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de Sitter Invariant Special Relativity

Some Physical Implications

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Abstract

Due to the existence of an invariant length at the Planck scale, Einstein special relativity breaks down at that scale. A possible solution to this problem is arguably to replace the Poincaré-invariant Einstein special relativity by a de Sitter invariant special relativity. Such replacement produces concomitant changes in all relativistic theories, including of course general relativity, which changes to what is called *de Sitter modified general relativity*, whose gravitational field equation is the *de Sitter modified Einstein equation*. A crucial property of this theory is that both the background de Sitter curvature and the gravitational dynamical curvature turns out to be included in the same curvature tensor. This means that the cosmological term Λ no longer explicitly appears in Einstein equation, and is consequently not restricted to be constant. In the first part of the thesis, a new definition for black hole entropy is defined. This new notion of entropy is strongly attached to the local symmetry, given the fact to be composed of two parts: the usual translational-related entropy plus an additional piece related to the proper conformal transformation. Also, it is obtained the de Sitter modified Schwarzschild solution, and using this solution we explore the consequences for the definition of entropy, as well as for the thermodynamics of the Schwarzschild-de Sitter system. In the second part the Newtonian limit of the de Sitter modified Einstein equation is obtained, and the ensuing Newtonian Friedmann equations are show to provide a good account of the dark energy content of the present-day universe. Finally, by using the same Newtonian limit, the circular velocity of stars around the galactic center is studied. It is shown that the de Sitter modified Newtonian force, which becomes effective only in the Keplerian region of the galaxy, could possibly explain the flat rotation curve of galaxies without necessity of supposing the existence of dark matter.

Keywords: de Sitter invariant special relativity; de Sitter modified gravitational theory; dark energy problem; dark matter problem.

Resumo

Devido à existência de um comprimento invariante na escala de Planck, a relatividade especial de Einstein deixa de ser válida naquela escala. Uma solução possível para esse problema é trocar a relatividade especial de Einstein, a qual tem o grupo de Poincaré como grupo de simetria, por uma relatividade especial invariante sob o grupo de de Sitter. Essa troca irá produzir mudanças concomitantes em todas as teorias relativísticas, incluindo naturalmente a teoria da relatividade geral. Essa teoria dá origem ao que denominamos *de Sitter modified general relativity*, cuja equação para o campo gravitacional foi chamada de *de Sitter modified Einstein equation*. Uma propriedade crucial dessa teoria é que tanto a curvatura de fundo de de Sitter como a curvatura dinâmica da gravitação estão ambas incluídas no mesmo tensor de curvatura. Isso significa que o termo cosmológico Λ não aparece explicitamente na equação de Einstein, e consequentemente não é restrito a ser uma constante. Trabalhando no contexto da *de Sitter modified general relativity*, na primeira parte da tese, uma nova definição de entropia para buraco negro é definido. Esta nova noção de entropia está fortemente ligado à simetria local, dado o fato de ser composto por duas partes:uma associada as translação e uma parte adicional relacionada com a transformação conformal. Assim mesmo, nós obtemos a solução de Schwarzschild modificada por de Sitter. Usando essa solução exploramos as consequências para a definição de entropia, bem como para a termodinâmica do sistema de Schwarzschild-de Sitter. Na segunda parte da tese obtemos o limite Newtoniano da *de Sitter modified Einstein equation*, e usamos as correspondentes equações de Friedmann Newtonianas para estudar o problema da energia escura. Mostramos que essas equações fornecem uma solução bastante razoável para a existência de energia escura do universo atual. Finalmente, usamos o mesmo limite Newtoniano para estudar a velocidade circular de estrelas ao redor do núcleo galáctico. Mostramos que a força Newtoniana modificada por de Sitter, a qual torna-se ativa apenas na região Kepleriana da galáxia, pode explicar as curvas de rotação planas sem necessidade de supor a existência de matéria escura.

Palavras chave: Relatividade especial de de Sitter; teoria gravitacional modificada por de Sitter; problema da energia escura; problema da matéria escura.

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...Y si llegué a este punto es porque mas una etapa terminó. Es mucho lo que he caminado hasta llegar aqui y es por eso, que no se puede cerrar éste capítulo sin mirar hacia atras y ver como lo he recorrido pero sobretodo quienes lo recorrieron conmigo.

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“Para el logro del triunfo siempre ha sido indispensable pasar por la senda de los sacrificios
Simón Bolivar”

Contents

Preamble

Our understanding of elementary particle physics is intimately related to both group representations and special relativity. In fact, all particles of Nature can be classified according to the irreducible representations of the Poincaré group $\mathcal{P}$, the kinematic group of special relativity. This property suggests that the symmetry of special relativity must be considered as an exact symmetry of Nature. In principle, therefore, there is no reason to replace Poincaré as the kinematic group of spacetime. However, when one tries to stick together, elementary particle physics or quantum field theory with gravitation, one faces conceptual problems related to the existence of a limit length scale, given by the Planck length. This scale, as is well known, shows up as the threshold of a new physics, represented by quantum gravity, means that the Planck length or some fundamental scale related to it must remain invariant under the relevant kinematics ruling the high-energy physics near to the Planck scale. Since in ordinary special relativity a length will contract by Lorentz transformations, the invariance requirement of such length scale seems to indicate that either the Lorentz symmetry must be broken down or the action of the Lorentz group $\mathcal{L}$—in particular the boosts—must in some way be modified, i.e near the Planck scale, the Poincaré group must be replaced by a more general group, which will preside over the high-energy kinematics.

The emergence of a new theory, or even a reformulation of one that already exists, is a direct consequence of the impossibility of the currents to bring statements in order to explain some experimental facts. Newton's theory enjoyed its greatest success when it was used to predict the existence of Neptune based on motions of Uranus, facts that could not be accounted for by the actions of the other planets. Nevertheless, a discrepancy in Mercury's orbit pointed out flaws in Newton's theory. By the end of the 19th century, it was known that its orbit showed slight perturbations that could not be accounted for entirely under Newton's theory, but all searches for another perturbing body (such as a planet orbiting the Sun even closer than Mercury) had been fruitless. Einstein in 1915 found the way to give the explanation to this fact with his theory of general relativity (GR), which accounted for the small discrepancy in Mercury's orbit.

From the kinematics point of view the Newton's theory is invariant under the Galilean transformation; a group of transformation that ruled a physics based on an absolute time and space. Nevertheless, at the end of the 19th century, the wave theory of light as a disturbance of a "light medium" or luminiferous ether was widely accepted, the theory reaching its most developed form in the work of James Clerk Maxwell. According to Maxwell's theory, all optical and electrical phenomena propagate through that medium, which suggested that it should be possible to experimentally determine motion relative to the hypothetical medium. The inconsistency around the Newtonian Mechanics with the Maxwell equations and the lack of prediction of the motion near to the speed of light, lead to correct the existing theory and develop the Einstein special relativity theory, which is adopted as a theory that allows to describe any movement at any speed scale, even close to the speed of light. However, one problem with the theory arises in the impossibility to deal with phenomena at the Planck scale due to the incompatibility between a scale-invariant length and the Lorentz transformation.

The Poincaré group is the group of symmetries of the Einstein special relativity which is a semidirect product of the Lorentz group $\mathcal{L}$ and the 4-dimensional translations T^4. The Lorentz group is responsible for the rotation, boosts and the equivalence between framework; nevertheless, the cornerstone of Einstein's special relativity is the Lorentz symmetry. The problem is that the Lorentz group is believed not to allow the existence of an invariant length parameter and given the existence of a physical fundamental scale determined by the Planck length $l_p \approx 10^{-33}cm$, it is perfectly understandable to think that, at the Planck scale the Lorentz symmetry must be broken or there must exist another relativistic theory near to that energy scale. So, the thing is how to get a theory that can describe physical phenomena at any energy scale, even at the Planck scale without violated the Lorentz symmetry.

In 1998, a work reported by Amelino-Camelia, Jonh Ellis *et al* [1], proposed that high-energy light from distant active galaxies could be used to check the effect of the gravity at some quantum scale. They gave a theoretical approach that at least some gamma-ray bursts (GRBs) at some cosmological distance increase the possibility that the observations could provide interesting constraints on the fundamental laws of physics, suggesting that at the Planck energy scale it must exist some interaction that disturbs in some way the structure of the spacetime.

In 2005, the Major Atmospheric Gamma-ray Imaging Cherenkov telescope (MAGIC), analyzed two flares from the Markarian 501 (MRK 501), between May and July 2005. The intense portions of the flares were quantified using four different energy bands. On June 30 the flare appeared in the energy band of 0.25–0.6 Tev, and the determinants results were found in the flare of July 9 in an energy band of 1.2–10 Tev with a time lag about 4 min; those results seemed to match with the facts proposed by Amelino-Camelia and Jonh Ellis [2].

Some scientist assumed that such a delay in the emission that came from the MRK 501 was a consequence of some perturbation made by something intrinsic of the source, for example, an acceleration by some magnetic field near of the center of the galaxy. So, it is interesting tries to look for some explanations about what could cause this delay. This experimental fact constitutes a clear evidence that at those energies regimens, there must exist a theory capable to describe such as interaction with the spacetime; a fact that cannot be answered under the statements of the Einstein special relativity. In the light of this one then must look for a modified special relativity.

The attempt to reformulated the Einstein special relativity was in the last years implemented by the work of Amelino-Camelia [3, 4] followed by João Magueijo and Lee Smolin [5], called *double special relativity*. This theory is obtained by introducing into the dispersion relation of the special relativity scale-suppressed term of high order in the momentum and like this, the existence of an invariant length at the Planck scale is allowed. The dispersion relation is of the type

$$E^2 - c^2p^2 - c^4m^2 + f(E, p, m, l_p) = 0. \quad (1)$$

Now, what is actually relevant in this theory is the fact that this high-order term is controlled by a parameter κ, which changes the kinematics group of special relativity from Poincaré to a κ-deformed Poincaré in which the Lorentz symmetry is explicitly violated. What happens is that, as far as the theory goes away from the Planck scale, the Lorentz symmetry is recovered and the relativity effect returns to be described by the ordinary special relativity.

So, in the light of the latter and having in mind the Planck length, how from the perspective of the Einstein special relativity can be answered fact like the one exposed? or how could l_p plays a role in the structure of the spacetime without violating Einstein special relativity?. The answer is twofold, or the Lorentz symmetry is violated from the beginning in order to consider the role of the Planck length in the quantum structure of the spacetime or the Einstein special relativity is reformulated with the purpose to consider such a facts, but *without violates the Lorentz symmetry*—given the relationship between the Lorentz symmetry and causality [6].

There exists a solution for this, it is known that the Lorentz transformations do not change the curvature of the homogeneous spacetimes in which they performed and considering that the scalar curvature R of any homogeneous spacetimes is of the form

$$R \sim \pm l^{-2}, \quad (2)$$

with l the pseudo-radius—of the hyperboloid in this case—then, the Lorentz transformations are founded to leave the length parameter l invariant. This is a geometric characteristic that is not notorious in Minkowski spacetime, because in this case, what is left invariant will be an infinite length. However, in de Sitter and anti de Sitter space—that is the reason of the ± signal—the pseudo-radius is finite and from these spacetimes, one then sees that contrary to the usual belief, the Lorentz transformation leaves invariant the length parameter related to the scalar curvature of the homogeneous spacetime. Now here is the thing, it could be possible to think that if the Planck length l_p is to be invariant under Lorentz transformations, then taking it to represent the pseudo-radius, the scalar curvature turns

$$R \sim \pm l_p^{-2} \sim \pm 10^{66} cm^{-2}, \quad (3)$$

predicting indeed, that even at the Planck scale the Lorentz symmetry and the causality [6] are preserved. So, as one moves away from the Planck scale the l pseudo-radius becomes greater and the physics pass to be described by the Einstein special relativity—which is ruled by the Poincaré group. So as it is possible to see the Einstein special relativity as a generalization of Galileo relativity for velocities close to the speed of light, de Sitter-ruled special relativity—the kinematics will be ruled by the de Sitter group—can be seen as a generalization of the Einstein special relativity for any energy scale, even at the Planck scale.

The de Sitter special relativity is a theory that seems to live between two limit case, specific scenarios determined by the behavior of the Eq.(2) in the limit for great and small values of l. This is traduced in the non-cosmological and the infinity cosmological limits, two scenarios where it is possible to study different physics configurations [7]. During the past ten years, this geometric-relativistic theory is gaining attention even when this approach is not as new as it is believed, the first ideas about de-Sitter special relativity are due to L. Fantappié, who in 1952 introduced what he called *Projective Relativity*, a theory that was further developed by G. Arcidiacono (for details [8]).

The large pseudo radius parametrization has already been shown to bear algebraic, geometric and thermodynamic properties that fit with what one would expect for the current cosmological observation [9–11]. Also in [12], it is explored the fact that the rescaling factor of the de Sitter metric can be used as refractive index and under the optical geometric approximation, this scenario gives an acceptable numerical estimation about the photons-delay reported by the MAGIC experiment; representing a crucial point in the sense that it could be used as an experimental fact of the de Sitter-invariant special relativity. But, the de Sitter special relativity has been used also in other fields. Some recent works [13, 14] show that the conformal geometry of the 4D-de Sitter space dS_4 represented by the hyperboloid helps to deal with the quark-confinement problem, taking into account the hypothetical relationship between the conformal symmetry and the color confinement.

By other hand, under the infinite cosmological term, the underlying de Sitter spacetime contracts to the four-dimensional homogeneous conic spacetime [15], such a spacetime seems also to present the geometric and thermodynamic properties that fit with what one expect for an initial condition of a big bang Universe, what actually plays an important role in the Penrose's Conformal Cyclic Cosmology [16].

Now, this work is entirely developed for large values of the pseudo-radius l and it is organized in two main parts. The first part will address the basis and foundations of the de Sitter invariant special relativity, as well as the construction of the modified general relativity theory. The second part is devoted basically to physical implication of the theory. One will highlight how this theory is capable to provide a good background for topics such as the concept of entropy, thermodynamic in horizons, the problem of the amount of dark energy in the universe and the galaxy rotation curve problem.

- In Chapter 1, it will be presented how the de Sitter invariant special relativity was formulated, with special attention to the geometric and algebraic development of the theory.

- The Chapter 2, will be devoted in gives details of how the changes in the translation sector of the symmetry group, re-defines the ordinary conservation of the stress-energy tensor, where it is going to be identified a new definition for the stress-energy tensor [17].

The original part of this work is developed in the last five chapters

- In Chapter 3, it is studied the replacement of the Poincaré-invariant Einstein special relativity by de Sitter-invariant special relativity on the construction of general relativity, letting the spacetime being described by the de Sitter modified general relativity [18]. It will study the non-relativistic limit of this theory, which it will leave a Newtonian potential with an extra repulsive term proportional to the cosmological term Λ and lineal in r, being mathematically and physically different to the Newton-Hooke potential.

- In Chapter 4, it will be made a brief review of the most relevant concepts of the thermodynamics in horizons, such as temperature, energy, and entropy. This part will highlight the new definition of entropy, which is attached to the new definition of diffeomorphism in a locally-de Sitter spacetime, named *proper conformal entropy* [19].

The Following three Chapters are dedicated to apply the modified Newtonian gravitational potential and to study the intrinsic changes that the de Sitter kinematics generates in the Schwarzschild solution.

- In Chapter 5, it will be redefined the Schwarzschild solution according to the changes that the de Sitter kinematics does to this solution. In this part of the work, it will be emphasized that there exist a new relationship between the horizons in the Schwarzschild de Sitter solution, obtaining a new thermodynamic constraint between the parameter of the metric [19].

- In Chapter 6, the Newtonian Friedman equations are going to be formulated based on the force generated by the modified potential. It will be possible to give account for the amount of dark energy in the current Universe obtaining values close to those observed [20].

- The Chapter 7, will analyze how the modified-Newtonian potential is able to give account for the flat behavior in the Keplerian range of the rotation curves of galaxies, without the need for inclusion of non-baryonic matter or dark matter [21]

Finally, there will be a section devoted to the conclusions, possible future approaches, and some technical issues.

Part I

Foundations and construction of the theory

Chapter 1
de Sitter special relativity, foundations and construction

"I was, at one time, greatly interested in establishing all linear equations which are invariant under the inhomogeneous Lorentz group . . . E. P. Wigner".

1.1 Double special relativity

The attempt to reformulated the Einstein special relativity was in the last years implemented by the work Amelino-Camelia [3, 4] followed by João Magueijo and Lee Smolin [5], who focused on the invariance of Planck length in a very high energy. In this kind of theory, Lorentz symmetry is deformed through of a dimensional parameter κ, proportional to the Planck length, with the purpose to construct a theory capable of achieving ultra-high energy regimes, *nevertheless at the moment to introduce such parameter automatically the theory loses the invariance under the Lorentz group*.

The Galilei group is one of the first wells studied group, given the fact that Newtonian mechanics is invariant under its transformation; nevertheless, by the end of the nineteenth century, inconsistencies between Newtonian mechanics and the theory of electromagnetism have triggered the search for another kinematics group, another relativistic theory. The search ends once the Einstein special relativity was established. Einstein special relativity is a theory developed in Minkowski space and whose kinematics is ruled by the Poincaré group—the Lorentz symmetry plus the 4-D translations. But, even when all the elementary particles of the nature are described by representations of this group, the physics is again facing intricate consistency problems, two examples are

- the impossibility for find a role of the Planck length in the gravitation theory
- the acceleration in the universe expansion rate, known as the dark energy problem.

Could these statements mean that a new relativistic theory is needed? Well, there exist arguments suggesting that the Poincaré symmetry might break down at ultrahigh energies.

An alternative to the double special relativity theory, is a propose where the Poincaré group is replaced by another symmetry group defining like this, a new special relativity theory with a new kinematics group, in order that in a natural way incorporates two invariant quantities: the speed of light c— which is a fundamental constant of the nature—and a Lorentz invariant length without breaking the Lorentz invariance preserving in that way, the causality of the theory [6], this theory is the de Sitter-ruled special relativity, a generalization of the ordinary special relativity for energies comparable to the Planck energy.

This approach is not as new as it is believed, the first ideas about de-Sitter special relativity are due to L. Fantappié, who in 1952 introduced what he called *Projective Relativity*, a theory that was further developed by G. Arcidiacono (for details [8]). In the next it will be shown that since the de Sitter special relativity, naturally incorporates an invariant length parameter, this theory can be interpreted as an example of the so-called doubly special relativity; nevertheless, there is a fundamental difference, though: whereas in all doubly special relativity models the Lorentz symmetry is violated, in de Sitter-ruled special relativity it remains as a physical symmetry*.

1.2 de-Sitter Cartan geometry

The geometry of the de Sitter space is part of a generalization of the Euclidean geometry names Klein geometry. It describes homogeneous spaces defined in terms of the symmetry of Lie groups, giving like this a starting point for the study of theories that will not require anymore the Minkowski space describes the local geometry †.

In his so-called *Erlanger Programm*, Felix Klein aimed to systematize all geometries/spaces known at the time. The idea is to investigate groups of transformations of a space onto itself or to adjoin to any geometry a group of transformations that leave the geometry invariant. Thus, the geometry of a manifold is characterized as the theory of invariants of a transformation group of that manifold. This yields an one-to-one relationship between a **symmetry group and a geometry/space**.

The symmetry group associated to a geometry is called the isotropy group or its group of motion. Bacry and Levy-Leblon [23] showed that all kinematical groups admit a four-dimensional spacetime interpretation. This is basically true because by assumptions rotations and boosts form a subgroup of each of the kinematical groups. Thus, for every kinematical group, one can define a four-dimensional homogeneous space as the quotient of the group by the six-dimensional subgroup. In the case of the de Sitter groups and the Poincaré group, this six-dimensional group is the Lorentz group.

Now, just to have an idea of those theories, in the following some general facts are going to be explored. The Cartan geometry generalizes the local geometry of Riemann;

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*For details of the relationship between doubly special relativity, de Sitter space and general relativity [22] by Derek Wise.

