

PAPER • OPEN ACCESS

Drinfel'd double structures for Poincaré and Euclidean groups

To cite this article: Ivan Gutierrez-Sagredo *et al* 2019 *J. Phys.: Conf. Ser.* **1194** 012041

View the [article online](#) for updates and enhancements.



IOP | ebooks™

Bringing you innovative digital publishing with leading voices to create your essential collection of books in STEM research.

Start exploring the [collection](#) - download the first chapter of every title for free.

Drinfel'd double structures for Poincaré and Euclidean groups

Ivan Gutierrez-Sagredo, Angel Ballesteros and Francisco J Herranz

Departamento de Física, Universidad de Burgos, E-09001 Burgos, Spain

E-mail: igsagredo@ubu.es, angelb@ubu.es, fjherranz@ubu.es

Abstract. All non-isomorphic three-dimensional Poisson homogeneous Euclidean spaces are constructed and analyzed, based on the classification of coboundary Lie bialgebra structures of the Euclidean group in 3-dimensions, and the only Drinfel'd double structure for this group is explicitly given. The similar construction for the Poincaré case is reviewed and the striking differences between the Lorentzian and Euclidean cases are underlined. Finally, the contraction scheme starting from Drinfel'd double structures of the $\mathfrak{so}(3,1)$ Lie algebra is presented.

1. Introduction

The search for a quantum theory of gravity is nowadays one of the most challenging problems in theoretical physics. With this motivation various either partial or schematic models have been proposed from different perspectives, with the hope of getting useful insights in certain important aspects of the full theory, in particular those related to the interplay between quantum and gravitational properties of spacetime at the Planck scale. One such a particularly interesting approach is that of (2+1)-dimensional quantum gravity [1]. In this case the main feature of the theory is provided by the fact that general relativity in (2+1)-dimensions is a topological theory, and solutions of the vacuum Einstein field equations are locally isometric to one of the three maximally symmetric spacetimes of constant curvature, thus allowing a description that strongly depends on the sign of the cosmological constant. This fact greatly simplifies the picture and allows for several quantization approaches. In some of them, quantum groups play a relevant role, like in state sum models or spin foams (see for instance [2, 3, 4, 5] and references therein).

The description of (2+1)-gravity as a Chern-Simons theory, in which the group of isometries of the corresponding spacetime model plays the role of the gauge group [6, 7], will be specially relevant in what follows. As it was proved in [8, 9], the phase space structure of this theory is related with the moduli space of flat connections on a Riemann surface. Symmetries on this phase space are described by Poisson-Lie groups, and this means that quantum groups turn out to be the natural candidates for symmetries once the quantization of the theory had been performed. In [9] it was proved that the relevant Poisson-Lie structures are those defined by classical r -matrices fulfilling the so-called Fock-Rosly condition, which was proven in [10] to be automatically verified if the chosen r -matrix is the canonical one coming from a Drinfel'd double (DD) structure of the corresponding model Lie group of isometries. Therefore, the study of DD structures and their quantization (see [11, 12, 13, 14, 15]) turns out to be a relevant subject in a quantum gravity context, and recent works have been devoted to the explicit construction and analysis of DD structures for the Poincaré group [16] and for the (anti-)de Sitter ones [10]. Once



the DDs are classified, the natural question that arises concerns which are the noncommutative spacetimes obtained as quantizations of the corresponding Poisson homogeneous spaces induced by the DDs, and which are their main structural features (see, for instance, [17, 18, 19, 20] and references therein). This question has been fully answered for the Poincaré case in [16], and in this work we provide the Euclidean counterparts of these results, given that spaces with Euclidean signature are also commonly employed in quantum gravity considerations, as shown in [21]. These results will be put into correspondence with the full classification of Euclidean r -matrices given in [22] (note that analogous classifications for the (anti-)de Sitter cases can be found in [23, 24]), and the analysis of the contraction to the Euclidean case from the DDs for the Lie algebra $\mathfrak{so}(3, 1)$, which were provided in [10], will be also given.

2. Drinfel'd doubles, Lie bialgebras and Poisson homogeneous spaces

In this section we quickly review the mathematical structures needed for the rest of the work. For further details see [16, 19] and references therein. The first structure we need is that of a (classical) *Drinfel'd double*: A $2d$ -dimensional Lie algebra \mathfrak{a} is said to be the Lie algebra of a (classical) DD Lie group [25] if there exists a basis $\{Y_1, \dots, Y_d, y^1, \dots, y^d\}$ of \mathfrak{a} in which the Lie bracket reads

$$[Y_i, Y_j] = c_{ij}^k Y_k, \quad [y^i, y^j] = f_k^{ij} y^k, \quad [y^i, Y_j] = c_{jk}^i y^k - f_j^{ik} Y_k. \quad (1)$$

We call $\mathfrak{g} = \text{span}\{Y_1, \dots, Y_d\}$ and $\mathfrak{g}^* = \text{span}\{y^1, \dots, y^d\}$. The (classical) DD is the unique connected and simply connected Lie group G with Lie algebra given by \mathfrak{a} . The Lie algebra \mathfrak{a} is said to be the double Lie algebra of \mathfrak{g} and of its dual Lie algebra \mathfrak{g}^* , where the duality is defined with respect to the nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{a} \times \mathfrak{a} \rightarrow \mathbb{R}$ given by

$$\langle Y_i, Y_j \rangle = 0, \quad \langle y^i, y^j \rangle = 0, \quad \langle y^i, Y_j \rangle = \delta_j^i, \quad \forall i, j, \quad (2)$$

which is ‘associative’, *i.e.* $\langle [X, Y], Z \rangle = \langle X, [Y, Z] \rangle$, $\forall X, Y, Z \in \mathfrak{a}$. Such a DD Lie algebra possesses a canonical quadratic Casimir

$$C = \frac{1}{2} \sum_i (y^i Y_i + Y_i y^i).$$

A given even-dimensional Lie group G with Lie algebra \mathfrak{a} can have several DD structures, *i.e.* several splittings of \mathfrak{a} of the form (1) that are nonisomorphic, or no DD structures at all.

