ČERENKOV RADIATION AND THE STABILITY OF BEAMS IN THE WAVE GUIDES OF SLOW WAVES USED IN LINEAR ACCELERATORS †

A. I. AKHIEZER, Ia. B. FAINBERG and G. Ia. LIUBARSKI

Ukrainian Academy of Sciences, Kharkov

1. As is known, the accelerating systems at present used in linear accelerators are metallic wave guides in which despite the absence of a dielectric the phase velocity of wave propagation v_p is less than the velocity of light in vacuum. Since the condition for the occurrence of the Čerenkov effect amounts to the requirement that the velocity of the particle v should be greater or equal to the phase velocity of the wave, a peculiar Čerenkov effect is possible in such wave guides. The Čerenkov radiation results in additional energy losses of the accelerated particles which were not taken into account by Schwinger in estimating the radiation losses in linear accelerators, his estimate being based on the assumption that the radiation takes place in free space, and hence the electron velocity is less than the phase velocity c of the wave.

Since the bunches of electrons moving through the accelerator are considerably smaller in size than the wavelength, the radiation intensity increases considerably due to coherence. There is also an additional increase in intensity due to motion through the accelerating system of the set of bunches the distances between which are equal to the wavelength.

Thus as the current increases the intensity of the Čerenkov radiation grows, and in the case of high currents may affect the process of acceleration.

It also should be noted that the Čerenkov radiation may be used to determine the position of any bunch and correct its radial and phase motion.

It is therefore interesting to consider the question of the radiation of a charged particle moving through the wave guides used as the accelerating system in electron accelerators.

Radiations of this kind play an important part in the generation and amplification of micro-radio waves. Together with the Doppler effect, the Čerenkov radiation is the underlying principle of a number of amplifiers and generators of radio micro-waves, such as the constant wave valve, the double-beam valve etc. Great attention also has been paid latterly to the development of new methods of amplification and generation of micro-waves using the Čerenkov and Doppler effects in the case of relativistic particles.

The wave guides used as accelerating systems in electron accelerators are periodic structures of the resonator chain type; we shall therefore consider the question of the radiation of a charged particle moving through coupled resonant cavities.

We shall assume linear periodic structures made of identical coupled elementary cells arranged along a certain axis (X axis).

2. To determine the field formed by the moving charge, we shall proceed from the wave equation for the vector potential

$$\Delta \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{C}\mathbf{j} \qquad (1)$$

where $\mathbf{j}$ is the density of the current connected with the moving particle. In the following, only the transverse part of the vector potential, which satisfies certain conditions, will concern us. As usual, we shall give the transverse part of the vector potential, simply denoted as $\mathbf{A}$, in the form of the series

$$\mathbf{A} = \sum_{\lambda} q_{\lambda}(t)\mathbf{A}_{\lambda}(\mathbf{r}) \qquad (2)$$

where $\mathbf{A}_\lambda(\mathbf{r})$ is a set of orthogonal functions depending only on the spatial co-ordinates and satisfying the equation

$$\Delta \mathbf{A}_\lambda + \frac{\omega^2{}_\lambda}{c^2}\mathbf{A}_\lambda = 0, \ \mathrm{div}\mathbf{A}_\lambda = 0$$

as well as certain boundary conditions. The values q_λ are certain unknown functions of time.

Depending on the periodicity condition, the functions $\mathbf{A}_\lambda(\mathbf{r})$ have the form

$$\mathbf{A}_\lambda = e^{ikx}\mathbf{a}_{ks}(\mathbf{r}),$$

† This paper was presented in title only.

where a_{ks} is a periodic function of the co-ordinate with the period l; k is a continuous parameter, enclosed in the interval $(-\pi/l, \pi/l)$; s represents one or several discrete parameters characterizing individual waves; and $\lambda \equiv (k, s)$. In order to eliminate the continuous parameter, we assume that our system contains N cells and is enclosed in a "periodicity box"; at a later stage, we shall make tend N to infinity. It may thus be considered that k takes only discrete values $k = 2\pi n/N$ $(n = 0, 1, 2...)$.

We normalise A_λ according to the condition

$$\int_{V_N} |A_\lambda|^2 dV = 4\pi c^2 N$$

where $W_N = N_{v_1}$ is the volume of N cells. Applying the condition of periodicity, we get

$$\int_{V_1} |a_{ks}|^2 dv = 4\pi c^2 \quad (3)$$

We now deduce a basic differential equation satisfied by q_λ. For this purpose, we substitute in (1) the sum of the transverse part of the vector potential determined by formula (2) and the longitudinal part of the potential. Multiplying both sides of the resulting equation by A_λ^* and integrating it over the volume V_N, we get a_{ks},

$$\ddot{q}_\lambda + \omega_\lambda^2 q_\lambda = \frac{1}{cN}\int_{V_N} \mathbf{j} A_\lambda^* \, dV \quad (4)$$

Thus q_λ satisfies the equation of an oscillator of eigen-frequency ω_λ under the action of the external force

$$f_\lambda(t) = \frac{1}{cN}\int_{V_N} \mathbf{j} A_\lambda^* \, dV \quad (5)$$

The energy of the transverse part of the field radiated by the particle is equal to

$$H(t) = \frac{N}{2}\sum_\lambda (\dot{q}_\lambda^2 + \omega_\lambda^2 q_\lambda^2)$$

With the radiation taking place, the energy is proportional to t when $t \to \infty$. This asymptotic behaviour of H(t) is possible only in the case of resonance between the eigen-oscillations of the oscillators q_λ and the external force $f_\lambda(t)$.

Let us determine the radiation of a particle moving uniformly along the axis of the system. The current density in this case is

$$j_x = ev\delta(x - vt), \; j_y = j_z = 0$$

and equation (3) takes the form

$$\ddot{q}_\lambda + \omega_\lambda^2 q_\lambda = ev e^{-ikvt} a_\lambda^* / cN$$

where a_λ is the projection of the vector $\mathbf{a}_{ks}$ on the axis of the system.

