MOVING BRANCH POINTS IN THE j-PLANE AND REGGE UNITARITY CONDITIONS

V. N. Gribov

A.F. Ioffe Physico-Technical Institute, Leningrad, USSR

I. Ya. Pomeranchuk and K. A. Ter-Martirosyan

Institute of Theoretical and Experimental Physics,
State Committee of Atomic Energy of the USSR
(Presented by V. N. GRIBOV)

INTRODUCTION

Hitherto there were no reasons to suggest the existence of moving singularities other than poles in the j-plane. Recently, however, Mandelstam [1] presented arguments in favor of the possibility that in the relativistic theory moving branch points may arise from singularities for integral negative l [2] and their displacement [3] in the case of particles with spin. These singularities correspond to the formation thresholds of several resonant states (reggeons) with integral negative orbital momenta $L = -1, -2, \ldots$ They can be regarded as the continuation of branch points to the complex j-plane which for integral physical j are located on nonphysical sheets of the t-plane and which correspond to the formation thresholds of several resonances with physical values of L.

The branch points were obtained by Mandelstam on the basis of an analysis of the asymptotic behaviors of a certain class of diagrams of the perturbation theory. From his study [1] it follows that they are related to many-particle intermediate states. The study of these singularities therefore requires analysis of the many-particle unitarity conditions, analytically continued to the complex j-plane.

This analytic continuation involves considerable difficulties and as yet has not been achieved

We used a definite assumption regarding the form of his analytic continuation only near those values of j which are singular for the amplitude.

To understand the structure of this analytic continuation, we consider the terms of the unitarity condition, which correspond to the formation in an intermediate state of two particles, one of which has a nonzero spin σ (and a mass M). They can be written in the form

$$\sum_{m=-\sigma}^{\sigma} \frac{2p\,(t, M^2, \mu^2)}{\sqrt{t}} \, \frac{\Gamma\,(j-m+1)}{\Gamma\,(j+m+1)} \, f_{jm}(t)\, f^*_{jm}(t),$$

where $f_{jm}(t)$ is the helical partial amplitude of the formation of the two particles; $p = p(t, M^2, \mu^2)$ is their relative momentum. As pointed out by Ya. I. Azimov [3], this expression has a pole for $m = j + 1$, and in particular for $j = \sigma - 1$ (due to the pole of the Γ-function). Near the pole $j = \sigma - 1$ it has the form

$$\frac{1}{\Gamma\,(2\sigma)} \cdot \frac{2p\,(t, M^2, \mu^2)}{\sqrt{t}} \cdot \frac{f_{j\sigma} f^*_{j\sigma}}{j+1-\sigma}\,.$$

At the same time, the contribution from the branch point found by Mandelstam to the unitarity condition can be represented [1] in the form

$$C \int_{4\mu^2}^{(\sqrt{t}-\mu)^2} \frac{2p\,(t,\,t_1,\,\mu^2)}{\sqrt{t}} \frac{f_{j\alpha} f^*_{j\alpha}}{j+1-a\,(t_1)}\,dt_1, \qquad (1)$$

where $p(t,\, t_1,\, \mu^2)$ is the relative momentum in the intermediate state of a particle and a pair of particles, having a Regge pole for $l = a\,(t_1)$.

Comparing the last two expressions, we see that a branch point results from integrating over the mass $M^2 = t_2$ of the state of a pair of particles having a variable spin $\sigma = \alpha(t_1)$.

This may be due to the presence of an Azimov singularity along the whole Regge trajectory.

From the above comparison it can be understood that the unitarity conditions are important for the investigation of the branch points in many -particle terms. It is necessary to know that part of these terms which contains the formation amplitudes of three particles for m close to $j + 1$ and for an orbital momentum l of the pair, which is equal or close to its pole value $l = \alpha(t_1)$.

A similar situation is possible in the formation not of three, but of a larger number of particles.

Accordingly, here we proposed and used in studying the singularities in the j-plane a method of analytic continuation of the unitarity conditions to complex j, which corresponds to the form (1) of the answer. We do not claim that this method is exact in general, but apparently, it correctly reflects the formation mechanism of branch points of the amplitude $f_j(t)$, observed by Mandelstam.

With the aid of the proposed method of analytic continuation the position of branch points and their character can be found.

Let us consider for simplicity only those branch points which arise from a vacuum pole.

