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INFRARED BEHAVIOR OF THE EFFECTIVE COUPLING IN QUANTUM CHROMODYNAMICS: A NON-PERTURBATIVE APPROACH*

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ABSTRACT

In this paper we examine a different viewpoint based on a self-consistent approach. This means that rather than attempting to identify any particular physical mechanism as dominating the QCD vacuum state we use the non-perturbative Schwinger-Dyson equations and Slavnov-Taylor identities of QCD as well as the renormalization group equation to obtain the self-consistent behavior of the effective coupling in the infrared region. We show that the infrared effective coupling behavior $\bar{g}\left(q^2/\mu^2, g_R(\mu)\right) = \left(\mu^2/q^2\right)^{\lambda/2} g_R(\mu)$ in the infrared limit $q^2/\mu^2 \to 0$ where μ^2 is the Euclidean subtraction point; $\lambda = (d-2)/2$ where d is the space-time dimension, is the preferred solution <u>if</u> a sufficient self-consistency condition is satisfied. Finally we briefly discuss the nature of the dynamical mass Λ and the 1/N expansion as well as an effective bound state equation.

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1. Introduction

Some time ago it was discovered that a non-Abelian gauge theory, also known as Quantum Chromodynamics (QCD), possessed the unique property in four space-time dimensions of controllable short distance behavior known as "asymptotic freedom" [1]. More specifically the high momentum or equivalently the asymptotic short distance behavior of the effective coupling $\bar{g}^2 \sim 1/\ell n\ (q^2/\mu^2)$; $q^2/\mu^2 >> 1$ indicated free-like particle behavior of the quark and gluon colored fields. Extrapolating this result into the asymptotic infrared region $q^2/\mu^2 >> 1$ indicated that the effective coupling might become quite large possibly leading to the permanent confinement of these fields and the appearance of the known color singlet particle spectrum of hadronic states. Such a conjecture became known as "infrared slavery" [2]. In the intervening period of time many interesting physical mechanisms have been proposed in an attempt to prove the "infrared slavery" conjecture from the QCD Lagrangian. Such attempts have so far been inconclusive [3].

QCD is a relativistic renormalizable quantum field theory of colored quarks and gluons based on the classical non-Abelian Lagrangian

$$\mathscr{L} = -(1/4)F^a_{\mu\nu}F^{a\mu\nu} + \psi_\alpha (i\not{D} - m)^{\alpha\beta}\ \psi_\beta \qquad (1)$$

where $F^a_{\mu\nu} = \partial_\mu A^a - \partial_\nu A^a_\mu + gf_{abc}A^b_\mu A^c_\nu$ and $D^{\alpha\beta}_\mu = \delta^{\alpha\beta}\partial_\mu - igA^a_\mu \lambda^{\alpha\beta}_a/2$ where $A^a_\mu(x)$ is a colored gauge field, $\psi_\alpha(x)$ is a quark field (quark flavor indices are suppressed) α are color indices, where repeated indices are summed and where m is the weak-electromagnetic quark mass; λ_a are SU(N) colored matrices $[\lambda^a,\lambda^b] = 2\,i\,f^{abc}\lambda^c$ with f^{abc} the structure constants of SU(N) and g the universal coupling constant.

In this paper we present and elaborate on results which were reported in an earlier letter [4]. Our method is based on a self-consistent approach which has also been pursued independently by several other theorists and in particular R. Delbourgo, M. Baker, J. Ball and F. Zachariasen [5,6]. This means that rather than attempting to identify any particular physical mechanism as dominating the QCD vacuum state, we use the exact non-perturbative Schwinger-Dyson equations and Slavnov-Taylor identities of QCD as well as the renormalization group equation to obtain the self-consistent behavior of the effective coupling in the asymptotic infrared region [7].

Let us briefly recall several non-perturbative infrared properties of QCD which set the context of our work. By non-perturbative properties, we mean results which are obtained independently of any conjectured physical mechanisms such as instanton or monopole gas models of the vacuum state [8].

In 1976, F. Strocchi showed that the proof of cluster decomposition of Wightman functions i.e., $w(x_1,x_2) \to w(x_1)w(x_2)$ where $x_2 = x_1 + \lambda$, $\lambda \to \infty$, that follows from local field commutator relations in QED fails for QCD [9]. What this meant was that it was possible to have a failure of the cluster decomposition property in QCD, i.e., a long-range potential, and still retain locality. Several years later, G. 't Hooft was able to show using an algebra of operators that several types of phases of QCD are possible [10]. He used the gauge invariant operator $A(c) = \mathrm{Tr}[P \exp ig\oint_c dx^\mu \times A_\mu(x)]$ where c is a directed closed path in four-dimensional Euclidean space-time whose vacuum expectation value is the Wilson electric order parameter and a magnetic operator B(c) which creates a thin magnetic flux tube coinciding with the closed path c. He was then able to show that a confinement phase was possible in QCD. In such a phase the expectation value of the A(c) operator, $\langle 0|A(c)|0\rangle \sim \exp(-\text{area})$, where the area is enclosed by path c, obeys the Wilson area criterion for quark confinement. What phase QCD decides to choose is, of course, a matter of dynamics and is not answered by this formulation. In our paper using the dynamical content of the Green's function equations, we shall show that if QCD is a self-consistent theory in the infrared, then the preferred solution is a confining solution (fig. 1).

Our paper is organized as follows: In Section 2 we sketch the derivation of the Slavnov-Taylor identities which are derived from the gauge invariance properties of the QCD Lagrangian. Next we briefly review the origin of the Schwinger-Dyson Green's function equations. Section 3 is devoted to a discussion of the renormalization group equation in QCD and its general solution. Using this result and the Slavnov-Taylor identity as well as general constraints such as Boson symmetry, Lorentz invariance and the absence of kinematical singularities, we construct the most general one particle irreducible (I.P.I.) triple gluon vertex. In Section 4 we use the Slavnov-Taylor identity to show the transversality of the Schwinger-Dyson vacuum polarization equation. We then derive using this equation both necessary and sufficient conditions which the effective coupling must satisfy in the infrared region. In Section 5 we discuss the nature of the hidden dynamical mass Λ as well as the 1/N topological expansion in QCD. Finally, in Section 6 we solve for the infrared behavior of the quark propagator in QCD using our self-consistency approximation in the covariant Landau gauge. We contrast this result with the unconfined infrared behavior of the electron propagator in QED. Using our quark propagator we derive in the ladder approximation an effective bound state equation.

2. QCD Equations

2.1 Slavnov-Taylor Identities

In this section we will review the derivation of the Slavnov-Taylor identities. Such identities are the non-Abelian gauge theory analog of the Ward-Takahashi identities of Quantum Electrodynamics (QED), an Abelian gauge theory. In essence these identities are consequences of local gauge invariance and relate various Green's functions of the theory through algebraic constraint equations. Such identities are exact constraints on any solution to QCD. They also imply as in QED where $Z_1 = Z_2$, Z_1 the electron-photon vertex and Z_2 the electron wave-function renormalization constants, that all of the ultraviolet renormalization constants are not independent. The literature pertaining to the detailed derivations of these identities, in general linear covariant gauges, is quite considerable and was essential in proving the ultraviolet renormalizability of QCD [3,11].

Our starting point is the path integral for the generating functional $Z(J,\eta,\bar{\eta})$ of Green's functions

$$Z(J,\eta,\bar{\eta}) = (1/\bar{N}) \int [dA][d\bar{\psi}] \det M[A]$$

$$\times \exp i \int d^4x \left[\mathscr{L}_{QCD} - (1/2\alpha) F_a^2[A] + J_i A^i + \bar{\eta}_i \psi_i + \eta_i \bar{\psi}_i\right] \tag{2.1}$$

where $1/\bar{N}$ is the overall constant normalization factor; J^a is the gauge field external source; η_i and $\bar{\eta}_i$ are quark external anticommuting sources, $F_a[A] = F_{ai}A^i$ is the linear gauge fixing term; det $M[A] = (\delta F_a/\delta A_i)\, D_i^b[A]$ the Faddeev-Popov determinant and where all fields have spatial and group indices which are suppressed. In our paper we shall choose the Landau covariant gauge $\mathscr{L}_g = -(1/2\alpha)(\partial_\mu A_a^\mu)^2$; $\lim \alpha \to 0$ which corresponds to $F_a^4 = \partial_\mu \delta^{ab}$. Later in the paper we shall discuss how some technical problems which arise in this gauge may be ameliorated by the axial gauge $\mathscr{L}_g = 1(1/2\alpha)(n_\mu A_a^\mu)^2$, where n_μ is an arbitrary four-dimensional vector which corresponds to the choice $F_a^i = n_\mu \delta^{ab}$.