Since a_λ (x, o, o) is a periodic function of x with a period l, it can be expressed as a Fourier series

$$a_\lambda(vt, 0, 0) = \sum_{n \to \infty}^{\infty} b_\lambda^{(n)} e^{2\pi i n vt/l}$$

Therefore

$$\ddot{q}_\lambda + \omega^2 q_\lambda = \frac{ev}{cN}\sum_{n=-\infty}^{\infty} b_\lambda^{(n)*} e^{-i(k+2\pi n/l)vt} \quad (6)$$

As was mentioned above, the radiation may be treated as a resonance between the eigen-oscillation and the external force. In our case the external force has the frequency spectrum $\Omega_{kn} = (k + 2\pi n/l)v$. Hence the condition of resonance has the form

$$\omega_\lambda \equiv \omega_{ks} = \Omega_{kn} = (k + 2\pi n/l)v \quad (7)$$

In the unbounded free space $\omega_\lambda = kc$; $\Omega = kv \cos\theta$ (θ is the angle between the direction of propagation of the wave and the velocity of the particle), $n = 0$; and, since $v < c$, equation (5) cannot be fulfilled. In an unbounded dielectric $\omega_\lambda = kc/\sqrt{\varepsilon}$ (ε is the dielectric constant) and (5) results in the well-known condition of the Čerenkov radiation

$$\cos\theta = c/v\sqrt{\varepsilon} \leqslant 1$$

For periodic structures, condition (5) may be fulfilled by a great variety of combinations of the values s, n, k.

For example, let us consider a linear chain of identical cylindrical resonators coupled weakly by apertures arranged along the common axis of the resonators. The frequency spectrum of such a chain consists of separated bands. The frequency in the lowest band is given by the following formula

$$\omega_\lambda = \omega_0 [1 + \alpha(1 - \cos kl)] \quad (7')$$

where ω_0 is the cut-off frequency of the resonator and α a small parameter characterizing the coupling between the resonators.

If v is greater than $\omega_0 l(1 + 2\alpha)/\pi$, condition (7) can be fulfilled and the particle will radiate.

An essential difference should be noted between the nature of the Čerenkov radiation in an unbounded dielectric and the radiation in periodic structures. In the first case the particle, at a given velocity, radiates a continuous frequency spectrum, each direction of radiation in the dispersed medium having its own corresponding frequency. In the second case, the particle, at a given velocity, will radiate only discrete frequencies.

A similar situation takes place in ordinary wave guides filled with a dielectric.

In this case $\omega_\lambda = \sqrt{\omega_0^2 + k^2 [u(\omega_\lambda)]^2}$, where $u = c/\sqrt{\varepsilon(\omega_\lambda)}$ and ω_0 is the cut-off frequency. If the particle moves along the axis, the condition of resonance (7) leads to the relation

$$\sqrt{\omega_0^2 + k^2u^2} = vk \tag{8}$$

(in a continuous medium $u = 0$)

The resonance condition (8) can be fulfilled if $v > u$ $\omega_0 > 0$, and in this case it determines the discrete values of k†. From (8) it follows that the radiated frequency may be found from the equation

$$\omega_\lambda = \frac{\omega_0}{\sqrt{1 - \frac{u^2(\omega_\lambda)}{v^2}}}$$

Now let us determine the intensity of the radiation. The solution of equation (6) satisfying the initial conditions $q_\lambda = 0$, $\dot{q}_\lambda = 0$ has the form

$$q_\lambda = \frac{ev}{cN} \sum_{n=-\infty}^{\infty} b_\lambda^{(n)*} \frac{F_{\lambda n}(t)}{\omega_\lambda^2 - \Omega_{kn}^2}$$

where

$$F_{\lambda n} = e^{-it\Omega_{kn}} + \frac{1}{2}\left(\frac{\Omega_{kn}}{\omega_\lambda} - 1\right)e^{i\omega_\lambda t} - \frac{1}{2}\left(\frac{\Omega_{kn}}{\omega_\lambda} + 1\right)e^{-i\omega_\lambda t};$$

$$\Omega_{kn} = \left(k + \frac{2\pi n}{l}\right) v.$$

Therefore

$$H(t) = \frac{e^2v^2}{2Nc^2} \sum_{\lambda, n, m} b_\lambda^{(n)*} b_\lambda^{(m)} \frac{\dot{F}_{\lambda n}\dot{F}^*_{\lambda m} + \omega_\lambda^2 F_{\lambda n}F^*_{\lambda m}}{(\omega_\lambda^2 - \Omega_{kn}^2)(\omega_\lambda^2 - \Omega_{kn}^2)}$$

Noting that in this sum the essential terms are those with $n = m$, and making N tend to infinity, we finally get

$$I = \frac{e^2 v^2 l}{4c^2} \left\{ \sum_{n, \lambda'_j} \frac{|b_\lambda^{(n)}|^2}{\left|\frac{\partial\omega_\lambda}{\partial k} - v\right|_{\lambda = \lambda'_j}} + \sum_{n, \lambda''_j} \frac{|b_\lambda^{(n)}|^2}{\left|\frac{\partial\omega_j}{\partial k} + v\right|_{\lambda = \lambda''_j}} \right\} \tag{9}$$

where λ_j' is the set of values (k, s) satisfying the equation $\omega_\lambda - \Omega_{kn} = 0$, and λ_j'' the set of values (ks) satisfying the equation $\omega_\lambda + \Omega_{kn} = 0$.

Thus the intensity of radiation is determined by the dispersion dependence $\omega_\lambda = \omega_s(k)$ and the Fourier components of the function $a_\lambda(x, o, o)$ describing the eigen-oscillation of the system.

It will be noted that if a set of particles is moving through the periodic structure, the fields they create will, under certain conditions, have an identical phase.

The distance a between each pair of neighbouring particles being the same, the particle radiation will be synphasal if $a = 2\pi vn/\omega_\lambda$ (n is an integer).