A Lie algebra \mathfrak{g} endowed with a skew-symmetric cocommutator map $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$, fulfilling the two following conditions:

- i) δ is a 1-cocycle, *i.e.*,

$$\delta([X, Y]) = [\delta(X), Y \otimes 1 + 1 \otimes Y] + [X \otimes 1 + 1 \otimes X, \delta(Y)], \quad \forall X, Y \in \mathfrak{g}. \quad (3)$$

- ii) The dual map $\delta^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ is a Lie bracket on \mathfrak{g}^* ,

is called a *Lie bialgebra* (\mathfrak{g}, δ) . There is a one-to-one correspondence between DD Lie algebras and Lie bialgebras on \mathfrak{g} , obtained by endowing the first Lie subalgebra \mathfrak{g} with a skew-symmetric cocommutator map defined by the structure constants for \mathfrak{g}^* , namely

$$[Y_i, Y_j] = c_{ij}^k Y_k, \quad \delta(Y_n) = f_n^{lm} Y_l \otimes Y_m.$$

This correspondence is due to the fact that the Jacobi identity for the DD Lie algebra (1) is equivalent to the 1-cocycle condition (3) for the cocommutator map δ with respect to the Lie algebra \mathfrak{g} .

Moreover, each DD structure for a Lie algebra \mathfrak{a} generates a solution r of the classical Yang-Baxter equation (CYBE) on \mathfrak{a} , which is provided by the canonical classical r -matrix

$$r = \sum_i y^i \otimes Y_i. \quad (4)$$

Therefore, if \mathfrak{a} is a finite-dimensional DD Lie algebra, then it can be endowed with the (quasitriangular) coboundary Lie bialgebra structure (\mathfrak{a}, δ_D) given by

$$\delta_D(X) = [X \otimes 1 + 1 \otimes X, r], \quad \forall X \in \mathfrak{a}.$$

It is in terms of this canonical r -matrix that the Fock-Rosly condition (see [9, 26, 27] and references therein) is found to be fulfilled [10].

To any coboundary Lie bialgebra, as the one defined by (4), a *Poisson-Lie (PL) group* structure (G, Π) on G is associated, and this PL structure is explicitly given by the so-called Sklyanin bracket

$$\{f, g\} = r^{ij} (\nabla_i^L f \nabla_j^L g - \nabla_i^R f \nabla_j^R g), \quad f, g \in \mathcal{C}^\infty(G), \quad (5)$$

where r^{ij} are the elements of the skew-symmetric part of (4) and ∇_i^L, ∇_i^R the left- and right-invariant vector fields on the Lie group G . Moreover, for G connected and simply connected, this structure is unique [28]. In the same manner as we can construct smooth manifolds from quotients of Lie groups by certain subgroups, a similar construction exists for PL groups where now the group action must be compatible with the Poisson structure.

A *Poisson homogeneous space* (PHS) of a PL group (G, Π) is a Poisson manifold (M, π) endowed with a transitive group action $\triangleright : G \times M \rightarrow M$ which is a Poisson map with respect to the Poisson structure π on M and the product of the Poisson structures $\Pi \times \pi$ on $G \times M$. In particular, we are interested in coset spaces $M = G/H$ of the group G with isotropy subgroup H , whose Lie algebra will be denoted by \mathfrak{h} . In the rest of the paper we will describe PL structures on G that can be canonically projected to the coset G/H , and this is guaranteed if the so-called *coisotropy condition* for the isotropy subalgebra \mathfrak{h} is satisfied, namely

$$\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{g}. \quad (6)$$

Furthermore, if H is a Poisson subgroup with respect to Π then \mathfrak{h} is a sub-Lie bialgebra structure and it satisfies $\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{h}$. This is a stronger condition and for some purposes the resulting spaces can have better behaviour, however the standard coisotropy condition (6) will be enough in order to have a well-defined PHS that can be obtained by projecting the Sklyanin bracket (5).

3. Drinfel'd double structures for the Poincaré group

In this section we review the relevant outcomes of the recent work [16], in which the Poincaré case was thoroughly analyzed. From now on, we will work on a kinematical basis $\{J, K_1, K_2, P_0, P_1, P_2\}$ corresponding to the generators of rotations, boosts, time translation and space translations, respectively. This basis is specially adapted to describe the role of the Poincaré algebra as the algebra of isometries of Minkowski spacetime. In this basis the commutation relations read

$$\begin{aligned} [J, K_1] &= K_2, & [J, K_2] &= -K_1, & [K_1, K_2] &= -J, \\ [J, P_0] &= 0, & [J, P_1] &= P_2, & [J, P_2] &= -P_1, \\ [K_1, P_0] &= P_1, & [K_1, P_1] &= P_0, & [K_1, P_2] &= 0, \\ [K_2, P_0] &= P_2, & [K_2, P_1] &= 0, & [K_2, P_2] &= P_0, \\ [P_0, P_1] &= 0, & [P_0, P_2] &= 0, & [P_1, P_2] &= 0. \end{aligned}$$

Table 1. The (2+1) Poincaré r -matrices and Poisson subgroup/coisotropy condition for the eight DD structures on $\mathfrak{p}(2+1)$ [16] as well as the corresponding class in the Stachura classification.