$\mathscr{L}_{QCD}$ is a non-Abelian gauge theory which is invariant under the infinitesimal gauge transformation

$$A_{\mu a} \to A_{\mu a} + \left[(1/g)\partial_\mu + t_{ab}^b\, A_{\mu c}\right]\varepsilon^b \tag{2.2a}$$

$$\psi_n \to \psi_n - i\left(\lambda_a^{nm}/2\right)\, \psi_m\, \varepsilon^a \tag{2.2b}$$

$$\bar{\psi}_n \to \bar{\psi}_n + i\, \psi_m\, \left(\lambda_a^{mn}/2\right)\, \varepsilon^a \tag{2.2c}$$

The usual derivation of the Slavnov-Taylor identities begins with the observation that the measure [dA]det M[A] remains invariant under the special choice of parameter ε^a

$$\varepsilon^a = M^{-1}[A]\lambda_b \tag{2.3}$$

where λ_b is arbitrary and infinitesimal.

Under this particular gauge transformation, $\delta[F_a(A)] = \lambda_a$. Next varying the variables of integration in Z(J) according to this gauge transformation, we note that $Z(J)$ remains independent of the change of variables $\delta Z(J)/\delta\lambda = 0$. This observation may be written out explicitly in terms of source derivatives and expresses the content of the Slavnov-Taylor identities

$$\left[\left\{-F_a\big((1/i)(\delta/\delta J)\big) + J_i D_i^a[(1/i)(\delta/\delta J)] - \left(i\bar{\eta}_i t^b_{ij}(\delta/\delta\bar{\eta}_j) - i\eta_i t^b_{ji}(\delta/\delta\bar{\eta}_j)\right)\right\} \times M^{-1}\big((1/i)(\delta/\delta J)\big)_{ba}\right] Z(J,\eta,\bar{\eta}) = 0 \tag{2.4}$$

Repeated differentiation of this equation with respect to external sources $J,\eta,\bar{\eta}$ will then give the Slavnov-Taylor identities of QCD. However, it is usually more convenient to examine proper one-particle irreducible vertices. One therefore introduces the Legendre transform

$$\Gamma\left(\underset{\sim}{A}\ \underset{\sim}{\bar{\psi}}\ \underset{\sim}{\psi}\right) = W(J,\eta,\bar{\eta}) - J_i \underset{\sim}{A}^i - \bar{\eta}_i \underset{\sim}{\psi}^i - \underset{\sim}{\bar{\psi}}^i \eta_i \tag{2.5}$$

where the new variables (classical fields) are defined as follows

$$\underset{\sim}{A}_i \equiv \left(\delta Z/\delta J_i\right) \qquad \underset{\sim}{\psi}_i = \left(\delta Z/\delta\bar{\eta}_i\right) \qquad \underset{\sim}{\bar{\psi}}_i = -\left(\delta Z/\delta\eta_i\right) \tag{2.6}$$

and where $W(J,\eta,\bar{\eta}) = i\,\ell n\, Z(J,\eta,\bar{\eta})$ is the generating functional of one-particle irreducible (I.P.I.) vertices. One can therefore derive Slavnov-Taylor identities in terms of I.P.I. vertices, where the I.P.I. triple gluon and gluon-quark-antiquark vertex in the Landau gauge $\alpha = 0$ is in momentum space [11]

$$\left(1 + b(q_1^2)\right) q_{1_{\mu_1}} \Gamma^{abc}_{\mu_1\mu_2\mu_3}(q_1,q_2,q_3) = g_R\, f_{abc}\left[D^{-1}_{\mu_2\mu_3}(q_3) - D^{-1}_{\mu_2\mu_3}(q_2)\right]$$

$$+ \underset{-}{\Lambda}^{abc}_{\mu_2\mu_3'}(q_1,q_2) D^{-1}_{\mu_3'\mu_3}(q_3) - \underset{-}{\Lambda}^{abc}_{\mu_3\mu_2'}(q_1,q_2) D^{-1}_{\mu_2'\mu_2}(q_2) \tag{2.7a}$$

$$\left(1+b(q_1^2)\right)q_{1_\mu}\Gamma^\mu_a(q_1,q_2,q_3) = g_R\ \lambda^a/2\left[S^{-1}(q_3) - S^{-1}(q_2)\right] + \overset{o}{\underline{\Lambda}}{}^a(q_1,q_2)\ S^{-1}(q_3) - \overset{o}{\underline{\Lambda}}{}^a(q_1,q_2)\ S^{-1}(q_2) \tag{2.7b}$$

and where $\underline{\Lambda}^{abc}$ and $\overset{o}{\underline{\Lambda}}{}^a$ are ghost scattering-like kernels as shown in figure 2(a,b) and where $b(q_1^2)$ is the ghost self energy. Now one observes that in the Landau gauge, ghost-gluon and ghost-quark scattering-like kernels which appear in the Slavnov-Taylor identities for the one-particle irreducible triple gluon and gluon-quark-antiquark vertices have been shown to vanish for incoming ghost momentum going to zero. This result follows from the observation that in the Landau gauge, the gluon propagator is transverse $(\ell^\lambda - q_1)D^{\lambda\mu}(\ell - q_1) = 0$ and therefore as $\lim q_{1_\mu} \to 0$ the function $\Lambda_\mu \sim q_{1_\mu}A$ vanishes [3].

Using this result as well as an initial approximation of dropping the ghost self-energy term $ib(q^2)$ in the infrared region, the Slavnov-Taylor identities considerably simplify in the infrared region. We obtain Abelian-like Ward identities for the one-particle irreducible (I.P.I.) triple gluon (figure 3(b)) and gluon-quark-antiquark (figure 3(c)) where internal color symmetry is taken to be unbroken

$$q_{1_{\mu_1}}\Gamma^{abc}_{\mu_1\mu_2\mu_3}(q_1,q_2,q_3) = g_R f_{abc}\left[D^{-1}_{\mu_2\mu_3}(q_2) - D^{-1}_{\mu_2\mu_3}(q_2)\right] \tag{2.8a}$$

$$q_{1_\mu}\Gamma^\mu_a(q_1,q_2,q_3) = g_R\ \lambda^a/2\left[S^{-1}(q_3) - S^{-1}(q_2)\right] \tag{2.8b}$$

2.2 Schwinger-Dyson Equations

The Schwinger-Dyson equations are an infinite coupled system of non-linear equations for the Green's functions of QCD. Such equations express the full non-perturbative content of the quantum field theory equations of motion. In principle they must contain all non-perturbative effects of the physical vacuum. Solving such equations is therefore equivalent to solving the quantum field theory.

Two distinct approaches have been used to obtain these equations. The Dyson approach sums all possible Feynman perturbation diagrams, thus expressing them as sums of an infinite set of graphs, lumped into one particle irreducible vertices and propagators, which obey non-linear equations. Such an approach although seeming to sum about the free field vacuum, finally leads to equations which make no reference to perturbation theory. More rigorously, Schwinger obtained these equations using his formulation of quantum field theory in the presence of external sources as in eq. (2) for the generating functional Z(J). Such an approach makes reference only to the exact physical vacuum and fully interacting fields. This derivation is however more difficult and less intuitively obvious than the Dyson graphical approach. They have however been obtained by Eichten and Feinberg for QCD using Schwinger's method [12,13]. One uses the Lagrangian equations of motion for the field as well as an action principle.