Moving branch points in the j-plane lead to the fact that the partial amplitude $f_j(t)$ for fixed j as a function of t has on the physical sheet besides the ordinary threshold singularities also branch points $t = t_n(j)$, whose position depends on j. Each of them is the formation threshold of a certain number n of reggeons. The unitarity conditions determining the discontinuities of $f_j(t)$ at these singularities were found. These Regge terms of the unitarity condition are similar in form to the usual one in the sense that they are determined by integrals of the product of the formation amplitude of several reggeons above the cut and the value of this amplitude below the cut (related to the given singularity).

In the present report we give only a brief account of the results, without discussing the analytic continuation problem.

We will start directly from expressions, similar to equation (1), for the singular part of the contribution of many -particle states to the unitarity condition.

MOVING BRANCH POINTS AND MANY-PARTICLE INTERMEDIATE STATES

The simplest branch point, observed by Mandelstam, which arises from the integral (1) has the form

$$j = \alpha\left[\left(\sqrt{t}-\mu\right)^2\right] - 1. \qquad (2)$$

In the t-plane, when j varies along the real axis from large to small values, this branch point emerges onto the physical sheet at the formation threshold of three particles $[t = (3\mu)^2]$ for $j = \alpha(4\mu^2) - 1$ and then moves along the real axis (Fig. 1).

To study this branch point it is therefore natural to consider a three-particle intermediate state. In the case of four-particle states, new branch points appear due to the formation of two reggeons. Thus, the discontinuity of the partial amplitude $f_j(t) - \Delta^4 f_j(t)$ at the

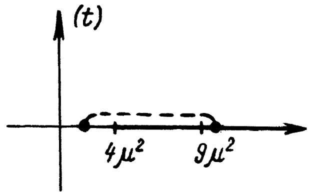


Fig.1.

threshold $t = (4\mu)^2$ has a singular part of the form

$$\Delta^4 f_j(t) = \frac{1}{2}\int\limits_{C_1}\frac{dt_1}{2i}\int\frac{dt_2}{2i}\times$$

$$\times\frac{N_{j\alpha_1\alpha_2}N^*_{j\alpha_1\alpha_2}}{j+1-\alpha(t_1)-\alpha(t_2)}\frac{2p(t,t_1,t_2)}{\sqrt{t}}, \qquad (3)$$

where $N_{j\alpha_1\alpha_2}$ is the formation amplitude of two reggeons. It appeared in equation (3), since the transformation amplitude $f_{jl_1m_1l_2m_2}$ (Fig. 2) of two particles with momentum j into four particles with momenta, helicities,

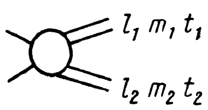


Fig.2.

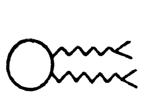


Fig.3.

and masses of the pairs l_1, m_1, t_1 and l_2, m_2, t_2, was written for the vicinity of the singularity for $l_1 = m_1 = \alpha(t_1)$ and $l_2 = m_2 = \alpha(t_2)$ in the form (Fig. 3)

$$f_{jl_1m_1l_2m_2}(t,\ t_1,\ t_2) = N_{j\alpha_1\alpha_2}\frac{1}{l_1-\alpha(t_1)}\cdot\frac{1}{l_2-\alpha(t_2)}\ . \qquad (4)$$

The contours C_1 and C_2 are given in Fig. 4. Expression (3) has singularities for

$$\left.\begin{aligned} &j = 2\alpha(4\mu^2)-1\\ &j = \alpha(4\mu^2)+\alpha\left[\left(\sqrt{t}-2\mu\right)^2\right]-1\\ &j = \alpha(4\mu^2)+\alpha^*\left[\left(\sqrt{t}-2\mu\right)^2\right]-1\\ &j = 2\alpha^*\left(\frac{t}{2}\right)-1\end{aligned}\right\} \qquad (5)$$

$$j = 2\alpha\left(\frac{t}{2}\right)-1. \qquad (6)$$

If with the aid of the dispersion integral we pass from $\Delta^4 f_j(t)$ to $f_j(t)$, it is found that for $t < 16\mu^2$, $f_j(t)$ has only the last branch point. All the other singularities are located on other sheets with respect to the singularity (6) if for $j < 2\alpha(4\mu^2) - 1$ we draw a cut in the t-plane as shown in Fig. 5 [t_2 is defined as the solution of the equation $j = 2\alpha(\frac{t}{4}) - 1$].