Case	Classical r -matrix r'_i	$\delta_D(\mathfrak{h})$	Class [22]
0	$\frac{1}{2}(-P_0 \wedge J - P_1 \wedge K_2 + P_2 \wedge K_1)$	$= 0$	(IV)
1	$K_1 \wedge J + K_1 \wedge K_2 + (-P_0 \wedge J - P_1 \wedge K_2 + P_2 \wedge K_1)$	$\subset \mathfrak{h} \wedge \mathfrak{h}$	(I)
2	$P_2 \wedge J - P_0 \wedge K_2 - P_2 \wedge K_2 + \frac{1}{2}(P_0 \wedge J - P_1 \wedge K_2 + P_2 \wedge K_1)$	$\subset \mathfrak{h} \wedge \mathfrak{g}$	(IIa)
3	$-P_2 \wedge J - P_0 \wedge K_2 - P_2 \wedge K_2 + \frac{1}{2}(-P_0 \wedge J + P_1 \wedge K_2 - P_2 \wedge K_1)$ $+ \frac{1}{\lambda}(P_0 \wedge P_1 + 2(P_0 \wedge P_2 + P_2 \wedge P_1))$	$\not\subset \mathfrak{h} \wedge \mathfrak{g}$	(IIa)
4	$P_2 \wedge J + \frac{1}{2}(P_0 \wedge J - P_1 \wedge K_2 + P_2 \wedge K_1) + \lambda P_0 \wedge P_2$	$\not\subset \mathfrak{h} \wedge \mathfrak{g}$	(IIIb)
5	$P_1 \wedge J + \frac{1}{2}(-P_0 \wedge J + P_1 \wedge K_2 - P_2 \wedge K_1) + \frac{1}{\lambda}P_1 \wedge P_0$	$\not\subset \mathfrak{h} \wedge \mathfrak{g}$	(IIIb)
6	$P_0 \wedge K_2 + \frac{1}{2}(-P_0 \wedge J + P_1 \wedge K_2 + P_2 \wedge K_1)$	$\subset \mathfrak{h} \wedge \mathfrak{g}$	(IIIb)
7	$P_2 \wedge J + \frac{1}{2}(-P_0 \wedge J + P_1 \wedge K_2 - P_2 \wedge K_1)$	$\subset \mathfrak{h} \wedge \mathfrak{g}$	(IIIb)

This Lie algebra has two quadratic Casimir elements, namely

$$C_1 = P_0^2 - P_1^2 - P_2^2, \quad C_2 = J P_0 + K_2 P_1 - K_1 P_2.$$

In this notation, the (2+1)-dimensional Minkowski spacetime M^{2+1} arises as the homogeneous space of the Poincaré group $ISO(2,1) \equiv P(2+1)$ having the Lorentz subgroup $H = SO(2,1)$ as the isotropy subgroup of the origin, that is, $M^{2+1} \equiv ISO(2,1)/SO(2,1)$. Therefore we have that $\mathfrak{a} = \mathfrak{iso}(2,1) = \mathfrak{p}(2+1) = \text{span}\{J, K_1, K_2, P_0, P_1, P_2\}$ and $\mathfrak{h} = \text{Lie}(H) = \mathfrak{so}(2,1) = \text{span}\{J, K_1, K_2\}$. In the classical case, the above coset construction induces a unique smooth structure on the coset manifold. However, the situation is quite different if we allow quantum deformations of the isometry groups because these deformations are far from being unique and so the resulting quantum (noncommutative) spacetimes are different. Of course as smooth manifolds they are diffeomorphic, but the induced Poisson structures are not equivalent, and will provide the semiclassical limit of the noncommutative spacetimes. Some of these Poisson Minkowski spacetimes have the extra property of being defined by an r -matrix that comes from a DD structure of the Poincaré Lie group and they are specially interesting due to their connections to (2+1) quantum gravity, as explained above. Here we sketch the main results given in [16], where a complete study of this matter was presented.

We recall that in [22] all the PL structures on the Poincaré Lie group were classified in terms of its defining r -matrix (given that for this group every PL structure is a coboundary one), and it was found that there exist eight (possibly multiparametric) classes of PL structures. On the other hand, there also exist eight non-isomorphic DDs on the Poincaré Lie group, each of them giving a canonical r -matrix and thus a different PL structure. In Table 1 these DD r -matrices are presented together with the Poisson subgroup/coisotropy condition for each of them and the class of the classification [22] they belong to.

The first r -matrix in Table 1 is the one corresponding to the ‘Lorentz double’, i.e. the one coming from the Lie bialgebra $(\mathfrak{g} = \mathfrak{so}(2,1) \simeq \mathfrak{sl}_2(\mathbb{R}), \delta \equiv 0)$ and the resulting Poisson Minkowski spacetime associated is of Lie algebraic type, in fact isomorphic to $\mathfrak{so}(2,1)$. Out of the other seven cases, four of them satisfy the coisotropy condition, although only the Case 1 is of Poisson subgroup type (as, obviously, the Lorentz double of Case 0). These five DDs fulfilling the coisotropy condition give rise to five different Poisson Minkowski spacetimes, which are displayed in Table 2. We remark that Cases 1 and 2 are written as a two-parametric generalization of the corresponding DD structure, which is properly recovered for $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$, respectively.

Table 2. The (2+1) Poisson Minkowski spacetimes arising from coisotropic DD structures [16].

