X}_-^0$, and the central charges are

$$\mathbf{C} = \frac{1}{\zeta} e^{-i\mathbf{X}_-^0}\left(e^{-i\mathbf{P}}-1\right), \tag{2.26}$$

and its conjugate $\mathbf{C}^\dagger$. $\mathbf{X}_-^0$ is the conjugate quantum operator of $\mathbf{P}_+$. If we consider a state $\mathbf{P}_+|p_+\rangle = p_+|p_+\rangle$, then a state $e^{i\alpha\mathbf{X}_-^0}|p_+\rangle$ obeys

$$\mathbf{P}_+ e^{i\alpha\mathbf{X}_-^0}|p_+\rangle = (\alpha+p_+)\, e^{i\alpha\mathbf{X}_-^0}|p_+\rangle\,. \tag{2.27}$$

Because P_+ acts as the length of the string, the operator $e^{i\alpha\mathbf{X}_-^0}$ will be the length changing operator. The Hilbert space of the theory will be a direct sum, $\mathscr{H} = \bigoplus_{p_+}\mathscr{H}_{p_+}$, of spaces of each of the eigenvalues of $\mathbf{P}_+$.

$\mathfrak{su}(2|2)$ *subsector and mode expansion*

The explicit form of the charges $Q_{\mathscr{M}}$ was determined in [66] . The algebra $\mathscr{J}$ includes two $\mathfrak{psu}(2|2)$ subalgebras. We will be focusing on the $\mathfrak{psu}(2|2)_R$.

The leading quadratic order of (2.24) can be read from the results in [66]. The fermionic charges are, at leading order:

$$\begin{aligned}
Q_a^\alpha &= -\frac{1}{2}\int d\sigma e^{-\frac{i}{2}x_-}\Big[i\theta^\alpha\left(2P^Y+iY\right)_a + \left(2P^Z-iZ\right)^\alpha\eta_a^\dagger - \theta^{\dagger\alpha}Y_a' - iZ'^\alpha\eta_a + \\
&\qquad + \varepsilon^{\alpha\beta}\varepsilon_{ab}\left(i\theta_\beta\left(2P^Y+iY\right)^b + \left(2P^Z-iZ\right)_\beta\eta^{\dagger b} - \theta_\beta^\dagger Y'^b - iZ_\beta'\eta^b\right)\Big], \qquad (2.28)\\
\bar{Q}_\alpha^a &= \frac{1}{2}\int d\sigma e^{\frac{i}{2}x_-}\Big[i\theta_\alpha^\dagger\left(2P^Y-iY\right)^a - \left(2P^Z+iZ\right)_\alpha\eta^a + \theta_\alpha Y'^a - iZ_\alpha'\eta^{\dagger a} + \\
&\qquad + \varepsilon_{\alpha\beta}\varepsilon^{ab}\left(i\theta^{\dagger\beta}\left(2P^Y-iY\right)_b - \left(2P^Z+iZ\right)^\beta\eta_b + \theta^\beta Y_b' - iZ'^\beta\eta_b^\dagger\right)\Big] \qquad (2.29)\\
&= \left(Q_a^\alpha\right)^\dagger.
\end{aligned}$$

We want to restrict ourselves to the $\mathfrak{su}(2|2)$ subsector of [55]. This corresponds to keeping only the 2 complex co-ordinates Y^a and the respective conjugate momenta P^y. These will be seen to correspond, in the SYM side, to some bosonic excitations ϕ^a, with $a=1,2$. In terms of the fermions we will be interested in only keeping $\theta^\alpha, \theta^\dagger_\alpha$, which will correspond to two fermionic fields $\psi_\alpha, \psi^{\dagger\alpha}$ from SYM. On the Super Yang-Mills side, the vacuum of this sector of the theory corresponds to a long string of Z bosonic fields, in direct correspondence to the vacuum of the string defined by [5]

$$\frac{1}{\sqrt{J}N^{J/2}}\mathrm{Tr}\left(Z^J\right) \leftrightarrow \left|0,p^+\right\rangle.$$

With these restrictions, the fermionic supercharges (2.28) and (2.29) become:

$$\begin{aligned}
S^\alpha_a &= -\frac{1}{2}\int d\sigma e^{-\frac{i}{2}x_-}\left(i\theta^\alpha\left(2P^Y+iY\right)_a - \varepsilon^{\alpha\beta}\varepsilon_{ab}\theta^\dagger_\beta Y'^b\right), \\
Q^a_\alpha &= \frac{1}{2}\int d\sigma e^{\frac{i}{2}x_-}\left(i\theta^\dagger_\alpha\left(2P^Y-iY\right)^a + \varepsilon_{\alpha\beta}\varepsilon^{ab}\theta^\beta Y'_b\right).
\end{aligned}$$

Before continuing, let us notice that the co-ordinate $x_-(\sigma)$ obeys:

$$x_-(\sigma) = \int_{-r}^{\sigma} d\sigma' x_-^{'}\left(\sigma'\right) + x_-(-r) = \int_{-r}^{\sigma} d\sigma' \pi_{ws}\left(\sigma'\right) + x_-^0,$$

where $x'_- = \pi_{ws}(\sigma)$ is the worldsheet momentum density. The total worldsheet momentum is given by $p_{ws} = \int_{-r}^{r} d\sigma \pi_{ws}(\sigma)$.

We now want to perform a mode expansion. To do so we will follow the notation of [64]. For the bosonic fields we have:

$$\begin{aligned}
Y_a &= \frac{1}{\sqrt{\omega}}\left(A_a+B^\dagger_a\right) \quad ; \quad P^a = \frac{\sqrt{\omega}}{4i}\left(A^{\dagger a}-B^a\right); \\
Y^a = \overline{Y}_a &= \frac{1}{\sqrt{\omega}}\left(A^{\dagger a}+B^a\right) \quad ; \quad P_a = \overline{P}^a = i\frac{\sqrt{\omega}}{4}\left(A_a-B^\dagger_a\right),
\end{aligned} \tag{2.30}$$

where $\omega = \sqrt{1+\frac{1}{2}\tilde{\lambda}\partial^2_\sigma}$, and $\tilde{\lambda}$ is the effective coupling constant (light-cone gauge) in the pp-wave limit $\tilde{\lambda} \equiv \frac{4\lambda}{P_+^2}$, kept finite when $P_+, \lambda \to \infty$. For the fermionic fields we have:

$$\theta^\alpha = \sqrt{\frac{1}{2}\left(1+\frac{1}{\omega}\right)}c^\alpha \quad ; \qquad \theta^\dagger_\alpha = \sqrt{\frac{1}{2}\left(1+\frac{1}{\omega}\right)}c^\dagger_\alpha. \tag{2.31}$$

With these expansions, we get the following results:

$$\begin{aligned} i\theta^{\alpha}\left(2P^{Y}+iY\right)_{a} &= -\sqrt{\frac{\omega+1}{2\omega}}c^{\alpha}\left\{\frac{\sqrt{\omega}}{2}\left(A_{a}-B_{a}^{\dagger}\right)+\frac{1}{\sqrt{\omega}}\left(A_{a}+B_{a}^{\dagger}\right)\right\}, \\ \theta_{\beta}^{\dagger}Y'^{b} &= \sqrt{\frac{\omega+1}{2\omega}}c_{\alpha}^{\dagger}\frac{\sqrt{\tilde{\lambda}}\partial_{\sigma}}{\sqrt{2\omega}}\left(A^{\dagger b}+B^{b}\right). \end{aligned}$$

We will be keeping $Y \approx B^{\dagger}$, dropping the oscillators $A, A^{\dagger}$. Then up to order $\mathscr{O}\left(\sqrt{\tilde{\lambda}}\right)$,[14]

$$Q_{\alpha}^{a}=\frac{1}{4}\int d\sigma e^{\frac{i}{2}x_{-}}\left(c_{\alpha}^{\dagger}B^{a}+\sqrt{2}\varepsilon_{\alpha\beta}\varepsilon^{ab}c^{\beta}\sqrt{\tilde{\lambda}}\partial_{\sigma}B_{b}^{\dagger}\right). \tag{2.32}$$

The same can be done for the supercharge S, which then becomes:

$$S_{a}^{\alpha}=\frac{1}{4}\int d\sigma e^{-\frac{i}{2}x_{-}}\left(c^{\alpha}B_{a}^{\dagger}+\sqrt{2}\varepsilon^{\alpha\beta}\varepsilon_{ab}c_{\beta}^{\dagger}\sqrt{\tilde{\lambda}}\partial_{\sigma}B^{b}\right). \tag{2.33}$$

For a comparison with the Super Yang-Mills supercharges, we need to discretize the above results. To do so recall that $r = P_{+}/2$, and $\int_{-r}^{r} d\sigma = P_{+}$. Then the lattice version of Q is:

$$\begin{aligned} Q_{\alpha}^{a} &= \frac{1}{4}\sum_{\ell=1}^{P_{+}} e^{ix_{-}^{0}/2}\left(\prod_{k=0}^{\ell} e^{\frac{i}{2}\pi(k)}\right)\left\{c_{\alpha}^{\dagger}(\ell)B^{a}(\ell)+\sqrt{2}\varepsilon_{\alpha\beta}\varepsilon^{ab}c^{\beta}(\ell)\sqrt{\tilde{\lambda}}\left(B_{b}^{\dagger}(\ell)-B_{b}^{\dagger}(\ell-1)\right)\right\} \\ &= \frac{1}{4}\sum_{\ell=1}^{P_{+}} e^{\frac{i}{2}x_{-}^{0}}e^{\frac{i}{2}p(\ell)}\left\{c_{\alpha}^{\dagger}(\ell)B^{a}(\ell)+\sqrt{2}\varepsilon_{\alpha\beta}\varepsilon^{ab}\sqrt{\tilde{\lambda}}\left(B_{b}^{\dagger}(\ell)-B_{b}^{\dagger}(\ell-1)\right)c^{\beta}(\ell)\right\}, \end{aligned} \tag{2.34}$$

where $p(\ell) = \sum_{k=1}^{\ell}\pi(k)$.

To continue, we need to write what $p(\ell)$ does to an excitation:

$$e^{\frac{i}{2}p(\ell)}\chi(\ell_{k})e^{-\frac{i}{2}p(\ell)} = \begin{cases} \chi(\ell_{k}) & \ell_{k} < \ell \\ \chi(\ell_{k}+1) & \ell_{k} > \ell \end{cases}.$$

[14]Considering $\omega = \sqrt{1+x}$, with $x = \tilde{\lambda}\partial^{2}$, then we have the following:

$$\begin{aligned} \sqrt{\frac{\omega+1}{2\omega}} &= 1-\frac{x}{8}+\mathscr{O}\left(x^{2}\right), \\ \frac{1}{\sqrt{\omega}} &= 1-\frac{x}{4}+\mathscr{O}\left(x^{2}\right), \\ \frac{2-\omega}{2\sqrt{\omega}} &= \frac{1}{2}-\frac{3x}{8}+\mathscr{O}\left(x^{2}\right). \end{aligned}$$

By performing the following change of variables, $c^{\dagger}_{\alpha}(\ell) \to e^{-\frac{i}{2}x^0_-} e^{-\frac{i}{2}p(\ell)} c^{\dagger}_{\alpha}(\ell)$, the charge becomes:

$$\begin{aligned} Q^a_\alpha &= \frac{1}{4}\sum_{\ell=1}^{P_+}\left\{c^{\dagger}_{\alpha}(\ell) B^a(\ell) + \sqrt{2}\varepsilon_{\alpha\beta}\varepsilon^{ab}\sqrt{\tilde{\lambda}} e^{\frac{i}{2}x^0_-} e^{\frac{i}{2}p(\ell)} c^{\beta}(\ell) e^{\frac{i}{2}p(\ell)} e^{\frac{i}{2}x^0_-}\left(B^{\dagger}_b(\ell) - B^{\dagger}_b(\ell-1)\right)\right\} \\ &= \frac{1}{4}\sum_{\ell=1}^{P_+}\left\{c^{\dagger}_{\alpha}(\ell) B^a(\ell) + \sqrt{2}\varepsilon_{\alpha\beta}\varepsilon^{ab}\sqrt{\tilde{\lambda}} e^{ix^0_-}\left(B^{\dagger}_b(\ell) - B^{\dagger}_b(\ell-1)\right) e^{ip(\ell)} c^{\beta}(\ell)\right\}. \end{aligned} \tag{2.35}$$

The other supercharge S^{α}_a can also be determined to be:

$$\begin{aligned} S^{\alpha}_a &= \frac{1}{4}\sum_{\ell=1}^{P_+} e^{-\frac{i}{2}x^0_-} e^{-\frac{i}{2}p(\ell)}\left(c^{\alpha}(\ell) e^{\frac{i}{2}p(\ell)} e^{\frac{i}{2}x^0_-} B^{\dagger}_a(\ell)\right. \\ &\qquad \left. + \sqrt{2}\varepsilon^{\alpha\beta}\varepsilon_{ab} e^{-\frac{i}{2}x^0_-} e^{-\frac{i}{2}p(\ell)} c^{\dagger}_{\beta}(\ell)\sqrt{\tilde{\lambda}}\left(B^b(\ell) - B^b(\ell-1)\right)\right) \\ &= \frac{1}{4}\sum_{\ell=1}^{P_+}\left(B^{\dagger}_a(\ell) c^{\alpha}(\ell) + \sqrt{2}\varepsilon^{\alpha\beta}\varepsilon_{ab}\sqrt{\tilde{\lambda}} e^{-ix^0_-} e^{-ip(\ell)} c^{\dagger}_{\beta}(\ell)\left(B^b(\ell) - B^b(\ell-1)\right)\right). \end{aligned} \tag{2.36}$$

Recall that in the above expressions x^0_- plays the part of the length changing operator, as it is the conjugate variable to P_+, the total light-cone momentum, which is in its turn related to the width of the worldsheet cylinder. For closed strings the level matching condition states that the total worldsheet momentum p_{ws} has to vanish (on-shell). As was mentioned before, if we relax this condition (off-shell) and take $P_+ \to \infty$, then we obtain the centrally extended algebra with extra central charges C, C^* added to the Hamiltonian H (the same as the generators of translations P and boosts K).

One other way of comparing results with the SYM side is by writing the supercharges in first quantized framework. Choosing again a state such that:

$$|\chi_1 \cdots \chi_K; P_+\rangle = \sum_{\{m_i\}=0}^{P^+} e^{ip_1 m_1 + \cdots + ip_K m_K} \chi_1(m_1) \cdots \chi_K(m_K) |0; P_+\rangle,$$

where $\chi_i(m_i) = b_z^{m_i} \chi_i b_z^{-m_i}$, with b_z being the oscillators equivalent to the field Z. Then

$$\begin{aligned} Q^a_\alpha |\chi_1 \cdots \chi_K; P_+\rangle &= \frac{1}{4}\sum_{k=1}^{K}\left(\prod_{m=1}^{k-1}(-1)^{F(m)}\right)\left\{\delta\left(\chi_k, B^{\dagger}_b\right)\delta^a_b |\chi_1 \cdots c^{\dagger}_{\alpha}(k) \cdots \chi_K; P_+\rangle + \right. \\ &\quad \left. + \sqrt{2\tilde{\lambda}}\,\delta\left(\chi_k, c^{\dagger}_{\beta}\right)\varepsilon^{ab}\varepsilon_{\alpha\beta}\left(\prod_{l=k+1}^{K} e^{ip_l} - \prod_{l=k}^{K} e^{ip_l}\right)\left|\chi_1 \cdots B^{\dagger}_b(k) \cdots \chi_K; P_+ + 1\right\rangle\right\}. \end{aligned} \tag{2.37}$$

Doing the same calculation for the S generator, one gets:

$$S_a^\alpha |\chi_1 \cdots \chi_K; P_+\rangle = \frac{1}{4}\sum_{k=1}^{K}\left(\prod_{m=1}^{k-1}(-1)^{F(m)}\right)\left\{\delta\left(\chi_k, c_\beta^\dagger\right)\delta_\beta^\alpha \chi_1(m_1)\cdots\left(B_a^\dagger(k)\right)\cdots\chi_K(m_K)|0;P_+\rangle + \right.$$
$$\left. + \sqrt{2\tilde{\lambda}}\delta\left(\chi_k, B_b^\dagger\right)\varepsilon^{\alpha\beta}\varepsilon_{ab}\left(\prod_{l=k+1}^{K} e^{-ip_l} - \prod_{l=k}^{K} e^{-ip_l}\right)\left|\chi_1 \cdots c_\beta^\dagger(k)\cdots\chi_K; P_+ - 1\right\rangle\right\}. \quad (2.38)$$

The structure of these supercharges will be compared to the results obtained on the SYM side of the correspondence, present in Section 2.5, in particular expressions (2.49) and (2.50). We will be able to see that the actions of the supercharges Q and S, have a similar structure at one-loop, on both sides of the correspondence. But while the results presented in this section are perturbative in $\tilde{\lambda}$ (BMN limit), the results presented in Section 2.5 are perturbative in the 't Hooft coupling λ, so one cannot perform a direct comparison.

In Section A.1 one can find a summary of the results of [65] on an oscillator formalism for the superalgebra $\mathfrak{psu}(2|2)$.

2.4 $\mathcal{N}=4$ SYM *on* $\mathbb{R}\times S^3$*: A review*

In this first section, we summarize the method of finding the supercharges of $\mathfrak{su}(2,2|4)$ up to 1-loop, as can be found in [75, 73].[15]

The action for $\mathcal{N}=4$ SYM in four dimensions can be obtained from dimensional reduction of the $\mathcal{N}=1$ 10 dimensional SYM on a 6−torus. Using the notation where the $D=10$ Dirac matrices split into $\mathrm{SO}(1,3)\times\mathrm{SO}(6)$, the action becomes:

$$S = \frac{2}{g_{YM}^2}\int d^4x\sqrt{|g|}\mathrm{Tr}\left\{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}D^\mu\phi_i D_\mu\phi_i - \frac{\mathscr{R}}{12}\phi_i^2 + \frac{1}{4}[\phi_i,\phi_j]^2 - 2i\lambda_A^\dagger\sigma^\mu D_\mu\lambda^A + \right.$$
$$\left. + (\rho_i)^{AB}\lambda_A^\dagger i\sigma^2[\phi_i,\lambda_B^*] - \left(\rho_i^\dagger\right)_{AB}\left(\lambda^A\right)^T i\sigma^2\left[\phi^i,\lambda^B\right]\right\}.$$

We have a vector field A_μ, six real scalars ϕ_i and four Weyl spinors $\lambda_{\alpha A}$ (all in the adjoint representation of the gauge group). The six scalars transform in a **6** of the $R-$symmetry group $SO(6)\equiv SU(4)_R$, while the spinors transform in a **4**. Co-ordinate indices are $x^\mu=(t,x^a)$,

[15] We will be following the notation of [73], in which a different basis for the $\gamma-$matrices is used. The same procedure could be done by following [75] choice of basis.

$\mu = 0,...,3$, with the spacial co-ordinates having (curved) indices $a = 1,2,3$. The metric is given by

$$ds^2 = -dt^2 + R^2\left(d\theta^2 + \sin^2\theta d\psi^2 + \sin^2\theta\sin^2\psi d\chi^2\right),$$

where R is the radius of S^3, and $\mathscr{R} = \frac{6}{R^2}$ is the Ricci scalar.

Some Notation

From this point on we will be considering $\boldsymbol{\sigma}^\mu \equiv (\mathbf{1}, \boldsymbol{\sigma}^a)$ and $\overline{\boldsymbol{\sigma}}^\mu = (-\mathbf{1}, \boldsymbol{\sigma}^a)$, where the $\boldsymbol{\sigma}^a$ are the usual Pauli matrices pulled back to S^3. Also, $\rho_i^{AB} \equiv \sigma_i^{AB}$ are the Clebsch-Gordan coefficients of $SU(4)$ that relate two **4** irreducible representations (irreps) with one **6**. These coefficients have several properties, in particular $\rho_i^{AB} = \frac{1}{2}\varepsilon^{ABCD}\left(\rho_i^\dagger\right)_{CD}$, and allow us to write

$$\phi_i = \frac{1}{2}\rho_i^{AB}\Phi_{AB} = \frac{1}{2}\left(\rho_i^\dagger\right)_{AB}\Phi^{AB}.$$

Finally, one comment about the Weyl spinors. We know that in $D = 10$ we start from a 32-component complex spinor, and by imposing a Majorana-Weyl condition, obtain a 16-component (after fixing the κ-symmetry) spinor L. This spinor can be written in terms of Weyl spinors as

$$L = \begin{pmatrix} \lambda^{\alpha A} \\ i\left(\sigma^2\right)^{\alpha\beta}\lambda^*_{\beta A} \end{pmatrix},$$

with $\alpha = 1,2$ and $A = 1,2,3,4$. The $\lambda^{\alpha A}$ are four 2-component Weyl spinors.[16]

SUSY transformations and corresponding charges

The SUSY transformations are given by:

$$\begin{aligned}
\delta_\eta A_\mu &= 2i\left(\lambda_A^\dagger\sigma_\mu\eta^A - \eta_A^\dagger\sigma_\mu\lambda^A\right), \\
\delta_\eta\Phi^{AB} &= 2i\left(-\lambda_E^\dagger i\sigma^2\varepsilon^{ABEF}\eta_F^* - \left(\lambda^A\right)^T i\sigma^2\eta^B - \left(\lambda^B\right)^T i\sigma^2\eta^A\right), \\
\delta_\eta\lambda^A &= \frac{1}{2}F_{\mu\nu}\sigma^{\mu\nu}\eta^A + 2D_\mu\Phi^{AB}\overline{\sigma}^\mu i\sigma^2\eta_B^* + \Phi^{AB}\overline{\sigma}^\mu i\sigma^2\nabla_\mu\eta_B^* - 2i\left[\Phi^{AC},\Phi_{CB}\right]\eta^B.
\end{aligned}$$

[16] In the basis used in [75], the separation of L into $L = (L_+\ L_-)^T$ becomes a separation into $SU(2)_L \times SU(2)_R$, for which one uses dotted/undotted indices $\dot{\alpha},\alpha$. In the basis used in [73] this separation is not obvious.

Our objective is to build the Noether charge $Q\eta$. To do so we need to take into consideration the pairs of canonical variables. From the action, we have the following (anti-)commutation relations:

$$\begin{aligned} \left[F_{0\mu}, A^{\nu}\right] &= \delta_{\mu}^{\nu}, \\ \left[D_0\phi_i, \phi_j\right] &= \delta_{ij} \quad \Rightarrow \left[D_0\Phi_{AB}, \Phi^{CD}\right] = \frac{1}{2}\left(\delta_A^D\delta_B^C - \delta_A^C\delta_B^D\right), \\ \left\{-i\left(\lambda_A^{\dagger}\sigma^0\right)_{\alpha}, \lambda^{B\beta}\right\} &= \delta_{\alpha}^{\beta}\delta_A^B. \end{aligned}$$

Also one has to take into consideration that $\eta^{\alpha A}$ are Killing spinors, which in $\mathbb{R}\times S^3$ obey the equation $\nabla_\mu \eta = \pm \frac{i}{2R}\sigma_\mu \eta$, and so will give us two solutions $\eta_\pm$. We will then obtain two charges $Q \equiv Q_L$ and $\overline{Q} \equiv Q_R$, corresponding to η_+ and η_- respectively.

The fermionic Noether charges are thus[17]

$$Q\eta = \frac{2}{g_{YM}^2}\int_{S^3} d\Omega\, \mathrm{Tr}\left\{-2i\lambda_A^{\dagger}\sigma^0\delta_\eta\lambda^A - 2i\left(\lambda^A\right)^T\sigma^0\delta_\eta\lambda_A^*\right\}.$$

For the purposes of this paper, we will simplify the calculations by setting the vector field to zero (we will be looking only at the sector of scalars and spinors). This truncation is consistent with the one-loop calculation we will be performing.

The non-vector sector of the charges $Q\eta$ is given by:

$$\begin{aligned} Q\eta \;=\; -\frac{2}{g_{YM}^2}\int_{S^3} d\Omega\, \mathrm{Tr}\Big\{ & 2i\lambda_A^{\dagger}\left(2\nabla_a\Phi^{AB}\overline{\sigma}^a i\sigma^2\eta_B^* + \Phi^{AB}\overline{\sigma}^{\mu} i\sigma^2\nabla_\mu\eta_B^* - 2i\left[\Phi^{AC}, \Phi_{CB}\right]\eta^B\right) + \\ & +2i\lambda_A^{\dagger}\left(2\Pi^{AB}\sigma^0 i\sigma^2\eta_B^*\right) + 2i\left(\lambda^A\right)^T\left(-2\Pi_{AB}\left(\overline{\sigma}^0\right)^T i\sigma^2\eta^B\right) + \\ & +2i\left(\lambda^A\right)^T\left(-2\nabla_a\Phi_{AB}\left(\overline{\sigma}^a\right)^T i\sigma^2\eta^B - \Phi_{AB}\left(\overline{\sigma}^{\mu}\right)^T i\sigma^2\nabla_\mu\eta^B - 2i\left[\Phi_{AC}, \Phi^{CB}\right]\eta_B^*\right)\Big\}, \end{aligned} \quad (2.39)$$

where Π_{AB} is the momentum conjugate to the bosonic field Φ^{AB}.

We now have the an expression for the supercharges. The next step is to evaluate it on $\mathbb{R}\times S^3$: we expand the four-dimensional fields in terms of the spherical harmonics of S^3, and then perform the integration of the sphere.

[17]For comparison purposes, one could also write this charge, in the $SU(2)_L \times SU(2)_R$ formalism, as

$$Q_\varepsilon = \frac{2}{g_{YM}^2}\int_{s^3} \mathrm{Tr}\left\{i\overline{\lambda}_A^{\dot{\alpha}}\overline{\sigma}^0_{\dot{\alpha}\alpha}\delta_\varepsilon\lambda^{\alpha A} + i\overline{\lambda}_\alpha^A\left(\sigma^0\right)^{\alpha\dot{\alpha}}\delta_\varepsilon\lambda_{\dot{\alpha}A}\right\}.$$

Harmonic Expansion on S^3 and the Plane-Wave Limit

Each field, defined by its spin, will have a decomposition in spherical harmonics on S^3. These spherical harmonics can be labeled by the irreducible representations (irreps) (m_L, m_R) of the isometry group $SO(4) \equiv SU(2)_L \otimes SU(2)_R$. As such, we have:

- Spin 0: We have scalar spherical harmonics $Y^{kI}_{(0)}$, in the irrep $(k+1, k+1)$. Their mass will be $(k+1)/R$.

- Spin $\frac{1}{2}$: In this case we'll use spinor spherical harmonics: $Y^{kI+}_{(1/2)}$, in the irrep $(k+2, k+1)$; $Y^{kI-}_{(1/2)}$, in the irrep $(k+1, k+2)$. Both have mass $(k+3/2)/R$.

As usual, k labels different irreducible representations, and I enumerates the elements of a particular irrep ($I = 1 \cdots d$), where d is the dimension of the irreducible representation.

The expansions of the fields in the corresponding harmonics are:

$$
\begin{aligned}
\phi_i(x^\mu) &= \sum_{k=0}^{\infty} \sum_{I=1}^{(k+1)^2} \phi_i^{kI}(t)\, Y^{kI}_{(0)}(x^a), \\
\lambda^A_\alpha(x^\mu) &= \sum_{k=0}^{\infty} \sum_{I=1}^{(k+1)(k+2)} \sum_{\pm} \lambda^{A,kI\pm}(t)\, Y^{kI\pm}_{(1/2)\,\alpha}(x^a).
\end{aligned}
$$

Note that spinor spherical harmonics are 2−dimensional commuting Weyl spinors. The Killing spinor η^A (parameter of the superconformal transformations) will have the same expansion as λ^A, with coefficients $\eta^{A,kI\pm}(t)$.

Plane-Wave Limit [74, 73]

We want to truncate the infinite tower of Kaluza-Klein modes to the lowest supermultiplet. One can then climb up the various states (with increasing masses) by acting with the two supercharges $Q_L = \left(\mathbf{2}, \mathbf{1}, \bar{\mathbf{4}}\right)$ and $Q_R = (\mathbf{1}, \mathbf{2}, \mathbf{4})$, where the numbers correspond to representations of $SU(2)_L \otimes SU(2)_R \otimes SU(4)$. Focusing on the zero modes of the Kaluza-Klein tower we find 6 scalar spherical harmonics, constant on S^3, and 4 lowest spinor spherical harmonics $S^{\hat{\alpha}\pm}_\alpha$, in irrep $(2,1) \oplus (1,2)$ of $SU(2)_L \otimes SU(2)_R$ (the hatted index refers to the degeneracy of the solution), solutions to the killing spinor equation for a Weyl spinor.

The fields with only these zero modes become:

$$\begin{aligned}
\phi_i(x^\mu) &= X_i(t), \\
\lambda^A_\alpha(x^\mu) &= \sum_{\hat{\alpha}=1}^{2}\left(\theta^{A+}_{\hat{\alpha}}(t)\, S^{\hat{\alpha}+}_{\alpha}(x^a) + \theta^{A-}_{\hat{\alpha}}(t)\, S^{\hat{\alpha}-}_{\alpha}(x^a)\right).
\end{aligned}$$

If we restrict ourselves to half of the supercharges Q_L, then these together with the bosonic symmetry generators will generate the subalgebra $\mathfrak{su}(2|4)$. The restriction to the Q_L charges leads us to consider only the zero modes that are $SU(2)_R$ singlets. Then we keep all the lowest scalar harmonics, and only two spinor harmonics $S^{\hat{\alpha}+}_{\alpha}$ (instead of the 4 if we included $S^{\hat{\alpha}-}_{\alpha}$). The conjugate momenta π_i will have the same expansion as its conjugate field ϕ_i, that is $\pi_i(x^\mu) = \Pi_i(t)$.

Now we can proceed to the actual integration on the supercharges. Going back to (2.39), we find that:[18]

$$\begin{aligned}
Q_L &= Q\eta^+ = \mathrm{Tr}\left\{\left(\frac{1}{R}X^{AB} + 2i\Pi^{AB}\right)\theta^{+\dagger}_{A} i\sigma^2 \eta^{+*}_{B} - \sqrt{2}\left[X_{AC}, X^{CB}\right]\theta^{+A}_{\hat{\alpha}}\varepsilon^{\hat{\alpha}\hat{\beta}}\eta^{+*}_{B\hat{\beta}}\right. \\
&\qquad \left. + \left(\frac{1}{R}X_{AB} - 2i\Pi_{AB}\right)\left(\theta^{+A}\right)^T i\sigma^2 \eta^{+B} - \sqrt{2}\left[X^{AC}, X_{CB}\right]\theta^{+\dagger}_{A\hat{\alpha}}\varepsilon^{\hat{\alpha}\hat{\beta}}\eta^{+B}_{\hat{\beta}}\right\} \\
&= Q_+\eta + S_+\eta^*.
\end{aligned}$$

The final expression for the supercharges is [19]

$$\begin{aligned}
Q^{\hat{\alpha}}_{A} &= \mathrm{Tr}\left\{-\theta^{B\hat{\alpha}}\left(\frac{1}{R}X_{BA} - 2i\Pi_{BA}\right) - \sqrt{2}\varepsilon^{\hat{\beta}\hat{\alpha}}\theta^{\dagger}_{B\hat{\beta}}\left[X^{BC}, X_{CA}\right]\right\}, \\
S^{A\hat{\alpha}} &= \mathrm{Tr}\left\{\theta^{\dagger}_{B\hat{\beta}}\left(\frac{1}{R}X^{BA} + 2i\Pi^{BA}\right) - \sqrt{2}\left[X_{BC}, X^{CA}\right]\theta^{B}_{\hat{\beta}}\right\}\varepsilon^{\hat{\beta}\hat{\alpha}}.
\end{aligned} \tag{2.40}$$

---

[18] In order to obtain the supercharges integrated over S^3, we used the properties of the spherical harmonics, as well as other properties of the Pauli matrices. These properties can be found in [75, 74, 73], and include $\overline{\sigma}^\mu i\sigma^2 \sigma^T_\mu = \left(\overline{\sigma}^\mu\right)^T i\sigma^2 \sigma_\mu = -2i\sigma^2$. In the same references one can find the expansion of spin 1 vector fields. We also used an identification between the radius of the sphere R and the Yang-Mills coupling constant g_{YM} such that $\frac{4\pi^2 R^3}{g^2_{YM}} \to 1$. This prefactor shows up when obtaining the action of the plane-wave matrix theory action from $\mathcal{N} = 4$ SYM action, and would also appear in the charges.

[19] Note that in our choice of basis the relation $S = Q^\dagger$ is not manifest.

2.5 The $\mathfrak{su}(2|3)$ subsector and its restriction to the $\mathfrak{su}(2|2)$

We'll continue by studying the sector $\mathfrak{su}(2|3)$, as in [67]. For that we reduce our fields as follows:

$$\theta^{\alpha} \equiv \theta^{4\hat{\alpha}},\ \phi^{a} \equiv X^{a4},\ \alpha = 1,2;\ a = 1,2,3.$$

By construction we have $\overline{\phi^a} \equiv \phi_a$, and $\pi_a = \Pi_{4a}$, as well as $X^{BC} = \frac{1}{2}\varepsilon^{BCAD}X_{AD}$. The supercharges restricted to this sector can then be written as:

$$\begin{aligned} Q^{\alpha}_{a} &= \mathrm{Tr}\left\{-\theta^{4\hat{\alpha}}\left(\frac{1}{R}X_{4a} - 2i\Pi_{4a}\right) - \sqrt{2}\theta^{\dagger}_{4\hat{\beta}}\varepsilon^{\hat{\beta}\hat{\alpha}}\left[X^{4C}, X_{Ca}\right]\right\} \\ &= \mathrm{Tr}\left\{\theta^{\alpha}\left(\frac{1}{R}\overline{\phi}_{a} + 2i\pi_{a}\right) - \sqrt{2}\theta^{\dagger}_{\beta}\varepsilon^{\alpha\beta}\varepsilon_{abc}\left[\phi^{c}, \phi^{b}\right]\right\}; \end{aligned} \tag{2.41}$$

$$\begin{aligned} S^{a\alpha} &= \mathrm{Tr}\left\{\theta^{\dagger}_{4\hat{\beta}}\left(\frac{1}{R}X^{4a} + 2i\Pi^{4a}\right) - \sqrt{2}\left[X_{4C}, X^{Ca}\right]\theta^{4}_{\hat{\beta}}\right\}\varepsilon^{\hat{\beta}\hat{\alpha}} \\ &= \mathrm{Tr}\left\{\theta^{\dagger}_{\beta}\left(\frac{1}{R}\phi^{a} - 2i\overline{\pi}^{a}\right) - \sqrt{2}\varepsilon^{abc}\left[\overline{\phi}_{c}, \overline{\phi}_{b}\right]\theta^{\gamma}\varepsilon_{\gamma\beta}\right\}\varepsilon^{\beta\alpha}. \end{aligned} \tag{2.42}$$

In order to continue, we will need to rewrite the fields in terms of creation/annihilation operators. First identify $\frac{1}{R} = \frac{m}{6}$, i.e. exchange the parameter R by a mass parameter m. [73] Then consider the expansion of the six scalars/momenta X_i, Π_i:

$$\left\{\begin{array}{l} a_i = \sqrt{\frac{3}{m}}\left(i\Pi_i + \frac{m}{6}X_i\right), \\ a^{\dagger}_{i} = \sqrt{\frac{3}{m}}\left(-i\Pi_i + \frac{m}{6}X_i\right), \end{array}\right. \qquad \Rightarrow \qquad \left\{\begin{array}{l} X_i = \sqrt{\frac{3}{m}}\left(a_i + a^{\dagger}_{i}\right), \\ \Pi_i = \frac{1}{2i}\sqrt{\frac{m}{3}}\left(a_i - a^{\dagger}_{i}\right). \end{array}\right.$$

The bosons X_{AB} are a combination of two real scalar fields such that $X_{a4} = \frac{1}{2}\left(X_a + iX_{a+3}\right), a = 1,2,3$. If we now define the creation/annihilation operators as $a^a \equiv a^a + ia^{a+3}$ and $b^{a\dagger} = a^{a\dagger} + ia^{a+3\dagger}$, with $a = 1,2,3$, we then have the following expansions for our (complex) fields:

$$\phi^{a} \quad \equiv X^{a4} = \quad \sqrt{\frac{3}{m}}\left(a^{a} + b^{\dagger a}\right); \quad \pi_{a} \equiv \Pi_{4a} == \frac{1}{4i}\sqrt{\frac{m}{3}}\left(a^{\dagger}_{a} - b_{a}\right), \tag{2.43}$$

with equivalent expressions for fields $\overline{\phi}_a$ and $\overline{\pi}^a$. Introducing also fermionic creation operators, the fermions become

$$\theta^{\dagger\alpha} = c^{\alpha} = \varepsilon^{\alpha\beta} c_{\beta}\,; \quad \theta^{\alpha} = c^{\dagger\alpha}. \tag{2.44}$$

We will be interested in action of the charges on the subspace of states that will only have excitations of $c^{\dagger}$ and $b^{\dagger}$, so we will drop the oscillators $a, a^{\dagger}$ in the bosonic fields. We find:

$$\begin{aligned} Q_{a}^{\alpha} &= \mathrm{Tr}\left\{ \sqrt{\frac{m}{3}}\, c^{\dagger\alpha} b_{a} - \frac{3\sqrt{2}}{m} \varepsilon^{\alpha\beta} \varepsilon_{abc} \left[b^{\dagger c}, b^{\dagger b} \right] c_{\beta} \right\}, \\ S_{\alpha}^{a} &= \mathrm{Tr}\left\{ -\sqrt{\frac{m}{3}}\, b^{\dagger a} c_{\alpha} - \frac{3\sqrt{2}}{m} \varepsilon_{\alpha\beta} \varepsilon^{abc} c^{\dagger\beta} \left[b_{c}, b_{b} \right] \right\}. \end{aligned} \tag{2.45}$$

As expected, these results are similar with the ones in [74], up to a change of basis for the gamma matrices.

The $\mathfrak{su}(2|2)$ *subsector: vacuum and excitations*

We shall now focus on states that transform in the $\mathfrak{su}(2|3)$ sector and are single trace (gauge invariant) operators of the fields (3 bosons and 2 fermions). This *spin-chain* arises from the large $N-$limit of the gauge theory. In this sector the action of the algebra generators can be found in [67]. Consider now the vacuum as a long string of $Z \equiv \phi^{3}$ fields. In oscillator notation, we have $Z = b^{3\dagger}$, and the vacuum state can be written as:

$$|0, J\rangle \equiv \left|Z^{J}\right\rangle \equiv \frac{1}{\sqrt{J} N^{J/2}} \mathrm{Tr}\left(b^{3\dagger J}\right) |0\rangle\,.$$

A generalization of this vacuum consists in an infinitely long string of Z fields (the asymptotic regime, $J \to \infty$), as in [55]. The excitations are now the other fields of the $\mathfrak{su}(2|3)$ algebra, $\chi \in \left\{\psi^{1}, \psi^{2} | \phi^{1}, \phi^{2}\right\}$, which corresponds to the $\mathfrak{su}(2|2)$ subsector of the algebra. The excitations can move through the chain on $Z's$ with some momentum p. Thus, in momentum space we can write

$$\chi = \sum_{n_k=1}^{N} e^{i p_k n_k} \chi\left(n_k\right) = \sum_{n_k=1}^{J} e^{i p_k n_k} \chi_k \equiv \chi\left(p_k\right),$$

where n denotes the position of the impurity/excitation χ on the vacuum string.

A general state with K impurities can then be written as:

$$|\chi_1,...\chi_K;J\rangle = \sum_{n_1,...,n_K=1} e^{ip_1n_1+\cdots+ip_Kn_K}\left|Z\cdots Z\chi_1 Z\cdots\chi_2\cdots\chi_K...Z\right\rangle.$$

For an asymptotic state ($J\to\infty$) we consider the dilute gas approximation, where the positions $n_1,\cdots,n_k$ of the impurities obey $n_1 \ll n_2 \ll \cdots \ll n_K$.

We should note that on-shell the physical states are cyclic (property of the trace), and so we must have $\sum_{k=1}^{K} p_k = 0$.

Now that we defined the states that the supercharges will be acting on, we can determine their action. The first step will be to check what the charges do to just one excitation on the vacuum. Then one can generalize to multi-excitation states of the $\mathfrak{su}(2|2)$ subsector of $\mathfrak{su}(2|3)$. Once we have the action of the charges on a multi-excitation state, we can determine the commutator of two supercharges, as a check of our results.

In this subsector the charges (2.45) become

$$\begin{aligned} Q_a^\alpha &= \sqrt{\frac{m}{3}}\mathrm{Tr}\left\{\psi^\alpha\frac{\partial}{\partial\phi^a} - \left(\sqrt{\frac{3}{m}}\right)^3\sqrt{2}\varepsilon_{ab}\varepsilon^{\alpha\beta}\left[Z,\phi^b\right]\frac{\partial}{\partial\psi^\beta}\right\}, \\ S_\alpha^a &= \sqrt{\frac{m}{3}}\mathrm{Tr}\left\{-\phi^a\frac{\partial}{\partial\psi^\alpha} - \left(\sqrt{\frac{3}{m}}\right)^3\sqrt{2}\varepsilon^{ab}\varepsilon_{\alpha\beta}\psi^\beta\left[\frac{\partial}{\partial Z},\frac{\partial}{\partial\phi^b}\right]\right\}, \end{aligned} \tag{2.46}$$

where we chose a coherent state basis, such that

$$\begin{aligned} c^{\dagger\alpha} &\to \psi^\alpha\,; \quad c_\alpha \to \frac{\partial}{\partial\psi^\alpha}; \\ b^{\dagger a} &\to \phi^a\,; \quad b_a \to \frac{\partial}{\partial\phi^a}. \end{aligned}$$

For $a=3$, we have the identification $\phi^3\equiv Z$. The factor $\sqrt{\frac{m}{3}}$ will appear as an overall factor in every charge calculated, and will be dropped, as we know that the quadratic terms come from the free theory $g_{YM}=0$.

We now proceed to determine the action of the supercharge Q (and equivalently S) on a *single excitation* state $|\chi;J\rangle = \sum_n e^{ipn}\left|Z^{n-1}\chi Z^{J-n+1}\right\rangle$. If the excitation is bosonic, $\chi_\ell = \phi^\ell$, then

$$Q_a^\alpha \left|\chi;J\right\rangle = \sum_n e^{ipn}\delta_a^\ell \left|Z^{n-1}\psi^\alpha(n)Z^{J-n+1};J\right\rangle,$$

while if the excitation is fermionic, $\chi^\beta = \psi^\beta$, we have

$$\begin{aligned}
Q_a^\beta \left|\chi;J\right\rangle &= -\left(\sqrt{\frac{3}{m}}\right)^3 \sqrt{2}\varepsilon_{ab}\varepsilon^{\alpha\beta}\sqrt{\frac{J+1}{J}}N^{1/2}\left|Z^{n-1}\left[Z,\phi^b\right]Z^{J-n+1};J+1\right\rangle \\
&= -\left(\sqrt{\frac{3}{m}}\right)^3 \sqrt{2}\varepsilon_{ab}\varepsilon^{\alpha\beta}\sqrt{\frac{J+1}{J}}N^{1/2}\left|Z^{n}\phi^b(n+1)Z^{J-n+1};J+1\right\rangle + \\
&\qquad + \left(\sqrt{\frac{3}{m}}\right)^3 \sqrt{2}\varepsilon_{ab}\varepsilon^{\alpha\beta}\sqrt{\frac{J+1}{J}}N^{1/2}\left|Z^{n-1}\phi^b(n)Z^{J-n+2};J+1\right\rangle \\
&\approx -\left(\sqrt{\frac{3}{m}}\right)^3 \sqrt{2}\varepsilon_{ab}\varepsilon^{\alpha\beta}\sum_n e^{ipn}\left(e^{-ip}-1\right)N^{1/2}\left|Z^{n-1}\phi^b(n)Z^{J-n+2};J+1\right\rangle.
\end{aligned}$$

It can be seen from the expression above that the insertion of a Z field before the excitation changes its phase by e^{-ip}, while the insertion after the excitation leaves that phase untouched. This is a property of the asymptotic state, for which an infinite number of Z fields exist after the (last) excitation. This was seen in [55] as being equivalent to "opening" the trace. In the above expression we also kept only the first order in $\frac{1}{J}$.

From the results shown above, we can easily determine the generalization to a *multi-excitation* state. First, rewrite the state as

$$\left|\chi;J\right\rangle \equiv \left|\chi_1...\chi_K;J\right\rangle = \sum_{\{l_i\}} e^{ip_1l_1+...+ip_Kl_K}\chi_1^\dagger\chi_2^\dagger\cdots\chi_K^\dagger\left|0;J\right\rangle. \tag{2.47}$$

The action of one charge on such state is (zeroth order in $\frac{1}{J}$):

$$\begin{aligned}
Q_a^\alpha\left|\chi_1...\chi_K;J\right\rangle &= \sum_{k=1}^K\sum_{\{l_i\}} e^{ip_1l_1+...+ip_Kl_K}\left(\prod_{m=1}^{k-1}(-1)^{F(m)}\right)\chi_1^\dagger\chi_2^\dagger\cdots\left(Q_a^\alpha\chi_k^\dagger\right)\cdots\chi_K^\dagger\left|0;J\right\rangle \\
&= \sum_{k=1}^K\sum_{\{l_i\}} e^{ip_1l_1+...+ip_Kl_K}\left(\prod_{m=1}^{k-1}(-1)^{F(m)}\right)\Big\{\delta\left(\chi_k^\dagger,\phi^b\right)\delta_a^b\chi_1^\dagger\chi_2^\dagger\cdots\psi^\alpha(l_k)\cdots\chi_K^\dagger\left|0;J\right\rangle - \\
&\quad -\frac{\sqrt{2N}}{M^3}\delta\left(\chi_k^\dagger,\psi^\beta\right)\left(\prod_{m=k+1}^{K}e^{-ip_m}\right)\left(e^{-ip_k}-1\right)\varepsilon_{ab}\varepsilon^{\alpha\beta}\chi_1^\dagger\chi_2^\dagger\cdots\phi^b(l_k)\cdots\chi_K^\dagger\left|0;J+1\right\rangle\Big\}.
\end{aligned} \tag{2.48}$$

and similarly for the S charge (noticing that the action of S on a bosonic excitation returns an extra factor of N). In here $\delta\left(\chi_k^\dagger,\phi^b\right)$ means that the excitation $\chi\left(l_k\right)$ is bosonic ϕ^b, while in $\delta\left(\chi_k^\dagger,\psi^\beta\right)$ the excitation $\chi\left(l_k\right)$ is fermionic ψ^β. The factor $(-1)^{F(m)}$ is equal to 1 if χ_m is bosonic and -1 if χ_m is fermionic. Finally we defined $M=\sqrt{\frac{m}{3}}$. When χ_k is a fermionic excitation, one gets the expected factor of $\left(e^{-ip_k}-1\right)$, which already showed up in the single excitation case, but one also gets an extra factor of $\prod_{m=k+1}^{K} e^{-ip_m}$. This last factor can also be explained by the insertion of the Z field. In fact, we saw that in the single excitation case Z changed the momentum when inserted before the excitation on the chain of fields. But now the field Z gets inserted before all of the excitations χ_m with $m>k$, hence the change of momenta of all these excitations.

The results of the action of Q and S on a multi-excitation state will be summarized next using a non local notation (see also [56]).

Twisted vs. non-local notations

The supercharges Q and S acting on a general state $|\chi;J\rangle$ can be written in a *non-local notation*:

$$
\begin{aligned}
Q_a^\alpha\,|\chi;J\rangle &= \sum_{k=1}^{K}\Big\{a_k\delta_a^b\,\delta\left(\chi_k^\dagger,\phi^b\right)|\chi_1\cdots\psi^\alpha\cdots\chi_K;J\rangle+\\
&\qquad +b_k\varepsilon_{ab}\varepsilon^{\alpha\beta}\,\delta\left(\chi_k^\dagger,\psi^\beta\right)\left|\chi_1\cdots\phi^b\cdots\chi_K;J+1\right\rangle\Big\}, \qquad (2.49)\\
S_\alpha^a\,|\chi;J\rangle &= \sum_{k=1}^{K}\Big\{c_k\varepsilon^{ab}\varepsilon_{\alpha\beta}\,\delta\left(\chi_k^\dagger,\phi^b\right)\left|\chi_1\cdots\psi^\beta\cdots\chi_K;J-1\right\rangle+\\
&\qquad +d_k\delta_\alpha^\beta\,\delta\left(\chi_k^\dagger,\psi^\beta\right)|\chi_1\cdots\phi^a\cdots\chi_K;J\rangle\Big\}, \qquad (2.50)
\end{aligned}
$$

where the coefficients are given by

$$
\begin{aligned}
a_k &= \prod_{m=1}^{k-1}(-1)^{F(m)},\\
b_k &= \frac{\sqrt{2N}}{M^3}\left[\prod_{m=1}^{k-1}(-1)^{F(m)}\right]\left(1-e^{-ip_k}\right)\left[\prod_{m=k+1}^{K}e^{-ip_m}\right]=\frac{\sqrt{2N}}{M^3}e^{-iP}\left(e^{ip_k}-1\right)\left[\prod_{m=1}^{k-1}(-1)^{F(m)}e^{ip_m}\right],\\
c_k &= \frac{\sqrt{2N}}{M^3}\left[\prod_{m=1}^{k-1}(-1)^{F(m)}\right]\left(e^{ip_k}-1\right)\left[\prod_{m=k+1}^{K}e^{ip_m}\right]=\frac{\sqrt{2N}}{M^3}e^{iP}\left(1-e^{-ip_k}\right)\left[\prod_{m=1}^{k-1}(-1)^{F(m)}e^{-ip_m}\right],\\
d_k &= -\prod_{m=1}^{k-1}(-1)^{F(m)}. \qquad (2.51)
\end{aligned}
$$

There is one other notation, introduced by Beisert in [56], called the *twisted notation*. In this local notation we have

$$
\begin{aligned}
Q^{\alpha}_{a,k}\left|\cdots\phi^{b}_{k}\cdots\right\rangle &= a'_{k}\delta^{b}_{a}\left|\cdots\mathscr{Y}^{+}\psi^{\alpha}_{k}\cdots\right\rangle, \\
Q^{\alpha}_{a,k}\left|\cdots\psi^{\beta}_{k}\cdots\right\rangle &= b'_{k}\varepsilon^{\alpha\beta}\varepsilon_{ab}\left|\cdots\mathscr{Z}^{+}\mathscr{Y}^{-}\phi^{b}_{k}\cdots\right\rangle, \\
S^{a}_{\alpha,k}\left|\cdots\phi^{b}_{k}\cdots\right\rangle &= c'_{k}\varepsilon^{ab}\varepsilon_{\alpha\beta}\left|\cdots\mathscr{Z}^{-}\mathscr{Y}^{+}\psi^{\beta}_{k}\cdots\right\rangle, \\
S^{a}_{\alpha,k}\left|\cdots\psi^{\beta}_{k}\cdots\right\rangle &= d'_{k}\delta^{\beta}_{\alpha}\left|\cdots\mathscr{Y}^{-}\phi^{a}_{k}\cdots\right\rangle.
\end{aligned}
\tag{2.52}
$$

We notice the presence of the markers $\mathscr{Z}^{\pm},\mathscr{Y}^{\pm}$. These markers have a simple explanation, up to one loop. The marker $\mathscr{Y}^{\pm}$ marks the position on the string of fields (the state) where a fermion field was inserted ($\mathscr{Y}^{+}$) or removed ($\mathscr{Y}^{-}$). In the twisted notation we are only given the action of the supercharge on the field k of the string. But in order for a supercharge to act on such field it will have to pass by the previous ones. If these are bosonic fields nothing happens, but if they are fermionic, a minus sign will appear (for each fermionic fields it passes). Thus, it is important to know where the supercharge acted, which is done by the marker. The marker is shifted around as follows:

$$
\left|\cdots\chi_{k}\mathscr{Y}^{\pm}\cdots\right\rangle = (\xi_{k})^{\pm 1}\left|\cdots\mathscr{Y}^{\pm}\chi_{k}\cdots\right\rangle,
$$

where

$$
\xi_{k} = (-1)^{F(k)} = \begin{cases} 1 & \text{if } \chi_{k} \text{ bosonic} \\ -1 & \text{if } \chi_{k} \text{ fermionic} \end{cases}. \tag{2.53}
$$

The marker $\mathscr{Z}^{\pm}$ marks a position where an extra Z field was inserted in the string. This changes the length of the vacuum spin chain, reflecting a change in the momenta of the excitation fields. But this change in momenta only affects the excitation fields after the position of the marker. The marker has the property

$$
\left|\cdots\chi_{k}\mathscr{Z}^{\pm}\cdots\right\rangle = \frac{x^{\pm}_{k}}{x^{\mp}_{k}}\left|\cdots\mathscr{Z}^{\pm}\chi_{k}\cdots\right\rangle, \text{ where } \frac{x^{\pm}_{k}}{x^{\mp}_{k}} = e^{\pm i p_{k}}, \tag{2.54}
$$

with p_k being the momenta of the excitation χ_k, as before.

In summary, the *twisted notation* is a local notation, since it only provides the action of the supercharge on the excitation field χ_k, plus a set of markers that allow us to rewrite it in a *non-local notation*, as found in (2.49, 2.50). We can go from the twisted notation to the non-local one by removing the markers from the first, i.e., shifting them so that they will be at the right (or left) of all the excitation fields.

In the local *twisted notation* we have[20]

$$d'_k \;=\; -d'_k = 1\,, \quad b'_k = \frac{\sqrt{2N}}{M^3}\left(1-e^{-ip_k}\right), \quad c'_k = -\frac{\sqrt{2N}}{M^3}\left(1-e^{ip_k}\right).$$

Comparison with Beisert at 1-loop

One can find the all-loop version of these coefficients in [56], for both the non-local and the twisted notation. In fact, we can expand the (non-local) coefficients given in that reference to order $\mathscr{O}(g)$, and compare them to our results. These coefficients are:

$$\begin{aligned}
a_k &= \gamma_k \prod_{j=1}^{k-1} (-1)^{F(j)}\,, \\
b_k &= g\frac{\alpha}{\gamma_k}\left(1-e^{ip_k}\right)\prod_{j=1}^{k-1}\left(e^{ip_k}(-1)^{F(j)}\right), \\
c_k &= i\frac{\gamma_k}{\alpha x_k^+}\prod_{j=1}^{k-1}\left(e^{-ip_k}(-1)^{F(j)}\right), \\
d_k &= g\frac{x_k^+}{i\gamma_k}\left(1-e^{-ip_k}\right)\prod_{j=1}^{k-1}(-1)^{F(j)}\,.
\end{aligned}$$

We used the identifications (2.53) and (2.54) into the transcribed coefficients, and also made a rescaling of the parameter $\gamma_k \to \sqrt{g}\gamma_k$. The expansion in g is hidden in the dependence of

[20] The coupling constant $M^6 = \left(\frac{m}{3}\right)^3$ is related to the Yang-Mills coupling constant g_{YM} in the following way

$$\frac{1}{M^6} = \frac{g_{YM}^2}{32\pi^2}.$$

This relation comes from matching the prefactor of the reduced SYM action with the prefactor of the matrix model action. In fact we had $m = \frac{6}{R}$, where R was the radius of S^3. Taking the radius small corresponds to $m \gg 1$ and consequently $g_{YM} \ll 1$.

x^+, x^- on the coupling constant:

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g}. \tag{2.55}$$

This last equation, together with (2.54), allows us to solve for $x^+(g)$:

$$x^+ = i\frac{1+\sqrt{1+16g^2\sin^2(p/2)}}{2g\left(1-e^{-ip}\right)}.$$

Then by expanding this expression up to order $\mathscr{O}(g)$, we obtain exact agreement with (2.51), as long as we identify $\gamma_k = (-1)^{F(k)}$ and $\alpha = e^{-iP}$. Note that the relation between the normalized 't Hooft coupling g and the Yang-Mills coupling constant g_{YM} is $g = \frac{g_{YM}}{4\pi}\sqrt{N_c}$, from the gauge group $SU(N_c)$.

The other charges that we are interested in determining are the Hamiltonian H and the central charges of the extended algebra P, K. These charges arise from commutation relations between the supercharges, which will be determined next.

Commutation Relations

At this moment we have calculated only the supercharges of the full extended algebra $\mathfrak{su}(2|2)$, up to $\mathscr{O}(g)$. We are interested in having the complete set of charges at this order, which comprises also the rotations generators L, R, the dilatation operator H, and also the central charges of the extended algebra P, K (bosonic generators of momentum and boosts, which have zero eigenvalues when applied to physical states). All of these generators can be obtained to $\mathscr{O}(g)$ from the commutation relations of the supercharges.

The central charges of the extended algebra receive no loop corrections , and as such, can be obtained exactly by the anti-commutation relations $\{Q, Q\} \sim P$ and $\{S, S\} \sim K$, by knowing the zeroth order of the supercharges. The other generators will be obtained from the last anti-commutator $\{Q, S\} \propto R + L + H$, but while the zeroth order supercharges will be enough to determine rotation generators L and R, the central charge H will only be known correctly up to $\mathscr{O}(g)$, as we'll see below.

In the anti-commutator of any two supercharges the only terms that will not vanish are the ones where the two supercharges are applied to the same excitation. The anti-commutator of two Q charges is:

$$\begin{aligned}
\left\{Q_b^{\beta}, Q_a^{\alpha}\right\}|\chi_1 \ldots \chi_K; J\rangle &= \sum_{k=1}^{K} \left|\chi_1 \cdots \left(\left\{Q_b^{\beta}, Q_a^{\alpha}\right\}\chi_k\right) \cdots \chi_K; J\right\rangle \\
&= \frac{\sqrt{2N}}{M^3} \sum_{k=1}^{K} \left[\left(1-e^{-ip_k}\right) \prod_{l=k+1}^{K} e^{-ip_l}\right] |\chi_1 \ldots \chi_K; J+1\rangle \\
&= \frac{\sqrt{2N}}{M^3} \left(1-e^{-i\sum_{k=1}^{K} p_k}\right) |\chi_1 \ldots \chi_K; J+1\rangle .
\end{aligned}$$

This is just the action of the central charge $\{Q,Q\} \propto P$ of the extended algebra on a multi-excitation state. The action of the other central charge of the extended algebra is obtained from $\{S,S\} \propto K$:

$$\left\{S_{\beta}^{b}, S_{\alpha}^{a}\right\}|\chi_1 \ldots \chi_K; J\rangle = \frac{\sqrt{2N}}{M^3}\left(1-e^{i\sum_{k=1}^{K} p_k}\right)|\chi_1 \ldots \chi_K; J-1\rangle .$$

We know from [56] that there is an outer automorphism relating H and the central charges of the extended algebra P,K, which corresponds to an $\mathfrak{sl}(2)$ algebra. Closure of this algebra on the original commutation relations of the supercharges requires that

$$H^2 - PK = \frac{1}{4}. \tag{2.56}$$

This relation should only hold when we consider the all loop H, and not only when we consider the first two orders. Using non-local notation, we find that the product PK is given by

$$PK = -\frac{2N}{M^6}\left(e^{-i\sum_{k=1}^{K} p_k}-1\right)\left(e^{+i\sum_{k=1}^{K} p_k}-1\right) = \frac{8N}{M^6}\sin^2\left(\sum_{k=1}^{K}\frac{p_k}{2}\right) = \frac{8N}{M^6}\sin^2\left(\frac{p}{2}\right),$$

and so $H^2 = \frac{1}{4} + PK = \frac{1}{4} + \frac{8N}{M^6}\sin^2\left(\frac{p}{2}\right)$, which implies

$$H = \pm\frac{1}{2}\sqrt{1+\frac{32N}{M^6}\sin^2\left(\frac{p}{2}\right)} = \pm\frac{1}{2}\sqrt{1+\frac{g_{YM}^2 N}{\pi^2}\sin^2\left(\frac{p}{2}\right)}.$$

This is the result expected at one loop. The identification of the matrix model mass parameter with the Yang-Mills coupling coupling holds at one loop but some mismatches were seen to appear at higher loop calculations, implying some kind of BMN scaling breakdown, and a substitution of the factor $\frac{32N}{M^6}$ for a function $f\left(\frac{N}{M^6}\right)$ [78].

We now calculate the anti-commutator of Q and S, which will be proportional to to $L^{\alpha}_{\beta}, R^{a}_{b}$ and the Hamiltonian H:

$$\begin{aligned}\left\{Q^{\beta}_{b}, S^{a}_{\alpha}\right\}|\chi_1 ... \chi_K; J\rangle = \sum_{k=1}^{K} \Big\{ & c_k b_k \varepsilon^{aa'} \varepsilon_{bb'} \delta^{\beta}_{\alpha} \delta\left(\chi^{\dagger}_k, \phi^{a'}\right) \left|\chi_1 \cdots \phi^{b'} \cdots \chi_K; J\right\rangle + \\ & + c_k b_k \varepsilon^{\beta\beta'} \varepsilon_{\alpha\alpha'} \delta^{a}_{b} \delta\left(\chi^{\dagger}_k, \psi^{\beta'}\right) \left|\chi_1 \cdots \psi^{\alpha'} \cdots \chi_K; J\right\rangle + \\ & + a_k d_k \delta^{\beta}_{\alpha} \delta\left(\chi^{\dagger}_k, \phi^{b}\right) |\chi_1 \cdots \phi^{a} \cdots \chi_K; J\rangle + \\ & + a_k d_k \delta^{a}_{b} \delta\left(\chi^{\dagger}_k, \psi^{\alpha}\right) \left|\chi_1 \cdots \psi^{\beta} \cdots \chi_K; J\right\rangle \Big\}.\end{aligned}$$

From equations (2.51) we have that:

$$a_k d_k = -1 \qquad ; \qquad b_k c_k = \frac{4N}{M^6}\left(1 - e^{-ip_k}\right)\left(e^{ip_k} - 1\right) = -\frac{16N}{M^6} \sin^2\left(\frac{p_k}{2}\right).$$

Also, we know from the algebra (2.1) that

$$\begin{aligned}\mathscr{L}^{\beta}_{\alpha}|\psi^{\gamma}\rangle &= \delta^{\gamma}_{\alpha}\left|\psi^{\beta}\right\rangle - \frac{1}{2}\delta^{\beta}_{\alpha}|\psi^{\gamma}\rangle, \\ R^{a}_{b}|\phi^{c}\rangle &= \delta^{c}_{b}|\phi^{a}\rangle - \frac{1}{2}\delta^{a}_{b}|\phi^{c}\rangle.\end{aligned}$$

For multi-particle states this generalizes to

$$\mathscr{L}^{\beta}_{\alpha}|\chi_1 ... \chi_K; J\rangle = \sum_{k=1}^{K} \chi^{\dagger}_1 \cdots \mathscr{L}^{\beta}_{\alpha}\left(\chi^{\dagger}_k\right) \cdots \chi^{\dagger}_K |0; J\rangle,$$

with a similar result for the charge R^{a}_{b}.[21]

[21]The charges $\mathscr{L}$ and R are the generators of the algebra that correspond to rotations of the ψ^{γ} $\mathfrak{su}(2)$ algebra and of the ϕ^{a} $\mathfrak{su}(2)$ algebras, respectively. As such, $\mathscr{L}^{\beta}_{\alpha}|\phi^{c}\rangle = 0$, and $R^{a}_{b}|\psi^{\gamma}\rangle = 0$.