The ingredients which go into these equations may be illustrated graphically in figure 3(a-f). They are the various one-particle irreducible vertices as well as the fully dressed propagators. The Schwinger-Dyson equations themselves, which will be of interest to us in this paper, are graphically represented in figure 4(a-g), the gluon vacuum polarization equation and the equation for the gluon propagator; figure 5, the I.P.I.

triple gluon vertex equation; figure 6, the skeleton expansions of kernels in figure 5; and figure 7, the quark self-energy and ghost self-energy equation. The infrared non-perturbative nature of these complicated non-linear equations is almost totally unknown. It is exactly this problem which we address in this paper.

3. QCD Green's Functions

3.1 Renormalization Group Equation

QCD has infrared singularities at zero momentum due to the masslessness of the gluon gauge fields. Therefore in order to avoid this infrared problem, renormalization constants as well as renormalized I.P.I. Green's functions are defined at a spacelike momentum point $p_i^2 = \mu^2$, where μ^2 is an abribrary mass parameter known as the renormalization point.

The I.P.I. ultraviolet renormalized Green's function with n external gluons ${}^{R}\Gamma^{(n)}_{\mu_1\mu_n} = Z_3^{n/2}\Gamma^{(n)}_{\mu_1\ldots\mu_n}$ can be shown to satisfy a renormalization group equation [3]. Such an equation is just the statement that before renormalization the bare amplitudes are independent of μ, the renormalization point

$$\mu(d/d\mu)\Gamma^{(n)}_{\mu_1\ldots\mu_n} = 0 \tag{3.1}$$

The total derivative $\mu(d/d\mu)$ can then be written in terms of the renormalized quantities which depend on μ implicitly. (For the purposes of this discussion we shall ignore the effect of the quark term in $\mathscr{L}_{QCD}$ and incorporate it self-consistently in a later section)

$$\mu(d/d\mu) = \mu(\partial/\partial\mu)+\beta\left(g_R(\mu)\right)\left(\partial/\partial g_R(\mu)\right) + \delta(g_R,\alpha)(\partial/\partial\alpha) \tag{3.2}$$

where $g_R(\mu) = Z_3^{\frac{3}{2}} \big/ Z_1 \; g_0$ is the renormalized coupling, Z_3 is the gluon wavefunction renormalization constant; Z_1 is the triple gluon renormalization constant; $\beta(g_R(\mu))=\mu(\partial g_R(\mu)/\partial\mu)$ is the important Callan-Symanzik β-function; $\delta(g_R(\mu),\alpha)=-2\alpha\gamma_G(g_R,\alpha)$ where $\gamma_G(g_R,\alpha)=\frac{1}{2}\mu(\partial/\partial\mu)\ell n\, Z_3$ the gluon anomalous dimension, and $\alpha=Z_3^{-1}\alpha_0$ the renormalized gauge parameter. Choosing $\alpha=0$, The Landau gauge $\delta(g_R,0)=0$, we find using equation (3.1) and equation (3.2) the renormalization group equation

$$[\mu(\partial/\partial\mu)+\beta(g_R(\mu))(\partial/\partial g_R)-n\gamma_G(g_R)]\Gamma^{(n)R}_{\mu_1\ldots\mu_n}(q_1\ldots q_n,g_R)=0 \qquad (3.3)$$

The renormalization constants Z_3 and Z_1 are defined by normalizing the gluon propagator and triple gluon vertex at the subtraction point μ^2 to their bare amplitudes (zero-th order perturbation terms). Upon mulitplying both left and right hand sides of equation (2) by the gluon propagators $D_{\mu_2\mu_3}(q_2)$ and $D_{\mu_2\mu_3}(q_3)$ and taking the limit $\alpha\to 0$, we obtain upon substituting the normalized forms, the simple Abelian-like relation $Z_1=Z_3$. Using this relation we find the important relation

$$\beta(g_R)=g_R\gamma_G(g_R) \qquad (3.4)$$

Therefore, only one independent function $\beta(g_R)$ remains in our infrared self-consistency scheme. Such a simplification makes it possible to obtain self-consistency conditions for the QCD vacuum polarization equation.

The effective coupling $\bar{g}(t,g_R)$ function is next introduced by the defining equation

$$\left.\frac{\partial\bar{g}(t,g_R)}{\partial t}\right|_{g_R}=\beta(\bar{g})\;;\qquad \bar{g}(0,g_R)=g_R \qquad (3.5)$$

where t is a dimensionless variable given implicitly by $t=\int_{g_R}^{\bar{g}}(dx/\beta(x))$.

More specifically if we take $t = \frac{1}{2}\ln\left(q^2/\mu^2\right)$, we obtain

$$\left(\mu(\partial/\partial\mu) + \beta(g_R)(\partial/\partial g_R) \right) \bar{g} = 0 \tag{3.6}$$

Using this equation and the relation $g_R\gamma_G(g_R) = \beta(g_R)$, we obtain a general solution to equation (3.3)

$$\Gamma_{\mu_1\ldots\mu_n} = g_R^n \sum_{i=1}^{n} \bar{F}\left(\bar{g}^{-2}(t_1)\ldots\bar{g}^{-2}(t_n)\right) T_i^{\mu_1\ldots\mu_i} \tag{3.7}$$

where $T_i^{\mu_1\ldots\mu_n}$ are tensors constructed out of tensor elements $g_{\mu_i\mu_j}$, q_i, $q_i \cdot q_j$; $i,j = 1\ldots n$ where, for example, the gluon propagator solution in the Landau gauge is

$$D_{\mu_1\mu_2}(q_1) = \frac{\bar{g}^2(q_1^2)}{q_1^2}\,\frac{1}{g_R^2}\left(g_{\mu_1\mu_2} - \frac{q_{1\mu_1}q_{1\mu_2}}{q_1^2}\right) + \frac{\alpha q_{1\mu_1}q_{1\mu_2}}{q_1^4} \; ; \tag{3.8}$$

$$\lim \alpha \to 0 \quad .$$

It is straightforward to show that solution equation (3.7) satisfies equation (3.3) by direct substitution using equation (3.6) and the definition $\beta(g_R) = \mu(\partial g_R/\partial\mu)$.

3.2 Construction of the I.P.I. Triple Gluon Green's Function

The fundamental strategy of our paper is to extract the infrared behavior of the effective coupling using the most general I.P.I. triple gluon vertex in our gluon vacuum polarization equation. We find that such a vertex can be constructed [6].

In order to construct an infrared effective triple gluon I.P.I. longitudinal vertex $\Gamma^{L\,abc}_{\mu_1\mu_2\mu_3}(q_1,q_2,q_3)$, we impose the following constraints: (a) Boson and Lorentz symmetry, (b) renormalization group equation solution, equation (3.7), (c) Abelian-like Ward identity equation (2.8a), and (d) absence of kinematical singularities. The transverse gluon vertex $\Gamma^{L\,abc}_{\mu_1\mu_2\mu_3}(q_1,q_2,q_3)$ is similarly determined by the homogeneous version of equation (2.8a) $q_{1_{\mu_1}}\Gamma^{L\,abc}_{\mu_1\mu_2\mu_3}(q_1,q_2,q_3) = 0$. We therefore construct the most general gluon vertex satisfying all of these constraints using the kinematically singularity-free elements $\rho_{ij} \equiv \left(\bar{g}^{-2}(q_i^2) - \bar{g}^{-2}(q_j^2)\right) / (q_i^2 - q_j^2)$; where $q_i^2/\mu^2 \ll 1$

$$\Gamma^{L\,abc}_{\mu_1\mu_2\mu_3}(q_1,q_2,q_3) = g_R^3 f^{abc}\Big\{ g_{\mu_1\mu_3}\left[q_{3_{\mu_2}}\,\bar{g}^{-2}(q_3^2) - q_{1_{\mu_2}}\,\bar{g}^{-2}(q_1^2)\right]$$

$$+ \rho_{13}\left[q_1\cdot q_3\, g_{\mu_1\mu_3} - q_{3_{\mu_1}} q_{1_{\mu_3}}\right](q_1 - q_3)_{\mu_2} + \text{cyclic permutations}\Big\}$$