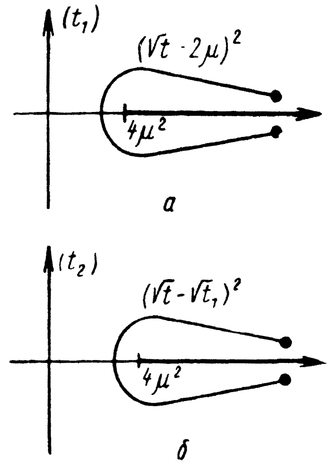


Fig.4.

When j varies along the real axis, $t_2(j)$ moves on to the nonphysical sheet at the point $t = 16\mu^2$ for $j = 2\alpha(4\mu^2) - 1$.

Many-particle states with a number larger than four can be considered in similar fashion. The singular part of the discontinuity of the partial wave at the formation threshold of n particles can be written in the form

$$\Delta^{2n} f_j(t) = \sum_k\frac{1}{2^{\nu_k}}\int\frac{N^k_{j\alpha_1\ldots\alpha_n}N^{k*}_{j\alpha_1\ldots\alpha_n}}{j+n-1-\sum\limits_i^n\alpha(t_i)}, \qquad (7)$$

where $N^k{}_{j\alpha_1,\ldots\alpha_n}$ is the formation amplitude of n reggeons. The index k has the following meaning. If more than two reggeons form, then the state of these reggeons is no longer characterized only by their spins l and helicities m (which are assumed equal to l).

In order to characterize the state, for example of three reggeons, we must assign a momentum l_{12} of any pair of reggeons and an energy t_{12}. The formation amplitude of three reggeons in reality is equal to $N_{jl_{12}\alpha_1\alpha_2\alpha_3}(t, t_n)$ (Fig. 6). It can be shown that the amplitude $N_{jl_{12}\alpha_1\alpha_2\alpha_3}(t, t_{12})$ appears in the singular part $\Delta^6 f_j$ only for $l_{12} = \alpha(t_1) + \alpha(t_2) - 1$ and therefore $N_{jl_{12}\alpha_1\alpha_2\alpha_3}(t, t_{12})$ depends only on t, t_{12}, and $t_1 t_2 t_3$. Since all reggeons are identical, it is immaterial which pair of reggeons is ascribed a definite momentum. States, for example, of four reggeons can be characterized in two different ways (Fig. 7).

One can define the momenta and energies of two reggeon paris or the momentum and energy of one pair and then the momentum and energy of three reggeons. Contributing to the singular part $\Delta^8 f_j(t)$ are transitions both to states with definite l_{12}, t_{12}, l_{34}, t_{34}, and to states with definite l_{12}, t_{12} and l_{123}, t_{123}. The index k, which takes on two values in this case, distinguishes the transition amplitudes in the two states. In equation (7) we assume $l_{12} = \alpha(t_1) + \alpha(t_2) - 1$, $l_{34} = \alpha(t_3) + \alpha(t_4) - 1$, a $l_{123} = l_{12} + \alpha(t_3) - 1$. In the case of formation of more than four reggeons, the number of different amplitudes will be larger and the index k takes on a larger number of values. The factor $1/2^{\nu} k$ is defined by the number of ways in which a given type of transition can be obtained by permutation of the reggeons.

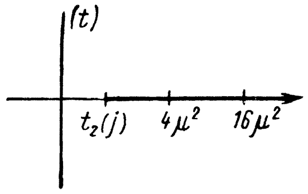


Fig. 5.

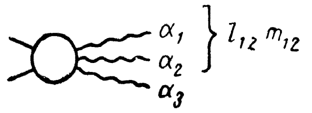
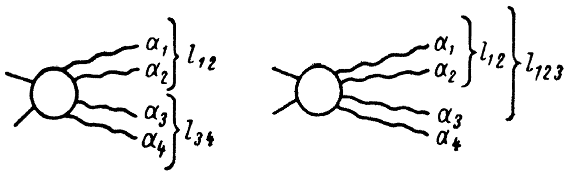


Fig. 6.



a *b*

Fig. 7.