One can now easily see that

$$\begin{aligned}
\left\{Q_b^{\beta}, S_{\alpha}^{a}\right\}|\chi_1 \ldots \chi_K; J\rangle &= \delta_{\alpha}^{\beta} R_b^a |\chi_1 \ldots \chi_K; J\rangle + \delta_b^a \mathscr{L}_{\alpha}^{\beta}|\chi_1 \ldots \chi_K; J\rangle \\
&\quad + \delta_{\alpha}^{\beta}\delta_b^a \sum_{k=1}^{K}\left(\frac{1}{2}a_k d_k + b_k c_k\right)|\chi_1 \ldots \chi_K; J\rangle \\
&\quad - \delta_{\alpha}^{\beta}\sum_{k=1}^{K} b_k c_k \delta\left(\chi_k^{\dagger}, \phi^b\right)|\chi_1 \cdots \phi^a \cdots \chi_K; J\rangle \\
&\quad - \delta_b^a \sum_{k=1}^{K} b_k c_k \delta\left(\chi_k^{\dagger}, \psi^{\alpha}\right)\left|\chi_1 \cdots \psi^{\beta} \cdots \chi_K; J\right\rangle . \qquad (2.57)
\end{aligned}$$

If we compare (2.57) with the expected results from commutation relations given in (2.1), the last two terms seem to be extra. But in fact this is the exact result! We (anti-)commuted only the order g^0 and order g^1 of the supercharges. That is, we calculated the nonzero anti-commutators $\{Q_0, S_0\} \propto R + L + H_0$ and $\{Q_1, S_1\}$. This last anti-commutator contributes to order g^2 of the Hamiltonian, H_2, but there will be another contribution to H_2: the two-loop terms of the supercharges, Q_2 and S_2, will have nonzero commutation relations with S_0 and Q_0, respectively, and contribute to $\mathscr{O}\left(g^2\right)$. So H_2(the energy central charge of order g^2) will be fully determined by:

$$H_2 \propto \{S_1, Q_1\} + \{S_2, Q_0\} + \{S_0, Q_2\} . \qquad (2.58)$$

Only considering all the above anti-commutators we will get the correct result for the H_2. For calculations see Appendix A.2, and also [67].

Supercharges as operators in momentum space

We now present a description of the supercharges in terms of operators in momentum space. Consider as before an infinite chain of fields Z. The vacuum state, written before as $|0;J\rangle = \mathrm{Tr}\left(Z^J\right)|0\rangle$, can be rewritten, in the "Hamiltonian formalism" introduced in [5] as $|0;J\rangle = \left(b_z^{\dagger}\right)^J|0\rangle$, where $b_z^{\dagger}$ creates an extra Z field in the string.[22] Then we can write a state

[22] The subscript z is used in this section, to distinguish the creation operator $b_z^{\dagger}$ for the boson Z from the creation operator $b^{a\dagger}$ for the two bosonic impurities.

with K impurities as:

$$|\Psi\rangle = \sum_{n_1,\ldots,n_K} e^{ip_j n_j} b^\dagger(n_1)\cdots b^\dagger(n_K)|0;J\rangle = b^\dagger(p_1)\cdots b^\dagger(p_K)|0;J\rangle\,.$$

We are imposing dilute gas approximation, in which we consider $n_1 \ll n_2 \ll \cdots \ll n_K$. We will now assume $p_1 < p_2 < \cdots < p_K$.

In the last expression for $|\Psi\rangle$ we used the creation operators $b^\dagger(n) = \left(b_z^\dagger\right)^n b^\dagger(b_z)^n$, which create a boson b at position n in the string of $Z's$. One can also introduce $c^\dagger(n) = \left(b_z^\dagger\right)^n c^\dagger(b_z)^n$ as a creation operator for a fermion at position n. The action of the Hamiltonian in this framework can be found in [5], and a further comparison with lattice strings can be found in [81].

To write the action of the supercharges in terms of these operators, we also need to introduce a partial momentum operator

$$\hat{\mathscr{P}}(p) = \int_0^p dp'\,p'\left[b^\dagger\left(p'\right)b\left(p'\right) + c^\dagger\left(p'\right)c\left(p'\right)\right],$$

or the discrete momentum version

$$\hat{\mathscr{P}}(p) = \sum_{k=0}^{p-1} k\left[b^\dagger(k)\,b(k) + c^\dagger(k)\,c(k)\right].$$

The total momentum operator is just $\hat{P} = \hat{\mathscr{P}}(p_{max})$, where p_{max} is either ∞ in the continuum case, or finite (but large) in the lattice. Also, define an operator $\hat{\Theta}$ conjugate to the “R-charge operator” $\hat{\mathscr{J}}$. In the spin-chain formalism, $\hat{\mathscr{J}}$ effectively measures the length of the chain of Z fields, and $\hat{\Theta}$ changes that length:

$$\hat{\mathscr{J}}\,e^{\pm i\hat{\Theta}}|0,J\rangle = (J\pm 1)\,e^{\pm i\hat{\Theta}}|0,J\rangle\,.$$

We can now proceed to the action of the supercharges. In momentum space, they become:

$$
\begin{aligned}
Q_b^{\beta} &= -\frac{\sqrt{2N}}{M^3}\varepsilon_{bb'}\varepsilon^{\beta\beta'}e^{i\hat{\Theta}}e^{-i\hat{P}}\sum_p b^{b'\dagger}(p)\left(e^{ip}-1\right)e^{i\hat{\mathscr{P}}(p)}c_{\beta'}(p)+\sum_p c^{\beta\dagger}(p)\,b_b(p)\,; \\
S_{\alpha}^{a} &= \frac{\sqrt{2N}}{M^3}\varepsilon_{\alpha\alpha'}\varepsilon^{aa'}e^{-i\hat{\Theta}}e^{i\hat{P}}\sum_p c^{\alpha'\dagger}(p)\left(1-e^{-ip}\right)e^{-i\hat{\mathscr{P}}(p)}b_{a'}(p)-\sum_p b^{a\dagger}(p)\,c_{\alpha}(p)\,. \quad (2.59)
\end{aligned}
$$

It is not hard to check that these definition give us the results obtained in the previous section. In the above expression the sum over momenta has increments of $\frac{2\pi}{J}$.[23] If we wrote the charges obtained from the string formalism (2.37) and (2.38) in momentum space, we would obtain the exact structure for the supercharges as was seen in (2.59), as long as we make the correspondence for the conjugate pair $\left(x_-^0, P_+\right) \leftrightarrow \left(\hat{\Theta}, \hat{\mathscr{J}}\right)$.

Commuting two central charges Q will give us the central charge $\mathscr{P}$:

$$
\{Q,Q\} = e^{i\hat{\Theta}}e^{-i\hat{P}}\sum_p b^{\dagger}(p)\left(e^{ip}-1\right)e^{i\hat{\mathscr{P}}(p)}b(p)+e^{i\hat{\Theta}}e^{-i\hat{P}}\sum_p c^{\dagger}(p)\left(e^{ip}-1\right)e^{i\hat{\mathscr{P}}(p)}c(p)=\mathscr{P}. \quad (2.60)
$$

One can show that the central charge takes the much more common form:[24]

$$
\mathscr{P} = e^{i\hat{\Theta}}e^{-i\hat{P}}\left(e^{i\hat{P}}-1\right) = e^{i\hat{\Theta}}\left(1-e^{-i\hat{P}}\right). \quad (2.61)
$$

To summarize, we found expressions for the supercharges as operators in momentum space, as well as for their commutation relations, in the large J limit. These expressions once applied to states with K impurities will result in the expressions obtained in the previous section.

Looking at the value of the central charge here (from the spin-chain formalism) and the one obtained from the string side (2.25), we can conclude that the results are correct up to an overall phase $e^{\pm ip_{ws}}$, as long as we match $\left\{Q,\overline{Q}\right\} \leftrightarrow \{S,Q\}$ and $\left\{C,C^{\dagger}\right\} \leftrightarrow \{K,P\}$. This overall phase found on the string side is natural, as different boundary conditions for x_- will

---

[23] The operator $e^{\pm i\hat{\Theta}}$ does not commute with the sum over the momenta, as it changes the increments in the sum. But in the limit J very large, this change will be negligible.

[24] This can be proven by using the property (valid for any power n, proven by induction, and for χ fermionic or bosonic)

$$
\hat{\mathscr{P}}^n(p)\,\chi^{\dagger}(p') = \chi^{\dagger}(p')\left[\theta\left(p-p'\right)p'+\hat{\mathscr{P}}(p)\right]^n.
$$

differ from each other by such a phase. Also the algebra (2.1) allows a $U(1)$ automorphism, which means we can always multiply all supercharges by some phase that can depend on all central charges.

.

2.6 Summary of the superalgebra results

In this chapter we studied in detail the Q,S generators of the extended algebra $\mathfrak{su}(2|2)$ in the plane-wave matrix theory formalism. By using a coherent basis we determined the supercharges in the non-local notation of Beisert [56] (as well as in the local twisted notation), and determined some of the coefficients in this notation up to order $\mathscr{O}(g_{YM})$.

We also determined the anti-commutation relations of these supercharges, and obtained the expected results for the central charges P,K and H. We saw that we needed to know the Hamiltonian up to two-loops in order to have a closed (anti-)commutation relation between Q and S.

Finally, we wrote a first quantized formulation of the supercharges obtained directly from the sigma model action for the string. Having the supercharges written in that way allowed us to compare their structure with the what we had previously calculated from gauge side.

The evidence seems to point to $\mathscr{N}=4$ SYM and IIB superstring theory being integrable models in the 't Hooft limit. It has been seen that the scattering matrix is completely defined by the underlying symmetry algebra $\mathfrak{psu}(2,2|4)$. In this limit, the S-matrix can be found to actually retain a symmetry algebra that is two copies of a central extension of the $\mathfrak{psu}(2|2)$ algebra, in particular: $\mathfrak{psu}(2|2)\ltimes\mathbb{R}^3=\mathfrak{su}(2|2)\ltimes\mathbb{R}^2$. From the properties of this S-matrix, one finds that it naturally satisfies the Yang-Baxter equation [55,56], which is more evidence towards having factorized scattering and integrability. Thus, this symmetry of the S-matrix is expected to be a Yangian symmetry, [61,84,85] and have an underlying Hopf algebra [86] (see also [87,88]). Having these new developments in minds, it would be interesting to apply the methods used in this paper to the study of the Hopf algebra related to the central extension, and get some results on the corresponding Yangian generators.

The sector of near 1/2 BPS operators in $\mathcal{N} = 4$ super Yang-Mills has been well studied by the use of collective methods [80, 81], and the same methods can be used to study the elements of the algebra in 1/4 BPS sector.

Chapter 3

Solitons in $\mathbb{R}\times S^5$: Giant Magnons & Single Spikes

Much has been learned about the AdS$_5$-CFT$_4$ correspondence [1] by looking at limits in which an $SO(6)$ charge J also becomes large. At large λ the theory is a theory of classical strings moving in $AdS_5 \times S^5$, with J an angular momentum on the sphere, while at small λ it is perturbative Yang–Mills theory in 4 dimensions, with J an R-charge of this theory [11,89]. This is the large-J sector.

The first well-studied example in this sector is the BMN limit [5, 90], as was mentioned before, which on the string side, consists of nearly point-like solutions orbiting the sphere, experiencing a pp-wave geometry. On the gauge theory side, the anomalous dimension $\Delta - J$ can be computed as the energy of a ferromagnetic spin chain [26, 67, 55]. These spin chains are integrable systems, allowing the use of Bethe ansatz techniques to compute the spectrum from the S-matrix for two-particle scattering [20, 91, 92, 17, 93, 56] (in some cases one can explicitly recover the string action from the spin-chain [13]).

The elementary excitations of spin chains are magnons, which to be scattered must have some momentum $p \neq 0$. Extending the theory to allow lone magnons with momentum has been seen to lead to the centrally extended algebras [55,66] on the gauge side. This subject was discussed in depth in the previous chapter. These lone magnons are dual to strings which do not close, called giant magnons [26]. Generalizations which have been explored include magnons with more than one large angular momentum [27, 94, 95] and magnons with finite J [82, 96, 97, 98].

Giant magnons are one type of rigidly rotating strings with cusps, moving on the sphere and made as large as they can be. In general these are called spiky strings, and they also exist in flat space [99, 100] and in AdS [28, 101]. In flat space T-duality leads to another class of spiky strings, with cusps pointing inwards, and these 'T-dual' solutions can also exist on the sphere. Starting with one of these and taking the same maximum-size limit used for the magnon then leads to the single spike solution which we study here. [48] Recent papers on the single spike include [102, 103, 104, 105, 106, 107, 108].

The giant magnon can be viewed as an excitation above a vacuum solution of a point particle orbiting along the equator [109] (the label 'giant' is meant to indicate that they explore much of the S^5 geometry, as the earlier giant gravitons did [110, 68]). Fluctuations of this vacuum have Hamiltonian $\Delta - J$ [12] (where $J \leq \Delta$ is the BPS bound). The single spike is similarly an excitation above a string wound around the equator, which we call the "hoop". In the Hamiltonian for fluctuations, the angular momentum J is replaced with a measure of the winding along the same direction, which we call Φ. This is almost T-duality, except that the circle involved is part of sphere. It is not clear whether this duality can be usefully related to the T-duality used in [111] and [112], in S^5 and AdS.

The single spike, and indeed the hoop, are not supersymmetric. Exploring the correspondence in sectors with less or no supersymmetry is of great interest, and it is our hope that the close relationship to the magnon case can be used as a tool for this. The gauge theory dual of the single spike is not known, but it is conjectured to be some excitation of an anti-ferromagnetic state of the spin chain [113, 114] in what has been named the large-winding sector of the correspondence [115]. In the absence of supersymmetry it is possible that integrability will help to find the dual of the spike solution.

Solitons have long been studied in field theory, and a set of tools called semi-classical quantization enables us to learn about the related objects in the quantum theory [116, 117, 118, 119, 120, 121, 122]. Many of these techniques have been revived to study solutions of classical string theory in $AdS_5 \times S^5$ [12, 123, 124, 125] (which is known to be integrable [19, 126]). The single spike case has the extra complication that it is an excitation of an unstable vacuum state (as the string wrapped around an equator of S^5 can slide off towards the pole) so what we aim to calculate by these methods is not an energy correction but a lifetime, as discussed in the text.

3.1 Semiclassical Giant Magnons and Giant Spikes in $\mathbb{R}\times S^2$

In the AdS/CFT correspondence, there are two kinds of operators/states that can be compared avoiding the problem of the strong/weak property of the duality. The first are just sets of chiral primary operators (and their descendants) in which one can use nonrenormalization theorems to make comparisons [1, 52, 53]. The second are sets of operators of SYM with large global charges, which are dual to semi-classical states on the string side. One example of the latter are the BMN operators mentioned before [5]. The exact string dual of a chiral primary operator is a point-like string orbiting a geodesic of S^5, with angular momentum J, which has to be large enough for the classical approximation to be correct. We will be studying excitations of this point-like "vacuum" solution, called the giant magnons, as well as related classical solutions called giant spikes. These live on the subspace $\mathbb{R}\times S^2$ of the full background.

String dynamics of bosonic degrees of freedom in the $AdS_5\times S^5$ space-time can be described by the bosonic $\sigma-$model action

$$S=\frac{\sqrt{\lambda}}{2\pi}\int \mathrm{d}\tau\,\mathrm{dx}\left\{\underbrace{\eta^{ab}\partial_a Y^{\mu}\partial_b Y_{\mu}+\alpha_1\left(Y^2+1\right)}_{AdS_5}+\underbrace{\eta^{ab}\partial_a X^{i}\partial_b X_{i}+\alpha_2\left(X^2-1\right)}_{S^5}\right\},$$

where one embeds both the sphere as the AdS space in $\mathbb{R}^6$ with the respective constraints. This action has a symmetry under $\mathfrak{su}(2,2)\times\mathfrak{so}(6)$. To consider magnons moving on the sphere one restricts the space-time to $\mathbb{R}\times S^5$, with $\mathbb{R}$ being one of the time directions of the AdS_5 space, the respective charges are the generators of rotations in S^5

$$J_{ij}=\frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi}dx\left(X_i\dot{X}_j-X_j\dot{X}_i\right),$$

and the generator of time translations

$$\Delta = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} dx\dot{Y}^0.$$

One considers the limit when the angular momentum $J = J_{12}$ in the direction $\varphi \equiv (12)$ of S^5 is very large, and look at states with $\Delta - J$ finite. The momentum of the excitation p is also kept fixed. The relevant limit to be taken here is to have $\Delta, J \to \infty$, while keeping $\Delta - J$, p and the 't Hooft coupling $\lambda = g_{YM}^2 N$ fixed. Then by varying the coupling λ we can reach both sides of the gauge/string correspondence. This limit differs from the previous BMN limit [5] in that the latter considered the coupling λ large, keeping g_{YM} fixed and small, and considered the momentum p to be small, keeping the quantity $n = pJ$ fixed instead. One of the major properties of this limit is that quantum effects (expansions in λ) become decoupled from finite-J effects, and we will be using this fact when studying both effects separately in the next chapters.

In this limit, we find that the string ground state has $\Delta - J = 0$, which consists of a point-particle with a light-like trajectory along the direction φ, time coordinate $Y^0 \equiv \tau$ obeying $\varphi - \tau =$ constant, and sitting at the origin of the spatial directions of AdS_5.

To find excitations above this ground state one looks at solutions rotating in the $Z_1 = X^1 + iX^2$ plane. The remaining four directions of the embedding space we call $\vec{X}$, and $Y^0 \equiv \tau$ is the time co-ordinate (ultimately from AdS). So the motion is all in the time direction of AdS space, and on the subspace $S^2 \subset S^5$: $\mathbb{R} \times S^2$.

Spiky strings in flat space

If instead of S^5 we consider flat space, one can find a spiky string solution [28] [99,100]

$$\begin{aligned}
Y^0 &= \tau, \\
X^1 &= A\cos\left(\frac{\tau + x}{2A}\right) + AB\cos\left(\frac{\tau - x}{2AB}\right), \\
X^2 &= A\sin\left(\frac{\tau + x}{2A}\right) + AB\sin\left(\frac{\tau - x}{2AB}\right),
\end{aligned} \tag{3.1}$$

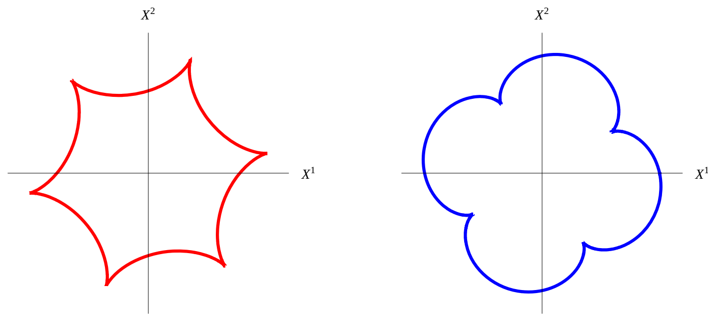

Figure 3.1: The original and T-dual spiky string in flat space. Both are for $B = 5$, leading to 6 and 4 spikes respectively.

with two parameters A, B. This solution is a rigidly rotating string with $n = B + 1$ cusps (spikes) pointing outwards and moving at the speed of light (see figure 3.1). The parameter A determines the overall size of the cusps.

We can perform a T-duality transformation to this solution, in particular in the X^2 direction. But because the solution has neither center-of-mass momentum nor winding, this T-duality only changes the sign of the left-movers in that direction, [48] giving

$$X^0 \text{ and } X^1 \text{ unchanged,} \qquad (3.2)$$
$$X^2 = A \sin\left(\frac{\tau + x}{2A}\right) - AB \sin\left(\frac{\tau - x}{2AB}\right).$$

This is again a rigidly rotating string, but now with $B - 1$ spikes pointing inwards, and also moving at the speed of light.

Note that this T-dual solution could also have been obtained by simply interchanging x and t in the spatial co-ordinates X^i. In fact this is a symmetry of the equations of motion

$$\left(-\partial_\tau^2 + \partial_x^2\right) X^i = 0$$

and of the Virasoro constraints (for $Y^0 = \tau$)

$$\left(\partial_\tau X^i\right)^2 + \left(\partial_x X^i\right)^2 = 1, \qquad \partial_\tau X^i \partial_x X^i = 0,$$

as these equations are unchanged by interchanging $x \leftrightarrow \tau$.

On the sphere

Similar solutions exist on the sphere, and when they are small they will reduce to those in flat space. In [48] it was shown that if the analogue of the original solution (3.1) becomes large, so that the spikes touch the equator, then each segment (between spikes) of it becomes a giant magnon. For the analogue of the T-dual solution (3.2), the limit in which the lobes touch the equator is the single spike.

By choosing a time-like $t = \tau$, conformal gauge (the induced metric is proportional to the standard metric, $\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \propto \eta_{ab}$) we are looking for solutions that solve the Virasoro constraints

$$\left(\partial_\tau X^i\right)^2 + \left(\partial_x X^i\right)^2 = 1, \qquad \partial_\tau X^i \partial_x X^i = 0,$$

and obey the conformal equations of motion

$$\left(-\partial_\tau^2 + \partial_x^2\right) X^i + X^i \left(-(\partial_\tau X^j)^2 + (\partial_x X^j)^2\right) = 0.$$

Solving these equations, Hofman and Maldacena [26] found the Giant Magnon solution:

$$\begin{aligned} Y^0 &= \tau, \\ Z_1 &= e^{i\tau}\left(c + i\sqrt{1-c^2}\tanh u\right), \\ \vec{X} &= \vec{n}\sqrt{1-c^2}\,\mathrm{sech}\, u, \end{aligned} \tag{3.3}$$

where $c = \cos(p/2)$ is the worldsheet velocity, and (u, v) are boosted worldsheet co-ordinates

$$\begin{aligned} u &= \gamma(x - c\tau), \\ v &= \gamma(\tau - cx), \qquad \text{with } \gamma = \frac{1}{\sqrt{1-c^2}} = \frac{1}{\sin(p/2)}. \end{aligned} \tag{3.4}$$

This is a rigidly rotating string along the equator of S^2, with cusps touching this equator and moving at the speed of light. Note that $-\infty < x < \infty$ covers only one of the curves between cusps. It is understood that the physical closed-string solution consists of several

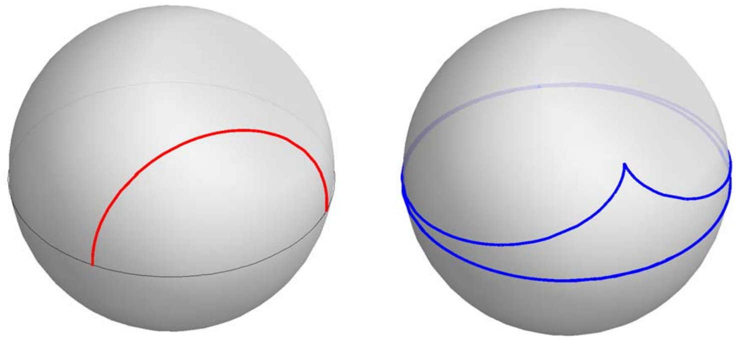

Figure 3.2: The giant magnon (left, $c = \cos(p/2) = 0.7$) and the single spike (right, $c = 0.8$). These are both are rigidly rotating along the equator shown, with their cusps moving at the speed of light.

giant magnons connected together. The case $c = 0$ (zero worldsheet velocity, $p = \pi$) is one of GKP's folded strings. [11] In the limit $p \to 0$ the magnon becomes a point particle moving along the equator.

As in flat space, the Virasoro constraints and the conformal equations of motion are unchanged by the interchange of x and τ. So there is another solution $X^i_{\text{spike}}(\tau, x) = X^i_{\text{magnon}}(x, \tau)$, which has been dubbed the single spike: [48]

$$\begin{aligned} Y^0 &= \tau, \\ Z_1 &= e^{ix}\left(c + i\sqrt{1-c^2}\tanh v\right), \\ \vec{X} &= \vec{n}\sqrt{1-c^2}\,\operatorname{sech} v\,. \end{aligned} \tag{3.5}$$

This solution is drawn in figure 3.2. We keep the same parameter $0 < c < 1$, although the worldsheet velocity is now $1/c$ in the x, τ co-ordinates.[1]

Both solutions are localized on the worldsheet. As $x \to \infty$, the magnon solution approaches the point particle $Z_1 = e^{i\tau}$ and $\vec{X} = 0$ while the single spike solution becomes instead the infinitely wound hoop $Z_1 = e^{ix}$. The point particle and the hoop are clearly related by the same $x \leftrightarrow \tau$ swop, and they are also the vacuum solutions needed to obtain the magnon or the single spike by the dressing method, which survives this interchange. [127,128] [109,104]

In the conformal, time-light gauge we started from, the relevant charges can be written

[1]This is related to the parameter θ_0 used in [48], which is the angle from the north pole to the spike, by $\sin\theta_0 = c = \cos(p/2)$. Also note that $\bar{\theta} = \frac{\pi}{2} - \theta_0 = p/2$.

as

$$\begin{aligned}
\Delta &= \frac{\sqrt{\lambda}}{2\pi}\int dx\, 1 && \text{time-translations,}\\
J &= \frac{\sqrt{\lambda}}{2\pi}\int dx\, \mathrm{Im}\left(\overline{Z}_1\partial_\tau Z_1\right) && \text{angular momentum in } Z_1 \text{ plane,}\\
p &= \frac{1}{i}\int dx \frac{d}{dx}\ln Z_1 && \text{worldsheet momentum.}
\end{aligned}$$

For the case of the Giant Magnon, Δ and J are infinite, with $\Phi = \frac{\sqrt{\lambda}}{2\pi}p$ and

$$E_{mag} = \Delta - J = \frac{\sqrt{\lambda}}{\pi}\sin(p/2). \tag{3.6}$$

For the single spike we have one further charge of interest, the winding charge:

$$\Phi = \frac{\sqrt{\lambda}}{2\pi}\int dx\, \mathrm{Im}\left(\partial_x \log Z_1\right) = \frac{\sqrt{\lambda}}{2\pi}\Delta\phi \qquad . \tag{3.7}$$

This Φ is a conveniently scaled version of the the opening angle $\Delta\phi$, where $\phi = \arg Z_1$ is the azimuthal angle.

For the single spike, it is Φ instead of J that is infinite, and we have

$$\begin{aligned}
\Delta - \Phi &= \frac{\sqrt{\lambda}}{2\pi}p,\\
J &= \frac{\sqrt{\lambda}}{\pi}\sin(p/2)\,.
\end{aligned} \tag{3.8}$$

3.2 Other Classical String Solutions: Finite-J solutions

Other classical solutions of the σ-model action in $AdS_5 \times S^5$ have been studied, in particular rigidly rotating solutions, such as spinning strings and pulsating strings [12,129,130,131,25, 123,132,133,134,135], and strings rigidly rotating in some direction of AdS_5 [28,101]. These string solutions are solitons of the worldsheet σ-model which correspond to semiclassical states that are high in the energy spectrum of the string, as well as very large angular momenta J.

But a lot remains to be done with respect to the solutions with finite charges. On the gauge side of the correspondence, some wrapping-interactions were seen to require corrections to the Bethe ansatz. On the string side, the finite J was seen to correspond on the light cone gauge to correspond to a finite size worldsheet: the theory is defined on a cylinder with radius proportional to the light-cone momentum, and taking the limit of infinite angular momentum corresponds to decompactifying the cylinder. Explicit calculations in a finite-size world-sheet effects to the energies of rigidly rotating strings and were seen to require an improved Bethe ansatz for the string [38, 136].

Finite-J corrections have been studied for the giant magnon solutions. The first treatment of giant magnons $AdS_5 \times S^5$ at finite J was by [82], who worked in uniform light-cone gauge (in which the worldsheet density of J is constant instead the one of Δ). They used a gauge parameter $a \in [0,1]$, and at $a=0$ (and in conformal gauge) they found the following correction to the dispersion relation:

$$\begin{aligned} \varepsilon \equiv \Delta - J &= \frac{\sqrt{\lambda}}{\pi} \sin\left(\frac{p}{2}\right) \left[1 - \frac{4}{e^2} \sin^2\left(\frac{p}{2}\right) e^{-2J/\varepsilon} + o(e^{-4J/\varepsilon})\right] \\ &= \frac{\sqrt{\lambda}}{\pi} \sin\left(\frac{p}{2}\right) \left[1 - 4\sin^2\left(\frac{p}{2}\right) e^{-2\Delta/\varepsilon} + \dots\right] . \end{aligned} \tag{3.9}$$

The gauge-dependence that seemed to exist in [82] was resolved by [97], making use of the fact that the solutions are periodic both on the worldsheet and in the azimuthal angle on the sphere to see these solutions as wound strings moving on an orbifold of S^5, in particular on $S^2/\mathbb{Z}_n$ [97, 137]. The scattering of finite-J magnons was studied in [98], through the relation to sine-Gordon theory in finite volume.

One can generalize the giant magnon solutions living on S^2 to dyonic bound states [27], and their exact solutions at any J (both in S^2 and in S^3) were studied by [96], where it was shown that they are connected by the Pohlmeyer map (to be described in the next chapter) to periodic soliton solutions of (complex) sine-Gordon theory. In [138] finite-size corrections to the dispersion relation of dyonic giant magnons were given. Other methods of determining these finite-size corrections for the dispersion relation can be used. One can use the field-theoretic Lüsher formulas [139], which depend on the world-sheet S-matrix [140]. One can also start directly from the algebraic curve description for the giant magnon [140, 141] and determine these corrections.

We will discuss the finite-J corrections to giant magnon solutions in the formalism of the algebraic curve in more detail in Chapters 5 and 7.

3.3 Zero and Non-zero modes for the Single Spike

In order to better understand the classical string theory, we have to develop further the semi-classical properties of the solitons of the theory: the giant magnons and the related giant spike. To do so we first characterize its zero and non-zero modes, focusing on the case of the single spike solution.

Zero modes

We first concentrate on the bosonic zero modes of the giant magnons and spikes, which are the variations due to changing collective co-ordinates

$$\delta_v X^i = -\frac{\partial X^i}{\partial v_0}\Big|_{v_0=0},$$

where v_0 is some modulus. The single spike solution (3.5) can be written with explicit parameters x_0 and v_0 and orientation $\vec{n}$

$$Z_1 = e^{i(x-x_0)}\left(c + i\sqrt{1-c^2}\tanh(v-v_0)\right),$$
$$\vec{X} = \vec{n}\sqrt{1-c^2}\,\mathrm{sech}(v-v_0).$$

We can then obtain the following modes:

- a rigid rotation of Z_1, δ_x:

$$\delta_x Z_1 = iZ_1\,,$$
$$\delta_x \vec{X} = 0\,;$$

- a reparametrization along v, δ_v:

$$\begin{aligned}\delta_v Z_1 &= e^{it} i\sqrt{1-c^2}\,\mathrm{sech}^2 v\,, \\ \delta_v \vec{X} &= -\vec{n}\sqrt{1-c^2}\,\mathrm{sech}\, v \tanh v\,;\end{aligned} \tag{3.10}$$

- three possible rotations of the orientation vector $\vec{n}$, δ_m:

$$\begin{aligned}\delta_m Z_1 &= 0\,, \\ \delta_m \vec{X} &= \vec{m}\sqrt{1-c^2}\,\mathrm{sech}\, v\,,\end{aligned}$$

  where $\vec{m}\cdot\vec{n} = 0$.

The reason for determining the reparametrization mode δ_v holding x fixed (and δ_x holding v fixed) instead of using one pair x,τ or u,v is that in this case we obtain a convenient linear combination of the modes, in which one is normalizable and the other is not. We can also determine δ_τ holding x fixed (and *vice versa*):

$$\begin{aligned}\delta_{\tau|x} X^i &= \gamma\delta_v X^i & \qquad \delta_{x|\tau} X^i &= \delta_x X^i - c\gamma\delta_v X^i\,, \\ \delta_{\tau|x} X^0 &= 1 & \qquad \delta_{x|\tau} X^0 &= 0\,,\end{aligned}$$

where we wrote both time and spacial components. The meaning of these two modes ($\delta_{\tau|x}$ and $\delta_{x|\tau}$) in spacetime is that at any point they are the two tangent vectors to the string: they correspond to the x,τ co-ordinate basis vectors, and will not generate physical modes, just reparametrizations. But in fact we are not studying the complete string solution, and to make a physical state we need to glue two solutions (in the same way as for giant magnons),[2] i.e., there has to be other solitons in the worldsheet, and the relative motion between them is physical. This is the reason why we keep one of these reparametrization modes. This mode together with the three modes δ_m makes a total of four zero modes.[3]

[2] We return to this question in section 3.4 below.

[3] The mode δ_v (3.10) is the analogue of (3.11) from [142] and (2.16) from [124]. In [142] this is derived from a translation of the sine-Gordon soliton.

Note that the other physical zero modes, the perpendicular rotations δ_m, are independent of u. Comparing to the giant magnon, the same modes can be found in (2.15) of [124]. These modes are independent of v, which is time boosted by c. This leads us to consider u as being the time co-ordinate for the purpose of identifying zero and non-zero modes.[4]

Non-zero modes

To determine non-zero modes, we allow fluctuations $X^i + \delta X^i$ and plug them into the equations of motion. The equations for the fluctuations then become

$$\partial_a \partial^a \delta X^i + \left(1 - 2\,\mathrm{sech}^2 v\right) \delta X^i - \left(X^j \partial_a \partial^a \delta X^j\right) X^i = 0.$$

The zero modes discussed above are solutions of this equations. To find non-zero modes, we look for solutions of the kind

$$\delta X^j = e^{ikv - i\omega u} f^j(v).$$

For the giant magnon, this problem of finding bosonic non-zero modes was solved in [124], through finding a scattering solution and analytically continuing it. Just like the background solution, the non-zero modes for the giant spike are related to the ones of the giant magnon by simply interchanging x and t. They are:

- one massless solution (i.e. $\omega^2 = k^2$):

$$\begin{aligned} \delta_r \vec{X} &= e^{ikv - i|k|u}\, \vec{n} \left(k + |k| \cos \frac{p}{2}\right) \mathrm{sech}\, v \tanh v\,, \\ \delta_r X^1 + i \delta_r X^2 &= -i\, e^{ikv - i|k|u} e^{ix} \left(k - |k| \sinh v \sinh(v + i\frac{p}{2})\right) \mathrm{sech}^2 v\,, \\ \delta_r X^1 - i \delta_r X^2 &= i\, e^{ikv - i|k|u} e^{-ix} \left(k - |k| \sinh v \sinh(v - i\frac{p}{2})\right) \mathrm{sech}^2 v\,. \end{aligned} \tag{3.12}$$

We drop this solution as it is pure gauge: at any given point (x,t), it is just a linear

---

[4] We could consider u as being the product of a boost by velocity $\frac{1}{c} > 1$

$$u = \gamma(x - c\tau) = -\frac{\tau - \frac{1}{c}x}{\sqrt{(\frac{1}{c})^2 - 1}}\,. \tag{3.11}$$

See also (4.6).

combination of the reparametrization zero modes δ_v and δ_x:[5]

$$\delta_r X^i = e^{ikv-i|k|u}\left(-(k+k\cos\frac{p}{2})\delta_v X^i + |k|\,\delta_x X^i\right).$$

- three orthogonal fluctuations, in directions $\vec{m}$ with $\vec{m}\cdot\vec{n}=0$:

$$\delta_\perp \vec{X} = e^{ikv-i\omega u}\,\vec{m}\,(k+i\tanh v)\,, \tag{3.13}$$
$$\delta_\perp X^1 = \delta X^2_\perp = 0,$$

and one parallel fluctuation, along the spike's orientation $\vec{n}$:

$$\delta_{||}\vec{X} = e^{ikv-i\omega u}\,\vec{n}\left(k+i\tanh v - \left(k+\omega\cos\frac{p}{2}\right)\operatorname{sech}^2 v\right), \tag{3.14}$$
$$\delta_{||}X^1 + i\delta_{||}X^2 = -i\,e^{ikv-i\omega u}e^{ix}\left(k\sinh v + \omega\sinh(v+i\frac{p}{2}) + i\cosh v\right)\operatorname{sech}^2 v\,,$$
$$\delta_{||}X^1 - i\delta_{||}X^2 = i\,e^{ikv-i\omega u}e^{-ix}\left(k\sinh v + \omega\sinh(v-i\frac{p}{2}) + i\cosh v\right)\operatorname{sech}^2 v\,.$$

These all have the dispersion relation for a particle of mass $m^2=1$: $\omega^2 = k^2+1$.

These modes appear massive in u,v, but they still represent an instability with respect to physical time. To see this, we write the modes in the original co-ordinates x,t, and define new variables K,W by:

$$\delta X^j = e^{ikv-i\omega u} f^j(v) = e^{iKx-iW\tau} f^j(\gamma(t-cx)). \tag{3.15}$$

We can see that the dispersion relation now is given by $W^2 = K^2-1$, that is, these modes are tachyonic (with $m^2=-1$) with respect to co-ordinates x,τ. In our gauge, $\tau = X^0$ is the target-space's time co-ordinate. Since there is no reason to exclude modes with $|K|<1$, we will have modes with imaginary W: they will not oscillate, instead they will exponentially grow or die in time.

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[5] Note that the breaking of translational symmetry on the worldsheet (discussed in section 3.3) affects only the zero modes.

Modes in AdS directions

The magnon and spike solutions live in the $\mathbb{R}\times S^5$ subspace of $AdS_5\times S^5$. Because they stay on the center of the AdS space, there are no zero modes in the AdS directions but there will be non-zero ones. These will be identical in both the giant magnon and single spike cases, and will correspond to the modes of a point particle about the center of Anti-de Sitter space.

If we write the AdS_5 part of the metric as

$$ds^2_{\text{AdS}} = -\left(\frac{1+\eta^2/4}{1-\eta^2/4}\right)^2 d\tau^2 + \frac{1}{(1-\eta^2/4)^2}d\eta_k d\eta_k\,,$$

where $k=1,2,3,4$, the modes are given by

$$\eta_k(x,t) = e^{iKx-iW\tau}f_k(K)$$

with $W^2=K^2+1$. The infinitely wound hoop also has identical AdS modes to the ones shown above.

A calculation of the single spike's fermionic fluctuations and zero modes can be found in appendix Chapter B. We will now use the results of bosonic and fermionic fluctuations (these last ones will be seen to drop out of the calculations) to determine the semiclassical corrections to the energy of the single spike.

3.4 Quantum Corrections

Corrections to what?

Having found the modes, it would be natural to use them to compute a first quantum correction, i.e. to perform 'semi-classical quantization'. For the giant magnon, this means finding quantum corrections to $\Delta-J$. The origin of this is as follows:

Frolov and Tseytlin [12] consider the 'vacuum' of the large-J sector, the point particle orbiting the sphere, which has $\Delta=J$. They add small perturbations to this, and show that $\Delta-J$ is (at leading order in $1/\sqrt{\lambda}$) the Hamiltonian of a 1+1-dimensional theory. The perpendicular fluctuations in both the sphere and AdS are non-interacting massive fields of this theory. So far this is classical. The semi-classical correction is to treat each mode of these fields as a harmonic oscillator, and their zero-point energies '$\frac{1}{2}\hbar\omega$' are corrections to $\Delta - J$. The magnon is interpreted as a 'giant perturbation' of this vacuum, tall enough to see the curvature of spacetime (and, it turns out, of high enough momentum to see that the 1+1-dimensional theory is a spin chain, with periodic dispersion relation).

Here we repeat their calculation, for the 'vacuum of the large-winding sector': the infinitely wound hoop. We find as Hamiltonian $\Delta - \Phi$, with the winding charge Φ replacing the angular momentum J. The single spike is similarly a 'giant perturbation' of this vacuum.

Recall from Section 3.1 that the flat-space versions of these two classes of spiky strings are related by T-duality, which famously exchanges winding and momentum around a compact direction. Clearly this change in the Hamiltonian is somehow a consequence of this duality. But notice that the compact direction here is part of a sphere, and that the radius of this sphere is unchanged.

Finding the Hamiltonian

Write the metric in the form[6]

$$ds^2_{\rm AdS} = -\left(\frac{1+\eta^2/4}{1-\eta^2/4}\right)^2 d\tau^2 + \frac{1}{(1-\eta^2/4)^2} d\eta_k d\eta_k \qquad k = 1,2,3,4$$

$$ds^2_{\rm S} = d\theta_1^2 + \cos^2\theta_1 \left(d\theta_2^2 + \cos^2\theta_2 \left(d\theta_2^2 + \cos^2\theta_2 \left(d\theta_3^2 + \cos^2\theta_3 \left(d\theta_4^2 + \cos^2\theta_4 d\phi^2\right)\right)\right)\right).$$

The action (in conformal gauge) is

$$S = -\frac{\sqrt{\lambda}}{2\pi}\int dx d\tau\, \mathscr{L}_B, \qquad \mathscr{L}_B = \frac{1}{2}\partial^a X^\mu \partial_a X^\nu G_{\mu\nu}. \tag{3.16}$$

[6]The azimuthal angle ϕ here is the same as used before, in (B.1), but $\theta_4 = \pi/2 - \theta$ is the elevation above the equator. The expansions of the metric components which we need are $G_{\tau\tau} = -1 - \eta^2 + \cdots$ and $G_{\theta\theta} = 1 - \theta^2 + \cdots$.

We write the perturbed the solution as $X^\mu = X^\mu_{\text{hoop}} + \tilde{X}^\mu/\lambda^{1/4}$:

$$\begin{aligned} t &= \tau + \frac{1}{\lambda^{1/4}}\tilde{t} & \phi &= x + \frac{1}{\lambda^{1/4}}\tilde{\phi} \\ \eta_k &= \frac{1}{\lambda^{1/4}}\tilde{\eta}_k & \theta_s &= \frac{1}{\lambda^{1/4}}\tilde{\theta}_s, \qquad s = 1,2,3,4. \end{aligned} \tag{3.17}$$

Expanding at large λ, the Lagrangian becomes

$$\begin{aligned} \mathscr{L}_B = 1 &+ \frac{1}{\lambda^{1/4}}\left(\partial_0\tilde{t} + \partial_1\tilde{\phi}\right) \\ &+ \frac{1}{2\sqrt{\lambda}}\left(-\partial^a\tilde{t}\partial_a\tilde{t} + \partial^a\tilde{\eta}_k\partial_a\tilde{\eta}_k + \partial^a\tilde{\phi}\partial_a\tilde{\phi} + \partial^a\tilde{\theta}_s\partial_a\tilde{\theta}_s + \tilde{\eta}_k\tilde{\eta}_k - \tilde{\theta}_s\tilde{\theta}_s\right) \\ &+ \frac{1}{\lambda^{3/4}}\left((\partial_0\tilde{t})\tilde{\eta}_k\tilde{\eta}_k - (\partial_1\tilde{\phi})\tilde{\theta}_s\tilde{\theta}_s\right) + \mathscr{O}(\frac{1}{\lambda}). \end{aligned} \tag{3.18}$$

In the quadratic piece, $\tilde{\eta}_k$ appears massive and $\tilde{\theta}_s$ tachyonic, matching what we found for the single spike's modes.

The Virasoro constraints are first $\gamma_{00} + \gamma_{11} = 2T_{00} = 0$:

$$\begin{aligned} 0 = &\frac{1}{\lambda^{1/4}}\left(-\partial_0\tilde{t} + \partial_1\tilde{\phi}\right) \\ &+ \frac{1}{2\sqrt{\lambda}}\left(-\partial_a\tilde{t}\partial_a\tilde{t} + \partial_a\tilde{\eta}_k\partial_a\tilde{\eta}_k + \partial_a\tilde{\phi}\partial_a\tilde{\phi} + \partial_a\tilde{\theta}_s\partial_a\tilde{\theta}_s - \tilde{\eta}_k\tilde{\eta}_k - \tilde{\theta}_s\tilde{\theta}_s\right) \\ &+ \mathscr{O}(\frac{1}{\lambda^{3/4}}), \end{aligned} \tag{3.19}$$

(writing $\partial_a\partial_a = \partial_0\partial_0 + \partial_1\partial_1$ in a temporary abuse of notation) and second $\gamma_{01} = T_{01} = 0$:

$$0 = \frac{1}{\lambda^{1/4}}\left(-\partial_1\tilde{t} + \partial_0\tilde{\phi}\right) + \frac{1}{2\sqrt{\lambda}}\left(-\partial_0\tilde{t}\partial_1\tilde{t} + \partial_0\tilde{\phi}\partial_1\tilde{\phi} + \partial_0\tilde{\eta}_k\partial_1\tilde{\eta}_k + \partial_0\tilde{\theta}_s\partial_1\tilde{\theta}_s\right) + \mathscr{O}(\frac{1}{\lambda^{3/4}}).$$

Now we expand the spacetime charges: the energy is the integral of the momentum density Π^0_τ:

$$\begin{aligned} \Delta &= \frac{1}{2\pi}\int dx\, \frac{\partial \mathscr{L}_B}{\partial\, \partial_0 t} \\ &= \frac{1}{2\pi}\int dx \left(\sqrt{\lambda} + \lambda^{1/4}\partial_0\tilde{t} + \tilde{\eta}_k\tilde{\eta}_k + \mathscr{O}(\frac{1}{\lambda^{1/4}})\right), \end{aligned}$$

and the winding charge defined in (3.7) is

$$\begin{aligned}\Phi &= \frac{\sqrt{\lambda}}{2\pi}\int dx\,\partial_1\phi \\ &= \frac{1}{2\pi}\int dx\left(\sqrt{\lambda}+\lambda^{1/4}\partial_1\tilde{\phi}\right).\end{aligned}$$

Subtracting these two charges, the two $\sqrt{\lambda}$ terms will cancel, leaving a finite result. The linear terms can then be replaced with quadratic terms using the first Virasoro constraint (3.19). To leading order in $1/\lambda$, we obtain:

$$\begin{aligned}\Delta-\Phi = \frac{1}{4\pi}\int dx\Big[&-(\partial_0\tilde{t}\partial_0\tilde{t}+\partial_1\tilde{\tau}\partial_1\tilde{\tau})+\left(\partial_0\tilde{\phi}\partial_0\tilde{\phi}+\partial_1\tilde{\phi}\partial_1\tilde{\phi}\right) \\ &+(\partial_0\tilde{\eta}_k\partial_0\tilde{\eta}_k+\partial_1\tilde{\eta}_k\partial_1\tilde{\eta}_k)+\left(\partial_0\tilde{\theta}_s\partial_0\tilde{\theta}_s+\partial_1\tilde{\theta}_s\partial_1\tilde{\theta}_s\right)+\tilde{\eta}_k\tilde{\eta}_k-\tilde{\theta}_s\tilde{\theta}_s\Big].\end{aligned} \tag{3.20}$$

This is the analogue of the result in [12]. The fields $\tilde{t}$ and $\tilde{\phi}$ correspond to transformations that are pure gauge, so we drop them. We can write $\Delta-\Phi$ in terms of the Hamiltonian one would obtain from only the quadratic part of the Lagrangian $\mathscr{L}_B$, which contains the transverse (physical) modes $\tilde{\eta}_k$ and $\tilde{\theta}_s$ and their conjugate momenta $\tilde{\Pi}_{\tilde{\eta}_k},\tilde{\Pi}_{\tilde{\theta}_s}$.

Quadratic 2-dimensional Hamiltonian

Starting from the Lagrangian for the fluctuations (3.18), we find its quadratic part to be (up to factors of λ):

$$\tilde{\mathscr{L}}^2 = \frac{1}{2}\left(-\partial^a\tilde{t}\partial_a\tilde{t}+\partial^a\tilde{\eta}_k\partial_a\tilde{\eta}_k+\partial^a\tilde{\phi}\partial_a\tilde{\phi}+\partial^a\tilde{\theta}_s\partial_a\tilde{\theta}_s+\tilde{\eta}_k\tilde{\eta}_k-\tilde{\theta}_s\tilde{\theta}_s\right).$$

By determining the conjugate momenta for each of the fluctuation fields, $\tilde{\Pi}_\mu = \frac{\partial\tilde{\mathscr{L}}}{\partial\left(\partial_0\tilde{X}^\mu\right)}$, we find

$$\begin{aligned}\tilde{\Pi}_{\tilde{\tau}} &= \partial_0\tilde{t} \\ \tilde{\Pi}_{\tilde{X}^\mu} &= -\partial_0\tilde{X}^\mu \quad \text{for } \tilde{X}^\mu=\tilde{\eta}_k,\tilde{\theta}_s,\tilde{\phi}.\end{aligned}$$

From these we can construct the corresponding Hamiltonian density in the usual way, obtaining

$$\tilde{\mathscr{H}}^2 = \frac{1}{2}\left(-\tilde{\Pi}_{\tilde{t}}^2 + \tilde{\Pi}_{\tilde{\phi}}^2 + \tilde{\Pi}_{\tilde{\eta}_k}^2 + \tilde{\Pi}_{\tilde{\theta}_s}^2 + \tilde{\eta}_k\tilde{\eta}_k - \tilde{\theta}_s\tilde{\theta}_s\right).$$

We want to check that the quantity $\Delta - \Phi$ is just this Hamiltonian. To do so we start by determining the Hamiltonian corresponding to the original bosonic Lagrangian (3.16), and expand it in fluctuations. The conjugate momenta for the fields are given by $\Pi_\mu = \frac{\partial \mathscr{L}}{\partial(\partial_0 X^\mu)}$ where $X^\mu = t, \phi, \eta_k, \theta_s$. To find the Hamiltonian for the fluctuations, we expand the fields as in (3.17), as well as the momenta:

$$\Pi_\mu \;=\; \Pi_\mu^{cl} + \lambda^{-\frac{1}{4}}\tilde{\Pi}_\mu \quad ; \qquad X^\mu = X^\mu_{cl} + \lambda^{-\frac{1}{4}}\tilde{X}^\mu,$$

where the classical values of the fields are $\Pi_\tau^{cl} = 1$, $\Pi_{X^\mu \neq \tau}^{cl} = 0$, $t_{cl} = \tau$, $\phi_{cl} = x$ and all other fields are zero. The expansion of the Hamiltonian then gives:

$$\begin{aligned}\mathscr{H}_b \;=\; & \frac{1}{2\sqrt{\lambda}}\left(-\tilde{\Pi}_{\tilde{t}}^2 + \tilde{\Pi}_{\tilde{\phi}}^2 + \tilde{\Pi}_{\tilde{\eta}_k}^2 + \tilde{\Pi}_{\tilde{\theta}_s}^2 - (\partial_1\tilde{t})^2 + \left(\partial_1\tilde{\phi}\right)^2 + (\partial_1\tilde{\eta}_k)^2 + \left(\partial_1\tilde{\theta}_s\right)^2\right) + \\ & + \frac{1}{2\sqrt{\lambda}}\left(\tilde{\eta}_k\tilde{\eta}_k - \tilde{\theta}_s\tilde{\theta}_s\right) - \frac{1}{\lambda^{\frac{1}{4}}}\left(\tilde{\Pi}_{\tilde{t}} - \left(\partial_1\tilde{\phi}\right)\right) + \mathscr{O}\left(\frac{1}{\lambda}\right).\end{aligned}$$

The Virasoro constraint (3.19) is equivalent to setting $\mathscr{H}_b = 0$.

It is easy to check that $\Delta - \Phi$ can be written in terms of the fields and conjugate momenta as

$$\Delta - \Phi = \frac{\sqrt{\lambda}}{2\pi}\int dx \left(\frac{1}{\lambda^{\frac{1}{4}}}\left(\tilde{\Pi}_{\tilde{t}} - \left(\partial_1\tilde{\phi}\right)\right)\right).$$

By using the Virasoro constraint in the form $\mathscr{H}_b = 0$, we finally find

$$\begin{aligned}\Delta - \Phi \;=\; & \int \frac{dx}{2\pi}\Big(-\tilde{\Pi}_{\tilde{t}}^2 + \tilde{\Pi}_{\tilde{\phi}}^2 + \tilde{\Pi}_{\tilde{\eta}_k}^2 + \tilde{\Pi}_{\tilde{\theta}_s}^2 - (\partial_1\tilde{t})^2 + \left(\partial_1\tilde{\phi}\right)^2 + (\partial_1\tilde{\eta}_k)^2 + \left(\partial_1\tilde{\theta}_s\right)^2 \\ & + \tilde{\eta}_k\tilde{\eta}_k - \tilde{\theta}_s\tilde{\theta}_s\Big),\end{aligned}$$

which returns the expected expression, when we drop the gauge fluctuations. We obtain at last:

$$\Delta - \Phi = \int \frac{dx}{2\pi} \mathscr{H}_{2d}\left(\tilde{t}, \tilde{\phi}, \tilde{\eta}_k, \tilde{\theta}_s\right)$$
$$= \sqrt{\lambda} \int \frac{dx}{4\pi} \left[\tilde{\Pi}^2_{\tilde{\eta}_k} + \tilde{\Pi}^2_{\tilde{\theta}_s} + \partial_1 \tilde{\eta}_k \partial_1 \tilde{\eta}_k + \partial_1 \tilde{\theta}_s \partial_1 \tilde{\theta}_s + \tilde{\eta}_k \tilde{\eta}_k - \tilde{\theta}_s \tilde{\theta}_s\right].$$

We are left with four massive fields from vibrations in the AdS directions and four tachyonic fields from the sphere directions. Then $\Delta - \Phi$ is the expected quadratic Hamiltonian for these 8 fields. One could perform a similar construction for the fermionic modes obtaining 16 massless fermionic fields [143, 144] [12].

First quantum correction

For each of the eight bosonic modes $\tilde{\eta}_k$ and $\tilde{\theta}_s$, we have a quadratic Hamiltonian of the kind

$$H_2 = \int dx \left[\frac{1}{2}\hat{\Pi}^2 + \hat{\phi}\left(-\partial_x^2 + V\right)\hat{\phi}\right].$$

Note that $V = \pm 1$ in our case, depending on whether the mode is massive or tachyonic. We can expand both $\hat{\Pi}$ and $\hat{\phi}$ eigenfunctions ψ_n of the differential operator $\left(-\partial_x^2 + V\right)\psi_n = \omega_n^2 \psi_n$, which we write $\hat{\phi} = \sum \hat{\phi}_n \psi_n$ and $\hat{\Pi} = \sum \hat{\Pi}_n \psi_n$. The Hamiltonian becomes a sum of decoupled harmonic oscillators

$$H_2 = \sum \frac{1}{2}\left(\hat{\Pi}_n^2 + \omega_n^2 \hat{\phi}_n^2\right).$$

By introducing creation and annihilation operators in the usual way, for each oscillator, we find that each of these contributes with $\frac{1}{2}\sum \hbar\omega_n$, with[7] $\omega_n = \sqrt{k_n^2 + m^2}$, for some mass m^2 and allowed momenta k_n.

For our solution the bosonic modes in 3.3 have $W(K) = \sqrt{K^2 \pm 1}$. Each of the fermionic modes will contribute $-\frac{1}{2}\sum \hbar W_{\text{fermi}}$, where the fermionic modes found in appendix B.1 have $W(K) = K$.

There are two important issues here:

- First, to obtain a finite first quantum correction for any solution, one must always

[7] In the literature, $\nu_n = T\omega_n$ (where T is some large time) is called a stability angle.

subtract the quantum correction for the corresponding vacuum solution. Both of these are normally UV divergent (and this subtraction is not the only renormalization usually needed). For the single spike, the relevant vacuum is the hoop solution. Note that the hoop has $\Delta - \Phi = 0$ classically, so this subtraction is only needed for the quantum corrections.

- Second, we are interested in studying those modes of the spike which result in its instability. To determine the decay time of this unstable solution, we are only interested in the imaginary part of the energy correction. None of the fermionic modes will contribute to this, as they are massless, nor will the 4 bosonic modes in AdS_5, as they are massive. The only contribution is from the 4 tachyonic modes on the sphere, which have $W(K) = \pm\sqrt{K^2-1}$, and here only from those modes with $|K| < 1$. This excludes the UV modes, and in fact no other renormalization will be needed.

Modes for the hoop (vacuum) solution

It is simple to solve the equations of motion from the bosonic Lagrangian $\mathscr{L}_B$ (3.18) in order to determine the modes for the hoop solution. The transverse modes are

$$\tilde{\eta}_k(x,\tau) = e^{iKx-iW\tau} f_k(K), \qquad W^2 = K^2+1\,,$$
$$\tilde{\theta}_s(x,\tau) = e^{iKx-iW\tau} g_s(K), \qquad W^2 = K^2-1\,,$$

i.e $m^2 = 1$ in the AdS directions, and $m^2 = -1$ on the sphere, the same masses as for the single spike's modes. The longitudinal modes are massless:

$$\tilde{\tau}(x,\tau) = e^{iKx-i|K|\tau} f(K),$$
$$\tilde{\phi}(x,\tau) = e^{iKx-i|K|\tau} g(K).$$

The same modes can also be obtained from those for the single spike, by going far away from the spike itself. The sphere modes $\delta_\perp$ (3.13) and $\delta_{||}$ (3.14) of section 3.3 become these simple ones $\tilde{\theta}_s$ in the limit $v \to \infty$, and the AdS modes are identical. The $\tilde{\phi}$ mode is the $v \to \infty$ limit of δ_r (3.12), now more obviously pure gauge. We did not write down the analogue of the $\tilde{\tau}$ mode (among the spike's non-zero modes) as we were focusing on the spatial part, but this too is pure gauge.

Performing the same limit $v \to \infty$ for the fermionic modes (B.13) and (B.15) leaves the following modes for the hoop:

$$\begin{aligned}
\Psi^1 &= \frac{i}{\sqrt{1-c}}\left[\Gamma_0\left(\cos\chi + \Gamma_{\phi\theta}\sin\chi\right) - \Gamma_\phi\left(\cos\chi - \Gamma_{\phi\theta}\sin\chi\right)\right] \\
&\qquad \times \left(\cos\beta\,\tilde{U}_0 + \sin\beta\,\Gamma_{\phi\theta}\tilde{U}_1\right), \\
\Psi^2 &= \frac{-1}{\sqrt{1+c}}\Gamma_*\Gamma_\theta\left[\Gamma_0\left(\cos\tilde{\chi} + \Gamma_{\phi\theta}\sin\tilde{\chi}\right) - \Gamma_\phi\left(\cos\tilde{\chi} - \Gamma_{\phi\theta}\sin\tilde{\chi}\right)\right] \\
&\qquad \times \frac{1}{1+4\omega^2}\left(\cos\tilde{\beta}\,U_0 + \sin\tilde{\beta}\,\Gamma_{\phi\theta}U_1\right).
\end{aligned}$$

Vacuum

The bosonic and fermionic modes for the hoop found above have the same masses as their counterparts for the single spike, in particular the sphere modes have $W(K) = \pm\sqrt{K^2-1}$. To discretize the momentum K, we put the solution in a box $-\frac{L}{2} < x < \frac{L}{2}$ and impose periodic boundary conditions $\delta X\left(-\frac{L}{2}\right) = \delta X\left(\frac{L}{2}\right)$. Then $K_n = \frac{2\pi n}{L}$, with $n \in \mathbb{Z}$, and the contribution of these modes to the vacuum energy is given by

$$\begin{aligned}
\Delta E_{\text{hoop}} &= 4\frac{1}{2}\sum_n \sqrt{K_n^2 - 1} \\
&\approx 2\frac{L}{2\pi}\int_{-1}^{1} dK\,\sqrt{K^2-1} \quad \text{as } L \to \infty \\
&= \frac{i}{2}L.
\end{aligned} \tag{3.21}$$

The integration is over $|K| < 1$ because we are looking for just the imaginary part. We do not encounter a UV divergence here.

Spike solution

Again we study only the bosonic modes on the sphere with $|K| < 1$. But the discrete momenta K allowed for the spike are not the same as those for the hoop K_n, as the modes have a phase shift at large x compared to the hoop. Looking at the bosonic sphere modes given in

(3.13) and (3.14), far away from the spike ($|v| \gg 1$) we have

$$\begin{aligned}\delta_{\perp}\vec{X}(x) &= e^{iKx-i\sqrt{K^2-1}\tau}\vec{m}\left[\gamma(cK-W)+i\tanh\left(\gamma(\tau-cx)\right)\right], \\ \delta_{||}\vec{X}(x) &= e^{iKx-i\sqrt{K^2-1}\tau}\vec{n}\left[\gamma(cK-W)+i\tanh\left(\gamma(\tau-cx)\right)\right],\end{aligned} \tag{3.22}$$

and $\delta X^1 = \delta X^2 = 0$ for both.[8] Fixing $t=0$ and evaluating at large distance $x = \pm\frac{L}{2}$, they both become

$$\delta\vec{X}\left(\pm\frac{L}{2}\right) = e^{\pm iK\frac{L}{2}\pm i\delta_{\pm}}A_{\pm},$$

where the phase shifts and amplitudes at the two ends are given by

$$\begin{aligned}\tan(\delta_{\pm}) &= \frac{-1\mp\gamma\sqrt{1-K^2}}{\gamma cK}, \\ A_{\pm} &= \sqrt{(\gamma cK)^2+\left(\gamma\sqrt{1-K^2}\pm 1\right)^2}.\end{aligned} \tag{3.23}$$

The next step would be to impose periodic boundary conditions on δX at $x=\pm\frac{L}{2}$. But here we encounter a problem, as the modes have different amplitudes at the two ends.[9] Instead we will demand only that the phases match at $x = \pm\frac{L}{2}$, and allow the amplitudes to be different. (We will discuss this further in the next section.) Then K has to obey

$$KL + \delta_{+}(K) + \delta_{-}(K) = K_n L,$$

where $K_n = \frac{2\pi n}{L}$ is still the discretized momentum of the vacuum solution. Taking L very

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[8] To obtain this, note that K, W and k, ω are related by $K = -\gamma\left(ck+\sqrt{k^2-1}\right)$ and $W = -\gamma\left(k+c\sqrt{k^2-1}\right)$, from (3.15) and (3.4).

[9] Recall that the worldsheet velocity of the single spike is $1/c > 1$. Thus $(x,\tau) = (\pm L/2, 0)$ might be better thought of as points before and after the spike, rather than left and right of it. Consider instead points (x,τ) with large $|\tau|$, for which both of the modes $\delta_{\perp}$ and $\delta_{||}$ in (3.22) become

$$\begin{aligned}\delta X &= e^{iKx-iW\tau}\left(\gamma(cK-W)+i\,\mathrm{sign}(\tau)\right) \\ &= e^{iKx+\sqrt{1-K^2}\tau}\left(\gamma cK - i\gamma\sqrt{1-K^2}+i\,\mathrm{sign}(\tau)\right).\end{aligned}$$

In the second line we've chosen to focus on the growing mode $W = +i\sqrt{1-K^2}$. Averaging over x by taking the modulus, we get

$$|\delta X| = e^{\sqrt{1-K^2}\tau}\sqrt{(\gamma cK)^2+\left(\mathrm{sign}(\tau)-\gamma\sqrt{1-K^2}\right)^2}.$$

This is an exponentially growing mode, but with a step in it where the spike happens.

large we can approximate K by

$$K = K_n - \frac{1}{L}\delta\,(K_n) + \mathcal{O}(\frac{1}{L^2})$$

where $\delta(K) \equiv \delta_+(K) + \delta_-(K)$. Finally we can determine the imaginary correction to the energy of the spike from the four tachyonic modes, by putting $L \to \infty$:

$$\begin{aligned} \Delta E_{\text{spike}} &= 4\sum_K \frac{1}{2} W\,(K) \\ &\approx 4\frac{L}{2\pi}\int_{-1}^{1} dK\,\frac{1}{2}W\left(K - \frac{1}{L}\delta(K)\right) \quad \text{as } L \to \infty \\ &= \Delta E_{\text{hoop}} - i2\sqrt{\frac{1-c}{1+c}}. \end{aligned} \tag{3.24}$$

In the expression above, $\Delta E_{\text{hoop}} = iL/2$ is the correction (3.21) to the vacuum solution. Thus in the difference $\Delta E_{\text{spike}} - \Delta E_{\text{hoop}}$ the IR divergence from $L \to \infty$ is canceled.