†It is important to note, when a homogeneous space, describes the spacetime of the physics theory, the group of symmetry is also called the kinematics group.

means that for each point p of the manifold M, the geometry of the manifold is not described anymore by the Riemannian geometry. The geometry instead of being represented locally by an open set defined for each $p \in M$; it passes to be represented by a symmetry group. In the present work physics quantities are going to be constructed taking into account that the spacetime-manifold is no longer anymore locally represented by a Minkowski space but for a de Sitter spacetime, so from now and on the theory is considered as *the de Sitter-Cartan geometry*.

1.3 Aspects of the de Sitter special relativity, definitions and construction

The de Sitter special relativity is a first principles theory based on the fact to replace the Poincaré-invariance group by the de-Sitter group like this, it is expected to explore aspects of the geometry and the group itself.

1.3.1 The de Sitter Space

The de Sitter Space is a homogeneous and a maximally symmetric spacetime, a quotient space defined as

$$dS(4,1) = SO(4,1)/\mathcal{L}, \tag{1.1}$$

with the Lorentz group as a sub-group of de Sitter group. The significant changes in contrast with the Poincaré group yield in the total replacement of how are defined or represented the translation in the kinematics group. The four-dimensional translations are replaced by a combination of the ordinary-Poincaré translations plus, the conformal transformations. The Lorentz group is maintained as the responsible for the isotropy and the equivalence between the inertial frames. Keeping the Lorentz group into the kinematics of the theory will provide some advantages, for example as homogeneous space the de Sitter spacetime has constant sectional curvature given by

$$R \sim l^{-2}, \tag{1.2}$$

where l is the de Sitter length parameter or pseudo-radius. Now, by definition the Lorentz transformations do not change the curvature of the homogeneous spacetime in which they are performed and by (1.2) *Lorentz transformations are found to leave the length parameter l invariant*, which gives to the relativistic theory invariant under the de Sitter group—the de Sitter special relativity—special and suitable conditions, this theory counts with the speed of light and a length parameter without violating Lorentz symmetry, which leaves a theory capable to reach even the Planck scale given the fact that, if the Planck length l_p is to be invariant under Lorentz transformation, then taking it to represent the pseudo-radius, the scalar curvature turns

$$R \sim l_p^{-2} \sim 10^{66} cm^{-2}, \tag{1.3}$$

predicting indeed, that even at the Planck scale the Lorentz symmetry and the causality [6] are preserved.

1.3.2 The geometry

The de Sitter spaces can be defined as a hyper-surfaces embedded into the pseudo-Euclidean spaces $\mathbb{E}^{4,1}$, with metric

$$\eta_{AB} = (1, -1, -1, -1, -1), \quad (A, B = 0, \cdots, 4), \tag{1.4}$$

whose points in Cartesian coordinates

$$(\chi^A) = (\chi^\mu \chi^4); \quad \mu = 0, ...3 \tag{1.5}$$

satisfies

$$\eta_{AB}\chi^A\chi^B = -l^2 \tag{1.6}$$

or equivalent in four dimensional coordinates

$$\eta_{\mu\nu}\chi^\mu\chi^\nu - \chi^4 = -l^2, \tag{1.7}$$

with l the de Sitter length parameter. Even when in the following the geometries and physical quantities will be express in stereographic coordinates—this represent just a fact of the interest in this part of the work—the de Sitter spacetime could be represents in more than one coordinate system [24].

Through the stereo-graphic projection, the 5-dimensional coordinates are carrying to 4-dimension as follow

$$\chi^a \equiv h^a_\mu x^\mu, \quad h^a_\mu = \Omega(x)\delta^a_\mu,^{\ddagger} \tag{1.8}$$

then

$$\chi^a = \Omega(x)x^\mu, \quad \chi^4 = -l^2\Omega(x)\left(1 + \epsilon\frac{\sigma^2}{4l^2}\right) \tag{1.9}$$

where

$$\Omega(x) = \frac{1}{1 + \sigma^2/4l^2} \tag{1.10}$$

and $\sigma^2 = \eta_{ab}x^a x^b$, is the Lorentz quadratic form. Finally in this coordinate system the metric turns to be represented as

$$g_{\mu\nu} = \Omega^2(x)\eta_{\mu\nu} \tag{1.11}$$

which leave us a conformally flat metric.

‡ h^a_μ, is a tetrad field, which allows to establish a bridging between the spacetime and the tangent space.

1.3.3 Curvature parameters in conformal space

Being de Sitter spacetime a curved spacetime, it is natural to look for how are defined the curvatures tensor, such as the Riemann tensors, Ricci tensor and the scalar curvature.

The corresponding Christoffel connection is [25]

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{\Omega}{2l^2}(\delta^{\lambda}_{\mu}\,\eta_{\nu\alpha}\,x^{\alpha} + \delta^{\lambda}_{\nu}\,\eta_{\mu\alpha}\,x^{\alpha} - \eta_{\mu\nu}\,x^{\lambda}), \tag{1.12}$$

the Riemann tensor given by

$$R^{\mu}{}_{\nu\rho\sigma} = \frac{\Omega^2}{l^2}\left(\delta^{\mu}_{\rho}\,\eta_{\nu\sigma} - \delta^{\mu}_{\sigma}\,\eta_{\nu\rho}\right), \tag{1.13}$$

and the Ricci and the scalar curvature are consequently,

$$R_{\nu\sigma} = \frac{3\Omega^2}{l^2}\,\eta_{\nu\sigma} \qquad \text{and} \qquad R = \frac{12}{l^2}. \tag{1.14}$$

1.4 The de Sitter group and the algebra

As Poincaré, the de Sitter group $SO(4,1)$, has also a well-defined algebra. In order to study the de Sitter algebra, one should start for the Lorentz generator L_{ab} $a,b = 0,...,3$ which follow the algebra

$$[L_{ab}, L_{cd}] = \delta_{bc}L_{ad} + \delta_{ad}L_{bc} - \delta_{bd}L_{ac} - \delta_{ac}L_{bd}, \tag{1.15}$$

By another hand, the generators of infinitesimal de Sitter transformations in Cartesian coordinates x^A are given by

$$J_{AB} = \eta_{AC}x^C\frac{\partial}{\partial x^B} - \eta_{BC}x^C\frac{\partial}{\partial x^A}, \tag{1.16}$$

which satisfied the commutations relations

$$[J_{AB}, J_{CD}] = \eta_{BC}J_{AD} + \eta_{AD}J_{BC} - \eta_{BD}J_{AC} - \eta_{AC}J_{BD}. \tag{1.17}$$

Again in stereographics coordinates $\{x^{\mu}\}$ (1.16) end up express as

$$J_{\mu\nu} = \eta_{\mu\rho}x^{\rho}P_{\nu} - \eta_{\nu\rho}x^{\rho}P_{\mu}, \qquad J_{\mu 4} = lP_{\mu} - (4l^2)^{-1}K_{\mu}. \tag{1.18}$$

where

$$P_{\mu} = \partial_{\mu} \quad \text{and} \quad K_{\mu} = \left(2\eta_{\mu\nu}x^{\nu}x^{\rho} - \sigma^2\delta^{\rho}_{\mu}\right)\partial_{\rho} \tag{1.19}$$

are the generators of ordinary translations and proper conformal transformations respectively [26]. At this point is important to emphasize that the $L_{\mu\nu}$ refers to the generators of the Lorentz transformation—means the Lorentz group remains unchanged—meanwhile the elements $L_{4\mu}$ defines the transitivity on the homogeneous space. So from Eq. (1.18) it follows that the de-Sitter spacetime is transitive under a combination of ordinary translations and proper conformal transformations — usually called de Sitter "translations".

In terms of these generators the de-Sitter algebra (1.17) assumes the form

$$[J_{\mu\nu}, J_{\rho\sigma}] = \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\mu\rho}J_{\nu\sigma}, \tag{1.20}$$

$$[J_{4\mu}, J_{\nu\rho}] = \eta_{\mu\nu}J_{4\rho} - \eta_{\mu\rho}J_{4\nu}, \quad [J_{4\mu}, J_{4\nu}] = l^{-2}J_{\mu\nu}. \tag{1.21}$$

reveling that the commutation rules related to the translation part are involved with the de Sitter length parameter.

Everything previously developed, represented basically the foundations of this relativistic theory, because now the local symmetry of the theory and in consequence, all the facts related to it will be changed according to this. The de-Sitter special relativity is a combination of two kinds of relativistic theory, one related to the symmetries of the Poincaré-translation and another related to some kind of *proper conformal transformations* as it is going to be shown in the following.

This theory lives between two limits, one of them describes a spacetime with a vanished cosmological constant and the another will be a limit related to an infinity cosmological constant; this last brings important cosmological implications, for more details [16].

1.5 Inönü Wigner contraction of de Sitter group

Among of the kinematics groups, the most discussed in the literature are the Galilean and Poincaré group. The group's structure of each one of them is totally described through its algebra. It is often mentioned that the Galilean group is the non-relativistic or low-velocity limit of the Poincaré group.

There exist a mathematical procedure name *The Inönü Wigner group contraction* [27], from which it is possible to obtain a group's algebra of some specific group through another one. In this method, the generators of the transformation of some group are re-defined according to the limit—*the contraction limit*—that wants to be performed. So, as it was already said, in the following, in order to perform the contraction limit from the de Sitter algebra, (For details see Appendix A) the generators must be re-defined being the de Sitter length l the parameter that will allow performing the contraction group.

1.5.1 The contraction limit ($l \to \infty$)

In order to explore the non-cosmological limit ($l \to \infty$), one should re-write (1.18), as follows

$$J_{\mu\nu} \equiv L_{\mu\nu} = \eta_{\mu\rho}x^{\rho}P_{\nu} - \eta_{\nu\rho}x^{\rho}P_{\mu}, \tag{1.22}$$

where $L_{\mu\nu}$, is identified as the Lorentz generators, which follows the commutation relations

$$[L_{\mu\nu}, L_{\rho\sigma}] = \eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\nu\sigma}L_{\mu\rho} - \eta_{\mu\rho}L_{\nu\sigma}. \tag{1.23}$$

By other hand defining a new quantity as

$$\Pi_\mu \equiv \frac{J_{\mu 4}}{l} = \left(P_\mu - \frac{1}{4l^2}K_\mu\right), \tag{1.24}$$

together with (1.23) the commutations relations (1.20, 1.21) turns

$$[\Pi_\mu, L_{\nu\rho}] = \eta_{\mu\nu}\Pi_\rho - \eta_{\mu\rho}\Pi_\nu, \quad [\Pi_\mu, \Pi_\nu] = l^{-2}L_{\mu\nu}. \tag{1.25}$$

Now, under the limit $l \to \infty$,

$$\lim_{l\to\infty} L_{\mu\nu} = L_{\mu\nu}, \quad \lim_{l\to\infty} \Pi_\mu = P_\mu, \tag{1.26}$$

means that, the Lorentz sector of the kinematics group does not change, and the modified translation operator goes to the Poincaré translational operator. Meanwhile, the commutation rules turn

$$[L_{\mu\nu}, L_{\rho\sigma}] = \eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\nu\sigma}L_{\mu\rho} - \eta_{\mu\rho}L_{\nu\sigma}, \tag{1.27}$$

$$[P_\mu, L_{\rho\sigma}] = \eta_{\mu\sigma}P_\rho - \eta_{\mu\rho}P_\sigma, \quad [P_\mu, P_\sigma] = 0. \tag{1.28}$$

What is actually happening is that under this contraction process the de-Sitter algebra is transformed into the Poincaré algebra, and de Sitter space is transformed into the Minkowski space

$$dS = SO(4,1)/\mathcal{L} \longrightarrow M = \mathcal{P}/\mathcal{L}, \tag{1.29}$$

which is transitive under ordinary translation.

As it is expected the Riemann, Ricci tensor and also the scalar curvature vanishes under thus limit [25]:

$$R^\mu{}_{\nu\rho\sigma} \to 0, \quad R_{\nu\sigma} \to 0, \quad R \to 0; \tag{1.30}$$

as it was expected, because, in this limit, the de Sitter space goes asymptotically to Minkowski spacetime and the de Sitter group is deformed to the Poincaré group—the respective algebra in the strict sense [§].

[§]There exist another limit group of de Sitter group, the non-relativistic limit which carries de Sitter group to Newton-Hooke group, for details see [28] Appendix A.

Chapter 2
Diffeomorphism and conservations laws on locally de Sitter spacetime

The goal of this Chapter is to explore and analyze the changes in the conserved quantities when the kinematics is ruled by the de Sitter group. Most of the concepts involved and the conservation laws were already explored [17], nevertheless, here again, they are going to be presented with some kind of details, without underestimating the previous work.

2.1 Symmetries and Killing vectors

In Chapter 1 it was already presented that, when the de Sitter group starts to rule the local kinematics in the theory, the physics turns out to be invariant under the so-called de Sitter translations, which in stereographics coordinates can be seen as a combination of ordinary translations and proper conformal transformations. So, it is expected—as the Noether's theorem established—that a new symmetry must imply a new conservation law.

In the next, the consequences of the latter are going to be explored; how this new way to connect any two points in de Sitter space modifies the ordinary invariance law under the Poincaré translation.

2.1.1 The Killing vector

Minkowski and de Sitter spacetimes are maximally symmetric space and they can lodge the highest possible number of Killing vectors.

Having Killing vector into the geometric description of the spacetime contributes in the definition of what it is a *spacetime symmetry*. A spacetime possesses a symmetry if it admits a vector field ξ^a, called a Killing Vector, which satisfies that

$$\mathcal{L}_\xi g_{ab} = 0, \tag{2.1}$$

which at the same time is related to the Killing equations as it follows

$$\mathcal{L}_\xi g_{ab} = \xi^c \nabla_c g_{ab} + g_{ac} \nabla_b \xi^c + g_{cb} \nabla_a \xi^c = 0, \tag{2.2}$$

so considering the compatibility of the metric with the Levi-Civita connection; $\nabla_a g_{bc} = 0$, where can be identified the Killing equation

$$\nabla_b \xi^c + \nabla_c \xi^b = 0. \tag{2.3}$$

By another hand; an infinitesimal coordinate transformations is given by

$$\delta x^a = \frac{1}{2}\varepsilon^{bc} J_{bc} x^a, \tag{2.4}$$

where J_{bc}, are the generator of the transformations. Again in stereographic coordinates (2.4) split out turns to be defined as

$$\delta x^\mu = \delta_L x^\mu + \delta_\Pi x^\mu; \tag{2.5}$$

where the first term represent the infinitesimal Lorentz transformation and the second term is the one related to the de Sitter translations, which are given respectively by

$$\delta_L x^\mu = \frac{1}{2}\varepsilon^{\nu\rho} L_{\nu\rho} x^\mu, \quad \delta_\Pi x^\mu = \varepsilon^\nu \Pi_\nu x^\mu; \tag{2.6}$$

which at the same time can be represented in terms of the Killing vector as

$$\delta_L x^\mu = \frac{1}{2}\xi^\mu_{(\nu\rho)}\varepsilon^{\nu\rho}, \tag{2.7}$$

with $\varepsilon^{\nu\rho} = -\varepsilon^{\rho\nu}$ the transformation parameter. The de Sitter translations on the another hand, assumes the form

$$\delta_\Pi x^\mu = \Delta^\mu_\nu \varepsilon^{(\nu)}; \tag{2.8}$$

with ε^ν the transformation parameter and

$$\xi^\mu_{\nu\rho} = \eta_{\nu\sigma} x^\sigma \delta^\mu_\rho - \eta_{\rho\sigma} x^\sigma \delta^\mu_\nu, \quad \Delta^\mu_\nu = \delta^\mu_\nu - \frac{1}{4l^2}\bar{\delta}^\mu_\nu, \quad \text{where} \quad \bar{\delta}^\mu_\nu = (2\eta_{\nu\sigma} x^\sigma x^\mu - \sigma^2 \delta^\mu_\nu), \tag{2.9}$$

represent the Lorentz Killing vectors, and the de Sitter Killing vectors respectively, which are defined in consistency with the proper conformal generator.

2.2 Diffeomorphism

In any theory with a general covariant action, the invariance of this action under infinitesimal coordinate transformation (*diffeomorphism*)

$$x^\gamma \to x^\gamma + q^\gamma(x), \tag{2.10}$$

leads to the conservation of a Noether current and it is right here where lies the importance of the above statements. When

$$q^{\gamma}(x) = \delta^{\gamma}_{\alpha}\,\varepsilon^{\alpha}(x) \tag{2.11}$$

where $\delta^{\gamma}_{\alpha}\,\varepsilon^{\alpha}(x)$ is the local Killing vector, the one corresponding with those spacetimes that reduces locally to Minkowski spacetime; a general diffeomorphism is written in the form

$$x^{\gamma} \to x^{\gamma} + \delta^{\gamma}_{\alpha}\,\varepsilon^{\alpha}(x). \tag{2.12}$$

Now, if it is considered the case when the Killing vector is defined by (2.9) the corresponding general diffeomorphism is defined by

$$x^{\gamma} \to x^{\gamma} + \Delta^{\gamma}_{\alpha}\,\varepsilon^{\alpha}(x). \tag{2.13}$$

Certainly, the re-definition of diffeomorphism will bring change in all the facts related to it, for example, one should expect that general relativity changes accordingly—implications of the latter will be explored forward.

2.2.1 Reprasing the notion of stress-energy tensor

In order to explore how are modified the conserved quantities under a new infinitesimal coordinate variation, let us consider a general matter field with a Lagrangian $\mathcal{L}_m$, where the respective action is

$$S_m = \frac{1}{c}\int d^4x\,\mathcal{L}_m. \tag{2.14}$$

Under local transformations δx^{ρ}, the variation

$$\delta S_m = \int d^4x\,\sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu}, \tag{2.15}$$

where

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}_m}{\delta g_{\mu\nu}} \tag{2.16}$$

is the symmetric energy momentum tensor.

At this point, it is important to remind the following. Through the Noether theorem, it is known that to each symmetry there is a conserved quantity—Killing vectors are those who carry on the information about symmetries. Reminding the standard procedure in the way to look at the corresponding conserved quantity, one usually does not see such expression related with the Killing vector because in most of the case that Killing vector is the one related to the Poincaré translation, it is mean δ^{μ}_{ν}. So, what is actually related to the symmetry at the end is, in some way hidden with the parameter $\varepsilon^{\rho}(x)$.

In the following it is going to be consider the variation of the Eq. (2.15) under the de Sitter translation, which as it was pointed, they will be related with the Killing vector of the de Sitter translations given by Eq. (2.9).

Now as the local diffeomorfism

$$\delta x^{\rho} = \varepsilon^{\alpha}(x)\Delta^{\rho}_{\alpha} \tag{2.17}$$

is able to detect the local structure of the spacetime, through the Noether theorem and considering the de-Sitter kinematics, we have

$$\delta S_m = \frac{1}{c}\int dx^4 \sqrt{-g} T^{\mu\nu}\Delta_{\alpha\nu}\nabla_\mu \varepsilon^\alpha, \tag{2.18}$$

and integrating by parts

$$\begin{aligned}
\nabla_\mu (T^{\mu\nu}\Delta_{\alpha\nu})\varepsilon^\alpha &= \nabla_\mu (T^{\mu\nu})\Delta_{\alpha\nu}\varepsilon^\alpha + T^{\mu\nu}\nabla_\mu \Delta_{\alpha\nu}\varepsilon^\alpha, &(2.19)\\
\nabla_\mu (T^{\mu\nu}\Delta_{\alpha\nu})\varepsilon^\alpha - \nabla_\mu (T^{\mu\nu})\Delta_{\alpha\nu}\varepsilon^\alpha &= T^{\mu\nu}\nabla_\mu \Delta_{\alpha\nu}\varepsilon^\alpha, &(2.20)\\
\nabla_\mu (T^{\mu\nu}\Delta_{\alpha\nu})\varepsilon^\alpha &= T^{\mu\nu}\nabla_\mu \Delta_{\alpha\nu}\varepsilon^\alpha. &(2.21)
\end{aligned}$$

So, the variation of the action turns

$$\delta S_m = \frac{1}{c}\int dx^4 \sqrt{-g}\nabla_\mu (T^{\mu\nu}\Delta_{\alpha\nu})\varepsilon^\alpha, \tag{2.22}$$

from where it is possible to identify a full divergency and defining

$$\Pi^{\mu\alpha} = T^{\mu\nu}\Delta^\alpha_\nu, \tag{2.23}$$

and using the definition of the de Sitter Killing vector, the latter equation ends as

$$\Pi^{\mu\alpha} = T^{\mu\nu}\delta^\alpha_\nu - \frac{1}{4l^2}K^{\mu\nu}, \tag{2.24}$$

where $K_{\mu\nu}$ is the proper conformal current, defined as

$$K_{\mu\nu} = (2\eta_{\mu\nu}x^\rho x^\alpha - \sigma^2\delta^\alpha_\mu)T_{\alpha\nu}, \tag{2.25}$$

with $\sigma^2 = \eta_{\mu\nu}x^\mu x^\nu$ the Lorentz invariant quadratic form.