$$+ (1/\alpha)\bar{\Gamma}^{L\,abc}_{\mu_1\mu_2\mu_3}(q_1,q_2,q_3)\ ; \qquad \lim\ \alpha \to 0 \tag{3.9a}$$

where

$$q_{1_{\mu_1}}\bar{\Gamma}^{L\,abc}_{\mu_1\mu_2\mu_3}(q_1,q_2,q_3) = g_R^3 f^{abc}\left(q_{2_{\mu_2}} q_{2_{\mu_3}} - q_{3_{\mu_2}} q_{3_{\mu_3}}\right)$$

and

$$\Gamma^{T\,abc}_{\mu_1\mu_2\mu_3}(q_1,q_2,q_3) = g_R^3 f^{abc}\Big\{ a_1\left[\rho_{12} + \rho_{23} + \rho_{31}\right]$$

$$\times\left[g_{\mu_1\mu_2}\left(q_2\cdot q_3 q_{1\mu_3} - q_1\cdot q_3 q_{2\mu_3}\right) + \tfrac{1}{3}\left(q_{1\mu_2}q_{2\mu_3}q_{3\mu_1} - q_{1\mu_3}q_{2\mu_1}q_{3\mu_2}\right)\right]$$

$$+ \left(a_2\rho_{13}\rho_{23} + a_3\rho_{12}\rho_{12}\right)\left(q_1\cdot q_2 g_{\mu_1\mu_2} - q_{1\mu_2}q_{2\mu_1}\right)\left(q_2\cdot q_3 q_{1\mu_3} - q_1\cdot q_3 q_{2\mu_3}\right)\Big\}$$

$$+ \text{cyclic permutations} \tag{3.9b}$$

where a_1, a_2, a_3 are unknown constants. In the limit of any one of the incoming momentum $q_{i\mu_i}$ going to zero with others held fixed $(q_1 + q_2 + q_3 = 0)$ we observe $\Gamma_{\mu_1\mu_2\mu_3}(0,-q_3,q_3) = \partial D^{-1}_{\mu_2\mu_3} \big/ \partial q_{3\mu_1}$; $\Gamma^T \to 0$ and where $\bar{\Gamma}^L \to 0$ by construction. The differential version of the Slavnov-Taylor identity equation (2.8a) is thus satisfied.*

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*Note also that the total triple gluon vertex is normalized at the Euclidean point $q_1^2 = q_2^2 = q_3^2 = \mu^2$, only insofar as the Slavnov-Taylor identity equation (2.8a) is satisfied.

In order to understand the general approach to constructing such a vertex, let us examine as an example the terms which have just a $g_{\mu_i\mu_j}$ tensor structure in the vertex. Writing down the general $g_{\mu_i\mu_j}$ gluon vertex terms, we have:

$$\Gamma' = f^{abc}\left\{\left[A_1 q_{1\mu_3} - A_2 q_{2\mu_3} + \bar{A}_{12}\left(q_{1\mu_3} - q_{2\mu_3}\right)\right] g_{\mu_1\mu_2}\right\}$$

$$+ \quad \text{cyclic permutations} \qquad (3.11)$$

where A_i, A_{ij} are unknown functions of q_i^2, $\bar{g}^{-2}(q_i^2)$ and where similarly $A_{ij} = A_{ji}$ is symmetric. Substituting this expression into the triple gluon Slavnov-Taylor identity equation (2.7) and equating $g_{\mu_2\mu_3}$ terms on both sides, we obtain:

$$A_2 q_1\cdot q_2 - A_3 q_1\cdot q_3 + \bar{A}_{23}(q_1\cdot q_2 - q_1\cdot q_3) = \bar{g}^{-2}(q_3^2)q_3^2 - \bar{g}^{-2}(q_2^2)q_2^2 \qquad (3.12)$$

A solution which has no kinematical singularities may be written down:

$$A_2 = \bar{g}^{-2}(q_2^2) \quad ; \quad A_3 = \bar{g}^{-2}(q_3^2) \quad ; \quad \bar{A}_{23} = -q_2\cdot q_3\ \rho_{23} \qquad (3.13)$$

Using this solution, one may verify by tedious inspection that our general vertex equation (3.9) may be constructed. Such a solution appears to be unique. It is also possible, although very tedious, to construct the I.P.I. four gluon vertex figure 2(d) which is needed for the gluon vacuum polarization terms figure 3(d) and 3(e). We shall find, however, that the essential behavior of the infrared behavior of the effective coupling may be extracted from the triple gluon vacuum polarization term.

4. Infrared Behavior of the Effective Coupling

4.1 Vacuum Polarization Equation

The Schwinger-Dyson equation for the gluon vacuum polarization tensor $\pi_{ab}^{\mu_3\mu_3'}(q_3,\alpha)$ is given in figs. 4(a) to 4(f) and obeys the transversality condition $q_3^{\mu_3}\pi_{ab}^{\mu_3\mu_3'}(q_3,\alpha)=0$ where $\pi_{ab}^{\mu_3\mu_3'}(q_3,\alpha)=\delta_{ab}\pi(q_3,\alpha)\left(g^{\mu_3\mu_3'}q_3^2-q_3^{\mu_3}q_3^{\mu_3'}\right)$. Unlike QED the vacuum polarization is gauge dependent. We will consider only the terms figures 4(a-b) and discuss the remaining terms, figures 4(c-f) later. Using equation (2.8a) it is then straightforward to show that the terms figures 4(a-b) will obey the transversality condition $q_{3_{\mu_3}}\pi^{\mu_3\mu_3'}(q_3,\alpha)=0$. This can be seen by either regulating the denominators with a small mass $\delta^2\to 0$ or by analytically continuing in space-time dimension n where integrals of the form $\int d^n k(k^2)^\beta=0$ [3]. Such a proof is analogous to that given in QED

$$q_{3_{\mu_3}}\pi^{\mu_3\mu_3'}(q_3)=\frac{g_R^2}{2!}\int d^d q_1\Gamma^{o\,cda}_{\mu_1'\mu_2'\mu_3'}D^{cc'}_{R\mu_1\mu_1'}(q_1)f^{abc}\times D^{dd'}_{R\mu_2\mu_2'}(q_2)$$

$$\left[D^{-1}_{\mu_1\mu_2}(q_2)-D^{-1}_{\mu_1\mu_2}(q_1)\right]=0 \tag{4.1}$$

After replacing unrenormalized coupling, propagators and vertices by their renormalized counterparts according to the standard renormalization prescription, i.e., $g_0 = Z_3^{-\frac{1}{2}} g_R$, $D = Z_3 D_R$, $\Gamma = Z_3^{-1}\Gamma_R$, etc., we obtain a vacuum polarization equation

$$\delta^{ab}\pi_R(q_3^2,\alpha)\left(q_3^2 g^{\mu_3\mu_3'} - q_3^{\mu_3} q_3^{\mu_3'}\right) = \frac{g_R}{2!}\int d^d q_1 \Gamma^{o\,cda}_{\mu_1'\mu_2'\mu_3'}(-q_1,-q_2,-q_3) D^{cc'}_{R\mu_1\mu_1'}(q_1)$$

$$\times\ D^{dd'}_{R\mu_2\mu_2'}(q_2)\ \Gamma^{c'd'b}_{R\mu_1\mu_2\mu_3}(q_1,q_2,q_3) \qquad (4.2)$$

where

$$\pi_R(q_3^2,\alpha) = Z_3^{-1} g_R^2 / \bar{g}^{-2}(q_3^2) - 1 \text{ and } Z_3^{-1} < \infty$$

4.2 Self-Consistency Conditions

Imposing the weak constraint condition $q_3^2 \bar{g}^{-2}(q_3^2) \to 0$ as $q_3^2/\mu^2 \to 0$ one observes that the left-hand side of eq. (4.2) goes to zero as $q_3 \to 0$. Substituting the limiting forms for the propagator eq. (3.8) and vertices eq. (3.9a) and eq. (3.9b) as $q_3 \to 0$ into the right-hand side of eq. (4.2), one obtains a self-consistency constraint after contracting indices

$$\int d^n q_1 \left[\left(\bar{g}^2(q_1^2) / q_1^2\right) - (d/dq_1^2)\ \bar{g}^2(q_1^2)\right] = 0 \qquad (4.3)$$