About these boundary conditions

We found that the amplitude of the mode (3.22) (for $|K| < 1$) is different at large positive and negative x. This is the obstruction to imposing periodic boundary conditions, which we avoided by matching only the phases. One should not be surprised that we cannot impose these boundary conditions. They amount to gluing the string to itself after some large number of windings, or rather, gluing the vibrations on it to themselves, and this might not be allowed.

For the giant magnon, one has to glue a series of magnons together with $\sum_i p_i = 0$ to obtain a valid closed string solution. But is not clear that this is a condition on the allowed series of single spikes. It would tell you about periodicity of the spatial $X^i(x,\tau)$ under t, but say nothing about their behaviour at large $|x|$.

Here we consider a solution of two widely separated spikes with opposite velocities $\frac{1}{c}$ and $-\frac{1}{c}$, because for this choice we can impose honest boundary conditions. In this case we recover the twice the energy correction (3.24) obtained above, one for each spike. This justifies our use of these unusual boundary conditions.

Two spikes

As $x \to \pm\frac{L}{2}$, the amplitude of the mode (3.22) becomes $A_\pm$, given in (3.23). This formula is valid for $c > 0$; for $c < 0$ the sign $\pm$ is reversed, and we have instead $\left|\delta X_{c<0}(\pm\frac{L}{2}, 0)\right| = A_\mp$.

This immediately suggests the following way to impose consistent boundary conditions: take two spikes, far apart, with parameters c and $-c$. Each is in a box of length L, and we connect these together. That is, consider

$$X^\mu(x,\tau) = \begin{cases} X_{\text{spike}(c)}\left(x - \frac{L}{2}, \tau\right) & \text{for} \quad 0 < x < L\,, \\ X_{\text{spike}(-c)}\left(x - \frac{3L}{2}, \tau\right) & \quad L < x < 2L \end{cases}$$

which is an approximate solution near $t = 0$. In fact it is a part of a scattering solution, since the two spikes have velocities $1/c$ and $-1/c$. It can be viewed as an excitation above a hoop of length $2L$.

Vibrations of this solution will be described by the same modes we have been using, and we again focus on the $|K| < 1$ sphere modes, which give the imaginary energy correction. For the boundary condition at $x = L$, both modes δX have amplitude A_+, so matching them sets their phases equal there. And at $x = 0, 2L$ we can impose periodic boundary conditions, since both modes have amplitude A_- there. The resulting condition on the allowed K is simply

$$K = K_n - \frac{1}{2L}\delta_{(c)}(K_n) - \frac{1}{2L}\delta_{(-c)}(K_n) + \mathscr{O}(\frac{1}{L^2})\,,$$

where $K_n = \frac{2\pi n}{2L}$ are now the allowed wave numbers for the vacuum in length $2L$. This leads to energy correction

$$\Delta E = \Delta E_{\text{spike}(c)} + \Delta E_{\text{spike}(-c)}\,,$$

i.e. we obtain the sum of the corrections we calculated in (3.24) by imposing our phase-only boundary condition at $x = \pm\frac{L}{2}$. The finite piece (after subtracting the vacuum's ΔE_{hoop}) is twice the finite piece for one spike.

3.5 Properties and semi-classical results for the single spike solution

We now present a summary of all of the properties and results found in this chapter for the single spike solution, and compare them to the known results of the giant magnon.

After determining the bosonic and fermionic modes of the single spike solution, we found a mismatch between the modes in these two sectors, both in number and in their masses. This is evidence that the spike solution is not supersymmetric. Some of the bosonic modes found were seen to be tachyonic, showing that the single spike is unstable, just like the relevant 'vacuum' solution which we referred to as the hoop.

In order to perform a semi-classical analysis of the spike, we started by determining the Hamiltonian for small fluctuations of the corresponding vacuum (the hoop), which was seen to be just $\Delta - \Phi$. In this case, the winding Φ has replaced the angular momentum J found in the Hamiltonian for the magnon case. This was expected because we saw that in flat space T-duality related similar solutions. We then used this result to calculate a semi-classical estimate of the lifetime of the single spike solution.

Another subject of interest is the comparison of the single spike solution to some operator on the Super Yang-Mills side. It had been conjectured that the single spike is dual to an excitation of an anti-ferromagnetic spin chain. [48, 113] There have been various attempts to find a full N-body description of the giant magnon, such as the Hubbard [145] and Ruijsenaars-Schneider models [51]. It is possible that the single spike solution will be another test case for such a description.

The periodicity in the parameter p of the dispersion relation for giant magnons (3.6) is the signature of discrete space. On the SYM side, this can be understood to be the position along a spin chain. Even though on the case of the single spike it seems that such periodicity is nonexistent (3.8), that shouldn't count as evidence against such discreteness. In [104], one allows p outside our range $0 < p < 2\pi$, and finds that $\Delta - \Phi$ becomes periodic (their figure 1). However the meaning of this parameter p is not well understood for the spike.

The single spike is an excitation of an unstable vacuum state, the hoop, which consists of a string wrapped around an equator of S^5. One can stabilize such loops of strings by making them rotate in other planes [25,133]. Such solutions carry large angular momentum by being wound many times around one equator. It is possible that adding these extra angular momenta may stabilize the spike solution as well, and it may be this object which has a more natural gauge theory dual.

CHAPTER 4

SCATTERING OF N MAGNONS

Recent progress in understanding the Gauge/String duality in $N = 4$ Super Yang-Mills theory resulted in complete specification of the worldsheet S-matrix and the associated spectrum [20, 92, 36, 93, 22, 17, 146, 55, 65, 39, 61, 147, 148]. The conjectured exact result received impressive confirmation in both weak coupling Yang-Mills theory calculations and also semi-classical string theory calculations at strong coupling [12, 25, 123, 91, 149, 89, 41, 124, 125, 35]. These successes were accomplished due to the integrability property characterizing the string dynamics and present in Yang-Mills theory through its spin-chain representation [20, 10, 31, 34, 29, 150]. At the spectrum level there is a complete classification of states in terms of magnon excitations. Their dispersion formula is again known from both weak and strong coupling studies [56, 26].

Even though all orders results have been accomplished, further study of the models and of their integrability structures is still desirable. For instance the spin chain Hamiltonian is reliably known only from weak coupling calculations, its comparison (and agreement) with the string theory Hamiltonian is to some degree purely accidental. In this chapter we will pursue the question of multi-magnon dynamics. Magnons scatter from each other with known computable phase shifts [36, 41, 151, 57, 152, 59, 153, 104] and it is of relevance to determine their interactions. We will do that in the simplest case of magnons moving on $\mathbb{R} \times S^2$ working at the the semiclassical level. The equations of motion in this case (in a timelike-conformal gauge) coincide with those of the $O(3)$ nonlinear sigma model. Multi-magnon solutions have been constructed in these case using several different techniques, involving the dressing and the inverse scattering methods [151, 127, 154, 109, 155].

One particular approach (Pohlmeyer reduction method) reduces the problem at the equation of motion level to a well known integrable field theory, the sine-Gordon model [49]. In this reduction the role of magnons is played by sine-Gordon solitons. Much is known about inter-soliton dynamics in the sine-Gordon theory [119]. In particular it can be exactly described through an N-body model generalizing the Calogero-Moser model [156, 157]. The relativistic Ruijsenaars-Schneider model [158] is completely integrable, it summarizes the N-soliton (and anti-soliton) dynamics for a given coupling and can be directly deduced from sine-Gordon theory itself [159]. In turn it can be used as a full dynamical theory, even at the quantum level [160]. It is our goal to establish a related dynamical description for string theory magnons.

The connection between string dynamics and sine-Gordon theory, is known to be highly nontrivial. The two theories coincide at the level of equations of motion, but that is where the comparison stops [26]. Physical quantities like the energies (of magnons and solitons) and the associated phase shifts are different and it is our intention to clarify somewhat this nontrivial relationship. The nontrivial dynamical connection between the two systems can be traced back to a (nonabelian) dual description of sigma models and the fact that it is in the dual formulation that the connection can be described in canonical terms. This was established in several works by Mikhailov [161, 50, 128] and remains to be pursued at the quantum level.

For the question of formulating the dynamical system describing multi-magnon dynamics we start from the fact that at the level of equations of motion it coincides with the soliton or rather the N-body RS model. We then require a further fact, namely that the string theory model ought to reproduce the correct magnon energies and the phase shifts, both of which differ from the soliton case. From the comparison of energies we suggest a Hamiltonian, as the $n = -1$ member of the infinite Hamiltonian sequence [162, 163, 164]. Requiring the correct phase shift we are led to a nontrivial Poisson structure representing the N- magnon dynamics.

In this chapter we will start by giving a brief summary of the relation between magnon solutions in $\mathbb{R} \times S^2$ and sine-Gordon solitons. In Section 4.2 we review the integrable dynamics of solitons in terms of the N-particle R-S description. In Section 4.3 we consider the analogous representation for magnons. From comparison of energy eigenvalues we are led to an N-body Hamiltonian given by the inverse of the lax matrix of the RS model. Elaborating on the phase shift we are led to suggest a need for an alternative symplectic form. This symplectic form is explicitly given in the limit of well separated magnons in Section 4.4.

4.1 From Classical Strings to sine-Gordon: Pohlmeyer Map

It is well-known that the theory of classical strings moving on $\mathbb{R}\times S^2$ is related to the sine-Gordon model at the level of the equations of motion. In fact, the original Pohlmeyer map [49] related the $O(3)$ σ-model co-ordinates to the sine-Gordon theory through a projection map. Let us summarize this map: start from the $O(3)$ σ-model Lagrangian density

$$\mathscr{L}(\sigma,\tau)=\frac{1}{2}\sum_{i=1}^{3}\left(\partial_\sigma X_i\,\partial_\sigma X_i-\partial_\tau X_i\,\partial_\tau X_i\right)+\frac{\lambda}{2}\left(\sum_{i=1}^{3}X_iX_i-1\right),$$

where X_i are the embedding co-ordinates on $\mathbb{R}^3$ and λ is a Lagrange multiplier that fixes the motion to be on an S^2 sphere. This Lagrangian is invariant under the action of the internal symmetry group $O(3)$. To simplify notation, define for a vector of $\mathbb{R}^3$ the following

$$p=(p_1,p_2,p_3)\quad,\quad (p,q)=\sum_{i=1}^{3}p_iq_i\quad,\quad p^2=|p|^2=\sum_{i=1}^{3}p_i^2.$$

Also we will work with light-cone co-ordinates

$$x_\pm=\frac{1}{2}(\sigma\pm\tau)\ ,\quad \partial_\pm=\partial_\sigma\pm\partial_\tau\,.$$

In these co-ordinates we have the identity $\partial_+\partial_-=\partial_\sigma^2-\partial_\tau^2$ (where we defined $\partial_\sigma\equiv\partial/\partial\sigma$, etc), and the equations of motion can be written as

$$\partial_+\partial_-X_i+(\partial_+X\cdot\partial_-X)\,X_i\ =\ 0\,, \tag{4.1}$$

with the constraints

$$X \cdot X = 1 \quad , \qquad \lambda = -\partial_+ X \cdot \partial_- X \, .$$

The sum and the difference of the energy and momentum densities are just given by $\frac{1}{2}(\partial_- X)^2$ and $\frac{1}{2}(\partial_+ X)^2$, respectively. Recalling that we have the constraint $X \cdot X = 1$, then we find

$$X \cdot \partial_\pm X = 0,$$

so $\partial_\pm X$ are vectors orthogonal to X. From the equations of motion (4.1), we can also see that $\partial_+ \partial_- X$ is parallel to X, and consequently orthogonal to $\partial_\pm X$. From these results we can derive the equations of conservation of energy-momentum, which are

$$\partial_+ \left[\frac{1}{2} (\partial_- X)^2 \right] = \partial_- \left[\frac{1}{2} (\partial_+ X)^2 \right] = 0.$$

These equations imply that

$$(\partial_+ X)^2 = h^2 (x_+) \ , \qquad (\partial_- X)^2 = k^2 (x_-) \ ,$$

where the functions h, k are determined completely by initial conditions. Looking back at the equations of motion (4.1), one can easily see that they are invariant under local scale transformations $(x_+, x_-) \to (x_+^{'}, x_-^{'})$ such that

$$dx_+^{'} = |H(x_+)| \, dx_+ \, , \quad dx_-^{'} = |K(x_-)| \, dx_-,$$

where H, K are non-vanishing functions. Choosing, without loss of generality, $|H| = |h|$ and $|K| = |k|$, it is easy to see that $\left(\frac{\partial X}{\partial x_+^{'}}\right)^2 = 1 = \left(\frac{\partial X}{\partial x_-^{'}}\right)^2$, and so the energy density is constant (1/2) while the momentum density vanishes in these new co-ordinates. We can then start from these "normalized" co-ordinates, and drop the primes. Re-writing the results we have so far, in the "normalized" co-ordinates the equations of motion (4.1) are still valid, and we have the following constraints

$$X^2 = 1 \, , \quad (\partial_+ X)^2 = (\partial_- X)^2 = 1 \, , \quad (\partial_\pm X \cdot X) = 0 \, . \tag{4.2}$$

Only one of the $O(3)$-invariant quantities formed by the X_i's (and their derivatives) is undetermined: $\partial_+ X \cdot \partial_- X$. This quantity obeys $-1 \leq \partial_+ X \cdot \partial_- X \leq 1$[1], so we define a field $\alpha(x_+, x_-)$ such that

$$\cos\alpha = (\partial_+ X \cdot \partial_- X)\,. \tag{4.3}$$

Summarizing, in the $O(3)$ model the vectors $X, \partial_+ X, \partial_- X$ span the whole $\mathbb{R}^3$ space, and any solutions of the equations of motion (4.1) obey the constraints (4.2). The converse is also true, that is, if we have a vector $X \in \mathbb{R}^3$ such that it obeys these constraints, it also obeys the equations of motion.

Because the 3 vectors $X, \partial_+ X, \partial_- X$ span the whole $\mathbb{R}^3$ space, we can write any other vector as a linear combination of these, in particular[2]

$$\begin{aligned}
\partial_+^2 X &= -X + (\partial_+\alpha)\cot\alpha\,\partial_+ X - \frac{(\partial_+\alpha)}{\sin\alpha}\,\partial_- X\,, \\
\partial_-^2 X &= -X - \frac{(\partial_-\alpha)}{\sin\alpha}\,\partial_+ X + (\partial_-\alpha)\cot\alpha\,\partial_- X\,.
\end{aligned}$$

These results allow us to write, from the definition of α and using the constraints (4.2),

$$\begin{aligned}
\partial_+\partial_-\alpha &= -\partial_-\left[\frac{\partial_+^2 X \cdot \partial_- X}{\sin\alpha}\right] \\
&= -\frac{1}{\sin\alpha}\left[\partial_+^2 X \cdot \partial_-^2 X + \partial_-\partial_+^2 X \cdot \partial_- X - (\partial_-\alpha)\cot\alpha\,\partial_+^2 X \cdot \partial_- X\right] \\
&= -\sin\alpha\,.
\end{aligned}$$

[1]This happens because if we define $\vec{a} = \partial_+ X$ and $\vec{b} = \partial_- X$, we have $|\vec{a}| = \left|\vec{b}\right| = 1$, and

$$0 \leq \left(\vec{a} + \vec{b}\right)^2 = |\vec{a}|^2 + \left|\vec{b}\right|^2 + 2\vec{a}\cdot\vec{b} \leq \left(|\vec{a}| + \left|\vec{b}\right|\right)^2 \Rightarrow -1 \leq \vec{a}\cdot\vec{b} \leq 1\,.$$

[2]To determine the coefficients of the linear combinations we write a general expansion

$$\partial_+^2 X = \phi X + \beta\partial_+ X + \gamma\partial_- X\,,$$

and then determine the projections of $\partial_+^2 X$ on each of the basis vectors

$$\partial_+^2 X \cdot X = \phi = -1\,, \quad \partial_+^2 X \cdot \partial_+ X = \beta + \gamma\cos\alpha = 0\,, \quad \partial_+^2 X \cdot \partial_- X = \beta\cos\alpha + \gamma = \partial_+(\cos\alpha).$$

These equations allow us to solve for the coefficients ϕ, β, γ. The same can be done for $\partial_-^2 X$.

To obtain the last line, we used the following identities (using (4.1) and (4.2))

$$
\begin{aligned}
\partial_+^2 X \cdot \partial_-^2 X &= 1-(\partial_+\alpha)(\partial_-\alpha)\cos\alpha\,; \\
\cot\alpha\,\partial_+^2 X \cdot \partial_- X &= -(\partial_+\alpha)\cos\alpha\,; \\
\partial_-\partial_+^2 X \cdot \partial_- X &= \partial_+\left(\partial_+\partial_- X\right)\cdot\partial_- X = -\cos^2\alpha\,.
\end{aligned}
$$

So the field $\alpha = \arccos\left(\partial_+ X \cdot \partial_- X\right)$ obeys the sine-Gordon equation $\partial_-\partial_+\alpha = -\sin\alpha$, which can be derived from the Lagrangian density

$$
\mathscr{L} = -\frac{1}{2}(\partial_\tau\alpha)^2 + \frac{1}{2}(\partial_x\alpha)^2 - U(\alpha) = \frac{1}{2}(\partial_+\alpha)(\partial_-\alpha) - U(\alpha),
$$

(and thus the Hamiltonian is $\mathscr{H} = \frac{1}{2}(\partial_\tau\alpha)^2 + \frac{1}{2}(\partial_x\alpha)^2 + U(\alpha)$, in our sign convention), with the potential

$$
U(\alpha) = 1-\cos\alpha = 2\sin^2\left(\frac{\alpha}{2}\right).
$$

Starting from any solution of the sine-Gordon theory, there is always a corresponding solution of the equations of motion (4.1), with constraints (4.2) and which obeys the map $\partial_+ X\cdot\partial_- X = \cos\alpha$. But because this is a projective map, if we have different solutions of the $O(3)$ σ-model that have the same projection $\partial_+ X \cdot \partial_- X$, these solutions will be mapped to the same sine-Gordon field.

Giant magnons and giant spike as sine-Gordon solitons

Our interest is the study of classical rigidly rotating string solution of $\mathbb{R}\times S^2$ such as the giant magnon and the giant spike. For a classical solution in conformal gauge with $Y^0 = t$, one has the Pohlmeyer [49] identification of a scalar field $\phi(x,t)$as

$$
\cos 2\phi = \partial_\tau X^i \partial_\tau X^i - \partial_x X^i \partial_x X^i \tag{4.4}
$$

obeying the sine-Gordon equation

$$
-\partial_\tau\partial_\tau\phi + \partial_x\partial_x\phi = \frac{1}{2}\sin 2\phi\,.
$$

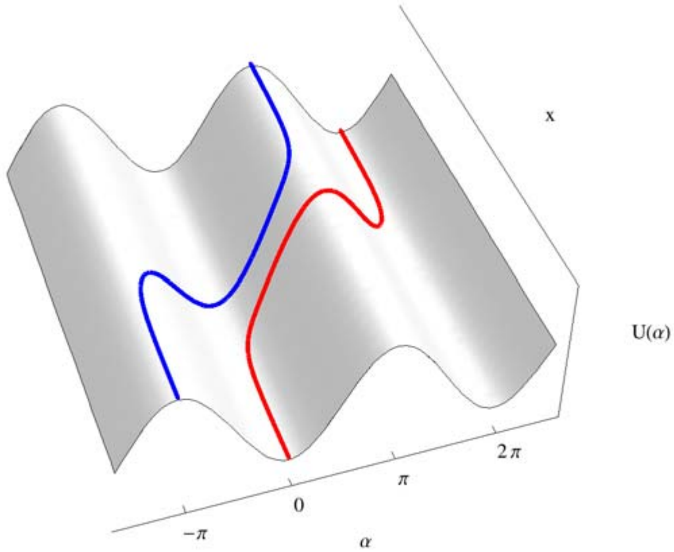

Figure 4.1: Under the Pohlmeyer map, the magnon is sent to the ordinary kink (in red) while the single spike is mapped to an unstable solution connecting the hilltops (in blue). The sine-Gordon field α is plotted left-to-right, x into the page, and $U(\alpha)$ vertically.

Comparing to the notation above we have the identification $\alpha = -2\phi$.

Under this map, the point particle is mapped to the vacuum $\alpha = 0$, with zero energy, while the giant magnon is mapped to the simple kink [26]

$$\alpha = 4 \arctan\left(e^{\gamma(x - c\tau)}\right),$$

connecting $\alpha = 0$ and $\alpha = 2\pi$ at $x = \pm\infty$ (note that α is defined up to $\bmod(2\pi)$). Its energy is ($\hat{\theta}$ is the asymptotic rapidity of the sine-Gordon soliton)

$$\varepsilon_{s.g} = \gamma = \frac{1}{\sin(p/2)} = \cosh\hat{\theta}\,. \tag{4.5}$$

The velocity c can be changed by boosting the kink, and the energy $E_{\text{s.g.}}$ changes as one would expect for a relativistic object.[3]

The comparison between sine-Gordon model and the classical string theory solutions was seen also for another type of "dual" solutions called single spike solutions (see for example

[3]However, giant magnons of different c are not related by worldsheet boosts (which are just reparametrizations) since $X^0 = t$ is held fixed.

the last Chapter and [104, 48, 165]). The single spike (3.5) is mapped instead to an unstable kink. From the map (4.4) it is clear that the effect of the $x \leftrightarrow t$ interchange is to shift the field by π:

$$\alpha(x,t) = \alpha_{\text{magnon}}(\tau,x) - \pi = 4\arctan\left(e^{\gamma(\tau - cx)}\right) - \pi\,.$$

This solution connects two adjacent maxima of $U(\alpha)$, rather than two minima: $\alpha = \pm\pi$ at $x = \pm\infty$, and is drawn in figure 4.1. If we choose the constant in $U(\alpha)$ to place these maxima at zero

$$U(\alpha) = -1 - \cos\alpha = -2\sin^2\left(\frac{\alpha+\pi}{2}\right),$$

then this unstable kink solution has energy

$$\varepsilon^{spike}_{\text{s.g.}} = c\gamma = \frac{\cos(p/2)}{\sin(p/2)} = \frac{1}{\sqrt{(\frac{1}{c})^2 - 1}}. \tag{4.6}$$

This direct connection between the two theories, sine-Gordon model and classical strings, holds only at the level of equations of motion [50], and it is non-trivial at the canonical level. Several physical properties are different, in particular the energies and the semiclassical phase shifts [26]. In fact the energies shown above in (3.6) (introduced in the previous Chapter) and (4.5) exhibit an inverse relationship

$$E_{magnon} = \frac{\sqrt{\lambda}}{\pi}\frac{1}{\gamma} = \frac{\sqrt{\lambda}}{\pi}\frac{1}{\varepsilon_{s.g}}\,.$$

That is, the energy coming from the sine-Gordon model's Hamiltonian is inverse to the spin-chain energy for magnons constructed out of target space charges $\Delta - J = \frac{\sqrt{\lambda}}{2}\sin(p/2)$.

This relation can be generalized further for the scattering solution of two magnons. The two-magnon scattering state, obtained via dressing method [109] (see figure 4.2), can can be mapped through Pohlmeyer's reduction to the scattering solutions of two solitons, whose scattering solution can found in [119]. It can be seen that the energy of the two-magnon scattering solution is related to the energy of each of the solitons in the following way

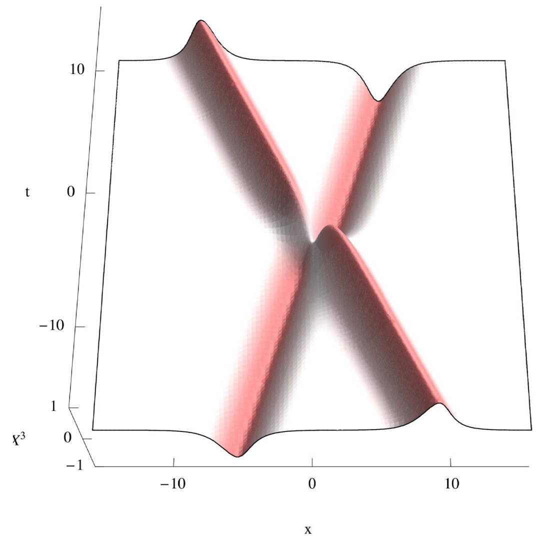

Figure 4.2: Scattering of two magnons. At earlier time the two magnons are separated in the worldsheet direction x. They cross each other, and retain the same shape while moving in opposite directions in the worldsheet, but with a time delay. In this figure $X_3 = \cos\theta$ is the height above the equator.

$$E_{2-mag} = \frac{1}{\varepsilon_{s.g,1}} + \frac{1}{\varepsilon_{s.g,2}}.$$

The scattering phase for two magnons is calculated in a very similar way to the scattering of two sine-Gordon solitons [166, 118]. See also [119] for a review on the classical and semiclassical behavior of sine-Gordon solitons. This is not surprising due to the equivalence of the two classical models through Pohlmeyer's map [49] (see also [161, 128]). The time-delay and phase shift of scattering of magnons was also studied through Bethe Ansatz techniques [41, 36, 151, 152, 153].

Since the string and the sine-Gordon equations share a common time t, it obviously follows that the time-delay of scattering of giant magnons (on the string worldsheet) and the time delay for the analogous scattering problem of solitons (in sine-Gordon theory) is the same

$$\Delta t_{sg} = \Delta t_{mag} = \frac{2}{m \sinh\theta} \ln\tanh\theta = \Delta\tau,$$

It does not mean however that the scattering phase shifts of the two problems are the same. In fact they differ, due to the difference in energies stated above. One has the well known relation, where the derivative of the phase shift with respect to energy equals the time delay, in the present case :

$$\frac{\partial \delta_{s.g}}{\partial \varepsilon_{s.g}} = \Delta\tau = \frac{\partial \delta_{mag}}{\partial E_{mag}} \quad \Rightarrow \quad \delta_{sg} \neq \delta_{mag}.$$

This will imply that a different interaction is responsible for the behaviour in the two cases.

4.2 Review of sine-Gordon Dynamics

The dynamics of sine-Gordon solitons can be summarized by a relativistic $N-$body model due to Ruijsenaars and Schneider. These class of models [158, 160] represent a relativistic generalization of the Calogero-Moser models [157, 156]. The relation between the field theoretic system of sine-Gordon solitons and the Lax matrix formulation of the Ruijsenaars-Schneider model was also thoroughly discussed in [159]. In this section we will review some of the aspects of this relation and give a summary of the needed notation. For more details and derivations the reader is directed to the original references.

For establishing the N-body description of soliton dynamics one starts with the N-soliton solution, written as:

$$e^{-i\phi} = \frac{\det(1+A)}{\det(1-A)}.$$

where A is a $N \times N$ matrix with components

$$A_{ij} = 2\frac{\sqrt{\mu_i \mu_j}}{\mu_i + \mu_j}\sqrt{X_i X_j}.$$

The μ_i are the rapidities, and the $X_i = a_i e^{2\left(\mu_i z_+ + \mu_i^{-1} z_-\right)}$ are related to the positions of the soliton through the a_i. Here we use light-cone co-ordinates $z_\pm = x \pm t$, and $\partial_\pm = \frac{1}{2}(\partial_x \pm \partial_t)$. Note that for a soliton or anti-soliton, μ is real and a pure imaginary. The breather solution corresponds to a pair of complex conjugated rapidities $(\mu, \overline{\mu})$ and positions $(a, -\overline{a})$.

The sine-Gordon equation can be described by a Hamiltonian system with the canonical symplectic form (π is the conjugate momentum to the s.G field ϕ)

$$\Omega_{sg} = \int \pi \wedge d\phi.$$

This, by direct substitution can be used to deduce the symplectic form of the soliton variables (a_i, μ_i), which can be seen to reduce to the usual symplectic form after a change of variables.

Considering the evolution of the system in terms of the null plane time $z_+ = \tau$ one has

$$A(\sigma,\tau) = e^{\sigma\mu^{-1}}\tilde{A}(\tau)\,e^{\sigma\mu^{-1}},$$

where $[\mu]_{ij} = \mu_{ij} = \mu_i\delta_{ij}$ is the matrix of rapidities, $\tilde{X}_i = a_i e^{\mu_i\tau}$ are the soliton co-ordinates. The matrix (co-ordinate)

$$\tilde{A}(\tau) = 2\frac{\sqrt{\mu_i\mu_j}}{\mu_i+\mu_j}\sqrt{\tilde{X}_i\tilde{X}_j}.$$

is then used to reconstruct the Lax matrix of the N-body system. Through diagonalization one has:

$$\begin{aligned} Q &= U^{-1}\tilde{A}U, \\ L &= U^{-1}\mu U. \end{aligned}$$

where $Q = \mathrm{diag}(Q_1,\cdots,Q_N)$ are the eigenvalues of $\tilde{A}$. The N-soliton solution is then written as $e^{-i\phi} = \prod_{i=1}^{N}\frac{1+Q_i}{1-Q_i}$, and the matrix L is the Lax operator, as its time evolution is given by a (Lax) equation

$$\dot{L} \equiv \frac{dL}{d\tau} = [M,L], \quad M = \dot{U}U^{-1}.$$

Consequently, the quantities $H_n = \mathrm{Tr}(L^n) = \sum_{i=1}^{N}\mu_i^n$ are conserved through the evolution of solitons.

Finally, if we define $\rho_i = \dot{Q}_i/Q_i$, and perform the change of variables $(\mu_i, a_i) \to (Q_i, \rho_i)$, which is a symplectic transformation, we find the Lax matrix to have the same form as the original $\tilde{A}$:

$$L = 2\frac{\sqrt{Q_iQ_j}}{Q_i+Q_j}\sqrt{\rho_i\rho_j}. \tag{4.7}$$

The Poisson brackets of these two variables Q_i, ρ_i are not canonical, so it is convenient to introduce a new set of variables θ_i conjugated to the variables q_i, given by

$$\rho_i = e^{\theta_i}\prod_{k\neq i}\frac{Q_i+Q_k}{Q_k-Q_i},\ Q_j = ie^{iq_j}.$$

In these new variables, the original symplectic form $\int \pi \wedge d\phi$ is simply given by the usual $\int \theta_i dq_i$, which corresponds to the canonical Poisson brackets. From the sequence of conserved quantities, or Hamiltonians, $H_n = \mathrm{Tr}\left(L^n\right)$ we are interested in particular in the $H_{\pm 1}$, which are the generators of the evolution in the light cone co-ordinates $\tau = z_+$ and $\sigma = z_-$. These are given by

$$H_{\pm 1} = \mathrm{Tr}\left(L^{\pm 1}\right) = \sum_j e^{\pm\theta_j} \prod_{k\neq j} \left| \coth\left(\frac{q_j - q_k}{2}\right)\right|.$$

The full Hamiltonian is given by $H = \frac{1}{2}\left(H_{+1} + H_{-1}\right) = \mathrm{Tr}\mathscr{L}_{rs}$, where we define

$$\mathscr{L}_{rs} = \frac{1}{2}\left(L + L^{-1}\right). \tag{4.8}$$

This system corresponds to a particular case of the $N-$particle relativistic Ruijsenaars-Schneider model [158]. The Lax matrix for the general case of RS model was constructed in [158]. This Lax matrix is defined by

$$L_j = V_i C_{ij} V_j, \tag{4.9}$$

where

$$V_i \equiv e^{\frac{1}{2}\theta_i}\left(\prod_{k\neq i} f\left(q_i - q_k\right)\right)^{1/2},$$

and $C_{ij}\left(q\right)$ is directly related to the choice of $f\left(q\right)$. For a family of interaction potentials of the type given below

$$f\left(q\right) = \left[1 + \alpha/\sinh^2\left(\frac{\mu q}{2}\right)\right]^{1/2}, \quad \mu, \alpha \in \left(0, \infty\right), \tag{4.10}$$

the components C_{ij} are just given by $C_{ij}\left(q\right) = \left[\cosh\left(\frac{\mu q}{2}\right) + ia\sinh\left(\frac{\mu q}{2}\right)\right]^{-1}$, with $\left(1+a^2\right)^{-1} = \alpha^2$.

The model has an infinite set of commuting conserved charges

$$H_n = \mathrm{Tr}\left(L^n\right),\; n \in \mathbb{N},$$

with the Hamiltonian (generator of time translations) and momentum (generator of space translations) given as [158]

$$
\begin{aligned}
H &= \frac{1}{2}\left(H_1 + H_{-1}\right) = mc^2 \sum_{j=1}^{N} \cosh\theta_j \prod_{k\neq j} f\left(q_k - q_j\right), && (4.11)\\
P &= \frac{1}{2}\left(H_1 - H_{-1}\right) = mc \sum_{j=1}^{N} \sinh\theta_j \prod_{k\neq j} f\left(q_k - q_j\right), && (4.12)
\end{aligned}
$$

where q_i are the positions of the solitons and θ_i the conjugate rapidities. The interaction between solitons is given by the even function $f\left(q_k - q_j\right)$, reducing to $f \equiv 1$ in the free theory. The RS model is relativistic, as the generators ($\mathscr{B}$ is the generator for Boosts)

$$
\mathscr{H}_k = \frac{1}{2}\left(H_k + H_{-k}\right), \quad \mathscr{P}_k = \frac{1}{2}\left(H_k - H_{-k}\right), \quad \mathscr{B} = -\frac{1}{c}\sum q_j,
$$

obey the two-dimensional Poincaré algebra:

$$
\{\mathscr{H}_k, \mathscr{P}_k\} = 0, \quad \{\mathscr{H}_k, \mathscr{B}\} = \mathscr{P}_k, \quad \{\mathscr{P}_k, \mathscr{B}\} = \mathscr{H}_k. \tag{4.13}
$$

Next let us discuss the question of the time delay (and phase shift) in the particle picture. Considering the two particle case, one goes to the center-of-mass frame, as in [158]:

$$
\begin{aligned}
s &\equiv q_1 + q_2 \quad , \qquad \varphi \equiv \frac{1}{2}\left(\theta_1 + \theta_2\right), \\
q &\equiv q_1 - q_2 \quad , \qquad \theta \equiv \frac{1}{2}\left(\theta_1 - \theta_2\right),
\end{aligned} \tag{4.14}
$$

The Lax matrix (4.9) and its inverse are then given by

$$
L = e^{\varphi} f(q) \begin{pmatrix} e^{\theta} & C_{12} \\ \bar{C}_{12} & e^{-\theta} \end{pmatrix} \quad ; \qquad L^{-1} = e^{-\varphi} f(q) \begin{pmatrix} e^{-\theta} & -C_{12} \\ -\bar{C}_{12} & e^{\theta} \end{pmatrix},
$$

where C_{ij} is a 2×2 matrix with entries $C_{11} = C_{22} = 1$, and

$$
C_{12} = \bar{C}_{21} = \left[\cosh\left(\frac{\mu}{2}q\right) + ia\sinh\left(\frac{\mu}{2}q\right)\right]^{-1}.
$$

Now it is simple to check that the Hamiltonian (4.11) becomes

$$H = 2\cosh\varphi\cosh\theta\, f(q) = (\cosh\theta_1 + \cosh\theta_2)\, f(q)\,. \tag{4.15}$$

The momentum given by (4.12) also becomes:

$$P = 2\sinh\varphi\cosh\theta\, f(q) = (\sinh\theta_1 + \sinh\theta_2)\, f(q)\,. \tag{4.16}$$

One comment should be made with respect to the interaction potential

$$f(q) = \left[1 + \alpha/\sinh^2\left(\frac{\mu q}{2}\right)\right]^{1/2}.$$

Going back to (4.10) one can see that for $\alpha = 1$ it reduces to the particular case of a repulsive soliton-soliton interaction in the sine-Gordon model, $f_r(q) = \left|\coth\left(\frac{q}{2}\right)\right|$. An extension of the interaction potential to $\alpha = -1$ leads to the attractive case of soliton-anti-soliton interaction of sine-Gordon, where $f_a(q) = \left|\tanh\left(\frac{q}{2}\right)\right|$.

With the Hamiltonian (4.15) and choosing certain interacting potentials f one can fully recover the properties of the sine-Gordon soliton- (anti)soliton scattering, such as time delay and phase shift. From (4.16) it is easy to see that the center of mass $P = 0$ corresponds to $\varphi = 0$. Then the center-of-mass Hamiltonian for two particles is:

$$H_{cm} = \cosh\theta\, f(q)\,.$$

Now we have the relation:

$$\dot{q}^2 + f^2(q) = H_{cm}^2.$$

Because H_{cm} is a constant of motion, so is the quantity $\varepsilon \equiv H^2 - 1$. Evaluating ε asymptotically, when $x \to \infty$, we obtain $\varepsilon = \sinh^2\hat{\theta}$, where $\hat{\theta}$ is the asymptotic center-of-mass rapidity. The time delay is then determined by the time taken along a trajectory from $-q$ to q, as

$|q| \to \infty$. For the repulsive soliton-soliton case f_r, we get

$$\int_{-q}^{q} \frac{dq}{\sqrt{\varepsilon - \operatorname{csch}^2\left(\frac{q}{2}\right)}} = \frac{4}{\sqrt{\varepsilon}} \cosh^{-1}\left(\sqrt{\frac{\varepsilon}{\varepsilon+1}} \cosh\left(\frac{q}{2}\right)\right)\Bigg|_{2\cosh^{-1}\sqrt{\frac{1+\varepsilon}{\varepsilon}}}^{q}$$
$$\underset{q\to\infty}{\longrightarrow} \frac{2q}{\sinh\hat{\theta}} + \frac{1}{\sinh\hat{\theta}} \ln\left(\tanh\hat{\theta}\right).$$

The first term is the time for each of the solitons to go from $-q$ to q if if it was free (no interaction). The second term is in fact the time delay due to having a repulsive interaction, and correctly reproduces the time delay for a soliton-soliton scattering in sine-Gordon theory, obtained through field theoretic methods.

4.3 An ansatz for the dynamics of a two-magnon system

Our aim is to describes the N-magnon dynamics in Hamiltonian terms. The appropriate dynamical system ought to be such that it reproduces the classical equations of motions, its energy, momentum and finally phase shift in agreement with the known magnon results [26]. We will begin by focusing on the two-magnon interactions.

We know that the sine-Gordon and the magnons have the same classical equations of motion, and as such the time delay for both systems agrees

$$\Delta t_m\left(E_m\right) = \Delta t_{sg}\left(\varepsilon_{sg}\right)\big|_{\varepsilon_{sG}=\frac{1}{E_m}}$$

but with different energies. This implies different Hamiltonians for the two systems. With the semiclassical phase shift obeying $\frac{\partial\delta(E)}{\partial E} = \Delta t$ one can try to deduce the (Hamiltonian) dynamics directly from the phase shift itself.

For the sine-Gordon system the center-of-mass Hamiltonian is $H_{sg} = \cosh\theta\, f(q) = \varepsilon_{sg}$, the equation of motion for the relative position q gives $\dot{q} = \sqrt{\varepsilon_{sg}^2 - f(q)^2}$, and the time delay in terms of the energy is simply given by

$$\Delta t_{sg} = \int \frac{dq}{\dot{q}} = \int \frac{dq}{\sqrt{\varepsilon^2 - f(q)^2}}.$$

and the scattering phase shift of two sine-Gordon solitons is just given by $\delta_{sg}(\varepsilon) = \int d\varepsilon\, \Delta t_{sg}$, while for the two-magnons

$$\delta_m(E_m) = \int dE_m\, \Delta t_{sg}(\varepsilon_{sg})|_{\varepsilon_{sg}=\frac{1}{E_m}} = \int dE_m\, E_m \int \frac{dq}{f(q)\sqrt{f^{-2}(q) - E_m^2}}. \tag{4.17}$$

In order to determine which Hamiltonian $H_{cm} \equiv E_m$ produces this phase shift we first perform a change of variables, introducing a new co-ordinate Q through $dQ = \frac{dq}{f(q)}$. Also define $F(Q) = \frac{1}{f(q(Q))}$. The interaction then follows (soliton-soliton interaction): for f$(q) =$ $\coth q$ we find $q = \cosh^{-1}\left(e^{Q}\right)$ and $F(Q) = \sqrt{1 - e^{-2Q}}$.

This means that the limit of relative position $q \to \infty$ corresponds to the new relative position doing the same $Q \to \infty$. Also in this limit, we have $f(q), F(Q) \to 1$ (the free theory limit).

After this change of variables, we rewrite the phase shift as

$$\delta(E_m) = \int dE_m \int dQ \sqrt{\frac{E_m^2}{F^2 - E_m^2}} = \int dE_m \int \frac{dQ}{\dot{Q}},$$

and want to find the center-of-mass Hamiltonian $H_{cm} \equiv E_m$ such that

$$\dot{Q} = \frac{\partial H_{cm}}{\partial \alpha} = \sqrt{\frac{F^2 - H_{cm}^2}{H_{cm}^2}}, \tag{4.18}$$

where α is the new relative rapidity, i.e. the conjugate variable to Q. The differential equation above can be solved to give

$$\sqrt{1 - \left(\frac{H_{cm}}{F}\right)^2} = -\frac{\alpha}{F}.$$

This result is only valid for $\alpha < 0$. Squaring this result, we can solve for H_{cm}, and find

$$H_{cm} = \sqrt{1 - \alpha^2 - e^{-2Q}}. \tag{4.19}$$

This two-body magnon Hamiltonian appears to be of relativistic (Toda) type. It faithfully reproduces the magnon scattering phase shift. But it is not directly recognizable as a known integrable system. Furthermore it is not obvious how to extend it to the N-body case. First one would need to find a two-body Hamiltonian that reduces to H_{cm} in the center-of-mass. In the limit $Q \to \infty$ we have that

$$H = \varepsilon_1 + \varepsilon_2 = \sin\frac{p_1}{2} + \sin\frac{p_2}{2},$$

where $\varepsilon_{1,2} = \sin\frac{p_{1,2}}{2}$ are the energies of each magnon in the free theory. For only one magnon the Hamiltonian would be given by $H = \sqrt{1-\alpha^2} = \sin\frac{p}{2}$, which means that the relation between the rapidity $\alpha < 0$ and the momentum p is $\alpha = -\cos\frac{p}{2} = \cos(\pi + \frac{p}{2})$. These results will hold in the free theory limit for each magnon. Then a good ansatz for the two-body Hamiltonian, which reproduces the correct result for the free limit, would be

$$H_2 = \sqrt{1-\alpha_1^2 - e^{-2Q}} + \sqrt{1-\alpha_2^2 - e^{-2Q}}, \tag{4.20}$$

with $Q = Q_1 - Q_2$, and the momentum of each magnon is given by $\frac{p_i}{2} = \arccos(\alpha_i) - \pi$.

This construction is non-unique because we do not have the expression of the total momentum. As mentioned we also have no information on the integrability properties of this system, which is crucial to generalize our results to the dynamics of N-magnon solutions. For these reasons we now pursue a different strategy, based on employing the known integrable structure of the RS model, in particular its Lax matrix L. Together with the classical equivalence between sine-Gordon solitons and giant magnons there was evidence that the poles of the S-matrix of scattering magnons were related to a Calogero type system in the non-relativistic limit [147], thus making us believe that the dynamics of magnons are intimately related to the dynamics of solitons in the RS model. In fact one would hope to describe the dynamics of magnons through a Lax pair formulation whose Lax matrix would be directly related to the Lax matrix of the relativistic RS model.

The N-magnon Hamiltonian

With the motivation for using the RS integrable structure described above we now proceed to the construction of the associated magnon dynamical system. This will involve specifying both the Hamiltonian and the symplectic structure. As we have emphasized before, the energies of the sine-Gordon solitons and the magnons are inverse of each other. This result leads us to the following ansatz for the N-magnon Hamiltonian:

$$H_m = \mathrm{Tr}\left[\mathscr{L}_{rs}^{-1}\right], \tag{4.21}$$

where $\mathscr{L}_{rs}$ is related to the Lax matrix of the RS model through (4.8). We will now study this Hamiltonian and consider the two-magnon interaction.

Recall that from (4.8)

$$\mathscr{L}_{rs} = \frac{L+L^{-1}}{2} = \frac{f(q)}{2}\left\{e^{\varphi}\begin{pmatrix} e^{\theta} & C_{12} \\ \bar{C}_{12} & e^{-\theta}\end{pmatrix} + e^{-\varphi}\begin{pmatrix} e^{-\theta} & -C_{12} \\ -\bar{C}_{12} & e^{\theta}\end{pmatrix}\right\}.$$

The RS Hamiltonian (4.11) is just the trace of the matrix above. This matrix has the following eigenvalues:[4]

$$h_{\pm} = \frac{f(q)}{2}\Big(\cosh(\varphi+\theta) + \cosh(\varphi-\theta) \\ \pm\sqrt{\left(\cosh(\varphi+\theta)-\cosh(\varphi-\theta)\right)^2 + 4\sinh^2\varphi\left(1-f(q)^{-2}\right)}\Big).$$

Then the Hamiltonian for the 2-magnon problem (4.21) will be just

$$H_m = h_+^{-1} + h_-^{-1} = \frac{1}{f(q)}\frac{2\cosh\theta\cosh\varphi}{\cosh^2\theta + f(q)^{-2}\sinh^2\varphi}.$$

Recall that if M is a diagonalizable matrix with $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$ the diagonal matrix of eigenvalues, then for a smooth function $g(M)$ the trace of $g(M)$, it will be given by

$$\mathrm{Tr}[g(M)] = \mathrm{Tr}[g(\Lambda)] = \sum_{i=1}^{N} g(\lambda_i). \tag{4.22}$$

It is easy to check that in the free theory $(f(q) \equiv 1)$ we have:

$$H_m^{free} = \frac{1}{\cosh\theta_1} + \frac{1}{\cosh\theta_2} = E_{mag,1} + E_{mag,2},$$

[4] For a 2×2 matrix $M = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$, its eigenvalues are simply given by $\lambda_{\pm} = \frac{a+d}{2} \pm \frac{1}{2}\sqrt{(a-d)^2 + 4bc}$.

which corresponds to the sum of the energy of the two magnons, as expected.

To have an ansatz for the momentum of the N-body magnon problem, we first look at the momentum for the magnons. In [26] we have that for one magnon the relation between the momenta p_m and the rapidity θ is given by $\cosh\theta = \left[\sin\frac{p_m}{2}\right]^{-1}$. But we know that for the sine-Gordon model the total momentum is $P = \sum_i p_i = \sum_i \sinh\theta_i$. Then a simple comparison allows us to conclude that the momenta for each magnon $p_{m,i}$ is related to the momenta of each soliton p_i by:

$$\sin\left(\frac{p_{m,i}}{2}\right) = \frac{1}{\sqrt{1+p_i^2}}. \tag{4.23}$$

Thus, a good ansatz for the momenta of the magnon $P_m = \sum_i p_{m,i}$ will be

$$P_m = 2\,\mathrm{Tr}\left[\arcsin\left(\mathbf{1}+\mathscr{P}_{rs}^2\right)^{-1/2}\right], \tag{4.24}$$

where we defined the momentum matrix for the RS model (whose trace gives the RS momenta given by (4.12)) to be $\mathscr{P}_{rs} = \frac{L-L^{-1}}{2}$. By knowing the eigenvalues of this last matrix, we can determine P_m by using the result (4.22):

$$P_m = 2\sum_{i=\pm}\arcsin\left(\frac{1}{\sqrt{1+p_i^2}}\right). \tag{4.25}$$

The eigenvalues of $\mathscr{P}_{rs}$ are

$$p_\pm = \frac{f(q)}{2}\left\{\sinh\theta_1+\sinh\theta_2 \pm\sqrt{(\sinh\theta_1-\sinh\theta_2)^2+4\cosh^2\varphi\left(1-f(q)^{-2}\right)}\right\}.$$

The magnon momentum will then be given by

$$\sin\frac{P_m}{2} = \frac{p_+ + p_-}{\sqrt{1+p_+^2}\sqrt{1+p_-^2}}.$$

In the limit of the free theory $f(q)\to 1$, we find the expected relation $P_m^{free} = p_1^m + p_2^m$, i.e. the magnon momentum is the sum of the momenta for each magnon.

Note that because all integrals of motion H_k Poisson commute with each other, so will

H_m and P_m:

$$\{H_m, P_m\} = 0.$$

The center of mass condition is given by

$$P_m = 0 \Rightarrow \quad \sin\frac{P_m}{2} = 0 \Rightarrow \quad \theta_1 + \theta_2 = 0.$$

In the center of mass, the Hamiltonian is simply

$$H_m \quad = \quad \frac{1}{f(q)}(\operatorname{sech}\theta_1 + \operatorname{sech}\theta_2) = \frac{2}{f(q)}\operatorname{sech}\theta,$$

with $f(q)$ the same as before.

A method for checking our ansatz is to determine the classical and semiclassical behaviors of our system, such as the time delay and phase shift for this two-body problem of scattering magnons, and compare them to the known results [26].

We start from the center-of-mass Hamiltonian determined above, and determine the classical equations of motion and time delay. But to do so, we need to choose a Poisson structure. Let us assume that the Poisson structure is the symplectic one. Then the equation of motion for q is just

$$\dot{q} \equiv \frac{\partial H}{\partial \theta} = -H_m \tanh\theta = H_m\sqrt{1 - \frac{1}{4}f(q)^2 H_m^2}.$$

The Hamiltonian is a conserved quantity, $H_m \equiv E$ and can be evaluated when $q \to \infty$, giving $E = \operatorname{sech}\hat{\theta}$, where $\hat{\theta}$ is the asymptotic rapidity. We find the time delay in this case to be

$$\Delta T_m = \int \frac{dq}{\dot{q}} = \cosh^2\hat{\theta}\,\Delta T_{RS}.$$

But this time delay is not correct, nor does it reproduce the right phase shift.

The phase shift is determined by WKB semiclassical methods to be given by the symplectic structure ω (the inverse of the Poisson structure). In phase space variables (x_i, p_i) we have:

$$\delta(E) \equiv \int p_i\,\omega_{ij}dx_j. \tag{4.26}$$

For the results in this section we have used a canonical Poisson brackets (the standard symplectic structure $\{p_i, x_j\} = \delta_{ij}$), so the phase shift is simply $\delta(E) = \int \theta \, dq$. By solving $H_m \equiv \frac{2}{f(q)} \operatorname{sech} \theta = E$, with respect to the rapidity, $\theta = \cosh^{-1}\left(\frac{2}{f(q)E}\right)$, we can determine the semiclassical phase shift to be

$$\delta(E) = \int \cosh^{-1}\left(\frac{2}{f(q)\,E}\right) dq. = \int dE\, \Delta T_m.$$

We find that even though our ansatz correctly reproduces the energies and momenta of the magnon system, it does not give the expected classical behaviour (time delay or equations of motion) nor the semiclassical phase shift. But in these calculations we have assumed the usual canonical Poisson brackets, which is equivalent to having a canonical symplectic form for q and p and which resulted in the usual form of the Hamilton-Jacobi equations, namely $\dot{q} = \frac{\partial H}{\partial p}$ and $\dot{p} = -\frac{\partial H}{\partial p}$. One can trace the difference in phase shifts to the different Poisson structures in the two cases. The semiclassical phase shift can be related to the symplectic form ω (the inverse of the Poisson structure) as follows :

$$\delta = \int p_i \omega_{ij} dq_j = \int p_i \omega_{ij} \dot{q}_j dt\,.$$

For the trivial symplectic structure, $\omega \equiv \mathrm{Id}$, and the phase shift has the usual form. But a non-trivial symplectic form is required for reproducing the correct phase shifts and for defining the full magnon N-body dynamics. This we will identify in the next section.

4.4 Poisson Structure for the N-Magnon Dynamics

We have identified above the N-body Hamiltonian for magnons as one member of the RS hierarchy. We have also understood that a new modified Poisson (and symplectic) structure is needed in order to obtain both the correct equations of motion and the correct magnon phase shift. In the present section we will be able to specify the modified symplectic structure in an approximation of well separated magnons. This approximation which we take just for the purpose of simplifying the problem involves a limit of the RS model when the solitons are far away from each other, called the Toda lattice. The relativistic Toda lattice was introduced in [163] as a relativistic version of the regular Toda lattice [167, 168, 169]. In these models the study of master symmetries [170, 171, 172] and of recursion relations [173, 174, 164] led to the discovery of a sequence of Hamiltonian/Poisson structures that return the same classical equations of motion, result of the existence of a bi-Hamiltonian system [162].

As seen in [163] we obtain the simpler model of relativistic Toda Lattice from the original Ruijsenaars-Schneider model (4.11) by considering that the particles are very far from each other $q_{i-1} \ll q_i$.[5] This allows us to keep only the nearest neighbour interactions and these interactions become exponential $f(q) = \sqrt{1+g^2 e^q}$. Note that we are studying the nonperiodic Toda lattice, for which $q_0 = -\infty$ and $q_{N+1} = \infty$.

The Hamiltonian for the relativistic Toda lattice is given by

$$H = \sum_{i=1}^{N} e^{\theta_i} V_i(q_1, ..., q_N). \tag{4.27}$$

But now the interaction potential is given by nearest neighbour interactions only

$$V_i(q_1, ..., q_N) = f(q_{i-1} - q_i) f(q_i - q_{i+1}), \quad i = 1, ...N. \tag{4.28}$$

Also, the symplectic form remains

$$\omega = \sum_{i=1}^{N} dq_i \wedge d\theta_i.$$

This system is integrable and has a Lax matrix formulation, inherited from the RS model (up to some similarity transformation) [163, 174, 164, 175, 176]. To write the Lax matrix we introduce the following change of variables

$$a_j = g^2 e^{q_j - q_{j+1} + \theta_j} \frac{f(q_{j-1} - q_j)}{f(q_j - q_{j+1})} \quad ; \quad b_j = e^{\theta_j} \frac{f(q_{j-1} - q_j)}{f(q_j - q_{j+1})}, \quad j = 1, ..., N.$$

[5] In fact Ruijsenaars introduced the limit $\varepsilon \to 0$ of the variables

$$q_j^{\varepsilon} \to q_j - 2j \ln \varepsilon, \quad j = 1, \cdots, N.$$

Note that $a_0 = a_N = 0$. The Lax matrix is then given by

$$L = \begin{pmatrix} a_1+b_1 & a_1 & & & \\ a_2+b_2 & a_2+b_2 & a_2 & 0 & \\ \vdots & \vdots & \ddots & \ddots & \\ a_{N-1}+b_{N-1} & a_{N-1}+b_{N-1} & \cdots & a_{N-1}+b_{N-1} & a_{N-1} \\ b_N & b_N & \cdots & b_N & b_N \end{pmatrix}.$$

The Hamiltonian $H_1(q,p)$ given in (4.27) can be written in the new variables:

$$h_1 = \mathrm{Tr}L = \sum_{i=1}^{N-1} a_i + \sum_{i=1}^{N} b_i, \tag{4.29}$$

The equations of motion in the (q,θ) co-ordinates are given by

$$\dot{q}_j = e^{\theta_j} V_j \quad ; \quad \dot{\theta}_j = -\sum_k e^{\theta_k} \frac{\partial V_k}{\partial q_j},$$

which can be obtained from the Hamiltonian (4.27) by using the symplectic Poisson bracket J_0, defined by $\left\{q_i, p_j\right\} = \delta_{ij}$. In the (a,b) variables, the symplectic Poisson bracket J_0 becomes a quadratic Poisson bracket π_2:

$$\{a_i, a_{i+1}\} = -a_i a_{i+1}\ , \quad \{a_i, b_i\} = a_i b_i\ , \quad \{a_i, b_{i+1}\} = -a_i b_{i+1}. \tag{4.30}$$

From this Poisson bracket and the Hamiltonian (4.29) one obtains the equations of motion in the (a,b) co-ordinates

$$\dot{a}_j = a_j\left(b_j - b_{j+1} + a_{j-1} - a_{j+1}\right) \quad ; \quad \dot{b}_j = b_j\left(a_{j-1} - a_j\right). \tag{4.31}$$

The Toda lattice is an integrable model. It also has a bi-Hamiltonian structure. Before continuing, let us summarize the properties of this structure..

Bi-Hamiltonian structure of Relativistic Toda Lattice

We now quickly summarize the properties of the bi-Hamiltonian structure of the relativistic Toda lattice, based on [174,164,175,176]. The relativistic Toda lattice is an integrable model, and has a sequence of conserved quantities

$$h_n = \frac{1}{n}\mathrm{Tr}\left(L^n\right).$$

To this sequence we have a corresponding set of Hamiltonian vector fields $\chi_1, ..., \chi_n$, with $\chi_i = [\pi, h_i]$, where π is some Poisson structure and $[,]$ is the Schouten bracket (Lie bracket). Also, we have a hierarchy of Poisson 2-tensors $\pi_1, ..., \pi_n$ (which are polynomial homogeneous of degree n), and a sequence of master symmetries $X_1, ..., X_n$, which obey the following properties (more information on the properties of these entities can be found in [176] and references therein):

1. the π_n tensors are all Poisson structures. The corresponding Poisson brackets are given by
$$\{f, g\} = \sum_{i,j} \pi^{ij} \frac{\partial f}{\partial x^i} \wedge \frac{\partial g}{\partial x^j},\ f, g \in C^\infty.$$
Note that π^{ij} are the matrix elements of the matrix π_n corresponding to this 2-tensor, and $\bar{x} = \left(x^1, ..., x^M\right)$ are the co-ordinates of the Hilbert space, in our case $(a_1, ..., a_{N-1}, b_1, ..., b_N)$;

2. functions h_n are in involution with all π_m;

3. $X_n\left(h_m\right) = (n+m)\, h_{m+n}$;

4. $\mathscr{L}_{X_n}\left(\pi_m\right) \equiv [X_n, \pi_m] = (m-n-2)\, \pi_{n+m}$, where $\mathscr{L}_X$ in the Lie derivative in the direction of the vector X;

5. $[X_n, X_m] = (m-n)\, X_{n+m}$;

6.
$$\pi_n \nabla h_m = \pi_{n-1} \nabla h_{m+1}, \tag{4.32}$$

where π_n now denotes the Poisson matrix of the tensor π_n, and $\nabla = \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{x^M}\right)$. It is known that once our system is bi-Hamiltonian [162], which means that we can identify two Hamiltonian functions h_1, h_2 and two compatible Poisson tensors π_1, π_2 satisfying

$$\pi_1 \nabla h_2 = \pi_2 \nabla h_1,$$

then we can find the whole hierarchy stated above, and the equations of motion are just given by

$$\frac{d\bar{x}}{dt} = \pi_1 \nabla h_2 = \pi_2 \nabla h_1 = \pi_0 \nabla h_3 = \cdots . \tag{4.33}$$

All of these properties are valid for $m, n > 0$ but can be seen to generalize to negative values as well.

In the case one of the Poisson brackets is symplectic, one can find a recursion operator which can then be applied to the initial symplectic bracket to determine the hierarchy [173, 164]. In our case, we will see that in the (a, b) co-ordinates the Poisson brackets are not symplectic (not even non-degenerate, as we don't have the same number of $a's$ and $b's$) and it is non-trivial to find an extra Poisson bracket in the (p, q) co-ordinates apart from the symplectic one, in order to form a bi-Hamiltonian system [175]. The construction of another Poisson bracket in the (p, q) variables, compatible with the symplectic one was done in [177], and has a structure highly non-trivial (not a polynomial dependence on the (p, q) variables).

Construction of the hierarchy of Poisson brackets

In order to construct this bi-Hamiltonian structure one needs to identify two Hamiltonian functions h_1, h_2 and two compatible Poisson tensors π_1, π_2 satisfying the same equations of motion, i.e. if $\nabla = \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{x^M}\right)$ where x^i are our phase space co-ordinates, then

$$\frac{d\bar{x}}{dt} = \pi_1 \nabla h_2 = \pi_2 \nabla h_1 . \tag{4.34}$$

We already have the Hamiltonian function $h_1 = \mathrm{Tr} L$ (4.29), and the corresponding quadratic Poisson bracket π_2 (4.30) such that $\pi_2 \nabla h_1$ gives the equations of motion (4.31)

[174, 164]. A compatible linear Poisson bracket π_1 was found in [178] such that

$$\{a_i, b_i\} = a_i \,; \quad \{a_i, b_{i+1}\} = -a_i \,; \quad \{b_i, b_{i+1}\} = a_i \,,$$

which together with the Hamiltonian $h_2 = \frac{1}{2}\mathrm{Tr}\left(L^2\right)$ also gives the equations of motion (4.31). These two pairs make a bi-Hamiltonian system with equations of motion given by (4.34). If we now construct the master symmetries that obey the properties shown in Section 4.4 [164, 176], it becomes possible to construct the hierarchy of Poisson brackets with the same equations of motion:[6]

$$\cdots = \pi_0 \nabla h_3 = \pi_1 \nabla h_2 = \pi_2 \nabla h_1 = \pi_? \nabla h_{-1} = \cdots .$$

Our final objective is to make an analogy with the system of N magnons. Recalling the RS Lax matrix $\mathscr{L}_{rs}$, the sine-Gordon model corresponds to the Hamiltonian $H \propto \mathrm{Tr}\mathscr{L}_{rs}$ with the canonical Poisson brackets, while the system of magnons was conjectured to correspond its "inverse" $H_m \propto \mathrm{Tr}\mathscr{L}_{rs}^{-1}$, with some other Poisson structure. In the limit we are considering (relativistic Toda), we have

$$H_{sg} \propto \mathrm{Tr}\mathscr{L}_{rs} \to h_1 \quad ; \quad H_{mag} \propto \mathrm{Tr}\mathscr{L}_{rs}^{-1} \to h_{-1} .$$

So, having started from the Hamiltonian $h_1 = \mathrm{Tr}L$, with a quadratic Poisson bracket π_2, we want to find the Poisson bracket corresponding to $h_{-1} = -\mathrm{Tr}\left(L^{-1}\right)$ that gives origin to the equations of motion (4.31), i.e.

$$\pi_2 \nabla h_1 = \pi_m \nabla h_{-1} .$$

To do so, we will restrict ourselves to $N = 2$.

[6] The Hamiltonian function h_0 is singular, reason why that point is skipped from the sequence. h_0 can only be defined as a limit

$$h_0 = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathrm{Tr}\left(L^\varepsilon\right) \sim \mathrm{Tr}\left(\ln L\right) .$$

But to continue the sequence after this point, one can just do power counting, followed by a direct verification of the equations of motion.

