

General Spherically Symmetric Gravity Interacting with Massive Complex Scalar Field

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Abstract

Interaction of gravity with complex massive scalar field is considered by coupling the scalar field equation to the Einstein field equation whose source term is given by the Energy Momentum tensor of the scalar field. Assuming general spherically symmetric space time, the scalar field results to depend only on time and radial coordinate. In turn this implies consistency of the scheme that is, both the vanishing of the divergence of the Energy Momentum tensor of the scalar field and the independence of its trace from the angular variables. The explicit scheme of coupled partial differential equations is partially integrated twice on general ground. This leads to a partial differential equation that can be interpreted as a generalized Kepler like equation whose solution is difficult. To have some information about explicit solutions, a method of variable separation integration is employed by considering suitable special assumptions. The cases considered are essentially those in which the scalar field is assumed to depend on one only variable. There result situations in which, among other, a homogenization effect is recovered. Also a cosmological interpretation of the results could be possible on condition of solving systems of coupled non-linear separated differential time equations

Keywords: General spherically symmetric space-time; Scalar field equation; Einstein field equation; Scalar field-gravity interaction

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1 Introduction

As is well known the interaction of gravity and fields is a problem of interest in physics. In General Relativity the problem can be formulated by coupling the Einstein field equation to the equation of a field of given spin. The interaction is generally formulated in a canonical way. The source of the Einstein field equation is assumed to be represented by the Energy Momentum tensor of the field in a context in which both the Einstein and the spin field equation rely on the same tensor metric $g_{\mu\nu}(x)$. The solution of the problem appears in general very difficult, not only by increasing the spin of the field, but also according to preliminary assumptions on $g_{\mu\nu}(x)$.

Applications of such scheme are very wide and run from collapsing field by gravity (e., g., [5, 16]), to cosmological implications of different kind such as black holes formation, cosmological inflation and accelerated expansion of the Universe (e., g., [8, 15, 7, 12, 1]). The mentioned interaction picture was already considered in the context of the Robertson-Walker (RW) space-time [20]. There it has been shown that the interaction of RW gravity and massive field of spin $s > 0$ has no standard solutions. The standard solutions proposed in [20] are the solution that are essentially obtained by an elementary variable separation of the equations. Even if they do not exhaust the totality of the solutions they represent anyway a very large class of “treatable” solutions.

A first step in generalizing the RW space time case is of relaxing the homogeneity condition on the space-time, e., g., by considering the Lemaitre-Tolman-Bondi (LTB) space-time [11, 14, 3] whose interest lies also in the fact that it is the base of the LTB cosmological model. A study of the interaction of LTB gravity with a massive (real or complex) scalar field $\phi(x)$ was done in [19, 21, 22, 23]. The result is that, under the assumption $\phi = \phi(r, t)$, factored solutions do indeed exist in such context but are very few. There are however non trivial solutions for mass-less scalar field. Instead factored solutions do not exist for massive real field such that $\phi = \phi(t)$ [19, 21, 22, 23].

In the present paper the interaction of the general spherically symmetric gravity with (minimally coupled) massive complex scalar field is considered. The study represents a generalization of [21] and it is performed similarly. Here it is expected to have solutions richer than in the LTB model. Indeed there are four functions to be determined, instead of three of the LTB model, satisfying the same number of differential equations. For what concerns the consistency of the scheme it is first shown that the spherical symmetry of the metric implies that the scalar field must be of the form $\phi = \phi(r, t)$. This is consequence of the form of the Ricci tensor $R_{\mu\nu}$ and Ricci scalar R (that both are explicitly calculated). It also simplifies the expression of the scalar field equation. In turn that property ensures the formal consistency of the Einstein field equation: it is checked that the divergence of the Energy Momentum tensor of the scalar

field vanishes and the expression of the Ricci tensor does effectively depend only on the r, t variables.

The explicit differential equations of the scheme are difficult to solve in general. It is however possible to perform two partial integration on general grounds. This leads to a Kepler-like equation that can be interpreted as a generalization of a similar equation in the LTB model [21] that in turn is a generalization of the Kepler-like equation of the LTB cosmological model [9]. Such equation greatly simplifies in case $\phi = \phi(t)$ or $\phi = \phi(r)$, but contrarily to the LTB case [21], the integration remains a difficult task. It is possible to proceed in the integration under further special assumptions. In fact, in the present paper, the case $\phi = \phi(t)$ is studied together with a factorization assumptions on the components of the metric tensor. In the study it is discriminating the value the separation constant c_o , relative to the R_{tr} equation, takes. An elementary separation of the remaining differential equations of the scheme is possible only for $c_o = 1$ or $c_o = 0$. For these two value, separated radial and time solutions exist for both mass less and massive scalar field. (In the LTB model solutions do not exist for massive real scalar field [21, 23]. Explicit solutions of the separated time equations not always are given. This may be a limitation of the study because it is not always possible to establish whether solutions exist suitable for a cosmological interpretation of the scheme, e., g., solutions admitting at least an initial inflationary and a late accelerated expansion of the universe.