The power of the radiation can be greatly increased if the separate charges moving at intervals a are substituted by bunches of particles of a size not exceeding one-quarter of the radiated wavelength (in this case the radiation of the individual particles of the bunch is coherent).

Using the above method, the radiation can be determined for uniform motion of a particle though an ordinary wave guide partly or completely filled with a dielectric. For example, we shall determine the radiation of a particle moving uniformly along the axis of a cylindrical wave guide filled with a homogeneous dielectric.

In this case, obviously, only those waves are essential which can propagate though the wave guide and possess an electrical field component E_x on the axis of the wave guide (the X-axis) differing from zero. The vector potential components A_x of such waves equal

$$A_{sx} = \xi_s e^{ikx} J_0\left(\frac{\mu_s r}{R}\right), \quad A_{sr} = ik\xi_s e^{ikx} J_0'\left(\frac{\mu_s r}{R}\right), \quad A_{s\varphi} = O$$

where R is the radius of the wave guide and μ_s ($s = 0, 1, 2, \ldots$) are the roots of the Bessel function $J_0(Z)$.

The potentials satisfy the normalizing condition

$$\int_{V_1} |A_\lambda|^2 \, dv = 4\pi u^2,$$

where V_1 is the volume of unit length of the wave guide and $u = u(\omega)$ is the phase velocity of a wave of frequency ω in an unbounded dielectric.

In our case

$$\xi_s = \frac{2n^2}{\omega_{ks} R |J_1(\mu_s)|}$$

The resonance conditions (7) take the following form in the case of a cylindrical wave guide :

$$\omega_{ks} = u\sqrt{k^2 + \frac{\mu_s^2}{R^2}} = kv, \quad u\sqrt{k^2 + \frac{\mu_s^2}{R^2}} = -kv \tag{10}$$

† This discreteness obviously becomes inessential if $|K| \gg 1/R$, where R is a value determining the cross-sectional dimensions of the wave guide.

These equations lead to real values of k if v is greater than $u(\omega)$. As the number s increases, the frequency ω_{ks} grows and the phase velocity $u(\omega)$ tends to c. Therefore, from a certain value of s on the condition $v > u(\omega)$ is violated. Hence only a finite number of waves are excited.

According to (9), the intensity of radiation is determined by the Fourier component $a_\lambda^{(n)}$ and the dispersion relation $\omega_{ks} = \omega(k)$. In our case the only coefficient which differs from zero is $a_{ks}^{(o)}$, equal to $\frac{\mu_s}{R}\xi_s$.. Hence the radiation intensity may be expressed as

$$I = \frac{2e^2v^2}{c^2R^4}\sum_s \frac{u^4(\omega)\mu_s^2}{I_1^2(\mu_s)\omega^2}\cdot\frac{1}{\left|\frac{dw}{dk}-v\right|_{k=k_s}} \tag{11}$$

where $\omega = u(\omega)\sqrt{k^2+\mu_s^2/R^2}$ and the summation extends over all the values of s for which $u(\omega_{ks}) \leqslant v$.

If we neglect dispersion, i. e. consider u constant,

$$d\omega/dk = u^2\, k/\omega = u^2/v$$

and expression (11) becomes

$$I = 2\,\frac{e^2v}{R^2}\left(\frac{u}{c}\right)^2\sum\frac{1}{I_1^2(\mu_s)}$$

3. Let us determine the radiation intensity when the particles move through a chain of coupled resonators. In the lower frequency band ω is determined by formula (7′) and the condition of resonance (7) has the following form

$$\omega_0[1+\alpha(1-\cos kl)] = (k+2\pi m/l)v$$

(m – is an integer). The components of the vector potential in the p-th resonator equal approximately

$$A_x = \eta J_0\left(\frac{\mu_0}{R}\right)e^{ip\psi},\; A_z = A_\varphi = 0$$

where $\psi = kl$. The condition of normality corresponds to the following value of the constant η:

$$\eta = \frac{1}{\sqrt{l}}\cdot\frac{2e}{R|J_1(\mu_0)|}$$

The function $a_{ks}(x, o, o)$ has the form ηe^{-ikx}. Its Fourier component equals:

$$a_{ks}^{(n)} = 2i\eta e^{-\frac{ikl}{2}}\,\frac{\sin\frac{kl}{2}}{kl+2\pi n}$$

Since $\alpha \ll 1$, in the first approximation, we have

$$k = \frac{\omega_0}{v} - \frac{2\pi m}{l}$$

and

$$d\omega/dk = \omega_0\alpha l\sin kl = \omega_0\alpha l\sin\omega_0 l/v$$

Hence the radiation intensity in the lower frequency band may be expressed as

$$I = \frac{2\eta^2 e^2 v^2 l^2}{c^2|v-\alpha\omega_0 l\sin\omega_0 l/v|}\sum_{n=-\infty}^{\infty}\frac{1}{(2\pi n+kl)^2}$$

Noting that

$$\left(\frac{\pi}{\sin\pi x}\right)^2 = \sum_{n=-\infty}^{\infty}\frac{1}{(n+x)^2}$$

we finaly get

$$I = \frac{16e^2v^3}{R^2l^2\omega^2}\sin^2\frac{\omega_0 l}{2v}\cdot\frac{1}{J_1^2(\mu_0)}\cdot\frac{1}{1-\alpha\frac{\omega_0 l}{v}\sin\frac{\omega_0 l}{v}} \tag{12}$$

4. Continuous acceleration in the field of a travelling wave requires that the phase velocity of the wave v_p be equal to the velocity of a synchronous particle. Under these conditions, the Čerenkov radiation considered above takes place. Unlike the unbounded medium, the spectrum of the radiation is here discrete. In this spectrum only those waves are represented for which the phase velocity of propagation coincides with the particle velocity. The radiated wave length λ is therefore equal to the wavelength of the external high-frequency accelerating field.

Equation (12) determines the intensity of radiation of the particle.