For example, in the case of four reggeons it is necessary, owing to the indistinguishability of the reggeons, to substitute a factor 1/4 ! into equation (7). The first type of transition (Fig. 7*a*), however, can be realized in three ways (one can define l_{12} ; l_{34}; l_{13}, l_{24}; l_{14}; l_{23}). Multiplying 1/4! by three, we obtain 1/8, and consequently $\nu_k = 3$. The second type of transition can be realized in twelve ways and, consequently, $\nu_k = 12$. The quantity $1/2^{\nu k}$ coincides with the coefficient which corresponds to a Feynman diagram for the proper energy, containing only simple loops. $\Gamma(t_i, t_{ik})$ is the phase volume of n particles with the masses and energies t_{ik} of pairs, triplets, etc., of particles. The integral of formula (7), like expression (3), has a singularity depending only on the trajectory $\alpha(t)$. This singularity is determined from the extremum condition for the denominator

$$\square\,[j, \alpha(t_i)] = j + n - 1 - \sum_{i=1}^{n} \alpha(t_i)$$

subject to the additional condition

$$\chi(t, t_i) = \sum_{i=1}^{n} \sqrt{t_i} - \sqrt{t} = 0.$$

It can be easily shown that such an extremum is attained for

$$t_i = \frac{t}{n^2},$$

and, consequently, a singularity exists for

$$j = n\alpha\left(\frac{t}{n^2}\right) - n + 1.$$

In the t-plane this singularity is located at $t = t_n(j)$, which is the solution of equation (8), and moves to the nonphysical sheet for $j = n\alpha(4\mu^2) - n + 1$ at the point $t = (2n\mu)^2$.

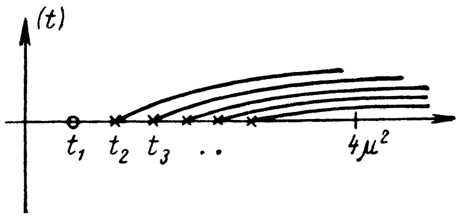


Fig. 8.

Other singularities of the integral (7), as in the two-reggeon case, do not lead to singularities of $f_j(t)$ for $t < (2n\mu)^2$. Thus, the partial amplitude $f_j(t)$ has in the t-plane for fairly small j, in addition to the usual threshold singularities $t = t_n(j)$ corresponding the the reggeon formation thresholds (Fig. 8).

REGGE UNITARITY CONDITION

Let us now consider the discontinuity of $f_j(t)$ at these singularities. Equations (3) and (7) enable us to calculate the discontinuity $\delta_j^{(n)}f_j(t)$ at a given singularity $t = t_n(j)$, or the discontinuity $\delta_j^{(n)}f_j(t)$ at a singularity $j = j_n(t)$ in the j-plane. It can be easily shown that $\delta_t^{(n)} f_j(t) = -\delta_j^{(n)} f_j(t)$. When calculating a discontinuity with the aid of expressions (3) and (7) it should be taken into account that in $\Delta^{2n}f_j(t) = \frac{1}{2i}[f_j(t) - f_j^*(t)]$ only $f_j(t)$ has a given singulari $j = n\alpha\left(\frac{t}{n^2}\right) - n + 1$. $f^*_j(t)$ has a singularity at the complex-conjugate point. Therefore

$$\delta_j^{(n)}\Delta^{2n}f_j(t) = \frac{1}{2i}\delta_j^{(n)} f_j(t).$$

In addition it should be taken into account that the quantity $N_{j\alpha_1\alpha_2\ldots}$ of the right-hand side of expression (3) or (7) has singularities at the same values of j as $f_j(t)$, since these quantities are related by the unitarity conditions. As a result, the discontinuity of $f_j(t)$ at a regge singularity is determined by the sum of the discontinuity resulting from integrating the denominator and the discontinuity of $N_{j\alpha_1\ldots\alpha_n}$. If we use the unitarity condition for $J_{j\alpha_1\ldots\alpha_n}$, writing its singular part in a form similar to expressions (3) and (7), it will contain transformation amplitudes of reggeons into reggeons. If then we write the unitarity condition for the transformation amplitudes of reggeons into reggeons, the resultant system of equations makes it possible to calculate the discontinuities of all the introduced amplitudes at the given Regge singularity. As a result we obtain for the discontinuity $\delta_t^{(2)} f_j(t)$ the straightforward result

$$\delta_t^{(2)} f_j(t) = \frac{\pi}{4i}\int dt_1 \int dt_2 N_{j\alpha_1\alpha_2} N^*_{j\alpha_1\alpha_2} \times$$
$$\times \frac{2p(t, t_1, t_2)}{\sqrt{t}}\,\delta[j+1-\alpha(t_1)-\alpha(t_2)], \qquad (9)$$

where

$N_{j\alpha_1\alpha_2} = N(t+i\varepsilon, t_1, t_2)$; $N^*_{j\alpha_1\alpha_2} = N(t-i\varepsilon, t_1, t_2)$.