At this point, it seems important to take at look closer (2.24). First, previously it was established the non cosmological limit case of the de Sitter special relativity and the fact to express the quantities involved in the right parametrization. Well (2.24) is suitable for large values of l and under the limit of non-cosmological constant ($l \to \infty$) the latter turns into the ordinary energy-momentum tensor $T_{\mu\nu}$ the one associated with the ordinary translation of the Einstein special relativity.

Second, the conservation law

$$\nabla_\mu \Pi^{\mu\nu} = 0, \tag{2.26}$$

does not imply the conservation of the each one of the quantities involved. Neither $T_{\mu\nu}$ nor $K_{\mu\nu}$ are conserved separately,

$$\nabla_\mu T^{\mu\nu} = \frac{2T^\rho_\rho x^\nu}{4l^2 - \sigma^2}, \quad \nabla_\mu K^{\mu\nu} = \frac{2T^\rho_\rho x^\nu}{1 - \sigma^2/4l^2}, \tag{2.27}$$

both will be zero separately if the trace of the ordinary stress energy tensor is zero.

Besides, it is important to emphasize that here the definition of the stress-energy tensor and the conservation law associated to it have been redefined; a fact that brings a direct violation of the energy conservation of the ordinary stress-energy tensor $T_{\mu\nu}$. From the moment the local symmetry is ruled by de Sitter group arises a new conservation law given by (2.26), but the non-conservation of the ordinary stress-energy tensor, is not an isolated case a similar scenario can be found in [29], where they consider the possibility of a physical theory allowing a violation of the stress-energy conservation law, in order to have a non-constant cosmological constant with the purpose to study problems like dark energy.

Finally, it is important to emphasize that in this work is maintained the fact that energy and its conservation can be re-defined, but one does not prefer to change are the quantities related to the Lorentz group given the dual relation between Lorentz symmetry/causality [6].

Chapter 3
The de Sitter-modified gravitational theory

Quantum field theories are geometrically described in Minkowski space; the local symmetry of the space manifold is ruled by the Poincaré group. So here comes the thing, as in this work it is proposed a change in the translation sector of the symmetry group that ruled the local symmetry and as it was showed in Chapter 2, there is a relation between the Killing vectors and the expression that defined the local diffeomorphism. In fact, this is something that it is expected because the local diffeomorphisms are able to detect the local structure of spacetime.

The replacement of the Poincaré invariant special relativity by the de Sitter-invariant special relativity brings changes in the construction of all relativistic theories, including general relativity. In this Chapter, the implications of this are going to be discussed. One important fact will be attached to the cosmological constant, which no longer will require being constant. This by fact will represent a concomitant change in standard general relativity and in cosmology.

3.1 The Einstein equation in locally-de Sitter spacetimes

The theory of general relativity is a geometric theory that studies how the curvature and matter interact with each other. This relationship is established in a suitable way through the Einstein's field equation,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{3.1}$$

The symmetries are determinant, both to define the new conserved quantities and to define how is locally represented the spacetime for each observed. The local symmetry of the theory of general relativity is described by the Poincaré group and the underlying spacetime is described by Minkowski. This, in very formal way to say, is what it is known as the *strong equivalence principle of general relativity*

In the presence of a gravitational field, it is always possible to find a local coordinate system in which the laws of physics reduces to those of special relativity,

it means that when it is chosen a frame where the local inertial effects can be compensated by the effects of the gravity; in that region, the spacetime is represented by the Minkowski space and the symmetry ruled by the Poincaré group.

On the other hand, under the observational facts [9–11] about the expansion and acceleration of the Universe, one should consider the cosmological constant Λ into the Eq. (3.1) *

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{3.2}$$

In this case, the second Bianchi identity implies that Λ must be constant:

$$\nabla_\mu \Lambda = \partial_\mu \Lambda = 0. \tag{3.3}$$

This fact represents a serious restriction in its prospective use for explaining the evolution of the of the Universe. For example, a possible explanation for the acceleration of the Universe expansion rate is to suppose that the cosmological term Λ is bigger today than it was a few billion years ago. However, since Einstein equation does not allow Λ to evolve, one cannot make use of it in cosmology. In order to circumvent this problem, many different models have been used in the literature for mimicking an evolving cosmological term. Examples of these models are the inflaton field, quintessence models, and also modified gravity models.

Solutions to Eq. (3.1) in the vacuum are spacetimes that outside of the strong regimen of the gravitational source of gravity, the spacetime is represented by Minkowski spacetime. Going back to the strong equivalence principle one should think of the following, what about if in those coordinates system, where those gravitational effects have suppressed the spacetime is not described anymore by Minkowski space. The replacement of the underlying Poincaré-ruled kinematics by a de Sitter-ruled kinematics does not change the dynamics of the gravitational field—the Einstein field equation for the spacetime-metric are the same—but what actually is changed is the strong equivalence principle, which in those coordinates system where the gravity effect is compensated by the inertial effect the spacetime will describe the de Sitter space. But, it could be perfectly natural to ask what else changes when such replacement is performed.

In Chapter 1, was showed that as homogeneous space the de Sitter space has the scalar curvature directly related to the length parameter l—as with the cosmological constant—so when the Einstein field equation is defined the cosmological parameter is already encoded into the equation—it does not appear explicitly in the equation—besides to the fact of counting with a new definition for the stress-energy tensor.

*Considering Λ to be responsible for the acceleration of the expanding Universe.

From the Einstein-Hilbert action of general relativity in a locally-de Sitter spacetime is written as

$$S_g = \int R \sqrt{-g} d^4 x, \tag{3.4}$$

with R the scalar curvature. Despite similarities to the Einstein-Hilbert lagrangian, there is a fundamental difference, *The Riemann tensor $R^{\alpha}{}_{\beta\mu\nu}$ represents now both the dynamical curvature of general relativity and the kinematic curvature of the underlying de Sitter spacetime*

Now, considering the total action integral

$$S = S_g + S_m, \tag{3.5}$$

the invariance of S under the diffeomorphism Eq. (2.13) yields the de Sitter-modified Einstein equation

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} \Pi_{\mu\nu}. \tag{3.6}$$

This is the equation that replaces the ordinary Einstein equation when the Poincaré-invariant special relativity is replaced by a de Sitter-invariant special relativity. At this point there is two important fact: first, in contrast with others modified-gravity theories where they change the way to define the general action field S_g, here what is present came entirely from first-principles: one has just to replace the Poincaré-invariant Einstein special relativity by a de Sitter-invariant special relativity—what is changing is the special relativity theory—and second, since both the dynamical curvature of general relativity and the kinematic curvature of the underlying de Sitter spacetime are now included in the Riemann tensor $R^{\alpha}_{\beta\mu\nu}$, the (contracted form of the second) Bianchi identity,

$$\nabla_\mu (R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R) = 0, \tag{3.7}$$

does not require Λ to be constant.

Certainly, the latter statement represents a changing in the way to think and to consider the cosmological constant. So, given the relationship between the way to define in homogeneous space the cosmological constant and the necessity of such local Λ to comply with the local symmetry of spacetime—now ruled by the de Sitter group— let us understand in a better way what does mean this local Λ.

3.2 Local value of the cosmological term Λ

According to the de Sitter special relativity, a physical system produces both a *dynamic* gravitational field described by general relativity, and a *kinematic* local cosmological term Λ in the underlying spacetime †. It is important to remark that this cosmological term is

†This idea was put forward for the first time by F. Mansouri [30].

different from the usual notion in the sense that it is not constant. In fact, outside the region occupied by the physical system, it vanishes. In a sense, it is possible to say that it represents an asymptotically flat de Sitter spacetime. The natural question then arises: given a physical system, how to obtain the local value of the cosmological term? In order to do this, for example, the smallest amount of an electromagnetic field, a photon, is determined by the Planck constant as a quantum of the field. In a similar fashion, the smallest possible length is the Planck length. Since in de Sitter relativity there is a free length parameter l, its minimum value will then be the Planck length $l_P = \sqrt{G\hbar/c^3}$. Let us then consider a de Sitter spacetime with $l = l_P$, for which the corresponding cosmological term is

$$\Lambda_P = \frac{3}{l_P^2}. \tag{3.8}$$

Considering that a cosmological term represents ultimately an energy density, it is defined the Planck energy density

$$\varepsilon_P = \frac{m_P\, c^2}{(4\pi/3) l_P^3}, \tag{3.9}$$

with $m_P = \sqrt{c\hbar/G}$ the Planck mass. In terms of ε_P, Eq. (3.8) assumes the form

$$\Lambda_P = \frac{4\pi G}{c^4}\, \varepsilon_P\, . \tag{3.10}$$

Now, the very definition of Λ_P can be considered an extremal particular case of a general expression relating the local "cosmological" term to the corresponding energy density of a physical system. Accordingly, to a physical system of energy density ε will be associated the "cosmological" term [31]

$$\Lambda = \frac{4\pi G}{c^4}\, \varepsilon = \frac{4\pi G}{c^2}\, \varepsilon_m \tag{3.11}$$

where $\varepsilon_m = \rho c^2$ has been used. It is important to note that the energy density ε_m appearing in this equation is not the dark energy density, but the energy density of ordinary matter. For small values of ε, the local cosmological term Λ will be small, spacetime will approach Minkowski, and de Sitter special relativity will approach ordinary special relativity, whose kinematics is governed by the Poincaré group

This local value for the cosmological constant will indeed bring some interesting fact for the overcoming studies and in order to do so, in the following it is going to be analyzed the Newtonian limit of the modified Einstein field equation (3.6).

3.3 The weak field limit of de Sitter-modified Einstein's equation

The non-relativistic Newtonian limit of the ordinary-Einstein field equation is the Poisson equation

$$\nabla^2 \phi = 4\pi G \rho \tag{3.12}$$

where ρ is the mass density. Its solution for a point particle of mass m is given by

$$\phi = -\frac{Gm}{r}. \tag{3.13}$$

which represent the static Newtonian gravitational potential.

As one should know the latter equation and in consequence, its solution is obtained under a small perturbation of the metric/spacetime. In the next, the modified-Einstein equation Eq. (3.6) is going to be linearized in order to look for the analogous Poisson equation and with that obtains what is going to be the modified Newtonian potential. It is expected that this new modified Newtonian potential provides a good approach to deal with problems such the dynamics of the rotation curve for galaxies—a problem frequently studied under the hypothesis of the existence of some non-baryonic matter—which apparently does not obey the Newtonian dynamics and to deal with the so-called *coincidence problem*.

3.3.1 The set up

In terms of the ambient space coordinates χ^A, again an infinitesimal de Sitter transformation is written as

$$\delta \chi^A = \tfrac{1}{2}\, \epsilon^{BC} \xi_{BC}{}^{A} \tag{3.14}$$

where $\epsilon^{BC} = -\epsilon^{CB}$ are the transformation parameters, and

$$\xi_{BC}{}^{A} = \chi_B\, \delta^A_C - \chi_C\, \delta^A_B. \tag{3.15}$$

are the Killing vectors of the de Sitter group. The components

$$\xi_{\beta\gamma}{}^{\alpha} = \chi_\gamma\, \delta^\alpha_\beta - \chi_\beta\, \delta^\alpha_\gamma \tag{3.16}$$

represent the Killing vectors of the Lorentz group, whereas the components

$$\Delta^\alpha_\beta \equiv l^{-1} \Delta_{\beta 4}{}^{\alpha} = l^{-1}(\chi_4\, \delta^\alpha_\beta - \chi_\beta\, \delta^\alpha_4) = l^{-1} \chi_4\, \delta^\alpha_\beta \tag{3.17}$$

represent the Killing vectors of the de Sitter "translations".

It will usefulness to work in static coordinates (ct, r, θ, φ). They can be obtained from the embedding coordinates χ^A through

$$\chi_0 = l\sqrt{1 - r^2/l^2}\,\sinh(ct/l), \quad \chi_1 = r\sin\theta\sin\varphi \tag{3.18}$$

$$\chi_2 = r\sin\theta\cos\varphi, \quad \chi_3 = r\cos\theta \tag{3.19}$$

$$\chi_4 = l\sqrt{1 - r^2/l^2}\,\cosh(ct/l). \tag{3.20}$$

The de Sitter metric in terms of the embedding coordinates is

$$ds^2 \equiv \eta_{AB}\,d\chi^A d\chi^B = (d\chi^0)^2 - (d\chi^1)^2 - (d\chi^2)^2 - (d\chi^3)^2 - (d\chi^4)^2\,. \tag{3.21}$$

Using Eq. (3.18-3.20), one can easily verify that it is

$$ds^2 = (1 - r^2/l^2)\,c^2dt^2 - \frac{dr^2}{1 - r^2/l^2} - r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right). \tag{3.22}$$

Similarly, it is possible to obtain the Killing vectors of the de Sitter group in static coordinates. In particular, the Killing vectors Eq. (3.17) associated to the de Sitter "translations" are found to be

$$\Delta^\alpha_\beta = (1 - r^2/l^2)^{1/2}\cosh(ct/l)\,\delta^\alpha_\beta\,. \tag{3.23}$$

In the contraction limit $l \to \infty$ they reduce to the Killing vectors δ^α_β of ordinary translations.

Now, since the Newtonian limit is static, the time dependence of Δ^α_β is neglected and rewrite it in the form

$$\Delta^\alpha_\beta = (1 - r^2/l^2)^{1/2}\,\delta^\alpha_\beta\,. \tag{3.24}$$

3.3.2 The weak field limit of the de Sitter-modified Einstein equation

In the Ricci form the field equation Eq. (3.6) turns

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(\Pi_{\mu\nu} - \tfrac{1}{2}\,g_{\mu\nu}\Pi\right). \tag{3.25}$$

In a de Sitter-Cartan geometry, in which the background spacetime is de Sitter instead of Minkowski, the spacetime metric is expanded in the form

$$g_{\mu\nu} = \hat{g}_{\mu\nu} + h_{\mu\nu}, \tag{3.26}$$

where $\hat{g}_{\mu\nu}$ represents the background de Sitter metric and $h_{\mu\nu}$ is the metric perturbation. The background connection, which corresponds to the zeroth-order connection, is

$$\hat{\Gamma}^\rho{}_{\mu\nu} = \tfrac{1}{2}\,\hat{g}^{\rho\lambda}(\partial_\mu\hat{g}_{\lambda\nu} + \partial_\nu\hat{g}_{\mu\lambda} - \partial_\lambda\hat{g}_{\mu\nu}). \tag{3.27}$$

The corresponding Riemann tensor $\hat{R}^{\alpha}{}_{\beta\mu\nu}$ represents the curvature of the (non-gravitational) de Sitter background.

The first-order connection, on the other hand, is given by

$$\Gamma^{\rho}_{(1)\mu\nu} = \tfrac{1}{2}\,\hat{g}^{\mu\nu}\left(\partial_\mu h^{\rho}{}_{\nu} + \partial_\nu h^{\rho}{}_{\mu} - \partial^{\rho} h_{\nu\mu}\right) - \tfrac{1}{2}\,h^{\rho\lambda}\left(\partial_\mu \hat{g}_{\lambda\nu} + \partial_\nu \hat{g}_{\mu\lambda} - \partial_\lambda \hat{g}_{\mu\nu}\right). \tag{3.28}$$

After some algebraic manipulation, it can be rewritten in the form

$$\Gamma^{\rho}_{(1)\mu\nu} = \tfrac{1}{2}(\hat{\nabla}_\mu h^{\rho}{}_{\nu} + \hat{\nabla}_\nu h^{\rho}{}_{\mu} - \hat{\nabla}^{\rho} h_{\nu\mu}), \tag{3.29}$$

with $\hat{\nabla}_\mu$ a covariant derivative in the de Sitter connection (3.27). The corresponding first-order Ricci tensor is

$$R^{(1)}_{\mu\nu} = \tfrac{1}{2}\hat{\nabla}_\rho\hat{\nabla}_\nu h^{\rho}{}_{\mu} + \tfrac{1}{2}\hat{\nabla}_\rho\hat{\nabla}_\mu h^{\rho}{}_{\nu} - \tfrac{1}{2}\hat{\nabla}^{\rho}\hat{\nabla}_\rho h_{\mu\nu} - \tfrac{1}{2}\hat{\nabla}_\mu\hat{\nabla}_\nu h, \tag{3.30}$$

where $h = h^{\rho}{}_{\rho}$. Using the identity

$$\hat{\nabla}_\rho\hat{\nabla}_\mu h^{\rho}{}_{\nu} - \hat{\nabla}_\mu\hat{\nabla}_\rho h^{\rho}{}_{\nu} = -\,h^{\sigma}{}_{\nu}\,\hat{R}_{\sigma\mu} + h^{\rho}{}_{\sigma}\,\hat{R}^{\sigma}{}_{\nu\rho\mu} \tag{3.31}$$

obtaining

$$R^{(1)}_{\mu\nu} = -\,\tfrac{1}{2}\hat{\Box}h_{\mu\nu} + \tfrac{1}{2}\hat{\nabla}_\mu(\hat{\nabla}_\rho h^{\rho}{}_{\nu} - \tfrac{1}{2}\hat{\nabla}_\nu h) + \tfrac{1}{2}\hat{\nabla}_\nu(\hat{\nabla}_\rho h^{\rho}{}_{\mu} - \tfrac{1}{2}\hat{\nabla}_\mu h) - h^{\sigma}{}_{(\nu}\,\hat{R}_{\sigma\mu)} + h^{\rho}{}_{\sigma}\,\hat{R}^{\sigma}{}_{(\mu\rho\nu)}, \tag{3.32}$$

with the parentheses indicating a symmetrization in the neighbor indices.

At the first order, the class of harmonic coordinates is obtained by imposing the condition

$$\hat{g}^{\mu\nu}\,\Gamma^{\rho}_{(1)\mu\nu} = 0. \tag{3.33}$$

After some algebraic manipulation, it can be recast in the form

$$\hat{\nabla}_\nu h^{\rho\nu} - \tfrac{1}{2}\hat{\nabla}^{\rho} h = 0. \tag{3.34}$$

Using this condition in (3.32), the first-order Ricci tensor is found to be

$$R^{(1)}_{\mu\nu} = -\,\tfrac{1}{2}\hat{\Box}h_{\mu\nu} - h^{\sigma}{}_{(\nu}\,\hat{R}_{\sigma\mu)} + h^{\rho}{}_{\sigma}\,\hat{R}^{\sigma}{}_{(\mu\rho\nu)}. \tag{3.35}$$

At this order the de Sitter-modified Einstein equation (3.6) assumes then the form

$$-\,\tfrac{1}{2}\hat{\Box}h_{\mu\nu} - h^{\sigma}{}_{(\nu}\,\hat{R}_{\sigma\mu)} + h^{\rho}{}_{\sigma}\,\hat{R}^{\sigma}{}_{(\mu\rho\nu)} = \frac{8\pi G}{c^4}\left(\Pi_{\mu\nu} - \tfrac{1}{2}\,g_{\mu\nu}\Pi\right). \tag{3.36}$$

3.4 Newtonian limit and de Sitter-modified Poisson equation

The Newtonian limit is achieved when the fields are weak and the velocities are small. Nevertheless, in this case an additional fact is added; the presence of the cosmological term Λ—which brings some subtleties related to the process of group contraction. For example the Galilei group is obtained from Poincaré under the contraction limit $c \to \infty$, the Newton-Hooke group, however, does not follow straightforwardly from the de Sitter group through the same limit. The reason is that, under such limit the boost transformations are lost. In order to obtain a physically acceptable result, one has to simultaneously consider the limits $c \to \infty$ and $\Lambda \to 0$, but in such a way that

$$\lim c^2 \Lambda = \frac{1}{\tau^2} \tag{3.37}$$

with τ a time parameter. This means that the usual weak field condition of Newtonian gravity must be supplemented by the small Λ condition [32]

$$\Lambda r^2 \ll 1, \tag{3.38}$$

which is equivalent to $r^2/l^2 \ll 1$. Accordingly, in what follows terms up to order r^2/l^2 are going to be keeping and terms of order r/l^2 will be discarded as they represent corrections to Newtonian limit.