By the same factored assumption it is finally noted that all the equations of the model separates also for $\phi = \phi(r)$ with $c_o = 0$ in case of massive scalar field and with $c_o = 1$ in case of mass less scalar field.

2 General formulation

In General Relativity the interaction of a massive complex scalar field $\phi(x^\mu)$ with the gravity of a space-time of metric tensor $g_{\mu\nu}$ can be described by coupling the scalar field equation (in the minimally coupled case [2]) to the Einstein field equation:

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = kT_{\mu\nu} \quad (|k| = 8\pi G/c^4) \quad (1)$$

$$D(\phi) \equiv \nabla^\alpha \nabla_\alpha \phi + m_o^2 \phi = 0 \quad (2)$$

$$T_{\mu\nu}(\phi) = \frac{1}{2} \left[\partial_\mu \phi \partial_\nu \bar{\phi} + \partial_\nu \phi \partial_\mu \bar{\phi} - g_{\mu\nu} \partial^\alpha \phi \partial_\alpha \bar{\phi} + m_o^2 g_{\mu\nu} \phi \bar{\phi} \right] \quad (3)$$

($\partial_x = \partial/\partial x$) where m_o is the mass of the field and $T_{\mu\nu}$ its energy momentum tensor [10]. By taking the trace in (1), the scalar R can be expressed as a

function of $\phi, \partial_\mu \phi$ so that the equation (1) can finally be written:

$$R_{\mu\nu} = \frac{k}{2} [\partial_\mu \phi \partial_\nu \bar{\phi} + \partial_\nu \phi \partial_\mu \bar{\phi} - m_o^2 g_{\mu\nu} \phi \bar{\phi}] \quad (4)$$

In the following the object is to study the interaction scheme of equations (1-4) in the general spherical symmetric metric. To that end we refer to [4, 9] Accordingly the metric tensor $g_{\mu\nu}$ is assumed to be defined by

$$ds^2 = e^{2\gamma(r,t)} dt^2 - e^{2\alpha(r,t)} dr^2 - Y^2(r,t) (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (5)$$

For the following considerations it is useful to know the explicit expressions of $R_{\mu\nu}$ and R . By setting $R_{\mu\nu} = -R_{\mu\nu\sigma}^\sigma$ where $R_{\mu\nu\sigma\alpha}$ are the expressions given in [4], the non vanishing components are:

$$-R_{tt} = \frac{\gamma'' + \gamma'^2 - \alpha'\gamma' + 2\gamma' \frac{Y'}{Y}}{e^{2(\alpha-\gamma)}} - \ddot{\alpha} - \dot{\alpha}^2 + \dot{\alpha}\dot{\gamma} - 2\frac{\ddot{Y}}{Y} + 2\dot{\gamma}\frac{\dot{Y}}{Y} \quad (6)$$

$$-R_{rr} = \frac{\ddot{\alpha} + \dot{\alpha}^2 - \dot{\alpha}\dot{\gamma} + 2\dot{\alpha}\frac{\dot{Y}}{Y}}{e^{2(\gamma-\alpha)}} - \gamma'' - \gamma'^2 + \alpha'\gamma' - 2\frac{Y''}{Y} + 2\alpha'\frac{Y'}{Y} \quad (7)$$

$$-R_{\theta\theta} = \frac{Y\ddot{Y} - \dot{\gamma}Y\dot{Y} + \dot{\alpha}\dot{Y}Y + \dot{Y}^2}{e^{2\gamma}} - \frac{YY'\gamma' + YY'' - \alpha'YY' + Y'^2}{e^{2\alpha}} + 1 \quad (8)$$

$$-R_{\varphi\varphi} = -\sin^2 \theta R_{\theta\theta} \quad (9)$$

$$-R_{tr} = -R_{rt} = -2\left[\frac{\dot{Y}'}{Y} - \dot{\alpha}\frac{Y'}{Y} - \frac{\dot{Y}}{Y}\gamma'\right] \quad (10)$$