Case	$\{x^0, x^1\}$	$\{x^0, x^2\}$	$\{x^1, x^2\}$
0	$-x^2$	x^1	x^0
1	$-\alpha_1 x^2(x^0 + x^1) + 2\beta_1 x^2$	$\alpha_1 x^1(x^0 + x^1) - 2\beta_1 x^1$	$\alpha_1 x^0(x^0 + x^1) - 2\beta_1 x^0$
2	0	$-\alpha_2(x^0 - x^2)$	$-\beta_2(x^0 - x^2)$
6	0	$-x^0 + x^1$	0
7	0	0	$-(x^0 + x^2)$

Four of these five Poisson Minkowski spacetimes are of Lie algebraic type (namely Cases 0, 2, 6 and 7) while the remaining Case 1 is the only quadratic one. Moreover, Cases 0 and 1 are the only ones that are of Poisson subgroup type, and Case 0 is the only one which is both of Lie algebraic and of Poisson subgroup type. These two conditions singularize this Poisson Minkowski spacetime from the rest and makes the construction of its associated quantum Poincaré group an interesting problem. Looking at Table 2 we see that the Poisson Minkowski spacetime obtained from Case 1 is a quadratic generalization of Case 0.

Further information on the structure of Poincaré DD r -matrices can be obtained by performing a contraction procedure on the r -matrices coming from the DD structures for the (anti-)de Sitter ((A)dS) groups, which were studied in [10]. In this way we obtain the answer to the question on which of the eight Poincaré DD r -matrices can be seen as the vanishing cosmological constant limit of the (A)dS ones. The result of this analysis, performed in [16], is that four of the seven different DD (A)dS r -matrices give rise to Poincaré DD r -matrices, namely Cases 0 and 2. Note that Case 0 is also special in this regard, while Case 2 is found to produce an r -matrix which is isomorphic to the ‘space-like’ κ -Poincaré r -matrix along with a twist [29]. In this respect, it should be noted that Table 1 indicates that a similar DD construction of the twisted ‘time-like’ κ -Poincaré deformation is not possible.

An interesting remark concerning DDs for the Poincaré group is that the (2+1)-dimensional situation is quite special, since in (3+1) and higher dimensions, the existence of any DD is precluded by the non-existence of a suitable non-degenerate bilinear form (2), both symmetric and associative. In (1+1)-dimensions there is obviously no DDs (given that the group has odd dimension), but it turns out that the non-trivially centrally extended Poincaré group in (1+1) dimensions has two non-isomorphic DD structures which were also explicitly constructed in [16].

4. Drinfel’d double Euclidean r -matrices and Poisson homogeneous spaces

Once the complete set of DDs for the Poincaré group has been clarified, it certainly makes sense to study the DDs for the Euclidean group in 3-dimensions, both because of its inherent interest related with the construction of Poisson Euclidean spaces and also for the possibility of comparing DD structures for these two closely related Lie algebras. So let us consider the Euclidean Lie algebra $\mathfrak{e}(3) = \mathfrak{iso}(3)$ in terms of generators of rotations J_i and translations P_i ($i = 1, 2, 3$). The commutation rules read

$$[J_i, J_j] = \varepsilon_{ijk} J_k, \quad [J_i, P_j] = \varepsilon_{ijk} P_k, \quad [P_i, P_j] = 0, \quad i, j, k = 1, 2, 3, \quad (7)$$

where ε_{ijk} is the totally skewsymmetric tensor with $\varepsilon_{123} = 1$. The two quadratic Casimir elements for this algebra are given by

$$C_1 = P_1^2 + P_2^2 + P_3^2, \quad C_2 = J_1 P_1 + J_2 P_2 + J_3 P_3. \quad (8)$$

The Euclidean space in three dimensions, E^3 , can be constructed as the homogeneous space of the Euclidean isometry group $ISO(3) = E(3)$ having the subgroup $H = SO(3)$ as the isotropy subgroup of the origin, that is, $E^3 \equiv ISO(3)/SO(3)$. Hence we have that $\mathfrak{a} = \mathfrak{iso}(3) = \mathfrak{e}(3) = \text{span}\{J_1, J_2, J_3, P_1, P_2, P_3\}$, and $\mathfrak{h} = \text{Lie}(H) = \mathfrak{so}(3) = \text{span}\{J_1, J_2, J_3\}$. If we denote $\mathfrak{t} = \text{span}\{P_1, P_2, P_3\}$, we have that $\mathfrak{e}(3) = \mathfrak{h} \oplus \mathfrak{t}$ as a vector space.

According to the classification [30] there is no ‘non-trivial’ DD structure for $E(3)$. However, $E(3)$ has the ‘trivial’ DD structure induced by its semidirect product form. Notice that $E(3) = SO(3) \ltimes \mathbb{R}^3$ is the semidirect product of the rotation subgroup and the translations, inherited by its Lie algebra $\mathfrak{e}(3) = \mathfrak{so}(3) \oplus_S \mathbb{R}^3$, and this is just the DD structure arising in correspondence with the Lie bialgebra structure $(\mathfrak{g}, \delta) = (\mathfrak{so}(3), \delta \equiv 0)$. It is straightforward to check that such unique DD structure for $\mathfrak{e}(3)$ is given by the isomorphism $Y_i = J_i$ and $y^i = P_i$, and thus we have $c_{ij}^k = \varepsilon_{ijk}$ and $f_j^{ik} = 0$. In this way we obtain

$$r = \sum_i P_i \otimes J_i, \quad C_2 = \sum_i J_i \otimes P_i,$$

so the skew-symmetric component of the r -matrix reads

$$r' = r - C_2 = \sum_i P_i \wedge J_i, \quad (9)$$

while the induced pairing has as non-vanishing entries $\langle P_i, J_i \rangle = 1$. This DD structure is the Euclidean analogue of the Case 0 in the Poincaré classification, and is directly related to the semidirect product structure of both Lie groups. The striking difference between the Euclidean and the Poincaré Lie groups is that while in the latter there is a plurality of DD structures (eight non-isomorphic ones), in the former a single one does exist.