For the $N=2$ case, the Lax matrix reduces to

$$L=\begin{pmatrix} a_1+b_1 & a_1 \\ b_2 & b_2 \end{pmatrix},$$

the Hamiltonian functions are given by

$$\begin{aligned} h_1 &= \mathrm{Tr}L = a_1+b_1+b_2; \\ h_2 &= \tfrac{1}{2}\ \mathrm{Tr}L^2 = \frac{1}{2}\left(a_1^2+2a_1b_1+2a_1b_2+b_1^2+b_2^2\right), \end{aligned}$$

and the corresponding Poisson bracket matrices are

$$\pi_1=\begin{pmatrix} 0 & a_1 & -a_1 \\ -a_1 & 0 & a_1 \\ a_1 & -a_1 & 0 \end{pmatrix} \quad ; \quad \pi_2=\begin{pmatrix} 0 & a_1b_1 & -a_1b_2 \\ -a_1b_1 & 0 & 0 \\ a_1b_2 & 0 & 0 \end{pmatrix}.$$

The equations of motion obtained from this bi-Hamiltonian system is

$$\begin{pmatrix} \dot{a}_1 \\ \dot{b}_1 \\ \dot{b}_2 \end{pmatrix} = \pi_1\nabla h_2 = \pi_2\nabla h_1 = \begin{pmatrix} a_1\left(b_1-b_2\right) \\ -a_1b_1 \\ a_1b_2 \end{pmatrix}.$$

The objective is to determine which π_m gives origin to the previous equations of motion with Hamiltonian function

$$h_{-1} = -\mathrm{Tr}L^{-1} = -\frac{1}{b_1b_2}\left(a_1+b_1+b_2\right).$$

We want to construct the next Poisson brackets given the master symmetries. The master symmetries X_1 and X_2 were determined in [176], such that: $X_1\left(h_n\right) = \left(n+1\right)h_{n+1}$ and $X_2\left(h_n\right) =$

$(n+2)\,h_{n+2}$. These are given by

$$\begin{aligned}
X_{1,2} &= r^1_{1,2}\frac{\partial}{\partial a_1} + s^1_{1,2}\frac{\partial}{\partial b_1} + s^2_{1,2}\frac{\partial}{\partial b_2}, \\
\text{with} \quad & r^1_1 = a_1^2 + 3a_1b_2\,;\; r^1_2 = a_1\left(a_1^2 + 5a_1b_1 + 4b_1^2 + 2b_1b_2 - b_2^2\right); \\
& s^1_1 = b_1^2 + 2a_1b_1\,;\; s^1_2 = b_1\left(-2a_1^2 - a_1b_1 - 2a_1b_2 + b_1^2\right); \\
& s^2_1 = b_2^2 - a_1b_2 \quad ;\; s^2_2 = b_2\left(2a_1^2 + 3a_1b_1 + 4a_1b_2 + b_2^2\right).
\end{aligned}$$

Then the next Poisson Brackets are given by property 4 (recall that the Poisson matrices are anti-symmetric):[7]

$$\pi_3 = -\mathscr{L}_{X_1}\pi_2 = \begin{pmatrix} 0 & a_1b_1(a_1+b_1) & -a_1b_2(a_1+b_2) \\ & 0 & -a_1b_1b_2 \\ & & 0 \end{pmatrix};$$

$$\pi_4 = -\frac{1}{2}\mathscr{L}_{X_2}\pi_2 = \begin{pmatrix} 0 & a_1b_1((a_1+b_1)^2 + a_1b_2) & -a_1b_2(a_1(a_1+b_1) + 2a_1b_2 + b_2^2) \\ & 0 & -a_1b_1b_2(a_1+b_1+b_2) \\ & & 0 \end{pmatrix}.$$

With these results we can easily see that $\pi_3\nabla h_{-1}$ does not give the right equations of motion, but $\pi_4\nabla h_{-1}$ does. So the Hamiltonian h_{-1} with Poisson bracket π_4 will give the same classical behavior than the Hamiltonian h_1 with Poisson bracket π_2. The hierarchy is given by (the point π_3, h_0 is not defined)

$$\cdots = \pi_0\nabla h_3 = \pi_1\nabla h_2 = \pi_2\nabla h_1 = \pi_4\nabla h_{-1} = \cdots.$$

All of these pairs generate the same equations of motion and time delay. In particular, the Hamiltonian h_{-1} with (quartic) Poisson bracket π_4 will give the same classical behavior

[7] To determine the Lie derivative of the 2-tensor π^{ij}, we use the rule for a general tensor

$$\begin{aligned}
\mathscr{L}_X T^{a_1\ldots a_r}{}_{b_1\ldots b_s} &= X^c\nabla_c T^{a_1\ldots a_r}{}_{b_1\ldots b_s} - \nabla_c X^{a_1} T^{c\ldots a_r}{}_{b_1\ldots b_s} - \cdots - \nabla_c X^{a_r} T^{a_1\ldots c}{}_{b_1\ldots b_s} + \\
&\quad + \nabla_{b_1} X^c T^{a_1\ldots a_r}{}_{c\ldots b_s} + \cdots + \nabla_{b_s} X^c T^{a_1\ldots a_r}{}_{b_1\ldots c}.
\end{aligned}$$

than the Hamiltonian h_1 with Poisson bracket π_2. Since we have found that (in the limit of well separated magnons) the Hamiltonian reduces to

$$H_{mag} = \mathrm{Tr}\mathscr{L}_{rs}^{-1} \to h_{-1},$$

it will reproduce the correct equations of motion (the same as the limiting case of sine-Gordon solitons) as long as we use the quartic Poisson structure π_4 defined above.

For a non degenerate Poisson structure, the phase shift is given by the corresponding symplectic form (the inverse of the Poisson tensor). The usual symplectic form is replaced with the following

$$\int p_i \dot{q}_i \to \int p_i \left(\pi^{-1}\right)_{ij} \dot{q}_j$$

(for a degenerate Poisson structure one has to check this more carefully). Consequently, if two different systems have the same equations of motion, the different Poisson Structures give origin to different phase shift.

4.5 Summary of results

In this chapter, we considered the question of an N-particle dynamics that would fully describe interacting magnons at the semiclassical level. For this we have specified the interacting Hamiltonian as a member of the RS hierarchy. This Hamiltonian had the property that it reproduces energies of magnons. We argued that an alternative symplectic form is needed in order to obtain the correct magnon phase shifts. We have considered the question of the modified symplectic form explicitly for the case of well separated magnons. In this limit one had the results of relativistic Toda theory where a sequence of symplectic forms was already established in the literature.

Altogether the new Hamiltonian and the modified symplectic form are defined so to reproduce correctly the original classical equations of motion and therefore the time delay. Regarding future interesting problems we mention the following. We have succeeded in establishing the necessary symplectic form in the limit of well separated magnons. For establishing an exact result one will have to give the multi-Poisson structure for the RS model itself. It is likely that this is definitely possible, although technically (and possibly conceptually) challenging. But one can definitely expect that a sequence of symplectic structures always follows for an integrable system. Generalization of the present construction to magnons moving on higher spheres [94] is also a challenging task. One would also want to define the dynamics in the periodic case appropriate for string motions with finite J [98].

Chapter 5

Algebraic curve formalism for strings in $AdS_5 \times S^5$

In the previous chapters we discussed some classical string solutions and comparison to their counterparts in the gauge theory, but we are far from having a full spectrum of solutions. Nevertheless, even if we don't know explicit form of the solutions, we can use the (classical) integrability of the string σ-model in $AdS_5 \times S^5$ to find a classification of the energy spectrum of classical strings. The first step towards this classification was done in [29] for bosonic string in $\mathbb{R} \times S^3$, where a correspondence was seen to exist between classical string solutions and hyperelliptic curves. From this algebraic formalism one could determine all of the conserved charges corresponding to each solution. So the problem of determining the set of classical string solutions becomes a problem of finding the moduli space of admissible curves.

The algebraic curve formalism is constructed from the classical string σ-model using the Lax connection, composed by a family of flat connections on the two-dimensional worldsheet, thus highlighting the importance of integrability in this formalism. In fact, for σ-models on group manifolds and coset spaces such a connection is known, and an infinite set of conserved charges [127,49,179,180,181] can be constructed, characteristic of integrable models. The search for the moduli space of curves that correspond to classical string solutions then becomes the issue of finding solutions to a spectral problem in terms of algebraic curves, which can in its turn be formulated as Riemann-Hilbert problem. This is done by representing the algebraic curves as Riemann sheets connected by branch cuts. These branch cuts correspond to fundamental particles, and are represented by contour integrals and densities on the complex plane. The quest of finding admissible curves is then summarized by some integral equations. These integral equations seem to describe a factorizable scattering problem, again pointing out the importance of integrability.

Solutions to the problem of finding curves corresponding to classical strings have been found in several (bosonic) subsectors of $AdS_5 \times S^5$ [30, 31, 32, 33], and the full spectrum of $AdS_5 \times S^5$ superstrings has been studied in [34] (at the classical level) and in [35] (quantum generalization).

In the gauge theory dual to strings in $AdS_5 \times S^5$, the $\mathcal{N} = 4$ super Yang-Mills theory, integrability allowed to determine a Bethe ansatz and to show an equivalence to quantum spin chains [20]. This Bethe ansatz gives a set of algebraic equations, whose solutions have a one-to-one correspondence with the eigenstates of the dilatation operator of the gauge theory [21, 23, 22]. These algebraic equations become integral in a particular limit [25, 91] and very similar to the integral equation obtained from the string side. A Bethe ansatz for quantum strings was first proposed in [36], and further studied in [37, 38, 39, 40, 41, 42], including comparisons to the Bethe ansatz equations from the gauge side.

Knowing the structure of the string algebraic curve and corresponding Bethe ansatz is an essential step in the effort of better understanding the correspondence between gauge and string theories. This chapter will be mainly a summary on the string Bethe ansatz and algebraic curve formalism for the string σ-model on $AdS_5 \times S^5$, and most of the results included in here can be found in [29, 31, 34]. A brief application of this formalism for the case of giant magnons is also discussed.

5.1 The sigma model of $\mathbb{R} \times S^{m-1}$

We start from the 2-dimensional σ-model on $\mathbb{R} \times S^{m-1}$ in conformal gauge (with Virasoro constraints) - classically equivalent to the truncation of the type IIB superstring in $AdS_5 \times S^5$ for $m = 6$. Consider, as before, the embedding co-ordinates $\vec{X} = (X_i)\,,\, i = 1, \cdots, m$ of S^{m-1}, such that $\vec{X}^2 = 1$, plus the time co-ordinate X_0. If $G_{\mu\nu}$ is the metric in $\mathbb{R} \times S^{m-1}$, and defining

$X^\mu = \left(X^0, X_i\right)$, then the classical action of the bosonic string is given by

$$S = -\frac{\sqrt{\lambda}}{4\pi}\int d\sigma d\tau \left(G_{\mu\nu}\partial_a X^\mu \partial^a X^\nu + \Lambda(\vec{X}^2 - 1)\right).$$

Recall that in conformal gauge, the string obeys the following equations of motion and Virasoro constraints

$$\begin{aligned} &\partial_+\partial_-\vec{X} + \left(\partial_+\vec{X}\cdot\partial_-\vec{X}\right)\vec{X} = 0; \\ &\partial_+\partial_- X^0 = 0; \\ &\left(\partial_\pm\vec{X}\right)^2 = \left(\partial_\pm X_0\right)^2, \end{aligned} \tag{5.1}$$

where now $\sigma_\pm = \frac{1}{2}(\tau \pm \sigma)$ and $\partial_\pm = \partial_\tau \pm \partial_\sigma$.[1] We further fix the residual gauge freedom to be "time-like", that is

$$X_0(\tau, \sigma) = \kappa\tau.$$

This system of equations can be equivalently represented by a system of orthogonal/chiral fields. Define the matrices h_v (vector representation of $\mathfrak{so}(m)$) and h_s (spinor representation), such as

$$h_v = 1 - 2\vec{X}\vec{X}^T \quad , \quad h_s = \vec{\gamma}\cdot\vec{X},$$

where γ_i is a basis of the Clifford algebra of $SO(m) \sim S^{m-1}$. The matrix h_v describes a reflection along the direction of $\vec{X}$, being an orthogonal, symmetric matrix, which obeys $\det h_v = -1$ and $h_v^{-1} = h_v^T = h_v$. On the other hand h_s can be seen as the spinor equivalent of h_v.

In the vector representation, we can construct two currents, left j_v and right ℓ_v, which are elements of $\mathfrak{so}(m)$ in the vector representation. They are given by

$$j_v \equiv h_v^{-1} dh_v\,, \qquad \ell_v \equiv -dh_v h_v^{-1}.$$

[1]We will keep this choice of conventions throughout this chapter, in order to limit the use of $\pm$ in later expressions. Also in this chapter x will be the spectral parameter. It is simple to go from this to the conventions used in other chapters, where we chose $\sigma_\pm = \frac{1}{2}(\sigma \pm \tau)$, it is enough in most places to change the sign of the $(\cdots)_-$ terms.

Because of the properties of the matrix h_v these two currents coincide and obey

$$j_v = \ell_v = 2\left(\vec{X}d\vec{X}^T - d\vec{X}\vec{X}^T\right), \tag{5.2}$$

or in components $(j_v)_{a,ij} = (\ell_v)_{a,ij} = 2(X_i\partial_a X_j - X_j\partial_a X_i)$. These currents are flat:

$$d\,j_v + j_v \wedge j_v = 0.$$

The Lagrangian of the sphere part of the σ-model can be re-written in terms of these currents as

$$S = -\frac{\sqrt{\lambda}}{4\pi}\int d\sigma d\tau\,\mathrm{Tr}\left(j_v \wedge * j_v\right).$$

Invariance of this Lagrangian under right shifts h_v is associated with the current j_v being conserved

$$d(*j_v) = 0 \quad \Leftrightarrow \quad \partial_a j_v^a = 0.$$

This conservation equation together with the flatness condition is equivalent to the equations of motion of the σ-model (5.1). One can also find equivalent currents in the spinor representation, $j_s = \ell_s$, starting from shifts produced by h_s.

From the flat, conserved current j (j can be either j_v or j_s), we can construct a family of flat currents parametrized by a spectral parameter x, called the Lax connection:

$$a(x) = \frac{1}{1-x^2}j + \frac{x}{1-x^2} * j,$$

and a Lax pair from the operator $d + a(x)$:[2]

$$\begin{aligned}
\mathscr{L}(x) &= \partial_\sigma + a_\sigma(x) = \partial_\sigma + \frac{1}{2}\left(\frac{j_+}{1-x} - \frac{j_-}{1+x}\right), \\
\mathscr{M}(x) &= \partial_\tau + a_\tau(x) = \partial_\tau + \frac{1}{2}\left(\frac{j_+}{1-x} + \frac{j_-}{1+x}\right).
\end{aligned}$$

Current conservation and flatness conditions are now given by $da + a\wedge a = 0$ in terms of $a(x)$ (that is, $a(x)$ is flat for all values of x), and $[\mathscr{L}, \mathscr{M}] = 0$ in terms of the Lax pair operators.

[2]Note that j (and $a(x)$) is a 1-form in the worldsheet, so $*(j_\tau, j_\sigma) = (j_\sigma, j_\tau)$, and $j_\pm = j_\tau \pm j_\sigma$.

We continue next by calculating the monodromy matrix of the operator $d+a$ around the closed string, which corresponds to the Wilson line along a (closed) curve γ that winds once around the string (starts and ends at point (τ,σ)):[3]

$$\Omega(x,\tau,\sigma) = P\exp\int_{\gamma}(-a(x)).$$

Because $a(x)$ is flat, Ω is independent of the path γ. Then choosing the path $\tau=0,\sigma\in[0,2\pi]$, the monodromy Ω only depends on the point $\gamma(0)=\gamma(2\pi)$, where the path is cut open. A shift on $\gamma(0)$ leads to a similarity transformation ($d\Omega+[a,\Omega]=0$). A change in the points of the path cannot be physical, so only the conjugacy class of the monodromy matrix will be physical. In particular, only its eigenvalues are invariant under this similarity transformation and thus physical. The monodromy with this particular path can be written as

$$\Omega(x) = P\exp\int_0^{2\pi} d\sigma\frac{1}{2}\left(\frac{j_+}{x-1}+\frac{j_-}{x+1}\right).$$

Vector representation If $j=j_v$, and noticing that $j_v^T=-j_v$, then Ω_v is complex (because x is complex) orthogonal $\Omega\in SO(m,\mathbb{C})$ and can be diagonalized as

$$\Omega_v(x) = \begin{cases} \mathrm{diag}\left(e^{iq_1(x)}, e^{-iq_1(x)}, \cdots, e^{iq_{[m/2]}(x)}, e^{-iq_{[m/2]}(x)}\right) & m \text{ even} \\ \mathrm{diag}\left(e^{iq_1(x)}, e^{-iq_1(x)}, \cdots, e^{iq_{[m/2]}(x)}, e^{-iq_{[m/2]}(x)}, 1\right) & m \text{ odd} \end{cases}.$$

These $q_i(x)$, $i=1,...[m/2]$ are the quasi-momenta, which are the physical quantities (apart from possible permutations).

Spinor representation In the spinor representation $j=j_s$, the monodromy matrix has a diagonalized form such as

$$\Omega_s = \mathrm{diag}\left(\exp\left(\pm\frac{i}{2}q_1\pm\frac{i}{2}q_2\pm\cdots\pm\frac{i}{2}q_{[m/2]}\right)\right),$$

[3] P in the monodromy matrix is the path ordering operator.

with all possible combinations of signs included ($2^{[m/2]}$ possibilities). In the case m is even, $SO(m)$ has a chiral representation, and Ω_s can be decomposed in its chiral/anti-chiral parts $\Omega_s^{\pm}$:

$$\Omega_s = \begin{pmatrix} \Omega_s^{+} & 0 \\ 0 & \Omega_s^{-} \end{pmatrix},$$

where in Ω_s^{+} we only allow an even number of plus signs, and in Ω_s^{-} we have odd numbers of plus signs in front of the eigenvalues. We then redefine $\Omega_s^{+} = \mathrm{diag}\left(e^{ip_1}, \cdots, e^{ip_{2^{m/2-1}}}\right)$.

Properties of the monodromy matrix

The monodromy matrix is analytic on x except at $x = \pm 1$, but its eigenvalues $q_i(x)$ present some more singularities. Assume that there is a point x_a^* at which two eigenvalues e^{iq_k} and e^{iq_l} degenerate. The restriction of $\Omega(x)$ to the subspace of these two eigenvalues then has the form

$$\Gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

One can calculate the eigenvalues of this sub-matrix to be

$$\gamma_{\pm} = \frac{1}{2}\left(a + d \pm \sqrt{(a-d)^2 + 4bc}\right).$$

The parameters a, b, c, d depend analytically on x. We define

$$f(x) = (\gamma_+ - \gamma_-)^2 = (a-d)^2 + 4bc = (\mathrm{Tr}\Gamma)^2 - 2\mathrm{Tr}\Gamma^2,$$

which vanishes at $x = x_a^*$, that is $f(x^*) = 0$. But its derivative need not be zero, $f'(x_a^*) \neq 0$. This implies that we can expand the square root above close to x_a^* and obtain $f(x) = f'(x_a^*)(x - x_a^*) + \mathscr{O}\left((x-x_a^*)^2\right)$, and the corresponding expansion for the quasi-momenta

$$e^{iq_{k,l}(x)} = e^{iq_k(x_a^*)}\left(1 \pm \alpha_a \sqrt{x - x_a^*} + \mathscr{O}(x - x_a^*)\right),$$

for some coefficient α_a, or

$$iq_{k,l}(x) = iq_{k,l}(x_a^*) \pm \alpha_a \sqrt{x - x_a^*} + \mathscr{O}(x - x_a^*).$$

A full circle around one of these square root singularities will result in an interchange of the two eigenvalues associated with the singularity: we introduce branch cuts $\mathscr{C}_a$ which connect the square root singularities. The function $q_k(x)$ is therefore analytic except at $\{\pm 1, \mathscr{C}_a\}$. [4] Recalling that the q_k's are only defined modulo 2π, that is, at the cuts the $q_k's$ can be permuted and shifted by multiples of 2π. Defining the principal part of $q_k(x)$ to be

$$\not{q}_k(x) = \frac{1}{2} q_k(x + i\varepsilon) + \frac{1}{2} q_k(x - i\varepsilon),$$

then at the cuts $\mathscr{C}_a$ where the eigenvalues q_k and q_l are permuted, we have:

$$\not{q}_k(x) \mp \not{q}_l(x) = 2\pi n_a, \text{for}\, x \in \mathscr{C}_a.$$

n_a is the mode number of $\mathscr{C}_a$. Without loss of generality, one can restrict the possible permutations to the permutations between $q_k$and q_{k+1} such that[5]

$$\not{q}_k(x) - \not{q}_{k+1}(x) = 2\pi n_{k,a}, \quad x \in \mathscr{C}_{k,a}. \tag{5.3}$$

The next step is to study the asymptotic behaviour of the monodromy matrix at $x \to \infty$, as well as its behaviour at the singularities $x = \pm 1$.

[4] Note that for the bosonic superstring [34], one uses another parameter $z^2 = \frac{x-1}{x+1}$, which has singularities at $\{0, \infty\}$ when $x \to \pm 1$. Other relations include $x \mapsto 1/x$ equivalent to $z \mapsto iz$ and $\frac{dx}{1-1/x^2} = \frac{dz}{z}$.

[5] This condition is complemented with, for m even $\not{q}_{[m/2]-1}(x) + \not{q}_{[m/2]}(x) = 2\pi n_{[m/2],a}$ and for m odd, $2\not{q}_{[m/2]}(x) = 2\pi n_{[m/2],a}$.

Asymptotic behaviour at $x \to \infty$

Expanding the connection $a(x)$ at infinity gives $a(x) = -\frac{1}{x} * j + \mathcal{O}\left(1/x^2\right)$. The monodromy matrix then becomes

$$\Omega(x) = P\exp\{\frac{1}{x}\int_0^{2\pi} d\sigma\, j_\tau + \mathcal{O}\left(1/x^2\right)\} = 1 + \frac{1}{x}\int_0^{2\pi} d\sigma\, j_\tau + \mathcal{O}\left(1/x^2\right).$$

We can see from the result above that the asymptotic behaviour of the monodromy matrix is directly related to a conserved charge of the σ-model

$$J = \frac{\sqrt{\lambda}}{4\pi}\int_0^{2\pi} d\sigma\, j_\tau,$$

being its first order in the expansion in $1/x$:

$$\Omega(x) = 1 + \frac{1}{x}\frac{4\pi J}{\sqrt{\lambda}} + \mathcal{O}\left(1/x^2\right).$$

In the vector representation $J = J_v$ has eigenvalues

$$J_v \simeq \begin{cases} \mathrm{diag}\left(iJ_1, -iJ_1, \cdots, iJ_{[m/2]}, -iJ_{[m/2]}\right) & m \text{ even} \\ \mathrm{diag}\left(iJ_1, -iJ_1, \cdots, iJ_{[m/2]}, -iJ_{[m/2]}, 0\right) & m \text{ odd} \end{cases}.$$

These eigenvalues are directly related to the Dynkin labels of $SO(m)$, $[s_1, s_2, \cdots]$ by

$$J_k = \sum_{j=k}^{[m/2]} s_j - \frac{1}{2}\left(s_{[m/2]-1} + s_{[m/2]}\right)$$

for m even, and

$$J_k = \sum_{j=k}^{[m/2]} s_j - \frac{1}{2}s_{[m/2]}$$

for m odd. Finally we can also write the asymptotic behaviour of the eigenvalues of the monodromy matrix, q_k's. The diagonalization matrix of Ω should also diagonalize J for large x, and consequently

$$q_k(x) = \frac{1}{i}\ln\left(1 + \frac{i}{x}\frac{4\pi J_k}{\sqrt{\lambda}} + \mathscr{O}\left(1/x^2\right)\right) \approx \frac{1}{x}\frac{4\pi J_k}{\sqrt{\lambda}} + \mathscr{O}\left(1/x^2\right). \tag{5.4}$$

To obtain this last result, we fixed the branch of the logarithms so that all q_k's vanished at $x = \infty$.

Asymptotic behaviour at singularities $x = \pm 1$

In [31] there is an argument for why it is enough to integrate the eigenvalues of $j_{v,\pm}$ for determining the leading singular behaviour of the eigenvalues of Ω_v. The argument goes as follows. Suppose that have matrices $u_\pm(\sigma)$ which diagonalize the currents $j_\pm$ for all σ at points $x = \pm 1$. That is

$$j_\pm^{\text{diag}}(\sigma) = u_\pm j_\pm u_\pm^{-1}.$$

Also one considers an analytic continuation of $u_\pm(\sigma)$ to $u_\pm(\sigma, x) = u_\pm(\sigma) + \mathscr{O}(x \mp 1)$. We can expand $u_\pm(x,\sigma) = \sum_{r=0}^\infty (x \mp 1)^r u_{\pm,r}(\sigma)$, as well as $\mathscr{L}(x) = \partial_\sigma + a_\sigma(x,\sigma)$ at the points $x = \pm 1$. Let us focus on the case $x = 1$. Then we want that each term in $u_+ = \sum_{r=0}^\infty (x-1)^r u_{+,r}(\sigma)$ diagonalizes the corresponding term in $\mathscr{L}(x)$.

The expansion of $a_\sigma(x)$ at $x = 1$ is given by

$$a_\sigma(x) = -\frac{1}{2}\frac{j_+}{x-1} - \frac{1}{2}\sum_{r=0}^\infty (x-1)^r j_{-,r}.$$

Thus we can assume the same behaviour of $\tilde{a}(x)$, defined at the singularity $x = 1$ by

$$\begin{aligned}
\partial_\sigma + \tilde{a}(x) &= u_+(\partial_\sigma + a_\sigma)u_+^{-1} \\
&= u_+\partial_\sigma u_+^{-1} - \frac{1}{2}\frac{1}{x-1}u_+ j_+ u_+^{-1} - \frac{1}{2}\sum_{r=0}^\infty (x-1)^r u_+ j_{-,r} u_+^{-1}.
\end{aligned}$$

We first note that $u_+\partial_\sigma u_+^{-1} = \partial_\sigma - (\partial_\sigma u_+)u_+^{-1}$, and consequently

$$\tilde{a}(x) = -(\partial_\sigma u_+)u_+^{-1} - \frac{1}{2}\frac{1}{x-1}u_+ j_+ u_+^{-1} - \frac{1}{2}\sum_{r=0}^\infty (x-1)^r u_+ j_{-,r} u_+^{-1} = \sum_{r=-1}^\infty (x-1)^r \tilde{a}_r(x).$$

Now we determine $u_{+,0}$ such that it diagonalizes the first order $\tilde{a}_{-1}$. The first term in the expression above is of order $\mathscr{O}\left((x-1)^0\right)$, as well as all of the last terms: the only term surviving at this order is the second one. So $u_{+,0}$ diagonalizes j_+ (as expected). One can construct the rest of the matrix $u_+(x,\sigma)$, and perform the same analysis for $x=-1$, thus obtaining $u_\pm(x)$ that completely diagonalizes $\mathscr{L}(x)$ for all σ. To first order we find:

$$u_\pm(x)\mathscr{L}(x)u_\pm^{-1}(x) = \partial_\sigma - \frac{1}{2}\frac{j_\pm^{\text{diag}}}{x\mp 1} + \mathscr{O}\left((x\mp 1)^0\right),$$

and the diagonalized leading order of the connection is $\tilde{a}_\pm(x) = -\frac{1}{2}\frac{j_\pm^{\text{diag}}}{x\mp 1}$ at the singularities $x=\pm 1$, for all σ. The monodromy matrix can then be written as

$$u_\pm(x,2\pi)\,\Omega(x)\,u_\pm^{-1}(x,0) = \exp\left(\frac{1}{2}\int_0^{2\pi} d\sigma\,\frac{j_\pm^{\text{diag}}}{x\mp 1} + \mathscr{O}\left((x\mp 1)^0\right)\right).$$

Looking back at the vector representation of j_ν (5.2), we can see that it is an antisymmetric matrix of the form $M = X_{[i}Y_{j]}$ where $\vec{X}$ and $\vec{Y}$ are independent vectors. Because of being antisymmetric real, its eigenvalues are pure imaginary and come in pairs $\pm i\lambda_k$. In particular one can show that M has maximum rank 2 (maximum two non-zero eigenvalues). Assume that u is an eigenvector of M with eigenvalue $i\lambda$. Then

$$M_{ij}u_j \quad = X_i\left(\vec{Y}\cdot\vec{u}\right) - Y_i\left(\vec{X}\cdot\vec{u}\right) = \quad i\lambda u_i.$$

This is equivalent to

$$\vec{u} \propto \vec{X}\left(\vec{Y}\cdot\vec{u}\right) - \vec{Y}\left(\vec{X}\cdot\vec{u}\right),$$

that is, for any eigenvector $\vec{u}$ related to a non-zero eigenvalue, it has components only in the directions of $\vec{X}$ and $\vec{Y}$. In other words, the eigenvectors associated with non-zero eigenvalues span a vector space of dimension 2 at most. Because all of the non-zero eigenvalues come in pairs, this matrix M has either rank zero or rank 2.[6]

[6] In the particular case of $\vec{X}$ and $\vec{Y}$ being orthogonal then one can use them as a basis of eigenvectors, and a diagonalized form of M would have the form $M = \text{diag}(i\lambda, -i\lambda, 0, ..., 0)$. This is the case of j_ν as $\vec{X} = \vec{X}$ and $\vec{Y} = d\vec{X}$.

By determining the trace of $(j_{v,a})^2$ we learn that the matrix j_v has in fact rank 2:

$$\begin{aligned}
\mathrm{Tr}\,(j_{v,\pm})^2 &= 4\left(X_i\partial_\pm X_j - X_j\partial_\pm X_i\right)\left(X_j\partial_\pm X_i - X_i\partial_\pm X_j\right) \\
&= -8\left(\left(\partial_\pm X_i X_i\right)^2 + \partial_\pm X_j\partial_\pm X_j\right) \\
&= -8\left(\partial_\pm \vec{X}\right)^2.
\end{aligned}$$

where we used the fact that we have $X_i\partial_a X_i = 0$ ($\vec{X}^2 = 1$). Now using the Virasoro constraint (5.1) we find

$$\mathrm{Tr}\,(j_{v,\pm})^2 = -8\left(\partial_\pm X_0\right)^2 = -8\kappa^2.$$

Thus the diagonalized current becomes (independent of σ)

$$j^{\mathrm{diag}}_{v,\pm} = 2i\kappa\,\mathrm{diag}\left(1,-1,0,\cdots\right).$$

We can now relate these eigenvalues with the quasi-momenta q_k:

$$iq_k = \frac{2\pi i\kappa}{x\mp 1}\delta_{k1} + \mathscr{O}\left(\left(x\mp 1\right)^0\right) \qquad \text{if}\, x\to\pm 1. \tag{5.5}$$

Note that κ is directly related to the energy of the classical string.

Inversion symmetry

In our σ-model analysis we found both left and right currents, $j = h^{-1}dh$ and $\ell = -dhh^{-1}$ respectively. We also found a family of flat currents $a_r(x) \equiv a(x)$ related to the right current. There is also a family of left flat currents $a_\ell(x)$ which is related to $a_r(x)$ by an inversion of the spectral parameter. The corresponding flat connections $d+a$ obey the following:

$$\begin{aligned}
h\left(d + a_r(x)\right)h^{-1} &= hdh^{-1} + \frac{1}{1-x^2}h\,jh^{-1} + \frac{x}{1-x^2}h*jh^{-1} \\
&= d - dhh^{-1} - \frac{1}{1-x^2}\ell - \frac{x}{1-x^2}*\ell \\
&= d + \frac{1}{1-1/x^2}\ell + \frac{1/x}{1-1/x^2}*\ell \\
&= d + a_\ell\left(1/x\right).
\end{aligned}$$

But for S^{m-1} the currents obey $j=\ell$ and consequently this relation becomes a symmetry: $h_v\mathscr{L}_v(x)h_v^{-1}=\mathscr{L}_v(1/x)$ and the same for $\mathscr{L}_s$. In terms of the monodromy matrix we have $h(2\pi)\Omega(x)h^{-1}(0)=\Omega(1/x)$ for both Ω_v and Ω_s. If m is even, then the reduction into chiral/anti-chiral components means that h_s inverts chirality (it has one gamma matrix $\vec{\gamma}$) while $j_s=h_s^{-1}dh_s$ preserves it (has the square of the gamma matrix). Consequently, $h_s(2\pi)\Omega^{\pm}(x)h_s^{-1}(0)=\Omega^{\mp}(1/x)$. For a closed string we have $h(0)=h(2\pi)$ and the last expression is just a similarity transformation. Thus $\Omega(x)$ and $\Omega(1/x)$ have the same eigenvalues: the set of eigenvalues q_k transforms into one other eigenvalue under the transformation $x\to 1/x$.

Recalling how q_1 behaves at $x\to\pm 1$. It is easy to see that $q_1(1/x)\to -q_1(x)$. In fact this is not the whole story, and the right expression is

$$q_1(1/x)=4\pi n_0-q_1(x).$$

The fact that there is such an integer n_0 is because when we determine the relation between eigenvalues of the monodromy matrix, these are of the form $e^{iq_k(x)}$, and to find a relation such as the one above requires the use of the complex logarithm function: there is an ambiguity in the choice of branch of the logarithm when $x\to 0$ and when $x\to\infty$, and choosing different branches will give rise to the term with n_0. But the monodromy matrix in the spinor representation has eigenvalues of the shape $e^{\pm\frac{i}{2}q_k(x)}$, which show this same ambiguity. Consequently the integer n_0 has to come with a factor of 4π.

What happens to the other q_k for $k\neq 1$? In the case m odd there are no restrictions, but for m even one finds that the interchange of chiral/anti-chiral representations on Ω gives rise to extra restrictions: one has to flip an odd number of signs of the q_k. As q_1 already flips sign, the rest must show an even number of flips. We will make a possible choice of all of the other q_k staying invariant:[7]

$$q_k(1/x)=(1-2\delta_{k1})q_k(x)+4\pi n_0\delta_{k1}. \tag{5.6}$$

[7] There might be other consistent choices, but one assumes this is the correct one. [31]

Restriction to $\mathbb{R}\times S^5$: Bethe ansatz

The isometry group of S^5 is $SO(4) \cong SU(4)$, which has a chiral spinor representation related to the quasi-momenta p_i as defined above. The chiral representation **4** of $SO(6)$ is equivalent to the fundamental of $SU(4)$: $\Omega_s^+ \sim \mathrm{diag}\left(e^{ip_1},\cdots,e^{ip_4}\right)$ can be seen as the monodromy matrix of $SU(4)$, with quasi-momenta defined by:

$$
\begin{aligned}
p_1 &= \frac{1}{2}(q_1+q_2-q_3), \\
p_2 &= \frac{1}{2}(q_1-q_2+q_3), \\
p_3 &= \frac{1}{2}(-q_1+q_2+q_3), \\
p_4 &= -\frac{1}{2}(q_1+q_2+q_3),
\end{aligned}
\tag{5.7}
$$

and $\sum_i p_i = 0$. The inversion symmetry found in terms of the q_k (5.6) becomes in terms of p_k:

$$p_{1,2}(1/x) = 2\pi n_0 - p_{2,1}(x) \quad , \qquad p_{3,4}(1/x) = -2\pi n_0 - p_{4,3}(x).$$

This inversion symmetry gives rise to the structure of cuts and mirror cuts shown in figure 5.1.[8]

The asymptotic behaviour of the p_k at the poles $x=\pm 1$ can be seen to be

$$p_{1,2}(x) = -p_{3,4}(x) = \frac{\pi\kappa}{x\mp 1} + \mathscr{O}\left((x\mp 1)^0\right) \qquad \text{if} x\to\pm 1.$$

Finally the asymptotic behaviour at $x=\infty$ is

$$
\begin{aligned}
p_1(x) &= \frac{1}{x}\frac{2\pi}{\sqrt{\lambda}}(J_1+J_2-J_3)+\cdots = \frac{1}{x}\frac{4\pi}{\sqrt{\lambda}}\left(\frac{3}{4}r_1+\frac{1}{2}r_2+\frac{1}{4}r_3\right)+\cdots, \\
p_2(x) &= \frac{1}{x}\frac{2\pi}{\sqrt{\lambda}}(J_1-J_2+J_3)+\cdots = \frac{1}{x}\frac{4\pi}{\sqrt{\lambda}}\left(-\frac{1}{4}r_1+\frac{1}{2}r_2+\frac{1}{4}r_3\right)+\cdots, \\
p_3(x) &= \frac{1}{x}\frac{2\pi}{\sqrt{\lambda}}(-J_1+J_2+J_3)+\cdots = \frac{1}{x}\frac{4\pi}{\sqrt{\lambda}}\left(-\frac{1}{4}r_1-\frac{1}{2}r_2+\frac{1}{4}r_3\right)+\cdots, \\
p_4(x) &= \frac{1}{x}\frac{2\pi}{\sqrt{\lambda}}(-J_1-J_2-J_3)+\cdots = \frac{1}{x}\frac{4\pi}{\sqrt{\lambda}}\left(-\frac{1}{4}r_1-\frac{1}{2}r_2-\frac{3}{4}r_3\right)+\cdots,
\end{aligned}
\tag{5.8}
$$

[8] In figure 5.1 both cuts and mirror cuts are shown outside the circle $|x|^2=1$ for easier visualization, even though to be exact one of the cuts should be placed inside this circle (due to inversion symmetry $x\to 1/x$).

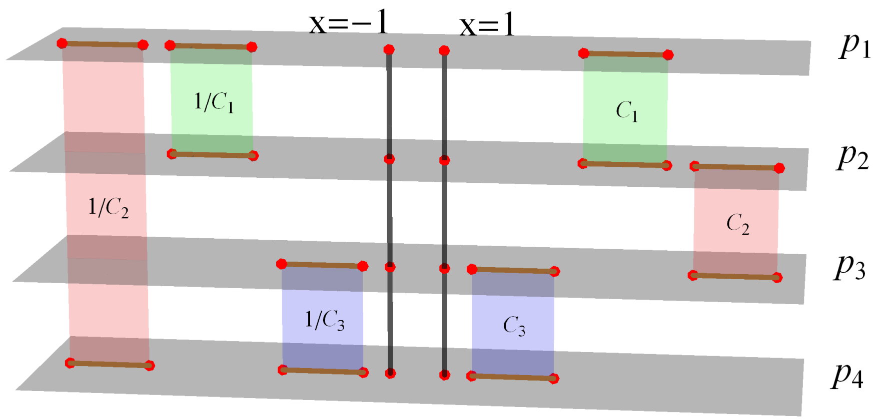

Figure 5.1: Algebraic curve of $\mathbb{R}\times S^5$ in the spinor representation of $SO(6)$. Bosonic excitations are shown: cut $\mathscr{C}_1$ between sheets 1,2 (and its mirror cut), cut $\mathscr{C}_2$ between sheets 2,3 (mirror cut between 1,4), and cut $\mathscr{C}_3$ between 3,4 (and mirror cut). The dots in red are singularities of the quasi-momenta, including $x=\pm 1$.

In the above expression $[r_1, r_2, r_3]$ are the Dynkin labels of $SU(4)$, related to the Dynkin labels of $SO(6)$ $[s_1, s_2, s_3]$[9] through:

$$r_1 \quad = \quad s_2 = J_2 - J_3\,, \quad r_2 = s_1 = J_1 - J_2\,, \quad r_3 = s_3 = J_2 + J_3\,.$$

Branch cuts

From the above results, plus the known relation connecting sheets k and $k+1$:

$$p_k(x) - p_{k+1}(x) = 2\pi n_a\,, \quad x \in \mathscr{C}_a,$$

we find that e^{ip} is a single valued holomorphic function of a Riemann surface with four sheets, except at the points $x = \pm 1, \infty$. But it is not an algebraic curve, because it has an essential singularity at $x = \pm 1$ of the type $\exp\left(\frac{i}{x\mp 1}\right)$. On the other hand $p(x)$ only has pole singularities, but it is only defined up to multiples of 2π. So we work with its derivative $p'(x)$, which has double poles at $x = \pm 1$, and no other poles, and will indeed be an algebraic

[9]And consequently to the J_k by inverting the the result of the J_k in terms of the s_j:

$$J_1 = s_1 + \frac{1}{2}s_2 + \frac{1}{2}s_3\,, \quad J_2 = \frac{1}{2}s_2 + \frac{1}{2}s_3\,, \quad J_3 = -\frac{1}{2}s_2 + \frac{1}{2}s_3.$$

curve of degree four.

Define a function $y(x)$ that will be analytic at $x=\pm1$:

$$y_k(x) = (x-1/x)^2 x p_k'(x).$$

The extra factor not only removes the poles but also simplifies the the symmetry $x \to 1/x$. This function y satisfies a quartic algebraic equation

$$F(y,x) = P_4(x) y^4 + P_2(x) y^2 + P_1(x) y + P_0(x) = P_4(x) \prod_{k=1}^{4} (y - y_k(x)) = 0. \tag{5.9}$$

For a solution that shows a finite number of cuts, one can assume that $P_k(x)$ is a polynomial in x. There is no term y^3 because we have $p_1+p_2+p_3+p_4=0$.

Noting that if $p(x) \sim \sqrt{x-x^*}$ at a branch point x^*, then $y \sim 1/\sqrt{x-x^*}$. Then at this point, solving $F(y,x)=0$ at $y \to \infty$ implies $(x-x^*) = -P_4(x)/P_2(x)$, that is we look for the roots of P_4/P_2. Considering a general P_2, and assuming P_4 to be of order $2A$, we look for roots of $P_4(x)$

$$P_4(x) \sim \prod_{a=1}^{A} (x-a_a)(x-b_a),$$

and we can see that A is just the number of cuts and a_a, b_a are the branch points.[10]

Asymptotics at $x \to \infty$

At $x=\infty$ we had $p(x) \sim 1/x$, so we now have $y(x) \sim x$. From (5.9) we can see that the highest order of each P_k will be related to the order of P_0 by $P_k(x) \sim x^{-k} P_0(x)$. Assuming P_4 is of order x^{2A}, then $P_2 \sim x^{2A+2}$, $P_1 \sim x^{2A+1}$ and $P_0 \sim x^{2A+4}$. Through the $x \to 1/x$ symmetry we have that at $x=0$ $y(x) \sim \frac{1}{x}$ (the inversion symmetry requires that $p_k \sim x$ at $x=0$). Then the lowest order term in P_k is related to the lowest order term of P_0 by $P_k(x) = x^k P_0(x)$, that is, if P_0's lowest order is x^0, then $P_k(x) \sim x^k$ at $x \sim 0$. In general $P_k(x)$ will be a polynomial

[10] The algebraic equation can have further cuts of the kind $y \sim \sqrt{x-x^*}$, which would lead to the unwanted behaviour $p(x) \sim (x-x^*)^{3/2}$. Their positions can be found as the roots of the discriminant R of the quartic equation [31]

$$R = -4P_1^2 P_2^3 + 16 P_0 P_2^4 - 27 P_1^4 P_4 + 144 P_0 P_1^2 P_2 P_4 - 128 P_0^2 P_2^2 P_4 + 256 P_0^3 P_4^2.$$

All solutions of $R=0$ with odd multiplicity originate the unwanted branch cuts, so we have to require that the discriminant be a perfect square $R(x) = Q(x)^2$ for some polynomial $Q(x)$.

of the type

$$P_k(x) = a_{k1}x^k + \cdots + a_{k,2A+4-2k}x^{2A+4-k}.$$

But for this expression to make sense, one would have to have $2A+4-k > k$ for all $k = 0,1,2,4$, requirement for the highest order to be not smaller that the lowest order, and we find that $A \geq 2$. We can redefine $A \to A-2$ (to have $A \geq 0$), and we then find[11]

$$P_k(x) = a_{k1}x^k + \cdots + a_{k,2A+8-2k}x^{2A+8-k}.$$

Inversion symmetry

We have seen how the inversion symmetry determines the behaviour of $y(x)$ at $x \sim 0$. For a finite x the function $y(x)$ obeys:

$$\begin{aligned} y_{1,2,3,4}(1/x) &= \left(x-\frac{1}{x}\right)^2 \frac{1}{x}\frac{d}{dy}p_{1,2,3,4}(y)\Big|_{y=\frac{1}{x}} \\ &= \left(x-\frac{1}{x}\right)^2 \frac{1}{x}\frac{d}{dy}\left(-p_{2,1,4,3}(1/y)\right)\Big|_{y=\frac{1}{x}} \\ &= \left(x-\frac{1}{x}\right)^2 xp'_{2,1,4,3}(x) = y_{2,1,4,3}(x). \end{aligned} \tag{5.10}$$

But for $y(x)$ to behave in this form, then the polynomials have to obey $P_k(1/x) = x^{-2A-8}P_k(x)$, that is, the coefficients of these polynomials are the same when read backwards and forwards.[12] For the particular case of P_4 and its expansion in branch points, the inversion symmetry requires that $a_a = 1/a_b$ and $b_a = 1/b_b$ for a pair of cuts $\mathscr{C}_{a,b}$ that are interchanged through this symmetry.

The symmetry found in the coefficients of P_k only states that $y_k(1/x) = y_{\pi(k)}(x)$ for some permutation $\pi(k)$. In fact the permutation chosen above in (5.10) is only allowed if we

[11] The discriminant becomes:

$$R(x) = \sum_{\ell=1}^{10A+24} r_\ell x^{8+\ell}.$$

[12] The discriminant obeys $R(1/x) = x^{-10A-40}R(x)$.

consider the four extra constraints

$$y_{1,3}(x) = y_{2,4}(x) \quad \text{for}\, x = \pm 1.$$

This can be seen from the inversion symmetry at the points $x = \pm 1$.[13]

Asymptotic behaviour at the singularities $x = \pm 1$

Because of the behaviour of the quasi-momenta at the singularities, one can assume that the most general expansion allowed for its derivative is

$$p_k^{'}(x) = \frac{\alpha_k^{\pm}}{(x \mp 1)^2} + \frac{\beta_k^{\pm}}{x \mp 1} + \mathscr{O}\left((x \mp 1)^0\right).$$

From requiring that $y_{1,3}(\pm 1) = y_{2,4}(\pm 1)$, we find that $\alpha_{1,3}^{\pm} = \alpha_{2,4}^{\pm}$. Now requiring $y_{1,3}(1/x) = y_{2,4}(x)$ gives us a condition for the β's. One can easily see that we have $\beta_{1,3}^{\pm} = -\beta_{2,4}^{\pm}$. But other constraints can be found. The sum of all sheets has to be zero $p_1 + p_2 + p_3 + p_4 = 0$, and the $p_k^{'}$ have to obey the same equation. Around the singularities, the condition $\sum p_k^{'} = 0$ requires that $\alpha_{1 or 2}^{\pm} = -\alpha_{3 or 4}^{\pm}$, with no extra requirements for the $\beta_k^{\pm}$. Consequently, there are three independent parameters that define the behaviour of all $p_k^{'}$ at each point $x = \pm 1$, for example $\alpha_1^{\pm} = \alpha_2^{\pm} = -\alpha_3^{\pm} = -\alpha_4^{\pm}$, $\beta_1^{\pm} = -\beta_2^{\pm}$ and $\beta_3^{\pm} = -\beta_4^{\pm}$. One other information we have is that the residues of p at $x = \pm 1$ are equal (proportional to κ, related to the energy), giving us one more constraint on the α's, and finally, if p does not have a logarithmic behaviour, then all β's to be zero (total of five extra constraints).

Cycles, periods and fillings

As was said before the eigenvalues e^{ip_k} of the monodromy matrix Ω_s^{+} are holomorphic in x, but the p_k can present at particular cuts jumps of multiples of 2π (and are smooth apart from that). These cuts can in general be originated from logarithmic or branch cut

[13] This seems to require that the discriminant has a quadruple pole at $x = \pm 1$, see [31]

$$R(x) = x^8 \left(x^2 - 1\right)^4 \left(\sum_{\ell=1}^{10A+16} r_\ell x^\ell\right).$$

singularities, but we will assume for now that p_k does not have the former. Such cuts show up when a closed integral around a singularity does not vanish. Considering $\mathscr{A}_a$-cycles to be closed paths around a cut $\mathscr{C}_a$, see figure 5.2, we have that

$$\oint_{\mathscr{A}_a} dp = 2\pi m_a,$$

where one can rearrange the cuts between (square root) branch points in order to set $m_a = 0$, except for the case when the cycle $\mathscr{A}_a$ crosses a condensate of poles. These condensates happen when there is constant density of poles (of p) in a certain area, and can be viewed as extra logarithmic cuts which connect pairs of original cuts $\mathscr{C}_l$.

In the same way as the quasi-momenta, the branch cuts and poles have to obey the inversion symmetry. So for each cut $\mathscr{C}_a$ there is an image or mirror cut $\mathscr{C}_{A+a} = 1/\mathscr{C}_a$, the independent cuts are then given by $\mathscr{C}_a$, $a = 1, \cdots, A$. The same is true for the $\mathscr{A}_a$ cycles. Assuming that there is a cut $\mathscr{C}_a$ connecting sheets $1,2$ (or sheets $3,4$, see figure 5.1), the cycle $\mathscr{A}_a$ circles the cut $\mathscr{C}_a$ in either of the sheets. Through inversion symmetry, the cycle $\mathscr{A}_b \equiv \mathscr{A}_{A+a} = 1/\mathscr{A}_a$ circles the cut $\mathscr{C}_a^{-1}$ on both of the sheets $1,2$ once more. Then

$$\oint_{\mathscr{A}_a} dp_1 = -\oint_{\mathscr{A}_b} dp_2 = \oint_{\mathscr{A}_{A+a}} dp_1.$$

One obtains this result by first applying inversion symmetry, through which $\mathscr{A}_a \to \mathscr{A}_b$ and $dp_1 \to -dp_2$, so while the first integral is in sheet 1, the second is in sheet 2. Also one can switch sheets 1,2 by using an involution property of solutions of the equation $F(x,y) = 0$.[14]

Considering another type of cut, between sheets $2,3$, its mirror cut is between sheets $1,4$, and we have

$$\oint_{\mathscr{A}_a} dp_2 = -\oint_{\mathscr{A}_b} dp_1.$$

We find then that for each pair of (mirror) cuts we have only one constraint $\oint_{\mathscr{A}_a} dp = 0$. Considering that there are A branch cuts, the $\mathscr{A}$-cycles would give us $\frac{1}{2}A$ constraints. Indeed the number of constraints is $\frac{1}{2}A - 2$ because the cycles around all poles will be trivially zero

[14] An involution $y \to -y$ of the Riemann surface can be used to move the contour between the two sheets, by interchanging the two sheets and flipping the sign of $y(x)$. If $y(x)$ changes sign (for the same values of x), so will the differential dp. This corresponds to going from Ω^+ to Ω^-, or from $q_{1,2,3}$ to $q_{4,5,6} = -q_{3,2,1}$.

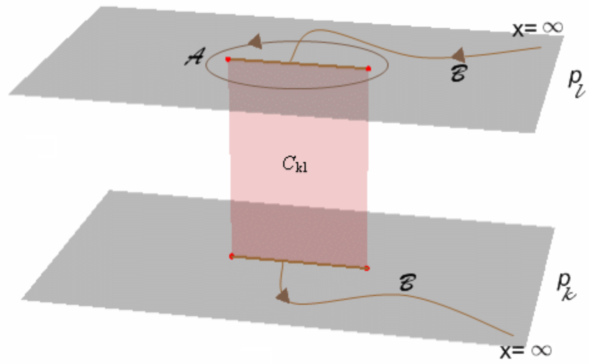

Figure 5.2: One can define an $\mathscr{A}$-cycle around a cut which connects sheets k,l, as well as a $\mathscr{B}$-period contour connecting points $x=\infty$ on both sheets k,l.

(there are no more single poles on any sheet). There are two independent such cycles (the ones from sheets 1 and 2 are related, so are the ones from 3 and 4).

We now concentrate on the fact that through a cut $\mathscr{C}_a$ the eigenvalues $p_k(x)$ can permute and be shifted by a multiple of 2π. This can be summarized by the integral of dp along a curve $\mathscr{B}_a$ connecting the points $x=\infty$ on the two sheets connected by the cut $\mathscr{C}_a$, see figure 5.2. $p(x)$ is analytic along the $\mathscr{B}_a$-period except at the point where $\mathscr{B}_a$ intersects $\mathscr{C}_a$. Recalling that we set the logarithm branch such that $p(\infty)=0$, we have that the shift of $p(x)$ at the cut $\mathscr{C}_a$ is

$$\int_{\mathscr{B}_a} dp = 2\pi n_a.$$

What happens with respect to the inversion symmetry? Because we have chosen the quasi-momenta to be zero at infinity $p_k(\infty)=0$, we automatically have $\int_0^\infty dp_k = -p_k(0)$. Then the inversion symmetry tells us that

$$p_{1,2}(0) = -p_{3,4}(0) = 2\pi n_0,$$

called the momentum constraint (reduces the degrees of freedom by one). Note that n_0 is an overall winding number. The $\mathscr{B}_a$-period between sheets 1,2 (or 3,4) crossing the cut $\mathscr{C}_a$

at the point x_a gives us, by inversion symmetry,

$$\begin{aligned}\int_{\mathscr{B}_a} dp &= \int_{\infty}^{x_a} dp_1 + \int_{x_a}^{\infty} dp_2 = -\int_0^{1/x_a} dp_2 - \int_{1/x_a}^{0} dp_1 \\ &= -\int_0^{\infty} dp_2 - \int_{\infty}^{x_b} dp_2 - \int_{\infty}^{0} dp_1 - \int_{x_b}^{\infty} dp_1 \\ &= p_2(0) - p_1(0) + \int_{\infty}^{x_b} dp_1 + \int_{x_b}^{\infty} dp_2 = \int_{\mathscr{B}_b} dp,\end{aligned}$$

where we have used the same arguments as for the $\mathscr{A}$-cycles, plus the fact that $\int_0^x = \int_0^\infty - \int_x^\infty$. For a $\mathscr{B}$-period between sheets 2,3 (mirror cut between 1,4) we get similarly:

$$\begin{aligned}\int_{\mathscr{B}_a} dp &= \int_{\infty}^{x_a} dp_2 + \int_{x_a}^{\infty} dp_3 = -\int_0^{1/x_a} dp_1 - \int_{1/x_a}^{0} dp_4 \\ &= p_1(0) - p_4(0) - \int_{\infty}^{x_b} dp_1 - \int_{x_b}^{\infty} dp_4 = 4\pi n_0 - \int_{\mathscr{B}_b} dp\,.\end{aligned}$$

We find that the $\mathscr{B}$-periods through a pair of (mirror) cuts are related, and together with the momentum constraint, the number of constraints from the $\mathscr{B}$-periods is $\frac{1}{2}A+1$.

Degrees of freedom: The algebraic curve is a solution $y(x)$ of the equation $F(x,y)=0$. This equation has a total of $8A+22$ coefficients, from all possible coefficients of the polynomials $P_k(x)$ (each has a total of $2A+8-2k$ coefficients). Using the inversion symmetry on the polynomials $P_k(x)$, one finds that $A+4-k$ of the coefficients of P_k are determined by the others, giving a total of $4A+9$ constraints. Because we are determining solutions of $F(x,y)=0$, the overall normalization of F doesn't matter, removing one more degree of freedom (total degrees of freedom up to now are $4A+12$). We have seen that branch points appear as roots of the discriminant $R(x)=0$ (defined in footnote 10on page 123) which have even multiplicity. The number of non-trivial (other than zero and ±1) roots of $R(x)$ after inversion symmetry is $5A+8$, and the even multiplicity cuts this number by 2: $R(x)=0$ fixes $\frac{5}{2}A+4$ coefficients of the P_k's. As was seen above, at the singularities $x=\pm1$, the values of the residues are related by inversion symmetry, and the absence of logarithmic singularities gives 5 more constraints added to another 4 from constraining the y_k to non-trivial permutations. The $\mathscr{A}$-cycles and $\mathscr{B}$-periods give $A-1$ more constraints. The total of degrees of freedom in solving the equation $F(x,y)=0$ is $\frac{1}{2}A$, which will correspond to one filling number for each pair of cuts.

The filling of a cut $\mathscr{C}_a$ is given by (using integration by parts)

$$K_a = -\frac{\sqrt{\lambda}}{8\pi^2 i}\oint_{\mathscr{A}_a} dx\left(1-\frac{1}{x^2}\right)p(x) = \frac{\sqrt{\lambda}}{8\pi^2 i}\oint_{\mathscr{A}_a}\left(x+\frac{1}{x}\right)dp. \tag{5.11}$$

We should introduce a further quantity L such that

$$L = \frac{\sqrt{\lambda}}{16\pi^2 i}\oint_{+1}\sum_{k=1}^{4}\varepsilon_k p_k + \frac{\sqrt{\lambda}}{16\pi^2 i}\oint_{-1}\sum_{k=1}^{4}\varepsilon_k p_k + \sum_{a=1}^{A/2}\frac{\sqrt{\lambda}}{8\pi^2 i}\oint_{\mathscr{A}_a}\frac{dx}{x^2}\sum_{k=1}^{4}\varepsilon_k p_k, \tag{5.12}$$

where $\varepsilon_k = (1,1,-1,-1)$ is defined by the inversion symmetry $p_k(1/x) = -p_{k'}(x) + 2\pi\varepsilon_k n_0$, with $k^{'} = (2,1,4,3)$ when $k=(1,2,3,4)$.We know the behaviour of p_k at the singularities:

$$p_k \sim \frac{\pi\kappa}{x\mp 1} + \mathscr{O}\left((x\mp 1)^0\right).$$

Then $\sum_{k=1}^{4}\varepsilon_k p_k = 4p_1(x)$ at these singularities, and the sum of the first two integrals of (5.12) just gives $\sqrt{\lambda}\kappa$. Among $\{L, K_a\}$ there are only $A/2$ independent continuous parameters, $A/2-1$ independent fillings K_a and the quantity L. This is true because L is related to the fillings by the constraint

$$n_0 L = \sum_{a=1}^{A/2} n_a K_a.$$

The proof of this constraint can be found in page 685 of [34] and page 641 of [31].

The filling numbers have been shown to be the action angle variables of the theory [182], and in comparison to the gauge side, these filling numbers correspond to an integer number of Bethe roots.

Global charges

Now we want to determine the global charges, or equivalently the Dynkin labels, at $x=\infty$, as functions of the fillings. From the relation (5.8) between the quasi-momenta p_k and the

Dynkin labels $r_{1,2,3}$ at infinity, we easily see that:

$$r_j = \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_\infty dx \left(p_j(x) - p_{j+1}(x)\right).$$

From this expression, and using inversion symmetry, we find[15]

$$\begin{aligned}
r_1 &= -\frac{\sqrt{\lambda}}{8\pi^2 i} \oint_\infty dx x \left(p_1'(x) - p_2'(x)\right) = -\frac{\sqrt{\lambda}}{8\pi^2 i} \oint_\infty dx x \frac{1}{x^2} \left(p_2'(1/x) - p_1'(1/x)\right) \\
&= -\frac{\sqrt{\lambda}}{8\pi^2 i} \oint_0 dy \left(-\frac{1}{y^2}\right) y \left(p_2'(y) - p_1'(y)\right) = \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_0 dy \frac{1}{y} \left(p_2'(y) - p_1'(y)\right) \\
&= \frac{\sqrt{\lambda}}{4\pi} \left(p_2'(0) - p_1'(0)\right),
\end{aligned}$$

and a similar result for r_3 and r_2:

$$\begin{aligned}
r_3 &= -\frac{\sqrt{\lambda}}{8\pi^2 i} \oint_\infty dx x \left(p_3'(x) - p_4'(x)\right) = -\frac{\sqrt{\lambda}}{4\pi} \left(p_3'(0) - p_4'(0)\right), \\
r_2 &= -\frac{\sqrt{\lambda}}{8\pi^2 i} \oint_\infty dx x \left(p_2'(x) - p_3'(x)\right) = -\frac{\sqrt{\lambda}}{4\pi} \left(p_4'(0) - p_1'(0)\right).
\end{aligned}$$

We want to re-write the Dynkin labels as functions of the fillings. Using the results of Appendix C.1 we can finally write:

$$\begin{aligned}
r_1 &= -\frac{\sqrt{\lambda}}{8\pi^2 i} \oint_\infty dx x \left(p_1'(x) - p_2'(x)\right) = \sum_{a=1}^{A_2/2} K_{2,a} - 2\sum_{a=1}^{A_1/2} K_{1,a}, \\
r_3 &= -\frac{\sqrt{\lambda}}{8\pi^2 i} \oint_\infty dx x \left(p_3'(x) - p_4'(x)\right) = \sum_{a=1}^{A_2/2} K_{2,a} - 2\sum_{a=1}^{A_3/2} K_{3,a}, \\
r_2 &= -\frac{\sqrt{\lambda}}{8\pi^2 i} \oint_\infty dx x \left(p_2'(x) - p_3'(x)\right) = L + \sum_{a=1}^{A_1/2} K_{1,a} - 2\sum_{a=1}^{A_2/2} K_{2,a} + \sum_{a=1}^{A_3/2} K_{3,a}.
\end{aligned}$$

One final useful relation, also obtained from these relations, is

$$\frac{1}{2} r_1 + r_2 + \frac{1}{2} r_3 = \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_\infty dx \left(p_1 + p_2\right) = L - \sum_{a=1}^{A_2/2} K_{2,a}.$$

[15] Recall that $p_1'(x) = \frac{1}{x^2} p_2'(x)$.

5.2 General Bethe Ansatz for a string in $\mathbb{R}\times S^{m-1}$

We want to be able to recover the group theory information into our Bethe Ansatz equations. To do so we introduce singular resolvents $\tilde{H}_k$, related to the simple roots of $\mathfrak{so}(m)$:

$$\tilde{H}_k = \sum_{j=1}^{k} q_j, \tag{5.13}$$

where for m even we also have

$$\tilde{H}_{[m/2]-1} = \sum_{j=1}^{[m/2]-1} \frac{1}{2}q_j - \frac{1}{2}q_{[m/2]}\,, \quad \tilde{H}_{[m/2]} = \sum_{j=1}^{[m/2]-1} \frac{1}{2}q_j + \frac{1}{2}q_{[m/2]},$$

and for m odd

$$\tilde{H}_{[m/2]} = \sum_{j=1}^{[m/2]} \frac{1}{2}q_j.$$

These singular resolvents are directly related to the roots of $\mathfrak{so}(m)$. At $x=\infty$ one can expand the resolvents and find a relation to the Dynkin labels $[s_1, s_2, \cdots]$. Considering the Cartan matrix M_{kj} of $\mathfrak{so}(m)$, one finds

$$\tilde{H}_k(x) = \frac{1}{x}\sum_{j=1}^{[m/2]} M_{kj}^{-1}\frac{4\pi s_j}{\sqrt{\lambda}} + \mathscr{O}(1/x^2). \tag{5.14}$$

This can easily be seen to be true for the $SO(6)$ case by using the expansion of the quasi-momenta q_j and the explicit form of the Cartan matrix M. At the singularities one finds

$$\tilde{H}_k = \sum_{j=1}^{k} \frac{2\pi\kappa}{x\mp 1}\delta_{j1} + \mathscr{O}\left((x\mp 1)^0\right) = \frac{2\pi\kappa}{x\mp 1}\sum_{j=1}^{[m/2]} M_{kj}^{-1} V_{v,j} + \mathscr{O}\left((x\mp 1)^0\right),$$

where $V_{v,j} = (1,0,0,\cdots)$ are the Dynkin labels in the vector representation [31]. The $\tilde{H}_k$ also have a inversion symmetry transformation:

$$\begin{aligned}\tilde{H}_k(1/x) &= \tilde{H}_k(x) - 2M_{k1}^{-1}\tilde{H}_1(x) + 4\pi n_0 M_{k1}^{-1} \\ &= \tilde{H}_k(x) - 2\sum_{j,\ell=1}^{[m/2]} M_{kj}^{-1} V_{v,j} V_{v,\ell}\tilde{H}_\ell(x) + 4\pi n_0 \sum_{j=1}^{[m/2]} M_{kj}^{-1} V_{v,j}\,.\end{aligned}$$

Assume that we have a function H_k such that $H_k(x)$incorporates the $A/2$ cuts, while $H_k(1/x)$ incorporates the mirror image cuts. Then $\tilde{H}_k$ will be a linear combination of these, but will also have components at the singularities $x = \pm 1$, as well as an arbitrary constant c_k to be able to fix $\tilde{H}_k = 0$ at infinity. Finally $\tilde{H}_k$ will have a constant corresponding to its value at $x = 0$. Then we can consider an ansatz such as[16]

$$\tilde{H}_k(x) = H_k(x) + H_k(1/x) - 2M_{k1}^{-1} H_1(1/x) + \frac{4\pi\kappa}{x - 1/x} M_{k1}^{-1} + c_k - c_1 M_{k1}^{-1} + 2\pi n_0 M_{k1}^{-1}. \tag{5.15}$$

One introduces a density function $\rho_k(x)$ describing the discontinuities across the cuts $\mathscr{C}_k = \bigcup \mathscr{C}_{k,a}$ ($\mathscr{C}_{k,a}$ are the connected components of the curves $\mathscr{C}_k$) as

$$\rho_k(x) = \frac{1 - 1/x^2}{2\pi i} \left(H_k(x - i\varepsilon) - H_k(x + i\varepsilon) \right) \qquad x \in \mathscr{C}_k,$$

where the weight $1 - 1/x^2$ will be essential for ρ_k to be a density. With this definition, the resolvent becomes

$$H_k(x) = \int_{\mathscr{C}_k} dy \frac{\rho_k(y)}{1 - 1/y^2} \frac{1}{y - x}. \tag{5.16}$$

In fact substituting ρ_k into this last expression, we find that the resulting integral can be represented by a contour integral, with the contours surrounding all the cuts. The integrand has only one singularity on the outside of the contour: it is a pole at $y = x$, with residue $H_k(x)$. Shrinking the contour around the cuts we find the that the above expression is true.