' = $\partial/\partial r$, $\dot{} = \partial/\partial t$. From these expressions the Ricci scalar $R = R_\mu^\mu$ can be calculated :

$$R = 2e^{-2\alpha} \left[\gamma'' + \gamma'^2 - \alpha'\gamma' + 2(\gamma' - \alpha') \frac{Y'}{Y} + 2\frac{Y''}{Y} + \frac{Y'^2}{Y^2} \right] \quad (11)$$

$$-2e^{-2\gamma} \left[\ddot{\alpha} + \dot{\alpha}^2 - \dot{\alpha}\dot{\gamma} + 2\frac{\ddot{Y}}{Y} + 2(\dot{\alpha} - \dot{\gamma}) \frac{\dot{Y}}{Y} + \frac{\dot{Y}^2}{Y^2} \right] \quad (12)$$

One can check that the $R_{\mu\nu}$'s and R obtained above, when specialized to the case of Lemâitre-Tolman-Bond metric, coincide with those obtained in [6, 19, 21]. The same holds for the special spherical symmetry case considered in [17].

3 Consistency conditions

One has to take into account that the Einstein tensor has zero divergence. Moreover, by taking trace of (4), and from (11), R can depend only on the r, t variables. Therefore the following constraints must be satisfied

$$\nabla^\mu T_{\mu\nu} = 0, \quad \nu = 0, 1, 2, 3 \quad (13)$$

$$\partial_j (\partial^\mu \phi \partial_\mu \bar{\phi} - 2m_o^2 \phi \bar{\phi}) = 0, \quad j = \theta, \varphi \quad (14)$$

The condition (14) is satisfied in the present scheme because, as it will be presently seen, the field ϕ is forced to depend only on r, t . Indeed from (4) one has

$$R_{t\theta} \equiv 0 = \frac{k}{2}(\phi_t \bar{\phi}_\theta + \bar{\phi}_t \phi_\theta) \quad (15)$$

$$R_{t\varphi} \equiv 0 = \frac{k}{2}(\phi_t \bar{\phi}_\varphi + \bar{\phi}_t \phi_\varphi) \quad (16)$$

$$R_{\theta\theta} \equiv R_{\theta\theta}(r, t) = \frac{k}{2}(2\phi_\theta \bar{\phi}_\theta + m_o^2 Y^2 \phi \bar{\phi}) \quad (17)$$

$$R_{\varphi\varphi} \equiv \sin^2 \theta R_{\theta\theta} = \frac{k}{2}(2\phi_\varphi \bar{\phi}_\varphi + m_o^2 Y^2 \phi \bar{\phi}) \quad (18)$$

$$R_{\theta\varphi} \equiv 0 = \frac{k}{2}(\phi_\theta \bar{\phi}_\varphi + \phi_\varphi \bar{\phi}_\theta) \quad (19)$$

($\phi_x = \partial\phi/\partial x$). From (15), (16) one obtains $\phi_\varphi \bar{\phi}_\theta = \bar{\phi}_\varphi \phi_\theta$ so that (19) implies $\phi_\varphi \bar{\phi}_\theta = 0$. If $\phi_\varphi = 0$, then by comparing (17), (18) there follows $\partial_\theta \phi = 0$. Similarly if $\phi_\theta = 0$ then $\partial_\varphi \phi = 0$. Therefore $\phi = \phi(r, t)$.

In order to prove that also condition (13) is fulfilled, it is useful to note that the angular dependence of the solution of the scalar field equation (2) can be separated. Indeed by setting $\phi(t, r, \theta, \varphi) = \phi(r, t)\chi(\theta, \varphi)$ in the general metric (5), by requiring regularity conditions for $\theta = 0, \varphi = 0, 2\pi$ (See, e.g., [18] with the identification $Y^2 \equiv \exp \beta(r, t)$), one is left with:

$$\chi(\theta, \varphi) = Y_{lm}(\theta, \varphi), \quad l = 0, 1, 2, \dots, \quad m = \pm l, \pm(l-1), \dots, \pm 1, 0 \quad (20)$$

$$e^{-2\gamma} \ddot{\phi} - e^{-2\alpha} \phi'' + e^{-2\alpha} \left(\alpha' - \gamma' - 2 \frac{Y'}{Y} \right) \phi' + e^{-2\gamma} \left(\dot{\alpha} - \dot{\gamma} + 2 \frac{\dot{Y}}{Y} \right) \dot{\phi} + \left(\frac{\lambda}{Y^2} + m_o^2 \right) \phi = 0, \quad \lambda = l(l+1) \quad (21)$$

the Y_{lm} 's being the spherical harmonics. The solution of (2) can then be represented by

$$\phi(t, r, \theta, \varphi) = \sum_{lm} \phi_{lm}(r, t) Y_{lm}(\theta, \varphi) \quad (22)$$

with ϕ_{lm} satisfying (21). Therefore

$$\phi = \phi(r, t) \quad \Rightarrow \quad l = 0 \quad \Rightarrow \quad \lambda = 0 \quad (23)$$

Hence the scalar field depends only on r and t and the scalar field equation, $D(\phi) = 0$, is given by equation (20) with $\lambda = 0$.