3.4.1 de Sitter-modified Poisson equation

In the Newtonian limit, only the component $R^{(1)}_{00}$ is needed. In this case, the last term on the right-hand side of Eq. (3.35) vanishes. Identifying furthermore

$$h_{00} = 2\phi/c^2, \tag{3.39}$$

with ϕ the gravitational scalar potential, it is obtained

$$R^{(1)}_{00} = \frac{2}{c^2}\left[-\frac{1}{2}\hat{\Box}\phi - \phi\,\hat{R}_{00}\right]. \tag{3.40}$$

Neglecting the time derivatives in the d'Alembertian, it is obtained

$$R^{(1)}_{00} = \frac{1}{c^2}\left[\hat{\Delta}\phi - 2\phi\,\hat{R}_{00}\right] \tag{3.41}$$

with $\hat{\Delta}$ the Laplacian in the de Sitter metric. Using this result the de Sitter-modified Einstein equation Eq. (3.36), it becomes

$$\hat{\Delta}\phi + 2\phi\,\hat{R}_{00} = \frac{4\pi G}{c^2}\,\Pi_{00} \tag{3.42}$$

where the fact that $\Pi = \Pi^0{}_0$ was used .

Now, in static coordinates, the component $\hat{R}_{00}$ of the Ricci tensor is

$$\hat{R}_{00} = \frac{3}{l^2}\,(1 - r^2/l^2) \simeq \frac{3}{l^2}, \tag{3.43}$$

where a term proportional to r^2/l^4 was discarded. The source current, on the other hand, is

$$\Pi_{\mu\nu} = \Delta_\mu^\alpha \, T_{\alpha\nu} \tag{3.44}$$

with ξ_μ^α the Killing vectors of the de Sitter "translations". Since only $T_{00} = \rho c^2$ contributes in the Newtonian limit, then

$$\Pi_{00} = \Delta_0^0 \, T_{00}. \tag{3.45}$$

Substituting the Killing vector ξ_0^0 as given by Eq. (B.4) of the Appendix B, the zero-zero component of the source turns

$$\Pi_{00} = \rho_\Pi \, c^2 \tag{3.46}$$

where

$$\rho_\Pi = \rho \, (1 - r^2/l^2)^{1/2}. \tag{3.47}$$

Then finally Eq. (3.42) assumes the form

$$\hat{\Delta}\phi - \frac{6\phi}{l^2} = 4\pi G \rho_\Pi, \tag{3.48}$$

where the Laplace operator $\hat{\Delta}$ in the background de Sitter metric $\hat{g}^{ij}$ is

$$\hat{\Delta} \equiv \hat{g}^{ij} \hat{\nabla}_i \hat{\nabla}_j = \frac{1}{\sqrt{\hat{g}}} \, \partial_i \big(\sqrt{\hat{g}} \, \hat{g}^{ij} \partial_j \big). \tag{3.49}$$

Now, using the space components of the metric Eq. (3.22), it is found to be

$$\hat{\Delta}\phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \left(1 - \frac{r^2}{l^2} \right) \frac{\partial \phi}{\partial r} \right] . \tag{3.50}$$

Equation (3.48) can then be rewritten in the form

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \left(1 - \frac{r^2}{l^2} \right) \frac{\partial \phi}{\partial r} \right] - \frac{6\phi}{l^2} = 4\pi G \rho_\pi \, . \tag{3.51}$$

The solution to this equation will be the de Sitter-modified gravitational potential.

3.4.2 de Sitter-modified gravitational potential

In the contraction limit $l \to \infty$ (which corresponds to $\Lambda \to 0$), equation (3.51) reduces to the usual Poisson equation

$$\Delta\phi \equiv \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) = 4\pi G \rho. \tag{3.52}$$

Its solution is given by

$$\phi(r) = - \int \frac{G}{r - r'} \, \rho(r') \, dV' \tag{3.53}$$

where r' is the distance from the volume element dV' to the point where we are determining the potential. For a point particle located at $\mathbf{r}'$, the mass density is given by $\rho(r') = M\delta(\mathbf{r} - \mathbf{r}')$ and it is obtains

$$\phi(r) = -\frac{GM}{r}, \tag{3.54}$$

which is the Newtonian potential. The same procedure should in principle be used to solve equation (3.51). However, this is not necessary because as an easy computation shows, if it is replaced

$$\phi(r) \to \left(1 - \frac{r^2}{l^2}\right)\phi(r) \tag{3.55}$$

the left-hand side of the ordinary Poisson equation (3.52), up to terms of order r/l^2 it transforms into the left hand side of the de Sitter-modified Poisson equation (3.51). If the solution of the ordinary Poisson equation is given by Eq. (3.54), the transformed potential $(1 - r^2/l^2)\phi(r)$ will be a solution of the de Sitter-modified Poisson equation (3.51) with the same Green function:

$$\left(1 - \frac{r^2}{l^2}\right)\phi(r) = -G\int d^3r' \frac{1}{r - r'}\rho(r')\left(1 - \frac{r'^2}{l^2}\right)^{1/2}. \tag{3.56}$$

For a point particle located at $\mathbf{r}'$, the mass density is given by $\rho(r) = M\delta(r - r')$, the latter equation turns

$$\phi(r) = -\left(1 - \frac{r^2}{l^2}\right)^{-1}\left(1 - \frac{r^2}{l^2}\right)^{1/2}\frac{GM}{r}, \tag{3.57}$$

and performing the suitable simplification

$$\phi(r) = -\frac{GM}{r}\left(1 - \frac{r^2}{l^2}\right)^{-1/2}; \quad \left(1 - \frac{r^2}{l^2}\right)^{-1/2} \simeq 1 + \frac{r^2}{2l^2} + ..., \tag{3.58}$$

the de Sitter-modified Newtonian potential can be re-written in the form

$$\phi = -\frac{GM}{r} - \frac{GM}{2l^2}r. \tag{3.59}$$

The corresponding gravitational force $F = -\partial\phi/\partial r$ is

$$F = -\frac{GM}{r^2} + \frac{GM}{2l^2}. \tag{3.60}$$

which using the relation $\Lambda \sim 3/l^2$, de Sitter-modified Newtonian force assumes the form

$$F = -\frac{GM}{r^2} + \frac{GM\Lambda}{6}. \tag{3.61}$$

The first term of the gravitational force Eq. (3.61), represents the Newtonian force. The de Sitter background is then found to contribute with an additional constant force. Far away from the sources, where ordinary Newtonian force becomes small, it may become relevant.

Given the result, it is not possible to not compare with the Newton-Hooke potential,

$$\phi_{NH} = -\frac{GM}{r} - \frac{r^2}{l^2}; \tag{3.62}$$

which in term of the cosmological term ends as

$$\phi_{NH} = -\frac{GM}{r} - \frac{\Lambda}{3} r^2. \tag{3.63}$$

The first thing to note in Eq. (3.63)—apart from the Newtonian term— is that the new term is quadratic in r, while in Eq. (3.59) is lineal in r. Another fact to emphasize is that even when the modified-de Sitter potential was obtained by changing the local symmetry at the moment to perform the linealization of the field equation, this new potential acquired a gravitational status gaining in the extra term the gravitational constant G as it can be observed from its expression.

Finally, the last Chapters of this work are devoted in the application of this modified-Newtonian potential in the discrepancy between the theoretical and the observation in the rotation curve of galaxies as well as the *cosmological constant problem*.

Part II

Some physical implications

Chapter 4
Thermodynamics and the de Sitter special relativity

The de Sitter special relativity has two main characteristics, the change in the translation symmetry and the fact that Lorentz symmetry is preserved at any energy scale, even close to the Planck scale. This Chapter is dedicated to studying some thermodynamics quantities that might arise as a consequence of the redefinition of the stress-energy tensor $\Pi_{\mu\nu}$, specifically the definition of entropy.

It is known that under a suitable coordinate system is possible to identify the horizon in the de Sitter space [33], which suggests that such spacetime will have non trivial thermodynamics features. Being able to define the notion of the horizon such as in the case of the Black hole, gives to de Sitter spacetime the ability to determine the notion of surface gravity which will relate to the temperature associated with this horizon. But actually, this is not new, the first work to analyze the thermodynamic quantities of the causal horizon was given by *Gibbons& Hawking 1976* [34].

So, having in mind that it could be possible to determine the notion of temperature, and giving the fact that we already have a new Noether charge, it is going to be defined a notion of entropy in according principally with the change in the local diffeormorfims [35].

4.1 Horizons, temperature and entropy

Besides to the fact that thermodynamics laws are well defined and study in the context of black holes event horizons [36, 37] *, there is so much less discussed for spacetimes where cosmological aspect are involved such as the de Sitter spacetime.

The Schwarzschild black hole can be considered as a thermodynamics system; temperature, energy, and entropy can be attributed to it; indeed Bardeen-Carter & Hawking summarized the four thermodynamics laws for black hole [37]. This four mechanics laws, suggests that one can identify the surface gravity κ of a black hole with temperature and

*The term *event horizon* will be used for the causal limit region on the black hole.

the area A of the event horizon with entropy S at least up to some multiplicative constants.

The concept of temperature of a black hole arrives when quantum mechanical effects are taken into account. One finds that black holes emit thermal radiation knows as Hawking radiation [38], at a temperature

$$T_{\mathrm{BH}} = \frac{\hbar\,\kappa}{2\pi\,c\,k_B},^{\dagger} \tag{4.1}$$

setting in this way the first relationship between a geometric parameter of the black hole and a thermodynamic one.

The Schwarzschild black hole has a compact spherical surface of radius $r = 2M$, which is what determines the well known event horizon. Since this horizon can hide "information"—which is connected with entropy—it is possible through the Bekenstein entropy [39], associates to this horizon such value for the entropy as

$$S_{BH} = \frac{k_B\,c^3\,A}{4\hbar G}. \tag{4.2}$$

which is actually proportional to the area of the event horizon, characterist of such definition as it was comment above.

Meanwhile, given the good work in the laws of thermodynamics for event horizons, the question will be, how are define those thermodynamics variables on cosmological horizons, such as the case for de Sitter causal horizons? From now, here the objective is to extend the above concepts—briefly studied—-for more general spacetime such as the de Sitter spacetime.

4.2 Thermodynamics variables on de Sitter spacetime

In Chapter 2, a new Noether current was found due to the change in the notion of a local transitivity. This new definition for the stress-energy must changes the quantities that are related to it, such as for example the entropy.

4.2.1 Temperature

Spacetimes with horizons possess a natural analytic continuation from Minkowski signature to the Euclidean signature with $\tau \to t = i\tau$. If the metric is periodic in τ, then one can associate a natural notion of a temperature to such spacetimes [40] ‡.

For example, from the de Sitter manifold with the metric

$$ds^2 = -c^2 d\tau^2 + l^2 \cosh(l^{-1}c\tau)[d\chi^2 + sin^2\chi(d\theta^2 + sin^2\theta d\phi^2)] \tag{4.3}$$

† For the case of the Schwarzschild black hole the the temperature is given by $T_{BH} = 1/8\pi\,M$, where $\kappa = 1/4M$

‡ See Appendix D for details of the relation between the Euclidean time and temperature.

it is possible to make such analytic continuation, defining a metric

$$ds^2 = c^2 dt^2 - l^2 \cos(l^{-1} c/t)[d\chi^2 + sin^2\chi(d\theta^2 + sin^2\theta d\phi^2)] \tag{4.4}$$

which is clearly periodic in t with period $2\pi l$ [§] . Like this the temperature associated to the de Sitter horizon is given by

$$T_{ds} = \frac{\hbar\, c}{2\,\pi\, l\, k_B}, \tag{4.5}$$

with l the de-Sitter radius. As it was seen, the de Sitter temperature is not so far from the notion of temperature in black hole; but the analogy remains until here.

Energy and entropy are less discussed in spacetimes more complicated as de Sitter space. A possible reason could be that in the simplest context of a Schwarzschild black hole of mass M, one can relate the energy with the matter as an equal, $E = M$, temperature $T = (8\,\pi\, M)^{-1}$ and entropy $S = (1/4)A_H/l_P$, where A_H is the area of the horizon and $l_P = (G\hbar/c^3)^{1/2}$ is the Planck length; quantities that are clearly related to the thermodynamic identity

$$TdS = dE, \tag{4.6}$$

usually called the first law of black hole thermodynamics. This result has been obtained in much more general contexts and has been investigated from many different points of view in the literature [40]. The simplicity of the result depends on the following features:

- The Schwarzschild metric is a vacuum solution with no pressure so that there is no PdV term in the first law of thermodynamics.
- The metric has only one parameter M, then the changes in all the physical parameters can be related to this parameter, changing in time i.e dM.
- Most importantly, there exists a well-defined notion of energy E to the spacetime and the changes in the energy dE can be interpreted in terms of the physical process of the black hole evaporation.

So, if the goal is to extend thermodynamics concepts as the mentioned before, it is important to recognize which is (are) the dynamical parameter, when it is considered the de Sitter special relativity approach.

So, the answer to the question above could be that, even when it is possible to define quantities such as temperature, energy, and entropy for de Sitter horizons; without a parameter that allows some kind of dynamics in it, there will be no *dynamics* between those variables. Indeed, even having a notion of energy, it is not clear how to write and interpreted an equation analogous to (4.6) in a spacetime which is locally de Sitter, because dE is attached to the notion of evaporation and how this is understood under de Sitter-ruled special relativity approach still remains unclear.

[§] It is important to remark that the spacetime described by Eq. (4.3) represents a 4-dimensional hyperboloid in the 5-dimensional space, while the Eq. (4.4) represents a 4-sphere, from where the notion of *period* is more natural

Commentary 4.1 In order to get a glimpse about how can be related those variables, in the next Chapter is going to be redefined the thermodynamics of horizons where these two variables are going to be related to each other, as a unique system. ◀

4.2.2 Energy as the new Noether charge

A new Noether current gives a new Noether charge. From (2.24), the generalized notion of energy will be

$$E \equiv \Pi^{00} = E_T - (2l)^{-2} E_K, \tag{4.7}$$

where E_T is the ordinary translational notion of energy and E_K is the proper conformal notion of energy. Again, in the formal limit for large values of l, it remained that the total energy is given by the ordinary translation energy E_T.

4.2.3 Entropy on locally Minkowski spacetime

As it was discussed, there is no problem in defines temperature on de Sitter horizon; so in the light of this, is reasonable to try to define the entropy associated to the causal de Sitter horizon as

$$S_{ds} = \frac{k_B A_H}{4 l_p^2} = \frac{\pi c^3 k_B l^2}{G \hbar}, \tag{4.8}$$

where $A_H = 4\pi l^2$ is the area of the causal horizons; but these quantities are related by the first law of thermodynamics (4.6), and with a new definition for the energy (4.7), there must be a new definition for the entropy and in a consequence a variation for the first law of thermodynamics at least into the de-Sitter special relativity approach.

The procedure to construct the definition for the entropy, according to with the de Sitter stress energy tensor $\Pi_{\mu\nu}$, is addressed practically, following the work by *Wald* [41] where the entropy is defined as a Noether charge. It is important to remind that the concept of *Entropy*, measures the lack of information of some physical system, and in this case, that information is attached to the causal-de Sitter horizon.

Under this, the event horizon of a stationary black hole is a Killing horizon $\mathcal{H}$[¶], a null surface to which a Killing vector field ζ^α is normal then, the surface gravity κ at any point of the Killing horizon, is defined by the condition

$$\zeta^\alpha \nabla_\alpha \zeta^\beta = \kappa \, \zeta^\beta, \tag{4.9}$$

being able to determine the physical temperature $T = \kappa/2\pi$ at the horizon. So, the quantity playing the role of black hole entropy in this formula is simply 2π times the integral over Σ of the Noether charge associated with the horizon Killing field (i.e., the Killing field which vanishes on Σ). Therefore, in order to define the entropy for a spacetime locally-de

[¶]For more information about causal theory, horizons and Killing horizons see Appendix C

Sitter first let see, how is the formalism for a spacetime locally Minkowski. The E_T, is the Noether charge relative to infinitesimal coordinate variation (2.11); its integral over a closed spacelike two-dimensional surface Σ will be referred to as the Noether charge of Σ relative to δ^γ.

In this case, it can be shown that the entropy S of a Killing horizon is related to Noether's charge according to [35, 41]

$$S = \frac{2\pi}{\kappa} \int_\Sigma E_T, \tag{4.10}$$

where Σ is a two-dimensional surface endowed with a positive-defined metric, allowing in this way the definition of length. It is actually a bifurcate Killing horizon, that is, a surface formed by two Killing horizons that intersect on the space-like surface Σ. One should remark that the subscript 'T' has been used to remind that the above result holds in a general spacetime that reduces locally to Minkowski, which is transitive under ordinary spacetime *translations*.

4.2.4 Entropy on de-Sitter Cartan geometry

Having in mind the fact of those spacetimes which the local symmetry will not be described anymore by the Poincaré group, the aim of this section is to introduce the definition of the entropy for the specific case of spacetimes that reduces locally to de Sitter spacetime.

Based on the case explained in the last section, now it is going to be considered a spacetime that reduces locally to de Sitter space, which is transitive under a combination of translations and proper conformal transformations and where the local diffeomorfism are determined by (2.13). As a consequence, the Noether charges associated with such transformation will acquire an additional piece related to the proper conformal transformations, as can be seen from Eq. (2.9).

This means that in such spacetimes the relation between entropy of a Killing horizon and Noether's charge assumes the form

$$S = \frac{2\pi}{\kappa} \int_\Sigma \left[E_T - (2l)^{-2} E_K \right] \tag{4.11}$$

where E_T represents the part of the Noether charge related to translations, and E_K the part related to proper conformal transformations, and specifically for this case $\kappa = 1/l$ [42]

Entropy is consequently made up of two parts, one connected to the translations which correspond to the usual gravitational notion of entropy and another connected to the proper conformal transformations, which it has been called *proper conformal entropy* [19]. Also notes that even when it could be possible to think in separates (4.11) in two integrals, the fact of being the whole quantity into the integral what is actually conserved made this not possible, what makes this concept to be strongly attached to the local new symmetry.

The proper conformal entropy is a concept that is not present in locally-Minkowski spacetimes. In fact, in the contraction limit $l \to \infty$, the underlying de Sitter spacetime reduces to Minkowski and the usual (or Riemannian) expression (4.10) for entropy is recovered [‖].

In this approach, the de Sitter symmetry allows introducing a new kind of degree of freedom. This new general definition of entropy presents a new term that is not considered in the usual definition of black hole entropy, even in the original work of Wald [41], the invariance of the general field action is under the Poincaré symmetry, not allowing to see the new term added in Eq. (4.11).

Meanwhile, it is interesting to look for the application of this new definition of entropy. One of the main problems around the thermodynamics in horizons is the problem or paradox—associated with the information lost once a particle crosses the horizon. Many theories have developed tools in order to try to explain this possible problem, string theories, quantum field theory in curved spacetime and even there are some theories that include modifications of quantum mechanics [43].