We can now use the asymptotic behaviour of the resolvents H_k at $x = \infty$ and $x = 0$ to determine some of the unknown constants. At $x = \infty$, the behaviour of H_k is given by

$$H_k(x) = -\frac{1}{x} \int_{\mathscr{C}_k} dy \frac{\rho_k(y)}{1 - 1/y^2} + \mathscr{O}\left(1/x^2\right) = -\frac{1}{x} \left(\int_{\mathscr{C}_k} dy \rho_k(y) + \int_{\mathscr{C}_k} dy \frac{\rho_k(y)}{y^2 - 1} \right) + \mathscr{O}\left(1/x^2\right).$$

Differentiating H_k, we easily see that the derivative of H_k at $x = 0$ is just $H_k'(0) = \int_{\mathscr{C}_k} dy \frac{\rho_k(y)}{y^2 - 1}$.

[16] This ansatz is defined up to an anti-symmetric function

$$H_k(x) \to H_k(x) + f_k(x) - f_k(1/x) + 2M_{k1}^{-1} f_1(1/x).$$

Also one can define the fillings (normalizations) of the densities as

$$K_k = \frac{\sqrt{\lambda}}{4\pi} \int_{\mathscr{C}_k} dy \rho_k(y) \, .$$

As a consequence we can rewrite $H_k(x)$ at $x = \infty$ in the following form

$$H_k(x) = -\frac{1}{x} \left(\frac{4\pi}{\sqrt{\lambda}} K_k + H_k^{'}(0) \right) + \mathscr{O}(1/x^2).$$

At $x = 0$, a simple Taylor expansion of $H_k(x)$ produces

$$H_k(x) = H_k(0) + x H_k^{'}(0) + \mathscr{O}(x^2).$$

Now we can relate these expansions with the analogue expansions for $\tilde{H}_k(x)$, in order to determine the constants in its expression (5.15). For $x = \infty$ (5.15) becomes:

$$\begin{aligned} \tilde{H}_k(x) = & \, c_k - c_1 M_{k1}^{-1} + 2\pi n_0 M_{k1}^{-1} + H_k(0) - 2M_{k1}^{-1} H_1(0) + \\ & + \frac{1}{x} \left(4\pi\kappa M_{k1}^{-1} - \frac{4\pi}{\sqrt{\lambda}} K_k - 2M_{k1}^{-1} H_1^{'}(0) \right) + \mathscr{O}\left(1/x^2\right). \end{aligned}$$

We also know the asymptotic behaviour of $\tilde{H}_k(x)$ as a function of the Dynkin labels (5.14). Comparing these two, we find a relation between fillings and Dynkin labels

$$K_k = \sqrt{\lambda} M_{k1}^{-1} \left(\kappa - \frac{1}{2\pi} H_1^{'}(0) \right) - \sum_{j=1}^{[m/2]} M_{kj}^{-1} s_j \, , \tag{5.17}$$

and also a relation for the constants in $\tilde{H}_k$:[17]

$$c_k = c_1 M_{k1}^{-1} - 2\pi n_0 M_{k1}^{-1} - H_k(0) + 2M_{k1}^{-1} H_1(0) \, .$$

For the particular case of $k = 1$ one finds the momentum constraint written as:

$$H_1(0) = 2\pi n_0. \tag{5.18}$$

[17]The constant c_1 cannot be fixed from these equations.

Finally substituting these results back into (5.15) we get

$$\tilde{H}_k(x) = H_k(x) + H_k(1/x) - H_k(0) + M_{k1}^{-1}\left\{-2H_1(1/x) + 2H_1(0) + \frac{4\pi\kappa}{x - 1/x}\right\}. \tag{5.19}$$

We found an expansion of the singular resolvents $\tilde{H}_k$ which respects the asymptotic behaviour of the quasi-momenta, and has branch cuts along the $\mathscr{C}_k$. These branch cuts don't exist at the level of the monodromy matrix, as the latter are analytic everywhere except special points. To make sure of this analyticity condition, one has to enforce that across a cut, only the labeling of the sheets and the branch of the logarithms can change. This condition is reflected in the following expression for the singular resolvents:

$$\sum_{j=1}^{[m/2]} M_{kj}\tilde{\not{H}}_j(x) = 2\pi n_{k,a} \quad , x \in \mathscr{C}_{k,a}\,, \tag{5.20}$$

with $\not{H}_k(x) = \frac{1}{2}H_k(x+i\varepsilon) + \frac{1}{2}H_k(x-i\varepsilon)$. The above equations (5.20) are the Bethe equations. Substituting the expression for $\tilde{H}_k(x)$ in the Bethe equations, these become[18]

$$2\pi n_{k,a} = \sum_{j=1}^{[m/2]} M_{kj}\left(\not{H}_j(x) + H_j(1/x) - H_j(0)\right) + \delta_{k1}\left(-2H_1(1/x) + 2H_1(0) + \frac{4\pi\kappa}{x - 1/x}\right),$$

when $x \in \mathscr{C}_{k,a}$.

For a given set of mode numbers $n_{k,a}$ and of fillings $K_{k,a} = \frac{\sqrt{\lambda}}{4\pi}\int_{\mathscr{C}_{k,a}} dy\rho_k(y)$, the Bethe equations only have a solution if κ has particular values. That is the same as saying that the energy has particular values, because in classical string solutions, κ is related to the spacetime energy through (a less trivial relation exists for infinite length strings)

$$\Delta = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma\partial_\tau X_0 = \sqrt{\lambda}\kappa.$$

One can rewrite some equations to get κ, or Δ, as a function of the fillings and the length

[18] For the cut $x \in \mathscr{C}_{k,a}$ the resolvent $H(x)$ notices the cut, but $H(1/x)$ does not: we are considering one cut at a time and $H(1/x)$ notices the mirror cut.

L. In fact using the definition of L (5.12) we find

$$\sqrt{\lambda}\kappa = L - \frac{\sqrt{\lambda}}{2\pi}\sum_{a=1}^{A/2}\frac{1}{2\pi i}\oint_{\mathscr{A}_a}\frac{dx}{x^2}q_1(x) = L + \frac{\sqrt{\lambda}}{2\pi}\sum_{a=1}^{A/2}\int_{C_{1,a}}\frac{dy\rho_1(y)}{1-1/y^2}\frac{1}{y^2}.$$

The u-plane

A final useful result is to determine the Bethe equations in the so called u-plane, related to the spectral parameter x by

$$x(u) = \frac{1}{2}u + \frac{1}{2}\sqrt{u^2-4} \quad \Leftrightarrow \quad u = x + 1/x.$$

In this new variable, one defines the singular resolvent to be

$$\tilde{H}_k(u) = \int dy \frac{\rho_k(y)}{y+1/y-u} = \int dv \frac{\rho_k(v)}{v-u},$$

where we have that $dx\rho_k(x) = du\rho_k(u)$, as expected for a density. $\tilde{H}_k(u)$ is related to the resolvents in the x-space by

$$\tilde{H}_k(x+1/x) = H_k(x) + H_k(1/x) - H_k(0),$$

and the resulting Bethe equations are written as

$$\sum_{j=1}^{[m/2]} M_{kj}\tilde{\not{H}}_j(u) + \delta_{k1}F_{string}(u) = 2\pi n_{k,a} \quad, u \in \mathscr{C}_{k,a},$$

where the function F_{string} is given by

$$F_{string}(u) = \frac{4\pi\kappa}{\sqrt{u^2-4}} + 2H_1(0) - 2H_1(1/x(u)).$$

The quasi-momenta for $\mathbb{R}\times S^5$

One can now write the quasi-momenta in terms of the resolvents $H_k(x)$. We will restrict ourselves to the case of S^5 $(m=6)$. From the relation of the singular resolvents $\tilde{H}_k$ and the quasi-momenta q_j (5.13), we can easily see that

$$\begin{aligned} q_1(x) &= \tilde{H}_1(x), \\ q_2(x) &= -\tilde{H}_1(x)+\tilde{H}_2(x)+\tilde{H}_3(x), \\ q_3(x) &= \qquad\qquad -\tilde{H}_2(x)+\tilde{H}_3(x). \end{aligned}$$

The Cartan matrix of $\mathfrak{so}(6)$ is given by

$$M = M_{\mathfrak{so}(6)} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix} \quad \Rightarrow \quad M^{-1} = \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 3/4 & 1/4 \\ 1/2 & 1/4 & 3/4 \end{pmatrix},$$

and we can write

$$\begin{aligned} q_1(x) &= \frac{4\pi\kappa x}{x^2-1} + H_1(x) - H_1(1/x) + H_1(0), \\ q_2(x) &= -H_1(x) - H_1(1/x) + H_1(0) + H_2(x) + H_2(1/x) - H_2(0) + H_3(x) + H_3(1/x) - H_3(0), \\ q_3(x) &= \qquad\qquad -H_2(x) + H_2(1/x) - H_2(0) + H_3(x) + H_3(1/x) - H_3(0). \end{aligned}$$

These once more obey the inversion symmetry

$$q_k(1/x) = (1-2\delta_{k1})\, q_k(x) + 4\pi n_0 \delta_{k1},$$

and recalling that H_k are analytic at $x=\pm 1$, the q_k have the right asymptotics at these singularities. The behaviour at $x\to\infty$ is (in terms of the $SO(6)$ charges and Dynkin labels $[s_1, s_2, s_3]$)

$$\begin{aligned} q_1(x) &\sim \frac{1}{x}\left(4\pi\kappa - \frac{4\pi K_1}{\sqrt{\lambda}} - 2H_1'(0)\right) = \frac{1}{x}\frac{4\pi J_1}{\sqrt{\lambda}} = \frac{1}{x}\frac{4\pi\left(s_1+\frac{1}{2}s_2+\frac{1}{2}s_3\right)}{\sqrt{\lambda}}, \\ q_2(x) &\sim \frac{1}{x}\left(\frac{4\pi K_1}{\sqrt{\lambda}} - \frac{4\pi K_2}{\sqrt{\lambda}} - \frac{4\pi K_3}{\sqrt{\lambda}}\right) = \frac{1}{x}\frac{4\pi J_2}{\sqrt{\lambda}} = \frac{1}{x}\frac{4\pi(s_2+s_3)}{2\sqrt{\lambda}}, \\ q_3(x) &\sim \frac{1}{x}\left(\frac{4\pi K_2}{\sqrt{\lambda}} - \frac{4\pi K_3}{\sqrt{\lambda}}\right) = \frac{1}{x}\frac{4\pi J_3}{\sqrt{\lambda}} = \frac{1}{x}\frac{4\pi(-s_2+s_3)}{2\sqrt{\lambda}}. \end{aligned}$$

Finally the analyticity condition states that

$$\not{q}_k(x) - \not{q}_{k+1}(x) = 2\pi n_{k,a} \quad , x \in \mathscr{C}_{k,a}, \tag{5.21}$$

together with the extra condition

$$\not{q}_2(x) + \not{q}_3(x) = 2\pi n_{3,a} \quad , x \in \mathscr{C}_{3,a}. \tag{5.22}$$

One can re-do these equations in terms of the $SU(4)$ quasi-momenta p_k. We know that the resolvents are related to the $\mathfrak{so}(6)$ Dynkin labels at $x = \infty$, $\tilde{H}_k \sim M^{-1}_{kj} s_j$. The resolvents related to the $SU(4)$ quasi-momenta p_k (representation $\mathbf{4}$) will have the corresponding relation $\tilde{G}_k \sim \left(M^{-1}_{su(4)}\right)_{kj} r_j$, and recalling that $r_{1,2,3} = s_{2,1,3}$, we have that

$$\tilde{G}_{1,2,3} = \tilde{H}_{2,1,3}.$$

Also from the definition of the p_k we find that the $SO(6)$ quasi-momenta in the $\mathbf{6}$ representation are given by:

$$\begin{aligned}
q_1 &= p_1 + p_2\,, \\
q_2 &= p_1 + p_3\,, \\
q_3 &= p_2 + p_3\,, \\
q_4 &= p_1 + p_4 = -q_3\,, \\
q_5 &= p_2 + p_4 = -q_2\,, \\
q_6 &= p_3 + p_4 = -q_1\,.
\end{aligned}$$

This means that a cut between sheets $1,2$ in the q_k's corresponds to a cut between sheets $2,3$ of the p_k's, or the earlier $\mathscr{C}_2$ cuts. Its mirror cut, between $2,6$ of the q_k is the mirror cut between the sheets $1,4$ in the p_k's. A cut between q_2 and q_3 will have a mirror cut between q_4, q_5, and will correspond to the cuts between sheets p_1 and p_2 (mirror is p_1, p_2), the $\mathscr{C}_1$ cuts. Finally a cut between q sheets $2,4$ (which corresponds to the Bethe equation (5.22)) and its mirror cut between sheets 3,5 (or 2,4 again) will correspond to the cuts between p_3 and p_4, called previously $\mathscr{C}_3$, see figure 5.1. With these results one can simply re-write the Bethe equations as:

$$\begin{aligned} 2\pi n_{1,a} &= \not{p}_1 - \not{p}_2 = 2\tilde{\not{G}}_1(x) - \tilde{G}_2(x) \,, && x \in \mathscr{C}_{1,a}\,, \\ 2\pi n_{2,a} &= \not{p}_2 - \not{p}_3 = 2\tilde{\not{G}}_2(x) - \tilde{G}_1(x) - \tilde{G}_3(x) \,, && x \in \mathscr{C}_{2,a}\,, \\ 2\pi n_{3,a} &= \not{p}_3 - \not{p}_4 = 2\tilde{\not{G}}_3(x) - \tilde{G}_2(x) \,, && x \in \mathscr{C}_{3,a}\,. \end{aligned}$$

In the above result we find that the singular resolvent $\tilde{G}_k(x)$ only feels the cuts of the kind $\mathscr{C}_k$.

Branch cuts and condensates

As was mentioned above, the quasi-momenta can have two kinds of singularities in the x complex plane: branch cuts along some contour $\mathscr{C}_{k,a}$ and condensate cuts $\mathscr{B}_{k,j}$. The first corresponds to cuts at which two of the quasi-momenta are interchanged on the two sides of the cut up to a multiple of 2π. The second case happens when we have (pairs of) singular points where the quasi-momenta changes by a factor of 2π when it goes around each of these singular point. If we consider a contour $\mathscr{B}_{k,j}$ connecting each pair of singular points, then the quasi-momentum $q_k(x)$ will change by a multiple of 2π when crossing this contour. These pairs of singular points, which are the the endpoints of the contour, are single poles of the differential of the quasi momenta (or $p^{'}(x)$). So the $\mathscr{A}$-cycle around each will pick up a multiple of 2π. From the point of view of the quasi-momenta themselves (and the resolvents), this contour is in fact a condensate of poles with constant density, equivalent to a logarithmic cut.[19]

Generalizing the density function $\rho_k(x)$ to describe the discontinuities across both the cuts $\mathscr{C}_k = \bigcup \mathscr{C}_{k,a}$ and the condensates $\mathscr{B}_k = \bigcup \mathscr{B}_{k,j}$, we find that the resolvent H_k in (5.16)

[19]Each of this singular points is an end point of a branch cut, and the condensate cut connects them. If the $\mathscr{A}$-cycle encircles the branch cut that the singular point belongs to, then it will cross the condensate cut and so jump by 2π. Because this point is a single pole of dp, the $\mathscr{A}$-cycle is just a contour integral around a single pole, resulting in the residue at that pole, which is proportional to the constant density.

can be generalized to be[20]

$$H_k = \sum_a \int_{\mathscr{C}_{k,a}} dy \frac{\rho_k(y)}{y-x} + \sum_j \int_{\mathscr{B}_{k,j}} dy \frac{\rho_k(y)}{y-x}.$$

This density ρ_k characterizes the distribution of Bethe roots in the complex plane, and the term condensate comes from the fact that it can be interpreted as a condensate of Bethe roots, and as such, the density becomes constant: it is just $\rho_k(y) = i n_{k,j}$ when $y \in \mathscr{B}_{k,j}$. Then one can readily see that the terms coming from the condensate cuts give a logarithmic behaviour to H_k: they are logarithmic cuts, in general connecting pairs of original contours $\mathscr{C}_k$.

The Bethe equations found above still have to be obeyed for solutions that admit branch cuts, but if we also allow condensates, in addition to the Bethe equations derived above we will have extra conditions on the solutions of these equations when they cross a logarithmic cut.

Different string solutions have been addressed in the literature, from circular and pulsating strings [29, 91] where the resolvent is given just by branch cuts, to giant magnon bound states [89] where one only considers condensate cuts. In the latter case (no branch cuts, only logarithmic cuts), the ending points $X^{\pm}_{k,j}$ of the condensate contour have to be complex conjugate of each other, in order to have real energy and angular momenta.

5.3 The full $AdS_5 \times S^5$ string Bethe ansatz

In this section we will present a brief summary of major results for the full $AdS_5 \times S^5$ algebraic curve formalism, which can be found in [34]. In this case we have eight sheets defined by the quasi-momenta

$$p(x) = \{\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4 | \hat{p}_5, \hat{p}_6, \hat{p}_7, \hat{p}_8\},$$

where $e^{i\tilde{p}_k}$ and $e^{i\hat{p}_l}$ are eigenvalues of the monodromy matrix, but corresponding to two different gradings, so while we can interchange each set of four quasi-momenta, we cannot

[20] We have dropped the weight factor $1-1/x^2$ on the definition of the density for convenience.

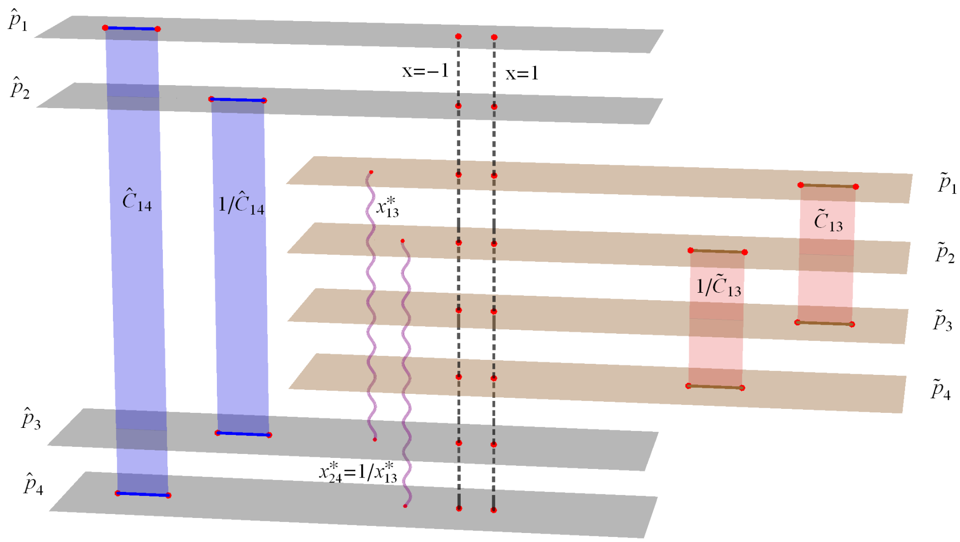

Figure 5.3: The algebraic curve, with $\hat{p}_k$ corresponding to AdS_5 and $\tilde{p}_l$ corresponding to S^5. Three physical modes are shown: the cut $\hat{C}_{14}$ (and its mirror cut, in blue) corresponds to a bosonic AdS_5 excitation; the cut $\tilde{C}_{13}$ (in red) corresponds to a bosonic S^5 excitation; the pole x^*_{13} corresponds to a fermionic excitation. The dots in red are singularities of the quasi-momenta, including $x = \pm 1$.

interchange between them by a simple bosonic similarity transformation. The $\tilde{p}_k$ are related to the S^5 part, and the $\hat{p}_k$ to the AdS_5 part. The momentum condition is in this case

$$\sum_{k=1}^{4} \tilde{p}_k - \sum_{l=1}^{4} \hat{p}_l = 2\pi n_0.$$

All quasi-momenta are analytic. They have single poles at $x = \pm 1$, where all sheets have equal residues. One has bosonic degrees of freedom at a collection of branch cuts $\tilde{\mathscr{C}}_a, \hat{\mathscr{C}}_b$ with $a = 1, \cdots, 2\tilde{A}$, and $b = 1, \cdots, 2\hat{A}$, where the branch cuts $\tilde{\mathscr{C}}_a$ connect sheets $\tilde{k}_a$ and $\tilde{l}_a$ of quasi-momenta $\tilde{p}(x)$, while branch cuts $\hat{\mathscr{C}}_b$ connect sheets $\hat{k}_b$ and $\hat{l}_b$ of quasi-momenta $\hat{p}(x)$. At the ends of all these cuts, $\tilde{x}^{\pm}_a$ and $\hat{x}^{\pm}_b$, we have square-root singularities. The fermionic degrees of freedom are poles x^*_a, with $a = 1, \cdots, 2A^*$, which "connect" one sheet k^*_a of $\tilde{p}(x)$ and one sheet l^*_a of $\hat{p}(x)$ with the same residue on both sheets.[21] This structure of cuts and poles is shown in Figure 5.3.

[21]To see that in the case of fermionic modes, we have poles instead of the square root singularities that give rise to the bosonic branch cuts, we use an argument similar to the one on page 114. This argument can be found in [34] and is summarized in appendix Section C.2.

At $x \to \infty$ the quasi-momenta approaches zero, and its expansion will be related to the conserved charges, or Dynkin labels, as before. The inversion symmetry for the quasi-momenta can be written as follows

$$\tilde{p}_k(1/x) = 2\pi m \varepsilon_k - \tilde{p}_{k'}(x) \ , \quad \hat{p}_l(1/x) = \hat{p}_{l'}(x) \, ,$$

where we defined for each sheet $k = (1,2,3,4)$, a permutation $k' = (2,1,4,3)$, and a sign change $\varepsilon_k = (1,1,-1,-1)$. Note that the shift by $2\pi m$ is related to the winding allowed in S^5, but not allowed in AdS_5 (there aren't any windings in the time direction). The cuts and poles also obey this inversion symmetry in the same way as before: $\tilde{\mathscr{C}}_{\tilde{A}+a} = 1/\tilde{\mathscr{C}}_a$, $\hat{\mathscr{C}}_{\hat{A}+a} = 1/\hat{\mathscr{C}}_a$ and $x^*_{A^*+a} = 1/x^*_a$.

Defining cycles $\tilde{\mathscr{A}}_a, \hat{\mathscr{A}}_b$ surrounding respective cuts $\tilde{\mathscr{C}}_a, \hat{\mathscr{C}}_b$ is done as before, but we have another cycle to define, the $\mathscr{A}^*_a$, which surrounds a fermionic pole x^*_a. Assuming no logarithmic singularities at any poles (including $x = \pm 1$) we have

$$\oint_{\tilde{\mathscr{A}}_a} d\tilde{p} = \oint_{\hat{\mathscr{A}}_b} d\hat{p} = \oint_{\mathscr{A}^*_a} d\tilde{p} = \oint_{\mathscr{A}^*_a} d\hat{p} = 0.$$

We again define the periods $\tilde{\mathscr{B}}_a$ ($\hat{\mathscr{B}}_b$) connecting $x = \infty$ on sheets $\tilde{k}_a$ and $\tilde{l}_a$ ($\hat{k}_b$ and $\hat{l}_b$) through the cuts $\tilde{\mathscr{C}}_a$ ($\hat{\mathscr{C}}_b$), and also define an extra period $\mathscr{B}^*_a$ that connects $x = \infty$ to $x = x^*_a$ on sheet k^*_a of $\tilde{p}(x)$, and then goes through the fermionic singularity to connect $x = x^*_a$ to $x = \infty$ on sheet l^*_a of $\hat{p}(x)$. On these periods we have

$$\int_{\tilde{\mathscr{B}}_a} d\tilde{p} = 2\pi \tilde{n}_a \, , \quad \int_{\hat{\mathscr{B}}_b} d\hat{p} = 2\pi \hat{n}_b \, , \quad -\!\!\!\!\!\!\int_{\mathscr{B}^*_a} dp = 2\pi n^*_a \, .$$

Another way of writing this is

$$\begin{aligned}
\not{\tilde{p}}_{\tilde{l}_a}(x) - \not{\tilde{p}}_{\tilde{k}_a}(x) &= 2\pi \tilde{n}_a \quad , \qquad x \in \tilde{\mathscr{C}}_a \, , \\
\not{\hat{p}}_{\hat{l}_b}(x) - \not{\hat{p}}_{\hat{k}_b}(x) &= 2\pi \hat{n}_b \quad , \qquad x \in \hat{\mathscr{C}}_b \, , \\
\not{p}_{\hat{l}_a}(x^*_a) - \not{p}_{\tilde{k}_a}(x^*_a) &= 2\pi n^*_a \, .
\end{aligned} \tag{5.23}$$

One final thing to note is that for a period connecting $x = 0$ and $x = \infty$, we once again obtain the winding of S^5 and a vanishing contributions from AdS_5: $\tilde{p}_{1,2}(0) = -\tilde{p}_{3,4}(0) = 2\pi m$ and $\hat{p}_l(0) = 0$.

Having summarized the properties of the 8 sheets quasi-momenta, we will now present the $AdS_5 \times S^5$ Bethe ansatz.

$AdS_5 \times S^5$ *string Bethe ansatz*

Consider a general cycle $\mathscr{A}_a$ and a period $\mathscr{B}_a$, where $a = 1, \cdots, 2A$, which includes all cuts and poles, with $A = \tilde{A} + \hat{A} + A^*$. We also introduce an integral form of bosonic resolvents in terms of densities as[22]

$$\tilde{G}_{kl}(x) = \int_{\tilde{\mathscr{C}}_{kl}} dy \frac{\tilde{\rho}_{kl}(y)}{1 - 1/y^2} \frac{1}{y - x}, \quad \hat{G}_{kl}(x) = \int_{\hat{\mathscr{C}}_{kl}} dy \frac{\hat{\rho}_{kl}(y)}{1 - 1/y^2} \frac{1}{y - x},$$

and a fermionic resolvent as a sum of (discrete) poles

$$G^*_{kl}(x) = \sum_{a=1}^{A^*_{kl}} \frac{\alpha^*_{kl}}{1 - 1/x^*_{kl,a}} \frac{1}{x^*_{kl,a} - x}.$$

In the above expression the subscript kl states that the two sheet being connected are sheets k and l of the corresponding cut or pole. The fillings are defined now in terms of the densities as

$$\tilde{K}_{kl,a} = \frac{\sqrt{\lambda}}{4\pi} \int_{\tilde{\mathscr{C}}_{kl,a}} dy\, \tilde{\rho}_{kl}(y), \quad \hat{K}_{kl,a} = \frac{\sqrt{\lambda}}{4\pi} \int_{\hat{\mathscr{C}}_{kl,a}} dy\, \hat{\rho}_{kl}(y), \quad K^*_{kl,a} = \frac{\sqrt{\lambda}}{4\pi} \alpha^*_{kl,a}.$$

Together with these fillings we had previously seen that in the $\mathbb{R} \times S^5$ case there were two other quantities: the length L and the winding m. In the full theory we will have yet another quantity, called the energy shift δE:

$$\delta E = \frac{\sqrt{\lambda}}{8\pi^2 i} \sum_{a=1}^{A} \oint_{\mathscr{A}_a} \frac{dx}{x^2} \sum_k \varepsilon_k (\hat{p}_k - \tilde{p}_k).$$

From the fillings written above, one can define some global fillings (both in S^5 and in AdS_5) by

$$K_j = \frac{\sqrt{\lambda}}{8\pi^2 i} \sum_{a=1}^{A} \oint_{\mathscr{C}_a} dx \left(1 - \frac{1}{x^2}\right) \sum_k a_{k,j}\, p_k(x),$$

[22] These resolvents have to be antisymmetric in exchange of sheets k, l.

for some linear combination of the quasi-momenta (this definition will become helpful later). There are 6 of these global filling, with 3 related to Dynkin labels $[\tilde{r}_1, \tilde{r}_2, \tilde{r}_3]$ for the S^5, and other 3 related to the $[\hat{r}_1, \hat{r}_2, \hat{r}_3]$ of the AdS_5 part:

$$\tilde{r}_1 = \tilde{K}_2 - 2\tilde{K}_1\,, \qquad \tilde{r}_2 = L - 2\tilde{K}_2 + \tilde{K}_1 + \tilde{K}_3\,, \qquad \tilde{r}_3 = \tilde{K}_2 - 2\tilde{K}_3\,,$$
$$\hat{r}_1 = \hat{K}_2 - 2\hat{K}_1\,, \qquad \hat{r}_2 = -L - \delta E - 2\hat{K}_2 + \hat{K}_1 + \hat{K}_3\,, \qquad \hat{r}_3 = \hat{K}_2 - 2\hat{K}_3\,,$$

where the Dynkin labels are just the residue of the quasi-momenta at infinity

$$\tilde{r}_j = \frac{\sqrt{\lambda}}{8\pi^2 i}\oint_\infty dx\,(\tilde{p}_j - \tilde{p}_{j+1})\,, \quad \hat{r}_j = \frac{\sqrt{\lambda}}{8\pi^2 i}\oint_\infty dx\,(\hat{p}_{j+1} - \hat{p}_j)\,.$$

In the full theory there is also one global fermionic filling K^*, related to the hypercharge eigenvalue r^* by

$$r^* \equiv -\frac{\sqrt{\lambda}}{8\pi^2 i}\sum_{a=1}^{A}\oint_{\mathscr{A}_a} dx \sum_{k=1}^{4}\frac{1}{2}(\tilde{p}_k + \hat{p}_k) = 2B - K^*,$$

where B a constant (hypercharge of the vacuum).

The full expression for the quasi-momenta in terms of the resolvents $\tilde{G}, \hat{G}, G^*$ is just given by

$$\begin{aligned}
\tilde{p}_k(x) &= \sum_{l=1}^{4}\left(\tilde{H}_{kl}(x) + H^*_{kl}(x)\right) + \varepsilon_k \tilde{F}(x) + F^*(x)\,, \\
\hat{p}_k(x) &= \sum_{l=1}^{4}\left(\hat{H}_{lk}(x) + H^*_{lk}(x)\right) + \varepsilon_k \hat{F}(x) + F^*(x)\,,
\end{aligned} \tag{5.24}$$

where as before we define an inversion symmetric function

$$H_{kl}(x) \equiv G_{kl}(x) + G_{kl}(1/x) - G_{kl}(0)\,,$$

and have some auxiliary potentials

$$\begin{aligned}
\tilde{F}(x) &= \left(\frac{2\pi L}{\sqrt{\lambda}} + \tilde{G}'_+(0)\right)\frac{x}{x^2-1} + \frac{\hat{G}_+(0)}{1-1/x^2} - \tilde{G}_+(1/x) + G_{mom}(0)\,,\\
\hat{F}(x) &= \left(\frac{2\pi L}{\sqrt{\lambda}} + \tilde{G}'_+(0)\right)\frac{x}{x^2-1} + \frac{\hat{G}_+(0)}{1-1/x^2} - \tilde{G}_+(1/x)\,,\\
F^*(x) &= \left(\frac{2\pi B}{\sqrt{\lambda}} + G^{*'}_+(0)\right)\frac{x}{x^2-1} + \frac{G^{*'}_+(0)}{1-1/x^2} - G^{*'}_+(1/x)\,.
\end{aligned}$$

The resolvents G_+ and G_{mom} are simply given by

$$\begin{aligned}
\tilde{G}_+(x) &= \frac{1}{2}\sum_{k,l=1}^{4} \varepsilon_k\left(\tilde{G}_{kl} + G^*_{kl}\right)(x)\,,\\
\hat{G}_+(x) &= \frac{1}{2}\sum_{k,l=1}^{4} \varepsilon_l\left(\hat{G}_{kl} + G^*_{kl}\right)(x)\,,\\
G^*_+(x) &= \frac{1}{2}\sum_{k,l=1}^{4} G^*_{kl}(x)\,,\\
G_{mom}(x) &= \tilde{G}_+(x) - \hat{G}_+(x)\,.
\end{aligned}$$

These resolvent are defined to take into account that only the cuts/poles with $\varepsilon_k \neq \varepsilon_l$ are physical and these are the only ones to contribute for G_{mom}.[23]

The behaviour of the quasi-momenta at $x = \infty$ gives us the conserved charges of the system, which can be described through the Dynkin labels or through the fillings K, the length L and the energy shift δE. Expanding the symmetric H at infinity we obtain

$$H(x) = -\frac{1}{x}\sum_{a=1}^{A}\frac{4\pi K_a}{\sqrt{\lambda}} + \mathscr{O}(1/x^2),$$

while the expansions of the auxiliary potentials are given by

$$\tilde{F}(x) = \frac{1}{x}\frac{2\pi L}{\sqrt{\lambda}} + \mathscr{O}(1/x^2)\,, \quad \hat{F}(x) = \frac{1}{x}\frac{2\pi(L+\delta E)}{\sqrt{\lambda}} + \mathscr{O}(1/x^2)\,,$$

where the energy shift is just $\delta E = \frac{\sqrt{\lambda}}{2\pi}G'_{mom}(0)$.

The integral Bethe equations (5.23) still have the same form, but with the additional

[23] The physical cuts connect sheets 1,2 or sheets 3,4 of of both $\tilde{p}$ and $\hat{p}$.

momentum constraint $G_{mom}(0) = 2\pi m$.

Moduli of string solutions

Performing a similar counting of degrees of freedom to the one done above for the $\mathbb{R}\times S^5$ case, we can determine the number of moduli of allowed curves. For each pair of cuts connecting sheets k_a, l_a we have one continuous modulus, the filling K_a and one discrete parameter n_a. The full theory still has the (continuous) length L and the (discrete) winding m parameters (and one constraint connecting all of these). Most of these have a direct analogous in the classification of physical excitation modes of classical strings in flat space (or a plane-wave geometry): in light-cone gauge each excitation mode has a mode number n_a, an amplitude K_a, and an orientation k_a, l_a (for the superstring we will have 8 physical bosonic modes plus 8 fermionic ones). The quantities corresponding the length L and winding do not exist in flat space, as they are related to the effective curvature of the space and the winding around S^5, respectively. As one expects the number of moduli to remain the same for closed strings in different backgrounds, then each string solution of the σ-model in $AdS_5\times S^5$ will have a corresponding curve in this formalism.

5.4 Restriction to $\mathbb{R}\times S^3$: The dyonic Giant Magnon solution

The restriction of the full σ-model to $\mathbb{R}\times S^3$ was extensively studied in [29]. In this case we are interested in the isometry group $SO(4)\sim SU(2)_L\times SU(2)_R$. In the spinor representation $(\mathbf{2}_L.\mathbf{2}_R)$ we have the monodromy matrix decomposed in two matrices, diagonalized as

$$\Omega_S^+ = \mathrm{diag}\left(e^{ip_L}, e^{-ip_L}\right), \quad \Omega_S^- = \mathrm{diag}\left(e^{ip_R}, e^{-ip_R}\right).$$

We then have two independent quasi-momenta p_L, p_R, related to the quasi-momenta of the vector representation by

$$p_L = \frac{1}{2}(q_1+q_2), \quad p_R = \frac{1}{2}(q_1-q_2).$$

By inversion symmetry one can easily see that $p_R(x) = -p_L(1/x) + 2\pi n_0$. This means that we can either choose two independent quasi-momenta with inversion symmetry relating them, or one quasi-momentum $p(x) = p_R(x)$ with no inversion symmetry. The behaviour of this $p(x)$ at the singularities $x = \pm 1$ is just

$$p(x) = \frac{\pi\kappa}{x \mp 1} + \mathscr{O}\left((x \mp 1)^0\right).$$

The asymptotic behaviour when $x \to \infty$ is

$$p(x) = \frac{2\pi (J_1 - J_2)}{\sqrt{\lambda}} \frac{1}{x} + \mathscr{O}\left(1/x^2\right),$$

while for $x \to 0$ (through inversion symmetry) we have

$$p(x) = 2\pi n_0 - \frac{2\pi (J_1 + J_2)}{\sqrt{\lambda}} x + \mathscr{O}\left(x^2\right).$$

Note that the Dynkin labels $[r_L, r_R]$ of $SU(2)_L \times SU(2)_R$ are related to the charges of S^3 by $r_L = J_1 + J_2$ and $r_R = J_1 - J_2$. Finally the analyticity conditions are just

$$2\not{p}(x) = 2\pi n_a, \quad x \in \mathscr{C}_a.$$

We can re-write these results in terms of the resolvent $H(x)$. The $SO(4)$ Cartan Matrix is just

$$M_{SO(4)} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

and so, from (5.13) we find

$$\tilde{H}_1 = \frac{1}{2}(q_1 - q_2) \equiv p_R, \quad \tilde{H}_2 = \frac{1}{2}(q_1 + q_2) \equiv p_L.$$

We keep $\tilde{H}_1$, related to $p(x)$, and define the resolvent $H(x)$ in a similar way to (5.15):

$$H(x) = \tilde{H}_1(x) - \frac{2\pi\kappa}{x - 1/x} = \sum_a \int_{\mathscr{C}_a} dy \frac{\rho(y)}{y - x} + \sum_j \int_{\mathscr{B}_j} dy \frac{\rho(y)}{y - x}. \tag{5.25}$$

In the above expression, the first sum is over all branch cuts, while the second is over condensates.

The asymptotics of $H(x)$ is just (note that $\Delta = \sqrt{\lambda}\kappa$, and that $p = 2\pi n_0$)

$$\begin{aligned} H(x) &= \frac{2\pi}{\sqrt{\lambda}}\left(-\Delta + J_1 - J_2\right)\frac{1}{x} + \mathscr{O}\left(1/x^2\right), \quad x \to \infty, \\ H(x) &= p - \frac{2\pi\left(J_1 + J_2\right)}{\sqrt{\lambda}}x + \mathscr{O}\left(x^2\right), \quad x \to 0\,. \end{aligned} \tag{5.26}$$

The Bethe integral equation for the resolvent $H(x)$ crossing a branch cut is just given by

$$H(x + i\varepsilon) + H(x - i\varepsilon) = 2 \fint dy \frac{\rho(y)}{y - x} = -\frac{2\pi\kappa x}{x^2 - 1} + 2\pi n_a\,, \quad x \in \mathscr{C}_a.$$

Giant magnons were first studied using the algebraic curve by [89], where it was shown that they correspond to logarithmic cuts (see also [183]). In fact, giant magnon solutions and their bound states are solutions made up of only condensates. The condition that we have a closed string requires that the total string momentum is $p = 2\pi n_0$. But the contribution of each condensate for the charges is additive, and as such we can consider each condensate separately with a momentum p that does not satisfy such condition, as long as the momentum of all the condensates satisfies it. For one condensate, we can go back to (5.25) and set the density to be constant, $\rho(x) = -in$, and the end points of the condensate to be complex conjugate $X^{\pm}$. Then we find an ansatz for the giant magnon solution

$$H(x) = -in \int_{X^-}^{X^+} \frac{dy}{y - x} = -in \log\left(\frac{x - X^+}{x - X^-}\right). \tag{5.27}$$

The factor n will give us a bound state of n magnons. We are interested at the moment to work with one magnon, so we set $n = 1$:

$$G_{\mathrm{mag}}(x) \equiv -i \log\left(\frac{x - X^+}{x - X^-}\right) \tag{5.28}$$

From the asymptotics of the resolvent (5.26), and the expansion of the expression for

$G_{mag}(x)$ (5.28), we find

$$
\begin{aligned}
p &= -i\log\left(\frac{X^+}{X^-}\right), \\
\Delta - J + Q &= -2ig\left(X^+ - X^-\right), \\
\Delta - J - Q &= 2ig\left(\frac{1}{X^+} - \frac{1}{X^-}\right).
\end{aligned}
\tag{5.29}
$$

In the above expression we redefined $J_1 \equiv J$, $J_2 \equiv Q$ to better compare to the known results for the dispersion relation of the giant magnon. Also, we redefined the coupling as $\lambda = 8g^2$. Solving these equations for $X^\pm$ in terms of p, Q, we have

$$
X^\pm = e^{\pm i\frac{p}{2}}\csc\left(\frac{p}{2}\right)\frac{Q + \sqrt{Q^2 + 16g^2\sin^2\left(\frac{p}{2}\right)}}{4g},
$$

and we find the dispersion relation for a dyonic giant magnon as expected

$$
\mathscr{E} \equiv \Delta - J = \sqrt{Q^2 + 16\,g^2\sin^2\frac{p}{2}}\,.
$$

Finite size corrections

In the particular case of the giant magnon solutions, we consider an new ansatz for the resolvent to determine finite-J corrections to the S^2 magnon [184], that is, we replace $G_{\text{mag}}(x)$ with the resolvent

$$
G_{\text{finite}}(x) = -2i\log\left(\frac{\sqrt{x - X^+} + \sqrt{x - Y^+}}{\sqrt{x - X^-} + \sqrt{x - Y^-}}\right)
\tag{5.30}
$$

where $Y^\pm$ are points shifted from the end points of the cut $X^\pm$ by a small amount $\delta \ll 1$:[24]

$$
Y^\pm = X^\pm\left(1 \pm i\delta e^{\pm i\phi}\right)
\tag{5.31}
$$

The giant magnon can be described as singular limit of a solution with two branch cuts $\mathscr{C}_X$ and $\mathscr{C}_Y$, with endpoints $X^\pm$ and $Y^\pm$ respectively, when one takes $Y^\pm \to X^\pm$. The finite-size correction to the giant magnon will then appear when we consider $Y^\pm$ very close to $X^\pm$ but

[24] This is a different choice of ϕ to that used in [141, 184, 185]. It was chosen to separate ϕ from the phase of $X^\pm$, which is $p/2$. The phase ϕ will be seen to give the orientation factor $\cos(2\phi)$ in $\delta\mathscr{E}$.

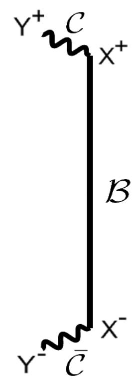

Figure 5.4: Structure of cuts and condensates for finite-J magnons

still apart. In [184] one finds that a $SL(2,\mathbb{Z})$ takes us from this picture to the one shown in figure 5.4, where we have one condensate cut between points $X^{\pm}$ and two square root branch cuts, one connecting X^{+},Y^{+} and the other connecting X^{-},Y^{-}. It is this picture that leads to the ansatz for the finite resolvent (7.9).

When $\delta = 0$ this new $G_{\text{finite}}(x)$ clearly reduces to the infinite-size magnon resolvent (5.28). The square root cuts of G_{finite} are such that the relative sign between square roots in the numerator (or denominator, depending on which cut we are crossing) changes when the contour crosses the respective cut $\mathscr{C}(X^{+},Y^{+})$ (or $\mathscr{C}(X^{-},Y^{-})$), see figure 5.4. We will be using the superscript G^{-} to indicate that this term is being evaluated on the other side of the cut from what G is being evaluated (and thus has the opposite sign between the terms of the numerator inside G).

To determine the leading finite-J correction to the giant magnon dispersion relation, we again calculate the asymptotic behaviour at $x \to \infty$ of the quasi-momenta using the ansatz (7.9) to determine the charges of the solution as functions of the end points $X^{\pm},Y^{\pm}$, and then solve the equations for these charges to second order in δ.

In Chapter 7 we will apply this method to some giant magnon solutions in the context of bosonic string in another background. We will then give a full description on how to obtain these finite-J leading corrections to the dispersion relation.

Chapter 6

Giant Magnons in $AdS_4 \times CP^3$

Classical string solutions in $AdS_5 \times S^5$ have played an important role in the study of the duality to $\mathcal{N} = 4$ SYM [11, 12, 26]. It seems that this pattern is being repeated in the new $\mathcal{N} = 6$ duality [6], in which planar superconformal Chern–Simons theory is dual to string theory on $AdS_4 \times CP^3$. Some of the most interesting recent papers study strings moving in an $AdS_2 \times S^1$ subspace, where although the classical solutions are identical to those used in the $\mathcal{N} = 4$ case, the quantum properties are different. The results from semiclassical quantization [43, 45, 186, 18, 187, 188] can be compared to those from the asymptotic Bethe ansatz, and there appeared to be some difficulties [47, 189, 190, 44, 191].

In this chapter, we study string solutions exploring primarily the CP^3 factor. One would expect to find analogues of the giant magnons [26] here, which in the $\mathcal{N} = 4$ case live in an $S^2 \subset S^5$. And indeed, it turns out that the same solutions exist in CP^3 [192, 193]. There are two inequivalent ways to embed the basic S^2 magnon, into either $CP^1 = S^2$ or $RP^2 = S^2/\mathbb{Z}_2$ [192], both two-dimensional subspaces of CP^3.

In either theory, the anomalous dimension can be calculated as the Hamiltonian of some spin chain [20, 194, 195, 192]. The giant magnons are dual to the elementary excitations of this spin chain, and have a periodic dispersion relation $\Delta - J = \sqrt{1 + f^2(\lambda)\sin^2(p/2)}$ which on the gauge side is an symptom of the discrete spatial dimension of the spin chain, and on the string side arises from p being an angle along an equator. The conformal dimension Δ and the R-charge J are mapped by AdS/CFT to energy and angular momentum of the string state. For the state dual to the (ferromagnetic) vacuum of the spin chain, which is a point particle, $\Delta - J$ becomes the Hamiltonian for small fluctuations. We confirm that in the $\mathcal{N} = 6$ case, the difference $\Delta - (J_1 - J_4)/2$ has the same property.

An important difference between the old $\mathcal{N} = 4$ case and the new $\mathcal{N} = 6$ case is the behaviour of the function $f(\lambda)$, the only part of the dispersion relation not fixed by supersymmetry [55, 26]. In the old case, calculations of $f(\lambda)$ at both large and small λ give $f(\lambda) = \sqrt{\lambda}/\pi$, and this is conjectured to be true for all λ. In the new case, however, the function (often called h instead) is $h(\lambda) = \lambda$ at small λ but $h(\lambda) \sim \lambda^{1/2}$ at large λ. Our knowledge of this function at large λ comes (in both cases) from studying classical string theory, and so depends on the correct identification of the relevant string solutions.

Dyonic giant magnons are those with more than one large angular momentum, dual to a large condensate of impurities on the spin chain. These are string solutions in S^3, and they can at least sometimes be embedded into CP^3 in much the same way as the basic magnon, generalizing the RP^2 magnons and living in an RP^3 subspace [196, 197]. There is room for dyonic solutions with other angular momenta, truly exploring CP^3, including those generalizing the CP^1 magnon.

Solutions of the string sigma-model should be in exact correspondence to algebraic curves [46]. Here too several giant magnon solutions were known (compared to one in the S^5 case [89]) named 'small' and 'big' [198]. However these could not be the same two solutions as those previously known in the sigma-model. In the S^5 case, and for the small magnon, one naturally obtains a dyonic (two-parameter, two-spin) solution. But the big magnon is something not seen in the S^5 case, a two-parameter solution with only one non-zero angular momentum, and thus cannot be the RP^3 magnon. There are also two distinct small giant magnons, and it has been observed that a pair of small magnons has all the properties we expect the RP^3 magnon to have [185].

The situation has improved with the recent publication of two new string solutions. The first was found using the dressing method, and, like the big giant magnon, is a two-parameter one-angular-momentum solution [199, 200, 201]. They have exactly the same dispersion relation, and in both cases we can take a non-dyonic limit and recover the RP^2/pair of small magnons. The second one is a dyonic generalization of the CP^1 magnon [202]. This is a solution which does not exist in S^5, exploring the four-dimensional subspace CP^2, and it has a dispersion relation matching that of the small giant magnon. It exists in two orientations, and like the small magnon has a third angular momentum which is $J_3 = \pm Q$ in these two cases. We have as yet only been able to find the $p = \pi$ case of this solution, but this is sufficient to see these properties.[1]

In this chapter we will introduce the CP^3 geometry, and discuss potentially interesting subspaces where one could find non-trivial solutions. Noteworthy is the fact that the subspace frequently called $S^2 \times S^2$ in the literature is in fact just RP^2, and while there is a genuine $S^2 \times S^2$ subspace, one cannot place arbitrary S^2 string solutions into each factor, because the equations of motion couple the two factors. This discussion will be followed by a description of the giant magnon solutions known in particular subspaces, and corresponding charges. In particular, their dispersion relations will allow us to set up the correspondence with the algebraic curve results, which will be given in the next chapter.

6.1 Groups in ABJM theory

The ABJM [6] $\mathcal{N} = 6$ superconformal Chern–Simons-matter theory[2] has a $U(N) \times U(N)$ gauge symmetry. We want to focus on its scalar sector A_i, B_i. The fields A_1, A_2 are matrices which transform under the $(N, \bar{N})$ representation of this group (one fundamental index, one anti-fundamental), while the fields B_1, B_2 transform under the $(\bar{N}, N)$. The R-symmetry can be decomposed in a manifest $SU(2)_A$, in which the A fields form a doublet, and $SU(2)_B$, which acts on the B fields. The theory is also invariant under the conformal group $SO(2,3)$, since we are in 2+1 dimensions. We consider spacetime to be $\mathbb{R} \times S^2$, we consequently restrict ourselves to fields in the lowest Kaluza–Klein mode on this S^2, i.e. in the singlet representation of $SO(3)_r$ - the spatial part of the conformal group.

It was seen in [212] that the R-symmetry group can be extended to the full $SU(4)$, with

[1] Note that we use the term dyonic to mean a two-parameter two-charge solution, but sometimes write 'dyonic' to mean a two-parameter, one-charge solution, like the big giant magnon. This last use of the term dyonic is to specify that we mean the $r \neq 1$ case, and when we talk about the non-dyonic limit, that means taking the second parameter $r \to 1$. This limit always takes us into some embedding of the HM magnon.

[2] These theories were discovered after the explorations of 3-dimensional superconformal theories with non-Lie-algebra gauge symmetry by BLG, [203, 204, 205, 206, 207] and built on earlier work on Chern–Simons-matter theories by [208, 209, 210, 211].

the following combination of scalars transforming in its fundamental representation:

$$Y^A = (A_1, A_2, B_1^\dagger, B_2^\dagger)\,, \tag{6.1}$$

while $Y_A^\dagger$ transforms in the anti-fundamental. Keeping only the scalars $(Y^1, Y^4) = (A_1, B_2^\dagger)$, only a subgroup called $SU(2)_{G'}$ remains from the full symmetry group. If we kept only $(Y^2, Y^3) = (A_2, B_1^\dagger)$ then the subgroup would be $SU(2)_G$.[3]

This theory was found to be dual to membranes on $AdS_4 \times S^7/\mathbb{Z}_k$, where $(k, -k)$ are the level numbers of the two Chern–Simons terms. In the 't Hooft limit $N \to \infty$ with $\lambda = N/k$ fixed we need to take $k \to \infty$, and in this limit the theory of membranes on $AdS_4 \times S^7/\mathbb{Z}_k$ reduces to type IIA strings on $AdS_4 \times CP^3$.

A spin-chain description of ABJM theory can be obtained [194, 192, 195], through the study of gauge invariant operators of length $2L$ of the form

$$\mathscr{O} = \chi^{B_1 B_2 \cdots B_L}_{A_1 A_2 \cdots A_L} \operatorname{tr} Y^{A_1} Y^\dagger_{B_1}\, Y^{A_2} Y^\dagger_{B_2}\, \ldots\, Y^{A_L} Y^\dagger_{B_L}.$$

When χ is fully symmetric (in the As, and in the Bs) and traceless, $\mathscr{O}$ is a chiral primary with protected scaling dimension $\Delta = L$, that is, the anomalous dimension defined by $D = \Delta - L$ will be zero.

The $SU(2) \times SU(2)$ sector of the theory corresponds to operators $\mathscr{O}$ in which only Y^1, Y^2 and $Y_3^\dagger$, $Y_4^\dagger$ appear.[4] The relevant vacuum is given by

$$\mathscr{O}_{\text{vac}} = \operatorname{tr} \left(Y^1 Y_4^\dagger \right)^L. \tag{6.2}$$

This has $\Delta = L$, and $J = L$, where J is the Cartan generator in $SU(2)_{G'}$: $J(Y^1) = \frac{1}{2}$ and $J(Y^4) = -\frac{1}{2}$, thus $J(Y_4^\dagger) = +\frac{1}{2}$.

The $SU(3)$ sector allows operators with Y^1, Y^2, Y^3 and $Y_4^\dagger$, and has the same vacuum (6.2) as the previous sector.

---

[3] These subscripts are the notation of [192], except that they have B_1 and B_2 the swapped: their spin chain vacuum is $\operatorname{tr}(A_1 B_1^\dagger)^L$ rather than the $\operatorname{tr}(Y^1 Y_4^\dagger)^L$ of [194] used here, (6.2).

[4] This equivalent to saying that only fields A_1, A_2, B_1 and B_2 appear The two factors of $SU(2)$ in the sector are just $SU(2)_A$ and $SU(2)_B$.

The two-loop anomalous scaling dimension of operators in the $SU(2)\times SU(2)$ sector can be determined by the sum of the Hamiltonians of two independent Heisenberg XXX spin chains, for the even and odd sites. The momentum constraint (from the $U(N)$ trace tr) is that the sum of both their momenta be zero (the condition in the $\mathcal{N}=4$ case was slightly stronger [20], as in that case there was one total momentum which had to be zero).

6.2 The geometry of CP^3

The string dual of ABJM theory in the 't Hooft limit is type IIA superstrings in $AdS_4\times CP^3$, with sizes specified by the metric

$$ds^2=\frac{R^2}{4}ds^2_{AdS_4}+R^2 ds^2_{CP^3} \tag{6.3}$$

where $R^2=2^{5/2}\pi\sqrt{\lambda}$. The large-$\lambda$ limit gives strongly coupled gauge theory, dual to classical strings. In addition to this (string-frame) metric, there is a dilaton and RR forms, given by [6], which do not influence the motion of classical strings.

The metric for CP^3 is given in [6] as

$$ds^2_{CP^3}=\frac{dz_i d\bar{z}_i}{\rho^2}-\frac{|z_i d\bar{z}_i|^2}{\rho^4}, \qquad \text{where } \rho^2=z_i\bar{z}_i \tag{6.4}$$

in terms of the homogeneous co-ordinates $\vec{z}\in\mathbb{C}^4$, where $\vec{z}\sim\lambda\vec{z}$ for any complex λ. The $SU(4)$ isometry symmetry is manifest here, with $\vec{z}$ in the fundamental representation. AdS/CFT identifies this isometry group with the $SU(4)$ R-symmetry group, so it is natural to take $\vec{z}$ to be in the same basis as the fields Y^A in (6.1) above.

Instead of using four complex numbers as co-ordinates for CP^3, we can use six real angles.

One set of these is defined by

$$\vec{z} = \begin{pmatrix} \sin\xi\, \cos(\vartheta_2/2)\, e^{-i\eta/2}\, e^{i\varphi_2/2} \\ \cos\xi\, \cos(\vartheta_1/2)\, e^{i\eta/2}\, e^{i\varphi_1/2} \\ \cos\xi\, \sin(\vartheta_1/2)\, e^{i\eta/2}\, e^{-i\varphi_1/2} \\ \sin\xi\, \sin(\vartheta_2/2)\, e^{-i\eta/2}\, e^{-i\varphi_2/2} \end{pmatrix}, \tag{6.5}$$

and in terms of these angles, the metric is [213, 214]:

$$\begin{aligned} ds^2_{CP^3} = d\xi^2 + \frac{1}{4}\sin^2 2\xi \left(d\eta + \frac{1}{2}\cos\vartheta_1\, d\varphi_1 - \frac{1}{2}\cos\vartheta_2\, d\varphi_2\right)^2 \\ + \frac{1}{4}\cos^2\xi \left(d\vartheta_1^2 + \sin^2\vartheta_1\, d\varphi_1^2\right) + \frac{1}{4}\sin^2\xi \left(d\vartheta_2^2 + \sin^2\vartheta_2\, d\varphi_2^2\right) \end{aligned} \tag{6.6}$$

where $\xi \in [0, \frac{\pi}{2}]$, $\vartheta_1, \vartheta_2 \in [0, \pi]$, $\varphi_1, \varphi_2 \in [0, 2\pi]$ and $\eta \in [0, 4\pi]$ (this can be obtained by building S^7 from $S^3 \times S^3$ with the seventh co-ordinate ξ controlling their relative sizes).

This parametrization in terms of angles is one of two angular parametrizations commonly used. The other one is given by [215]:

$$\begin{aligned} ds^2_{CP^3} = d\mu^2 + \frac{1}{4}\sin^2\mu\cos^2\mu \left[d\chi + \sin^2\alpha\, (d\psi + \cos\theta\, d\phi)\right]^2 \\ + \sin^2\mu \left[d\alpha^2 + \frac{1}{4}\sin^2\alpha \left(d\theta^2 + \sin^2\theta\, d\phi^2 + \cos^2\alpha\, (d\psi + \cos\theta\, d\phi)^2\right)\right] \end{aligned} \tag{6.7}$$

with ranges $\alpha, \mu \in [0, \frac{\pi}{2}]$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$ and $\psi, \chi \in [0, 4\pi]$.

In Appendix D.1 we give the maps between these angles and the homogeneous co-ordinates.

6.3 The string sigma-model for $AdS_4 \times CP^3$

The full metric of $AdS_4 \times CP^3$ is then

$$ds^2 = \frac{R^2}{4} ds^2_{AdS} + R^2 ds^2_{CP} = R^2 \left(\frac{dy_\mu dy^\mu}{-4\vec{y}^2} + \frac{dz_i d\bar{z}_i}{|\vec{z}|^2} - \frac{|z_i d\bar{z}_i|^2}{|\vec{z}|^4} \right)$$

where we have embedded $AdS_4 \subset \mathbb{R}^{2,4}$ and $CP^3 \subset \mathbb{C}^4$, parametrized by $\vec{y}$ and $\vec{z}$ respectively. To study strings in this space, we constrain the lengths of these embedding co-ordinate vectors: $\vec{y}^2 = y_\mu y^\mu = -(y^{-1})^2 - (y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2 = -1$ and $|\vec{z}|^2 = z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3 = +1$. In addition to these constraints, points in $\mathbb{C}^4$ differing by an overall phase are identified in CP^3. This can be dealt with by introducing a gauge field: write the conformal gauge Lagrangian as

$$2\mathscr{L} = \frac{1}{4} \partial_a \vec{y} \cdot \partial^a \vec{y} - \Lambda(\vec{y}^2 + 1) + \overline{D_a \vec{z}} \cdot D^a \vec{z} - \Lambda'(\bar{\vec{z}} \cdot \vec{z} - 1)$$

where the covariant derivative is $D_a = \partial_a - A_a$. The equation of motion for the gauge field fixes $A_a = \bar{\vec{z}} \cdot \partial_a \vec{z}$. We can write the equations of motion for $\vec{y}$ and $\vec{z}$ as[5]

$$\partial_a \partial^a \vec{y} + (\partial_a \vec{y} \cdot \partial^a \vec{y}) \vec{y} = 0, \qquad D_a D^a \vec{z} + \left(\overline{D_a \vec{z}} \cdot D^a \vec{z} \right) \vec{z} = 0.$$

The AdS and CP components are coupled by the Virasoro constraints, which read:

$$\begin{aligned} \frac{1}{4} \partial_\tau \vec{y} \cdot \partial_\tau \vec{y} + \overline{D_\tau \vec{z}} \cdot D_\tau \vec{z} + \frac{1}{4} \partial_x \vec{y} \cdot \partial_x \vec{y} + \overline{D_x \vec{z}} \cdot D_x \vec{z} &= 0 \\ \frac{1}{4} \partial_\tau \vec{y} \cdot \partial_x \vec{y} + \mathrm{Re} \left(\overline{D_\tau \vec{z}} \cdot D_x \vec{z} \right) &= 0. \end{aligned}$$

We now restrict to solutions in $\mathbb{R} \times CP^3$, with $y^{-1} + iy^0 = e^{2i\tau}$ and $y^1 = y^2 = y^3 = 0$. We will always work in a gauge in which this τ is worldsheet time (timelike, or static, conformal

[5]Note that the CP^3 equation here reduces to that derived in Appendix D.2, where instead of treating the total phase as a gauge symmetry it was fixed to a constant using another Lagrange multiplier.

gauge).[6] The metric then reduces to

$$ds^2_{\mathbb{R}\times CP^3} = -d\tau^2 + |d\vec{z}|^2 - \left|\bar{\vec{z}}\cdot d\vec{z}\right|^2 .$$

In writing the Lagrangian, and this metric, we have pulled out the large radius factor $R^2 = 2^{5/2}\pi\sqrt{\lambda}$ to give a prefactor to the action:

$$S = \int \frac{dx\,d\tau}{2\pi} R^2 \mathscr{L} = 2\sqrt{2\lambda}\int dx\,d\tau\,\mathscr{L}.$$

This same factor appears when calculating conserved charges. The one from time-translation (which we define with respect to AdS time, $\tan t_{AdS} = y^0/y^{-1}$) is simply

$$\Delta = 2\sqrt{2\lambda}\int dx\,\frac{\partial\mathscr{L}}{\partial\,\partial_\tau t_{AdS}} = \sqrt{2\lambda}\int dx\,1 \tag{6.8}$$

where we used the fact that $t = 2\tau$ for the the solutions we're studying. The charges from rotations of CP^3's embedding co-ordinate planes are:

$$\begin{aligned} J_i \equiv J(z_i) &= 2\sqrt{2\lambda}\int dx\,\frac{\partial\mathscr{L}}{\partial\,\partial_\tau(\arg Z_i)} = 2\sqrt{2\lambda}\int dx\left[\mathrm{Im}(\bar{z}_i\partial_\tau z_i) - |z_i|^2\sum_j \mathrm{Im}(\bar{z}_j\partial_\tau z_j)\right] \\ &= 2\sqrt{2\lambda}\int dx\,\mathrm{Im}\left(\bar{z}_i D_\tau z_i\right) \qquad (\not{\Sigma}_i) \end{aligned} \tag{6.9}$$

Only three of the four $J(z_i)$ are independent, since $\sum_{i=1}^4 J(z_i) = 0$. These three are the charges from the Cartan generators of $\mathfrak{su}(4)$, and the charges from all of the generators can be obtained using their Lie-algebra matrices $T^a = (T^a)_{ij}$:

$$J[T^a] = 2\sqrt{2\lambda}\int dx\,\mathrm{Im}\left(\bar{\vec{z}}\cdot T^a D_\tau\vec{z}\right)$$

The matrices T^a are Hermitian and traceless, and the charges $J(z_i)$ are those generated by

[6] This implies the the length of the worldsheet cannot be held fixed to 2π. Instead it is proportional to the energy Δ, and thus infinite for the giant magnon. Taking this to be finite makes Δ and J finite too, thus we use 'finite-J' and 'finite-size' interchangeably.

diagonal T^a. The charges we will need for the giant magnons are

$$\begin{aligned} J &= J(z_1) - J(z_4) = J\left[\mathrm{diag}(1,0,0,-1)\right] \\ Q &= J(z_2) - J(z_3) = J\left[\mathrm{diag}(0,1,-1,0)\right] \\ J_3 &= J\left[\mathrm{diag}(-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2})\right] \end{aligned} \tag{6.10}$$

The charges of interest can also be written using the angles defined in (6.5): writing $J_{\varphi_2} = 2\sqrt{2\lambda}\int dx \, \frac{\partial \mathscr{L}}{\partial\, \partial_\tau \varphi_2}$ etc., we have $J = 2J_{\varphi_2}$, $Q = 2J_{\varphi_1}$ and $J_3 = J_\eta$

The Penrose limit describes the geometry very near to a null geodesic [216] and has been very important in AdS/CFT [5]. This has been studied in $AdS_4 \times CP^3$ by [192], where the particle travels along $\chi = 4\tau$ with $\alpha = 0$, $\mu = \pi/4$ in terms of the angles in (6.7), and by [217, 193], who use co-ordinates (6.6), expanding near $\vartheta_1 = \vartheta_2 = 0$, $\xi = \pi/4$ with distance along the line $\tilde{\psi} = \eta + (\varphi_1 - \varphi_2)/2 = -2\tau$. In all cases, the test particle moves along the path[7]

$$\vec{z} = \tfrac{1}{\sqrt{2}}\left(e^{i\tau}, 0, 0, e^{-i\tau}\right). \tag{6.11}$$

This has large angular momentum in opposite directions on the z_1 and z_4 planes, as one would expect for the state dual to the operator (6.2). This led to write this state down as the string state dual to the vacuum $\mathscr{O}_{\mathrm{vac}}$ [194].