Accordingly it is now possible to see the validity of the consistency equation (13). Indeed one has

$$\nabla^\mu T_{\mu\nu} = \nabla^\mu [\partial_\mu \phi \partial_\alpha \bar{\phi} + \partial_\mu \bar{\phi} \partial_\alpha \phi] - \nabla^\mu [g_{\mu\alpha} (\partial_\beta \phi \partial^\beta \bar{\phi}) - m_o^2 \phi \bar{\phi}] \quad (24)$$

$$= [\partial^\mu \phi (\partial_\mu \partial_\alpha \bar{\phi} - \Gamma_{\alpha\mu}^k \partial_k \bar{\phi}) + \partial_\alpha \bar{\phi} (\partial_\mu \partial^\mu \phi + \Gamma_{\mu k}^\mu \partial^k \phi)] + [\phi \leftrightarrow \bar{\phi}] - \partial_\alpha (\partial^\beta \partial_\beta \bar{\phi} - m_o^2 \phi \bar{\phi}) \quad (25)$$

The calculations can be performed by means of the explicit expression of the Christoffel symbols in the metric (5) (see e.g., [4]). With some care in grouping the terms one finally gets

$$\nabla^\mu T_{\mu\alpha} = 0, \quad \alpha = \theta, \varphi \quad (26)$$

$$\nabla^\mu T_{\mu r} = \bar{\phi}' D(\phi) + \phi' D(\bar{\phi}) \quad (27)$$

$$\nabla^\mu T_{\mu t} = \bar{\phi} \dot{D}(\phi) + \dot{\phi} D(\bar{\phi}) \quad (28)$$

where $D(\phi)$ is now the right hand side of (20) with $\lambda = 0$. Therefore $\nabla^\mu T_{\mu\nu} = 0$ on account of assumption (2). These properties follows from the spherical symmetry assumptions and therefore holds also in the LTB model as explicitly noted in [21].

4 Explicit interacting scheme

According to the previous results the study of the interaction of the complex scalar field with LTB gravity can be reduced to the study of the system of equations:

$$\begin{aligned} R_{tt} &\equiv \ddot{\alpha} + \dot{\alpha}^2 - \dot{\alpha}\dot{\gamma} + 2\frac{\ddot{Y}}{Y} - 2\dot{\gamma}\frac{\dot{Y}}{Y} - e^{2(\gamma-\alpha)}[\gamma'' + \gamma'^2 - \alpha'\gamma' + 2\gamma'\frac{Y'}{Y}] \\ &= k(\dot{\phi}\bar{\phi} - \frac{1}{2}m_0^2\phi\bar{\phi}e^{2\gamma}) \end{aligned} \quad (29)$$

$$\begin{aligned} R_{rr} &\equiv \gamma'' + \gamma'^2 - \alpha'\gamma' + 2\frac{Y''}{Y} - 2\alpha'\frac{Y'}{Y} - e^{2(\alpha-\gamma)}[\ddot{\alpha} + \dot{\alpha}^2 - \dot{\alpha}\dot{\gamma} + 2\dot{\alpha}\frac{\dot{Y}}{Y}] \\ &= k(\phi'\bar{\phi}' + \frac{1}{2}e^{2\alpha}m_0^2\phi\bar{\phi}) \end{aligned} \quad (30)$$

$$\begin{aligned} R_{\theta\theta} &\equiv e^{-2\alpha}[YY'\gamma' + YY'' - \alpha'YY' + Y'^2] - 1 \\ &\quad - e^{-2\gamma}[Y\ddot{Y} - \dot{\gamma}Y\dot{Y} + \dot{\alpha}\dot{Y}Y + \dot{Y}^2] = \frac{k}{2}m_0^2\phi\bar{\phi}Y^2 \end{aligned} \quad (31)$$

$$R_{tr} \equiv 2\left[\frac{\dot{Y}'}{Y} - \dot{\alpha}\frac{Y'}{Y} - \frac{\dot{Y}}{Y}\gamma'\right] = \frac{k}{2}(\dot{\phi}\bar{\phi}' + \bar{\phi}\phi') \quad (32)$$

$$\frac{\ddot{\phi}}{e^{2\gamma}} - \frac{\phi''}{e^{2\alpha}} + \frac{(\alpha' - \gamma' - 2\frac{Y'}{Y})\phi'}{e^{2\alpha}} + \frac{(\dot{\alpha} - \dot{\gamma} + 2\frac{\dot{Y}}{Y})\dot{\phi}}{e^{2\gamma}} + m_0^2\phi = 0 \quad (33)$$