On the other hand, there is an interesting approach pointed by Penrose, calls Conformal Cyclic Cosmology (CCC) [**]. In [44], is exposed the need of an explanation to the extraordinary evidence of the thermal equilibrium presented in the Cosmic Microwave Background. The problem is in the fact that thermal equilibrium corresponds to a maximum entropy state and if this is true, how could there be a maximum value for the entropy at the beginning of the evolution of the Universe? Clearly, there must be something wrong in the way that this topic needs to be treated or perhaps the value of the entropy to which it reference is made is not related to the whole matter content by that time.

One of the hypothesis is that for that moment, the gravitational degrees of freedom potentially available in the Universe are not being excited at all. As the time progress, the entropy rise as the initially uniform distribution of matter begins to clump allowing the star to be formed, contributing to an extra degree of freedom to the total matter content. This dynamic way to pass to *one entropy to another* is the positive aspect of the Eq. (4.11); what is by fact missing in the usual approach or definition of entropy in horizons—because the way of the spacetime-local symmetry is defined— and could be used in the paradox problem without having to change for example quantum mechanics (for example).

Commentary 4.2 The *Wald entropy* is a concept established for spacetimes which are asymptotically flat. The following Chapters will show that this definition also fits for locally-de Sitter spacetime as long as

[‖]It should be remarked that the entropy usually assumed to satisfy the second law of thermodynamics is the translational entropy given by Eq. (4.10). How to interpreted the second law of thermodynamics in terms of the new entropy (4.11) is an open question yet to be studied.

[**]Also the de-Sitter special relativity achieve to gives to the CCC a mathematical framework [16].

the matter content vanishes; also this point will clear a new total aspect of the cosmological constant which actually will not require being constant [††]. ◄

[††]New roles of the cosmological constant has been found recently in the literature gaining more attention each day [45, 46].

Chapter 5
The Schwarzschild solution in locally de Sitter spacetime

The goal of this part of the work is to explore how is redefined— in the case of being modified—black hole solution in presence of a cosmological constant. In order to do that, let us consider a black hole immersed in a universe with an effective cosmological term Λ. At the end of the Chapter, the thermodynamic of the horizons is going to be analyzed, noting that there exist a unique system formed by the two horizons l and $2M$ [19]

5.1 The Schwarzschild in the presence of a background Λ

Consider a gravitational field possessing central symmetry, and produced by a centrally symmetric distribution of matter. Using arguments based on the symmetry of the solution, its metric can be written in the form

$$ds^2 = e^\nu\, c^2\, dt^2 - e^\lambda\, dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)\,, \tag{5.1}$$

where $\nu(r,t)$ and $\lambda(r,t)$ functions of the coordinates r and t. Denoting by $\{x^0, x^1, x^2, x^3\}$, respectively, the coordinates $\{ct, r, \theta, \phi\}$, the non-zero components of the metric tensor are

$$g_{00} = e^\nu, \qquad g_{11} = -e^\lambda, \qquad g_{22} = -r^2, \qquad g_{33} = -r^2\,\sin^2\theta\,. \tag{5.2}$$

On the other hand, consider this spacetime immersed in a universe with an overall energy density ε_m, which induces an effective cosmological term

$$\Lambda = \frac{4\pi G}{c^4}\varepsilon_m\,. \tag{5.3}$$

In this case, therefore, the de Sitter modified Einstein equation must be written with an *external cosmological term*—that is, a cosmological term not generated by the masses producing the black hole—to its right-hand side:

$$R^{\mu}{}_{\nu} = \frac{8\pi G}{c^4}\left(\Pi^{\mu}{}_{\nu} - \tfrac{1}{2}\delta^{\mu}_{\nu}\Pi\right) + \Lambda\,\delta^{\mu}_{\nu}\,, \tag{5.4}$$

where $\Pi^{\mu}{}_{\nu} = \xi^{\mu}_{\alpha}\,T^{\alpha}{}_{\nu}$ is the source for the Schwarzschild solution only. As in the previous case, it is going to be consider the vacuum solution, that is, a solution that holds outside of the masses producing the black hole field, where the energy-momentum tensor $T^{\alpha}{}_{\nu}$ vanishes. The field equation (5.4) assumes then the form

$$R^{\mu}{}_{\nu} = \Lambda\,\delta^{\mu}_{\nu}\,. \tag{5.5}$$

A straightforward computation yields the following equations:

$$-\,e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} = \Lambda\,\delta^{1}_{1} \tag{5.6}$$

$$-\,\tfrac{1}{2}e^{-\lambda}\left(\nu'' + \frac{\nu'^2}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu'\lambda'}{2}\right) + \tfrac{1}{2}e^{-\nu}\left(\ddot{\lambda} + \frac{\dot{\lambda}}{2} - \frac{\dot{\lambda}\dot{\nu}}{2}\right) = \Lambda\,\delta^{2}_{2} = \Lambda\,\delta^{3}_{3} \tag{5.7}$$

$$-\,e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2} = \Lambda\,\delta^{0}_{0} \tag{5.8}$$

$$-\,e^{-\lambda}\,\frac{\dot{\lambda}}{r} = \Lambda\,\delta^{0}_{1} = 0\,. \tag{5.9}$$

All other components of (5.5) vanish identically.

Considering now that in this case Eq. (5.7) is redundant in the sense that it can be obtained from the other three equations, it is obtained:

$$e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = -\,\Lambda \tag{5.10}$$

$$e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = \Lambda \tag{5.11}$$

$$\dot{\lambda} = 0\,. \tag{5.12}$$

It follows from Eq. (5.12) that λ does not depend on the time. Furthermore as in the case before, adding Eqs. (5.10) and (5.11), it is found $\lambda' + \nu' = 0$, that is,

$$\lambda + \nu = f(t)\,, \tag{5.13}$$

where $f(t)$ is an arbitrary function of the time. However, when we chose the quadratic interval ds^2 in the form (5.1), there still remained the possibility of an arbitrary transformation of the time of the form $t = f(t')$. Such a transformation is equivalent to adding to ν

an arbitrary function of the time. With this process we can always make $f(t)$ in (5.13) to vanish. Without loss of generality, therefore,

$$\lambda + \nu = 0 \,. \tag{5.14}$$

In this case, equation (5.11) is easily integrated and gives

$$e^{-\lambda} = e^{\nu} = 1 + \frac{\kappa}{r} - \frac{\Lambda}{3} r^2 \,, \tag{5.15}$$

with κ an integration constant. Considering that for a vanishing Λ the solution must reduce to the Schwarzschild solution in locally Minkowski spacetimes, the value of the integration constant κ is that given by $2GM/c^2$. Using furthermore the relation

$$\Lambda = 3/l^2 \,, \tag{5.16}$$

the solution assumes the form

$$e^{-\lambda} = e^{\nu} = 1 - r_s/r - r^2/l^2 \,, \tag{5.17}$$

with r_s the Schwarzschild radius. The corresponding metric is the so-called Schwarzschild-de Sitter metric

$$ds^2 = \left(1 - r_s/r - r^2/l^2\right) c^2\, dt^2 - \frac{dr^2}{(1 - r_s/r - r^2/l^2)} - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \,. \tag{5.18}$$

Commentary 5.1 It is important to remark that the Schwarzschild-de Sitter solution (5.18) can be obtained directly from the ansatz (5.1) with the identification

$$g_{00} = 1 + 2\phi/c^2 \,, \tag{5.19}$$

where ϕ is the Newtonian potential, provided the gravitational potential is given by

$$\phi = -\frac{GM}{r} - \frac{\Lambda c^2}{6} r^2 \,. \tag{5.20}$$

This is the so-called Newton-Hooke potential [47]. ◄

5.2 Thermodynamics of horizons

The aim of what is next is to explore the thermodynamics aspects of the Schwarzschild-de Sitter solution. Consider a black hole and a positive cosmological constant, means that the spacetime will be describe necessary by the metric mentioned before; just that as it was already seen, the way to obtain and the origin of such cosmological constant here changes. Even this, there is some works that attempt to give to the cosmological constant another connotation when it is consider together with a black hole ”*system*”; such connotation is a thermodynamic one [45, 46].

The Comment 4.1, remarked that it could be possible some kind of *dynamic* relation between the two horizon determined by M and l; here this will be taking into consideration in order to explore if the variation of the parameter l affects or not—in some way—the Schwarzschild radius. Finally in the light of the last mentioned in [48] is mentioned that could be a relationship between the dynamic of the black hole and the dynamic of the Universe, and if the universe is taking to be describe by the de Sitter space, then, the following could attempt to deal with such assumption.

The Schwarzschild solution immersed in a universe with an effective cosmological term, is a spacetime that ends represented by the Schwarzschild-de-Sitter solution. It is on this metric that concepts such as entropy, temperature and energy are going to be studied for each one of the horizons; the goal is to analyzed the possible relation between them. Remembering that $r_s = 2M$ *

$$ds^2 = f(r)\, dt^2 - f(r)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) . \tag{5.21}$$

with

$$f(r) = \left(1 - r_s/r - r^2/l^2\right) . \tag{5.22}$$

As can be easily checked, in a locally inertial frame, where gravitation is eliminated by inertial effects, it reduces to the de Sitter metric. Considering furthermore that this metric represents ultimately the Schwarzschild solution, the function $f(r)$ must keep its Schwarzschild form

$$f(r) = 1 - \frac{2M}{\mathcal{R}} \tag{5.23}$$

with

$$R = \frac{r}{1 + r^3/2Ml^2}. \tag{5.24}$$

Seen from this perspective, the background de Sitter kinematics is found to produce a change in the Schwarzschild radius, which by equating $R = 2M$ turns out to be defined by the solutions of the cubic polynomial equation [49]

$$\frac{r_{SdS}^3}{l^2} - r_{SdS} + 2M = 0, \tag{5.25}$$

where r_{SdS} denotes the radius of the horizons present in the solution.

From now on, for the sake of simplicity, the de Sitter pseudo-radius is considered much larger than the Schwarzschild radius: $l \gg r_S = 2M$. One of the three roots of the cubic polynomial equation (5.25) is negative, and for this reason it will be neglected. The other two, when expanded in powers of M/l, are given by

$$r_{SdS} = 2M\left(1 + \frac{4M^2}{l^2} + \cdots\right) \tag{5.26}$$

*Using units where $G = \hbar = c = k_B = 1$

and

$$r'_{SdS} = l\Big(1 - \frac{M}{l} - \frac{3M^2}{2l^2} + \cdots\Big). \tag{5.27}$$

The first root represents the Schwarzschild horizon, now modified by the underlying de Sitter kinematics. The second solution, on the other hand, represents the de Sitter horizon, which in turn appears modified by the presence of the black hole. *This means that the black hole and the de Sitter horizons are connected to each other and cannot be considered separately. In particular, when studying their thermodynamic properties, one has necessarily to consider both horizons as a unique entangled system.* Of course, they have different temperatures, and for this reason they require independent thermodynamic analysis. In the remaining of this Section, it begins with the black hole and then the de Sitter case.

5.2.1 Black hole thermodynamics

The entropy $S = A/4$ defined by the de Sitter-modified Schwarzschild radius (5.26) is

$$S = 4\pi M^2\Big(1 + \frac{8M^2}{l^2} + \cdots\Big). \tag{5.28}$$

The first term on the right-hand side is the usual entropy associated with the translational part of the spacetime local transitivity. The remaining terms, as discussed in Section 4.2.4, represent the contribution to the entropy coming from the proper conformal part of the spacetime local transitivity.† In the formal limit $l \to \infty$, the underlying de Sitter spacetime contracts to Minkowski, and the usual entropy $S = 4\pi M^2$ of an isolated black hole horizon is recovered. Observe that now all thermodynamic quantities are functions of M and l, which represent the two thermodynamic variables of the system [46]. The differential of the entropy (5.28) is consequently given by

$$dS = 8\pi M\Big(dM + \frac{16M^2}{l^2}dM - \frac{8M^3}{l^3}dl + \cdots\Big). \tag{5.29}$$

The horizon temperature, on the other hand, is defined by

$$T = \frac{\kappa}{2\pi}, \tag{5.30}$$

where κ is the surface gravity, which in the case of ordinary Schwarzschild solution has the form

$$\kappa = \frac{1}{4M} \equiv \frac{1}{2\,r_S}. \tag{5.31}$$

†One should note that in the exact case only terms proportional to l^{-2} would appear in the entropy expression, as well as in all other thermodynamic variables. In the present case, however, because all variables were expanded in powers of M/l, they turn out to be expressed also by an infinite series in powers of M/l.

In the case of a de Sitter-modified Schwarzschild solution, the surface gravity turns out to be

$$\kappa \equiv \frac{1}{2\,r_{SdS}} = \frac{1}{4M}\Big(1 - \frac{4M^2}{l^2} + \cdots\Big). \tag{5.32}$$

The horizon temperature assumes then the form

$$T = \frac{1}{8\pi M}\Big(1 - \frac{4M^2}{l^2} + \cdots\Big). \tag{5.33}$$

Similarly to the entropy, the first term on the right-hand side represents the temperature of the usual black hole horizon, whereas the remaining terms represent the change induced by the underlying de Sitter spacetime, which has already discussed is part of the system Schwarzschild-de Sitter system.

The de Sitter-modified energy of a black hole can be obtained from the first law of black hole thermodynamics

$$dE = T\,dS. \tag{5.34}$$

Using expressions (5.29) and (5.33), it assumes the form

$$dE = dM + \frac{12M^2}{l^2}dM - \frac{8M^3}{l^3}dl + \cdots \,. \tag{5.35}$$

An integration yields

$$E = M + \frac{8M^3}{l^2} + \cdots \,. \tag{5.36}$$

The first term on the right-hand side is the usual energy of a black hole. The remaining terms represent the contribution to the energy coming from the underlying de Sitter spacetime. In the limit $l \to \infty$, the background de Sitter spacetime contracts to Minkowski, and the usual black hole energy is recovered.

5.2.2 de Sitter thermodynamics

There is naturally a de Sitter horizon present in spacetime. Now, here is described its thermodynamic evolution [34]. The entropy $S' = A'/4$ of the de Sitter horizon with radius r'_{SdS}, given by Eq. (5.27), is

$$S' = \pi l^2\Big(1 - \frac{2M}{l} - \frac{2M^2}{l^2} + \frac{3M^3}{l^3} + \cdots\Big). \tag{5.37}$$

The first term on the right-hand side is the usual entropy associated with the de Sitter horizon. The remaining terms, as discussed in Section 4.2.4, represent the contribution to the entropy coming from the proper conformal part of the spacetime local transitivity.

In the limit $M \to 0$, which represents absence of black hole, these terms vanish and one obtains back the entropy $S = \pi l^2$ of an isolated de Sitter horizon. The differential of S' is

$$dS' = 2\pi\Big(l dl - l dM - M dl - 2M dM + \frac{9M^2}{2l} dM + \cdots\Big). \tag{5.38}$$

In the usual case, the temperature of the de Sitter horizon is $T' = 1/2\pi l$, with l the horizon radius. In the presence of a black hole, the de Sitter horizon turns out to be given by r'_{SdS}, and the temperature assumes the form

$$T' \equiv \frac{1}{2\pi r'_{SdS}} = \frac{1}{2\pi l}\Big(1 + \frac{M}{l} + \frac{3M^2}{2l^2} + \cdots\Big). \tag{5.39}$$

The first term on the right-hand side represents the temperature of an isolated de Sitter horizon. The remaining terms represents the change induced by the presence of the black hole.

The black hole-modified energy of a de Sitter horizon can be obtained from the thermodynamic equation

$$dE' = T' \, dS'. \tag{5.40}$$

Of course, since the cosmological term is interpreted as a purely kinematic entity, and not a solution of Einstein's equations with a source possessing negative pressure, no $P'dV'$ appears in the above thermodynamic equation. Using expressions (5.38) and (5.39), that equation assumes the form

$$dE' = dl - dM - \frac{3M}{l} dM + \frac{M^2}{2l^2} dl + \frac{M^2}{l^2} dM - \frac{3M^3}{2l^3} dl + \cdots . \tag{5.41}$$

Integrating

$$E' = l - M - \frac{2M^2}{l} + \frac{13M^3}{12l^2} \cdots . \tag{5.42}$$

The first term on the right-hand side represents the usual energy of an isolated de Sitter horizon. The remaining terms represent the contribution to the energy coming from the presence of a black hole.

Finally, the temperature of the two horizons are different, which means that the system is not thermodynamically stable [50]. This suggests that there can exist a heat flow from the dynamic black hole horizon to the kinematic de Sitter horizon, and vice versa. In fact, as discussed in Chapter 2, in a locally-de Sitter spacetime there is an additional freedom—not present in locally Minkowski spacetimes—that allows ordinary energy-momentum current to transform into proper conformal current, while keeping the total energy $E + E'$ constant, where E is the black hole energy (5.36) and E' is the de Sitter energy (5.42).

This conservation law can be written in the equivalent form

$$dE + dE' = 0,$$

from where the constraint

$$dl = \left(\frac{3M}{l} - \frac{13M^2}{l^2} + \cdots \right) dM \tag{5.43}$$

between the thermodynamic variables of the de Sitter and of the black hole horizons can be easily obtained. Such constraint says that every change in the radius of a black hole horizon—either by emitting or absorbing a particle—will produce concomitant changes in the radius of the de Sitter horizon. This result provides a new scenario for the study of cosmology, and in particular for the study of Penrose's Conformal Cyclic cosmology [44,51]. Finally, under a speculatively context, it is known that recent experimental results indicate that the universe expansion became accelerated in the last few billion years [9–11]. A possible explanation for this late time acceleration is to suppose that the cosmological term Λ is bigger today than it was a few billion years ago. Considering that $\Lambda \sim l^{-2}$, this is equivalent to say that the de Sitter parameter l is becoming smaller, which implies that $dl < 0$. Since the leading-order term of the expansion within parentheses in the constraint (5.43) is positive, this implies that $dM < 0$. In this context, therefore, the recent accelerated expansion of the universe can be explained by supposing that in the last few billion years the black holes inside the de Sitter causal horizon—all of them exchanging energy (heat) with the de Sitter horizon—are preponderantly emitting more Hawking radiation than absorbing baryonic matter.

Chapter 6
On the dark energy problem

6.1 The dark energy problem

In the standard cosmology there exist a problem—among other—which is, that most of the content of matter in the Universe has never been directly detected in the laboratory. The mayor among of the total energy in the Universe is the form of what is called dark energy almost 69 %, the remaining contributions are composed of matter (dark and baryonic matter). Into the scenarios in cosmology, the most accepted candidate for being that mysterious component is the cosmological constant Λ—which is strongly projected as responsible for the accelerated expansion of the Universe.

The cosmological constant was originally introduced by Einstein in 1917 to achieve a static Universe. After Hubble's discovery of the expansion of the Universe in 1929, it was dropped by Einstein as it was no longer required. From the point of view of particles physics, the cosmological constant arises as an energy of the vacuum but if Λ originates from a vacuum energy density, then this must require a fine-tuning to adjust, given the discrepancy between the observed value and the theoretical one. That is to say, observationally it is known that the cosmological term is of order the present value of the Hubble parameter H_0, which corresponds to a critical density of

$$\epsilon_\Lambda \sim 10^{-47} GeV^4. \tag{6.1}$$

On the other hand, another possibility to approach to the critical density associated to the cosmological constant is thinking, that such value could arise as the vacuum energy density and like this, evaluating by the sum of the zero-point energies of quantum fields with mass m,

$$\epsilon_\Lambda = \frac{1}{2}\int_0^\infty \frac{d^3\mathbf{k}}{(2\pi)^3}\sqrt{k^2+m^2} \tag{6.2}$$

$$= \frac{1}{4\pi^2}\int_0^\infty dk\, k^2\sqrt{k^2+m^2} \tag{6.3}$$

so, It is expect the validity of quantum field theory up to some cut-off k_{max}, in which the last integral is finite

$$\epsilon_\Lambda \approx \frac{k_{max}^4}{16\pi^2}; \tag{6.4}$$

for some extreme case of general relativity, one expects to be valid to just below the Planck scale $m_{pl} \sim 10^{19}$, then if this scale is considered as the k_{max}, it is found that

$$\epsilon_\Lambda \approx 10^{74} GeV^4 \tag{6.5}$$

which is about 10^{121} orders of magnitude larger that the observed, revealing what is the so-called *the cosmological constant problem.*

Returning to the gravitational role that may play the cosmological constant, it is known that from the Einstein field equation with cosmological constant for a universe homogenous and isotropic, it is obtained the Friedman equation, from which it is possible to read the following

$$\varepsilon_c = \varepsilon_m + \frac{\Lambda}{8\pi G} \tag{6.6}$$

where $\varepsilon_c = 3H^2c^2/8\pi G$ and ε_m is the Friedman critical density and the mass density respectively. The another term is what it might represent the density associated to the cosmological constant.