In the $AdS_5 \times S^5$ case, the spin chain vacuum $\mathrm{tr}(\Phi_1 + i\Phi_2)^L$ is dual to a point particle defined by $X = (\cos\tau, \sin\tau, 0, 0, 0, 0)$. This string state has large angular momentum $J = \Delta$ in the 1-2 plane. The study of small fluctuations around this string state, one can show that $\Delta - J$ is a Hamiltonian for the physical modes [12]. In fact, semiclassical quantization views these modes as quantum fields with energy $\Delta - J$. As giant magnons are excitations above this vacuum, their semiclassical quantization is directly related to calculating quantum corrections to this energy [142, 124].

For the present case of $AdS_4 \times CP^3$, given the point particle state (6.11) and the dual vacuum (6.2), one would guess that $\Delta - (J_1 - J_4)/2$ will play the same role. In Appendix

[7] We stress that there are not different Penrose limits for the different giant magnon sectors. To get precisely this path $\vec{z}$, using our conventions given in (D.2) and (D.3), we fix in addition $\theta = \pi$ (in the first case) and $\varphi_1 = \varphi_2$ (in the second), and also swop $z_2 \leftrightarrow z_4$ in the second case.

D.3 we explicitly derive the fluctuation Hamiltonian for this case, and confirm that $\Delta - (J_1 - J_4)/2$ is indeed a Hamiltonian for the physical states.

6.4 Placing giant magnons into CP^3

Recall that the Hofman–Maldacena giant magnon [26] is a rigidly rotating classical string solution in $\mathbb{R} \times S^2$, given in timelike conformal gauge by

$$\begin{aligned} \cos\theta_{\text{mag}} &= \sin\frac{p}{2}\,\text{sech}\,u \\ \tan\left(\phi_{\text{mag}} - \tau\right) &= \tan\frac{p}{2}\tanh u \end{aligned} \tag{6.12}$$

where $u = (x - \tau\cos\frac{p}{2})/\sin\frac{p}{2}$ is the boosted spatial co-ordinate for a soliton with worldsheet velocity $\cos(p/2)$ (we will be using worldsheet space and time to be x, τ). The spacetime is $ds^2 = -d\tau^2 + d\theta^2 + \sin^2\theta\, d\phi^2$ — by timelike gauge we mean that the target-space time is also worldsheet time.[8]

We define conserved charges here as follows:

$$\Delta = \sqrt{2\lambda}\int dx\, 1 \tag{6.13}$$

$$J_{\text{sphere}} = \sqrt{2\lambda}\int dx\, \text{Im}\left(\bar{W}_1\partial_\tau W_1\right). \tag{6.14}$$

This Δ matches (6.8) used in Appendix D.3 when the AdS fluctuations $\tilde{t}$ and $\tilde{\vec{r}}$ are turned off. Note that we keep the same prefactor $\sqrt{2\lambda}$ here, which is not the one we would use in the $AdS_5 \times S^5$ case. Finally, we write the complex embedding co-ordinates $S^2 \subset \mathbb{C}^2$ and we find[9]

$$\vec{w} \equiv \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} e^{i\tau}\left[\cos\frac{p}{2} + i\sin\frac{p}{2}\tanh u\right] \\ \sin\frac{p}{2}\,\text{sech}\,u \end{pmatrix} = \begin{pmatrix} e^{i\phi_{\text{mag}}(x,\tau)}\sin\theta_{\text{mag}}(x,\tau) \\ \cos\theta_{\text{mag}}(x,\tau) \end{pmatrix} \tag{6.15}$$

[8] What we call timelike conformal gauge is sometimes called static conformal gauge. In our conventions, AdS time t is given by $t = 2\tau$. However, because of the factor $\frac{1}{4}$ in the metric (D.8), it is τ rather than t which is physical time.

[9] Our notation is that (w_1, w_2) are complex embedding co-ordinates for the sphere, while z_i are for CP^3. Capital letters indicate a string solution in this space.

Both Δ and J_{sphere} are infinite for the solution (6.12), but their difference is finite:

$$\Delta - J_{\text{sphere}} = 2\sqrt{2\lambda}\,\sin\left(\frac{p}{2}\right).$$

The parameter p is the (absolute value of the) momentum of the spin chain excitation in the dual gauge theory, which is why this is called a dispersion relation. It is also equal to the opening angle $\Delta\phi_{\text{mag}}$ of the string solution on the equator $\theta_{\text{mag}} = \frac{\pi}{2}$.

We now turn to solutions in $\mathbb{R}\times CP^3$, with metric $ds^2 = -d\tau^2 + ds^2_{CP^3}$. All solutions will be in conformal gauge, and with worldsheet time τ related to AdS time t by $t = 2\tau$, so we will continue to use the definition of Δ from (6.13), although for J we must now use (6.9) and (6.10). We will also continue to use the parameter $p \in [0, 2\pi]$ in all the cases below, and while this should still be a momentum in the dual theory, we make no comment here on the precise factors involved.

The subspace CP^1

If we set $z_2 = z_3 = 0$, or in terms of angles (6.7) $\alpha = 0$, then we obtain the space $CP^1 = S^2$ with metric

$$ds^2 = \frac{1}{4}\left[d(2\mu)^2 + \sin^2(2\mu)d\left(\frac{\chi}{2}\right)^2\right]. \tag{6.16}$$

This is a sphere of radius $\frac{1}{2}$, so to place the magnon solution (6.12) here (as was done by [192]) maintaining conformal gauge we need to set

$$\begin{aligned} 2\mu &= \theta_{\text{mag}}(2x, 2\tau) \\ \frac{\chi}{2} &= \phi_{\text{mag}}(2x, 2\tau)\,. \end{aligned} \tag{6.17}$$

Using the map (D.3), given in appendix D.1, and choosing $\theta = \pi$, we obtain

$$\vec{Z}(x,\tau) = \begin{pmatrix} e^{\frac{i}{2}\phi_{\text{mag}}(2x,2\tau)}\sin\left(\frac{1}{2}\theta_{\text{mag}}(2x,2\tau)\right) \\ 0 \\ 0 \\ e^{-\frac{i}{2}\phi_{\text{mag}}(2x,2\tau)}\cos\left(\frac{1}{2}\theta_{\text{mag}}(2x,2\tau)\right) \end{pmatrix}. \tag{6.18}$$

Calculating charges for this solution, using definitions (6.10) for J and (6.13) for Δ, we recover the dispersion relation[10]

$$\mathscr{E}(p) = \Delta - \frac{J}{2} = \sqrt{2\lambda}\,\sin\left(\frac{p}{2}\right). \tag{6.19}$$

One should check that this subspace is a legal one, meaning that solutions found here are guaranteed to be solutions in the full space. This can be done by finding the conformal gauge equations of motion coming from the Polyakov action with the metric (6.7), and confirming that α's equation is solved by $\alpha = 0$.[11] But in this case it is easier to note that $z_2 = z_3 = 0$ trivially solves their equations of motion, (D.5), which we derive in appendix D.2.

The subspace RP^2

A second embedding of the S^2 solution was first used by [193][12]

$$\vec{Z}(x,\tau) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\phi_{\text{mag}}(x,\tau)}\sin\theta_{\text{mag}}(x,\tau) \\ \cos\theta_{\text{mag}}(x,\tau) \\ \cos\theta_{\text{mag}}(x,\tau) \\ e^{-i\phi_{\text{mag}}(x,\tau)}\sin\theta_{\text{mag}}(x,\tau) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} W_1 \\ W_2 \\ \bar{W}_2 \\ \bar{W}_1 \end{pmatrix}. \tag{6.20}$$

This solution lives in an RP^2 subspace, as can be seen by simply rotating some of the planes in $\mathbb{C}^4 = \mathbb{R}^8$ by $\frac{\pi}{4}$: in terms of new co-ordinates $\vec{w}$ defined by

$$\begin{aligned} w_1 &= \tfrac{1}{\sqrt{2}}(z_1 + \bar{z}_4) \qquad & w_4 &= \tfrac{1}{\sqrt{2}}(z_1 - \bar{z}_4) \\ w_2 &= \tfrac{1}{\sqrt{2}}(z_2 + \bar{z}_3) \qquad & w_3 &= \tfrac{1}{\sqrt{2}}(z_2 - \bar{z}_3), \end{aligned} \tag{6.21}$$

[10] Note that if you were to omit the second term in (6.9) when calculating J, thus effectively using (6.14) appropriate for the sphere, you would get instead $\Delta - J/2 = \sqrt{2\lambda}\,p\cos\left(\frac{p}{2}\right)$. In the RP^2 and RP^3 subspaces discussed below, this second term vanishes.

[11] In addition to solving the conformal gauge equations of motion, a string solution must be in conformal gauge, i.e. must solve the Virasoro constraints. If the solution on the subspace is in conformal gauge, then it follows trivially that the solution in the full space is too: the induced metric $\gamma_{ab} = \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$ is influenced only by those directions the solution explores, and in these directions the metric $G_{\mu\nu}$ is the same in both the full space and the subspace.

[12] We discuss the equations of motion used by [193] for strings in CP^3 in Appendix D.2.

this solution has $w_3 = w_4 = 0$ and is precisely the original giant magnon in the other two co-ordinates:

$$(w_1, w_2) = \left(e^{i\phi_{\text{mag}}} \sin\theta_{\text{mag}}\,, \cos\theta_{\text{mag}}\right).$$

The reason this is RP^2 rather than S^2 is that sending $(w_1, w_2) \to -(w_1, w_2)$ gives an overall sign change on $\vec{z}$, and these two points are identified in CP^3.[13]

The subspace which this magnon explores can also be obtained from the metric (6.6), by fixing $\vartheta_1 = \frac{\pi}{2}$, $\vartheta_2 = \frac{\pi}{2}$, $\varphi_1 = 0$ and $\eta = 0$. The metric then becomes

$$ds^2 = d\xi^2 + \sin^2\xi\, d\left(\frac{\varphi_2}{2}\right)^2$$

and the magnon (6.20) is simply $\xi = \theta_{\text{mag}}(x,\tau)$, $\varphi_2 = 2\phi_{\text{mag}}(x,\tau)$. This can be checked to be a legal restriction from the equations of motion for the four angles fixed.

This subspace is sometimes, rather misleadingly, referred to as $S^2 \times S^2$. It is true that $|z_1|^2 + |z_2|^2 = \frac{1}{2}$ and $|z_3|^2 + |z_4|^2 = \frac{1}{2}$, and $\operatorname{Im} z_2 = 0 = \operatorname{Im} z_3$. These restrictions alone would describe a subspace of $\mathbb{C}^4$, namely $S^2 \times S^2 \subset \mathbb{C}^2 \times \mathbb{C}^2$. But we are in CP^3, not $\mathbb{C}^4$, and the space described by θ, ϕ (or by ξ, φ_2) has only two dimensions — these two S^2 factors are not independent. In Section 6.5 below we discuss a genuine four-dimensional $S^2 \times S^2$ subspace.

The charges of this solution are very simply related to those of the magnon on the sphere, since the extra term in the CP^3 angular momentum (6.9) compared to the that for the sphere vanishes: $J_{\text{sphere}} = \frac{1}{2}(J_1 - J_4) = \frac{J}{2}$, and we get simply

$$\mathscr{E} = \Delta - \frac{J}{2} = 2\sqrt{2\lambda}\, \sin\left(\frac{p'}{2}\right). \tag{6.22}$$

One difference from the magnon on S^2 is that when $p' = \pi$ the magnon becomes a single closed string. Its cusps, at opposite points on the equator of S^2, are in fact at the same point in RP^2. In general the magnon connects two points a distance $\Delta\varphi_2 = 2\Delta\phi_{\text{mag}} = 2p'$ apart on

[13] In S^2, the standard co-ordinates have ranges $\theta \in [0,\pi]$ and $\phi \in [0,2\pi]$, and changing $\theta \to \pi - \theta$ and $\phi \to \phi + \pi$ simultaneously moves you to the antipodal point on S^2. But performing this change in the subspace of CP^3 parametrized by (6.20) changes $\vec{z} \to -\vec{z}$, and these two points are identified by the definition of CP^3. This is what makes the subspace $RP^2 = S^2/\mathbb{Z}_2$ instead of S^2. To obtain co-ordinates which cover this subspace only once, we can shorten the range of either θ or ϕ, and in figure 6.1 we choose to restrict to $\phi \in [0,\pi]$ while keeping $\theta \in [0,\pi]$.

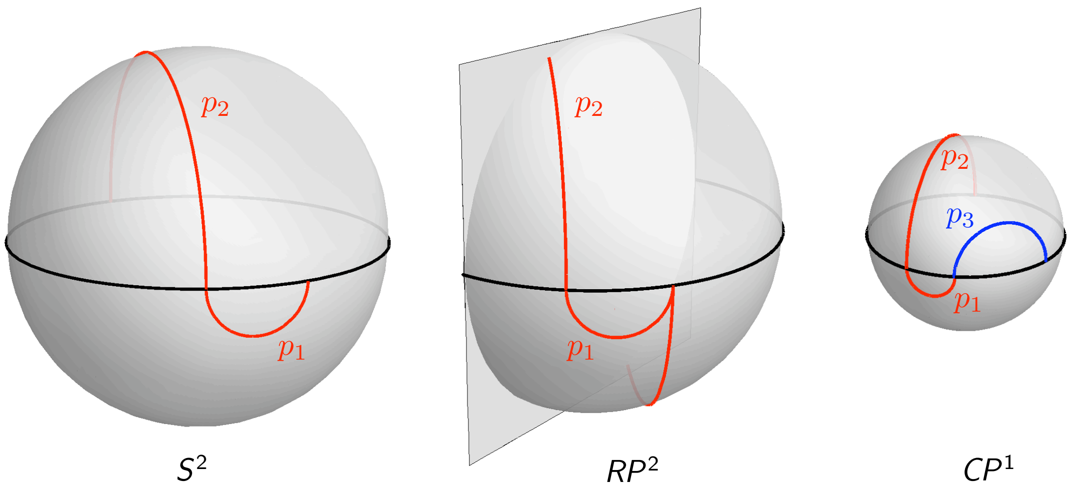

Figure 6.1: Two giant magnons are shown (in red) on the unit sphere S^2 (left), on RP^2 (center, drawn here as half a sphere) and on CP^1, a sphere of radius $\frac{1}{2}$ (right). In all cases they have $p_1 = \frac{1}{2}$ and $p_2 = \pi - \frac{1}{2}$, which leads to a closed string in the RP^2 case, but not in the S^2 or CP^1 cases.

In both the RP^2 and CP^1 cases, the equator is of length π, and we parametrize it by $\beta \in [0, \pi]$. The magnon with $p_1 = \frac{1}{2}$ spans $\Delta\beta = \frac{1}{2}$ in the RP^2 case, but only $\Delta\beta = \frac{1}{4}$ in the CP^1 case. On CP^1 we have also drawn a third magnon (in blue) with $p_3 = 1$, which spans the same length of equator $\Delta\beta = \frac{1}{2}$ as does the p_1 magnon on RP^2.

the equator, but $\varphi \sim \varphi + 2\pi$ so $p' = \delta$ and $p' = \pi + \delta$ both connect the same two points. As was noted by [192], this can be viewed as giving rise to a second class of magnons, with

$$\Delta - \frac{J}{2} = 2\sqrt{2\lambda}\, \sin\left(\frac{\pi + \delta}{2}\right) = 2\sqrt{2\lambda}\, \cos\left(\frac{\delta}{2}\right).$$

Figure 6.1 shows two magnons on S^2 and then on RP^2, one with $p = \frac{1}{2}$ and another with $p = \pi - \frac{1}{2}$. In the RP^2 case they have opposite opening angles $\delta = \pm\frac{1}{2}$, thus form a single closed string, while in the S^2 case the total opening angle is π. Note that between the parameter p in the CP^1 case and the parameter p' here we have the relation $p = 2p'$.

The subspace RP^3

In the $AdS_5 \times S^5$ case, Dorey's giant magnons with a second large angular momentum $J' \sim \sqrt{\lambda}$ allow one to see that the dispersion relation is $\Delta - J_{\text{sphere}} = \sqrt{J'^2 + \frac{\lambda}{\pi^2}\sin^2(p/2)}$ [27,94]. These necessarily live in S^3 rather than S^2. They are called dyonic magnons, and (embedding

$S^3 \subset \mathbb{C}^2$) can be written as

$$
\begin{aligned}
W_1 &= e^{i\tau}\left(\cos\frac{p}{2} + i\sin\frac{p}{2}\tanh U\right)\\
W_2 &= e^{iV}\sin\frac{p}{2}\,\mathrm{sech}\,U
\end{aligned}
$$

where

$$
\begin{aligned}
U &= (x\cosh\beta - \tau\sinh\beta)\cos\alpha, & \cot\alpha &= \frac{2r}{1-r^2}\sin\frac{p}{2},\\
V &= (\tau\cosh\beta - x\sinh\beta)\sin\alpha, & \tanh\beta &= \frac{2r}{1+r^2}\cos\frac{p}{2}.
\end{aligned}
$$

The parameter p is still the opening angle along the equator in the W_1 plane, although $\cos(p/2)$ is clearly no longer the worldsheet velocity. Sending the new parameter $r \to 1$ reproduces the original giant magnon.

The second method of embedding S^2 solutions into CP^3, given by (6.20), points out a way to embed S^3 solutions:

$$\vec{Z} = \tfrac{1}{\sqrt{2}}(W_1, W_2, \bar{W}_2, \bar{W}_1). \tag{6.23}$$

As before, this is in fact a subspace RP^3 rather than S^3, thanks to the identification of $(w_1, w_2) \sim -(w_1, w_2)$ implied.[14]

Embedding a dyonic giant magnon in this way gives a CP^3 solution with charges[15]

$$
\begin{aligned}
\Delta - \frac{J}{2} &= 2\sqrt{2\lambda}\,\frac{1+r^2}{2r}\sin\left(\frac{p'}{2}\right),\\
\frac{Q}{2} &= 2\sqrt{2\lambda}\,\frac{1-r^2}{2r}\sin\left(\frac{p'}{2}\right),
\end{aligned}
$$

where the angular momenta J and Q were defined in (6.10). The third angular momenta

[14] Note that the rotation from $\vec{z}$ to $\vec{w}$ given by (6.21) is not an isometry, and in particular that the identification $\vec{z} \sim \lambda\vec{z}$ which defines CP^3 does not apply afterwards: $\vec{w} \nsim \lambda\vec{w}$ for complex λ. If $w_3 = w_4 = 0$, as is implied by (6.23), then the phases of w_1 and w_2 are both physical. (Which is good if we're claiming that the dyonic magnon has momentum along both of them.)

However, the relation $\vec{w} \sim \lambda\vec{w}$ is true for real λ, and since we have fixed $w_1^2 + w_2^2 = 1$ by starting with a string solution on S^2, the identification $(w_1, w_2) \sim -(w_1, w_2)$ is all that survives.

[15] In calculating these charges from (6.9), the same cancellation of the second term happens here as happened in the previous section. Thus using the charges one would expect for $S^7 \subset \mathbb{C}^4$ gives the right answer here. This does not work in the CP^1 case, see footnote 10.

from (6.10) is still zero. These charges satisfy the relation[16]

$$\mathscr{E} = \Delta - \frac{J}{2} = \sqrt{\left(\frac{Q}{2}\right)^2 + 8\lambda \sin^2\left(\frac{p'}{2}\right)}. \tag{6.24}$$

Notice that the second angular momentum here, $Q \equiv J_2 - J_3$, is that carried by Y^2 and $Y_3^\dagger$, which are the impurities we insert into the vacuum (6.2) to make magnons in the $SU(2) \times SU(2)$ sector.

This subspace can also be obtained from (6.6), by fixing $\vartheta_1 = \frac{\pi}{2}$, $\vartheta_2 = \frac{\pi}{2}$ and $\eta = 0$. The metric becomes

$$ds^2 = d\xi^2 + \sin^2\xi\, d\left(\frac{\varphi_2}{2}\right)^2 + \cos^2\xi\, d\left(\frac{\varphi_1}{2}\right)^2.$$

This restriction can be checked to be a legal one from the equations of motion for the angles ϑ_1, ϑ_2 and η. The dyonic giant magnon in this space was re-derived by [197], using exactly these angles. It was also re-derived by [196] using co-ordinates $\vec{z}$.

Like the RP^2 magnons above, at $p' = \pi$ these form single closed strings, and beyond this ($\pi < p' < 2\pi$) give a second class of magnons connecting the same two points on the equator as the magnon with $\tilde{p} = p' - \pi$.

Comparison of CP^1 and RP^2 magnons

In Section 6.4 we looked at two different ways to embed the basic single-charge giant magnon (6.15), into either CP^1 or RP^2 [192, 193]. This CP^1 is a two-sphere of radius $\frac{1}{2}$, while RP^2 is half a two-sphere, so both have an equator of length π. We lined up the embeddings into $\mathbb{C}^4$ such that, in both cases, the equator is the line

$$\vec{z} = \tfrac{1}{\sqrt{2}}\left(e^{i\beta}, 0, 0, e^{-i\beta}\right)$$

where we name the angle $\beta \in [0, \pi]$, as in (D.7), to avoid confusion.

[16] We again used the prime p' to distinguish from the parameter $p = 2p'$ in CP^1.

Since the basic magnon (6.15) has opening angle $\Delta\phi_{\text{mag}} = p$, these two solutions have

$$\begin{aligned} CP^1: &\qquad \beta = \chi/4 = \phi_{\text{mag}}/2 &&\implies \quad \Delta\beta = p/2 \\ RP^2: &\qquad \beta = \varphi_2/2 = \phi_{\text{mag}} &&\implies \quad \Delta\beta = p' \end{aligned}$$

(where we write p' for the parameter of the RP^2 magnon, to distinguish it from the CP^1 case's p). A single giant magnon is not a closed string solution, one must join a set of them together at their endpoints on the equator. The condition for a set p_i of CP^1 magnons or p'_j of RP^2 magnons to close is that the total opening angle $\Delta\beta$ should be a multiple of π, that is,

$$\begin{aligned} CP^1: &\qquad \sum_i p_i = 2\pi n \\ RP^2: &\qquad \sum_j 2p'_j = 2\pi n\,, \qquad n \in \mathbb{Z}. \end{aligned} \tag{6.25}$$

Another way of putting this last result is that for the CP^1 magnon, like the original S^2 solution, the condition for the set of giant magnons to close is $\sum_i p_i = 0 \bmod 2\pi$, since p is the opening angle along the equator. However, for the RP^2 magnon, the $p' = \pi$ magnon is also a closed string, due to the $\mathbb{Z}_2$ identification in this space, and thus we have $\sum_i p'_i = 0 \bmod \pi$ instead.

The point particle (6.11) moves along the same equator too, and by calculating fluctuations of this solution, we checked in Appendix D.3 that $\Delta - \frac{J_1 - J_4}{2}$ is indeed a Hamiltonian for them, just as $\Delta - J$ is in the S^5 case. Calculating the same difference of charges for the two magnon embeddings, we obtained dispersion relations (6.19) and (6.22), which we now write also in terms of the opening angle $\Delta\beta$:

$$\begin{aligned} CP^1: &\qquad \Delta - \frac{J_1 - J_4}{2} = \sqrt{2\lambda}\sin\left(\frac{p}{2}\right) &&= \sqrt{2\lambda}\sin(\Delta\beta) \\ RP^2: &\qquad \Delta - \frac{J_1 - J_4}{2} = 2\sqrt{2\lambda}\sin\left(\frac{p'}{2}\right) &&= 2\sqrt{2\lambda}\sin\left(\frac{\Delta\beta}{2}\right). \end{aligned}$$

Notice that these agree at small $\Delta\beta$. The limit $p \to 0$ takes you from giant magnons to the Penrose limit (via the interpolating case of [218], studied here by [219]). Finite-J effects in the Penrose limit were studied by [220].

As noted in Section 6.4, there is also a second magnon on RP^2 for any given opening angle $\Delta\beta$, which has charges [192]

$$RP^{2\prime}: \qquad \Delta - \frac{J_1 - J_4}{2} = 2\sqrt{2\lambda}\cos\left(\frac{\Delta\beta}{2}\right).$$

For small $\Delta\beta$ this is almost a circular string, with its ends slightly offset along the equator — see figure 6.1 on page 163.

6.5 Some larger subspaces

All of the string solutions we have discussed so far are known from the $AdS_5 \times S^5$ case, and explore known subspaces S^2 or $S^3 \subset S^5$. In this section we look into CP^2 and $S^2 \times S^2$ subspaces of CP^3, in search of possible string solutions which explore the full geometry. In the first of these subspaces, the CP^2 subspace, such solutions have been found, and are discussed in the next Section 6.6.

We also study restrictions of this $S^2 \times S^2$ subspace (in Section 6.5), since these restrictions have been used in the literature.

The subspace CP^2

The simplest nontrivial subspace one can find is CP^2, obtained by setting $z_3 = 0$. This is certainly a legal subspace, for the same reason as given for CP^1: setting $z_3 = 0$ solves the z_3 equation of motion. The corresponding metric can be obtained as a restriction of (6.6) to $\vartheta_2 = 0$ (and $\varphi_2 = 0$, since now φ_2 is redundant):

$$ds^2 = d\xi^2 + \tfrac{1}{4}\cos^2\xi\left(d\vartheta_1^2 + \sin^2\vartheta_1\, d\varphi_1^2\right) + \tfrac{1}{4}\sin^2 2\xi\,\left(d\eta + \tfrac{1}{2}\cos\vartheta_1\, d\varphi_1\right)^2.$$

We have two manifest isometries, along φ_1 and η. For $\xi = 0$ we recover an S^2 equivalent to (6.16) (once $z_2 \leftrightarrow z_4$ are exchanged). Allowing $\xi \neq 0$ will allow us to determine a new dyonic solution (in section 6.6), generalizing the CP^1 solution (6.18) in the same way the dyonic

RP^3 solution generalizes the RP^2 one.

The subspace $\mathrm{S}^2 \times \mathrm{S}^2$

Setting $\varphi_1 = \varphi_2$ and $\vartheta_1 = \vartheta_2$ in the metric (6.6) leads us to the four-dimensional space

$$ds^2 = \frac{1}{4}\left[d(2\xi)^2 + \sin^2(2\xi)\, d\eta^2\right] + \frac{1}{4}\left[d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right] \tag{6.26}$$

which is a $S^2 \times S^2$ (possibly up to co-ordinate ranges). The new angles are defined by $\vartheta \equiv (\vartheta_1 + \vartheta_2)/2$ and $\varphi \equiv (\varphi_1 + \varphi_2)/2$.

To obtain this space, the restrictions on the angles are that $\vartheta_- \equiv \vartheta_1 - \vartheta_2 = 0$ and $\varphi_- \equiv \varphi_1 - \varphi_2 = 0$. Unfortunately, the equations of motion for ϑ_- and φ_- are not automatically solved by this choice: they have complicated relations between the other co-ordinates. For ϑ_- we have

$$\begin{aligned} 0 = &-\partial_\tau\left(\cos 2\xi\, \partial_\tau \vartheta\right) + \partial_x\left(\cos 2\xi\, \partial_x \vartheta\right) + \frac{1}{2}\cos 2\xi \sin 2\vartheta \left(\partial_\tau^2 \varphi - \partial_x^2 \varphi\right) \\ &- \sin^2 2\xi \sin\vartheta \left(\partial_\tau \eta\, \partial_\tau \varphi - \partial_x \eta\, \partial_x \varphi\right) \end{aligned}$$

and for φ_- the equation is

$$\begin{aligned} 0 = &-\partial_\tau\left(\sin^2 2\xi \cos\vartheta\, \partial_\tau \eta + \cos 2\xi \sin^2\vartheta\, \partial_\tau \varphi\right) \\ &+ \partial_x\left(\sin^2 2\xi \cos\vartheta\, \partial_x \eta + \cos 2\xi \sin^2\vartheta\, \partial_x \varphi\right). \end{aligned}$$

These equations do not rule out the existence of solutions on this subspace, but because these equations couple the variables ξ, η of one of the factors to the variables ϑ, φ of the other factor, just placing an arbitrary S^2 solution into each of the factors is not likely to give a solution.

The subspace $\mathrm{S}^2 \times \mathrm{S}^1$

We can further restrict the above subspace by holding one of the angles fixed, giving origin to a $S^2 \times S^1$ subspace (again up to identifications). This space, studied by [197], can be obtained by putting $\vartheta = \frac{\pi}{2}$:

$$ds^2 = \tfrac{1}{4}\left[d(2\xi)^2 + \sin^2(2\xi)\, d\eta^2 + d\varphi^2\right].$$

The equation of motion for ϑ is solved simply by $\vartheta = \frac{\pi}{2}$, which simplifies the constraints imposed by $\vartheta_- = 0$ and $\varphi_- = 0$ to

$$0 = -\partial_\tau \eta\, \partial_\tau \varphi + \partial_x \eta\, \partial_x \varphi \tag{6.27}$$

$$0 = -\partial_\tau \left(\cos 2\xi\, \partial_\tau \varphi\right) + \partial_x \left(\cos 2\xi\, \partial_x \varphi\right). \tag{6.28}$$

These constraints were not taken into account by [197], who sets $\vartheta_- = 0$ before calculating the equation of motion for ϑ (which is indeed solved) but never checks if the constraint coming from $\vartheta_- = 0$ is obeyed.[17] In that paper, the magnon ansatz used sets $\eta = \omega\tau + f(u)$, $\varphi = \nu\tau$ and $\xi = g(u)$, in terms of boosted worldsheet co-ordinates $u = \beta\tau + \alpha x$. The first constraint (6.27) then implies $\beta f'(u) = -\omega$, while one would expect $f(u) \propto \tanh u$ for a magnon solution. The second constraint (6.28) then gives $\beta = 0$, and together they imply $\omega = 0$.

In the other case studied by [197] one doesn't find the same problem, because the ϑ_- equation is solved by $\eta = 0$, and there is no φ_- constraint because $\varphi_1 \neq \varphi_2$. The subspace consequently found is the RP^3 subspace of Section 6.4.

The subspace CP^1, *again*

If we restrict the subspace $S^2 \times S^2$ of (6.26) even further, by setting ξ and η to be constants, it reduces to the space

$$ds^2 = \tfrac{1}{4}\left[d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right]$$

which is a sphere of radius $\frac{1}{2}$, equivalent to the CP^1 of Section 6.4. This is a legal subspace: the equations of motion for ξ and η are trivially solved (a stationary particle anywhere on the sphere is a solution) and the constraints arising from $\vartheta_- = 0$ and from $\varphi_- = 0$ become the equations of motion for ϑ and φ.

[17] The constraint (6.27) can be derived without using ϑ_-, if we set $\vartheta_1 = \frac{\pi}{2}$ and $\vartheta_2 = \frac{\pi}{2}$ in their respective equations of motion.

Setting $\xi = \frac{\pi}{2}$, this space can be embedded in homogeneous co-ordinates by (see Appendix D.1 for more details)

$$\vec{z} = \left(e^{i\varphi/2} \cos\frac{\vartheta}{2}\,, 0, 0, e^{-i\varphi/2} \sin\frac{\vartheta}{2} \right),$$

which is just the same subspace CP^1 as in (6.16), even though we had started from the other set of angles (6.7) and fixed $\alpha = 0$. Setting ξ to any other value will just rotate the 1-2 and 3-4 planes, still giving the subspace $S^2 = CP^1$.

These co-ordinates of the CP^1 space were used in [221] to study finite-J effects on the CP^1 giant magnon. These results can be found in (6.33).

6.6 New solutions in CP^3

Until now, we have focused on giant magnon solutions which are "trivial" embeddings of already known string solutions moving $\mathbb{R} \times S^2$ and $\mathbb{R} \times S^3$. But there is great interest in finding solutions that truly explore the geometry of CP^3. In this section we present two such solutions, the first being a dyonic generalization of the CP^1 magnon, and the second being some kind of "bound state" of two magnons (we will have a better interpretation for it when comparing to the results in the algebraic curve formalism) – that is, a two-parameter, one angular momentum string solution. Both of these solutions explore the full CP^2 space.

Dyonic generalization of the CP^1 magnon

Consider the subspace CP^2 obtained by fixing $z_3 = 0$, or in terms of the angles, $\theta_1 = 0$ and $\eta = 0$, leaving

$$\vec{z} = \begin{pmatrix} \sin\xi\,\cos(\vartheta_2/2)\,e^{i\varphi_2/2} \\ \cos\xi\,e^{i\varphi_1/2} \\ 0 \\ \sin\xi\,\sin(\vartheta_2/2)\,e^{-i\varphi_2/2} \end{pmatrix}. \tag{6.29}$$

The metric for this subspace can be written as

$$ds^2 = \frac{1}{4}\sin^2\xi\left[d\vartheta_2^2 + \sin^2\vartheta_2\, d\varphi_2^2\ + \cos^2\xi\,(d\varphi_1 - \cos\vartheta_2\, d\varphi_2)^2\right] + d\xi^2.$$

At $\xi = \frac{\pi}{2}$ the space is CP^1, described by ϑ_2 and φ_2 only, but away from this value there is a second isometry direction $d\varphi_1$. It was proposed in [222] that the dyonic generalization of the CP^1 magnon might have momentum along this direction, but to do so, it must in addition have $\xi \neq \frac{\pi}{2}$ except at the endpoints of the string, at $x = \pm\infty$, where it must touch the same equator as the CP^1 solution.

We have not yet been able to find the full solution, but can find a GKP-like dyonic solution (i.e. a $p = \pi$ magnon[18]) using the ansatz:

$$\begin{aligned} \varphi_2 &= 2\tau, & \varphi_1 &= -2\omega\tau, \\ \cos\vartheta_2 &= \operatorname{sech}\left(\sqrt{1-\omega^2}2x\right), & \xi &= \frac{\pi}{2} - e(x). \end{aligned}$$

This amounts to assuming that the back-reaction on the original solution in ϑ_2, φ_2 created by giving it new momentum along φ_1 is exactly as for the S^3 dyonic solution, but unlike the S^3 case, there is one extra function $e(x)$.

With this ansatz the equations of motion for φ_1 and φ_2, and the second Virasoro constraint, are already solved. The equation of motion for ϑ_2 can be written

$$\partial_x\left(\cos^2 e(x)\,\operatorname{sech}X\right) = -2\frac{\cos^2 e(x)}{\sqrt{1-\omega^2}}\tanh X\left\{\operatorname{sech}X\cos^2 e(x) + \omega\sin^2 e(x)\right\},$$

where $X = \sqrt{1-\omega^2}2x$. Using a change of variables $y(x) = \ln\left(\cos^2 e(x)\right)$, this equation can be written as

$$y'(x) = e^{y(x)} f(x) + g(x)\,, \tag{6.30}$$

[18]GKP [11] studied rotating folded strings, which at $J = \infty$ are the $p = \pi$ case of the HM magnon [26]. Two-spin folded string solutions were studied by F&T [25], and are the $p = \pi$ case of Dorey's dyonic giant magnon [27, 94].

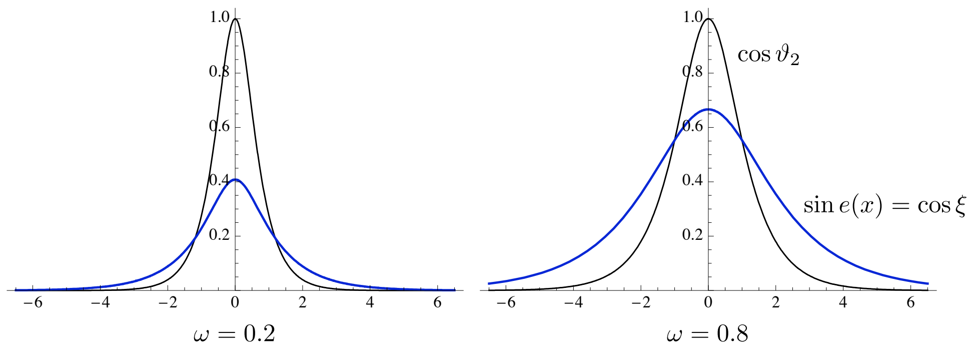

Figure 6.2: Profiles of the CP^2 solution. $\cos\vartheta_2$ is the same as for Dorey's S^3 dyonic giant magnon, but the solution also spreads away from $\xi = \frac{\pi}{2}$ as we increase ω. This is shown for $\omega = 0.2$ and 0.8.

where

$$f(x) = \frac{2}{\sqrt{1-\omega^2}}\left(\omega - \operatorname{sech} X\right)\sinh X,$$
$$g(x) = -f(x) - \frac{2\omega^2}{\sqrt{1-\omega^2}}\tanh X.$$

The equation (6.30) has solutions of the form $y(x) = -\ln(-F(x)) + G(x)$, where $G'(x) = g(x)$ and $F'(x) = f(x)e^{G(x)}$. After some algebra, we find the following form for the solution:

$$\cos^2(e(x)) = \sin^2\xi = \frac{1}{1+\omega\cos\vartheta_2} = \frac{1}{1+\omega\operatorname{sech}\left(\sqrt{1-\omega^2}\,2x\right)} \tag{6.31}$$

where $\omega \geq 0$.

Calculating charges J and Q for this solution, we find

$$\Delta - \frac{J}{2} = \sqrt{2\lambda}\frac{1}{\sqrt{1-\omega^2}}, \qquad \frac{Q}{2} = -\sqrt{2\lambda}\frac{\omega}{\sqrt{1-\omega^2}}$$

and therefore the dispersion relation is

$$\Delta - \frac{J}{2} = \sqrt{\frac{Q^2}{4} + 2\lambda}\,.$$

	$\mathscr{E}=\Delta-\frac{J}{2}$	$\delta\mathscr{E}$ (finite J)	Q	J_3
Vacuum	0			
CP^1 giant magnon	$\sqrt{2\lambda}\sin(\frac{p}{2})$	$-4\mathscr{E}\sin^2(\frac{p}{2})e^{-2\Delta/\mathscr{E}}$	0	0
Dyonic version in CP^2	$\sqrt{\frac{Q^2}{4}+2\lambda}$ when $p=\pi$	(Use 'small' curve, (7.12))	Q	$\pm Q$
RP^2 giant magnon	$2\sqrt{2\lambda}\sin(\frac{p'}{2})$	$-4\mathscr{E}\sin^2(\frac{p'}{2})e^{-2\Delta/\mathscr{E}}$	0	0
Dyonic version in RP^3	$\sqrt{\frac{Q^2}{4}+8\lambda\sin^2(\frac{p'}{2})}$	Like S^5 result, (6.34)	Q	0
HM/KSV/S dressed solution	$\sqrt{Q_f^2+8\lambda\sin^2(\frac{p'}{2})}$	(Use 'big' curve, (7.15))	0	0

Table 6.1: Summary of giant magnons in the string sigma-model. The dressed solution of [199, 200, 201] also lives in CP^2 (the RP^2 solution has often been called $SU(2)\times SU(2)$, 'big' and $S^2\times S^2$ in the literature). To match the curves we want $p'=p/2$.

We conjecture that for the general case (allowing $p\neq\pi$) the dispersion relation is

$$\mathscr{E}=\Delta-\frac{J}{2}=\sqrt{\frac{Q^2}{4}+2\lambda\sin\left(\frac{p}{2}\right)}$$

matching the one for the 'small giant magon' in the algebraic curve.

Unlike the RP^3 dyonic magnon, this one is charged not only under Q but also under J_3, with $J_3=Q$. There is a second CP^2 solution, in the subspace with $z_2=0$ instead of $z_3=0$, which has $J_3=-Q$ but is otherwise similar. In the limit $\omega\to 0$ both kinds become the same CP^1 solution. All of these properties match those of the two kinds of small giant magnons in the algebraic curve perfectly.

Dressing method solution in CP^2

One other solution which does not exist in S^5 was recently constructed by several groups using the dressing method [199, 200, 201].

The dressing method is used to generate multi-soliton solutions above a given vacuum in the principal chiral model, and is closely related to the Bäcklund transformation [154, 223]. We start from the 'bare' vacuum solution Ψ_0, and construct the so called 'dressed' solution Ψ by setting $\Psi=\chi(0)\Psi_0$, where $\chi(\lambda)$ is the dressing matrix and λ is a spectral parameter. Each independent pole λ_1 of the dressing matrix $\chi(\lambda)$[19] leads to one soliton, whose characterizing parameters are given by λ_1's position in the complex plane.

This method was first used to generate giant magnons in S^3 by [109]. In there, the string sigma-model was mapped to an $SU(2)$ principal chiral model and an $SO(3)$ vector model. For the former case, it was seen that a pole at $\lambda_1 = re^{ip/2}$ gave rise to a dyonic giant magnon with charges

$$\Delta - J_1 = \frac{\sqrt{\lambda}}{\pi}\frac{1+r^2}{2r}\sin\left(\frac{p}{2}\right), \qquad J_2 = \frac{\sqrt{\lambda}}{\pi}\frac{1-r^2}{2r}\sin\left(\frac{p}{2}\right)$$

which combine to give the usual dispersion relation (J_2 can be viewed as the second parameter of the solution, instead of r). In the $SO(3)$ case, only the non-dyonic giant magnon, $r = 1$ case, can be obtained.

The solution that was constructed in CP^3 makes use of the map to an $SU(4)/U(3)$ model. The position of the dressing pole $\lambda_1 = re^{ip'/2}$ provides two parameters, but unlike the S^3 case, there is only one nonzero angular momentum[20]

$$J = 2\Delta - 4\sqrt{2\lambda}\frac{1+r^2}{2r}\sin\left(\frac{p'}{2}\right)$$

Nevertheless, if we replace the parameter r with a new parameter defined by

$$Q_f = 2\sqrt{2\lambda}\frac{1-r^2}{2r}\sin(\frac{p'}{2}),$$

then the dispersion relation can be re-written as

$$\Delta - \frac{J}{2} = \sqrt{Q_f^2 + 8\lambda\sin^2\left(\frac{p'}{2}\right)}.$$

The factor in front of Q_f has been chosen to make the above expression for the dispersion relation match the dispersion relation for the big giant magnon in the algebraic curve (after setting $p = 2p'$). In fact, the big giant magnon is also a two-parameter single-momentum

[19] And sometimes also an image pole at $1/\lambda_1$.

[20] As in RP^2, we will call the "opening angle" parameter p'.

solution, as will be seen in Section 7.1 below.

Choosing the same basis as [199, 200],[21] the solution in the GKP[22] case $p' = \pi$ is given by:[23]

$$\vec{z}' = N \begin{pmatrix} (1+r^2)\cos(\tau) + \cos\left(\frac{1-3r^2}{1+r^2}\tau\right) + r^2\cos\left(\frac{3-r^2}{1+r^2}\tau\right) + i(1-r^2)\sin(\tau)\sinh\left(\frac{4r}{1+r^2}x\right) \\ -(1+r^2)\sin(\tau) + \sin\left(\frac{1-3r^2}{1+r^2}\tau\right) - r^2\sin\left(\frac{3-r^2}{1+r^2}\tau\right) - i(1-r^2)\cos(\tau)\sinh\left(\frac{4r}{1+r^2}x\right) \\ 2(1-r^2)\left[\sin\left(\frac{1-r^2}{1+r^2}\tau\right)\sinh\left(\frac{2r}{1+r^2}x\right) - i\cos\left(\frac{1-r^2}{1+r^2}\tau\right)\cosh\left(\frac{2r}{1+r^2}x\right)\right] \\ 0 \end{pmatrix}$$

where N is a normalization factor ensuring $|\vec{z}|^2 = 1$. The vacuum related to this solution, $\vec{z}'_{\text{vac}} = (\cos(t), \sin(t), 0, 0)$, has a large charge under $J' = J[\sigma_2 \oplus 1]$, and all other charges $J[T^a]$ zero. In the presence of this magnon solution, the value of the non-zero charge J' changes, but all the other charges remain zero. If we perform a rotation in the vacuum $\vec{z}'_{\text{vac}}$ so that it matches our $\vec{z}_{\text{vac}} = \frac{1}{\sqrt{2}}\left(e^{it}, 0, 0, e^{-it}\right)$, the same transformation will also rotate J' into the charge J used for the other magnons.

Taking the limit $r \to 1$, or $Q_f \to 0$ in this solution, we recover the dispersion relation of the RP^2 giant magnon. The same limit will naturally take the solution (in this basis) to the embedding of the ordinary magnon (6.12) given by $\vec{z}' = (\text{Re}\, w_1, \text{Im}\, w_1, \text{Re}\, w_2, 0) \in \mathbb{R}^4$.

Finite-J corrections to this solution have not been done in the string sigma-model (except trivially at $Q_f = 0$, where it coincides with the RP^2 magnon) but we compute them through the algebraic curve formalism in the next Chapter, Section 7.2.

We summarize all the properties of the various string solutions in table 6.1.

6.7 Finite-J corrections

All of the giant magnons solutions we have been discussing so far have both infinite energy and infinite angular momentum. One can easily see from (6.13) that in the timelike conformal

[21] In [201] the basis used to get the solution was the same as ours.

[22] See footnote 18 about this name.

[23] At $p' = \pi$ this solution is a closed string, just as it happened for the RP^2 and RP^3 magnons.

gauge we are using, this corresponds to having infinite worldsheet length.

The finite-J generalizations of the basic giant magnon are still solutions moving on S^2, and so one can embed them into CP^3 using either of the maps given in sections 6.4 and 6.4 above. For the RP^2 giant magnon, the corrections to the dispersion relation were derived in [224] to be

$$\Delta - \frac{J}{2} = 2\sqrt{2\lambda}\sin\left(\frac{p}{2}\right)\left[1 - 4\sin^2\left(\frac{p'}{2}\right)e^{-2\Delta\big/2\sqrt{2\lambda}\sin(\frac{p'}{2})} + \dots\right] \tag{6.32}$$

The CP^1 giant magnon case was studied in [221] where they found the corrected dispersion relation to be[24]

$$\Delta - \frac{J}{2} = \sqrt{2\lambda}\sin\left(\frac{p}{2}\right)\left[1 - 4\sin^2\left(\frac{p}{2}\right)e^{-2\Delta\big/\sqrt{2\lambda}\sin(\frac{p}{2})} + \dots\right]. \tag{6.33}$$

We observe that, even at finite J, two CP^1 magnons have the same dispersion relation as one RP^2 magnon, provided all three have the same value of the parameter p.[25]

For the RP^3 dyonic giant magnon, we can similarly embed the results from S^3. The corrections to the dispersion relation of the dyonic giant magnon in S^3 were originally computed by [138] , from the all-J solutions of [96]. The embedded solutions in CP^3 were then studied by [196, 225], and the result is:

$$\Delta - \frac{J}{2} = \sqrt{\frac{Q^2}{4} + 8\lambda\sin^2\left(\frac{p'}{2}\right)} - 32\lambda\cos(2\phi)\frac{1}{\mathscr{E}}\sin^4\left(\frac{p'}{2}\right)e^{-\Delta\mathscr{E}\big/2S} \tag{6.34}$$

---

[24] Here is a brief note about deriving these two results from the original S^2 case. The integrals defining the charges are now over a finite length $-L < x < L$, so write $J(L)$ and $\Delta(L)$. Note that $\Delta(2L) = 2\Delta(L)$. To get the charges for one magnon, we must integrate from one cusp to the next: choose L such that $\theta_{\text{mag}}(x = \pm L, \tau = 0)$ are at the first cusps.

For the RP^2 case, the relationship we used before $J_{\text{sphere}}(L) = J_1(L) = (J_1(L) - J_4(L))/2$ still holds, leading to (6.32). We wrote the S^2 result (3.9) on page 61 using the prefactor appropriate for $AdS_5 \times S^5$, so to get this result for the $AdS_4 \times CP^3$ theory one has to replace $\sqrt{\lambda}/\pi \to 2\sqrt{2\lambda}$.

For the CP^1 case, the cusp at $\theta_{\text{mag}}(L,0)$ is at $\vec{Z}_{CP^1}(\frac{L}{2},0)$, thanks to the scaling (6.17). The relationship between charges is that

$$\frac{J_1(\frac{L}{2}) - J_4(\frac{L}{2})}{2} = \frac{1}{2}J_{\text{sphere}}(L).$$

Thus $\Delta(\frac{L}{2}) - (J_1(\frac{L}{2}) - J_4(\frac{L}{2}))/2 = \Delta(\frac{L}{2}) - \frac{1}{2}J_{\text{sphere}}(L) = \frac{1}{2}\left(\Delta(L) - J_{\text{sphere}}(L)\right)$. In the result (6.33), it is the energy for one magnon $\Delta(\frac{L}{2})$ which appears both on the left hand side and in the exponent.

[25] One can see that all the properties of the two CP^1 magnons seem to add up to give those of the single RP^2 magnon: energy Δ, angular momentum $J/2$, the worldsheet length L and the opening angle along the equator (which we have called $\Delta\beta$ previously).

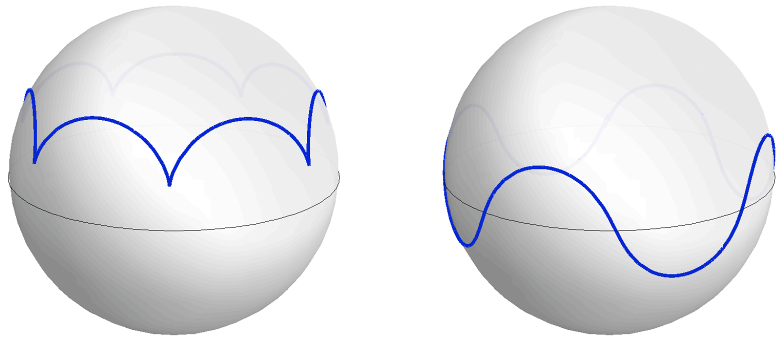

Figure 6.3: The two classes of finite-J magnons found by [96].

where $\mathscr{E}$ was given in (6.24) and we also define [26][27]

$$S = \frac{Q^2}{16\sin^2(\frac{p'}{2})} + 2\lambda \sin^2\left(\frac{p'}{2}\right). \tag{6.35}$$

We should clarify the meaning of the factor $\cos(2\phi)$. In [138], this factor is set to be $+1$ for 'type (i) helical strings', and -1 for 'type (ii)' strings. These are two kinds of finite-J solutions, which in the non-dyonic case in S^2 have the property that adjacent magnons either have the same or opposite orientations. Type (i) strings will then have a cusp not touching the equator, while type (ii) strings cross the equator at less than a right angle – see figure 6.3. Everything points to the interpretation of 2ϕ being the same in the CP^3 solutions: 2ϕ should be the angle between the two magnons' orientation vectors (the same factor could be included in the non-dyonic cases (6.32) and (6.33)).

As was mentioned before, in the $AdS_5 \times S^5$ case the finite-J corrections can also be calculated using algebraic curves [29, 31, 32, 34, 89, 183, 35, 141, 226, 184] or using the Lüscher formula [139, 227, 140, 228, 229]. The corrections obtained through these methods agree with the string σ-model result, which can be found in (3.9) on page 61. For calculations on the gauge theory side of the correspondence see [23, 230, 231, 136, 232, 233, 234, 235].

In $AdS_4 \times CP^3$ one can also use any of the methods mentioned above. In Chapter 7,

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[26] Note that $S(\frac{p'}{2}) \to \frac{1}{4}\mathscr{E}^2$ as $Q \to 0$. Comparing to [138], we should mention that $\cosh(\theta/2) = \mathscr{E}/2\sqrt{2\lambda}\sin(\frac{p'}{2})$. In terms of our notation for the next chapter, this θ is defined $r = e^{\theta/2}$.

[27] $\mathscr{E} \equiv \Delta - J/2$ is defined at infinite J.

we will discuss the application of the algebraic curve formalism to determine these finite-J corrections for the giant magnon solutions presented in this chapter.

Chapter 7

Algebraic curves in CP^3 and finite-J corrections

Classical and semi-classical strings allow us to explore some sectors of the $\mathcal{N}=6$ ABJM / $AdS_4 \times CP^3$ duality [6], and these are much richer than their well-known counterparts in the $\mathcal{N}=4$ SYM / $AdS_5 \times S^5$ duality [1]. It was known very early on that there are at least two kinds of giant magnons, created by placing the HM giant magnon [26] into various S^2-like subspaces, namely CP^1 and RP^2 [192, 193]. It is equally easy to place Dorey's S^3 dyonic giant magnon [27, 94] into RP^3, giving a two-spin generalization of the RP^2 magnon [196]. Other solutions truly living in CP^3 have also been studied, one being a two-parameter one-angular-momentum solution [199,200,201] generalizing the RP^2 magnon, and the other being a dyonic generalization of the CP^1 magnon [202], living in CP^2.

As was mentioned in the last Chapter, the solutions of the classical string sigma-model should be in exact correspondence to algebraic curves [46]. After having descibed the known giant magnon solutions on the string side, we can now review the solutions known from the algebraic curve formalism, and write the dictionary of solutions on both sides of this correspondence.

Finite-J corrections are of increasing importance in the study of gauge and string integrability. They can sometimes be computed directly on the string side by finding solutions with $J < \infty$, and all existing finite-J giant magnon solutions are embeddings of well-known S^5 results of this type [82,96,97,138]. Other methods that have been used to calculate finite size corrections includes the construction of corresponding algebraic curves [141, 226, 184, 185] and the Lüscher formulae [227, 185, 138, 236].

In this Chapter we will start by setting up the algebraic curve formalism for $AdS_4 \times CP^3$. We then extend the algebraic curve calculations of [185], by calculating finite-J corrections not only for a pair of small giant magnons, but also for a single small magnon and the big magnon. In the non-dyonic case, all of these give the result AFZ [82] found for a magnon in S^2. Likewise the dyonic pair of giant magnons matches the S^3 result: this too is a simple embedding of that string solution. But for the dyonic small and big magnons, which correspond to string solutions not found in S^5, we find new formulae for these energy corrections. We will finally conclude with the dictionary between strings and curves, summarized in two tables, on pages 173 and 187.

7.1 The algebraic curve for $AdS_4 \times CP^3$

In Chapter 5 we studied classical strings in $AdS_5 \times S^5$ from the point of view of algebraic curves, following the works of [29, 30, 31, 32, 34, 146]. One can take the same approach for the $AdS_4 \times CP^3$ case, starting from the σ-model given by [187, 18]. This was first studied by Gromov and Vieira [46]. We will start by summarizing the bosonic algebraic curve on $AdS_4 \times CP^3$ and some of its properties, which can be found in [46].[1]

The eigenvalues of the monodromy matrix in this background are $e^{i\tilde{p}_1}, e^{i\tilde{p}_2}, e^{i\tilde{p}_3}, e^{i\tilde{p}_4}$ for the CP^3 part, and $e^{i\hat{p}_1}, e^{i\hat{p}_2}, e^{i\hat{p}_3}, e^{i\hat{p}_4}$ for the AdS part, where the functions $\tilde{p}_i$ and $\hat{p}_i$ as are the quasi-momenta, as before. The continuity in the complex plane of the function $\mathrm{eig}(\Omega(\mathrm{x}))$ demands that when a branch cut C_{ij} connects sheets i and j, we must have

$$p_i^+ - p_j^- = 2\pi n\,, \quad \text{for}\, x \in C_{ij}\,.$$

The superscript $\pm$ indicates that the function is being evaluated immediately above/below

[1] In this section we use x to be the spectral parameter of the curves, and σ, τ to be the worldsheet co-ordinates which we called x, τ before.

the cut. These quasi-momenta also obey the traceless condition of the monodromy matrix

$$\sum_{k=1}^{4} \tilde{p}_k(x) = 0\,, \quad \hat{p}_1(x) + \hat{p}_4(x) = \hat{p}_2(x) + \hat{p}_3(x) = 0.$$

With respect to these quasi-momenta, one can then define fillings, and relate them to the relevant Dynkin labels or conserved charges, by performing an expansion at $x \to \infty$:

$$\begin{aligned}
(\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4) &\sim \frac{1}{gx}(L+\Delta, S, -S, -L-\Delta)\,, \\
(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4) &\sim \frac{1}{2gx}(L-M_u, M_u-M_r, M_r-M_v, -L+M_r)\,,
\end{aligned}$$

where L is the length operator charge, Δ is the energy (time-translations) charge, S is an angular momentum in the AdS factor, and the excitation numbers $M_{u,v,r}$ are related to the $SU(4)$ Dynkin labels by

$$\begin{bmatrix} p_1 \\ q \\ p_2 \end{bmatrix} = \begin{bmatrix} L-2M_u+M_r \\ M_u+M_v-2M_r \\ L-2M_v+M_r \end{bmatrix} \in \mathbb{Z}^3_{\geq 0}\,. \tag{7.1}$$

In the singularities $x = \pm 1$ the quasi-momenta can be expanded as

$$\begin{aligned}
(\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4) &\sim \frac{1}{x \mp 1}(\alpha_\pm, 0, 0, -\alpha_\pm)\,, \\
(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4) &\sim \frac{1}{2}\frac{1}{x \mp 1}(\alpha_\pm, 0, 0, -\alpha_\pm)\,.
\end{aligned}$$

Finally, under the inversion symmetry $x \to 1/x$, these quasi-momenta behave as

$$\hat{p}_k(1/x) = \varepsilon_k \hat{p}_k(x)\,, \quad \tilde{p}_k(1/x) = \tilde{p}_{k'}(x) + \varepsilon_k 2\pi m\,,$$

where $\varepsilon_k = (-1, 1, 1, -1)$ and $k' = (4, 2, 3, 1)$ when $k = (1, 2, 3, 4)$.

The quasi-momenta $\tilde{p}_i$ and $\hat{p}_i$ describe the bosonic sector of type IIA string theory on $AdS_4 \times CP^3$, and are the analogous quantities to the ones used in Chapter 5. Nevertheless, to perform calculations it is more convenient to work in a formalism with explicit $OSp(2,2|6)$

symmetry. To do this we define ten new quasi-momenta q_i as [46]

$$\{q_1,q_2,q_3,q_4,q_5\} = \frac{1}{2}\{\hat{p}_1+\hat{p}_2,\hat{p}_1-\hat{p}_2,\tilde{p}_1+\tilde{p}_2,-\tilde{p}_2-\tilde{p}_4,\tilde{p}_1+\tilde{p}_4\}$$

and

$$\{q_6,q_7,q_8,q_9,q_{10}\} = \{-q_5,-q_4,-q_3,-q_2,-q_1\}.$$

The functions q_i now define a ten-sheeted Riemann surface.

Algebraic curve with manifest $OSp(2,2|6)$ symmetry

In this ten-sheeted Riemann surface, the quasi-momenta q_i have to obey the following relations [46]:

1. Only five of the quasi-momenta are independent:

$$\{q_6,q_7,q_8,q_9,q_{10}\} = \{-q_5,-q_4,-q_3,-q_2,-q_1\}.$$

2. If we have a square-root branch cut C_{ij} between sheets i,j then:

$$q_i^+(x) - q_j^-(x) = 2\pi n_{ij},\; x\in C_{ij}.$$

3. Synchronized poles: the residues at poles $x=\pm 1$ are the same $(\alpha_\pm/2)$ for q_1,q_2,q_3,q_4, while q_5 does not have a pole at $x=\pm 1$.

4. Inversion symmetry:

$$\begin{pmatrix} q_1(1/x) \\ q_2(1/x) \\ q_3(1/x) \\ q_4(1/x) \\ q_5(1/x) \end{pmatrix} = \begin{pmatrix} -q_2(x) \\ -q_1(x) \\ -q_4(x)+2\pi m \\ -q_3(x)+2\pi m \\ +q_5(x) \end{pmatrix}.$$

   This $m\in\mathbb{Z}$ gives the momentum condition: $p=2\pi m$.

5. Asymptotic behaviour as $x \to \infty$:

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{pmatrix} = \frac{1}{2gx} \begin{pmatrix} \Delta + S \\ \Delta - S \\ L - M_r \\ L + M_r - M_u - M_v \\ M_v - M_u \end{pmatrix} + o\left(\frac{1}{x^2}\right) = \frac{1}{2gx} \begin{pmatrix} \Delta + S \\ \Delta - S \\ J_1 \\ J_2 \\ J_3 \end{pmatrix} + \ldots \tag{7.2}$$

where, as before, $\lambda = 8g^2$ (i.e. $4g = \sqrt{2\lambda}$).

In [185] they used the following ansatz for solutions mostly in CP^3:

$$\begin{aligned}
q_1(x) =& \ \frac{\alpha x}{x^2-1} \\
q_2(x) =& \ \frac{\alpha x}{x^2-1} \\
q_3(x) =& \ \frac{\alpha x}{x^2-1} \quad +G_u(0) - G_u(\tfrac{1}{x}) \quad +G_v(0) - G_v(\tfrac{1}{x}) \quad +G_r(x) - G_r(0) + G_r(\tfrac{1}{x}) \\
q_4(x) =& \ \frac{\alpha x}{x^2-1} \quad +G_u(x) \quad +G_v(x) \quad -G_r(x) + G_r(0) - G_r(\tfrac{1}{x}) \\
q_5(x) =& \quad G_u(x) - G_u(0) + G_u(\tfrac{1}{x}) \quad -G_v(x) + G_v(0) - G_v(\tfrac{1}{x})
\end{aligned} \tag{7.3}$$

From q_1 and q_2 we can conclude that $S = 0$ (that is, we have zero AdS angular momentum) and that $\alpha = \Delta/2g$.

The resolvents G_u, G_v, G_r control the CP^3 part of the curve, q_3, q_4, q_5. Consequently, their asymptotic expansion will be related to the $SU(4)$ excitation numbers M_u, M_v, M_r. The values of these resolvents at $x = 0$ will in turn control the momentum[2]

$$p = 2\pi m = q_3\left(\frac{1}{x}\right) + q_4(x). \tag{7.4}$$

[2]For a closed string $m \in \mathbb{Z}$, however, we want to consider a single giant magnon which in general is not a closed string. Hence we will relax this condition and consider general p. To get a physical state this momentum condition should be imposed. This can be done by considering multi-magnon states [141, 185].

The Dynkin labels of $SU(4)$ are directly related to the excitation numbers $M_{u,v,r}$ through (7.1), and thus can be combined into the $SO(6)$ charges:

$$\begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} = \begin{pmatrix} q+(p_1+p_2)/2 \\ (p_1+p_2)/2 \\ (p_2-p_1)/2 \end{pmatrix} = \begin{pmatrix} L-M_r \\ L+M_r-M_u-M_v \\ M_u-M_v \end{pmatrix},$$

which are in turn combined into the magnons' major and minor charges:

$$\begin{aligned} J = J_1+J_2 &= 2L-M_u-M_v &&= p_1+q+p_2\,, \\ Q = J_1-J_2 &= M_u+M_v-2M_r &&= q\,. \end{aligned}$$

Giant magnons in the curve

This discussion now follows the introduction of giant magnon solutions in the algebraic curve formalism, in Section 5.4 on page 145, where we saw that a condensate of poles gave rise to giant magnon solutions in $\mathbb{R}\times S^3$. For the CP^3 case, two different kinds of giant magnons were seen to exist in [198], who named them 'small' and 'big'. These can be constructed by setting some of the resolvents in the above ansatz to the giant magnon resolvent (5.28):

$$G_{\text{mag}}(x) = -i\log\left(\frac{x-X^+}{x-X^-}\right). \tag{7.5}$$

Recall that for solutions only with logarithmic cuts, $X^\pm$ are complex conjugates.