The complete solution of the equations (29)-(33) seems very difficult to be performed exactly. Integration steps can however be performed in general. Indeed, by following [19, 21], a time integration of equation (31) can be evaluated. There results that the function α can be expressed in terms of the other ones:

$$e^{2\alpha} = \frac{Y'^2}{1 + 2E(r)} e^{-\int dt \frac{Y(r,t)}{Y'(r,t)} A(r,t)}, \quad A = (k/2)[\dot{\phi}\bar{\phi}' + \bar{\phi}\phi'] + 2\dot{Y}\gamma' \quad (34)$$

$1+2E(r)$ the integration constant. By using the expression of α obtained from (34), the R_{tt} equation (29) can be recast into the form:

$$\ddot{Y}Y^2 = -M(r, t) \equiv - \int dr Y^2 Y' \left\{ \dot{\gamma} \frac{(Y^2 \dot{Y})'}{Y^2 Y'} + \frac{1}{2} \frac{d}{dt} \left(\frac{YA}{Y'} \right) - \frac{1}{4} \left(\frac{YA}{Y'} \right)^2 + \right. \\ \left. + \frac{Y \dot{Y}'}{Y'^2} A - \frac{\left[\gamma'' + \gamma'^2 - \gamma' \left(\alpha' - \frac{2Y'}{Y} \right) \right]}{e^{2(\alpha-\gamma)}} + k \left[\dot{\phi} \bar{\phi} - \frac{m_o^2}{2} \phi \bar{\phi} e^{2\gamma} \right] \right\} \quad (35)$$

By combining the equations (29-31) and taking into account the result (34) one finally obtains:

$$\frac{\dot{Y}^2}{2} - \frac{M(r, t)}{Y} - Y \dot{Y} \dot{\gamma} + \frac{1}{2} e^{2\gamma} = \frac{1}{2} e^{2(\gamma-\alpha)} Y'^2 + Y Y' \gamma' e^{2(\gamma-\alpha)} + \\ + \frac{k}{4} Y^2 [\dot{\phi} \bar{\phi} + \phi' \bar{\phi}' e^{2(\gamma-\alpha)} - m_o^2 e^{2\gamma} \phi \bar{\phi}] \quad (36)$$

The last equation can be interpreted as a generalized Kepler like equation as in the case of Lemâitre Tolman Bondi (LTB) gravity interacting with scalar field (e.g., [19, 21]) that in turn is a generalization of the Kepler like equation to which the LTB cosmological model can be reported [9].

The general integration steps of the scheme have the advantage of allowing the physical interpretation of eq. (36). An analytical solution of the resulting equations is still a difficult task. The form of (36) seems however suitable in case ϕ depends on only one of the variables t, r , as it already appears in (32). Indeed in such case the equation greatly simplifies, but it still remains of difficult solution. If one sets $\gamma = 0$ the interaction scheme falls into the context of the LTB space time that was discussed in [19, 21, 23]. There, "factorized" form for both metric coefficients and scalar field have been determined for $m_o = 0$ and real or complex scalar field. Instead in case $m_o \neq 0$ a particular solution exists for complex scalar field, but none for real scalar field. However determination of non "factorized" solutions was left unsolved.

In the present paper there is the same number of differential equations of the LTB model, but there is the further degree of freedom given by the presence of the non trivial g_{tt} metric coefficient. The explicit solution of the differential equations remains difficult and we are able to proceed only in the search of factorized solutions.

5 Mass-less time dependent scalar field

The equations (29)-(33) are first studied under the assumptions:

$$m_o = 0, \quad \phi = \phi(t) \quad (37)$$

$$\alpha(t, r) = a(r) + \tau(t), \quad \gamma(t, r) = c(r) + \eta(t), \quad Y(t, r) = y(r)\xi(t) \quad (38)$$

The assumptions generalizes what done in LTB space time [13,14].