This constraint follows from the longitudinal vertex solution eq. (3.9a) as the transverse vertex eq. (3.9b) vanishes when any single momentum is taken to zero. Next observe that if we rescale variables $q_1 \to \underset{\sim}{\lambda}^{\frac{1}{2}} q_1$, then in order for the left-hand side of the equation to remain $\underset{\sim}{\lambda}$ independent, we must have $\bar{g}^2(\underset{\sim}{\lambda} q^2) = \underset{\sim}{\lambda}^{((d-2)/2)} \bar{g}^2(q^2)$. This implies that only a polynomial solution is possible

$$\bar{g}^2(q^2/\mu^2, g_R) = (\mu^2/q^2)^\lambda g_R^2 \qquad \lambda = \left((d-2)/2\right) \tag{4.4}$$

Such a solution is strictly only valid for the region $q^2/\mu^2 \ll 1$, as we know from the "asymptotic freedom" short distance behavior $\bar{g}^2(q^2) \sim 1/\ell n(q^2/\mu^2)$ for $q^2/\mu^2 \gg 1$. We have assumed implicitly that the short distance region is decoupled from the asymptotic long distance region and thus U.V. divergencies as expressed as $1/(n-4)$ singularities have already been subtracted (see figure 1).

Integrating the second term in equation (4.3) by parts, we observe that a surface term must vanish

$$(q_1^2)^{(d/2)-1}\bar{g}^2(q_1^2)\Big|_0^\infty = 0 \tag{4.5}$$

Our solution for the effective coupling equation (4.4) satisfies this condition. Substituting our solution, equation (4.4), the remaining terms which are integrals of the polynomial form may be dimensionally regulated to zero as before $\int d^n q_1 \, (q_1^2)^\beta = 0$ [3].

Next we substitute our infrared singular solution equation (4.4) into the vacuum polarization equation (4.2) using equations (3.9a) and (3.9b) in order to ascertain its self-consistency for small but non-vanishing values of q_3. We find after straightforward although rather tedious algebra and after evaluating Feynman type integrals via the standard n-dimensional regularization techniques the expression

$$\left(\mu^2/q_3^2\right)\left(c_0/(n-4) + c_0' + a_1 c_1\right) + a_2 c_2 + a_3 c_3 = -1 \tag{4.6}$$

where $c_0 \neq 0$, c_0', c_1, c_2 and c_3 are dimensionless constants; and where we have set $d = 4$, i.e., $\lambda = 1$ and have already factored out transverse tensor $(g^{\mu_3\mu_3'}q_3^2 - q_3^{\mu_3}q_3^{\mu_3'})$ from both sides, Explanatory comments on eq. (4.6) are appropriate here. In obtaining eq. (4.6) we have taken the lim $\alpha \to 0$ and subtracted out a term with coefficient $1/\alpha$. The remaining term, i.e., the left-hand side of eq. (4.6) which is independent of α can be shown to remain transverse. The term $(\mu^2/q_3^2)\left(c_0/(n-4)\right)$ which arises from the sum of infrared divergent integrals of the form $\int d^n q_1 \left(1/(q_1^2)^\alpha\right)\left(1/[(q_1+q_3)^2]^\beta\right)$; $\alpha+\beta > 2$ originates from the longitudinal vertex eq. (3.9a) contribution to eq. (4.2). It is clear that eq. (4.6) cannot be consistent unless the terms in the parenthesis multiplying μ^2/q_3^2 disappear.

Let us suppose that we are able to eliminate the $c_0/(n-4)$ term, which leads to an infrared divergent vacuum polarization. It is then possible to adjust the underdetermined transverse parameters a_1, a_2 and a_3, i.e., $a_1 = c_0'/c_1$, $a_2 = 0$, $a_3 = -1/c_3$, etc., so that eq. (4.6) is satisfied. Knowledge of the longitudinal vertex solution eq. (3.9a) is therefore <u>not sufficient</u> in order to satisfy the vacuum polarization eq. (4.2) with solution eq. (4.4) for $\bar{g}^2$. Using the vacuum polarization equation fig. 4(a-f) one can therefore only show that the ansatzed solution eq. (4.4) for $\bar{g}^2$ satisifies a necessary condition eq. (4.5) although not necessarily a sufficient condition eq. (4.6). In order to determine a_i from QCD itself it is necessary to understand the global properties of the total triple gluon vertex. Such information however is obtained in the infinite hierarchy of coupled Schwinger-Dyson equations for the triple gluon, quadruple gluon vertex, et cetera. A truncation procedure for these equations, similar to our analysis of the vacuum polarization function may help determine the a_i parameters. More specifically one should consider substituting our I.P.I. triple gluon vertex into its Schwinger-Dyson equation retaining only the one loop skeleton terms on the right-hand side of figures 5 and 6. We are presently investigating this possibility.

To understand the origin of the infrared divergent 1/(n-4) term let us examine the infrared content of the right-hand side of the vacuum polarization eq. (4.2). This can be done by looking at the small integration momentum $q_1 \ll q_3$ region where q_3 is also small but finite

$$\pi_{\mu_3\mu_3'}(q_3) \approx (1/2!) \int_0^\sigma d^n q_1 \Gamma^o_{\mu_1'\mu_2'\mu_3'}(0,+q_3,-q_3) D_{\mu_1\mu_1'}(q_1) D_{\mu_2\mu_2'}(-q_3)$$

$$\times \Gamma_{\mu_1\mu_2\mu_3}(0,-q_3,q_3) \tag{4.7}$$

where $q_1 \ll \sigma \ll q_3$ restricts the integration strictly to the infrared region. Thus in order that $\pi_{\mu_3\mu_3}(q_3)$ contains no infrared divergent 1/(n-4) term we find the constraint $\int_0^\sigma d^n q_1 D_{\mu_1\mu_1'}(q_1) < \infty$. Substituting the Landau gauge propagator equation [Eq. (3.8)] and $\bar{g}^2$ equation [Eq. (4.4)] we observe that this conditon is not met, giving rise to a 1/(n-4) infrared divergent term in our simplified Landau gauge formulation.

In order to understand this problem, let us examine the axial gauge $\mathscr{L}_g = -(1/2\alpha)(n_\mu A^\mu_a)^2$ where n_μ is the gauge direction vector, lim $\alpha \to 0$ as in Baker, et al [6]. In this gauge the Slavnov-Taylor identities equations (2.8a) and (2.8b) are exact as Faddeev-Popov ghosts are absent. This is easily seen by observing that det M(A) becomes independent of A and can therefore be integrated out of the path integral.