'Small giant magnon'

The first curve in [198] is the 'small giant magnon' with

$$G_v(x) = G_{\text{mag}}(x), \qquad G_u = G_r = 0.$$

Reading off the charges from this curve we find:

$$p = -i \log \frac{X^+}{X^-}, \qquad Q = -i2g\left(X^+ - X^- + \frac{1}{X^+} - \frac{1}{X^-}\right),$$
$$J = 2\Delta + i2g\left(X^+ - X^- - \frac{1}{X^+} + \frac{1}{X^-}\right), \qquad J_3 = Q. \tag{7.6}$$

We can use the above relations to solve for $X^\pm$ and obtain the dispersion relation

$$\Delta - \frac{J}{2} = \sqrt{\frac{Q^2}{4} + 16g^2 \sin^2\left(\frac{p}{2}\right)}.$$

Another kind of small magnon can be obtained from G_u instead of G_v. The only change is in the sign of $J_3 = -Q$.

'Big giant magnon'

There is also the 'big giant magnon', which has

$$G_u(x) = G_v(x) = G_r(x) = G_{\mathrm{mag}}(x)$$

We then obtain the charges

$$p = -2i \log \frac{X^+}{X^-}, \qquad Q = 0,$$
$$J = 2\Delta + i4g\left(X^+ - X^- - \frac{1}{X^+} + \frac{1}{X^-}\right), \qquad J_3 = 0. \tag{7.7}$$

This is the second curve used by [198]. In the dispersion relation, it is not the total Q but rather Q_u, the contribution from just the u part (which is canceled by the v part in the full solution), that appears. This is the same function of $X^\pm$ as for the small giant magnon (7.6) above. The result is

$$\Delta - \frac{J}{2} = \sqrt{Q_u^2 + 64g^2 \sin^2\left(\frac{p}{4}\right)}.$$

For this solution, $\mathscr{E} = \Delta - J/2$ is a function of two parameters, Q_u and p, but Q_u is not an asymptotic charge of the full solution. Here we have only one angular momentum, plus another parameter Q_u, unlike the case of the ordinary dyonic giant magnons, which are two-parameter two-momentum solutions.

Pair of small magnons

We can also put one small magnon into each sector, $G_v(x) = G_u(x) = G_{\text{mag}}(x)$ keeping $G_r = 0$. All of the charges (including both Δ and p) are just the sum of those of each of the constituent small giant magnons, and we write $Q = Q_u + Q_v$ etc. Thus we get the dispersion relation

$$\begin{aligned}\Delta - \frac{J}{2} &= \sqrt{\frac{Q_u^2}{4} + 16g^2 \sin^2\left(\frac{p_u}{2}\right)} + \sqrt{\frac{Q_v^2}{4} + 16g^2 \sin^2\left(\frac{p_v}{2}\right)} \\ &= \sqrt{\frac{Q^2}{4} + 64g^2 \sin^2\left(\frac{p}{4}\right)}\end{aligned}$$

If we were to use the momentum p_u of one constituent magnon, rather than the total p, then we would have $\sin^2(p_u/2)$, as was found in [185]. Note that this solution has total $J_3 = 0$ (like the big magnon).

We summarize all of these properties, and more, in table 7.1.

Coalescence of non-dyonic solutions

In the non-dyonic limit $Q \ll g$, and $Q_u \ll g$, the dispersion relations for the pair of small magnons and the big magnon agree. This result is not limited to just the dispersion relation, in fact, $X^{\pm} = e^{\pm ip/4}$ in this limit (in both cases), and we have

$$G_{\text{mag}}(x) - G_{\text{mag}}(0) + G_{\text{mag}}\left(\frac{1}{x}\right) = 0. \tag{7.8}$$

Looking at the ansatz (7.3), we can see that (7.8) is equivalent to setting $G_r = 0$. Consequentely, in this limit, the big giant magnon becomes the same algebraic curve as the pair of small magnons.

For the small giant magnon, the same identity in the non-dyonic limit implies that $q_5 = 0$. This removes the difference between curves for the u and v small giant magnons in (7.3).

M_u, M_v, M_r	$[p_1,q,p_2]$	$\mathscr{E}=\Delta-\frac{J}{2}$	$\delta\mathscr{E}$ (finite J)	Q	J_3
Vacuum					
0, 0, 0	$[L,0,L]$	0	—	0	0
Small giant magnon					
1, 0, 0	$[L-2,1,L]$	$\sqrt{2\lambda}\sin(\frac{p}{2})$	$-4\mathscr{E}\sin^2\left(\frac{p}{2}\right)e^{-2\Delta/\mathscr{E}}$	1	1
Q, 0, 0	$[L-2Q,Q,L]$	$\sqrt{\frac{Q^2}{4}+2\lambda\sin^2(\frac{p}{2})}$	$\propto Q/\mathscr{E}\sqrt{S}$, see (7.12)	Q	Q
... and similar with $u\leftrightarrow v$:					
0, Q, 0	$[L,Q,L-2Q]$	(same)	(same)	Q	$-Q$
Big giant magnon					
1, 1, 1	$[L-1,0,L-1]$	$2\sqrt{2\lambda}\sin(\frac{p}{4})$	$-4\mathscr{E}\sin^2\left(\frac{p}{4}\right)e^{-2\Delta/\mathscr{E}}$	0	0
Q_u,Q_u,Q_u	$[L-Q_u,0,L-Q_u]$	$\sqrt{Q_u^2+8\lambda\sin^2(\frac{p}{4})}$	$\propto S/\mathscr{E}Q_u^2$, see (7.15)	0	0
Pair of small giant magnons					
1, 1, 0	$[L-2,2,L-2]$	$2\sqrt{2\lambda}\sin(\frac{p}{4})$	$-4\mathscr{E}\sin^2(\frac{p}{4})e^{-2\Delta/\mathscr{E}}$	2	0
$\frac{Q}{2}$, $\frac{Q}{2}$, 0	$[L-Q,Q,L-Q]$	$\sqrt{\frac{Q^2}{4}+8\lambda\sin^2(\frac{p}{4})}$	Like S^5 case, see (7.14)	Q	0

Table 7.1: Summary of giant magnons in the algebraic curve. In each case we list dyonic solutions, meaning $Q\sim\sqrt{\lambda}$, below the non-dyonic case. We write these using λ rather than g for comparison with the string sigma-model results on page 173; the relation is $\sqrt{2\lambda}=4g$. (Note that the AFZ-like result for the pair of small magnons is not new to this paper, it was found by [185].)

7.2 Finite-size corrections in the curve

Once more, we continue the discussion started in Section 5.4, on page 148. The finite-size corrections to algebraic curves in the CP^3 case were studied by [185], where one replaced the basic magnon's resolvent by the ansatz (7.9):

$$G_{\text{finite}}(x) = -2i\log\left(\frac{\sqrt{x-X^+}+\sqrt{x-Y^+}}{\sqrt{x-X^-}+\sqrt{x-Y^-}}\right). \tag{7.9}$$

As was explained in that Section, to determine the leading finite-J correction to the giant magnon dispersion relation, we calculate the asymptotic behaviour at $x\to\infty$ of the quasi-momenta using the above ansatz (7.9) to determine the charges of the solution as functions of the end points $X^\pm, Y^\pm$, and then solve the equations for these charges to second order in δ. Recall that $Y^\pm = X^\pm\left(1\pm i\delta e^{\pm i\phi}\right)$ are points shifted from the end points of the cut $X^\pm$ by a small amount $\delta \ll 1$, as in (5.31). We will now present the explicit calculations for the finite-J corrections of dispersion relations for the giant magnon solutions presented above (small, big and pair cases).

Finite-size small giant magnon

We want to study the magnon created by setting $G_u(x) = G_{\text{finite}}(x)$ in the general ansatz (7.3), keeping $G_v = G_r = 0$. We will discuss this case in the most detail, as subsequent examples are similar. We write the end points of the logarithmic cut as

$$X^\pm = r\,e^{ip_0/2}$$

in terms of which $p = p_0 + \delta p_{(1)} + \delta^2 p_{(2)} + o(\delta^3)$ and

$$\begin{aligned}\mathscr{E} = \Delta - \frac{J}{2} &= 4g\frac{r^2+1}{2r}\sin\left(\frac{p}{2}\right) - \frac{\delta}{2}J_{(1)} - \frac{\delta^2}{2}J_{(2)} + o(\delta^3)\\ Q &= 8g\frac{r^2-1}{2r}\sin\left(\frac{p}{2}\right) + \delta\,Q_{(1)} + \delta^2 Q_{(2)} + o(\delta^3)\end{aligned} \tag{7.10}$$

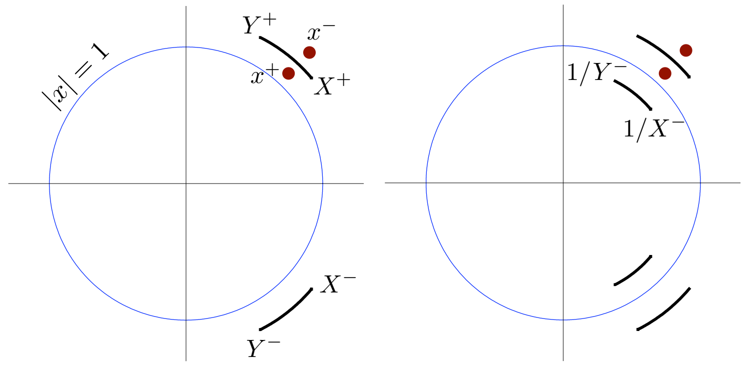

Figure 7.1: Branch cuts & evaluation points. On the left, the situation for the pair of small magnons, where only the cuts for $G_{\text{finite}}(x)$ appear. The evaluation points $x^\pm$ straddle the cut from X^+ to Y^+, which is at radius $|x| = r$. On the right, the cuts in $G_{\text{finite}}(1/x)$ are drawn too, which is the situation encountered in the 'small' and 'big' magnons. The evaluation points are on the same side of the cut from $1/X^+$ to $1/Y^+$, and remain so even when we take the non-dyonic limit $r \to 1$. (These are drawn for $\phi = 0$.)

We give formulae for these expansions in Appendix D.4 on page 234. From the full asymptotic charges, we can determine the energy correction in terms of δ. The first nonzero contribution is at order δ^2:

$$\begin{aligned}
\delta\mathscr{E} &= \left(\Delta - \frac{J}{2}\right) - \sqrt{\frac{Q^2}{4} + 16 g^2 \sin^2\left(\frac{p}{2}\right)} \\
&= -\delta^2 \frac{g}{4} \cos(2\phi) \frac{2r}{1+r^2} \sin\left(\frac{p}{2}\right) + o(\delta^3)
\end{aligned} \tag{7.11}$$

The resolvent $G_{\text{finite}}(x)$ has a square-root branch cut from X^+ to Y^+, which in the curve (7.3) we can choose to connect sheets q_4 and $q_6 = -q_5$. We can then fix δ using the branch cut condition:

$$\begin{aligned}
2\pi n &= q_4(x^+) - q_6(x^-) \\
&= \frac{2\alpha x}{x^2 - 1} + G^+_{\text{finite}}(X^+) + G^-_{\text{finite}}(X^+) - G_{\text{finite}}(0) + G_{\text{finite}}\left(\frac{1}{X^+}\right)
\end{aligned}$$

As explained in Section 5.4, the superscript G^- indicates that this term is evaluated on the other side of the cut from the others (thus having the opposite sign between the terms of the numerator inside G). Once we take this into account, we may take both evaluation points $x^\pm$ to be at $x = X^+$. Figure 7.1 shows the cuts and the evaluation points used. We finally obtain

$$\delta = \frac{8i\, e^{-ip/4} e^{i\pi n} e^{-i\phi} \sqrt{r^2-1}\, \sin(\frac{p}{2})}{\sqrt{e^{-ip/2} - r^2 e^{ip/2}}} \exp\left(\frac{i\Delta r/4g}{e^{-ip/2} - r^2 e^{ip/2}} \right) = e^{i\psi}\, |\delta|$$

In order to have a real energy correction, we require a real δ. We then find the correction to be

$$\begin{aligned} \delta\mathscr{E} &= -32g\cos(2\phi)\frac{r^2-1}{r^2+1}\frac{\sin^3\left(\frac{p}{2}\right)}{\sqrt{r^2+\frac{1}{r^2}-2\cos(p)}} e^{-\Delta\mathscr{E}/S(\frac{p}{2})} + o(\delta^3) \\ &= -32g^2\cos(2\phi)\frac{Q}{\mathscr{E}\sqrt{S(\frac{p}{2})}}\sin^3\left(\frac{p}{2}\right) e^{-\Delta\mathscr{E}/S(\frac{p}{2})} + o(\delta^3) \end{aligned} \tag{7.12}$$

where we define

$$S(\tfrac{p}{2}) = 4g^2\frac{(r^2-1)^2}{r^2} + 16g^2\sin^2\left(\frac{p}{2}\right).$$

Note that for the case of the 'small' magnon, $S(\frac{p}{2}) = \frac{Q^2}{4\sin^2(\frac{p}{2})} + 16g^2\sin^2\left(\frac{p}{2}\right) \to \mathscr{E}^2$ when $r \to 1$.

Three comments

- The result (7.12) is valid for the dyonic case $Q/g \sim 1$. As written it appears that $\delta\mathscr{E} \to 0$ in the non-dyonic limit $r \to 1$. In fact, this is not correct, as we've implicitly assumed, when expanding in δ, that $\delta \ll r-1 \sim \sqrt{Q/g}$, and this forbids taking $r \to 1$. Nevertheless, we can derive the correction for the non-dyonic case by writing $r = 1+k\delta$ before assuming that δ is small. We then expand in δ, fix δ using the branch cut, and only after we take the limit $k \to 0$. The result is the AFZ form:

$$\delta\mathscr{E}_{r=1} = -16g\cos(2\phi)\sin^3\left(\frac{p}{2}\right) e^{-2\Delta/\mathscr{E}} + o(\delta^3). \tag{7.13}$$

- The reality condition of δ is equivalent to its phase ψ being 0 or π:

$$\psi = n\pi - \frac{p}{4} - \phi - \frac{\Delta\, Q\cot(\frac{p}{2})}{4S(\frac{p}{2})} - \frac{1}{2}\arctan\left(\frac{2\mathscr{E}}{Q}\tan(\tfrac{p}{2})\right) = 0 \text{ or } \pi.$$

Like the energy correction (7.12), this expression is valid for the dyonic case; the phase of δ in the non-dyonic limit $r\to 1$ is instead

$$\psi_{r=1} = 2\pi n - \frac{p}{2} = 0 \text{ or } \pi$$

where we have assumed $\cos(2\phi) = \pm 1$. This implies that the momentum obeys $p = 0 \bmod 2\pi$, exactly the usual condition for a closed string.

- Finally, notice that the the same factor $\cos(2\phi)$ appears in both these results and the sigma-model results (6.34). We interpreted this factor there as a geometric angle between adjacent magnons. Here we can observe that for the identity (7.8) to hold at the evaluation point $x = X^+$ (in the limit $\delta\to 0$, as well as $r\to 1$) , we must set $\cos(2\phi) = \pm 1$.

Finite-size pair of small magnons

The non-dyonic ($r = 1$) pair of small magnons was studied in [185], who obtained

$$\delta\mathscr{E}_{r=1} = -32g\cos(2\phi)\sin^3\left(\frac{p}{4}\right)e^{-2\Delta/\mathscr{E}} + \dots.$$

The same result can be obtained by adding together all the charges of two small magnons, giving twice the correction (7.13). The dyonic case, however, is not as trivially obtained by adding together two dyonic finite-J small magnons, because they interact with each other. We must then perform a similar analysis to the one presented above for the small giant magnon.

The curve of interest is now $G_u = G_v = G_{\text{finite}}$ and $G_r = 0$. We now set

$$X^\pm = re^{\pm ip_0/4},$$

giving $p = p_0 + \dots$ and $\mathscr{E} = 8g\frac{r^2+1}{2r}\sin\left(\frac{p}{4}\right) + \dots$, $Q = 16g\frac{r^2-1}{2r}\sin\left(\frac{p}{4}\right) + \dots$.[3] The energy cor-

[3]As before, we give these expansions in δ in appendix D.4 on page 234.

rection in terms of δ is

$$\delta\mathscr{E} = -\delta^2 \frac{g}{2} \cos(2\phi) \frac{2r}{r^2+1} \sin\left(\frac{p}{4}\right) + \dots.$$

We continue by using the branch cut condition connecting sheets[4] q_4 and $q_7 = -q_4$ to fix δ:

$$\begin{aligned} 2\pi n &= q_4(x^+) - q_7(x^-) \\ &= \frac{2\alpha x}{x^2-1} + 2G^+_{\text{finite}}(X^+) + 2G^-_{\text{finite}}(X^+). \end{aligned}$$

The final energy correction is then:

$$\delta\mathscr{E} = -256 g^2 \cos(2\phi) \frac{1}{\mathscr{E}} \sin^4\left(\frac{p}{4}\right) e^{-\Delta\mathscr{E}/2S(\frac{p}{4})} + \dots. \tag{7.14}$$

This correction has the same form as the S^5 string result, and exactly matches the RP^3 magnon's correction (6.34).

In this case there is no issues about taking the $r \to 1$ limit, where it reduces to the RP^2 correction (6.32) (note that in this limit $S(\frac{p}{4}) \to \frac{1}{4}\mathscr{E}^2$, instead of $\mathscr{E}^2$ as in the small case).

The phase of δ is, in this case,

$$\psi = \frac{n\pi}{2} - \frac{p}{4} - \phi - \frac{\Delta Q \cot(\frac{p}{4})}{8S(\frac{p}{4})} = 0 \text{ or } \pi.$$

When $Q = 0$ and $\phi = 0$, we have $p/2 = p' = n'\pi$, $n' \in \mathbb{Z}$, exactly matching the condition for the RP^2 magnon to be a closed string.

Finite-size big magnon

In this case, the curve of interest is $G_u = G_v = G_r = G_{\text{finite}}$. We write $X^\pm = r\, e^{ip_0/4}$, and consider the 'dyonic' case in the sense that $r > 1$ (even though $Q = 0$). We also define Q_u to be the

[4] In [185] a condition for the G_v component to connect sheets q_4 and q_5 is used instead. This involves separating the two cuts slightly, so that $q_5 \neq 0$. The G_u component has instead a cut connecting q_4 and q_6, which gives the same equation. The resulting condition is the same as that given here except n is replaced by $2n$.

Q from the small magnon. Due to this choice of p_0, $Q_u = 8g\frac{r^2-1}{2r}\sin\left(\frac{p}{4}\right)+\ldots$, and we have $\mathscr{E} = 8g\frac{1+r^2}{2r}\sin\left(\frac{p}{4}\right)+\ldots$. With the expansions in δ of the asymptotic charges, we get

$$\delta\mathscr{E} = -\delta^2\frac{g}{2}\cos(2\phi)\frac{2r}{r^2+1}\sin\left(\frac{p}{4}\right)+\ldots.$$

We continue to set the branch cut condition. We will connect sheets q_3 and $q_7 = -q_4$, at evaluation points $x^{\pm}$ on either side of the cut from $X^+$to Y^+, but both $x^{\pm}$on the same side of the cut from $1/X^+$ to $1/Y^+$, thus obtaining the matching condition

$$\begin{aligned}
2\pi n &= q_3(x^+) - q_7(x^-) \\
&= 2\frac{\alpha x}{x^2-1} + G^+_{\text{finite}}(x) + G^-_{\text{finite}}(x) + 2G_{\text{finite}}(0) - 2G_{\text{finite}}\left(\frac{1}{x}\right)
\end{aligned}$$

As before, this equation fixes δ, and after demanding that it be real, we obtain the energy correction:

$$\begin{aligned}
\delta\mathscr{E} &= -64g\cos(2\phi)\frac{r^2+\frac{1}{r^2}-2\cos(p/2)}{(r^2+1)(r^2-1)^2}r^3\sin^3\left(\frac{p}{4}\right)e^{-\Delta\mathscr{E}\big/S(\frac{p}{4})}+\ldots \\
&= -1024g^2\cos(2\phi)\frac{S(\frac{p}{4})}{\mathscr{E}Q_u^2}\sin^6\left(\frac{p}{4}\right)e^{-\Delta\mathscr{E}\big/S(\frac{p}{4})}+\ldots. \qquad (7.15)
\end{aligned}$$

This expression is valid only in the dyonic case. The non-dyonic limit $r \to 1$ can be approached in the same way as for small magnon case, by setting $r = 1+k\delta$ before expanding in δ. The limit $k \to 0$ then gives the result

$$\delta\mathscr{E}_{r=1} = -32g\cos(2\phi)\sin^3\left(\frac{p}{4}\right)e^{-2\Delta/\mathscr{E}}+\ldots$$

matching the $r = 1$ limit of the pair of small magnons (7.14), and thus the RP^2 string result (6.32).

The phase of δ in this case is

$$\psi = n\pi - \frac{p}{2} - \phi - \frac{\Delta Q_u\cot(\frac{p}{4})}{2S(\frac{p}{4})} + \arctan\left(\frac{\mathscr{E}}{Q_u}\tan(\tfrac{p}{4})\right).$$

This expression is once more not valid in the non-dyonic case. In this limit $(r \to 1)$ we have instead (in the case $\cos(2\phi) = \pm 1$)

$$\psi_{r=1} = \frac{n\pi}{2} - \frac{p}{4} = 0 \text{ or } \pi.$$

i.e. $p = 0 \bmod 2\pi$. Consequently, the momentum condition is $p' = 0 \bmod \pi$, which is the condition for a closed string in RP^2, and matches the 'pair' case above.

7.3 Dictionary between strings and curves: summary of results

Let us summarize the dictionary of string and curve solutions which we have found:

- The small giant magnon in the curve matches the CP^1 giant magnon, and its dyonic generalization in CP^2.
- The RP^2 magnon is to be identified with the pair of small magnons. In the dyonic case this becomes the RP^3 magnon solution.
- The dressed solution is identified with the big giant magnon. Both are two-parameter one-charge solutions, and when the additional parameter (Q_f or Q_u) is sent to zero, they become the RP^2 solution / pair of small magnons, respectively.

The non-dyonic RP^2 and CP^1 string solutions seem to have multiple descriptions in the algebraic curves: the big and pair of small magnons differ in their excitation numbers M_u, M_v, M_r, as do the two kinds of small magnons. However these numbers are all of order $1 \ll 4g = \sqrt{2\lambda}$ and so, just like Q, they are invisible in the sigma-model. In the limit $Q \to 0$ the curves forget these distinctions too.

Finite-size corrections to these magnons are summarized as follows:

- In the non-dyonic cases, the corrections are always of the AFZ form. These can be calculated in both the string and curve pictures.
- For the RP^3 / 'pair' magnon, the corrections are the same as those for S^3 dyonic giant magnons, and can again be calculated in both pictures.

- For the dressed / big magnon, and also for the CP^2 / small magnon, we have calculated corrections in the algebraic curve. These do not have the same form as in S^5.

The above result for the finite-J corrections to the small giant magnon differs from that of the algebraic curve calculation in [185]. This difference can be seen to arise from an order-of-limits problem. As noted in section 7.2, we need to be careful in the non-dyonic case with how we take limits $Q \to 0$ ($r \to 1$) and $\delta \to 0$. However, the result of [185] is confirmed by the Lüscher-calculations in [185, 236]. It would be instructive to see if these results can be explained in a similar manner.

While the overall picture is now clear, there are various details which one would like to analyze explicitly in the string sigma-model. First, our CP^2 solution should certainly exist at $p \neq \pi$, but so far we have not been able to find such solutions. Second, it would be interesting to understand exactly how the two different CP^2 solutions join (and interact) to form one RP^3 magnon. Finally, finite-J versions of both this CP^2 solution and the dressed solution should exist as string solutions, and would provide confirmation of the energy corrections calculated here.

We conclude by noting that our results fit well into the context of the integrable alternating spin-chain for operators in the ABJM gauge theory [192, 194, 195, 237, 238, 47]. The two small giant magnons correspond to simple magnons in either the fundamental or anti-fundamental part of the spin-chain. The big magnon, on the other hand, carry the same charges as the heavy scalar excitation first discussed in [192].[5] In a recent paper Zarembo [239] showed that in the BMN limit these heavy modes disappear from the spectrum as soon as quantum corrections are taken into account. It would be very interesting to understand to what extent these arguments carries over to the giant magnon regime.

[5] In [200] the authors write the big giant magnon solution, and also a second solution, equation (4.14). This solution has the same angular momentum as the big giant magnons, but lives just in CP^1 and is a trivial embedding of the bound state solution (5.14) of [155], by setting $r = e^{-q/2}$. This bound state is an analytic continuation of two CP^1 magnons. Comparing to the results of the curve, it is possible that this is a bound state of two CP^1 magnons of the same kind (u of v kind), while the big giant magnon is some kind of bound state of two CP^1 magnons of different polarizations (one u and one v).

Chapter 8

Conclusions and Future Directions

The main focus of this dissertation is the study of the symmetries and integrability properties of two major examples of the AdS/CFT correspondence.

Symmetry algebra and semi-classical string solutions

In Chapter 2 we started by introducing the algebra of symmetries $\mathfrak{psu}(2,2|4)$ of the AdS_5/ CFT_4 duality from the classical string sigma-model in $AdS_5 \times S^5$, and restricted our attention to the action of the fermionic generators on the $\mathfrak{su}(2|2)$ sector for both the sigma-model and the $\mathscr{N} = 4$ super Yang-Mills gauge theory at one-loop. This study was done through the use of a matrix model formalism. The bosonic part of this sector, $\mathfrak{su}(2)$ was then taken into account on the string side, and on Chapter 6 we studied giant magnon solutions and the related giant spikes, and presented a semi-classical analysis of the giant spike solution.

The symmetry algebra of AdS_5/CFT_4 is very well understood in the $\frac{1}{2}$ BPS sector of the correspondence, but developments on the $\frac{1}{4}$ BPS sector have been harder to achieve [240, 241]. The set of $\frac{1}{4}$ BPS operators in SYM can be described by a two-matrix quantum mechanics, but the large N limit has both single trace and double trace operators, unlike the $\frac{1}{2}$ BPS sector which had only the former. It would be interesting to obtain a collective field theory description of the $\frac{1}{4}$ BPS correlator functions from SYM, in order to derive an effective string field theory Hamiltonian for the interactions of these correlators, thus connecting to the known results derived by A. Donos in supergravity [241]. Such a collective description comes as an application of the results found in Chapter 2.

In the framework of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence [6], there is a new superconformal group of symmetries, the $OSp(6|4)$, shared by the gauge and string sides of the duality. Another generalization of the work found in Chapter 2 is the study of the common algebra of these dual theories using matrix model techniques. In particular, the study of the fermionic generators is of great interest, as it can lead to a better understanding of this algebra and to the observation of a Yangian structure, typical of integrable systems.

Giant magnon solutions living in the space $\mathbb{R}\times CP^3$ have been extensively studied. Most of these are trivial embeddings of the known giant magnon solutions from $\mathbb{R}\times S^3$, in particular the RP^2, the RP^3 and the CP^1 magnons. But two other solutions which truly explore the CP^3 space have been found. In Chapter 6 we review the embeddings of giant magnons into CP^3 and a two parameter, one angular momentum giant magnon solution that lives in CP^2, found through dressing method. We also present the derivation a new dyonic magnon solution living in CP^2, which is a dyonic generalization of the CP^1 magnon.

Integrability and dynamics of string solutions

Integrability was intensely studied on both sides of the $\mathrm{AdS}_5/\mathrm{CFT}_4$ duality. On the gauge theory side the study of integrability through Bethe Ansatz techniques led to the discovery of an equivalent description in terms of Heisenberg spin chains and a relation to the Hubbard model (integrable short-range model of strongly correlated electrons) [20,145]. On the string theory side, the classical equivalence of (dyonic) giant magnons and (complex) sine-Gordon solitons through Pohlmeyer reduction, gave a very promising hint towards finding an integrable N-body description of string solutions. The N-body descriptions of gauge operators and of string solutions is important to the complete understanding of dynamics of these objects, and a discussion of the dynamics of giant magnon solutions from an N-body perspective can be found in Chapter 4. In this Chapter, we find a relation between the scattering of magnons and the Ruijsenaars-Schneider (RS) N-body model, motivated by the Poisson hierarchy of a limiting case of this N-body model.

With respect to the relation between giant magnons and the RS model, the next step in checking this relation is finding the Poisson hierarchy in the RS model and calculate the Poisson bracket related to the magnon Hamiltonian. It is also of great interest to generalize the work done in Chapter 4 for string solutions with more than one angular momentum (on $\mathbb{R} \times S^3$), called dyonic giant magnons. To do so one needs to find a Lax formulation for the interacting complex sine-Gordon solitons, a challenge still to be overcome.

The giant magnon solutions, defined in an infinite worldsheet volume $\sigma \in (-\infty, \infty)$, were seen to have finite-size corrections by considering the range of the worldsheet to be finite. These finite-size giant magnons are classically equivalent to the kink train solution of the sine-Gordon theory [98], and it would be interesting to find a similar N-body description of the dynamics of these finite-size giant magnons.

Another important aspect to analyze is the relation between string solutions in the CP^3 subspace and generalized sine-Gordon models. The RP^2, RP^3 and CP^1 magnons are trivially related to the sine-Gordon solitons, but the small and big magnons, both living in CP^2, are mapped to a non-trivial generalization of the sine-Gordon model [242].

The integrability of the gauge theories and string sigma models can be approached with the algebraic curve formalism. Chapter 5 is a review of this approach for the string sigma-model in $AdS_5 \times S^5$ and the derivation of the classical string Bethe ansatz for the $\mathfrak{so}(6)$ sector. While this formalism cannot be used directly to obtain the explicit form of string solutions, it still gives us a complete classification of the spectrum of those solutions, as the solutions of the classical string sigma-model should be in exact correspondence to algebraic curves [46]. This can be restricted to giant magnon solutions, by restricting the type of singularities in the algebraic curve, as in Chapter 5, on page 145. In Chapter 7 we review the algebraic curve formalism for $\mathbb{R} \times CP^3$, and the expected giant magnon solutions in this space. With these results, we can then write a dictionary between these and the string solutions in Chapter 6:

- The small giant magnon in the curve corresponds to the CP^1 magnon, and its dyonic generalization is the CP^2 magnon;

- The RP^2 giant magnon is identified with the pair of small magnons, and the RP^3 giant magnon is its dyonic generalization;

- The two parameter, one charge dressed solution is the so called big magnon in the curve, which becomes the RP^2 magnon in the taking a limit on the second parameter;

In this last chapter we also use the curve formalism to determine finite-J corrections to the energy of these solutions, which on the non-trivial cases of the small and big magnons were not known from string calculations.

There is still a lot to understand about the solutions living in the CP^3 space. For example, the CP^2 dyonic solution found in Chapter 6 was only determined for the GKP-like case of $p=\pi$, and one would have great interest in generalizing this case. Also, the solution called big magnon is still not very well understood. The fact that it has a second parameter but just one angular momentum points to it being some kind of bound state but it is not obvious how to construct it as such.

Appendix A

Expansions of the Supercharges of $\mathfrak{su}(2|2)$

In this appendix we present two different expansions of the fermionic charges of $\mathfrak{su}(2|2)$. The first uses an oscillator representations of the string degrees of freedom. The second one uses a regular expansion up to two loops in the coupling on the gauge side, and is followed by a discussion of the commutation relations of these supercharges.

A.1 The Oscillator Representation

This summary follows [65] closely. We start by redefining the degrees of freedom that are left after fixing light-cone gauge. Again we have the same choice of metric on the target space, and the fields are t, z^a from AdS_5 and ϕ, y^s from S^5. The transverse bosonic fields y^s, z^a transform under a representation of $SU(2)^2$ out of the possible four bosonic factors $SU(2)^4$. These fields can then be represented as bi-spinors of the relevant SU^2. For that we introduce Pauli matrices $\sigma_a = (\mathbb{I}, i\vec{\sigma})$ and $\sigma_s = (\mathbb{I}, i\vec{\sigma})$ for each of the two copies of $SO(4)$, and write:

$$Y_{a\dot{a}} = (\sigma_s)_{a\dot{a}} y^s , \qquad Z_{\alpha\dot{\alpha}} = (\sigma_a)_{\alpha\dot{\alpha}} z^a .$$

Spin Index of each $su(2)$	$J,\ a$	$\dot{J},\ \dot{a}$	$\dot{S},\ \dot{\alpha}$	$S,\ \alpha$
$Y_{a\dot{a}}$	2	2	1	1
$Z_{\alpha\dot{\alpha}}$	1	1	2	2
$\Psi_{a\dot{\alpha}}$	2	1	2	1
$\Upsilon_{\alpha\dot{a}}$	1	2	1	2

Table A.2: Physical degrees of freedom of $\mathfrak{su}(2)^4$ and their quantum numbers.The first two $su(2)$ are from S^5 while the two last ones are from AdS_5. Also $a, \dot{a}, \dot{\alpha}, \alpha = 1, 2$.

After fixing κ-symmetry we have the fermionic fields:

$$\Psi_{a\dot{\alpha}}\,, \quad \Upsilon_{\alpha\dot{a}}$$

which are also bi-spinors of some $SU(2)^2$ (they are charged under different $SU(2)$ factors than the bosonic fields).

We lower/raise indices in the following way

$$x^a = \varepsilon^{ab} x_b\,; \quad x_a = x^b \varepsilon_{ba},$$

where $\varepsilon^{12} = \varepsilon_{12} = 1$. The same rule applies to all other indices. Complex conjugation changes the position of the indices $(Y_{a\dot{a}})^* \equiv Y^{*a\dot{a}}$, which is different from $Y^*_{a\dot{a}} \equiv Y^{*b\dot{b}} \varepsilon_{ba} \varepsilon_{\dot{b}\dot{a}}$. Finally, the bosonic fields satisfy the reality condition $Y^*_{a\dot{a}} = Y_{a\dot{a}}$ and $Z^*_{\alpha\dot{\alpha}} = Z_{\alpha\dot{\alpha}}$.

The quantum numbers of these fields with respect to $SU(2)^4$ can be found in [65] and are summarized in table A.2. We have seen before that bosons and fermions together form the bi-fundamental representation of $\mathfrak{psu}(2|2)_L \times \mathfrak{psu}(2|2)_R$, where the bosonic subgroup of each $\mathfrak{psu}(2|2)$ factor consists of two $\mathfrak{su}(2)$. If we define the super-indices $A = (a|\alpha)$ and $\dot{A} = (\dot{a}|\dot{\alpha})$ (where lower-case Latin indices are Grassmann even and greek indices are Grassmann odd) then the fields combine into a single bi-fundamental supermultiplet of $\mathfrak{psu}(2|2)_L \times \mathfrak{psu}(2|2)_R$, denoted by $\Phi_{A\dot{A}}$.

We can write the gauge fixed Lagrangian in terms of the fields Y, Z, Ψ, Υ, and determine the corresponding equations of motion, as shown in [65]. These were solved by introducing mode expansions in momentum space:

$$\begin{aligned}
Y_{a\dot{a}}(\vec{x}) &= \int \frac{dp}{2\pi}\frac{1}{\sqrt{2\varepsilon}}\left(a_{a\dot{a}}(p)\,e^{-i\vec{p}\cdot\vec{x}} + a^{\dagger}_{a\dot{a}}(p)\,e^{+i\vec{p}\cdot\vec{x}}\right), \\
Z_{\alpha\dot{\alpha}}(\vec{x}) &= \int \frac{dp}{2\pi}\frac{1}{\sqrt{2\varepsilon}}\left(a_{\alpha\dot{\alpha}}(p)\,e^{-i\vec{p}\cdot\vec{x}} + a^{\dagger}_{\alpha\dot{\alpha}}(p)\,e^{+i\vec{p}\cdot\vec{x}}\right), \\
\Psi_{a\dot{\alpha}}(\vec{x}) &= \int \frac{dp}{2\pi}\frac{1}{\sqrt{\varepsilon}}\left(b_{a\dot{\alpha}}(p)\,u(p)\,e^{-i\vec{p}\cdot\vec{x}} + b^{\dagger}_{a\dot{\alpha}}(p)\,v(p)\,e^{+i\vec{p}\cdot\vec{x}}\right), \\
\Upsilon_{\alpha\dot{a}}(\vec{x}) &= \int \frac{dp}{2\pi}\frac{1}{\sqrt{\varepsilon}}\left(b_{\alpha\dot{a}}(p)\,u(p)\,e^{-i\vec{p}\cdot\vec{x}} + b^{\dagger}_{\alpha\dot{a}}(p)\,v(p)\,e^{+i\vec{p}\cdot\vec{x}}\right),
\end{aligned}$$

where $\varepsilon = \sqrt{1+p^2}$ is the energy, and $u(p) = \cosh\frac{\theta}{2}$ and $v(p) = \sinh\frac{\theta}{2}$ are the wave functions. The rapidity θ is related to momentum by $p = \sinh\theta$, and to energy by $\varepsilon = \cosh\theta$. Also note that $\vec{p}\cdot\vec{x} \equiv \varepsilon\tau + p\sigma$.

The mode operators have canonical commutation relations:

$$\begin{aligned}
\left[a^{a\dot{a}}(p), a^{\dagger}_{b\dot{b}}(p')\right] &= 2\pi\delta^{a}_{b}\delta^{\dot{a}}_{\dot{b}}\delta(p-p'), \quad \left\{b^{a\dot{\alpha}}(p), b^{\dagger}_{b\dot{\beta}}(p')\right\} = 2\pi\delta^{a}_{b}\delta^{\dot{\alpha}}_{\dot{\beta}}\delta(p-p'), \\
\left[a^{\alpha\dot{\alpha}}(p), a^{\dagger}_{\beta\dot{\beta}}(p')\right] &= 2\pi\delta^{\alpha}_{\beta}\delta^{\dot{\alpha}}_{\dot{\beta}}\delta(p-p'), \quad \left\{b^{\alpha\dot{a}}(p), b^{\dagger}_{\beta\dot{b}}(p')\right\} = 2\pi\delta^{\alpha}_{\beta}\delta^{\dot{a}}_{\dot{b}}\delta(p-p').
\end{aligned}$$

Note that the oscillators $a, a^{\dagger}$ are conjugate to each other, which comes from the reality condition for the bosonic fields. But also for the fermionic oscillators $b, b^{\dagger}$ are conjugate to each other, even though fermionic fields $\Psi, \Psi^{\dagger}$ are independent. This comes from the equations of motion.

Algebra Generators

We have found the generators for the off-shell symmetry algebra in the light-cone gauge-fixed theory, $\mathfrak{psu}(2|2)_L \times \mathfrak{psu}(2|2)_R \ltimes \mathbb{R}^2$ in Chapter 7. These generators were found in terms of the worldsheet fields. The total momentum is measured by one of the central generators C of the extended algebra, while the total energy is measured by the other central charge H.

We are now interested in knowing the generators in terms of the oscillator representation. We will leave the nonlocal nature of the generators for discussion elsewhere and focus on the local part,[1] which corresponds to determining the currents $\tilde{J}$. Considering integration over fixed time slices, the oscillator representation for the generators of $\mathfrak{psu}(2|2)_L$ is (up to quadratic order)

$$\begin{aligned}
\ell_a^b &= \int \frac{dp}{2\pi}\frac{1}{2}\left[c^\dagger_{a\dot{C}}c^{b\dot{C}} - c^{\dagger b\dot{C}}c_{a\dot{C}}\right], \\
\mathbf{r}_\alpha^\beta &= \int \frac{dp}{2\pi}\frac{1}{2}\left[c^\dagger_{\alpha\dot{C}}c^{\beta\dot{C}} - c^{\dagger\beta\dot{C}}c_{\alpha\dot{C}}\right], \\
\mathbf{Q}_\alpha^b &= \int \frac{dp}{2\pi}(-1)^{[\dot{C}]}\left[u\,c^\dagger_{\alpha\dot{C}}c^{b\dot{C}} - v\,c^{\dagger b\dot{C}}c_{\alpha\dot{C}}\right], \\
\mathbf{S}_a^\beta &= \int \frac{dp}{2\pi}(-1)^{[\dot{C}]}\left[u\,c^\dagger_{a\dot{C}}c^{\beta\dot{C}} - v\,c^{\dagger\beta\dot{C}}c_{a\dot{C}}\right],
\end{aligned}$$

and the generators of $\mathfrak{psu}(2|2)_R$ are similarly:

$$\begin{aligned}
\dot{\ell}_{\dot{a}}^{\dot{b}} &= \int \frac{dp}{2\pi}\frac{1}{2}\left[c^\dagger_{C\dot{a}}c^{C\dot{b}} - c^{\dagger C\dot{b}}c_{C\dot{a}}\right], \\
\dot{\mathbf{r}}_{\dot{\alpha}}^{\dot{\beta}} &= \int \frac{dp}{2\pi}\frac{1}{2}\left[c^\dagger_{C\dot{\alpha}}c^{C\dot{\beta}} - c^{\dagger C\dot{\beta}}c_{C\dot{\alpha}}\right], \\
\dot{\mathbf{Q}}_{\dot{\alpha}}^{\dot{b}} &= \int \frac{dp}{2\pi}\left[u\,c^\dagger_{C\dot{\alpha}}c^{C\dot{b}} - v\,c^{\dagger C\dot{b}}c_{C\dot{\alpha}}\right], \\
\dot{\mathbf{S}}_{\dot{a}}^{\dot{\beta}} &= \int \frac{dp}{2\pi}\left[u\,c^\dagger_{C\dot{a}}c^{C\dot{\beta}} - v\,c^{\dagger C\dot{\beta}}c_{C\dot{a}}\right].
\end{aligned}$$

The two central charge generators are given by

$$\mathbf{C} = \int \frac{dp}{2\pi}p\,c^\dagger_{A\dot{A}}c^{A\dot{A}} \quad , \qquad \mathbf{H} = \int \frac{dp}{2\pi}\varepsilon\,c^\dagger_{A\dot{A}}c^{A\dot{A}}.$$

Note that in the above notation, the oscillator c can be either a or b, depending on the value of the super-index $\dot{C}, C$.

From the (anti) commutation relations for the oscillators, it is easy to see that these generators obey the centrally extended $\mathfrak{psu}(2|2)^2$ algebra. The supercharges Q and S transform as components of a Lorentz spinor, from which one can conclude that this representation of

[1]The nonlocal behaviour of the symmetry generators in light-cone gauge comes from the presence of the light-cone field x^-. The $\mathfrak{psu}(2,2|4)$ super-currents whose supercharges generate $\mathfrak{psu}(2|2)^2$ depend on x^-. We can then write

$$J_{Q^A_B} = e^{i\sigma_{AB}x^-/2}\tilde{J}_{Q^A_B}, \qquad Q^A_B = \int d\sigma J_{Q^A_B}(\sigma),$$

where $\sigma_{AB} = [A]-[B]$, with $[A]$ being the grade of the index A, i.e. $[a]=0$ and $[\alpha]=1$. $\tilde{J}$ is a local combination of the transverse fields, and these local generators are the ones will will be working with.

generators will be related to a representation of the non-centrally extended algebra ($C=0$) by a Lorentz boost.

The algebra $\mathfrak{psu}(2|2)_L$

The rotation generators $\mathbf{r},\ell$ act on a generic generator J canonically as was seen before:

$$\begin{aligned} \left[\ell_a^b, J_c\right] &= \delta_c^b J_a - \tfrac{1}{2}\delta_a^b J_c\,, & \left[\ell_a^b, J^c\right] &= -\delta_a^c J^b + \tfrac{1}{2}\delta_a^b J^c\,, \\ \left[\mathbf{r}_\alpha^\beta, J_\gamma\right] &= \delta_\gamma^\beta J_\alpha - \tfrac{1}{2}\delta_\alpha^\beta J_\gamma\,, & \left[\mathbf{r}_\alpha^\beta, J^\gamma\right] &= -\delta_\alpha^\gamma J^\beta + \tfrac{1}{2}\delta_\alpha^\beta J^\gamma\,. \end{aligned}$$

The fermionic charges obey

$$\begin{aligned} \left\{\mathbf{Q}_\alpha^a, \mathbf{Q}_\beta^b\right\} &= -\frac{1}{2}\varepsilon_{\alpha\beta}\varepsilon^{ab}\mathbf{C}\,, \\ \left\{\mathbf{S}_a^\alpha, \mathbf{S}_b^\beta\right\} &= -\frac{1}{2}\varepsilon_{ab}\varepsilon^{\alpha\beta}\mathbf{C}^*\,, \\ \left\{\mathbf{Q}_\alpha^a, \mathbf{S}_b^\beta\right\} &= \delta_\alpha^\beta \ell_b^a + \delta_b^a \mathbf{r}_\alpha^\beta + \frac{1}{2}\delta_\alpha^\beta \delta_b^a \mathbf{H}\,. \end{aligned}$$

Actually in the quadratic approximation, one finds that the central charge appearing in the anti-commutators of the Qs and the Ss is the same, and only including higher orders we would find that they are actually conjugate to each other. The algebra $\mathfrak{psu}(2|2)_R$ would be identical.

A.2 Commuting the $\mathfrak{su}(2|2)$ *supercharges up to two-loops*

The expressions found here are restrictions to the $\mathfrak{su}(2|2)$ subsector of the full sector $\mathfrak{su}(2|3)$ found in [67]. The supercharges at order g^0, Q_0, S_0, at order g^1, Q_1, S_1 and at order g^2, Q_2, S_2 in the dilute gas approximation can be written as follows:

$$\begin{aligned} (Q_0)_\beta^b &= \left\{ \begin{array}{c} b \\ \beta \end{array} \right\}, \\ (S_0)_a^\alpha &= \left\{ \begin{array}{c} \alpha \\ a \end{array} \right\}; \end{aligned}$$

$$
\begin{aligned}
(Q_1)^b_\beta &= \frac{A}{\sqrt{2}}\varepsilon_{\beta\beta'}\varepsilon^{bb'}\left(\left\{\begin{array}{c}\beta'\\ b'3\end{array}\right\}-\left\{\begin{array}{c}\beta'\\ 3b'\end{array}\right\}\right),\\
(S_1)^\alpha_a &= \frac{A}{\sqrt{2}}\varepsilon_{aa'}\varepsilon^{\alpha\alpha'}\left(\left\{\begin{array}{c}a'3\\ \alpha'\end{array}\right\}-\left\{\begin{array}{c}3a'\\ \alpha'\end{array}\right\}\right);
\end{aligned}
$$

$$
\begin{aligned}
(Q_2)^b_\beta &= \left(\frac{A^2}{4}-\frac{i}{2}\gamma_3+\frac{i}{2}\gamma_4\right)\left(\left\{\begin{array}{c}b3\\ \beta3\end{array}\right\}+\left\{\begin{array}{c}3b\\ 3\beta\end{array}\right\}\right)+\left(-\frac{A^2}{4}-i\gamma_1\right)\left(\left\{\begin{array}{c}b3\\ 3\beta\end{array}\right\}+\left\{\begin{array}{c}3b\\ \beta3\end{array}\right\}\right),\\
(S_2)^\alpha_a &= \left(\frac{A^2}{4}+\frac{i}{2}\gamma_3-\frac{i}{2}\gamma_4\right)\left(\left\{\begin{array}{c}\alpha3\\ a3\end{array}\right\}+\left\{\begin{array}{c}3\alpha\\ 3a\end{array}\right\}\right)+\left(-\frac{A^2}{4}+i\gamma_1\right)\left(\left\{\begin{array}{c}\alpha3\\ 3a\end{array}\right\}+\left\{\begin{array}{c}3\alpha\\ a3\end{array}\right\}\right).
\end{aligned}
$$

We will be using the notation used in [67]. The index 3 above means an insertion of a field Z. The action of $\left\{\begin{array}{c}\alpha bc\\ c\alpha b\end{array}\right\}$ on a state looks for a sequence of a fermion followed by two bosons, and permutes them in the order 2nd boson-fermion-1st boson. As an example in $\mathfrak{su}(2|3)$, where indices $1,2,3$ correspond to bosons and indices $4,5$ correspond to fermions, we have

$$
\left\{\begin{array}{c}\alpha bc\\ c\alpha b\end{array}\right\}|142334452\rangle = |134234452\rangle + |242334415\rangle\,.
$$

Determining the anti-commutation relations, we have:

$$
\begin{aligned}
\frac{2}{A^2}\left\{(S_1)^\alpha_a,(Q_1)^b_\beta\right\} &= \delta^b_a\delta^\alpha_\beta\frac{1}{A^2}H_2-\delta^b_a\left[2\left\{\begin{array}{c}\alpha\\ \beta\end{array}\right\}-\left\{\begin{array}{c}3\alpha\\ \beta3\end{array}\right\}-\left\{\begin{array}{c}\alpha3\\ 3\beta\end{array}\right\}\right]\\
&\quad -\delta^\alpha_\beta\left[2\left\{\begin{array}{c}b\\ a\end{array}\right\}-\left\{\begin{array}{c}3b\\ a3\end{array}\right\}-\left\{\begin{array}{c}b3\\ 3a\end{array}\right\}\right];
\end{aligned}
$$

$$
\begin{aligned}
\frac{2}{A^2}\left\{(S_2)^\alpha_a,(Q_0)^b_\beta\right\}+\frac{2}{A^2}\left\{(S_0)^\alpha_a,(Q_2)^b_\beta\right\} &= 2\left[\delta^\alpha_\beta\left\{\begin{array}{c}b\\ a\end{array}\right\}+\delta^b_a\left\{\begin{array}{c}\alpha\\ \beta\end{array}\right\}\right]-\delta^\alpha_\beta\left[\left\{\begin{array}{c}3b\\ a3\end{array}\right\}+\left\{\begin{array}{c}b3\\ 3a\end{array}\right\}\right]\\
&\quad -\delta^b_a\left[\left\{\begin{array}{c}3\alpha\\ \beta3\end{array}\right\}+\left\{\begin{array}{c}\alpha3\\ 3\beta\end{array}\right\}\right].
\end{aligned}
$$

Then the sum of these anti-commutators gives:

$$\left\{(S_1)^{\alpha}_{a},(Q_1)^{b}_{\beta}\right\}+\left\{(S_2)^{\alpha}_{a},(Q_0)^{b}_{\beta}\right\}+\left\{(S_0)^{\alpha}_{a},(Q_2)^{b}_{\beta}\right\}=\frac{1}{2}\delta^{b}_{a}\delta^{\alpha}_{\beta}H_2,$$

where the two loop contribution for the Hamiltonian (dilute gas approx) is:

$$\frac{1}{A^2}H_2=2\left\{\begin{array}{c} a \\ a \end{array}\right\}+2\left\{\begin{array}{c} \alpha \\ \alpha \end{array}\right\}-\left\{\begin{array}{c} a3 \\ 3a \end{array}\right\}-\left\{\begin{array}{c} 3a \\ a3 \end{array}\right\}-\left\{\begin{array}{c} \alpha 3 \\ 3\alpha \end{array}\right\}-\left\{\begin{array}{c} 3\alpha \\ \alpha 3 \end{array}\right\}.$$

From the results presented above, we can see that we can only get the complete order g^2 of the Hamiltonian from the commutation of the supercharges if we consider their two-loops contributions.

Appendix B

Fermionic Sector of the Single Spike

The calculation of the fermionic fluctuations is important to determine whether the solution has any supersymmetry. We proceed to calculate these fermionic fluctuations for the single spike, and we will find that they are all massless, while to have 2D supersymmetry these fluctuations would have to have the same masses as the bosonic modes previously calculated. Another argument for the lack of supersymmetry is that we will find twice as many fermionic modes as there are bosonic ones, while supersymmetry requires equally many.

B.1 Fermionic Sector

This calculations follow closely the procedure done for the giant magnon in [142] (zero modes) and in [124] (non-zero modes).

Setup and definitions

We will use the notation from [142] as much as possible, except for the worldsheet co-ordinates: we use non-boosted co-ordinates(x, τ), and boosted co-ordinates (u, v), boosted by c (instead of (σ, t) and boost by v to (x, ξ)). Indices $a, b = 0, 1$ are worldsheet directions, μ, ν curved spacetime, A, B, C flat spacetime, and $I, J = 1, 2$ number fields.

We will work on the subspace $\mathbb{R}\times S^2$, where the unperturbed single spike solution (3.5) lives. The metric for this subspace, in terms of co-ordinates t and angles θ and ϕ, is given by

$$g_{\mu\nu} = E^A_\mu E^B_\nu \eta_{AB} = \begin{bmatrix} -1 & & \\ & 1 & \\ & & \sin^2\theta \end{bmatrix} \quad \text{for } \nu = \begin{matrix} t \\ \theta \\ \phi \end{matrix} \,, \tag{B.1}$$

where the vielbein's components are $E^t_t = E^\theta_\theta = 1$ and $E^\phi_\phi = \sin\theta$ (we are using labels t,θ,ϕ for both curved and flat indices). Writing the single spike (3.5) in these co-ordinates gives

$$\begin{aligned} X^0 &= \tau\,, \\ X^\theta &= \theta = \arccos\left(\frac{1}{\gamma\cosh v}\right), \qquad\qquad \text{i.e. } \cos\theta = \sqrt{1-c^2}\,\mathrm{sech}\, v\,, \\ X^\phi &= \phi = x + \arctan\left(\frac{\tanh v}{c\gamma}\right), \end{aligned}$$

where u,v,γ are still given by (3.4).

The fermionic fluctuations are two Majorana–Weyl fields Θ^I, which obey the action given by Metsaev and Tseytlin [7][1]

$$S = 2\frac{\sqrt{\lambda}}{4\pi}\int d\tau dx\, \mathscr{L}_F \qquad \text{where} \qquad \mathscr{L}_F = i(\eta^{ab}\delta^{IJ} + \varepsilon^{ab}\eta^{IJ})\,\overline{\Theta}^I \rho_a D_b \theta^J\,.$$

The covariant derivative introduced above is defined by

$$D_a\Theta^I = \left(\partial_a + \frac{1}{4}\omega_a^{AB}\Gamma_{AB}\right)\delta^{IJ}\Theta^J - \frac{i}{2}\Gamma_\star\rho_a\varepsilon^{IJ}\Theta^J$$

where $\Gamma_\star = i\Gamma_{01234} = i\Gamma_{[0}\Gamma_1\Gamma_2\Gamma_3\Gamma_{4]}$ (these are the AdS directions) obeys $\Gamma_\star^2 = 1$. From this action we obtain the equations of motion:

$$\begin{aligned} (\rho_0 - \rho_1)\,(D_0 + D_1)\,\Theta^1 &= 0\,, \\ (\rho_0 + \rho_1)\,(D_0 - D_1)\,\Theta^2 &= 0\,. \end{aligned}$$

[1] We use ε and η with two kinds of indices: $\varepsilon^{ab=01} = 1 = \varepsilon^{AB=12}$, and $\eta^{ab=00} = -1 = \eta^{IJ=11}$. The gamma-matrices are in the all imaginary basis: $\Gamma_{A\neq 0}$ are Hermitian and Γ_0 is anti-Hermitian. $\Gamma_{AB} = \Gamma_{[A}\Gamma_{B]}$, so $\Gamma_{\phi\theta} = \Gamma_\phi\Gamma_\theta$.

The projections of the gamma matrices $\rho_a = \Gamma_A E^A_\mu \partial_a X^\mu$ and the spin connection $\omega_a^{AB} = \omega_\mu^{AB}\partial_a X^\mu$ are:[2]

$$
\begin{aligned}
\rho_0 &= \Gamma_0 + c\gamma^2 \frac{\cos^2\theta}{\sin\theta}\Gamma_\phi + \gamma^2 \frac{\cos\theta}{\sin\theta}\sqrt{\sin^2\theta - c^2}\,\Gamma_\theta && = \Gamma_0 + r(\theta)\Gamma_\phi + s(\theta)\Gamma_\theta\,,\\
\rho_1 &= \gamma^2 \frac{\sin^2\theta - c^2}{\sin\theta}\Gamma_\phi - c\gamma^2 \frac{\cos\theta}{\sin\theta}\sqrt{\sin^2\theta - c^2}\,\Gamma_\theta && = p(\theta)\Gamma_\phi + q(\theta)\Gamma_\theta\,,\\
\omega_0 &= -\omega_0^{\phi\theta} = -c\gamma^2 \frac{\cos^3\theta}{\sin^2\theta}\,,\\
\omega_1 &= -\omega_1^{\phi\theta} = -\gamma^2 \frac{\cos\theta}{\sin^2\theta}(\sin^2\theta - c^2)\,.
\end{aligned}
$$

To simplify the equations of motion, start by replacing $\partial_0 = \partial_\tau$ and $\partial_1 = \partial_x$ with the boosted derivatives $\partial_u = \gamma(\partial_1 + c\partial_0)$ and $\partial_v = \gamma(\partial_0 + c\partial_1)$, such that

$$\partial_0 \pm \partial_1 = (1 \mp c)\gamma\{\partial_u \pm \partial_v\}\,.$$

Also, define G and $\tilde{G}$ as follows:

$$
\begin{aligned}
\omega_0^{\phi\theta} + \omega_1^{\phi\theta} &= (1-c)\gamma\frac{1}{2}G, &\quad \text{where}\quad G &= \gamma\frac{\cos\theta}{\sin^2\theta}(c+\sin^2\theta),\\
\omega_0^{\phi\theta} - \omega_1^{\phi\theta} &= (1+c)\gamma\frac{1}{2}\tilde{G}, & \tilde{G} &= \gamma\frac{\cos\theta}{\sin^2\theta}(c-\sin^2\theta).
\end{aligned}
$$

The equations of motion can then be written as

$$
\begin{aligned}
(\rho_0 - \rho_1)\left[(1-c)\gamma\left\{\partial_v + \partial_u + \frac{1}{2}G\Gamma_{\phi\theta}\right\}\Theta^1 - \frac{i}{2}\Gamma_\star(\rho_0+\rho_1)\Theta^2\right] &= 0,\\
(\rho_0 + \rho_1)\left[(1+c)\gamma\left\{\partial_v - \partial_u + \frac{1}{2}\tilde{G}\Gamma_{\phi\theta}\right\}\Theta^2 + \frac{i}{2}\Gamma_\star(\rho_0-\rho_1)\Theta^1\right] &= 0.
\end{aligned}
\tag{B.2}
$$

If we define the operators

$$\mathscr{D}_v = \partial_v + \frac{1}{2}G\Gamma_{\phi\theta}\,, \qquad \tilde{\mathscr{D}}_v = \partial_v + \frac{1}{2}\tilde{G}\Gamma_{\phi\theta}\,,$$

then the curly brackets in the equations of motion (B.2) become these operators plus or

[2]These projections, as functions of θ, are intimately related to the ones from the giant magnon case: $\rho_0 = \rho_1^{\text{magnon}} + \Gamma_0$, $\rho_1 = \rho_0^{\text{magnon}} - \Gamma_0$, $\omega_0 = \omega_1^{\text{magnon}}$ and $\omega_1 = \omega_0^{\text{magnon}}$. Our conventions for the spin connection can be found in [243]. Functions p, q, r, s introduced here will be useful in what follows.

minus the time derivative ∂_u.

The next step is to use kappa-symmetry fixed fields in the equations of motion [244] [142, 124], which we define by

$$\Psi^1 = -i(\rho_0 - \rho_1)\Theta^1\,, \qquad \text{(B.3)}$$
$$\Psi^2 = i(\rho_0 + \rho_1)\Theta^2\,.$$

Note that Γ_{11} anti-commutes with $i(\rho_0 \pm \rho_1)$, and that these operators are real. Consequently, the field Θ^I is Majorana–Weyl if and only if the field Ψ^I also is, so we can impose the Majorana–Weyl conditions on Ψ^I directly.

To re–write the equations of motion in terms of these symmetry-fixed fields, we will need several identities, identical in form to the ones found for the giant magnon case [142].[3] We have two nilpotent operators:

$$(\rho_0 \pm \rho_1)^2 = 0$$

(thus $(\rho_0 - \rho_1)\Psi^1 = 0$ and $(\rho_0 + \rho_1)\Psi^2 = 0$) which commute with the curly derivatives

$$[\mathscr{D}_v, (\rho_0 - \rho_1)] = 0\,, \qquad \left[\tilde{\mathscr{D}}_v, (\rho_0 + \rho_1)\right] = 0$$

(they trivially commute with ∂_u as well). The dagger of ρ_0 is given by:

$$\overline{\rho}_0 \equiv \Gamma_\star \rho_0 \Gamma_\star = -\rho_0^\dagger = \Gamma_0 - r\Gamma_\phi - s\Gamma_\theta\,,$$

with which we can write two more nilpotent operators

$$(\overline{\rho}_0 \pm \rho_1)^2 = 0$$

as well as a non-singular operator $(\overline{\rho}_0 - \rho_0) = -2r\Gamma_\phi - 2s\Gamma_\theta$, whose square is proportional

[3] Relations such as $(\rho_0 \pm \rho_1)^2 = -1 + (r \pm p)^2 + (s \pm q)^2$ and $(\overline{\rho}_0 \pm \rho_1)^2 = -1 + (-r \pm p)^2 + (-s \pm q)^2$ are needed to derive the identities.

to the unit matrix:

$$(\overline{\rho}_0 - \rho_0)^2 = 4\gamma^2 \cos^2\theta .$$

We can now re–write the equations of motion (B.2), by pulling the operators $(\rho_0 \pm \rho_1)$ to the right, using the identities above, until they are acting directly on Θ^I to give Ψ^I. The equations of motion become:

$$\begin{aligned} (1-c)\gamma\{\mathscr{D}_v + \partial_u\}\Psi^1 + \frac{i}{2}\Gamma_\star(\overline{\rho}_0 + \rho_0)\Psi^2 = 0\,, \\ (1+c)\gamma\{\tilde{\mathscr{D}}_v - \partial_u\}\Psi^2 - \frac{i}{2}\Gamma_\star(\overline{\rho}_0 - \rho_0)\Psi^1 = 0\,. \end{aligned} \tag{B.4}$$

Non-zero modes

From (B.4), we start by solving the equation for Ψ^2:

$$\Psi^2 = \frac{(\overline{\rho}_0 - \rho_0)}{4\gamma^2\cos^2\theta}\Gamma_\star\frac{2}{i}(1-c)\gamma\{\mathscr{D}_v + \partial_u\}\Psi^1 . \tag{B.5}$$

We can then eliminate Ψ^2 from the other equation, thus obtaining a second-order equation for Ψ^1 alone:

$$\{\tilde{\mathscr{D}}_v - \partial_u\}\frac{(\overline{\rho}_0 - \rho_0)}{\gamma^2\cos^2\theta}\{\mathscr{D}_v + \partial_u\}\Psi^1 + (\overline{\rho}_0 - \rho_0)\Psi^1 = 0.$$

Using the identity

$$\{\tilde{\mathscr{D}}_v - \partial_u\}\frac{(\overline{\rho}_0 - \rho_0)}{\cos\theta} = \frac{(\overline{\rho}_0 - \rho_0)}{\cos\theta}\{\mathscr{D}_v - \partial_u\}$$

and pulling the $(\rho_0 - \rho_1)$ from Ψ^1's definition to the left of the equation, we obtain

$$(\rho_0 - \rho_1)\left(\frac{1}{\gamma\cos\theta}\{\mathscr{D}_v - \partial_u\}\frac{1}{\gamma\cos\theta}\{\mathscr{D}_v + \partial_u\} + 1\right)\Theta^1 = 0\,. \tag{B.6}$$

This equation is analogous to equation (3.7) of [124]. Using a similar method to the one found in [124] we can solve this equation. We start by temporarily dropping the kappa-symmetry projection $(\rho_0 - \rho_1)$, and solve the remainder of the equation for Θ^1. Once we have a solution for Θ^1, we then apply the projection once more to recover Ψ^1, and find Ψ^2

using (B.5).