The R_{tr} equation (32) gives then: $\gamma'(y/y') = c_o = 1 - \dot{\tau}\dot{\xi}/\dot{\xi}$ or:

$$e^{\tau(t)} = \tau_o(\xi(t))^{1-c_o}, \quad e^{c(r)} = y_o(y(r))^{c_o} \quad (39)$$

c_o, y_o, τ_o the separation and integration constants. Hence, for $c_o \neq 0, 1$ both τ and c are not constant. By these assumptions and preliminary results the scalar field equation (33) can be integrated:

$$\dot{\phi} = \phi_o \frac{e^{\eta-\tau}}{\xi^2} \quad (40)$$

ϕ_o the constant of integration. By proceeding similarly, the R_{tt} eq. (29) separates into:

$$[\ddot{\tau} + \dot{\tau}^2 + 2\frac{\ddot{\xi}}{\xi} - \dot{\eta}(\dot{\tau} + \frac{\dot{\xi}}{\xi}) - k|\dot{\phi}|^2] = c_2 e^{2(\tau-\eta)} \quad (41)$$

$$[c'' + c'^2 + c'(2\frac{y'}{y} - a')]e^{2(c-a)} = c_2 \quad (42)$$

The R_{rr} eq. (30) separates into:

$$e^{2(\tau-\eta)}[\ddot{\tau} + \dot{\tau}^2 - \dot{\tau}\dot{\eta} + 2\dot{\tau}\frac{\dot{\xi}}{\xi}] = c_3 \quad (43)$$

$$e^{2(c-a)}[c'' + c'^2 - a'(c' + 2\frac{y'}{y}) + 2\frac{y''}{y}] = c_3 \quad (44)$$

c_2, c_3 the separation constants. As to $R_{\theta\theta}$, eq. (31), one obtains

$$e^{2(c-a)}[c'\frac{y'}{y} + \frac{y''}{y} - a'\frac{y'}{y} + \frac{y'^2}{y^2}] - e^{2(\tau-\eta)}[\frac{\ddot{\xi}}{\xi} + (\dot{\tau} - \dot{\eta})\frac{\dot{\xi}}{\xi} + \frac{\dot{\xi}^2}{\xi^2}] = \frac{e^{2\tau+2c}}{\xi^2 y^2} \quad (45)$$

that cannot be separated in elementary manner (on account of the right hand term) unless $c_o = 1$ or $c_o = 0$. By taking into account (38) for these two choices, eq. (45) separates into:

$$y_o^2 e^{-2a}[\frac{y''}{y} - a'\frac{y'}{y} + \frac{y'^2}{y^2}] - \frac{y_o^2 \tau_o^2}{y^2} = c_4 \quad (c_o = 0) \quad (46)$$

$$\ddot{\xi}\xi + \dot{\xi}\dot{\xi}(\dot{\tau} - \dot{\eta}) + \dot{\xi}^2 = \frac{c_4}{\tau_o^2} e^{2\eta} \quad (c_o = 0) \quad (47)$$

$$y_o^2 y^2 e^{-2a}[c'\frac{y'}{y} + \frac{y''}{y} - a'\frac{y'}{y} + \frac{y'^2}{y^2}] = c_4 \quad (c_o = 1) \quad (48)$$

$$\tau_o^2 e^{-2\eta}[\frac{\ddot{\xi}}{\xi} - \dot{\eta}\frac{\dot{\xi}}{\xi} + \frac{\dot{\xi}^2}{\xi^2}] + \frac{\tau_o^2 y_o^2}{\xi^2} = c_4 \quad (c_o = 1) \quad (49)$$

5.1 The special choice $c_o = 1$.

In the present case $c_o = 1$ one has so that $e^\tau = \tau_o$, $e^c = y_o y$ and $\dot{\phi} = (\phi_o \exp \eta)/(\tau_o \xi^2)$. As to the separated radial equation one has, by (43), $c_3 = 0$. Then (44) with $c' = y'/y$ give

$$y'' - a' y' = 0, \quad \Rightarrow \quad e^a = c_5 y' \quad (50)$$

(c_5 integration constant). Then (42) (48) are automatically satisfied and (by using (50)) if the constants are related by

$$2y_o^2 = c_2 c_5^2, \quad 2y_o^2 = c_4 c_5^2 \quad (51)$$

Hence $c_2 = c_4$. Therefore, so far, the solution takes the form

$$ds^2 = y_o^2 e^{2\eta} y^2 dt^2 - \tau_o^2 c_5^2 y'^2 dr^2 - y^2 \xi^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (52)$$

$$= e^{2\eta(t)} y^2 dt^2 - dy^2 - y^2 \xi^2(t) (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (53)$$

by suitable choice of the constants of integration and by interpreting y as the new radial coordinate, being it still an arbitrary function.