The full gluon propagator in the axial gauge is however more complicated (ignoring color indices)

$$D_{\mu_3\mu_3'}(q_3^2, q_3\cdot n) = A_{ren}(q_3^2, q_3\cdot n)P^{\mu_3\mu_3'} + B_{ren}(q_3^2, q_3\ n)g^{T_{\mu_3\mu_3'}}$$

$$+ \alpha q_3^{\mu_3} q_3^{\mu_3'} / (n\cdot q_3)^2 , \qquad \lim \alpha \to 0 \tag{4.8}$$

where

$$P^{\mu_3\mu_3'} \equiv (1/q_3^2)\Big(g^{\mu_3\mu_3'} - \big(q_3^{\mu_3} n^{\mu_3'} + q_3^{\mu_3'} n^{\mu_3}\big)/(n\cdot q_3) + \big(n^2 q_3^{\mu_3} q_3^{\mu_3'}\big)/(n\cdot q_3)^2\Big) \tag{4.9a}$$

and

$$g^{T_{\mu_3\mu_3'}} = \Big(g^{\mu_3\mu_3'} - \big(n^{\mu_3} n^{\mu_3'}\big)/n^2\Big) \tag{4.9b}$$

It is straightforward to show that by choosing the solution $A = \bar{g}^{2}(q_3^2)/g_R^2$, $B = 0$, the Slavnov-Taylor identity eq. (2.8a) is satisfied with the triple gluon vertex eq. (3.9a) and eq. (3.9b) (after dropping the $\bar{\Gamma}^{Labc}_{\mu_1\mu_2\mu_3}$ term. In terms of the general vacuum polarization term

$$\pi^{\mu_3\mu_3'} = \pi_1\big(q_3^2 g^{\mu_3\mu_3'} - q_3^{\mu_3} q_3^{\mu_3'}\big) + \pi_2\Big(q_3^{\mu_3} q_3^{\mu_3'} - q_3^2 \ \big(q_3^{\mu_3} n^{\mu_3'} + q_3^{\mu_3'} n^{\mu_3}\big)/(n\cdot q_3)$$

$$+ \big((q_3^2)^2\big) / \big((n\cdot q_3)^2\big) \ n^{\mu_3} n^{\mu_3'}\Big) \tag{4.10}$$

our solution corresponds to $\pi_1 = 1/A$; $\pi_2 = 0$. Our propagator also obeys the same renormalization group eq. (3.3). It can also be shown in the axial gauge that all renormalization group functions are independent of n_μ [14].

Substituting our axial propagator into the infrared convergence criterion $\int_0^\sigma d^n q_1 D^{\mu_1\mu_1'}(q_1) < \infty$ lim $n \to 4$ we observe that this criterion is satisfied due to the vanishing of the angular integral $\int d\Omega_{q_1} P^{\mu_3\mu_3'} = 0$. This can be seen by evaluating the integral [6]

$$\int d\,\Omega_{q_1}\left(n_{\mu_1} q_{1\mu_1'} + n_{\mu_1'} q_{\mu_1}\right) / (n\cdot q_1) = 2\, n_{\mu_1} n_{\mu_1'} / n^2 \tag{4.11a}$$

and

$$n^2 \int d\,\Omega_{q_1} q_{1\mu_1} q_{1\mu_1'} / (n\cdot q_1)^2 = -g_{\mu_1\mu_1'} + 2\, n_{\mu_1} n_{\mu_1'} / n^2 \tag{4.11b}$$

The rest of the arguments leading to our self-consistency condition eq. (4.6) can basically be retained in the axial gauge. The right-hand side of the vacuum polarization equation however does induce a new π_2 term. We conjecture that such an induced term will not give rise to an infrared singlular B term propagator which could potentially violate our infrared convergence criterion. We also note that $(q_3 \cdot n)$ type terms which are basically kinematic in nature are induced in the vacuum polarization tensor. Such complications are probably resolvable by assuming a better initial ansatz for A and B which involves some kind of $(q_3 \cdot n)$ dependent function.

The remaining quadruple gluon terms figs. 4(d) and 4(e) may be treated in a similar manner to our I.P.I. triple gluon term through the use of the I.P.I. quadruple gluon Slavnov-Taylor identity. The transverse I.P.I. triple gluon Green's function however already determines a sufficiency condition. It thus appears pointless to examine the even more complicated four gluon vacuum polarization terms until this condition is determined. We expect however, such terms {figures 4(d) and 4(e)} to be self-consistent with our solution equation (4.4). Finally returning to our $q_3 \to 0$ consistency condition we find in axial gauge with our propagator solution the same surface term self-consistency condition as in the Landau gauge, equation (4.5).

The Schwinger-Dyson equation for the ghost self-energy figure 7(a) may be solved in the infrared region $k^2 \to 0$ by making use of the effective coupling's equation (4.4) infrared singular behavior and the Landau gauge's simplifying properties for the ghost-ghost gluon vertex [3]. Solving we obtain $b_{ren}(k^2) \sim \tilde{Z}_3(k^2,\delta^2)$, $\lim k^2 \to 0$; $\tilde{Z}_3$, the ghost renormalization constant $\lim \delta^2 \to 0$ where $\tilde{Z}_3(k^2,\delta^2) = \tilde{Z}_3 \; \ell n \; k^2/\delta^2$. One therefore observes that our initial approximation of dropping $b_{ren}(k^2)$ in the Slavnov-Taylor identities is inconsistent (due to its infrared divergent behavior as $\delta^2 \to 0$) confirming again the lack of a simple self-consistent infrared scheme in the Landau gauge. We conjecture that retaining the ghost propagator self-energy in our vertex solution and in the vacuum polarization figure 4(c) will ameliorate both this difficulty and the $c_0/(n-4)$ problem. We are currently investigating this approach.

5. Dynamical Mass Λ and the Nature of the 1/N Topological Expansion

The function in the infrared region is obtained by substituting $\bar{g}^2 = (\mu^2/g^2)^{\lambda} g_R^2$ into equation (3.5) to obtain

$$\beta\left(g_R(\mu)\right) = -(\lambda/2) \; g_R(\mu) \tag{5.1}$$

Solving for $g_R(\mu)$ we observe that the quantity $g_R(\mu)\mu^{\lambda/2} = g_R(\mu_0)\mu_0^{\lambda/2}$ is a renormalization group invariant where μ_0 is an arbitrary mass point.

In particular for $d=4$, $g_R(\mu_0)$ is dimensionless and $\Lambda = g_R^2(\mu_0)\mu_0$ may be identified as a dynamical mass. This may also be seen explicitly by noting that $\Lambda = \mu \times \exp - \int^{g_R} \left(dg/\beta(g)\right)$ is the dimensionally transmuted mass

$$\mu(d/d\mu)\ \Lambda = \left(\mu(\partial/\partial\mu) + \beta(g_R)(\partial/\partial g_R)\right)\ \Lambda \ = 0 \qquad (5.2)$$

Note also that by setting $\mu_0 = \Lambda$ we see that Λ is defined as the scale for which $g_R^2(\Lambda) = 1$. Substituting our expression for $g_R(\mu)$ in terms of Λ we may rewrite $\bar{g}^2$ as follows

$$\bar{g}^2(q^2) \ = \ \mu\Lambda/q^2 \ ; \ \mu \le \Lambda \quad , \quad \lim q^2/\mu^2 << 1 \qquad (5.3)$$

Note that the choice of the Euclidean subtraction point μ is arbitrary and does not affect the values of physical quantities such as hadronic masses. In consequence with this observation some convenient choices for μ are $\mu = \Lambda$ or $\mu = \Lambda/N$. We therefore note that the role of the effective coupling in QCD is not that of a physical observable, as it is renormalization dependent. Rather, such a function serves to define the Green's functions of QCD which in turn themselves must be convoluted to determine physically observable quantities such as scattering cross-sections or hadronic masses.

We argue heuristically that the choice of $\mu = \Lambda/N$ corresponds to the well-known 't Hooft topological expansion of diagrams in the asymptotic $1/N$, $N \to \infty$ limit, whose leading terms are planar [15]. In order to see this as simply as possible, let us draw the parallel for the four-dimensional QCD theory when $\mu = \Lambda/N$ is chosen, with the two-dimensional QCD of 't Hooft where the coupling parameter $g^2 \to 1/N$, $N \to \infty$ and which has the dimensions M^2 . Noting that the propagator factor is $\left(\bar{g}^2(a^2)/q^2\right) = (\Lambda^2/N)\ 1/q^4$ in four dimensions we find that when lines are color contracted internally in diagrams, a factor of N will accompany them. A simple table then illustrates our correspondence

4 - d		2 - d	
$\mu\Lambda$	← coupling / dim $[M^2]$ →	g^2	
$\mu\Lambda \to (\Lambda^2/N)$	← choice $\mu = \Lambda/N$ →	$g^2 \to 1/N$	(5.4)
$1/q^4$	← I.R. Propagator →	$1/q^2$	

What this brief discussion points out is that in renormalized QCD the 1/N expansion merely corresponds to a particular renormalization scheme which arranges diagrams according to its topological characteristics, i.e., planar, one handle graphs, etc. We see however that in the two dimensional case, as there is no μ dependency, i.e., the choice $g^2 \to 1/N$ lim $N \to \infty$ is in fact a specific type of theory and does not correspond to an arbitrary renormalization scheme. In contrast to previous speculations, we therefore argue that the 1/N expansion is merely a convenient renormalization choice and does not give rise to any dynamical origins of the confinement phase of QCD [15].