Now we proceed to determine Θ^1. To do so, we decompose the second-order equation (B.6) into a system of two coupled first-order equations, defining an intermediate field $\tilde{\Theta}$:

$$\begin{bmatrix} \mathscr{D}_v + \partial_u & -i\,\mathrm{sech}\, v \\ -i\,\mathrm{sech}\, v & \mathscr{D}_v - \partial_u \end{bmatrix} \begin{pmatrix} \Theta^1 \\ \tilde{\Theta} \end{pmatrix} = 0\,.$$

We then expand the spinor in a Fourier series for u:

$$\begin{pmatrix} \Theta^1\,(u,v) \\ \tilde{\Theta}\,(u,v) \end{pmatrix} = e^{-i\omega u}\vec{\Theta}(v,\omega)\,, \qquad \text{(B.7)}$$

and into a sum of eigenspinors of $\Gamma_{\phi\theta}$: $\vec{\Theta} = \vec{\Theta}_+ + \vec{\Theta}_-$ with

$$\left(1_{2\times 2} \otimes \Gamma_{\phi\theta}\right) \vec{\Theta}_\pm = \pm i \vec{\Theta}_\pm .$$

The system of coupled linear equations can then be written as

$$(\partial_v - V_\pm)\,\vec{\Theta}_\pm = 0, \qquad \text{with} \quad V_\pm = \begin{bmatrix} i\left(\omega \mp \frac{G}{2}\right) & i\,\mathrm{sech}\, v \\ i\,\mathrm{sech}\, v & -i\left(\omega \pm \frac{G}{2}\right) \end{bmatrix}.$$

To solve this system of equations, we need to diagonalize it by a change of basis.

Diagonalization

We define $\vec{\Theta}'_\pm = S\vec{\Theta}_\pm$, such that

$$\partial_v \vec{\Theta}'_\pm = (\partial_v S + S V_\pm) S^{-1} \vec{\Theta}'_\pm = H_\pm \vec{\Theta}'_\pm \qquad \text{(B.8)}$$

(thus defining $H_\pm$). We want to determine S that makes $H_\pm$ diagonal. Writing

$$S = \begin{bmatrix} a(v) & b(v) \\ c(v) & d(v) \end{bmatrix},$$

and then setting the off-diagonal elements of $H_\pm$ to zero gives

$$0 = i(a^2 - b^2)\operatorname{sech} v - 2i\omega ab - ba' + ab',$$
$$0 = i(d^2 - c^2)\operatorname{sech} v + 2i\omega cd - cd' + dc'.$$

The above equations were obtained for both H_+ and H_-, which means that S diagonalizes both simultaneously. Because we have only two equations and four parameters, we choose two additional constraints[4]

$$a' = -ib\operatorname{sech}(v)\,, \tag{B.9}$$
$$c' = -id\operatorname{sech}(v)\,,$$

thus obtaining a second-order equation for a (c obeys the same equation):

$$-a'' - \tanh v\, a' + 2i\omega a' - \operatorname{sech}^2 v\, a = 0. \tag{B.10}$$

There are two independent solutions to (B.10):

$$a_1(v) = \frac{2i\omega}{1+4\omega^2} + \frac{\tanh v}{1+4\omega^2}\,,$$
$$a_2(v) = e^{2i\omega v}\operatorname{sech} v\,,$$

where $a(v)$ and $c(v)$ will be (different) linear combinations of these. The other functions $b(v)$ and $d(v)$ are then fixed by (B.9). The general solution for S is then given by

$$S = S_0 \begin{bmatrix} a_1(v) & b_1(v) \\ a_2(v) & b_2(v) \end{bmatrix}$$

[4]These extra relations can be imposed by multiplying S by a non-singular diagonal matrix, which is always allowed as it does not change the equations of motion (B.8).

where S_0 is a non-singular constant matrix, and

$$b_1(v) = i\frac{\operatorname{sech} v}{1+4\omega^2}\,,$$
$$b_2(v) = i\,(2i\omega - \tanh v)\,e^{2i\omega v}\,.$$

The determinant of this change of basis is

$$\det S = -ie^{2i\omega v}\det S_0\,,$$

different from zero, as expected.

This change of basis gives a simple (diagonal) form for $H_\pm$:

$$H_\pm = i\left(\omega \mp \frac{G}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{B.11}$$

Solving the diagonalized system

Using $H_\pm$ just obtained (B.11), the diagonalized system (B.8) can now be solved. The equations simplify to

$$\left(\partial_v - i\left(\omega \mp \frac{G}{2}\right)\right)f(v) = 0\,.$$

In the magnon case [142] (and also for the zero modes in appendix B.2) we found a very similar equation. It has solution $f(v) = e^{\pm i\chi}e^{i\omega v}$, where

$$e^{i\chi} = \left(\frac{\sinh v + ic}{\sinh v - ic}\right)^{1/4}\sqrt{\tanh v + i\operatorname{sech} v}.$$

The field $\vec{\Theta}'_\pm$ will then be given by the product of this phase and a spinor:

$$\vec{\Theta}'_\pm = e^{\pm i\chi}e^{i\omega v}\vec{U}_\pm\,,$$

where $\vec{U}_\pm$ is any eigenspinor of $\left(1 \otimes \Gamma_{\phi\theta}\right)$ with eigenvalues $\pm i$. Finally we need to undo the change of basis to obtain $\vec{\Theta}_\pm$:

$$\vec{\Theta}_\pm = S^{-1}\vec{\Theta}_\pm = e^{\pm i\chi} e^{i\omega v} S^{-1}\vec{U}_\pm.$$

Absorbing the constant matrix S_0^{-1} into the arbitrary spinor $\vec{U}_\pm$, we define new spinors $U_\pm$ and $\tilde{U}_\pm$

$$S_0^{-1}\vec{U}_\pm = \frac{1}{\sqrt{1-c}} \begin{pmatrix} U_\pm \\ \tilde{U}_\pm \end{pmatrix},$$

where the $\sqrt{1-c}$ factor is introduced for later convenience.

Recalling from (B.7) that our original spinor Θ^1 is the first component of $e^{-i\omega u}(\vec{\Theta}_+ + \vec{\Theta}_-)$, we can now write[5]

$$\begin{aligned}\Theta^1(u,v) &= \frac{1}{\sqrt{1-c}} e^{-i\omega u} \sum_\pm \frac{e^{i\omega v \pm i\chi}}{-ie^{2i\omega v}} \left[b_2(v) U_\pm - b_1(v)\tilde{U}_\pm\right] \\ &= \frac{-1}{\sqrt{1-c}} e^{-i\omega u} \sum_\pm e^{\pm i\chi} \left[e^{-i\omega v} \frac{\operatorname{sech} v}{1+4\omega^2} U_\pm + (\tanh v - 2i\omega) e^{i\omega v}\tilde{U}_\pm\right].\end{aligned}$$

To determine the symmetry-fixed field $\Psi^1 = -i(\rho_0 - \rho_1)\Theta^1$ we use the identity $e^{\pm 2i\chi} = (p - r) \mp i(q-s)$, and find the positive-frequency solution to be:

$$\begin{aligned}\Psi_p^1 &= \frac{i\, e^{-i\omega u}}{\sqrt{1-c}} \sum_\pm \left(e^{\pm i\chi}\Gamma_0 - e^{\mp i\chi}\Gamma_\phi\right) \left[e^{-i\omega v} \frac{\operatorname{sech} v}{1+4\omega^2} U_\pm + (\tanh v - 2i\omega) e^{i\omega v}\tilde{U}_\pm\right] \\ &= \frac{i}{\sqrt{1-c}} \sum_\pm \left(e^{\pm i\chi}\Gamma_0 - e^{\mp i\chi}\Gamma_\phi\right) \left[e^{i\alpha} \frac{\operatorname{sech} v}{1+4\omega^2} U_\pm + \sqrt{\tanh^2 v + 4\omega^2}\, e^{i\beta}\tilde{U}_\pm\right] \end{aligned} \tag{B.12}$$

where the phases α and β are defined by

$$\begin{aligned} e^{i\alpha} &= e^{-i\omega(u+v)}, \\ e^{i\beta} &= e^{-i\omega(u-v)} e^{-i\arctan(2\omega\coth v)}. \end{aligned}$$

[5] The second entry of $e^{-i\omega u}\left(\vec{\Theta}_+ + \vec{\Theta}_-\right)$ will be given by

$$\tilde{\Theta}(u,v) = -ie^{-i\omega u} e^{\pm i\chi} \left[e^{i\omega v} \operatorname{sech} v\, U_\pm - \frac{(\tanh v + 2i\omega)}{1+4\omega^2} e^{-i\omega v}\tilde{U}_\pm\right].$$

This will be useful in determining Ψ^2.

Majorana condition

We now want to impose the Majorana condition on the spinors: Ψ^I should be real $\Psi^{I*} = \Psi^I$. To do so, consider a superposition of positive and negative frequencies ω:

$$\begin{aligned}\Psi^1 &= 2\,\mathrm{Re}\,\Psi^1_p = \Psi^1_p + \Psi^{1*}_p \\ &= \frac{i}{\sqrt{1-c}} \sum_{\pm} \left(e^{\pm i\chi}\Gamma_0 - e^{\mp i\chi}\Gamma_\phi\right) \left[\frac{\operatorname{sech} v}{1+4\omega^2}\left(e^{i\alpha}U_\pm + e^{-i\alpha}U^*_\mp\right)\right. \\ &\qquad \left. + \sqrt{\tanh^2 v + 4\omega^2}\left(e^{i\beta}\tilde{U}_\pm + e^{-i\beta}\tilde{U}^*_\mp\right)\right].\end{aligned}$$

Note that $U^*_\mp$ is an eigenspinor of $\Gamma_{\phi\theta}$ of eigenvalue $\pm i$ (the Γ matrices are imaginary, thus $\Gamma_{\phi\theta}$ is real).

If we combine the four $\pm$ eigenspinors into two spinors $U = U_+ + U_-$ and $\tilde{U} = \tilde{U}_+ + \tilde{U}_-$ (we can reverse this with projection operators $U_\pm = \frac{i \pm \Gamma_{\phi\theta}}{2i} U$, and similarly for the others), we can then write

$$\begin{aligned}\Psi^1 &= \frac{i}{\sqrt{1-c}}\left[\Gamma_0\left(\cos\chi + \Gamma_{\phi\theta}\sin\chi\right) - \Gamma_\phi\left(\cos\chi - \Gamma_{\phi\theta}\sin\chi\right)\right] \\ &\qquad \times \left\{\frac{\operatorname{sech} v}{1+4\omega^2}\mathrm{Re}(e^{i\alpha}U) + \sqrt{\tanh^2 v + 4\omega^2}\,\mathrm{Re}(e^{i\beta}\tilde{U})\right\} \\ &= \frac{i}{\sqrt{1-c}}\left[\Gamma_0\left(\cos\chi + \Gamma_{\phi\theta}\sin\chi\right) - \Gamma_\phi\left(\cos\chi - \Gamma_{\phi\theta}\sin\chi\right)\right] \\ &\qquad \times \left\{\frac{\operatorname{sech} v}{1+4\omega^2}\left(\cos\alpha\, U_0 + \sin\alpha\,\Gamma_{\phi\theta}U_1\right)\right. \\ &\qquad\quad \left. + \sqrt{\tanh^2 v + 4\omega^2}\left(\cos\beta\,\tilde{U}_0 + \sin\beta\,\Gamma_{\phi\theta}\tilde{U}_1\right)\right\},\end{aligned} \tag{B.13}$$

where the new spinors are

$$\begin{aligned} U_0 &= 2\,\mathrm{Re}\left(U_+ + U_-\right), & \tilde{U}_0 &= 2\,\mathrm{Re}\left(\tilde{U}_+ + \tilde{U}_-\right), \\ U_1 &= 2\,\mathrm{Re}\left(U_+ - U_-\right), & \tilde{U}_1 &= 2\,\mathrm{Re}\left(\tilde{U}_+ - \tilde{U}_-\right), \end{aligned} \tag{B.14}$$

where $U_0 = 2\,\mathrm{Re}\,U$, and $U_1 = 2\Gamma_{\phi\theta}\,\mathrm{Im}\,U$ (and similarly with tildes).

Having found Ψ^1, Ψ^2 can be determined from (B.5). The final, Majorana, field Ψ^2 is

$$\begin{aligned}\Psi^2 = \frac{1}{\sqrt{1+c}}\Gamma_*\Gamma_\theta &\left[\Gamma_0\left(\cos\tilde{\chi}+\Gamma_{\phi\theta}\sin\tilde{\chi}\right)-\Gamma_\phi\left(\cos\tilde{\chi}-\Gamma_{\phi\theta}\sin\tilde{\chi}\right)\right] \\ &\times\Big\{\operatorname{sech}v\left(\cos\tilde{\alpha}\,\tilde{U}_0+\sin\tilde{\alpha}\,\Gamma_{\phi\theta}\tilde{U}_1\right) \\ &\qquad -\frac{\sqrt{\tanh^2 v+4\omega^2}}{1+4\omega^2}\left(\cos\tilde{\beta}\,U_0+\sin\tilde{\beta}\,\Gamma_{\phi\theta}U_1\right)\Big\},\end{aligned}\tag{B.15}$$

where the new phases are

$$\begin{aligned} e^{i\tilde{\chi}} &= \sqrt{\frac{\sinh v - ic}{\sinh v + ic}}e^{i\chi} = \left(\frac{\sinh v - ic}{\sinh v + ic}\right)^{1/4}\sqrt{\tanh v + i\operatorname{sech} v}, \\ e^{i\tilde{\alpha}} &= e^{-i\omega(u-v)}, \\ e^{i\tilde{\beta}} &= e^{-i\omega(u+v)}e^{i\arctan(2\omega\coth v)}. \end{aligned}$$

Mass and Counting

If we analyze the solution Ψ^1 far from the spike ($|v|\gg 1$), we observe that the phases obey $i\alpha = -i\omega u - i\omega v$ and $i\beta = -i\omega u + i\omega v$, which means that the fermionic modes are massless, $\omega^2 = k^2$. Consequently there is no supersymmetry, as we found in Section 3.3 the corresponding bosonic modes to be massive.

It is also interesting to determine the number of fermionic modes. There are four spinors $U_\pm$ and $\tilde{U}_\pm$, which are $\Gamma_{\phi\theta}$ eigenspinors, and thus have 16 complex components each. These must also be Γ_{11} eigenspinors, so that Ψ is Weyl, which cuts the number by half. Finally, from (B.14) we saw that the Majorana spinor depends only on the real part of each of the original spinors, cutting it in half again. The final counting is 16 complex degrees of freedom, twice the number for the ones found for the giant magnon [124].

The bosonic modes for the spike were determined by simply by interchanging $x \leftrightarrow t$ in the magnon modes. This means that the number of bosonic modes is the same as for the magnon case: there are 8 non-zero modes (4 on the sphere and 4 in AdS). Once more, the fact that there are two fermionic modes for each bosonic mode is more evidence against supersymmetry.

As we will see below, there are also twice as many fermionic zero modes (8 complex) as bosonic ones (4, same as the magnon). Because the non-zero modes are massless, $\omega = 0$ is part of the continuum of frequencies, and one can obtain expressions for the zero modes just by setting $\omega = 0$ in (B.13) and (B.15) (which sets $\alpha = \beta = 0$). But the counting of fermionic zero modes has to be done more carefully: the zero modes appear to have the same dependence on $\mathrm{Re}\, U_\pm$ and $\mathrm{Re}\, \tilde{U}_\pm$ as the non-zero modes, suggesting that there are also 16 of them. However, the same argument that happens in the magnon case [142] kills half of these modes, leaving 8 fermionic zero modes. Below we give the analogous calculation for the single spike that can be found in [142] for the magnon zero modes.

B.2 Fermionic Zero modes

In this appendix we determine the zero modes for the single spike, following [142]. It will be shown that from this calculation it is much easier to see why the single spike has twice as many fermionic modes as the magnon, even though the result is identical to simply setting $\omega = 0$ in the section above.

The zero modes obey $\partial_u \Psi^I = 0$, which simplifies the second-order equation (B.6) to

$$\left(\frac{1}{\gamma \cos\theta} \mathscr{D}_v \frac{1}{\gamma \cos\theta} \mathscr{D}_v + 1 \right) \Psi^1 = 0.$$

This equation factorizes, making the calculations much easier than the previous case.

The above equation implies that $(\mathscr{D}_v - \eta\, i\gamma \cos\theta)\Psi^1 = 0$ with $\eta = \pm 1$. Pulling the factor $(\rho_0 - \rho_1)$ to the left we find:

$$(\rho_0 - \rho_1) \left\{ \partial_v + \frac{1}{2} G \Gamma_{\phi\theta} + \eta\, i\gamma \cos\theta \right\} \Theta^1 = 0.$$

As for the non-zero modes, we only fix the κ-symmetry projection in the end, solving first for Θ^1 alone. The matrix part of this equation involves only $\mathbf{1}$ and $\Gamma_{\phi\theta}$, which can be simultaneously diagonalized. The solution is of the form

$$\Theta^1 = \Theta_+ + \Theta_- = f_+(v) U_+ + f_-(v) U_- \,,$$

where the spinors $U_\pm$ (and so $\Theta_\pm$) are $\Gamma_{\phi\theta}$ eigenvectors, with eigenvalues $\pm i$ respectively. All that is left to solve is

$$\left\{\partial_v \pm \frac{i}{2}G + \eta\, i\gamma\cos\theta\right\} f_\pm(v) = 0.$$

The solutions of this equation are pure phase,

$$f_\pm(v) = e^{\pm i\chi} e^{i\eta\chi_2} \qquad \text{where} \quad e^{i\chi} = \left(\frac{\sinh v + ic}{\sinh v - ic}\right)^{1/4} \sqrt{\tanh v + i\,\mathrm{sech}\, v},$$
$$e^{\pm i\chi_2} = \mathrm{sech}\, v \pm i \tanh v.$$

Comparing these solutions to the giant magnon ones [142], we find that instead of a modulating factor $\mathrm{sech}\, u$, we get an extra phase $e^{i\eta\chi_2}$. In the magnon case, it was this modulating factor that made one of the solutions normalizable, and allowed Minahan to reject the other sign of η for producing a solution which diverges at large u. In our case both signs lead to non-normalizable solutions, and the most general solution is a linear combination of the $\eta = +1$ and $\eta = -1$ cases:

$$\Psi^1 = -i(\rho_0 - \rho_1)\frac{1}{\sqrt{1-c}} \sum_\pm e^{\pm i\chi} \sum_\eta e^{i\eta\chi_2} U_\pm^\eta$$

(we've introduced a factor of $\sqrt{1-c}$ for later convenience). Using the identity $e^{\pm 2i\chi} = (p - r) \mp i(q-s)$, we obtain for Ψ^1:

$$\Psi^1 = \frac{i}{\sqrt{1-c}} \left[\Gamma_0 \left(\cos\chi + \Gamma_{\phi\theta} \sin\chi\right) - \Gamma_\phi \left(\cos\chi - \Gamma_{\phi\theta}\sin\chi\right)\right] \left(\mathrm{sech}\, v\, U_0 + \tanh v\, \tilde{U}_0\right)$$

where we defined new linear combinations of the arbitrary spinors $U_\pm^\eta$ as

$$U_0 = -\left(U_+^{\eta=1} + U_-^{\eta=1}\right) - \left(U_+^{\eta=-1} + U_-^{\eta=-1}\right),$$
$$\tilde{U}_0 = -i\left(U_+^{\eta=1} + U_-^{\eta=1}\right) + i\left(U_+^{\eta=-1} + U_-^{\eta=-1}\right).$$

The reason for this choice of linear combinations is that the Majorana condition $\Psi^* = \Psi$ now simply requires that U_0 and $\tilde{U}_0$ be themselves Majorana spinors (the Γ-matrices are all imaginary, thus $\Gamma_{\phi\theta}$ is real).

We can now completely determine Ψ^2 from (B.5), as an operator acting on Ψ^1. We then write:

$$\Psi^2 = \frac{\Gamma_\star\Gamma_\theta}{\sqrt{1+c}}\left[\Gamma_0\left(\cos\tilde{\chi}+\Gamma_{\phi\theta}\sin\tilde{\chi}\right)-\Gamma_\phi\left(\cos\tilde{\chi}-\Gamma_{\phi\theta}\sin\tilde{\chi}\right)\right]\left(\operatorname{sech} v\,\tilde{U}_0-\tanh v\,U_0\right)$$

where as before $e^{i\tilde{\chi}}=e^{-i\chi+i\chi_2}$, and we used the identity $(r\pm is)=\pm i\gamma\cos\theta e^{\pm i(\tilde{\chi}-\chi)}$.

Comparing these zero modes with the non-zero modes (B.13) and (B.15), we can easily see that they are just the $\omega=0$ case of the non-zero modes (i.e. $\alpha=\beta=0$). This differs from the case of the supersymmetric giant magnon, where the zero modes of [142] are disconnected from the massive non-zero modes of [124].

The counting of modes is now much easier: the four spinors $U^\eta_\pm$ are $\Gamma_{\phi\theta}$-eigenspinors, having 16 complex components each. As they are also Weyl spinors, i.e. Γ_{11}-eigenspinors, the number of components is cut in half. Requiring U_0 and $\tilde{U}_0$ to be Majorana cuts it in half again, to 16 components in total. This is the number of components we got from the non-zero modes, by setting $\omega=0$, as was expected. But at this stage the giant magnon had only 8 complex components [142], half of what we found for the single spike. The argument given below cuts the number of components by another factor of 2 in both cases, leaving the giant spike with just 8 zero modes, and the giant magnon with 4.

Slow-motion

In [142], we can find the following argument: we regard the spinors U_0 and $\tilde{U}_0$ as a moduli of the solution, and allow them to become time-dependent, $\partial_u U\neq 0$. If we plug a zero mode back into the action we will get zero, but plugging this 'slowly-moving' mode needn't do so. The zero modes whose related slowly-moving modes give a non-zero action are 'real' zero modes, while the others are pure gauge [245].

When substituting the slowly-moving mode $\Theta=\sum F(v)U(u)$ into the Lagrangian, the equations of motion force everything except the ∂_u terms to vanish, leaving

$$\mathscr{L}_F = -i\gamma(1-c)\,\overline{\Theta}^1(\rho_0-\rho_1)\partial_u\Theta^1+i\gamma(1+c)\,\overline{\Theta}^2(\rho_0+\rho_1)\partial_u\Theta^2\,,$$

where $\overline{\Theta} = \Theta^\dagger \Gamma_0$, as before. Using the identities $2\Gamma_0(\rho_0 \pm \rho_1) = -(\rho_0 \pm \rho_1)^\dagger(\rho_0 \pm \rho_1)$ the Lagrangian becomes

$$\mathscr{L}_F = i\gamma \frac{1-c}{2} \Psi^{1\dagger} \partial_u \Psi^1 - i\gamma \frac{1+c}{2} \Psi^{2\dagger} \partial_u \Psi^2.$$

Plugging in Ψ^I from above, we obtain

$$\begin{aligned}\mathscr{L}_F = \frac{i\gamma}{2} &\left[(\Gamma_0 - \Gamma_\phi) \left(\operatorname{sech} v\, U_0 + \tanh v\, \tilde{U}_0 \right) \right]^\dagger \partial_u \left[(\Gamma_0 - \Gamma_\phi) \left(\operatorname{sech} v\, U_0 + \tanh v\, \tilde{U}_0 \right) \right] \\ - \frac{i\gamma}{2} &\left[(\Gamma_0 - \Gamma_\phi) \left(\operatorname{sech} v\, \tilde{U}_0 - \tanh v\, U_0 \right) \right]^\dagger \partial_u \left[(\Gamma_0 - \Gamma_\phi) \left(\operatorname{sech} v\, \tilde{U}_0 - \tanh v\, U_0 \right) \right].\end{aligned}$$

Both U_0 and $\tilde{U}_0$ always appear acted on by $(\Gamma_0 - \Gamma_\phi)$. Using Minahan's words, below (2.32) of [142], only those modes satisfying $(\Gamma_0 + \Gamma_\phi)U_0$ and $(\Gamma_0 + \Gamma_\phi)\tilde{U}_0$ contribute. The situation is identical to the giant magnon in that only half of the modes appearing in Ψ^I contribute (they 'are real,' meaning true, zero modes). But since there are two constant Majorana–Weyl spinors U_0 and $\tilde{U}_0$ for the single spike, instead of only one for the giant magnon, there are twice as many modes: 8 instead of 4 complex degrees of freedom. Summarizing, in the fermionic sector, for both the non-zero modes and the zero modes we find twice as many modes as in the giant magnon case.

Appendix C

Fillings and Fermionic Poles

We will now review some properties of the algebraic curve quasi-momenta for $AdS_5 \times S^5$. These include some explicit calculations of the filling fractions in the bosonic subsector $\mathbb{R} \times S^5$, as well as a brief discussion of the fermionic poles in the full superstring theory.

C.1 Properties of fillings of branch cuts

In this appendix we present some properties of the filling functions for $\mathbb{R} \times S^5$, which can be used to relate them to the relevant Dynkin labels. Using the fact that

$$\begin{aligned}
\frac{\sqrt{\lambda}}{8\pi^2 i}\sum_{a=1}^{A_1/2}\oint_{\mathscr{A}_a}\frac{1}{x}dp_1 &= \frac{\sqrt{\lambda}}{8\pi^2 i}\sum_{a=1}^{A_1/2}\oint_{\mathscr{A}_a}\frac{1}{x^3}p_2^{'}(1/x)\,dx = -\frac{\sqrt{\lambda}}{8\pi^2 i}\sum_{b=1}^{A_1/2}\oint_{\mathscr{A}_b=1/\mathscr{A}_a} y p_2^{'}(y)\,dy \\
&= \frac{\sqrt{\lambda}}{8\pi^2 i}\sum_{b=1}^{A_1/2}\oint_{\mathscr{A}_b=1/\mathscr{A}_a} y p_1^{'}(y)\,dy\,; \\
\frac{\sqrt{\lambda}}{8\pi^2 i}\sum_{a=1}^{A_1/2}\oint_{\mathscr{A}_a} x\,dp_1 &= \frac{\sqrt{\lambda}}{8\pi^2 i}\sum_{a=1}^{A_1/2}\oint_{\mathscr{A}_a}\frac{1}{x}p_2^{'}(1/x)\,dx = -\frac{\sqrt{\lambda}}{8\pi^2 i}\sum_{b=1}^{A_1/2}\oint_{\mathscr{A}_b=1/\mathscr{A}_a}\frac{1}{y}p_2^{'}(y)\,dy \\
&= -\frac{\sqrt{\lambda}}{8\pi^2 i}\sum_{a=1}^{A_1/2}\oint_{\mathscr{A}_a} x p_2^{'}(x)\,dx\,,
\end{aligned}$$

and recalling that the total number of cuts between sheets $1,2$, the $\mathscr{C}_{1,a}$ and $\mathscr{C}_{1,b}=1/\mathscr{C}_{1,a}$, is given by A_1, then

$$\begin{aligned}\sum_{a=1}^{A_1/2} K_{1,a} &\equiv \frac{\sqrt{\lambda}}{8\pi^2 i}\sum_{a=1}^{A_1/2}\oint_{\mathscr{A}_a}\left(x+\frac{1}{x}\right)dp_1 = \frac{\sqrt{\lambda}}{8\pi^2 i}\sum_{a=1}^{A_1}\oint_{\mathscr{A}_a} x p_1^{'}(x)\,dx \\ &= -\frac{\sqrt{\lambda}}{8\pi^2 i}\sum_{a=1}^{A_1}\oint_{\mathscr{A}_a} x p_2^{'}(x)\,dx\,.\end{aligned}$$

A very similar result can be found for cuts between sheets $3,4$ (total A_3). In the same way considering now cuts between sheets $2,3$, which total A_2 ($\mathscr{C}_{2,a}$ and $\mathscr{C}_{2,b}=1/\mathscr{C}_{2,a}$), we easily see that

$$\frac{\sqrt{\lambda}}{8\pi^2 i}\sum_{a=1}^{A_2/2}\oint_{\mathscr{A}_a}\frac{1}{x}dp_2 \quad = \tfrac{\sqrt{\lambda}}{8\pi^2 i}\textstyle\sum_{a=1}^{A_2/2}\oint_{\mathscr{A}_a}\frac{1}{x^3}p_1^{'}(1/x)\,dx = \quad -\frac{\sqrt{\lambda}}{8\pi^2 i}\sum_{b=1}^{A_2/2}\oint_{\mathscr{A}_b=1/\mathscr{A}_a} y p_1^{'}(y)\,dy\,,$$

and consequently

$$\sum_{a=1}^{A_2/2} K_{2,a} \equiv \frac{\sqrt{\lambda}}{8\pi^2 i}\sum_{a=1}^{A_2/2}\oint_{\mathscr{A}_a}\left(x+\frac{1}{x}\right)dp_2 = \frac{\sqrt{\lambda}}{8\pi^2 i}\sum_{a=1}^{A_2/2}\oint_{\mathscr{A}_a} x dp_2 - \frac{\sqrt{\lambda}}{8\pi^2 i}\sum_{b=1}^{A_2/2}\oint_{\mathscr{A}_b=1/\mathscr{A}_a} y p_1^{'}(y)\,dy.$$

Now one should notice that all of the cuts existing in sheet 1 are A_1 connecting sheets $1,2$ and $A_2/2$ connecting $1,4$. If we sum over all possible $\mathscr{A}$-cycles around these cuts, this is equivalent to integrating over a closed contour at infinity (with all of the singularities inside) if we add the singularities at $x=\pm 1$. Then

$$\begin{aligned}\sum_{a=1}^{A_1/2} K_{1,a} &= \frac{\sqrt{\lambda}}{8\pi^2 i}\oint_{\infty} x p_1^{'}(x)\,dx - \frac{\sqrt{\lambda}}{8\pi^2 i}\sum_{b=1}^{A_2/2}\oint_{\mathscr{A}_b} y p_1^{'}(y)\,dy - \frac{\sqrt{\lambda}}{8\pi^2 i}\oint_{\pm 1} x p_1^{'}(x)\,dx \\ &= \frac{\sqrt{\lambda}}{8\pi^2 i}\oint_{\infty} x p_1^{'}(x)\,dx - \frac{\sqrt{\lambda}}{8\pi^2 i}\sum_{b=1}^{A_2/2}\oint_{\mathscr{A}_b} y p_1^{'}(y)\,dy + \frac{\sqrt{\lambda}}{8\pi^2 i}\oint_{\pm 1} x\frac{\pi\kappa}{(x\mp 1)^2}dx.\end{aligned}$$

For sheet 2 we have A_1 cuts connecting it to sheet 1, and $A_2/2$ cuts connected to sheet 3:

$$\begin{aligned}
\sum_{a=1}^{A_2/2} K_{2,a} &= \frac{\sqrt{\lambda}}{8\pi^2 i}\oint_{\infty} x p_2^{'}(x)\, dx - \frac{\sqrt{\lambda}}{8\pi^2 i}\sum_{a=1}^{A_1}\oint_{\mathscr{A}_a} x\, dp_2 \\
&\qquad - \frac{\sqrt{\lambda}}{8\pi^2 i}\oint_{\pm 1} x p_2^{'}(x)\, dx - \frac{\sqrt{\lambda}}{8\pi^2 i}\sum_{b=1}^{A_2/2}\oint_{\mathscr{A}_b} y p_1^{'}(y)\, dy \\
&= \frac{\sqrt{\lambda}}{8\pi^2 i}\oint_{\infty} x p_2^{'}(x)\, dx + \sum_{a=1}^{A_1/2} K_{1,a} + \frac{\sqrt{\lambda}}{8\pi^2 i}\oint_{\pm 1} x \frac{\pi\kappa}{(x\mp 1)^2} dx - \frac{\sqrt{\lambda}}{8\pi^2 i}\sum_{b=1}^{A_2/2}\oint_{\mathscr{A}_b} y p_1^{'}(y)\, dy\,.
\end{aligned}$$

For sheet 3 we have A_3 cuts connecting it to sheet 4, and $A_2/2$ connecting it to sheet 2:

$$\begin{aligned}
\sum_{a=1}^{A_3/2} K_{3,a} &= \frac{\sqrt{\lambda}}{8\pi^2 i}\sum_{a=1}^{A_3/2}\oint_{\mathscr{A}_a}\left(x + \frac{1}{x}\right) dp_3 = \frac{\sqrt{\lambda}}{8\pi^2 i}\sum_{a=1}^{A_3}\oint_{\mathscr{A}_a} x\, dp_3 \\
&= \frac{\sqrt{\lambda}}{8\pi^2 i}\oint_{\infty} x p_3^{'}(x)\, dx - \frac{\sqrt{\lambda}}{8\pi^2 i}\sum_{b=1}^{A_2/2}\oint_{\mathscr{A}_b} y p_3^{'}(y)\, dy - \frac{\sqrt{\lambda}}{8\pi^2 i}\oint_{\pm 1} x \frac{\pi\kappa}{(x\mp 1)^2} dx\,.
\end{aligned}$$

Finally, for sheet 4 we have A_3 cuts connecting it to sheet 3 and $A_2/2$ connecting sheets 1 and 4, and the relations can be written as the previous ones.

C.2 The behaviour of the quasi-momenta at a fermionic pole

In page 114 we reviewed the behaviour of the monodromy matrix eigenvalues at the points $\tilde{x}_a$ where two eigenvalues corresponding to S^5 coincide, and found that these points were square root singularities of the eigenvalues. The same behaviour is expected from points $\hat{x}_a$ where two eigenvalues corresponding to AdS_5 coincide. But the eigenvalues of the monodromy matrix will have a different behaviour at points x_a^* where eigenvalues of opposite gradings coincide, that is, points at which an eigenvalues of the S^5 part of the monodromy matrix $e^{i\tilde{p}_k}$ coincides with an eigenvalue of the AdS_5 part $e^{i\hat{p}_l}$.

The restriction of $\Omega(x)$ to the subspace of these two eigenvalues then has the form

$$\Gamma = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right),$$

where the lines above separate the two gradings. One can calculate the eigenvalues of this super-matrix to be[1]

$$\gamma_1 = a + \frac{bc}{d-a},\ \gamma_2 = d + \frac{bc}{d-a}$$

The parameters a,b,c,d depend analytically on x, but in this case it is the supertrace of this sub-matrix, $f(x) = \mathrm{str}\Gamma = a - d$, which vanishes at $x = x_a^*$, that is $f(x_a^*) = 0$. This is the denominator present in the eigenvalues $\gamma_{1,2}$. The numerator, $\alpha_a^* = -bc|_{x=x_a^*}$ does not need to be zero, and so we have a simple pole at $x = x_a^*$ and obtain $f(x) = \frac{\alpha_a^*}{x-x_a^*} + \mathscr{O}(x - x_a^*)$. Thus expanding around the singularity we find

$$e^{i\tilde{p}_k(x)} = e^{i\hat{p}_l(x)} = e^{i\check{p}_k(x_a^*)}\left(\frac{\alpha_a^*}{x-x_a^*} + 1 + \mathscr{O}(x - x_a^*)\right),$$

for some coefficient $\alpha_a^* = -bc$. Note that this coefficient is nilpotent, i.e., $(\alpha_a^*)^2 = 0$.

[1] This can be easily seen from the fact that $\mathrm{str}\Gamma \equiv \gamma_1 - \gamma_2 = a - d$ and $\mathrm{sdet}\Gamma = \frac{\gamma_1}{\gamma_2} = \frac{a}{d} - \frac{bc}{d^2}$, and by recalling that b,c are Grassmann-odd variables, that is, $(bc)^2 = 0$.

Appendix D

On the CP3 Geometry And Conserved Charges

In this appendix we review some properties of the CP^3 geometry, and the constraints that a string solution living in this space has to obey. We also perform semi-classical quantization around the vacuum (6.11), are recover the quantity $\Delta - \frac{J_1 - J_4}{2}$ as being the Hamiltonian for the physical excitations.

We finalize by giving a summary of the finite-J expansion of conserved charges for the several magnon solutions in CP^3.

D.1 More about the geometry of CP^3

The complex projective space CP^3 is defined by

$$CP^3 = \frac{\mathbb{C}^4}{\vec{z} \sim \lambda \vec{z}}$$

where the co-ordinates $\vec{z} = z_a$ are called homogeneous co-ordinates, and λ is complex. The identification $\vec{z} \sim \lambda \vec{z}$ can be separated in two parts: $\vec{z} \sim r\vec{z}$ and $\vec{z} \sim e^{i\phi}\vec{z}$ (for any $r, \phi \in \mathbb{R}$). Then, setting the first one to be $|\vec{z}|^2 = 1$, we obtain a sphere with one identification

$$CP^3 = \frac{S^7}{\vec{z} \sim e^{i\phi}\vec{z}} = \frac{S^7}{U(1)}\,.$$

The isometry group is $SU(4)$, acting in the natural way on $\vec{z}$. The stabilizer group of, for example, the point $z_4 = 1$ is $U(3)$, thus we can also write

$$CP^3 = \frac{SU(4)}{U(3)} \,.$$

The standard Fubini–Study metric for this space is, in the infinitesimal form:

$$\begin{aligned} ds^2_{CP^3} &= \frac{dz_i d\bar{z}_i}{\rho^2} - \frac{|z_i d\bar{z}_i|^2}{\rho^4} \\ &= ds^2_{\text{sphere}} - d\gamma^2 \\ &= \frac{ds^2_{\text{flat}} - d\rho^2}{\rho^2} - d\gamma^2 \end{aligned} \tag{D.1}$$

where $\rho^2 = z_i \bar{z}_i$.[1] In the equation above $ds^2_{\text{flat}} = dz_i d\bar{z}_i$ is the Euclidean metric for $\mathbb{C}^4$, and ds^2_{sphere} is a metric for S^7 in terms of the homogeneous co-ordinates. Instead of fixing $\rho = 1$, we subtract from the sphere the component coming from radial motion (and scale the rest appropriately). Consequently, we obtain CP^3 from the sphere S^7 by fixing the total phase $\gamma = \arg \prod_i z_i$ (this can be understood as subtracting the total phase component from the sphere). These radial and total phase components are given by

$$\begin{aligned} d\rho &= \frac{1}{2\rho}\left(z_i d\bar{z}_i + \bar{z}_i dz_i\right) = \frac{1}{\rho}\operatorname{Re}\left(\bar{z}_i dz_i\right) \\ d\gamma &= \frac{i}{2\rho^2}\left(z_i d\bar{z}_i - \bar{z}_i dz_i\right) = \frac{1}{\rho^2}\operatorname{Im}\left(\bar{z}_i dz_i\right). \end{aligned}$$

We now recall the two forms of the CP^3 metric in terms of real angles, and present the maps between these angles and the homogeneous co-ordinates. These maps can be found in [217] and [215], up to a relabeling of the z_i. For the metric (6.6) (η is often called ψ)

$$\begin{aligned} ds^2_{CP^3} = d\xi^2 &+ \frac{1}{4}\sin^2 2\xi \left(d\eta + \frac{1}{2}\cos\vartheta_1 \, d\varphi_1 - \frac{1}{2}\cos\vartheta_2 \, d\varphi_2\right)^2 \\ &+ \frac{1}{4}\cos^2\xi \left(d\vartheta_1^2 + \sin^2\vartheta_1 \, d\varphi_1^2\right) + \frac{1}{4}\sin^2\xi \left(d\vartheta_2^2 + \sin^2\vartheta_2 \, d\varphi_2^2\right) \end{aligned}$$

[1] In some conventions [246, 192] the metric has an extra factor of 4, making CP^1 (6.16) a unit sphere.

the map was given in the main text (6.5), and it is:

$$
\begin{aligned}
z_1 &= \sin\xi\ \cos(\vartheta_2/2)\, e^{-i\eta/2}\, e^{i\varphi_2/2} \\
z_2 &= \cos\xi\ \cos(\vartheta_1/2)\, e^{i\eta/2}\, e^{i\varphi_1/2} \\
z_3 &= \cos\xi\ \sin(\vartheta_1/2)\, e^{i\eta/2}\, e^{-i\varphi_1/2} \\
z_4 &= \sin\xi\ \sin(\vartheta_2/2)\, e^{-i\eta/2}\, e^{-i\varphi_2/2}.
\end{aligned}
\tag{D.2}
$$

For the other set of angular variables (6.7)

$$
\begin{aligned}
ds^2_{CP^3} = d\mu^2 &+ \frac{1}{4}\sin^2\mu\cos^2\mu\left[d\chi + \sin^2\alpha\ (d\psi + \cos\theta\ d\phi)\right]^2 \\
&+ \sin^2\mu\left[d\alpha^2 + \frac{1}{4}\sin^2\alpha\left(d\theta^2 + \sin^2\theta\ d\phi^2 + \cos^2\alpha\,(d\psi + \cos\theta\ d\phi)^2\right)\right]
\end{aligned}
$$

the map can be written as:

$$
\begin{aligned}
z_1/z_4 &= \tan\mu\ \cos\alpha\ e^{i\chi/2} \\
z_2/z_4 &= \tan\mu\ \sin\alpha\ \sin(\theta/2)\ e^{i\chi/2}\ e^{i(\psi-\phi)/2} \\
z_3/z_4 &= \tan\mu\ \cos\alpha\ \cos(\theta/2)\ e^{i\chi/2}\ e^{i(\psi+\phi)/2}.
\end{aligned}
\tag{D.3}
$$

The ratios z_i/z_4 are called inhomogeneous co-ordinates, and cover the patch $z_4 \neq 0$ with no identifications [246]. Taking the ranges of the angles (given in the main text, on page 155) into account, the trigonometric functions controlling the amplitudes of these rations are always positive in both (D.2) and (D.3). Analyzing the phases of the inhomogeneous co-ordinates of z_i/z_4 one can confirm that ranges of the remaining angles are correct.

D.2 Strings in homogeneous co-ordinates

It has been seen that it is convenient to use embedding co-ordinates for $\mathbb{R}^{n+1}$ to study bosonic string theory in S^n, using a constraint for the radius to be 1. This way one avoids the trigonometric algebra related with angular co-ordinates, and (in AdS/CFT) the correspondence between the R-symmetry generators and the rotations of the space becomes natural. We want to do an analogous construction for CP^3, by means of the homogeneous co-ordinates $\vec{z}$. In this case, we will need two constraints: $\rho^2 = 1$ and $\gamma = 0$.

Using Lagrange multipliers

We start by re-writing the metric for $\mathbb{R} \times CP^3$ in the form:

$$ds^2 = -\left(dX^0\right)^2 + d\bar{z}_i G_{ij} dz_j \qquad \text{with } \; G_{ij} = \frac{\delta_{ij}}{\rho^2} - \frac{z_i \bar{z}_j}{\rho^4}$$

In a conformal time-like gauge ($X^0 = \kappa\tau$), the Polyakov action becomes

$$\begin{aligned} S &= \int \frac{dx\, d\tau}{2\pi} R^2 \mathscr{L} \\ &= 2\sqrt{2\lambda} \int dx\, d\tau\, \mathscr{L} \\ 2\mathscr{L} &= \kappa^2 + \partial^a \bar{Z}_i G_{ij} \partial_a Z_j + \Lambda_\rho \left(\bar{Z}_i Z_i - 1\right) + i\Lambda_\gamma \left(Z_1 Z_2 Z_3 Z_4 - \bar{Z}_1 \bar{Z}_2 \bar{Z}_3 \bar{Z}_4\right), \end{aligned} \tag{D.4}$$

where $\Lambda_\gamma \in \mathbb{R}$, since the term in brackets is proportional to $2i \sin\gamma$, thus imaginary. In order to determine the Euler–Lagrange equations, we first set $\rho = 1$, simplifying $\partial G_{ij} / \partial Z_i$ etc. The Lagrange multipliers are simply determined from the parallel component of the equations (i.e. the product of $\bar{Z}_i$ with the equation of motion for Z_i), and the result is:

$$\Lambda_\rho - 4i \left(Z_1 Z_2 Z_3 Z_4\right) \Lambda_\gamma = \partial_\tau \bar{Z}_i \partial_\tau Z_i - 2 \left|\bar{Z}_i \partial_\tau Z_i\right|^2 - \partial_x \bar{Z}_i \partial_x Z_i + 2 \left|\bar{Z}_i \partial_x Z_i\right|^2,$$

where the factor 4 above comes from the number of complex embedding co-ordinates. The right-hand side of the equation is real, which implies $\Lambda_\gamma = 0$. Using this result, we can write the equation of motion for Z_i as

$$-\partial_\tau \left(G_{ij} \partial_\tau Z_j\right) + \partial_x \left(G_{ij} \partial_x Z_j\right) = Z_i \Lambda_\rho - \left(\bar{Z}_j \partial_\tau Z_j\right) \partial_t Z_i + \left(\bar{Z}_j \partial_x Z_j\right) \partial_x Z_i \,. \tag{D.5}$$

The Virasoro constraints are given by

$$-\kappa^2 + \partial_\tau \bar{Z}_i \, G_{ij} \, \partial_\tau Z_j + \partial_x \bar{Z}_i \, G_{ij} \, \partial_x Z_j = 0$$

$$\mathrm{Re}\left(\partial_\tau \bar{Z}_i\, G_{ij}\, \partial_x Z_j\right) = 0\,.$$

We should expand some more on the fact that $\Lambda_\gamma = 0$. Going back to the simple example of strings on the sphere, if we used a similar metric (in fact exactly ds^2_{sphere} from (D.1)):

$$2\mathscr{L} = 1 + \partial^a X_i \partial_a X_j g_{ij} + \Lambda(X^2 - 1), \qquad \text{with } g_{ij} = \frac{\delta_{ij}}{\rho^2} - \frac{X_i X_j}{\rho^4}$$

then we would also have $\Lambda = 0$, even though the equations of motion are the same as the ones obtained with $g_{ij} = \delta_{ij}$ (i.e. using ds^2_{flat}). In some sense the metric is directly enforcing the constraint, and the reason we had the other Lagrange multiplier $\Lambda_\rho \neq 0$ in the CP^3 case was because we set $\rho = 1$ in the beginning of the calculation.

Constraining S^7 *solutions*

In [193] (and others) they set up the problem of finding string solutions on CP^3 by first finding solutions on the sphere $S^7 \in \mathbb{C}^4$, and then demanding that the two Noether charges from ∂_γ vanish:

$$0 = C_0 \equiv \sum_{i=1}^{4} \mathrm{Im}\left(\bar{Z}_i \partial_\tau Z_i\right)\,, \qquad 0 = C_1 \equiv \sum_{i=1}^{4} \mathrm{Im}\left(\bar{Z}_i \partial_x Z_i\right).$$

This in fact happens for the RP^2 solution (6.20) given by [193], and for any solution on the larger RP^3 subspace of Section 6.4. Using the co-ordinates $\vec{w}$ from (6.21), the condition $w_3 = w_4 = 0$ which defines this subspace RP^3 implies $C_0 = C_1 = 0$, and at the same time reduces the equations of motion (D.5) to those for a sphere S^3 embedded in (w_1, w_2).

But more general solutions, like the CP^1 solution (6.18), will solve neither these constraints nor the equations of motion for $S^7 \subset \mathbb{C}^4$. Thus, the above conditions (solution on S^7, and $C_0 = C_1 = 0$) are certainly not necessary for a solution. Whether they are sufficient or not is not entirely clear.[2]

[2] A similar approach can be done to strings on the sphere. It consists on finding solutions in flat (embedding) space and then reject all of the ones which do not have $\rho = 1$. In this sphere case, solving the flat space equations and having $\rho = 1$ is a sufficient, not necessary, condition to find a solution.

For example, one can study loops of string rotating in S^3, where we find one critical speed at which they are solutions in unconstrained $\mathbb{R}^4$ as well [113]. But faster and slower motions are possible on the sphere which are not solutions in $\mathbb{R}^4$, such as the extreme cases of a point particle and a stationary hoop.

In Section 6.4 it was seen that in the subspace RP^3, the second term in the definition of charges J_i (6.9) vanishes, leaving just the conserved charge from rotational symmetry of the z_i plane one would expect in S^7. This term that vanishes is just $|Z_i|^2 C_0/\rho^4$, which does not vanish for the CP^1 case (6.18), see footnote 10 on page 161. Instead of the constraint $C_0 = 0$, the constraint that holds is $\sum_{i=1}^4 J_i = 0$, which follows immediately from the definition of the charges J_i (6.9).

D.3 Fluctuation Hamiltonian for the point particle

Start with the metric for the AdS_4 factor in the following form

$$ds^2_{AdS_4} = -\left(\frac{1+\vec{r}^2}{1-\vec{r}^2}\right)^2 dt^2 + \frac{4}{(1-\vec{r}^2)^2} d\vec{r}^2 \tag{D.6}$$

where $\vec{r} = r_i,\ i = 1,2,3$ are zero at the center of AdS, and t is AdS time (we will be using worldsheet space and time to be x, τ). For the CP^3 sector we will use yet other co-ordinates:[3]

$$\vec{z} = \left(e^{i\beta}\frac{1+\varepsilon}{\sqrt{2}},\, y_1 + iy_2,\, y_3 + iy_4,\, e^{-i\beta}\frac{1-\varepsilon}{\sqrt{2}}\right), \tag{D.7}$$

in terms of which $\rho^2 = \bar{z}_i z_i = 1 + \varepsilon^2 + \vec{y}^2$ (where we use the notation $\vec{y}^2 = y_j y_j$). The metric (6.4) can then be written as

$$\begin{aligned} ds^2_{CP^3} = &\frac{(1+\varepsilon^2)d\beta^2 + d\varepsilon^2 + d\vec{y}^2}{1+\varepsilon^2+\vec{y}^2} \\ &- \frac{(\varepsilon d\varepsilon + \vec{y}\cdot d\vec{y})^2 + (2\varepsilon d\beta + y_1 dy_2 - y_2 dy_1 + y_3 dy_4 - y_4 dy_3)^2}{(1+\varepsilon^2+\vec{y}^2)^2}. \end{aligned}$$

Joining the AdS and the CP^3 parts of the metric, and dropping R^2 in (6.3) (it will become

[3]The convenience of these co-ordinates (as opposed to the angles) is that we can easily identify the charges J_i in (6.9) with the ones for the magnons in section 6.4 and those for the dual gauge theory in section 6.1.

In order to cover the whole space, these co-ordinates will have to obey $\beta \in [0,\pi]$ and $\varepsilon \in [-1,1)$. This can be seen by connecting these co-ordinates with the inhomogeneous ones $z_1/z_4 = e^{i2\beta}(1+\varepsilon)/(1-\varepsilon)$ and z_2/z_4, z_3/z_4 (similar co-ordinates were used in [18]).

a prefactor in the action) the full metric becomes

$$
\begin{aligned}
ds^2 &= \frac{1}{4} ds^2_{AdS_4} + ds^2_{CP^3} \\
&= \left(-\tfrac{1}{4} - \vec{r}^2\right) dt^2 + d\vec{r}^2 + (1 - 4\varepsilon^2 - \vec{y}^2) d\beta^2 + d\varepsilon^2 + d\vec{y}^2 + \dots,
\end{aligned}
\tag{D.8}
$$

where we have expanded near $\vec{r} = \vec{y} = 0$, $\varepsilon = 0$ and present only the terms that will be needed. The point particle travels on the line $t = 2\tau$, $\beta = \tau$, and perturbations about this solution are given by:

$$
\begin{aligned}
t &= 2\tau + \tfrac{1}{\lambda^{1/4}} \tilde{t} & \vec{r} &= \tfrac{1}{\lambda^{1/4}} \tilde{\vec{r}} \\
\beta &= \tau + \tfrac{1}{\lambda^{1/4}} \tilde{\beta} & \varepsilon &= \tfrac{1}{\lambda^{1/4}} \tilde{\varepsilon} \\
& & \vec{y} &= \tfrac{1}{\lambda^{1/4}} \tilde{\vec{y}}.
\end{aligned}
\tag{D.9}
$$

The perturbations $\tilde{t}$ and $\tilde{\beta}$ will lead to modes which are pure gauge, but are needed for now to maintain conformal gauge.

In terms of the induced metric γ_{ab}, the Lagrangian is $\mathscr{L} = \frac{1}{2}(-\gamma_{00} + \gamma_{11})$ and the Virasoro constraints are $\gamma_{00} + \gamma_{11} = 0$ and $\gamma_{01} = 0$. The components we will need are:

$$
\begin{aligned}
\gamma_{00} &= G_{\mu\nu} \partial_\tau X^\mu \partial_\tau X^\nu \\
&= \frac{1}{\lambda^{1/4}} \left[-\partial_\tau \tilde{\tau} + 2 \partial_\tau \tilde{\beta} \right] \\
&\quad + \frac{1}{\sqrt{\lambda}} \left[-\frac{(\partial_\tau \tilde{t})^2}{4} + (\partial_\tau \tilde{\vec{r}})^2 + (\partial_\tau \tilde{\beta})^2 + (\partial_\tau \tilde{\varepsilon})^2 + (\partial_\tau \tilde{\vec{y}})^2 - 4 \tilde{\vec{r}}^2 - 4 \tilde{\varepsilon}^2 - \tilde{\vec{y}}^2 \right] \\
&\quad + \frac{1}{\lambda^{3/4}} \left[-4 \tilde{\vec{r}}^2 \partial_\tau \tilde{t} + \partial_\tau \tilde{\beta} \, (\dots) + \partial_\tau \tilde{\vec{y}} \cdot (\dots) \right] + o(\frac{1}{\lambda})
\end{aligned}
$$

(where $(\dots)$ indicates terms that won't be needed for this calculation) and

$$
\begin{aligned}
\gamma_{11} &= G_{\mu\nu} \partial_x X^\mu \partial_x X^\nu \\
&= \frac{1}{\sqrt{\lambda}} \left[-\frac{(\partial_x \tilde{t})^2}{4} + (\partial_x \tilde{\vec{r}})^2 + (\partial_x \tilde{\beta})^2 + (\partial_x \tilde{\varepsilon})^2 + (\partial_x \tilde{\vec{y}})^2 \right] + o(\frac{1}{\lambda}).
\end{aligned}
$$

We proceed to define the conserved charges of the string: the charge generated by time translation, Δ

$$\begin{aligned}\Delta &= 2\sqrt{2\lambda}\int dx\,\frac{\partial\mathscr{L}[\tau,\vec{r},\beta,\varepsilon,\vec{y}]}{\partial\,\partial_t\tau}\\ &= 2\sqrt{2}\lambda^{3/4}\int dx\,\frac{\partial\tilde{\mathscr{L}}\left[\tilde{\tau},\tilde{\vec{r}},\tilde{\beta},\tilde{\varepsilon},\tilde{\vec{y}}\right]}{\partial\,\partial_t\tilde{\tau}}\end{aligned}$$

and the charge generated by rotation of the z_i complex plane, J_i [4]

$$\begin{aligned}J_1 &= 2\sqrt{2\lambda}\int dx\,\frac{\partial\mathscr{L}}{\partial\,\partial_\tau(\arg Z_1)}\\ &= 2\sqrt{2\lambda}\int dx\left[\frac{\mathrm{Im}\,(\bar{Z}_1\partial_\tau Z_1)}{\rho^2}-\frac{|Z_1|^2\sum_i\mathrm{Im}\,(\bar{Z}_i\partial_\tau Z_i)}{\rho^4}\right]\end{aligned} \tag{D.10}$$

$$J_4 = 2\sqrt{2\lambda}\int dx\left[\frac{\mathrm{Im}\,(\bar{Z}_4\partial_\tau Z_4)}{\rho^2}-\frac{|Z_4|^2\sum_i\mathrm{Im}\,(\bar{Z}_i\partial_\tau Z_i)}{\rho^4}\right].$$

Using the above co-ordinates (D.7) and mode expansion (D.9), we get

$$\begin{aligned}\Delta &= \sqrt{2}\int dx\left[\sqrt{\lambda}+\frac{\lambda^{1/4}}{2}\partial_\tau\tilde{t}+4\tilde{\vec{r}}^2+o(\frac{1}{\lambda^{1/4}})\right] \qquad \text{(D.11)}\\ J_1 &= \sqrt{2}\int dx\left[\sqrt{\lambda}+\lambda^{1/4}\partial_\tau\tilde{\beta}-4\tilde{\varepsilon}^2-\tilde{\vec{y}}^2+(\tilde{y}_2\partial_\tau\tilde{y}_1-\tilde{y}_1\partial_\tau\tilde{y}_2+\tilde{y}_4\partial_\tau\tilde{y}_3-\tilde{y}_3\partial_\tau\tilde{y}_4)+o(\frac{1}{\lambda^{1/4}})\right]\\ J_4 &= \sqrt{2}\int dx\left[-\sqrt{\lambda}-\lambda^{1/4}\partial_\tau\tilde{\beta}+4\tilde{\varepsilon}^2+\tilde{\vec{y}}^2+(\tilde{y}_2\partial_\tau\tilde{y}_1-\tilde{y}_1\partial_\tau\tilde{y}_2+\tilde{y}_4\partial_\tau\tilde{y}_3-\tilde{y}_3\partial_\tau\tilde{y}_4)+o(\frac{1}{\lambda^{1/4}})\right].\end{aligned}$$

These charges diverge as $\lambda\to\infty$, but for the particular linear combination used below, the $o(\sqrt{\lambda})$ terms cancel. The $o(\lambda^{1/4})$ terms, linear in the fluctuations, can be re-written as

[4] In the derivation of these charges we treated $Z_1,\ldots,Z_4$ as independent fields, even though they are in fact related through $\vec{Z}\sim\lambda\vec{Z}$, which defines CP^3 from $\mathbb{C}^4$. We do this before using the parametrization (D.7), in which we have fixed some of this gauge freedom by writing only six (and not eight) real co-ordinates.

quadratic $o(1)$ terms using the Virasoro constraint $\gamma_{00}+\gamma_{11}=0$. We finally obtain

$$\begin{aligned}\Delta-&\frac{J_1-J_4}{2}\\ &=\frac{\sqrt{2}}{2}\int dx\Big[(\partial_\tau\tilde{\vec{r}})^2+(\partial_x\tilde{\vec{r}})^2+4\tilde{\vec{r}}^2+(\partial_\tau\tilde{\varepsilon})^2+(\partial_x\tilde{\varepsilon})^2+4\tilde{\varepsilon}^2+(\partial_\tau\tilde{\vec{y}})^2+(\partial_x\tilde{\vec{y}})^2+\tilde{\vec{y}}^2\\ &\qquad\qquad-\frac{(\partial_\tau\tilde{t})^2}{4}-\frac{(\partial_x\tilde{t})^2}{4}+(\partial_\tau\tilde{\beta})^2+(\partial_x\tilde{\beta})^2\Big]+o(\frac{1}{\lambda^{1/4}}).\end{aligned}$$

The last line of the above expression includes all of the gauge modes, which generate infinitesimal reparametrizations, and will be dropped in semiclassical quantization. We are then left with the Hamiltonian[5] $\Delta-\frac{J_1-J_4}{2}=\sqrt{2}\int dx\,\mathscr{H}$, where[6]

$$\mathscr{H}=\frac{1}{2}\left[\,(\partial_\tau\tilde{\vec{r}})^2+(\partial_x\tilde{\vec{r}})^2+4\tilde{\vec{r}}^2\;+\;(\partial_\tau\tilde{\varepsilon})^2+(\partial_x\tilde{\varepsilon})^2+4\tilde{\varepsilon}^2\;+\;(\partial_\tau\tilde{\vec{y}})^2+(\partial_x\tilde{\vec{y}})^2+\tilde{\vec{y}}^2\,\right].$$

This Hamiltonian describes eight massive modes: the three $\tilde{r}_i$ in AdS_4, together with the $\tilde{\varepsilon}$ and the four $\tilde{y}_i$ in CP^3. Note that one of the CP^3 modes, $\tilde{\varepsilon}$, has acquired the same mass as the AdS modes $\tilde{\vec{r}}$ [18]. The same list of masses was also found by [217, 192, 193] in the Penrose limit, and by [43, 45, 186, 18] when studying modes of spinning strings in the $AdS_2\times S^1$ subspace.

D.4 Expansions of charges at finite-J for CP^3 magnons

When working out finite-J corrections to the various algebraic curve magnons, we expanded the asymptotic charges in δ, defined by (5.31) $Y^\pm=X^\pm(1\pm i\delta e^{\pm i\phi})$. We used these expansions to work out the correction $\delta\mathscr{E}$, for example (7.11). Here we give the expansions of these charges explicitly.

We write all three cases at once, by setting $m=1$ for the 'small' magnon and $m=2$ for

[5] $\mathscr{H}$ is the two-dimensional Hamiltonian that one would obtain from the quadratic part of the fluctuation Lagrangian $\mathscr{L}=\frac{1}{2}(-\gamma_{00}+\gamma_{11})$, by dropping terms linear in time derivative and reversing the signs of the terms quadratic in the time derivative. But without dropping these $o(\lambda^{1/4})$ terms, the (actual) string Hamiltonian is fixed to zero by the Virasoro constraint $\gamma_{00}+\gamma_{11}=0$, condition that we have used to derive $\mathscr{H}$.

[6] There are some obvious charges one could add to $\Delta-(J_1-J_4)/2$, still keeping it finite, such as J_2 and J_3. These would add terms like $\tilde{y}_2\partial_\tau\tilde{y}_1-\tilde{y}_1\partial_\tau\tilde{y}_2$ to $\mathscr{H}$.

the ‘pair’ and ‘big’. Thus we always have $X^{\pm} = r e^{\pm ip/2m}$.

First, the momentum is $p = p_0 + \delta p_{(1)} + \delta^2 p_{(2)} + o(\delta^3)$, where

$$\begin{aligned}
p_{(1)} &= m\cos(\phi) \\
p_{(2)} &= \frac{3m}{8}\sin(2\phi).
\end{aligned}$$

Next, the angular momentum is $J = J_{(0)} + \delta J_{(1)} + \delta^2 J_{(2)} + o(\delta^3)$, with

$$\begin{aligned}
J_{(0)} &= 2\Delta - 4gm\frac{r^2+1}{r}\sin\left(\frac{p_0}{2m}\right) \\
J_{(1)} &= -\frac{2gm}{r}\left[r^2\cos\left(\frac{p_0}{2m}+\phi\right)+\cos\left(\frac{p_0}{2m}-\phi\right)\right] \\
J_{(2)} &= \frac{3gm}{2r}\sin\left(\frac{p_0}{2m}-2\phi\right).
\end{aligned}$$

Finally, the second angular momentum is $Q = Q_{(0)} + \delta Q_{(1)} + \delta^2 Q_{(2)} + o(\delta^3)$, where for the small and pair cases we have

$$\begin{aligned}
Q_{(0)} &= 4gm\frac{r^2-1}{r}\sin\left(\frac{p_0}{2m}\right) \\
Q_{(1)} &= \frac{2gm}{r}\left[r^2\cos\left(\frac{p_0}{2m}+\phi\right)-\cos\left(\frac{p_0}{2m}-\phi\right)\right] \\
Q_{(2)} &= \frac{3gm}{2r}\sin\left(\frac{p_0}{2m}-2\phi\right).
\end{aligned}$$

For the big magnon, $Q = 0$ to this order in δ. We used in the dispersion relation instead Q_u which (as a function of $X^{\pm}$) is the Q from the small magnon. For the purpose of these expansions (functions of r and p) it is easier to think of this as $Q_u = \frac{1}{2}Q_{\text{pair}}$ since the big and pair cases both have $m = 2$.

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