For what concerns the solution of the separated time equations one has that (43) is automatically satisfied because of $c_3 = 0$ and $\dot{\tau} = 0$. One is then left with (41) and (49), namely two equation in the two functions $\eta(t)$, $\xi(t)$ a priori compatible. It is possible to make some further comments in the special case $\eta = 0$. For such condition the equations become with ($\tau_o = 1$):

$$2\ddot{\xi} - k|\dot{\phi}|^2 \xi = c_2 \xi, \quad \ddot{\xi} \xi + \dot{\xi}^2 + y_o^2 = c_4 \xi^2 \quad (\eta = 0) \quad (54)$$

By putting $\dot{\xi} = u(\xi)$, both equation can be integrated once to get respectively

$$u^2 \equiv \dot{\xi}^2 = -\frac{k|\phi_o|^2}{2\xi^2} + \frac{c_2}{2}\xi^2 + D \quad (55)$$

$$u^2 \equiv \dot{\xi}^2 = \frac{C}{\xi^2} + \frac{c_4}{2}\xi^2 - y_o^2 \quad (\eta = 0) \quad (56)$$

Therefore the two equations coincide by a suitable choice of the integration constants. The solution will then follows by performing the quadrature integration of anyone of (55), (56):

$$t - t_0 = \frac{1}{2} \int \frac{dz}{\sqrt{C + \frac{c_4}{2}z - y_o^2 z^2}} \quad (z = \xi^2) \quad (57)$$

Note that if $z \gg 1$ the contribution to the integral becomes complex.

5.2 The special choice $c_o = 0$.

Under the assumption $c_o = 0$ the following relations are fulfilled:

$$e^{\tau(t)} = \tau_o \xi(t), \quad e^{c(r)} = y_o, \quad \dot{\phi}(t) = (\phi_o/\tau_o) e^{\eta(t)} \xi^{-3}(t) \quad (58)$$

Hence $c' = 0$ and comparing eqs. (44), (46), (that in fact are the same equation) it must be respectively

$$c_2 = 0, \quad c_3 = c_4 \quad (59)$$

For what concerns the solution of the separated radial equations, the eq. (44) reads:

$$-2a' e^{-2a} \frac{y'}{y} + 2 \frac{y''}{y} e^{-2a} = \frac{c_3}{y_o^2} \quad (60)$$

By setting $u(r) = \exp(-2a)$ into (60) one obtains a first order differential equation for u whose solution gives

$$\exp(-2a) = \frac{1}{y'^2} \left[C + \frac{c_3}{2y_o^2} y^2 \right] \quad (61)$$

Compatibility of (44) and (46) then implies $C = \tau_o^2$. Therefore, so far, the solution of the scheme gives:

$$ds^2 = e^{2\eta(t)} dt^2 - \frac{\tau_o^2 \xi(t)^2 y'^2(r)}{\tau_o^2 + \frac{c_3}{2y_o^2} y^2(r)} dr^2 - y^2(r) \xi^2(t) (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (62)$$

where again, y being an arbitrary function, can be interpreted as the new radial coordinate. For what concerns the independent separated time dependence one is left with the coupled equations

$$\ddot{\xi} - \dot{\eta} \dot{\xi} \xi = (k|\phi_o|^2/\tau_o^2) e^{2\eta}/\xi^4, \quad \ddot{\xi} - \dot{\eta} \dot{\xi} \xi + 2\dot{\xi}^2 = (c_3/\tau_o^2) e^{2\eta} \quad (63)$$

that, a priori, should give both $\xi(t)$ and $\eta(t)$. One can note that, by setting $t' = \int^t e^{2\eta(t)} dt$, the equation (62) reads

$$ds^2 = dt'^2 - \xi^2(t') \left[\frac{dy^2}{1 + \frac{c_3}{2y_o^2 \tau_o^2} y^2} + y^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right] \quad (64)$$

that can be interpreted, by a suitable choice of the integration constants, as a Robertson Walker like element of line. However, while y is free, $\xi(t)$ (or $\xi(t')$) are determined by the solution of the separated time equations. If the $\xi(t')$ solution would show appropriate behavior, namely an initial inflation and a late accelerated expansion, then a cosmological interpretation of the scheme could be given.