5. Classification of the coboundary Poisson Euclidean spaces

In [22] the complete classification of r -matrices for the three-dimensional Euclidean Lie algebra was obtained. In order to relate such a classification with our results we rename the Euclidean generators in [22] as $e_i = P_i$ and $k_i = J_i$ ($i = 1, 2, 3$) so satisfying the commutation rules (7). In such classification, the Euclidean classical r -matrices are expressed in the form $r = a + b + c$, with

$$a \subset \mathfrak{t} \wedge \mathfrak{t}, \quad b \subset \mathfrak{t} \wedge \mathfrak{h}, \quad c \subset \mathfrak{h} \wedge \mathfrak{h},$$

where $\mathfrak{t} = \text{span}\{P_1, P_2, P_3\}$ and $\mathfrak{h} = \text{span}\{J_1, J_2, J_3\} = \mathfrak{so}(3)$. It turns out that the classification is ruled by two real parameters μ and p according to the following relations:

$$[[a, b]] = p \tilde{\eta}, \quad 2[[a, c]] + [[b, b]] = \mu \Omega, \quad [[b, c]] = [[c, c]] = 0, \quad \tilde{\eta} \in \bigwedge^3 \mathfrak{t}, \quad \Omega \in \bigwedge^2 \mathfrak{t} \otimes \mathfrak{h},$$

where $[[\cdot, \cdot]]$ denotes the Schouten bracket. The cases with $p = 0$ correspond to r -matrices leading to coisotropic Lie bialgebras with respect to $\mathfrak{h} = \text{span}\{J_1, J_2, J_3\}$. It turns out that there exist three equivalence classes of three-dimensional Euclidean r -matrices, all of them with $c = 0$, which in our notation read:

Class (I)

$$b = \alpha(P_1 \wedge J_2 - P_2 \wedge J_1) + \rho P_3 \wedge J_3, \quad a = a_{12}P_1 \wedge P_2 + a_{13}P_1 \wedge P_3 + a_{23}P_2 \wedge P_3,$$

with $\alpha = \{0, 1\}$, $\rho \geq 0$, $\alpha^2 + \rho^2 \neq 0$, $\mu = -2\alpha^2$, $p \in \mathbb{R}$, and where from now on $\{a_{12}, a_{13}, a_{23}\}$ denote three free real parameters.

Class (II)

$$b = P_1 \wedge J_1 + P_2 \wedge J_2 + P_3 \wedge J_3, \quad a = 0, \quad (10)$$

with $\mu = 2$ and $p = 0$.

Class (III)

$$b = 0, \quad a = a_{12}P_1 \wedge P_2 + a_{13}P_1 \wedge P_3 + a_{23}P_2 \wedge P_3, \quad (11)$$

with $\mu = 0$ and $p = 0$.

With this classification at hand, we can easily identify the only three-dimensional Euclidean DD r -matrix (9) with the one in Class (II) (10) in [22]. Quite interestingly, exactly as it happened for the Poincaré case, the ‘trivial’ DD r -matrix is the one corresponding to the only non-parametric family of coboundary PL structures on the three-dimensional Euclidean group.

Regarding different dimensions, no DD structure exists for the Euclidean group. In higher dimensions than three, this is due to the lack of existence of a non-degenerate associative symmetric bilinear form. In the two-dimensional case, the statement follows because there are only three non-isomorphic DDs [30], two of them isomorphic to the non-trivially centrally extended Poincaré group and the other one isomorphic to the non-trivially centrally extended AdS group. Therefore, no centrally extended Euclidean group can be endowed with a DD.

6. Contraction of Drinfel’d double r -matrices from $\mathfrak{so}(3, 1)$

The complete study of DDs for the Lie algebra $\mathfrak{so}(3, 1)$ was carried out in [10]. In what follows we analyze the contraction of such structures to the Euclidean case. This contraction procedure can be understood in geometric terms as the zero-curvature limit of the three-dimensional hyperbolic space whose isometry group is just $SO(3, 1)$.

The classification of $\mathfrak{so}(3, 1)$ DD r -matrices in [10] was performed in the usual Chern-Simons basis $\{J_0, J_1, J_2, P_0, P_1, P_2\}$. It turns out that there are four DDs for $\mathfrak{so}(3, 1)$, called cases A, B, C and D in [10]. For the cases A and C the relationship with the geometrical basis used throughout the present paper is established by means of the isomorphism given by

$$J_0 \rightarrow -J_1, \quad J_1 \rightarrow \frac{1}{\eta} P_3, \quad J_2 \rightarrow \frac{1}{\eta} P_2, \quad P_0 \rightarrow P_1, \quad P_1 \rightarrow \eta J_3, \quad P_2 \rightarrow \eta J_2, \quad (12)$$

where η is a non-zero real parameter. And for the cases B and D the isomorphism reads

$$J_s \rightarrow J_{s+1}, \quad P_s \rightarrow P_{s+1}, \quad s = 0, 1, 2. \quad (13)$$