Bunches of charged particles move simultaneously through the accelerator. Since the size of a bunch is much smaller according to the conditions of acceleration, than the wavelength in the wave guide $\beta_p\lambda$, the radiation of the bunch is coherent. Under such conditions, the radiation intensity can rise N^2 times (N is the number of particles in the bunch) and the effective radiation field acting on the particles of the bunch N times. The intensity of the Čerenkov radiation may increase still further due to the interference among the radiations of bunches, the distance between them being $\beta_p\lambda$. In this case the radiation intensity increases n^2 times more where n is the number of bunches moving in a section of the accelerating system.

Thus the total intensity of radiation equals $I = I_p(nN)^2$, where I_p is the radiation intensity of a single particle.†

Taking into account that the current intensity of the accelerated particles J equals $J = Ne/T$ and the number of bunches over the length L of a section of the accelerator $L = n\beta_p\lambda$; $n = L/\beta_p\lambda$, we get, for the total intensity of radiation per accelerator section, a value of the order of

$$I = I_p(JT/e \cdot L/\beta_p\lambda)^2 \qquad (12')$$

The Čerenkov radiation should occur in the sections of the accelerator with $\beta_p < 1$, since it is only under such conditions that the velocity of the particle v can be equal to the phase velocity of the wave. In the sections where $\beta_p = 1$ the Čerenkov radiation cannot occur.

Actually, due to the finite conductivity of the walls of the wave guide and the proximity of the particle velocity in the main sections to c, this radiation will occur also in sections with $\beta_p = 1$.

With currents of the order of several ampères, the effective radiation field may become comparable with the Coulomb fields created by the charges of the bunch.

Relation (12′) can be used to estimate the intensity of radiation in micro-wave generators based on the Čerenkov effect.

5. Let us consider the radiation of an oscillator moving uniformly along the axis of a periodic structure. In this case, just as when the oscillator is moving in a medium or is a channel in the dielectric, Doppler effect arises.

If a dipole parallel to the x axis moves with uniform velocity v along the x axis, the current density vector equals

$$j_x = \Omega d \cos\omega t\,\delta(x - vt), j_y = j_z = 0$$

where $d = d_0\sqrt{1 - v^2/c^2}$, $\Omega = \Omega_0\sqrt{1 - v^2/c^2}$ (d_0 is the dipole moment and Ω the frequency of the oscillator in the co-ordinate system connected with it).

Substituting j_x into (6), we get

$$\ddot{q}_\lambda + \omega_\lambda^2 q_\lambda = \frac{\Omega d}{2c}\left(e^{i\Omega t} + e^{-i\Omega t}\right) e^{-ikvt} \times \sum_{n=-\infty}^{\infty} b_\lambda^{(n)} e^{-2\pi invt/l}$$

It was mentioned above that radiation occurs only in the case of resonance. In the case in question the resonance condition is as follows:

$$\omega_\lambda = v(k + 2\pi n/l) \pm \Omega \qquad (13)$$

This relation determines the Doppler effect in periodic structures.

Thus the spectrum of the radiated frequencies is discrete.

When the oscillator is moving in an unbounded homogeneous dielectric, the condition of resonance should obviously be written as

$$\omega_\lambda = c|k|/\sqrt{\varepsilon} = v|k|\cos\theta \pm \Omega \qquad (14)$$

where θ is the angle between the direction of propagation of the wave and the velocity of the oscillator. Hence it follows †† that

$$\omega = \frac{\Omega}{1 - \frac{\sqrt{\varepsilon(\omega)}}{c} v\cos\theta}, \quad \frac{\sqrt{\varepsilon(\omega)}}{c} v\cos\theta < 1,$$

$$\omega = \frac{\Omega}{\frac{\sqrt{\varepsilon(\omega)}}{c} v\cos\theta - 1}, \quad \frac{\sqrt{\varepsilon(\omega)}}{c} v\cos\theta > 1$$

Returning to the general formula (13), we write it as

$$\omega_\lambda = \frac{\Omega}{|1 - v/v_p|} \qquad (15)$$

where v_p is the phase velocity of propagation of the waves in the periodic structure, equal to $\omega_\lambda/(k + 2\pi n/l)$.

The Doppler effect may be treated as a kind of frequency transformation. If the velocity of the oscillator is close to the phase velocity of propagation of the waves in the medium or periodic structure, the radiated frequency may be considerably greater than the natural frequency of the oscillator.

This phenomenon may be utilized, in principle, for purposes of micro-wave generation. This possibility was pointed out by V. Ginsburg for an oscillator moving in a channel through a dense medium.

The use of periodic structures has some advantages over the use of dielectrics.

In order to lower the phase velocity of the wave in a channel through a dielectric to any considerable extent, the dielectric used must have a high enough dielectric constant.

In periodic structures there is no difficulty in lowering the phase velocity to values considerably below the velocity of light in vacuum, and oscillators moving with comparatively small velocities can accordingly be employed.

† The total intensity is actually smaller than the above value in view of the peculiarities of coherence of the Čerenkov radiation.
†† These formulae were obtained earlier by Frank in another way.

In conclusion, we give an expression for the intensity of radiation of a moving oscillator.

$$I = \frac{\Omega^2 d^2 l^2}{16c^2}\left\{\sum_{n,\lambda'_j}\frac{|b_\lambda^{(n)}|^2}{\left|\frac{d\omega_\lambda}{dk} - v\right|_{\lambda=\lambda'_j}} + \sum_{n,\lambda''_j}\frac{|b_\lambda^{(n)}|^2}{\left|\frac{d\omega_\lambda}{dk} + \omega\right|_{\lambda=\lambda''_j}}\right\},$$

where λ'_j is the set of values of (k, s) determined from the equations

$$\omega_\lambda = v(k + 2\pi n/l) \pm \Omega,$$

and λ'_j the set of values of (k, s) determined from the equations

$$\omega_\lambda = -v(k + 2\pi n/l) \pm \Omega.$$

6. We have considered the Čerenkov radiation of an individual particle and a bunch of particles in wave guide systems used in linear micro-wave accelerators, generators and amplifiers. For the elaboration of amplifiers and generators of micro-waves for feeding linear accelerators and the investigation of the influence of a beam of charged particles on the electrodynamic properties of accelerating systems of coupled resonators, we shall consider the problem of the interaction between the coupled resonators and the beam of charged particles.