6. Quark Propagator and an Effective Bound State Equation

It is instructive to examine the quark self-energy equation fig. 7(b) in our Landau gauge approximation (albeit its technical difficulties) and in the infrared region

$$\Sigma(p) = i\, g_R^2 \int d^d \bar{q}\, D_R^{\mu\nu}(q) \Gamma_{\mu R}^a(p, P+Q)\; S_R(p+q)\; \gamma_\nu \tag{6.1}$$

We can take advantage of the effective coupling's infrared singular behavior by making use of the renormalized version of equation (2.8b) $\Gamma_{\mu a}^a(p,p+q) = g_r t_a \left(\partial S_R^{-1}(p)/\partial p_\mu\right)$; where $S = Z_2 S_R$ and $\Gamma_{aR}^\mu = Z_2^{-1}\Gamma_a^\mu$. We obtain in the case of zero weak-E.M. mass quarks the equation

$$\Sigma(p) \approx kC_2(R)(\partial S_R^{-1}/\partial p_\mu) S_R(p)\gamma_\mu \int_{\delta^2}^{p^2} \left(\bar{g}^2(q^2)/q^2\right)\left(g^{\mu\nu} - (q^\mu q^\nu/q^2)\right)$$

$$\times \left(q^2\right)^{(d/2)-1} dq^2 \quad , \quad \lim \delta^2 \to 0 \tag{6.2}$$

where $C_2(R) = (N^2 - 1)/2N$, k is an angular integration constant and δ^2 is an infrared cutoff regulator. Substituting the quark propagator $S_R = i\left(\not{p} A_R(p^2) + B_R(p^2)\right)$, one obtains uncoupled first order differential equations if we use the approximation $\Sigma(p) \approx S^{-1}(p) = Z_2^{-1} S_R^{-1}$, which ignores the kinetic term $\not{p}$ [5(a)].

$$\beta_1\, 2p^2 \left(dA(p^2)/dp^2\right) + \beta_2 A(p^2) = \text{const.}/\ell n(p^2/\delta^2) \tag{6.3a}$$

$$\left(dB(p^2)/dp^2\right) = 0\,; \qquad \beta_1,\ \beta_2 > 0 \tag{6.3b}$$

One easily solves these equations obtaining the solutions

$$A_R(p^2,\delta^2) = (\rho/\beta_1)\, \log\left| \log p^2/\delta^2 \right| \Big/ (p^2/\delta^2)^{\beta_2/\beta_1} \tag{6.4a}$$

$$B_R(p^2) = \text{constant} \tag{6.4b}$$

where $\rho = \left(1/(2\pi)^4 \times Z_2^{-1}\right) \big/ k\, C_2(R) \mu^2 g_R^2$ and $\beta_1, \beta_2 > 0$ are constants.

To understand this result, let us compare the electron propagator's infrared behavior in QED $S_{unren} = Z_2 \not{p} / (p^2)^{1+\gamma}$; $\gamma = \alpha/2\pi + \ldots$. One can define $Z_{2\,I.R.}(p) \equiv Z_2/(p^2)^{1+\gamma}$, thus absorbing the soft coherent infrared photon cloud into the wave function renormalization, leaving a renormalized single particle pole state. By analogy our QCD quark propagator can be rewritten as $S_{unren} = Z_{2\,QCD}(p)\, S_R$, where $S_R = Z^{-1}_{2\,QCD}(p)\, \not{p}/p^2$ and where $Z^{-1}_{2\,QCD}(p) = p^2 A_R(p^2,\delta^2)$. One therefore observes that unlike QED the soft coherent gluon cloud cannot be renormalized away, but in fact confines as $\delta^2 \to 0$, leading to no on-shell renormalized quark state. The second important property to note is that the solution $B_R(p^2) = $ constant, violates Chiral symmetry $[\gamma_5, S^{-1}]_+ \neq 0$ and therefore realizes the PCAC phase. One thus obtains a dynamical Goldstone boson in the axial vertex Γ^5_μ as a consequence of our infrared solution [5(a)].

Next, using the product $S_R \Gamma^\mu_R = -i\gamma_\mu/\not{p}\ \left(\lim q \to 0\,,\ \delta^2 \to 0\right)$, it is straightforward to observe that the quark term fig. 4(f) vanishes in the $\delta^2 \to 0$. Inserting an explicit bare quark mass, i.e., $\not{p} \to \not{p} - m$ does not alter this conclusion. We expect the quark propagator in the axial gauge to behave in a similar way to the preceding results.

Using again the result $S_R \Gamma^\mu_R = -i\gamma_\mu (1/\not{p}) \left(\lim q_\mu \to 0,\ \delta^2 \to 0\right)$, one may derive an effective Bethe - Salpeter bound state equation for the quark-antiquark scalar channel in the ladder approximation. Using our result for the infrared effective coupling, we obtain the infrared effective kernel

$$K_{I.R.} = N\, \bar{g}^2(q^2) / g^2 \sim \Lambda^2/q^4 \ ; \quad \lim q^2/\Lambda^2 \ll 1 \tag{6.5}$$

where we have chosen $\mu = \Lambda / N$. Next using the effective coupling in the short distance $\bar{g}^2(q^2) \sim 1/\ell n\, q^2/\mu^2$; $q^2/\mu^2 \gg 1$, one finds via the renormalization group equation (3.3)

$$D^{\mu_1\mu_2}(q) \sim 1/q^2\left(g^{\mu_1\mu_2} - (q^{\mu_1}q^{\mu_2}/q^2)\right)\left(\ell n(q^2/\mu^2)\right)^{c/b} \quad (6.6a)$$

$$S(q) \sim 1/\not{q} \quad (6.6b)$$

$$\Gamma^\mu_a \sim \left(\lambda^a/2\right)\gamma^\mu \, \ell n(q^2/\mu^2)^{(-b/2c)-\frac{1}{2}} \quad (6.6c)$$

where b anc c are calculable constants [3].

Inserting these asymptotic Green's functions into the ladder kernel, we obtain the effective short-distance kernel

$$K_{U.V.} = N\bar{g}^2(q^2)/q^2 \sim (N/b)(1/q^2)\left(1/\ell n(q^2/\Lambda^2)\right), \ \lim q^2/\Lambda^2 >> 1 \quad (6.7)$$

Substituting these kernels, we obtain an effective bound-state equation $\lim \delta^2 \to 0$

$$(\not{p}_1 - m_1)(\not{p}_2 - m_2)\psi(p_1,p_2) = \int \psi(p_1+q, p_2-q) \ N\left(\bar{g}^2(q^2)/q^2\right) d^4q \quad (6.8)$$

Using the normalization condition for the Bethe - Salpeter $\psi(p_1,p_2)$ wavefunction, one can show that an interesting constraint condition on the number of flavored quarks $n_F < 9$ arises for SU(3) QCD [15]. Such a constraint follows from the short-distance asymptotic $1/(\ell n \, p^2/\mu^2)$ behavior of the bound-state wavefunction and the demand for an integrable wavefunction. Although our equation is constructed in the Landau gauge, we expect such a constraint to hold in other covariant gauges.

This equation also gives rise to a simple interpretation of free-like quarks $1/(\not{p} - m)$ interacting through a relativistic linear- or string-like confining potential Λ^2/q^4 at long-distances $q^2/\Lambda^2 << 1$ and Coulomb-like $(1/q^2)\ \ell n(q^2/\Lambda^2)$ at short distances $q^2/\Lambda^2 >> 1$. This is somewhat surprising, as the quark propagator has no on-shell state, i.e., threshold. In some respects this resembles behavior discovered by 't Hooft in his two-dimensional 1/N theory [15]. A similar equation to our meson effective equation can also be derived using our triple gluon vertex for the Baryon spectrum.