Here $\pm\, i\,\epsilon$ in $N_{j\alpha_1\alpha_2}$ and $N^*_{j\alpha_1\alpha_2}$ refer to paths going round the given Regge singularity $t = t_n(j)$.

The δ-function on the right-hand side of (9) is symbolic, since the integration is generally performed over the complex region of t_1 and t_2. Similarly, the discontinuity at an n-reggeon singularity can be written in the form

$$\delta_t^{(n)} f_j(t) = 2i\pi \sum_k \frac{1}{2^{\nu_k}} \int N^k_{j\alpha_1\ldots\alpha_n} N^{k*}_{j\alpha_1\ldots\alpha_n} \times$$
$$\times\, \delta\Big(j+n-1-\sum \alpha(t_i)\Big)\,\Gamma(t_i, t_{ik}) \times$$
$$\times \prod \frac{dt_i}{2i} \times \prod dt_{ik}, \qquad (10)$$

where N^* has the same meaning as in the two-reggeon case. Expression (10) represents the Regge unitarity condition. It is almost completely similar to the usual unitarity condition, defining the partial wave discontinuity at the formation threshold of n particles.

Reggeon formation amplitudes satisfy a unitarity condition of similar form

$$\delta_t^{(n)} N^q_{j\alpha_1\ldots\alpha_p} = 2i\pi \sum_k \frac{1}{2^{\nu_k}} \times$$

$$\times \int N^k_{j\alpha'_1\ldots\alpha'_n} H^{k,q}_{\alpha'_1\ldots\alpha'_n;\,\alpha_1\ldots\alpha_p} \times$$

$$\times \delta\Big(j+n-1-\sum_1^n \alpha'_i\Big)\Gamma(t'_i, t'_{ik})\frac{d\tau}{(2i)^n}, \quad (11)$$

where $H^{k,q}_{\alpha'_1\ldots\alpha'_n;\,\alpha_1\ldots\alpha_p}$ is the transformation amplitude of reggeons into reggeons, satisfying the unitarity condition

$$\delta_t^{(n)} H^{p,q}_{\alpha_1\ldots\alpha_p;\,\alpha'_1\ldots\alpha'_q} = 2i\pi \sum_k \frac{1}{2^{\nu_k}} \times$$

$$\times \int H^{p,k}_{\alpha_1\ldots\alpha_p;\,\alpha''_1\ldots\alpha''_n} \times$$

$$\times H^{k,q}_{\alpha''_1\ldots\alpha''_n;\,\alpha'_1\ldots\alpha'_q}\,\delta(\square)\,\Gamma \times \frac{d\tau}{(2i)^n}. \quad (12)$$

BEHAVIOR OF BRANCH POINTS

Expression (10) for the discontinuity at an n-reggeon singularity enables us to determine its behavior. Near the singularity, i.e., for $j \to j_n = n\alpha\left(\frac{t}{n^2}\right) - n + 1$, all the internal energies t_i tend to the value t/n^2. In this case the relative energies of pairs, triplets, etc., tend to zero $\left(t_{12} = 4\frac{t}{n^2},\quad t_{123} = 9\,\frac{t}{n^2}\right)$, i.e., the reggeons are at rest.

In order to find out the character of the singularity, it is necessary to know the threshold behavior of the formation amplitudes of the reggeons for reggeon momenta tending to zero.

First we consider the formation amplitude of two reggeons. If each reggeon is regarded as a group of particles with momenta α_1 and α_2 and masses t_1 and t_2, then the amplitude $N_{j\alpha_1\alpha_2}$ should have a threshold behavior corresponding to

$$N_{j\alpha_1\alpha_2} \sim [p(t, t_1, t_2)]^{j-\alpha_1-\alpha_2}, \quad (13)$$

i.e., in our case

$$N_{j\alpha_1\alpha_2} = \frac{c(j, t)}{p(t, t_1, t_2)}. \quad (14)$$

Expression (13) corresponds to reggeons in a state with an orbital momentum $L = -1$.