The wave propagation through the chain of coupled resonators was investigated in a number of theoretical papers, in particular, in a neatly constructed paper by V. V. Vladimirski in which the author considers in its general form the question of wave propagation in a chain of resonators coupled by small apertures. We shall investigate the wave propagation through a chain of coupled resonators through which a beam of charged particles is moving. The main problem is whether the state of the beam characterized by a certain constant density ρ_0 and constant particle velocity v_0 is stable. It will be shown that under certain conditions this state of the beam is not stable and that small density and velocity fluctuations will be propagated through the beam with increasing amplitude. Such instability arises if the unperturbed particle velocity v_0 exceeds a certain critical value s. When this happens, the electromagnetic field arising in the system also spreads in the form of waves with growing amplitude.

In the case of low densities this instability is closely connected with the Čerenkov effect, which may occur not only when the particle moves through a dielectric but when the particle moves through a periodic structure without a dielectric. The chain of resonators under consideration is a particular case of such a periodic structure. In this case the Čerenkov radiation appears if $v_0 > s$. Thus instability of low-density beam and growth of the amplitude of the electromagnetic waves occur under the same condition as the Čerenkov radiation of an individual particle moving in a chain of coupled resonators.

Let us consider a system of identical cylindrical resonators coupled through round apertures in the bases of the cylinders, the contres of the apertures lying on the common axis of the resonators. The walls of the resonators are assumed to be ideal conductors. A beam of charged particles focused by means of a sufficiently large constant magnetic field passes through the apertures. The aperture radius b is assumed to be small in comparison with both the radius of the cylinder base a and the length of the cylinder l.

If the apertures are covered with ideally conducting screens, we get a system of independent resonators. Let us assume the eigenfunctions and eigenfrequencies of such resonators to be known. It may be considered that with weak coupling between the resonators perceptible resonance phenomena occur only for close frequencies in the case to be considered below. We therefore consider only one fundamental vibration in each resonator.

Let $\mathbf{E}_0$, $\mathbf{H}_0$ be the field eigenfunctions in the isolated resonators, ω_0 the eigenfrequency. They satisfy the Maxwell equations

$$\text{rot}\,\mathbf{E}_0 = i\frac{\omega_0}{c}\mathbf{H}_0,\ \text{rot}\,\mathbf{H}_0 = -i\frac{\omega_0}{c}\mathbf{E}_0$$

$$\text{div}\,\mathbf{E}_0 = 0,\ \text{div}\,\mathbf{H}_0 = 0$$

The field in the coupled resonators with a beam passing through their apertures, which we shall denote by **E**, **H**, satisfies the equations

$$\text{rot}\,\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}$$

$$\text{rot}\,\mathbf{H} = \frac{4\pi}{c}\rho\mathbf{v} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

$$\text{div}\,\mathbf{H} = 0$$

where ρ is the density and **V** the velocity of the beam. They are related by the equation :

$$\text{div}\,\rho\mathbf{v} + \frac{\partial\rho}{\partial t} = 0$$

The equation of motion of the particle under the condition $v \ll c$ has the following form :

$$\frac{d\mathbf{v}}{dt} = \frac{e}{m}\mathbf{E}$$

(m is the mass of the particle and e is the charge).

Our task is to solve the system consisting of the Maxwell equation, the continuity equation and the equation of motion.

We assume that the beam density ρ and the particle velocity v differ little from the given unperturbed values

ρ_0, v_0

$$\rho = \rho_0 + \rho', \; v = v_0 + v', \; \rho' \ll \rho_0, \; v' \ll v_0$$

Assuming the focusing axial magnetic field to be sufficiently large, we may consider that the additional velocity v' is parallel to the unperturbed velocity v_0 directed along the axis of the resonators (the z axis).

Considering ρ' and v' to be small and neglecting their squares and products, we get the following linearized system of equations:

$$\text{rot}\,\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t},$$

$$\text{rot}\,\mathbf{H} = \frac{4\pi}{c}(\rho_0 \mathbf{v} + \mathbf{v}_0 \rho) + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t},$$

$$\text{div}\,\mathbf{H} = 0,$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(\rho_0 v_z + v_0 \rho) = 0$$

$$\frac{\partial v_z}{\partial t} + v_0 \frac{\partial v_z}{\partial z} = \frac{e}{m} E_z$$

(the primes of the values ρ and v are dropped for simplicity).

Assuming that all the values are proportional to $e^{-i\omega t}$, we rewrite (16) as follows:

$$\text{rot}\,\mathbf{E} = i\frac{\omega}{c}\,\mathbf{H}$$

$$\text{rot}\,\mathbf{H} = \frac{4\pi}{c}(\rho_0\,\mathbf{v} + \mathbf{v}_0\,\rho) - \frac{i\omega}{c}\,\mathbf{E}$$

$$-i\omega v_z + v_0 \frac{\partial v_z}{\partial z} = \frac{e}{m} E_z,$$

$$-i\omega\rho + \frac{\partial}{\partial z}(\rho_0 v_z + v_0 \rho) = 0 \qquad (17)$$

Since the coupling between the resonators is weak, a strong perturbation of the field occurs only near the apertures. Far from the apertures the field is the same as in the unperturbed problem. Hence it may be considered that throughout the entire volume of each resonator except the regions immediately adjacent to the apertures, the field has the form

$$E_n = q_n E_0 \qquad (18)$$

where n is the number of the resonator and q_n are certain constants. We shall see that the values of q_n satisfy a difference equation with constant coefficients and therefore have the form

$$q_n = q_0 e^{in\psi}$$

The value which characterizes the wave in the chain of resonators is similar to the wave vector in a plane wave and may be called the propagation constant.