So here comes the thing, for a universe with a flat spatial section $k = 0$ what take us to a cosmological parameter $\Omega = 1$—into this parameter it is expected all kind of energy, matter and even dark energy—it is obtained that the energy density associated to Λ is of the order of 10^{-52}, generating a coincidence for the critical and the mass density, given its small value [52, 53].

What really happens is the fact, that at 1998, *Pelmutter et al* [9] found that for a universe with $k = 0$,

$$\Omega_m + \Omega_\Lambda = 1 \tag{6.7}$$

where $\Omega_m = 0,28^{+0,09}_{-0,08}$ (1σ-statistical), letting a value for $\Omega_\Lambda \sim 70\%$.

The reason for the above is to expose in a better way that there is not a clear role for the cosmological constant—it is dynamic, it could change according to Eq. (3.11)—-and also there is not a unique procedure to obtain the value of the density associated with it. But what it is unclear is that indeed there is a discrepancy in such value which by fact does not help to conciliate the results in Eq. (6.7). There have been a number of attempts to solve this problem, quantum gravity, supergravity, string theory and even changing gravity, nevertheless, no clear solution is known up to date. But from the cosmological point of view and having in mind the role that might have the cosmological constant in the inflation, how the cosmological constant and inflation are conciliated?

There are two ways either supplementing the energy-momentum tensor by an exotic form of matter such cosmological constant or scalar field or even changing the theory of gravity. In inflationary cosmology, the cosmological constant is considered as *fluid with a constant equation of state* $w = -1$. So for example, in the case of the accelerating cosmic expansion being considered driven by a new form of energy such as a scalar field ϕ with potential $V(\phi)$; in the limit of $\frac{1}{2}\dot{\phi}^2 << V(\phi)$ the scalar field acts like a cosmological constant, with

$$P_\phi \sim -\rho_\phi, \tag{6.8}$$

which at the end it will require the existence of an exotic fluid with such characteristic.

On the other side, there are the theories that attempt to modify the geometry itself. Into the modifies gravity theories, the $f(R)$-gravity are the most discussed. These theories replace the Ricci scalar R by a function of $f(R)$ in the Einstein-Hilbert action

$$S_{EH} = \int dx^4 \sqrt{-g}\, f(R). \tag{6.9}$$

Just for mention one example [54] , here is considered $f(R) = R - \alpha^2/R$—a $1/R$ function—where in contrast to the Einstein-Hilbert action gives a non-trivial second order equation in the metric whose solution approach to a de Sitter Universe, given like this an alternative explanation for the cosmological acceleration.

In this part of the work is presented another alternative. It was already studied the de Sitter modified gravitation theory, and calculated the Newtonian limit of this new theory, the purpose of the next is to use this new modified potential and explores how the Newtonian Friedman equation provides a good account of the dark energy content of the present-day Universe [20]— giving a positive contribution to the problem exposed before—nevertheless, it seems important to remark that even when the Friedman equation coming from the Einstein general relativity coincides with the ones coming from the Newtonian gravity, these equations are an analogous of the Friedman equation just that they are obtained under the ”Newtonian gravitation”—in this case, a modified Newtonian gravitation.

Finally, in the light of the two alternative theories briefly exposed before, the modified-de Sitter gravitational theory nor does not require the existence of any fluid with specific characteristics to emulate the cosmological constant nor appeals for redefinition of the geometry in the Einstein-Hilbert action; this theory appeal entirely to first principles: to replace the Poincaré invariant special relativity by a de Sitter invariant relativity.

6.2 The Newtonian Friedmann equations

The derivation of the relativistic Friedmann equations is usually done by inserting the Friedmann-Robertson-Walker metric into Einstein equations. Nevertheless, it is possible to obtain the Friedmann equations from the Newtonian gravity—under some limits, for example, the absence of pressure. Even the surprising result in the way of these two approaches coincided, there are fundamental differences between them. For example, whereas in the Newtonian view the universe expands in flat Euclidian space under the influence of Newtonian gravity, in the relativistic view the whole universe consists of an expanding curved space. The purpose here is not to study the time evolution of the universe, but just to explore the consequences of the de Sitter-invariant special relativity for the present-day universe, and for that, the Newtonian Friedmann equations should suffice.

Considering a sphere of radius $\mathcal{R} = \mathcal{R}(t)$ and mass M undergoing an isotropic and homogeneous expansion. The equation of motion for $\mathcal{R}$ can be obtained from the gravitational acceleration at the border of the sphere

$$\frac{d^2\mathcal{R}}{dt^2} = -\frac{GM}{\mathcal{R}^2} + \frac{GM\Lambda}{6}, \tag{6.10}$$

where the de Sitter modified force Eq. (3.61) were consider. Multiplying both sides by $d\mathcal{R}/dt$ and integrating, the energy equation is given by

$$\frac{1}{2}\left(\frac{d\mathcal{R}}{dt}\right)^2 = \frac{GM}{\mathcal{R}} + \frac{GM\Lambda\mathcal{R}}{6} + E, \tag{6.11}$$

where the integration constant E represents the total energy per unit mass at the surface of the expanding sphere. Now the radius turn into the form

$$\mathcal{R}(t) = r\, a(t), \tag{6.12}$$

with $a(t) \equiv a$ the scale function parameter, and r the comoving radius of the sphere. Recalling that the mass of the sphere is

$$M = \frac{4\pi}{3}\mathcal{R}^3\rho, \tag{6.13}$$

with $\rho \equiv \rho(t)$ the mass density, after some algebraic manipulation the energy equation (6.11) assumes the form

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{4\pi G\Lambda\rho\mathcal{R}^2}{9} + \frac{2E}{\mathcal{R}^2\, a^2}. \tag{6.14}$$

Now, in order to make contact with the Friedmann equations, the mass density ρ must be replaced by the total density ε_m/c^2, where the subscript 'm' denotes all forms of matter (or source) energy, in addition to the mass energy. Furthermore, the energy E must be related to the curvature of space. If it is consider

$$E = -\frac{kc^2}{2}, \tag{6.15}$$

with k the curvature parameter, Eq. (6.14) acquires the usual form of the Friedmann equations

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\,\varepsilon_m + \frac{4\pi G\Lambda\mathcal{R}^2}{9c^2}\,\varepsilon_m - \frac{kc^2}{R^2\, a^2}, \tag{6.16}$$

where $H = \dot{a}/a$ is the Hubble parameter. Assuming a universe with a flat space section ($k = 0$), it becomes

$$\frac{3H^2c^2}{8\pi G} = \left(1 + \frac{\Lambda\mathcal{R}^2}{6}\right)\varepsilon_m\,. \tag{6.17}$$

One should note that the Friedmann critical energy density for a vanishing Λ, given by

$$\varepsilon_c = \frac{3H^2c^2}{8\pi G}\,, \tag{6.18}$$

is modified due to the presence of the de Sitter background:*

$$\varepsilon_c = \left(1 + \frac{\Lambda\mathcal{R}^2}{6}\right)\varepsilon_m\,. \tag{6.19}$$

Considering that $\varepsilon_c = \varepsilon_m + \varepsilon_\Lambda$, immediately can be identify the dark energy density

$$\varepsilon_\Lambda = \frac{\Lambda\mathcal{R}^2}{6}\,\varepsilon_m\,. \tag{6.20}$$

The results in Eq. (6.20) differs from the obtained under the *standard approach*— presented in Eq. (6.6)—where exist an equivalent behavior between the mass-energy density and the dark energy density. Here instead, there is a coupling relationship between both densities, making clear that in this model, that the existence of one determines the other. According to the de Sitter invariant special relativity, any physical system with energy density ε_m produces a cosmological term Λ, with a dark energy density ε_Λ which is necessary to comply with the local symmetry of spacetime, now ruled by the de Sitter group. This is exactly what it was commented at Comment 4.2, the presence of a local cosmological term Λ is then a natural consequence of the presence of ordinary matter.

For example, using the approximate present-day values of Λ and $\mathcal{R}$, given respectively by

$$\Lambda \simeq 10^{-52}\ \mathrm{m}^{-2} \qquad \text{and} \qquad \mathcal{R} \simeq 4 \times 10^{26}\ \mathrm{m}\,, \tag{6.21}$$

it is obtain

$$\varepsilon_\Lambda = 2.7\ \varepsilon_m\,. \tag{6.22}$$

The corresponding density parameters are found to be

$$\Omega_m \equiv \frac{\varepsilon_m}{\varepsilon_c} \simeq 0.24 \qquad \text{and} \qquad \Omega_\Lambda \equiv \frac{\varepsilon_\Lambda}{\varepsilon_c} \simeq 0.76\,, \tag{6.23}$$

which are close to the current values obtained from observations. In this way the present theory provides a good explanation for the so-called *coincidence problem*.

Commentary 6.1 Finally, it is important to remark that the values for the cosmological parameter Ω_m depends on the cosmological model used. To have a full description of the model presented here, it is expected to extract this values from the own model by fitting the modified de Sitter Friedman equation without any approximation. ◄

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*Note that, by using the identity $\Lambda = 3/l^2$, relation (6.19) can be rewritten in the form $\varepsilon_c = \xi_0^0\,\varepsilon_m$, with ξ_0^0 the Killing vector (B.5). This relation between the energy densities is then a direct consequence of the underlying de Sitter symmetry.

Chapter 7
On the dark matter problem

7.1 The missing mass problem

Since the 1930s, astronomical observations have accumulated evidence that our understanding of the dynamics of galaxies and groups of galaxies is grossly incomplete, and by today it is a fact that there is something missing in the way that such astrophysical systems are studied.

The discrepancy between the observed and the theoretical around the subject mentioned is known as the ”missing mass problem” was first identified in clusters by Swiss astronomer Fritz Zwicky in 1933 (who studied the Coma cluster), and subsequently extended to include spiral galaxies by the 1939 work of Horace Babcock on Andromeda. These early studies were augmented and brought to the attention of the astronomical community in the 1960s and 1970s by the work of Vera Rubin at the Carnegie Institute in Washington, who mapped in detail the rotation velocities of stars in a large sample of spirals. While the Laws in the Newton theory predict that stellar rotation velocities *should decrease with distance from the galactic centre*, Rubin and collaborators found instead that they remain almost constant; the rotation curves are said to be ”flat”. This observation necessitates at least one of the following:

1. There exists in galaxies large quantities of unseen matter which boosts the stars' velocities beyond what would be expected on the basis of the visible mass alone, or

2. Newton's Laws do not apply to galaxies.

The first statements lead to the dark matter hypothesis and the latter to the modified gravity theories. This part of the work is addressed to look for some new dynamics for this problem instead of considering some kind of exotic missing mass. In what follows, an attempt will be made to address the problem of the missing mass in galaxy rotation curve and for that, it will be explored briefly one of the modified theories, the one proposed by Mordehai Milgrom follows by the proposed under the de Sitter-invariant special relativity.

7.1.1 Galaxy rotation curve

Galaxies are found in a wide range of shapes, sizes, and masses, but usefully divided into four main types according to Hubbles's classification system, they are, Elliptical, Lenticular, Spiral, and irregular galaxies. The following analysis is made on spiral galaxies.

A spiral galaxy is a system composed of gas, dust and stars with three main components, a flat-rotating disk, a central bulge and a near-spherical galactic halo; each one of this region is composed of a specific kind of population of star. One of the subject studied in the dynamics of these objects is the rotating velocity curve of a star or cloud of gas. This latter is tightly related to the gravitational potential in it.

The Fig. 7.1, is a pictorial manifestation of what has been exposed. The curve **A** falls with the distance as it is expected, nevertheless the curve **B**, as long as the distance increases from the galactic center, the curve begins to grow and start to become flat.

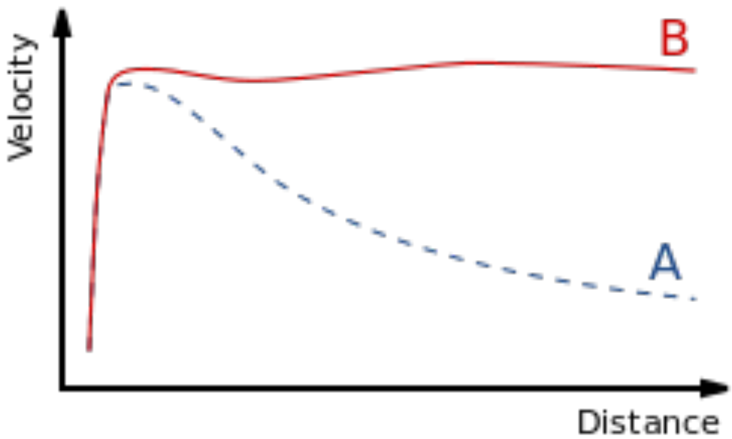

Figure 7.1: Theoretical and observational galaxy rotation curves

The first approach address to the existence of a non-baryonic matter, the so called *Dark matter*, a kind of exotic matter that might compose the halo region of the galaxy. It is important to note that each one of the region of the galaxy has a characteristic-curve, for example, in the inner region of the galaxy; the disk, the rotation curve is expected to behave following the Newton law, but it is out of this region where the matter content behaves different. In the Fig. (7.2), one can see a different curve for each one of the regions, even the speculative curve, predicting the existence of a dark matter in the galactic halo.

7.1.2 The set up for a galaxy rotating curve

In this case, of Newtonian gravitational force

$$F(r) = -\frac{G\,M(r)}{r^2}\,. \tag{7.1}$$

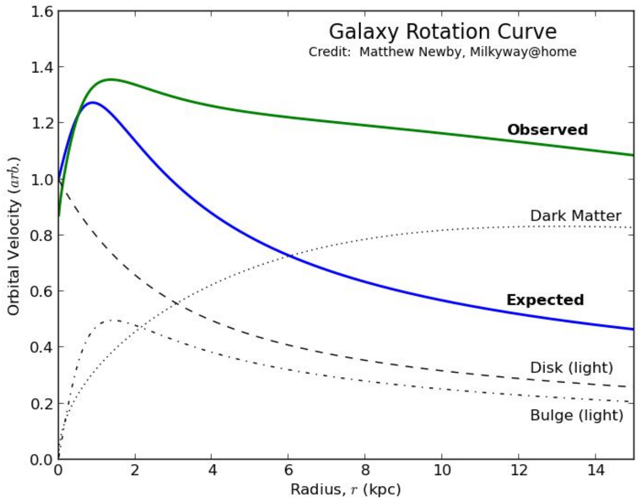

Figure 7.2: The dynamic for each sector of the galaxy

The circular velocity $v_c(r)$ of a star at a distance r from the galactic center is [55]

$$v_c^2(r) \equiv r\,|F(r)| = \frac{G\,M(r)}{r}\,, \tag{7.2}$$

where

$$M(r) = 4\pi \int_0^r \rho(r')\,r'^2 dr' \tag{7.3}$$

is the galaxy inner mass, with $\rho(r)$ the mass density in function of the radius. Given the mass density profile $\rho(r)$ of a galaxy, it is then possible to determine the galaxy rotation curve. Generically, the rotation curve can be divided into three regions: (i) an inner region in which the speed rises linearly with the distance from the center; (ii) a region where the speed reaches a maximum and then begins to decrease—it is important to take into account that this regions is observable in some galaxies—-(at the so-called *turn over radius*); (iii) a Keplerian region—which a region that is not observable—in which the gravitational force resembles that of a point mass force,

$$F = -\frac{GM}{r^2}, \tag{7.4}$$

which made the rotation speed (7.2) falls as $v_c \sim r^{-1/2}$ in this region, as depicted in the curve A of Figure 7.1. However, instead of such behavior some galaxies show a flat-like rotation curve, as depicted in curve B of Figure 7.1.

7.1.3 The Tully-Fisher relation

The Tully-Fisher relation is a correlation that holds for galaxies with disks stabilized by rotation, between the intrinsic luminosity L of the galaxy in optical or near-infrared bands and the rate of rotation V,

$$L_\star \sim V^\alpha. \tag{7.5}$$

Such luminosity profile is built from the stellar bright, allowing to infer the mass distribution of the galaxy. For the aim of this work, the importance of this relation is around to the MOND formalism, because is one of the strong experimental statement that the theory has. In this particular case, it is found that

$$L_\star \sim V^4 \tag{7.6}$$

7.2 Modified gravity theories

7.2.1 Modified Newtonian Dynamic, MOND

At the begging of this Chapter was discussed the problem of the discrepancy in the dynamics of the galaxies. In 1983, was postulated the Modified Newtonian dynamics (MOND). The motivation of this theory was to explain the fact that the rotation curve in galaxies was observed to be larger than the expected on the Newtonian dynamics.

The MOND theory is based on the fact that the discrepancy can be solved *if the gravitational force between the star in the galaxy in the outer-disk region is proportional to the square of the centripetal acceleration* [56]. The basic premise of MOND is that while Newton's laws have been extensively tested in high-acceleration environments (in the Solar System and on Earth), they have not been verified for objects with extremely low acceleration, such as stars in the outer parts of galaxies. The theory postulates a new effective gravitational force law (sometimes referred to as "Milgrom's law") that relates the true acceleration of an object to the acceleration that would be predicted for it on the basis of Newtonian mechanics. This law is the keystone of MOND, is chosen to reduce to the

Newtonian result at high acceleration, but lead to different (”deep-MOND”) behavior at low acceleration; in other words, the dynamics of the outer-disk region must be described differently—which is the region where the velocity curve starts to be flat.

Such a new force is defined as

$$\mathbf{F}_{New} = m\,\mu\Big(\frac{a}{a_0}\Big)a, \tag{7.7}$$

a is the gravitational acceleration, μ is the so-called interpolating function, and a_0 the new constant that establishes the transition between the Newtonian regimen to the deep-MOND one. Defining the interpolating function as

$$\mu\Big(\frac{a}{a_0}\Big) = \Big(1 + (a/a_0)^2\Big)^{-1/2} \tag{7.8}$$

and for $a \ll a_0$, finally helps to turn Eq. (7.7)in

$$\mathbf{F}_{New} = m\frac{a^2}{a_0}. \tag{7.9}$$

By other hand, the centripetal acceleration is given by,

$$a = \frac{v^2}{r}, \tag{7.10}$$

then applying this to an object of mass m in circular orbit around a point mass M, one finds

$$\frac{GM\,m}{r^2} = \frac{m\,(v^2/r)^2}{a_0}, \tag{7.11}$$

getting at the final

$$v^4 = GMa_0, \tag{7.12}$$

the rotation velocity is proportional this transition acceleration. An interesting point here is the fact that this expression for the velocity is independent of its distance r from the center of the galaxy, like this the rotation curve from some region become flat, matching with the observations.

The MOND theory is an alternative to LCDM model. An alternative based on changes in the dynamic of the galaxies. One of the powerful characteristics is that it provides a theoretical basis for the Tully-Fisher relation. Nevertheless, the theory is constructed in order to achieve these results and for this, the theory is kind of a *ad hoc* empirically motivated model.

7.2.2 The de Sitter invariant and the galaxy rotation curve

As it was seen, the replacement of the Poincaré symmetry by the de Sitter symmetry in the construction of general relativity allowed to define a modified Einstein equation and through the linearization of it a modified Newtonian potential and force were obtained [20].

In the following, is going to be presented how this potential and force give an understandable theoretical approach for the galaxy rotation curve problems.