By applying these two isomorphisms, we find that the commutation relations for the Lie algebra $\mathfrak{so}(3, 1)$ adopt the form

$$[J_i, J_j] = \varepsilon_{ijk} J_k, \quad [J_i, P_j] = \varepsilon_{ijk} P_k, \quad [P_i, P_j] = -\eta^2 \varepsilon_{ijk} J_k, \quad i, j, k = 1, 2, 3. \quad (14)$$

In this basis, the two quadratic Casimir elements for $\mathfrak{so}(3, 1)$ can be written as

$$C_1 = P_1^2 + P_2^2 + P_3^2 - \eta^2 (J_1^2 + J_2^2 + J_3^2), \quad C_2 = J_1 P_1 + J_2 P_2 + J_3 P_3. \quad (15)$$

Now we consider the three-dimensional hyperbolic space as the homogeneous space of the isometry group $SO(3, 1)$ with isotropy subgroup $H = SO(3)$, $H^3 \equiv SO(3, 1)/SO(3)$, provided that $\mathfrak{a} = \mathfrak{so}(3, 1) = \text{span}\{J_1, J_2, J_3, P_1, P_2, P_3\}$ and $\mathfrak{h} = \text{Lie}(H) = \mathfrak{so}(3) = \text{span}\{J_1, J_2, J_3\}$. The hyperbolic space H^3 has negative constant sectional curvature equal to $-\eta^2$, so that the parameter η is related with the radius of the space R through $\eta = 1/R$. The ‘flat’ contraction to the Euclidean algebra and space thus corresponds to applying the limit $\eta \rightarrow 0$ ($R \rightarrow \infty$). In this manner, the commutation rules (14) and Casimirs (15) reduce to the Euclidean ones (7) and (8), respectively.

Next, by using (12) and (13) in the results given in [10] we obtain the following four DD r -matrices for $\mathfrak{so}(3, 1)$:

$$\begin{aligned} r'_A &= \frac{1}{\eta} P_3 \wedge P_2 + \frac{1}{2} (P_1 \wedge J_1 - P_2 \wedge J_2 - P_3 \wedge J_3), \\ r'_B &= -\eta J_2 \wedge J_3 + \frac{1}{2} (P_1 \wedge J_1 + P_2 \wedge J_2 + P_3 \wedge J_3), \\ r'_C &= \frac{1}{2} \left(\frac{1}{\eta} P_3 \wedge P_1 + \eta J_1 \wedge J_3 + P_2 \wedge J_2 \right), \\ r'_D &= J_1 \wedge P_2 - J_2 \wedge P_1 + \frac{(1 + \mu^2)}{2\mu} P_3 \wedge J_3 + \frac{(\mu^2 - 1)}{2\eta\mu} (\eta^2 J_1 \wedge J_2 - P_1 \wedge P_2), \quad \mu > 0. \end{aligned} \quad (16)$$

In principle, only the r -matrix r'_B has a well defined flat limit $\eta \rightarrow 0$. Nevertheless, we can scale the remaining cases in order to obtain four contracted Euclidean r -matrices; these are

$$\begin{aligned} \lim_{\eta \rightarrow 0} \eta r'_A &= P_3 \wedge P_2, & \lim_{\eta \rightarrow 0} r'_B &= \frac{1}{2} (P_1 \wedge J_1 + P_2 \wedge J_2 + P_3 \wedge J_3), \\ \lim_{\eta \rightarrow 0} \eta r'_C &= \frac{1}{2} P_3 \wedge P_1, & \lim_{\eta \rightarrow 0} \eta r'_D &= \frac{(1 - \mu^2)}{2\mu} P_1 \wedge P_2, \quad \mu > 0. \end{aligned}$$

Consequently, the cases A, C and D give rise to Euclidean r -matrices belonging to Class (III) (11) of [22], meanwhile the case B gives exactly the complete Class (II) (10). Note that this Class is the one obtained from the DD structure for the Euclidean group (9). We recall that in [16], it was found that four (A)dS (two for dS and two for AdS) DD r -matrices contract to two DD Poincaré r -matrices (Cases 0 and 2).

We remark that the initial pairing for the cases A, B and C for the $\mathfrak{so}(3, 1)$ DDs in the basis (16) has the non-vanishing entries $\langle P_i, J_i \rangle = 1$, which does not depend on η , so that it remains unchanged under contraction. In contrast, the pairing for case D diverges under the limit $\eta \rightarrow 0$.

7. Poisson homogeneous Euclidean spaces

In the previous sections we have studied the DD structures for the Euclidean group in three dimensions and we have identified to which of the coboundary PL structures they correspond. Now we present the full construction of Poisson homogeneous Euclidean spaces, based in the classification presented in section 5. In order to perform that, we first need to introduce suitable coordinates on the Euclidean group and we write a generic element Q of the Lie algebra $\mathfrak{e}(3)$ as

$$Q = x^1 P_1 + x^2 P_2 + x^3 P_3 + \theta^1 J_1 + \theta^2 J_2 + \theta^3 J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ x^1 & 0 & -\theta^3 & \theta^2 \\ x^2 & \theta^3 & 0 & -\theta^1 \\ x^3 & -\theta^2 & \theta^1 & 0 \end{pmatrix}.$$