In accordance with the assumed form of the field (18), we shall seek the additions to the density and velocity also in the form of waves travelling through the chain of resonators and proportional to $e^{in\psi}$. Since we are not interested in the precise values of ρ and v in the vicinity of the apertures, we may proceed from certain mean values of ρ and v within each resonator. If these mean values of ρ and v , which we denote by and ρ_n and v_n, change slowly from one resonator to another, ρ_n and v_n can be determined by using the following difference equations, in analogy to the differential equations (17):

$$-i\omega \frac{l}{2}(v_{n-1} + v_n) + v_0 (v_n - v_{n-1}) = \frac{e}{m}\frac{l}{2}(E_{n-1} - E_n)$$

$$-i\omega \frac{l}{2}(\rho_{n-1} + \rho_n) + v_0(\rho_n - \rho_{n-1}) + \rho_0(v_n - v_{n-1}) = 0 \qquad (19)$$

These equations have solutions proportional to $e^{in\psi}$. Our approximation is valid, as it follows from the above consideration, if $|\psi| \ll 1$.

Now we deduce the main difference equation which determines the value q_n.

For this purpose we consider the n-th resonator and integrate the expression $\mathbf{E}$ rot rot $\mathbf{E}_0{}^* - \mathbf{E}_0{}^*$ rot rot $\mathbf{E}$ over to its volume. As a result we get

$$(k^2 - k_0^2)\int_{V_n} \mathbf{E}\mathbf{E}_0^*\,dv + \frac{4\pi}{c} ik \int_{V_n} (\rho_n \mathbf{V} + \mathbf{V}_0\,\rho)\,\mathbf{E}_0^*\,dv + ik_0 \times \int_{S_n' + S_n''} [\mathbf{E}\mathbf{H}_0^*]\,ds = 0$$

where $k_0 = \omega_0/c$ and $k = \omega/c$.

Equation (18) and the solutions of the equations (19) for the mean values ρ_n and v_n may be employed to compute the volume integrals in this expression.

The volume integral may be determined as follows. Since the dimensions of the hole have been assumed to be small in comparison with the wave length, the field in the vicinity of the hole is of an electrostatic type. The corresponding electrostatic problem has been solved by V. V. Vladimirski in the absence of a beam. This problem is formulated as follows: far from the aperture $E_x E_y \to 0$, and E_z tends to $q_n E_{0z}(0)$ on one side of the aperture and to $q_{u+1} E_{0z}(0)$ on the other (the zero argument denotes the centre of the hole).

When there is a beam of particles passing through the aperture, the field in the vicinity of the aperture is also of an electrostatic type but the asymptotic field values just mentioned can no longer be used. However, there is no need to find the precise solution of the Poisson equation with space charges, since in the selfconsistent field under consideration, while the mean charge density (for one resonator) changes as the field, the presence of space charges is equivalent to the introduction of a certain dielectric constant.

To find this equivalent dielectric constant of a periodic structure with a space charge, we put in equations (10):

$$E_n = E_0 e^{in\psi},\ v_n = Ve^{in\psi},\ \rho_n = Re^{in\psi}$$

It can be easily seen that

$$V = \frac{1}{i}\frac{el}{mc}\frac{E_0}{2\beta \mathrm{tg}\frac{\psi}{2} - kl},$$

$$R = -\frac{2}{i}\frac{el\rho_0}{mc^2}\frac{E_0 \mathrm{tg}\frac{\psi}{2}}{(2\beta \mathrm{tg}\frac{\psi}{2} - kl)^2},$$

where $\beta = v_0/c$

Introducing the mean polarization (for the volume of one resonator) P_n, we obtain, in analogy with the equation div $\mathbf{P} = \rho$, the difference equation

$$P_n - P_{n-1} = -l/2 \cdot (\rho_{n-1} + \rho_n)$$

the solution of which has the form:

$$P_n = P_0 e^{in\psi},\quad P_0 = -\frac{e\rho_0 l^2}{mc^2}\frac{E_0}{(2\beta \mathrm{tg}\frac{\psi}{2} - kl)^2}$$

The electric displacement $\mathbf{D}$ equals the sum of $\mathbf{E}$ and $4\pi\mathbf{P}$. Hence the dielectric constant may be expressed as follows:

$$\varepsilon = 1 - \frac{4\pi e\rho_0 l^2}{mc^2}\frac{1}{(2\beta \mathrm{tg}\frac{\psi}{2} - kl)^2} \tag{20}$$

We now formulate the electrostatic problem we are interested in as a problem of electrostatics in a vacuum. We must obviously require that far from the aperture $E_xE_y \to 0$, and that E_z tend to $\varepsilon q_n E_{0z}(0)$ on one side of the aperture and to $\varepsilon q_{n+1}E_{0z}(0)$ on the other, where ε is determined by formula (20).

Hence it follows that the surface integral equals:

$$ik_0 \int_{S_n' + S_n''} [\mathbf{EH}_0^*]\, ds = -\alpha\varepsilon k_0^2 (2q_n - q_{n-1} - q_{n+1}) \tag{21}$$

where

$$\alpha = \frac{2}{3\pi J_1^2(\mu_1)}\frac{b^3}{a^2 l}$$

is the first root of the Bessel function $J_0(x)$.