The gravitational potential

$$\phi(r) = -\frac{GM}{r} - \frac{GM\Lambda(r)}{6}\, r\,, \tag{7.13}$$

with Λ given by Eq. (3.11). Considering that the galaxy mass density ρ_m is not constant, Λ is not constant either, and the corresponding gravitational force $F = -d\phi(r)/dr$ assumes the form

$$F = -\frac{GM}{r^2} + \frac{GM\Lambda(r)}{6} + \frac{GM}{6}\, r\frac{d\Lambda(r)}{dr}\,. \tag{7.14}$$

The first term on the right-hand side represents the usual attractive Newtonian force. The background de Sitter spacetime contributes with an additional repulsive force proportional to $\Lambda(r)$, as well as with a force proportional to the radial derivative of $\Lambda(r)$, which will be attractive or repulsive depending on the sign of $d\Lambda(r)/dr$. In what follows the gravitational force (7.14) is going to be used in order to study the circular velocity of a star around the galactic center.

The Fig 7.3, represented how the problem is going to be study. First the inner region of the galaxy $r \ll r_0$, where the circular velocity of the stars rises almost linearly with r.

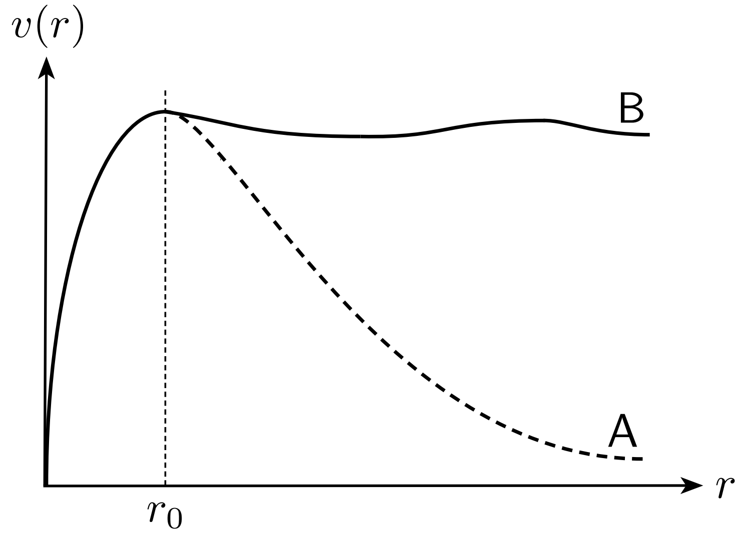

Figure 7.3: Here it is denoted the region on where the analysis is performed, for $r \ll r_0$, $r = r_0$ and $r \gg r_0$

This means that in this region only the Newtonian force is in action, and the mass density $\rho_m(r)$ of the galaxy decreases slowly with the radius r. In fact, for a nearly constant mass density $\rho_m(r) \simeq \rho_0$, the inner mass $M(r)$ assumes the form $M(r) = (4/3)\pi\rho_0 r^3$, and the Newtonian star velocity is easily seen to grow linearly with r

$$v(r) = \sqrt{(4/3)\pi G\rho_0}\, r\,, \tag{7.15}$$

in agreement with observations.

At the turn over point r_0, on the other hand, using the Milky Way values as the typical values for the mass density and radius of the bulge [**?**],

$$\rho_0 \simeq 10^{-12}\ \mathrm{Kg\,m^{-3}} \quad \text{and} \quad r_0 = 3\ \mathrm{kpc} \simeq 10^{20}\ \mathrm{m}\,, \tag{7.16}$$

the corresponding bulge cosmological term is found to be

$$\Lambda_0 = \frac{4\pi G}{c^2}\rho_0 \simeq 10^{-40}\ \mathrm{m^{-2}}\,. \tag{7.17}$$

Considering furthermore that around r_0 the mass density $\rho_m(r)$ still decreases slowly with the radius r, the same will happen to cosmological term $\Lambda(r)$, and consequently the last term of the gravitational force (7.14) can be neglected. In this case it is easy to see that, at the turn over point r_0, the first two terms of the force (7.14) are of the same order:

$$|GM_0/r_0^2| \simeq |GM_0\Lambda_0/6|\,. \tag{7.18}$$

The region around the turn over radius r_0 can accordingly be considered a transition region from the Newtonian to the de Sitter modified Newtonian regimes.

Now, the interesting part in terms of dynamics will be for $r \gg r_0$, which defines the Keplerian region*, the Newtonian force becomes negligible and the relevant force assumes the form

$$F = \frac{GM_0}{6}\Lambda(r) + \frac{GM_0}{6} r\frac{d\Lambda(r)}{dr}\,, \tag{7.19}$$

where it is assumed that in this region the whole mass of the galaxy can be represented by the bulge mass M_0. The squared circular velocity of a star at a distance r from the galactic center is now given by

$$v^2(r) \equiv r|F(r)| = \frac{GM_0}{6}\left[\Lambda(r)r + r^2\frac{d\Lambda(r)}{dr}\right]. \tag{7.20}$$

Then comes the point: the above expression has a solution in which $v^2(r)$ is constant. Such solution is obtained when $\Lambda(r)$ satisfies the first-order differential equation

$$r^2\frac{d\Lambda(r)}{dr} + \Lambda(r)r = \beta\,, \tag{7.21}$$

with β a constant.

*Even when from the experimental point of view, we are aware that this region is not visible, here in this model we tried to infer some analytical statements after all

It is convenient at this point to introduce the dimensionless coordinate $r' = r/r_0$, in terms of which the differential equation (7.21) assumes the form

$$r'^2 \frac{d\Lambda(r)}{dr'} + \Lambda(r)r' - \frac{\beta}{r_0} = 0\,. \tag{7.22}$$

In terms of the original variables, its solution is

$$\Lambda(r) = \frac{\beta}{r} \ln\left(\frac{r}{r_0}\right) + \gamma\,\frac{r_0}{r}\,, \tag{7.23}$$

with γ an integration constant. Since at $r = r_0$ the cosmological term has the value $\Lambda(r) = \Lambda_0$, from where immediately it is possible to infer, that $\gamma = \Lambda_0$. Imposing furthermore that $d\Lambda(r)/dr = 0$ at $r = r_0$, it is found that that $\beta = \Lambda_0 r_0$. The final form of the solution is consequently

$$\Lambda(r) = \Lambda_0 \left[\frac{r_0}{r} \ln\left(\frac{r}{r_0}\right) + \frac{r_0}{r}\right]. \tag{7.24}$$

On account of the relation (3.11), in terms of the mass density the solution is written as

$$\rho(r) = \rho_0 \left[\frac{r_0}{r} \ln\left(\frac{r}{r_0}\right) + \frac{r_0}{r}\right]. \tag{7.25}$$

The combination of this fiducial mass density profile with the de Sitter modified Newtonian force yields a flat rotation curve for the galaxy without necessity of supposing the presence of dark matter. It should be noted that this mass density profile represents a small correction to the power law $\rho(r) \simeq \rho_0(r_0/r)$, which is within the class of physically acceptable profiles [55].

Furthermore, note also that, from Eqs. (7.20) and (7.21), the squared velocity of the flat portion of a galaxy rotation curve is given by

$$v_0^2 \equiv \frac{GM_0}{6}\,\beta = \frac{GM_0}{6}\,\Lambda_0 r_0\,. \tag{7.26}$$

Using the average values for the mass and radius of a typical galaxy bulge, given respectively by [**?**]

$$M_0 = 10^{10} M_\odot \simeq 2\times 10^{40}\ \text{Kg} \qquad \text{and} \qquad r_0 = 3\ \text{kpc} \simeq 10^{20}\ \text{m}\,,$$

as well as Λ_0 given by Eq. (7.17), the squared circular velocity of a star around the galactic center is found to be

$$v_0^2 = \frac{GM_0}{6}\,\Lambda_0 r_0 \simeq 10^{10}\ \text{m}^2\text{s}^{-2}\,, \tag{7.27}$$

which is of the order of magnitude of the flat portion of a typical galaxy rotation curve. For example, the squared velocity of the Sun around the galactic center is $v^2 \simeq 5\times 10^{10}\ \text{m}^2\text{s}^{-2}$.

In addition to explain the galaxy rotation curve of galaxies without necessity of assuming the presence of dark matter, the de Sitter modified Newtonian force gives the correct order of magnitude for the circular velocities of the stars.

Finally, in addition, not all galaxies show a perfectly flat rotation curve. In spite of this fact, these galaxies can still be studied in this approach: one has simply to replace Eq. (7.21) by

$$r^2 \frac{d\Lambda(r)}{dr} + \Lambda(r) r = \beta(r) , \tag{7.28}$$

with $\beta(r)$ a function describing the behavior of the galaxy rotation curve in the Keplerian region. The solution to this equation is easily found to be

$$\Lambda(r) = \frac{1}{r} \int \frac{1}{r} \beta(r) dr + \gamma \frac{r_0}{r} . \tag{7.29}$$

Considering that the explicit form of $\beta(r)$ could be inferred from observations, one can then find the explicit form of $\Lambda(r)$, or equivalently, the explicit form of the mass density profile $\rho(r)$ that gives rise to the observed rotation curve.

As an illustration, let us consider the specific case in which, instead of a flat rotation curve, the circular velocity of the stars rises slowly and linearly with the distance r. This corresponds to a $\beta(r)$ of the form

$$\beta(r) = a \, \Lambda_0 r , \tag{7.30}$$

where a is a constant that determines how fast the circular velocities rise. In this case, and using the boundary condition $\Lambda(r_0) = \Lambda_0$, solution (7.29) assumes the form

$$\Lambda(r) = \Lambda_0 \Big[a + (1 - a) \frac{r_0}{r} \Big] , \tag{7.31}$$

which corresponds to the mass density profile

$$\rho(r) = \rho_0 \Big[a + (1 - a) \frac{r_0}{r} \Big] . \tag{7.32}$$

Conversely, given a specific mass density profile, we can proceed backwards to find $\beta(r)$, which determines the corresponding galaxy rotation curve through Eq. (7.20). This allows the theory to be applied individually for each galaxy, taking into account their different specificities. It is worth mentioning finally that our results are in agreement with recent theoretical and experimental evidences favoring a gravitational solution to the missing mass problem [57], as well as with the lack of experimental signs of particles that could play the role of dark matter [58–61].

Chapter 8
Conclusions and outlook

Physical motivations required Galileo relativity to be generalized to Einstein special relativity, in order to have a theory capable to deal with velocities comparable to the speed of light. However, Einstein special relativity cannot deal with phenomena at the Planck scale, where there is an invariant length represented by the Planck length l_P. The problem was—as it was already pointed out—the Lorentz symmetry, given the fact that the Lorentz symmetry is believed not to allow the existence of an invariant length parameter. Nevertheless, this is not a problem if it is considered that the Lorentz transformations do not change the curvature of the homogeneous spacetime in which they are performed. Being the de Sitter spacetime a homogeneous space, *Lorentz transformations are found to leave the length parameter l invariant*. Moreover—what is truly interesting—-if the Planck length l_P is to be invariant under Lorentz transformations, it must then represent the pseudo-radius of spacetime at the Planck scale, which will be a de Sitter space with a Planck cosmological term

$$\Lambda_P = 3/l_P^2 \simeq 1.2 \times 10^{66}\,\mathrm{m}^{-2}$$

and therefore in de Sitter-invariant special relativity the existence of an invariant length-parameter at the Planck scale does not clash with Lorentz invariance—the Lorentz symmetry remains at all scales even at the Planck scale–and then the theory can be thought of as a generalization of Einstein special relativity for energies near to the Planck energy.

From the mathematical point of view, when the local Riemannian geometry is generalized to the Cartan geometry important facts related to the local symmetry are attached. As the de Sitter spacetime is transitive under a combination of translations and proper conformal transformations the definition of diffeomorphism in this spacetimes will change accordingly and as a consequence a new Noether charge is defined. Under the latter and supporting by the *Wald entropy's approach*, the *proper conformal entropy*

$$S = \frac{2\pi}{\kappa} \int_\Sigma \left[E_T - (2l)^{-2}\, E_K \right];$$

the relation between the entropy of a Killing horizon and Noether's charge was consequently made up of two parts, one connected to the translations E_T—which corresponds to the usual gravitational notion of entropy—and another connected to the proper conformal transformations E_K. This new definition of entropy represents a new paradigm, given the extra term that carries on. It is important to mention that this characteristic is something that is not present in spacetime which reduces to Minkowski spacetime.

On the other hand, certainly, the replacement of the Poincaré-invariant Einstein special relativity by a de Sitter-invariant special relativity produces concomitant changes in all relativistic theories included general relativity. Considering de Sitter space as a fundamental background instead of Minkowski, when general relativity is constructed on it, the underlying spacetime will be described by de Sitter, and the local kinematic is ruled by the de Sitter group, turning the spacetime to be described by a Cartan geometry and gravitation turns out to be represented by a *de Sitter-modified general relativity*. As a consequence, the kinematical and dynamical curvature were both included in the same Riemann tensor making possible that the cosmological term no longer appears explicitly in the Einstein equation and consequently the second Bianchi identity does not require to be constant

$$\nabla_\mu (R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R) = 0.$$

One of the first point explored under the above was, how the de Sitter kinematics modified the Schwarzschild solution even in the presence of an overall effective cosmological constant. It was found that, that the Schwarzschild solution in locally de Sitter coincide with the usual solution and in the case to count with an external cosmological constant the Schwarzschild de Sitter solution brought important statements in the thermodynamics in horizons.

The de Sitter modified general relativity also was used in order to approach to some mainstream problems in cosmology and astrophysics. Observationally, the corresponds density parameter for mass and dark energy in the Universe are well known. Assuming a flat Universe Perlmutter *et al* found $\Omega_m = 0,28 \pm 0,04$, thus showing that about 70% of the energy density of the present Universe consists of dark energy. Nevertheless, from the theoretical point of view, the critical density associated to Λ does not report the same. According to the de Sitter-modified general relativity, any physical system with energy-momentum tensor induces in spacetime a local cosmological constant term

$$\Lambda = \frac{4\pi G}{c^4}\, \varepsilon_m \,,$$

in regions where no matter is present, both the mass density ε_m and the cosmological term will not contribute. Now thinking about the observational values reported in the Supernova cosmology project, for a universe as a whole, ε_m produces a cosmological term Λ, with a dark energy density ε_Λ, which in the Newtonian limit was found to be related as

$$\varepsilon_\Lambda = \frac{\Lambda \mathcal{R}^2}{6}\, \varepsilon_m \,.$$

At the moment to be applied to the present-day universe, gave a good account of the observed relation between those, seeing that the present theory provided a natural explanation for the coincidence problem.

Having performed the Newtonian limit of the modified Einstein field equation also, one could verify their prediction for the galactic rotation curves problem. The dependence of the cosmological constant on the local matter content and its relation to the extra term of the modified potential, gave a constraint about what it is the expression for the cosmological function, in order to achieve a flat rotation curve of galaxy—in the Keplerian region—- without the necessity of supposing the existence of dark matter. Through the de Sitter modified Newtonian force it was found a mass density profile presenting a small correction to the power law $\rho(r) \sim \rho_0(r_0/r)$—which is within the class of physically acceptable profiles. The flat portion of the galaxy rotation curve was found directly related to the local value of the cosmological constant and giving the correct order of magnitude for the circular velocities of the stars, without the constraint of the existence of dark matter and even more now, given the latest result in the $XENON100$-*experiment* where it was reported—so far—the non-evidence for dark matter [58].

It goes without saying that the ideas presented along this work left some open problems. Certainly the proper conformal entropy may represent a new model in the analysis of the information paradox problem in horizon, also here is highlighted the role that may has in Penrose's conformal cyclic cosmology with respect to the uniformity of the CMB in the primordial universe and its relation to the value of the entropy at that time.

Also, it is important to note that, in order to access all properties of how the de Sitter modified general relativity brings full advantage to cosmology the relativistic Friedmann equations for the de Sitter-modified Einstein equation should be obtained and studied. Based on the preliminary results obtained in this work, such approach may constitute a new paradigm for the study of cosmology. Finally, it is appropriate— and it is the intention—-to put the model under astrophysical model and sees if the approach gives a good account also for cluster galaxy.

Appendix A
Some group contractions

A.1 The Inönü-Wigner group contraction

The symmetries are behind of what characterizes a physical system. Knowing the transformations that leaves invariants the quantities that are involve in some physical theory will allow to know the symmetries of the theory. Those transformation are often associated with the group of symmetry and as it was showed in the first Chapter of this work, it is possible that from the algebra of some group obtains another through the contraction process.

Here besides to the the contraction group studied from the de Sitter algebra, it is going to be present the group contraction from Poincaré algebra and the another limit of the de Sitter group.

A.1.1 Contraction of the Poincaré group

As it was said that the Poincaré group is a direct product of the Lorentz group and the 4-dimensional translation, also it was emphasized that the contraction is made on the algebra that represents each group. In the case of the Poincaré its algebra defined as

$$[J_i, J_j] = i\epsilon_{ijkl} J_k, \quad [J_i, K_j] = i\epsilon_{ijkl} K_k, \quad [K_i, K_j] = -i\epsilon_{ijkl} J_k, \tag{A.1}$$

$$[J_i, P_j] = i\epsilon_{ijkl} P_k, \quad [K_i, P_j] = i\delta_{ij} H, \quad [J_i, H] = 0, \tag{A.2}$$

$$[K_i, H] = iP_i, \quad [P_i, P_j] = 0, \quad [P_i, H] = 0. \tag{A.3}$$

where

$$J_i, K_i, P_i, H, \quad (i = 1, 2, 3) \tag{A.4}$$

are the generators of rotations, inertial transformations, space and time translations respectively.

The first question around this algebraic process respects with the Poincaré group is what is the process involved in order to obtain the algebra that represents the Galilei group and if there is another kind of contraction from the algebra in discussion. For instance, the Galilei group is obtained by contraction the Poincaré group with respect to the direct product of the rotation and the time translation subgroup: i.e, leaving unchanged the generators **J** and H of these subgroups. Redefining

$$\mathbf{P} \to \epsilon \mathbf{P}, \quad \mathbf{K} \to \epsilon \mathbf{K}, \tag{A.5}$$

and under the $\epsilon \to 0$, the Poincaré algebra yields

$$[J_i, J_j] = i\epsilon_{ijkl} J_k, \quad [J_i, K_j] = i\epsilon_{ijkl} K_k, \quad [K_i, K_j] = 0, \tag{A.6}$$

$$[J_i, P_j] = i\epsilon_{ijkl} P_k, \quad [K_i, P_j] = i\delta_{ij} H, \quad [J_i, H] = 0, \tag{A.7}$$

$$[K_i, H] = 0, \quad [P_i, P_j] = 0, \quad [P_i, H] = 0. \tag{A.8}$$

The physical meaning of the contraction is very simple: the factor ϵ has affected the generator **P** and s**K**, so the contracted group will describe a situation where the velocities—parameter associated to **K**—and space translation are small. Moreover, if it is considers the speed of light as the unit speed and the spacelike intervals small compared with the timelike intervals, the latter means

$$v << 1, \quad \Delta s << \Delta t, \tag{A.9}$$

This is why this process is called *speed-space contraction*.

But contrary to the usual belief, the Poincaré group does has another limit group. In this case, leaving unchanged the generator of rotations and space translations, **J** and **P**,

$$\mathbf{H} \to \epsilon \mathbf{H}, \quad \mathbf{K} \to \epsilon \mathbf{K}, \tag{A.10}$$

this *The speed-time* contraction is made with respect to the three-dimensional Euclidean group, turning the algebra of Poincaré in

$$[J_i, J_j] = i\epsilon_{ijkl} J_k, \quad [J_i, K_j] = i\epsilon_{ijkl} K_k, \quad [K_i, K_j] = 0, \tag{A.11}$$

$$[J_i, P_j] = i\epsilon_{ijkl} P_k, \quad [K_i, P_j] = 0, \quad [J_i, H] = 0, \tag{A.12}$$

$$[K_i, H] = iP_i, \quad [P_i, P_j] = 0, \quad [P_i, H] = 0. \tag{A.13}$$

The interpretation, low velocities and large spacelike intervals—where lies the unusual. This contraction has an unphysical meaning [23], given the fact that it describes intervals that are not causally connected, for this reason it was named the *Carroll group*,* [62].