Next we introduce the coordinates on the Lie group as the ones associated to each Lie algebra generator through the exponential map

$$g = \exp(x^1 P_1) \exp(x^2 P_2) \exp(x^3 P_3) \exp(\theta^1 J_1) \exp(\theta^2 J_2) \exp(\theta^3 J_3),$$

and then we compute left- and right-invariant vector fields in these coordinates, thus obtaining

$$\begin{aligned} X_{J_1}^L &= \frac{\cos \theta^3}{\cos \theta^2} (\partial_{\theta^1} - \sin \theta^2 \partial_{\theta^3}) + \sin \theta^3 \partial_{\theta^2}, & X_{J_2}^L &= \frac{\sin \theta^3}{\cos \theta^2} (-\partial_{\theta^1} + \sin \theta^2 \partial_{\theta^3}) + \cos \theta^3 \partial_{\theta^2}, & X_{J_3}^L &= \partial_{\theta^3}, \\ X_{P_1}^L &= \cos \theta^2 \cos \theta^3 \partial_{x^1} + (\sin \theta^1 \sin \theta^2 \cos \theta^3 + \cos \theta^1 \sin \theta^3) \partial_{x^2} - (\cos \theta^1 \sin \theta^2 \cos \theta^3 - \sin \theta^1 \sin \theta^3) \partial_{x^3}, \\ X_{P_2}^L &= -\cos \theta^2 \sin \theta^3 \partial_{x^1} - (\sin \theta^1 \sin \theta^2 \sin \theta^3 - \cos \theta^1 \cos \theta^3) \partial_{x^2} + (\cos \theta^1 \sin \theta^2 \sin \theta^3 + \sin \theta^1 \cos \theta^3) \partial_{x^3}, \\ X_{P_3}^L &= \sin \theta^2 \partial_{x^1} - \cos \theta^2 (\sin \theta^1 \partial_{x^2} - \cos \theta^1 \partial_{x^3}), \end{aligned}$$

$$\begin{aligned}
X_{J_1}^R &= -x^3 \partial_{x^2} + x^2 \partial_{x^3} + \partial_{\theta^1}, & X_{P_1}^R &= \partial_{x^1}, \\
X_{J_2}^R &= x^3 \partial_{x^1} - x^1 \partial_{x^3} + \cos \theta^1 \partial_{\theta^2} + \frac{\sin \theta^1}{\cos \theta^2} (\sin \theta^2 \partial_{\theta^1} - \partial_{\theta^3}), & X_{P_2}^R &= \partial_{x^2}, \\
X_{J_3}^R &= -x^2 \partial_{x^1} + x^1 \partial_{x^2} + \sin \theta^1 \partial_{\theta^2} + \frac{\cos \theta^1}{\cos \theta^2} (-\sin \theta^2 \partial_{\theta^1} + \partial_{\theta^3}), & X_{P_3}^R &= \partial_{x^3}.
\end{aligned}$$

Thus we have all the ingredients to study explicitly the Poisson homogeneous Euclidean spaces. The final result is as follows.

Class (II). This is the only r -matrix coming from a DD. Its cocommutator reads

$$\delta(J_i) = 0, \quad \delta(P_1) = 2P_2 \wedge P_3, \quad \delta(P_2) = -2P_1 \wedge P_3, \quad \delta(P_3) = 2P_1 \wedge P_2,$$

which shows that its associated PHS is of Poisson subgroup type in a trivial way (this is consistent with the fact that the r -matrix (10) is the analogue of the Poincaré Case 0 studied in [16], see Table 1). The associated PHS is given by the fundamental Poisson bracket

$$\{x^i, x^j\} = 2\varepsilon_{ijk}x^k, \quad (17)$$

which is so isomorphic to the $\mathfrak{so}(3)$ Lie algebra, while its Poincaré counterpart was isomorphic to $\mathfrak{so}(2, 1)$, as shown in Table 2. As a matter of fact, if we compute the full Sklyanin bracket we get that the remaining group coordinates Poisson commute $\{x^i, \theta^j\} = \{\theta^i, \theta^j\} = 0$.

Class (III). This family of r -matrices are solutions of the CYBE, and its cocommutator reads

$$\begin{aligned}
\delta_a(J_1) &= -a_{13}P_1 \wedge P_2 + a_{12}P_1 \wedge P_3, \\
\delta_a(J_2) &= -a_{23}P_1 \wedge P_2 + a_{12}P_2 \wedge P_3, & \delta_a(P_i) &= 0, \\
\delta_a(J_3) &= -a_{23}P_1 \wedge P_3 + a_{13}P_2 \wedge P_3.
\end{aligned}$$

This cocommutator is not coisotropic with respect to the isotropy subgroup of rotations (apart from the trivial case $r = a = 0$), and we shall not write down the Poisson brackets for the group coordinates as we are only interested in describing coisotropic PHS.

Class (I). This multiparametric family of r -matrices is composed by solutions of the form of Class (III) plus some new terms. They satisfy the CYBE iff $\alpha = 0$. The cocommutator reads

$$\begin{aligned}
\delta(J_1) &= \alpha(-P_3 \wedge J_1 + P_1 \wedge J_3) - \rho(P_3 \wedge J_2 + P_2 \wedge J_3) + \delta_a(J_1), \\
\delta(J_2) &= \alpha(-P_3 \wedge J_2 + P_2 \wedge J_3) + \rho(P_3 \wedge J_1 + P_1 \wedge J_3) + \delta_a(J_2), & \delta(J_3) &= \delta_a(J_3), \\
\delta(P_1) &= \alpha P_1 \wedge P_3 + \rho P_2 \wedge P_3, & \delta(P_2) &= \alpha P_2 \wedge P_3 - \rho P_1 \wedge P_3, & \delta(P_3) &= 0,
\end{aligned}$$

proving that they are coisotropic deformations if the a -terms vanish, but they are never of Poisson subgroup type. So, in the case with $a = 0$ the associated Poisson Euclidean spaces read

$$\{x^1, x^2\} = 0, \quad \{x^1, x^3\} = \alpha x^1 - \rho x^2, \quad \{x^2, x^3\} = \alpha x^2 + \rho x^1.$$

The distinguished behavior of the third coordinate becomes evident both in the cocommutator and in the Poisson bracket, in contrast what happens in the DD noncommutative space (17).