Substituting (21) in the main integral relation results in a difference equation for q_n:

$$(k^2 - k_0^2)\, q_n + \frac{4\pi}{c} ik \int_{V_n} (\rho_0 V_n + V_0 \rho_n)\, E_0^*\, dv - \alpha\varepsilon \times (2q_n - q_{n-1} - q_{n+1})\, k_0^2 = 0$$

where ρ_n and V_n are determined by equations (19). Assuming $q_n = q_0^{in\psi}$, we get finally the following formula relating the frequency $\omega = kc$ to the propagation constant ψ:

$$\left[\frac{k^2}{k_0^2} - 2\alpha(1 - \cos\psi)\right]\left[1 - \frac{4\pi e\rho_0 l^2}{mc^2}\frac{1}{(2\beta \mathrm{tg}\frac{\psi}{2} - kl)^2}\right] = 1 \tag{22}$$

7. We now turn to an analysis of the dispersion equation (22).

Since our assumption, strictly speaking, holds good, only for small values of $|\psi|$, the equation should be written as:

$$\left[\frac{k^2}{k_0^2} - \alpha\psi^2\right]\left[1 - \frac{\delta}{(\beta\psi - kl)^2}\right] = 1, \tag{23}$$

where $\delta = \frac{4\pi e\rho_0 l^2}{mc^2}$.

Considering the frequency given, we have an equation of the fourth order with respect to the propagation constant ψ.

Let us consider the case when the beam density ρ_0 is sufficiently small, so that the condition $\delta \ll 1$ is fulfilled, and find whether the equation has complex roots. If $\delta \to 0$, this equation degenerates into two:

$$\frac{k^2}{k_0^2} - \alpha\psi^2 = 1,\ (\beta\psi - kl)^2 = 0$$

It can be easily seen that if δ differs from zero but is sufficiently small in value, no complex roots appear in the vicinity of $\psi_1 = \alpha^{-1/2}(k^2/k_0^2 - 1)$. As to the vicinity of the value $\psi_0 = kl/\beta$, a complex root may appear here under certain conditions. In fact, assuming that $\delta \ll 1$, let us seek a root of the equation close to ψ_0:

$$\psi = \frac{kl}{\beta} + i\eta, \ |\eta| \ll \frac{kl}{\beta}$$

Hence it can be seen that η is real if

$$\beta > kl_0\sqrt{\alpha}$$

The imaginary part of ψ equals

$$\eta = \frac{\sqrt{\delta}\,\dfrac{\omega}{\omega_0}\sqrt{1-\dfrac{s^2}{v_0^2}}}{\beta\sqrt{1-\left(\dfrac{\omega}{\omega_0}\right)^2\left(1-\dfrac{s^2}{v_0^2}\right)}} \tag{24}$$

where $s = \omega_0 l\sqrt{\alpha}$

Thus complex roots appear in the dispersion equation if the unperturbed velocity of the beam exceeds a certain critical velocity s, which is smaller the weaker the coupling between the resonators. At the same time, the frequencies must satisfy the condition

$$\omega \leqslant \frac{\omega_0}{\sqrt{1-\dfrac{s^2}{v_0^2}}} \tag{25}$$

If equation (25) possesses the sign of equality, equation (24) is not valid. Below we shall consider this case separately and it will be seen that with $\omega = \omega_0(1 - s^2/v_0^2)^{-1/2}$ the value η is finite and reaches its maximum magnitude.

The existence of this complex root means, from the physical point of view, that charge density waves and electromagnetic waves with growing amplitude propagate in the chain of resonators if $v > s$. (The ratio of the amplitudes in two adjacent resonators equals e^{η}). This means that under the condition $v_0 > s$ the state of a beam of constant density and velocity becomes unstable.

It will be made clear below that if $|\psi| \ll 1$ the set of resonators under consideration is equivalent to a wave guide with critical frequency ω_0 filled with a dielectric in which the phase velocity of wave propagation is equal to s.

An individual charged particle moving in such a medium radiates electromagnetic waves if the condition $v_0 > s$ is fulfilled. Thus instability of the beam arises under the same condition as the Čerenkov radiation of individual particles.

It should be noted that the conclusion reached as to the existence of growing solutions for charge density waves and the field can be used to explain the mechanism of excitation and amplification of vibrations in a multi-cavity magnetron. The fact that we have to deal in this case with a closed circular chain of resonators, rather then with linear one, is inessential. The infinite linear chain we have considered is, as it were, a straightened out magnetron. As the chain is closed, the propagation constant ψ takes only discrete values equal to $\psi = 2\pi n/N$ where N is the total number of chambers and n is an integer not greater than N. In this case ψ should be considered as given in the dispersion equation and the frequency ω sought. The latter becomes complex if $v_0 > s$, its imaginary part being equal to

$$\xi = \frac{c}{l}\sqrt{\delta}\,\frac{\psi\sqrt{1-\dfrac{s_0^2}{v_0^2}}}{\sqrt{\dfrac{\omega_0^2 l^2}{v_0^2}-\left(1-\dfrac{s^2}{v_0^2}\right)\psi^2}}, \ \psi = \frac{2\pi}{N}n \tag{26}$$

It is clear that the infinite growth of amplitudes obtained is connected with the linear nature of our scheme: only a non-linear treatment can give the true amplitude.

8. We shall show that the chain of weakly coupled resonators with a beam of particles passing through them is equivalent to a wave guide filled with a dielectric, with a beam of particles passing through it. If the velocity of propagation of the waves in an unbounded dielectric equals s, then, we get growing solutions just as in a resonator chain, if the condition $v_0 > s$ is fulfilled.

Let us consider a cylindrical wave guide filled with a dielectric having a dielectric constant ε. A beam of charged particles passes through the wave guide along its axis. In order not to complicate the problem by taking into account the boundary conditions between the beam and the dielectric, we shall consider that the beam completely fills the space inside the wave guide.

The linearized equations for the vector and scalar potentials are as follows

$$\Delta\mathbf{A} - \frac{1}{s^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}(\rho_0\mathbf{v} + \mathbf{v}_0\rho)$$

$$\Delta\Phi - \frac{1}{s^2}\frac{\partial^2\Phi}{\partial t^2} = -\frac{4\pi}{\varepsilon}\rho$$

where $s = c/\sqrt{\varepsilon}$.