6 Massive time dependent real scalar field

Suppose now the factored conditions (38) are fulfilled and $m_o \neq 0$, $\bar{\phi} = \phi = \phi(t)$ are assumed to hold. Accordingly the result (39) holds again. Moreover one can check that the R_{tt} equation (29) is of the form:

$$F(t) + G(r) = k \left[\dot{\phi}^2 - \frac{m_o^2}{2} \tau_o^2 y_o^2 \phi^2 \xi^{2(1-c_o)} y^{2c_o} \right] \quad (65)$$

that separates for $c_o = 0$ but not for $c_o = 1$. The separated radial equations coincide then with those of the previous case $m_o = 0$, $\phi = \phi(t)$. Hence the solution is again of the form (62). For what concerns the separated time dependence, one has:

$$\left[\ddot{\tau} + \dot{\tau}^2 + 2\frac{\ddot{\xi}}{\xi} - \dot{\eta}(\dot{\tau} + \frac{\dot{\xi}}{\xi}) \right] - k|\dot{\phi}|^2 e^{2(\tau-\eta)} - \frac{m_o^2}{2} y^2 \phi^2 = c_2 \quad (66)$$

$$e^{2(\tau-\eta)} \left[\ddot{\tau} + \dot{\tau}^2 - \dot{\tau}\dot{\eta} + 2\dot{\eta}\frac{\dot{\xi}}{\xi} \right] + k\frac{m_o^2}{2} y_o^2 \phi^2 e^\tau = c_3 \quad (67)$$

$$\tau_o^2 e^{-2\eta} \left[\xi\ddot{\xi} + \dot{\xi}\xi(\dot{\tau} - \dot{\eta}) + \dot{\xi}^2 \right] + k\frac{m_o^2}{2} \tau_o^2 y_o^2 \xi^2 \phi^2 = c_4 \quad (68)$$

$$\ddot{\phi} + \left(3\frac{\dot{\xi}}{\xi} - \dot{\eta} \right) \dot{\phi} + \frac{m_o^2}{y_o^2} \phi e^{2\eta} = 0 \quad (69)$$

Since $c_2 = 0$ (see (59)) and $\dot{\tau} = \dot{\xi}/\xi$, one can check that equations (67), (68) coincide. We are therefore left with three independent equation, e.g. (66), (68), (69), for three independent functions ξ, η, ϕ , in contrast to what happens in the LTB space time model [23] where the time solution is not possible. There is therefore the possibility of the existence of a non trivial time solution of the problem whose explicit form would give, also here, the possibility of testing whether the model is suitable for a cosmological interpretation satisfying some basic experimental features.

In absence of a general integration procedure, the method of looking for separated solutions can be applied in many other special situations. In case ϕ is assumed to depend only on r or only on t , the result (39) may be again of help. One can check that, e.g., if $\phi = \phi(r)$ is assumed, then equations (29)-(33) separate for $c_o = 1$ if $m_o = 0$ and for $c_o = 0$ if $m_o \neq 0$.

7 Remarks and Comments.

The study of the interaction of a general spherically symmetric gravity with complex massive scalar field has been consistently reduced to the set of coupled equation (29)-(33). The partial integration of the scheme leading to equation

(36), even if it allows the physical interpretation in terms of a Kepler like equation, it does not seem to facilitate very much a further integration step except the case of ϕ depending only on r or only on t . Therefore the study has been developed by special assumptions at the very beginning. If $\phi = \phi(t)$ and the factorization property (37) are assumed, the equations separate under both values $c_o = 0, 1$ if $m_o = 0$, but only for $c_o = 1$ if $m_o \neq 0$ and the field is real. Instead if $\phi = \phi(r)$ the equations separate for $c_o = 1$ if $m_o = 0$ and for $c_o = 0$ if $m_o \neq 0$. Contrarily to the LTB model, factorized solutions exist for massive purely time dependent scalar field. Indeed here there are the same constraint equations but a further degree of freedom with respect to the LTB model.

A case of interest is the one of Section 5.2. The resulting metric tensor in (64) can be interpreted as due to a homogenization effect of the interaction of spherical gravity with a homogeneous scalar field. It is the analog of similar result already discussed in the LTB metric (see e.g., [13, 21]). In this connection it remains also open the problem of establishing whether the time solution $\xi(t')$ in (64) shows both an initial and a late inflationary phase. In such case the resulting space time model could be suitable for a cosmological interpretation. The interaction scheme has been studied under very special assumptions. It would be interesting to establish and possibly determine other solutions in which ϕ depends on both r and t in a non factored and in a factorized form. (As far as the author knows this is an open problem already in the LTB model in case of non factored solutions). From a mathematical point of view, it would be desirable to have both particular solutions as well as general properties of the solutions of the interaction scheme. From a physical point of view the knowledge of those solutions would be again of interest to establish whether they are suitable for the interpretation of the expansion cosmological data. If so, one could also investigate whether the introduction of the notion of dark matter in the conventional cosmological model is strictly necessary.

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