8. Concluding remarks

Since the complete construction of DDs for the Poincaré Lie group was recently presented in [16], it seemed natural to perform the analogous analysis for the three-dimensional Euclidean group, since all these DDs are relevant for (2+1) quantum gravity, both in its Lorentzian and Euclidean facets. Also, in this paper we have presented the construction of the full set of Poisson

homogeneous Euclidean spaces, and it has been shown how two of the three classes of coboundary PL structures on the Euclidean group generate coisotropic Poisson homogeneous spaces.

Regarding DDs, it is worth stressing that the large plurality of nonisomorphic DDs for the Poincaré group is lost in its Euclidean counterpart, and this is clearly due to the flexibility of the Lorentz sector in order to give rise to DDs. However, while differences are striking it should be noted that the most studied DD structure (the ‘Lorentz double’ [11]) has an Euclidean analogue (the ‘rotation double’ or the ‘ $\mathfrak{su}(2)$ double’ [12, 13, 14, 15]) with similar algebraic properties, and both are canonically induced by the semidirect product structure of the group of isometries. Also, it should be noted that the corresponding contraction procedures applied to the (A)dS r -matrices gives similar results for the Lorentz and rotation doubles.

Acknowledgments

This work was partially supported by Ministerio de Ciencia, Innovación y Universidades (Spain) under grant MTM2016-79639-P (AEI/FEDER, UE), by Junta de Castilla y León (Spain) under grant BU229P18 and by the Action MP1405 QSPACE from the European Cooperation in Science and Technology (COST). IG-S acknowledges a predoctoral grant from Junta de Castilla y León and the European Social Fund.

References

- [1] Carlip S 1998 *Quantum gravity in 2+ 1 dimensions* (Cambridge: Cambridge University Press)
- [2] Turaev V G and Viro O Y 1992 *Topology* **31** 865–902
- [3] Alekseev A Y, Grosse H and Schomerus V 1995 *Commun. Math. Phys.* **172** 317–358
- [4] Buffenoir E and Roche P 1995 *Commun. Math. Phys.* **170** 669–698
- [5] Meusburger C and Schroers B J 2003 *Adv. Theor. Math. Phys.* **7** 1003–1043
- [6] Achúcarro A and Townsend P K 1986 *Phys. Lett. B* **180** 89–92
- [7] Witten E 1988 *Nucl. Phys. B* **311** 46–78
- [8] Alekseev A Y and Malkin A Z 1995 *Commun. Math. Phys.* **169** 99–119
- [9] Fock V V and Rosly A A 1999 *Am. Math. Soc. Transl.* **191** 67–86
- [10] Ballesteros A, Herranz F J and Meusburger C 2013 *Class. Quantum Gravity* **30** 155012
- [11] Matschull H J and Welling M 1998 *Class. Quantum Gravity* **15** 2981–3030
- [12] Bais F A and Muller N M 1998 *Nucl. Phys. B* **530** 349–400
- [13] Batista E and Majid S 2003 *J. Math. Phys.* **44** 107–137
- [14] Majid S 2005 *J. Math. Phys.* **46** 103520
- [15] Joung E, Mourad J and Noui K 2009 *J. Math. Phys.* **50** 052503
- [16] Ballesteros A, Gutierrez-Sagredo I and Herranz F J 2018 The Poincaré group as a Drinfel’d double. *Class. Quantum Gravity*, in press (2018) DOI: 10.1088/1361-6382/aaf3c2
- [17] Ballesteros A, Herranz F J and Meusburger C 2010 *Phys. Lett. B* **687** 375–381
- [18] Ballesteros A, Herranz F J and Meusburger C 2014 *Phys. Lett. B* **732** 201–209
- [19] Ballesteros A, Meusburger C and Naranjo P 2017 *J. Phys. A: Math. Theor.* **50** 395202
- [20] Ballesteros A, Gutierrez-Sagredo I, Herranz F J and Meusburger C 2017 *J. Phys. Conf. Ser.* **880** 012023
- [21] Majid S and Schroers B J 2009 *J. Phys. A: Math. Theor.* **42** 425402
- [22] Stachura P 1998 *J. Phys. A: Math. Gen.* **31** 4555–4564
- [23] Borowiec A, Lukierski J and Tolstoy V N 2016 *Phys. Lett. B* **754** 176–181
- [24] Borowiec A, Lukierski J and Tolstoy V N 2017 *Phys. Lett. B* **770** 426–430
- [25] Drinfeld V 1987 Quantum Groups. In *Proc. Int. Congr. Math. (Berkeley 1986)*, pp 798–820 (Providence: American Mathematical Society)
- [26] Meusburger C and Schroers B J 2003 *Class. Quantum Gravity* **20** 2193–2233
- [27] Osei P K and Schroers B J 2018 *Class. Quantum Gravity* **35** 075006
- [28] Drinfeld V 1983 *Sov. Math. Dokl.* **27** 68–71
- [29] Ballesteros A, Herranz F J, Meusburger C and Naranjo P 2014 *SIGMA. Symmetry, Integr. Geom. Methods Appl.* **10** 052
- [30] Gomez X 2000 *J. Math. Phys.* **41** 4939–4956