Now, as the Poincaré group, the de Sitter group does have others contracted group [16]. Nevertheless, here is going to be analyzed the process that take the de Sitter group to the non-relativistic Newton-Hooke group

*Named after the author of Alice in Wonderland, Lewis Carrol

A.1.2 Contraction of the de Sitter group

The Newton-Hooke group, is the non-relativistic limit with non-vanish cosmological constant of the de Sitter group. This limit can be obtained from the non-relativistic limit ($c \to \infty$), but keeping the cosmological factor given by the geometric parameter l. In this limit the translations of the spacetime are involved with a global effect inherent from the de Sitter curvature. Again, the generators must to be re-written, in order to take the convenient limit and avoid divergency. It is useful to consider the generators of the infinitesimal J_{AB} Eq. (1.16) been separated in space and time components obtaining explicit forms for L_{ab}, L_{a0}, L_{a4} and L_{04}, where $a, b = 1, 2, 3$. The generator ens redefiend as

$$L_{ab} \equiv L_{ab}, \quad L_{a0} \equiv L_{a0}/c, \quad T_a \equiv \epsilon L_{a4}/c\tau, \quad T_0 \equiv \epsilon L_{04} \equiv /\tau, \tag{A.14}$$

where it was took $\tau = l/c$ as the parameter that will allow to take the correct limit—actually is a kind of time-parameter. So, under the respective re-definitions of the generator, the commutations rules turns

$$[L_{ab}, L_{de}] = \delta_{bc}L_{ad} + \delta_{ad}L_{bc} - \delta_{bd}L_{ac} - \delta_{ac}L_{bd}, \quad [L_{ab}, L_{d0}] = \delta_{bd}L_{a0} - \delta_{ad}L_{b0} \tag{A.15}$$

$$[L_{0b}, L_{0e}] = \frac{1}{c^2}L_{be}, \quad [L_{ab}, T_d] = \delta_{bd}T_a - \delta_{ad}T_b \tag{A.16}$$

$$[L_{a0}, T_b] = \frac{1}{c^2}\delta_{ab}T_0, \quad [L_{a0}, T_0] = -T_a \tag{A.17}$$

$$[T_a, T_b] = -\frac{\epsilon}{\tau^2 c^2}L_{ab}, \quad [T_a, T_0] = -\frac{\epsilon}{\tau^2}L_{a0}, \quad [T_0, T_0] = 0. \tag{A.18}$$

Once it is considers the non-relativistic limit $c \to \infty$—which reminds the process for the Galilean algebra with the difference that there is not commutativity in the translations between the space and time—the relevant facts turns to be

$$[T_a, T_0] = \frac{1}{\tau^2}L_{a0} \neq 0. \tag{A.19}$$

The interpretation; this non-commutative is a consequence of the curvature, which came from the de Sitter space—given the definition of the parameter τ. Actually, this is the correct way to perform this contraction, in order to maintain the boots transformations [32] in the groups and in that way, keeps what is was pointed at the begging of this part

Appendix B
Newtonian limit of the Killing vectors

In the following the technical aspect of the calculus of the Killing vectors associated with the de Sitter "translations" are going to be presented. Even when at the begging of this work those Killing vectors were presented in stereographic coordinates, here the calculus is made in static coordinates but just for an specific needs on where these vectors were used.

From the Section 3.3.1, the Killing vectors associated to the de Sitter are

$$\xi^{\alpha}_{\beta} = (1 - r^2/l^2)^{1/2} \cosh(ct/l)\, \delta^{\alpha}_{\beta}\,. \tag{B.1}$$

As is well-known, the Newtonian limit is achieved by taking the limit $c \to \infty$. However, in the presence of a cosmological constant, such limit has physical meaning only if $l \to \infty$, but in such a way that [32]

$$\lim_{c,\, l \to \infty} \frac{c}{l} = \frac{1}{\tau}\,, \tag{B.2}$$

with τ an arbitrary time parameter. This means that, in the limit $c \to \infty$, the Killing vectors (B.1) assumes the form

$$\xi^{\alpha}_{\beta} = (1 - r^2/l^2)^{1/2} \cosh(t/\tau)\, \delta^{\alpha}_{\beta}\,, \tag{B.3}$$

which does not depend on c. Now, since the Killing vector in the Newtonian limit should be static, one can choose t such that $\cosh(t/\tau) = 1$, which yields

$$\xi^{\alpha}_{\beta} = (1 - r^2/l^2)^{1/2}\, \delta^{\alpha}_{\beta} \simeq (1 - r^2/2l^2)\, \delta^{\alpha}_{\beta}\,. \tag{B.4}$$

These are the Newtonian Killing vectors. In particular, the component ξ^0_0 is

$$\xi^0_0 \simeq (1 - r^2/2l^2)\,. \tag{B.5}$$

Appendix C

Causal properties of the spacetime

C.1 Causality theory

In de Sitter spacetime, an event or observer is equipped of one past and one future; means that, for each observer it is possible to define a particular causal structure. It is exactly what determines that, in the de Sitter spacetime the concepts are observer-dependent.

In those kind of causal structure, like for example Minkowski spacetime it is possible to define three types of vectors depending of the sing of the inner product ($(0, 2)$, g-form acting on some vector basic) of those vectors.

Let be M, the spacetime manifold and a vector $X \in T_{\mathbf{P}}M$, X is

- *Spacelike*, if $g(X, X) > 0$
- *Null*, if $g(X, X) = 0$
- *Timelike*, if $g(X, X) < 0$

An important remark, is that if X is either timelike or null, then is called *causal*. If $X = (t, x^1, x^2, x^3)$ is null vector at $\mathbf{p}$, then $t^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$, and hence X lies on cone with vertex at $\mathbf{p}$, know as the double null cone. It is denoted by $S_{\mathbf{p}}$

$$S_{\mathbf{p}} = X \in T_{\mathbf{p}M^{3+1}} : g(X, X) > 0, \tag{C.1}$$

the set of spacelike vector at $\mathbf{p}$, by $\mathcal{I}_{\mathbf{p}}$

$$\mathcal{I}_{\mathbf{p}} = X \in T_{\mathbf{p}M^{3+1}} : g(X, X) < 0, \tag{C.2}$$

the set of timelike vector at $\mathbf{p}$, and by $\mathcal{N}_p$,

$$\mathcal{N}_{\mathbf{p}} = X \in T_{\mathbf{p}M^{3+1}} : g(X, X) = 0, \tag{C.3}$$

the set of null vector at $\mathbf{p}$. An important topological characteristic arise from those set; for example $\mathcal{I}_{\mathbf{p}}$, is an open set consisting of two components which may denote by $\mathcal{I}^+$ and

$\mathcal{I}^-$ (resp.$\mathcal{N}_\mathbf{p}$). In that sense, $\mathcal{N}_\mathbf{p}^+$ (resp. $\mathcal{N}_\mathbf{p}^-$) is the topological boundary of $\mathcal{I}_\mathbf{p}^+$ (resp. $\mathcal{I}_\mathbf{p}^-$). So it is allow to ask about the time orientation respect of some point **p** of the manifold **p**, and how it helps to define the notion of past and future of it.

The time orientation is a choice of the positive component $\mathcal{I}_\mathbf{p}^+$ at each point **p**. Then $\mathcal{I}_\mathbf{p}^+$ is called, the set of the future-direct timelike (resp null) vector at **p**. The next is to extend the causality properties for geodesic.

C.1.1 Observers and particles

It is possible to extend the causal characterizations for curves. In particular, a curve α is called future-direct timelike if $\dot{\alpha}(t)$ is a future-direct timelike vector at $\alpha(t)$ for all t. The worldline (the observer path) of an observer is represented by a timelike curve; then if the observer is inertial, then he/she moves on a timelike geodesic. On the other hand, photons move on null geodesic, so that is means that the information propagates along the null geodesics. The Figure(C.1.1), illustrates the fact that from the point **p**, going the

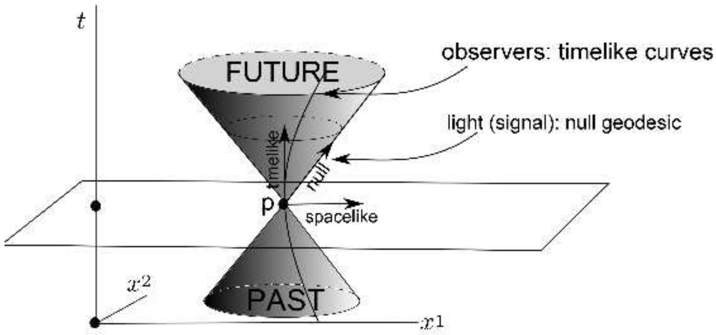

Figure C.1: Timelike and null curves.

worldline of some specific observer and also the propagating path of the light.

Those curves or geodesic (in the case of one inertial observer); can in some way fill out some region of the space, getting in that way, what is defined in the following

C.1.2 Hypersurfaces

In addition to the curves, it is important to discuss the causality of some sub-manifold.

- A hypersurface $\mathcal{H}$ is called spacelike, if the normal vector $\mathbf{N}_x$, at each point $x \in \mathcal{H}$ is timelike. In this case the metric restricted to the tangent plane at x, is positive-definite; in other words the hypersurface is a Riemannian manifold.

- A hypersurface $\mathcal{H}$ is called null, if the normal vector $\mathbf{N}_x$, at each point $x \in \mathcal{H}$ is null.
- A hypersurface $\mathcal{H}$ is called timelike, if the normal vector $\mathbf{N}_x$, at each point $x \in \mathcal{H}$ is spacelike

Remark 1 *In the case of a spacelike hypersurface, the metric restricted to the tangent plane at some x, is positive-definite; in other words the hypersurface is a Riemannian manifold. This result allows to have a well defined notion of length of curves, angles and specially areas. At this point is important to remind that macroscopic definition for the entropy of black holes is function of the area of the event horizon.*

The spacelike hypersurface are not the only that keeps some interesting causal characteristic, the null hypersurfaces plays an important role in General Relativity as they represent the horizons of various sort. For example the event horizon in the Schwarzschild spacetime is a null hypersurface. A null hypersurface has its null cone tangent to each of its points; also it is possible to say that the generator of this special hypersurface are null geodesic whose tangent vector is normal to the hypersurface. An ilustration of those kind of curves, and the hypersurfaces generated by them has been showed in the Figure (C.1.2)

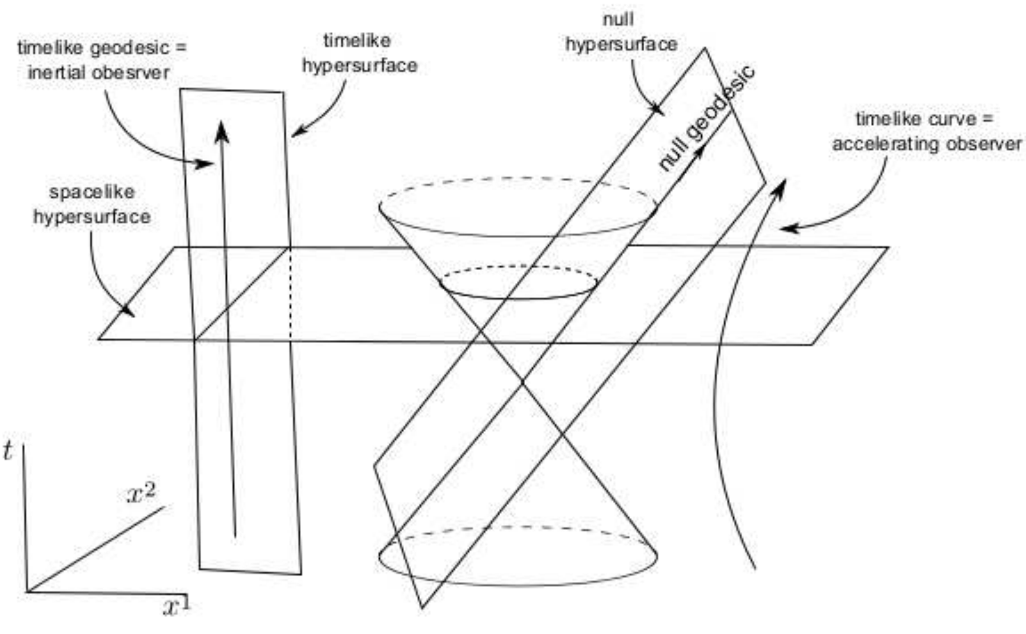

Figure C.2: Hypersurfaces

C.2 The Killing Horizons

First of all, a horizon it could be define as the boundary of the set of the future direct point $\mathcal{I}_{\mathbf{p}}^{+}$; is a limit part of the spacetime. For example in the Schwarzschild spacetime there exist an event horizon at $r = 2M$, which is a null hypersurface. It is worth it to know more about those. From Fig (C.2), can be observed, that there exist a limit part until one

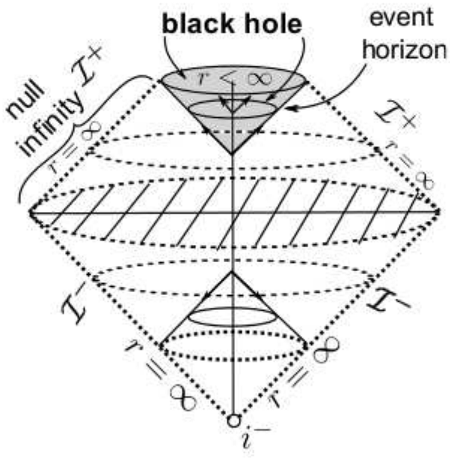

Figure C.3: The shaded region is known as the black hole region

observer can reach, that limit part is exactly the horizons of events, the one beyond that the black hole region.

Besides of the one mentioned there is a special and very important class of null hypersurface.

Definition 1 *Let us considerer a Killing vector field ξ^a, a Killing horizons is a null hypersurface on which the Killing vector is normal.*

For those metric that are static and stationary; the Killing vector is timelike and orthogonal to a family of hypersurface.

Finally, another important remark about those kind of horizon is mentioned in the work of Racz and Wald, [63], they strongly suggests that *all physically relevant Killing horizons are either of bifurcate type or degenerate*. The following is focus on the bifurcate type.

C.3 The causality on general spacetimes

In the article [41] Wald developed a formalism that allow to define the notion of entropy for a static black hole. A metric is called statics if it possesses a timelike Killing vector which is orthogonal to a family of hypersurfaces; well metrics like Schwarzschild has timelike Killing vector ∂_0.

Spacetimes with stationary and static metrics have Killing horizons, just for mention some, Minkowski and de Sitter spacetimes has Killing horizons. Inside the causal theory, sometimes it is convenient to apply conformal transformations on the spacetimes; note that such as transformation preserves the causal structure, since they "send" timelike curves in timelike curves, spacelike and null to, spacelike, null curves respectively.

In this context, let us take for example in Minkowski spacetime, the null coordinates

$$U = t - r \quad V = t + r \tag{C.4}$$

where (t, r) are the time and radial of the spherical coordinates of the null cone; note that this two straight lines bifurcate a region of the spacetime. Now, it is possible to considerer a pair of null plane, h_A and h_B, that separate the spacetime in four causal sector, and intersecting at Σ.

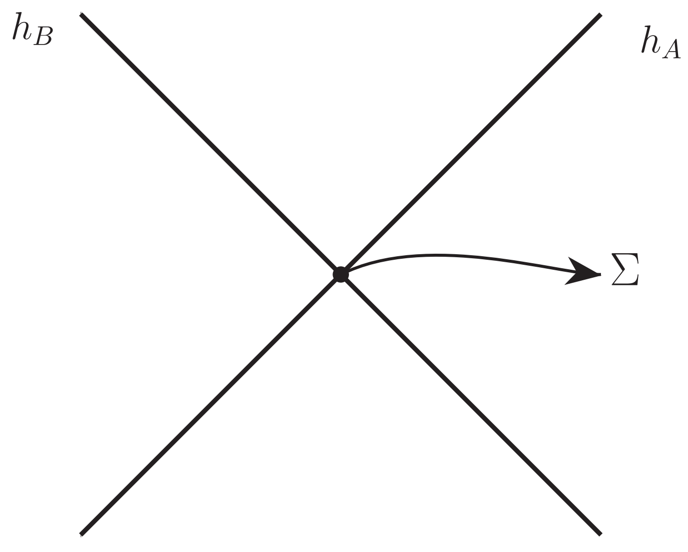

Figure C.4: The pair of null surfaces h_A and h_B, intersecting at Σ, is called the Bifurcate Killing Horizon

Exactly the same analysis applies for any n-dimensional manifold, either Riemannian or with a Lorentzian signature. In the Lorentzian case, those pair of null plane, are now considerer as a pair of null hypersurfaces h_A and h_B generated by null geodesic orthogonal to the Σ. In 4-dimensions (4-D), Σ, will be a 2-D spacelike surface, called Bifurcation surface. In the case to consider some Killing vector field ξ^a * defined over the null hypersurfaces, the sector comprised between h_A and h_B, is called *Bifurcate Killing Horizon* and the 2-D surface perpendicular to those horizons it is called *Killing bifurcation surface* [42].

Finally, it seems important to point out that, having a Bifurcation surface (a 2-D spacelike), gives a constraint that allow to have a well defined quantity such as entropy given the need to measure the area of the horizon. [63]

*The Killing vector ξ^a is normal to h_A and h_B.

Appendix D
Euclidean time and temperature

The goal of this Appendix is to provide the theoretical facts in order to relate the Euclidean extension of a spacetime with the notion of temperature, it is important to remark the facts in the following is a standard procedure in statistic mechanics and quantum field theory [64]. To see this connection, let us recall that the mean value of some dynamical variable $f(q)$ in quantum statistical mechanics can be expressed in the form

$$f = \frac{1}{Z}\sum_{E}\int \phi_E^*(q)f(q)\phi_E(q)e^{-\beta E}dq \tag{D.1}$$

where $\phi_E(q)$ is the stationary state eigen function of the Hamiltonian with $H\phi_E = E\phi_E$, $\beta = (1/T)$ is the inverse temperature and $Z(\beta)$ is the partition function. This expression calculates the mean value $< E|f|E >$ in a given energy state and then averages over a Boltzmann distribution of energy states with the weightage $Z^{-1}e^{-\beta E}$. On the other hand, the quantum mechanical kernel giving the probability amplitude for the system to go from the state q at time $t = 0$ to the state q' at time t is given by

$$K(q', t; q, 0) = \sum_{E}\phi_E^*(q')\phi_E(q)e^{-itE} \tag{D.2}$$

Now, comparing Eq. (D.1) and Eq. (D.2), it is possible to find that the thermal average in (D.1) can be obtained by

$$f = \frac{1}{Z}\int dqK(q', -i\beta; q, 0)f(q). \tag{D.3}$$

In the latter some considerations were taking into account

- The time coordinate has been analytically continued to imaginary values with $it = \tau$.
- The system is assumed to exhibit periodicity in the imaginary time tau with period beta in the sense that the state variable q has the same values at $\tau = 0$ and at $\tau = \beta$.

These considerations continue to hold even for a field theory with q denoting the field configuration at a given time. If the system, in particular the Greens functions describing the dynamics, are periodic with a period p in imaginary time, then one can attribute a temperature $T = (1/p)$ to the system. Also the partition function $Z(\beta)$ can be express as follows

$$Z(\beta) = \sum_{E} e^{\beta\, E} \int dqK(q', -i\beta; q, 0) = \int D_q exp[-A_E(q, \beta; q, 0)] \tag{D.4}$$

The first equality is the standard definition for $Z(\beta)$; the second equality follows from Eq. (D.2) and the normalization of $\phi_E(q)$; the last equality arises from the standard path integral expression for the kernel in the Euclidean sector (with A_E being the Euclidean action) and imposing the periodic boundary conditions. (It is assumed that the path integral measure curly $D_q$includes an integration over q.) Finally Eq. (D.3) and Eq. (D.4) represent the relation between the periodicity in *Euclidean time and temperature.*

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