The values ρ and $\mathbf{v}$ satisfy the equations of continuity and motion. The field $\mathbf{E}$ is determined as

$$-\operatorname{grad}\Phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t},$$

Φ and $\mathbf{A}$ being related by the equation:

$$\operatorname{div}\mathbf{A} + \frac{\varepsilon}{c}\frac{\partial\Phi}{\partial t} = 0$$

We seek the solution of the equations in the form of a monochromatic wave travelling along the z axis, i.e. in the form $e^{-i(\omega t-\gamma z)} f(\gamma, \eta)$. The function f satisfies the equation

$$\Delta f = -(\gamma^2 + \gamma_0^2) f$$

where $\gamma_0 = \omega_0/s$ and ω_0 is the cut-off frequency of the wave guide.

Taking into account (27) and noting that

$$E_z = -\frac{i}{k}\left(\gamma^2 - \frac{\omega^2}{s^2}\right)\Phi$$

we find the two relations

$$\left(-\omega^2 + 2\gamma v_0 \omega - \gamma^2 v_0^2 + \frac{4\pi e \rho_0}{m\varepsilon}\right)\rho = \frac{e\rho_0}{m}\gamma_0^2 \Phi$$

$$\left(-\frac{\omega^2}{s^2} + \gamma^2 + \gamma_0^2\right)\Phi = \frac{4\pi}{\varepsilon}\rho$$

from which follows the dispersion equation

$$\left(\frac{\omega^2}{s^2} - \gamma^2\right)\left(1 - \frac{4\pi e \rho_0}{\varepsilon m (\gamma v_0 - \omega)^2}\right) = \gamma_0^2 \qquad (28)$$

This equation is of exactly the same form as equation (23). Thus we have shown that the problem of the propagation of waves in the presence of a beam of charged particles through a chain of weakly coupled resonators is equivalent to that of their propagation through a wave guide filled with a dielectric. It is necessary only that the critical velocity value s deduced earlier coincides with the phase velocity of propagation of the waves through an unbounded dielectric † (2).

Let us now return to a consideration of the imaginary part of the propagation constant or wave vector.

The modulus of the imaginary part γ, which we denote by $\varkappa$, equals

$$\varkappa = \frac{\omega}{v_0}\frac{\frac{\Omega}{\omega_0}\sqrt{1-\frac{s^2}{v_0^2}}}{\sqrt{1-\left(\frac{\omega}{\omega_0}\right)^2\left(1-\frac{s^2}{v_0^2}\right)}}, \quad \Omega^2 = \frac{4\pi e \rho_0}{m\varepsilon} \qquad (29)$$

This formula becomes incorrect if the denominator approaches zero. Therefore, we specially consider the case where

$$\omega = \frac{\omega_0}{\sqrt{1-\frac{s^2}{v_0^2}}} \qquad (30)$$

Introducing a new variable $x = (\gamma v_0 - \omega)/\omega$, we get the following equation for determining x :

$$x^4 + 2x^3 - \frac{\Omega^2}{\omega_0^2}\left(1-\frac{s^2}{v_0^2}\right)(x^2+2x) + \frac{s^2}{v_0^2}\left(\frac{v_0^2}{s^2}-1\right)\frac{\Omega^2}{\omega_0^2} = 0$$

If $\Omega/\omega_0 \ll 1$ the first and third terms of this equation may be neglected. The complex roots we are interested in have the form

$$x = \left\{\frac{s^2}{2v_0^2}\left(\frac{v_0^2}{s^2}-1\right)\frac{\Omega^2}{\omega_0^2}\right\}^{1/3}\frac{1 \pm i\sqrt{3}}{2}$$

whence

$$\gamma = \frac{\omega}{v_0}(1+k) = \frac{\omega_0}{s\sqrt{\frac{v_0^2}{s^2}-1}}\left\{1 + \frac{1 \pm i\sqrt{3}}{2^{4/3}} \times \left[\frac{s}{v_0}\left(\frac{v_0^2}{s^2}-1\right)\frac{\Omega}{\omega_0}\right]^{2/3}\right\}$$

We see that the real part of the wave vector acquires a small addition compared to the value ω/s. This means that the unpertubed velocity of the beam exceeds the phase velocity of propagation of a wave with growing amplitude.

The modulus of the imaginary part γ determines the growth of the amplitude of the wave and equals

$$\varkappa = \frac{3^{1/2}}{2^{4/3}}\frac{\omega_0}{s}\left(\frac{s}{v_0}\right)^{2/3}\left(\frac{v_0^2}{s^2}-1\right)^{1/6}\left(\frac{\Omega}{\omega_0}\right)^{2/3} \qquad (31)$$

This is the maximum value of the imaginary part of the wave vector as a function of ω with given v_0.

The physical sense of the frequency determined by equation (28) can easily be found. If an isolated charged particle moves along the axis of the wave guide with velocity v_0, then if $v_0 > s$ Čerenkov radiation occurs. This radiation may be treated as a resonance between the eigenvibrations of the field in the wave guide and the external force caused by the moving charge. The frequency of the eigenvibrations in the waveguide equals $\omega = \sqrt{\omega_0^2 + \gamma^2 s^2}$ and the frequency of the inducing force equals $\omega = \gamma v_0$. Equating these values, we get the frequency determined by equation (28).

† Space charge waves and velocities were studied by S. Ramo, who deduced equation (28) for the case when $S = c$. However, Ramo did not consider the case of a wave guide filled with a dielectric ($s < c$) and for that reason did not obtain growing solutions. The latter are similar to those found by Pearce and Gayev in investigating various methods of amplifying and generating microwaves.

The above considerations stress once more the close relation between the phenomena of the growing field and charge density and the Cerenkov radiation of an individual particle.

The value $\varkappa$, determined by equation (31), reaches a maximum value

$$\varkappa_m = \frac{3^{1/2}\, 2^{1/3}}{4} \frac{\omega_0}{s} \left(\frac{\Omega}{\omega_0}\right)^{2/3} \tag{32}$$

if $(v_0/s) = \sqrt{2}$.

The corresponding most amplified frequency equals $\sqrt{2}\,\omega_0$.