According to expression (13), the amplitude $N_{j\alpha_1\alpha_2} \to \infty$ for $p \to 0$. If the reggeons were true particles with fixed masses, then such an amplitude increase would contradict the unitarity condition. The true threshold behavior of the formation amplitude of two particles with an orbital momentum has the form

$$f_L = \frac{p^L}{\Lambda + (-ip)^{2L+1}\frac{1}{\cos \pi L}}. \quad (15)$$

This threshold behavior arises from summation of the diagrams given in Fig. 9. For $L = -1$, $f_L \to$ const. In the case of the formation amplitude of two groups of particles with masses t_1 and t_2, the summation of diagrams similar to Fig. 9 does not result in the occurrence of $p(t, t_1, t_2)$ in the denominator, since one performs the integration over t'_1, t'_2 in an intermediate state. As a result, the denominator does not at all depend on t_1 and t_2 but depends only on t and the masses of the real particles.

Consequently, it may be assumed that the Regge amplitudes of interest have the threshold behavior (14). In the initial version of the investigation this threshold behavior was not allowed for, so that the physical results obtained on the basis of this behavior [4] are probably incorrect.

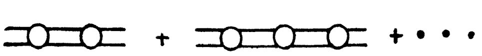


Fig. 9.

If we substitute expression (14) into the Regge unitarity condition (9), we obtain

$$\delta_t^{(2)} f_j(t) = \gamma c(j, t)\, c^*(j^*, t)] \quad (16)$$

and, consequently,

$$f_j(t) = A + \gamma c(j, t) c^*(j, t) \ln(j - j_2). \qquad (17)$$

We obtained $f_j(t) \to \infty$ for $j \to j_2$, if $c(j, t) \neq 0$ for $j = j_2$.

In fact, the tending of $f_j(t)$ to infinity for $j \to j_2$ contradicts the Regge unitarity condition (11), (12) for $n = 2$. Using these unitarity conditions, it can be easily shown that $c(j, t) \to 0$ for $j \to j_2$ and that near the singularity the amplitudes of interest have the form

$$\left.\begin{aligned} f_j(t) &= A + \frac{\nu^2(t)}{\Lambda - \gamma \ln(j - j_2)}; \\ N_{j\alpha_1\alpha_2} &= \frac{\nu(t)}{\Lambda - \gamma \ln(j - j_2)} \cdot \frac{1}{p(t, t_1, t_2)}; \\ H^j_{\alpha_1'\alpha_2';\, \alpha_1\alpha_2} &= \frac{1}{\Lambda - \gamma \ln(j - j_2)} \times \\ &\times \frac{1}{p(t, t_1', t_2')} \cdot \frac{1}{p(t, t_1, t_2)}. \end{aligned}\right\} \qquad (18)$$

In the n-reggeon case the pairs, triplets, etc., of reggeons should be assumed to be in states with orbital momentum $L = -1$. Therefore, for example, the formation amplitudes of three reggeons, $N^k{}_{j\alpha_1\alpha_2\alpha_3}$, have the threshold behavior

$$[p(t, t_{12}, t_3)\, p(t_{12}, t_1, t_2)]^{-1}.$$

Using this, it can be shown that near an n-reggeon singularity

$$f_j(t) = A + B\, \frac{(j - j_n)^{n-2}}{\ln(j - j_n)}. \qquad (19)$$

REFERENCES

1. Mandelstam S. Nuovo cimento, 30, 1113, 1127, 1143 (1963).
2. Gribov V. N., Pomeranchuk I. Ya. JETP 43, 1556 (1962); Phys. Lett., 2, 232 (1962).
3. Azimov Ya. JETP 43, 2321 (1962).
4. Gribov V. N. et al. Phys. Lett., 9, 269 (1964).

DISCUSSION

A. P. Contogouris

In your considerations you have been restricted, I think, to moving branch points in complex j of the Mandelstam type. However, Polkinghorne, in a Physics Letters, has shown that there is also another class of diagrams which produces moving branch points at the same position but with different discontinuity. Is there any motivation in neglecting these diagrams?

V. N. Gribov

We did not consider definite diagrams. Our consideration, if it is correct, includes diagrams of the Mandelstam and of the Polkinghorne types.

E. S. Fradkin

In dynamic models there is a definite relationship between the asymptotic behavior of the matrix elements on the energy and the extra-energy surfaces, and therefore the asymptotic behavior in the physical region dictates a definite behavior for the ρ-function in the spectral representations for one-particle Green's functions. Does the reporter have any considerations in connection with the values of the renormalization constants $Z = \int \rho(x)\, dx$ when allowance is made for the asymptotic behaviors, found in the work, in the physical domain of the momenta?

V. N. Gribov

In the work I mentioned the asymptotic behaviors in the physical domain were not found. If reference is made to the asymptotic behavior due to branch points, then I do not know to which renormalization constants they lead, if this behavior is related to them.