Clearly, our effective equation also has several problems. The ladder approximation is not a gauge-invariant approximation, and in a theory of infrared singular relativistic propagators, one cannot really ignore higher-order terms in the expansion of the Bethe - Salpeter kernel. It is also unclear whether a solution exists for the infrared region of such an infrared singular bound-state equation. Despite these difficulties, such an effective equation by itself may prove an interesting description of the hadronic spectrum.

7. Conclusions

We have shown in this paper that if a self-consistency sufficient condition is satisfied, then the preferred solution is an infrared singular effective coupling. In obtaining this solution, albeit several technical difficulties, we have also been made to realize its renormalization dependent presecription. Therefore one should not look upon such a function as a classical type of potential as in QED. Rather we view the infrared behavior of the effective coupling as a signal of the color confining nature of $\mathscr{L}_{QCD}$ and the possible development of extended spatial structures such as strings, bags, etc., in the theory.

An analogy that one may draw to this situation is the role of electron Cooper-pairing in the superconductivity phenomena. It provides a non-perturbative signal for the development of a new condensed phase. The extended structure of superconductivity, helium vortices, surface waves, etc., however, are not directly attainable from such a mechanism. Using the original local B.C.S. superconductivity Lagrangian and the new condensed phase, one is able to generate an effective Landau-Ginzburg Lagrangian having a long-range order parameter $|\psi|$. This prescription is then capable of describing the extended structure of superconductivity.

Perhaps using $\bar{g}^2$ as a function of F^2 where F^2 is treated as a long-distance order parameter, an effective Lagrangian for $\mathscr{L}_{QCD}$'s extended structure can be generated. Another approach one might consider are Schwinger-Dyson equations for path-dependent gauge invariant Green's functions, i.e., $\langle 0|T\bar{\psi}(x)\,P\exp(-\int_x^y A^\mu dx_\mu)\,\psi(y)|0\rangle$ in coordinate space. Such an approach may lead more directly both to a space-time picture as well as to a gauge invariant description of the spectrum.*

*Added in proof: After this paper was completed, we became aware of a preprint U.C.B.-PTH-79/8, July 79, S. Mandelstam, "Approximation Scheme for Q.C.D." Using a different approach to solving the Schwinger-Dyson equations, he obtains the same $1/q^4$ behavior for the gluon propagator as well as several other similar results.

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Figure Captions

[1] Effective coupling $\bar{g}^2(q^2/\mu^2)$.

[2] (a) Ghost-gluon scattering-like kernel.

(b) Ghost-quark scattering-like kernel.

(c) Ghost-ghost-gluon vertex.

[3] (a) Full gluon propagator.

(b) Triple gluon I.P.I. vertex.

(c) Gluon-quark-antiquark I.P.I. vertex.

(d) Four gluon I.P.I. vertex.

(e) Full ghost propagator.

(f) Full quark propagator.

[4] (a) to (g) Schwinger-Dyson equation for gluon vacuum polarization tensor and gluon propagator.

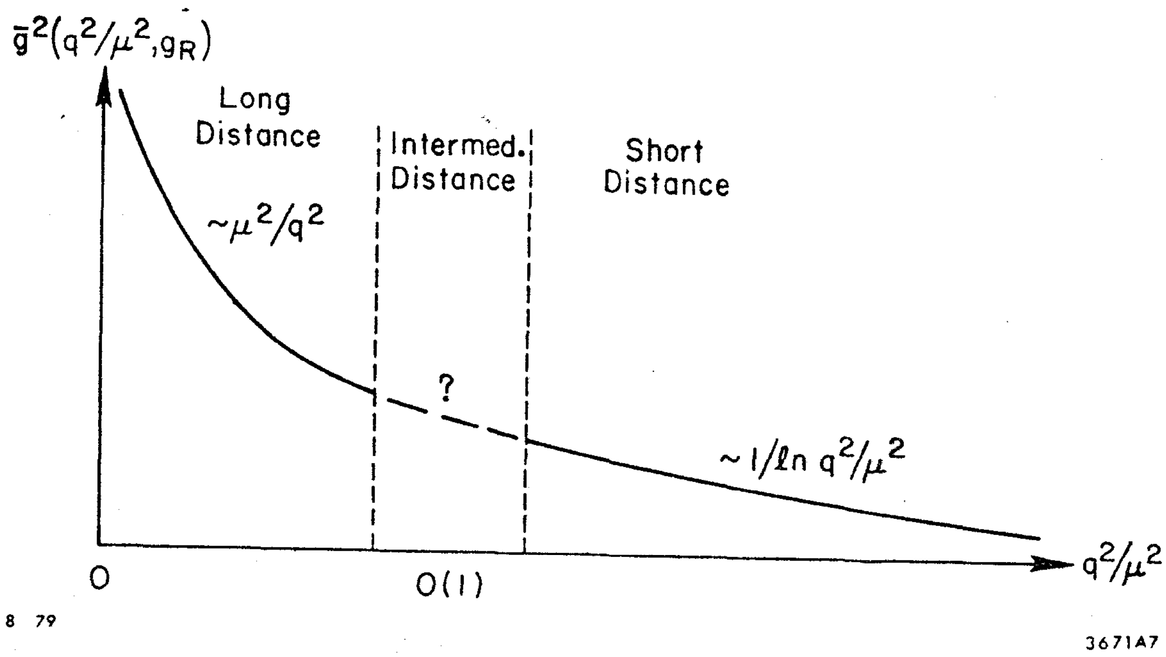
[5] I.P.I. triple gluon vertex equation.

[6] Skeleton expansions of kernels in fig. 5.

[7] (a) Schwinger-Dyson equation for ghost self-energy.

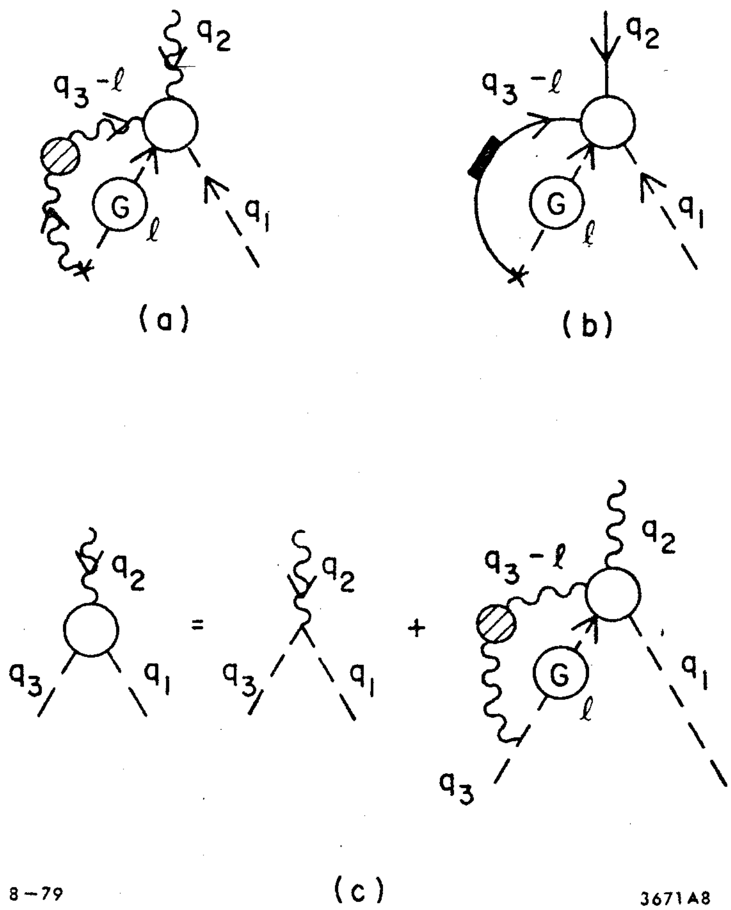
(b) Schwinger-Dyson equation for quark self-energy.

[8] Effective Bethe-Salpeter Bound State Equation.



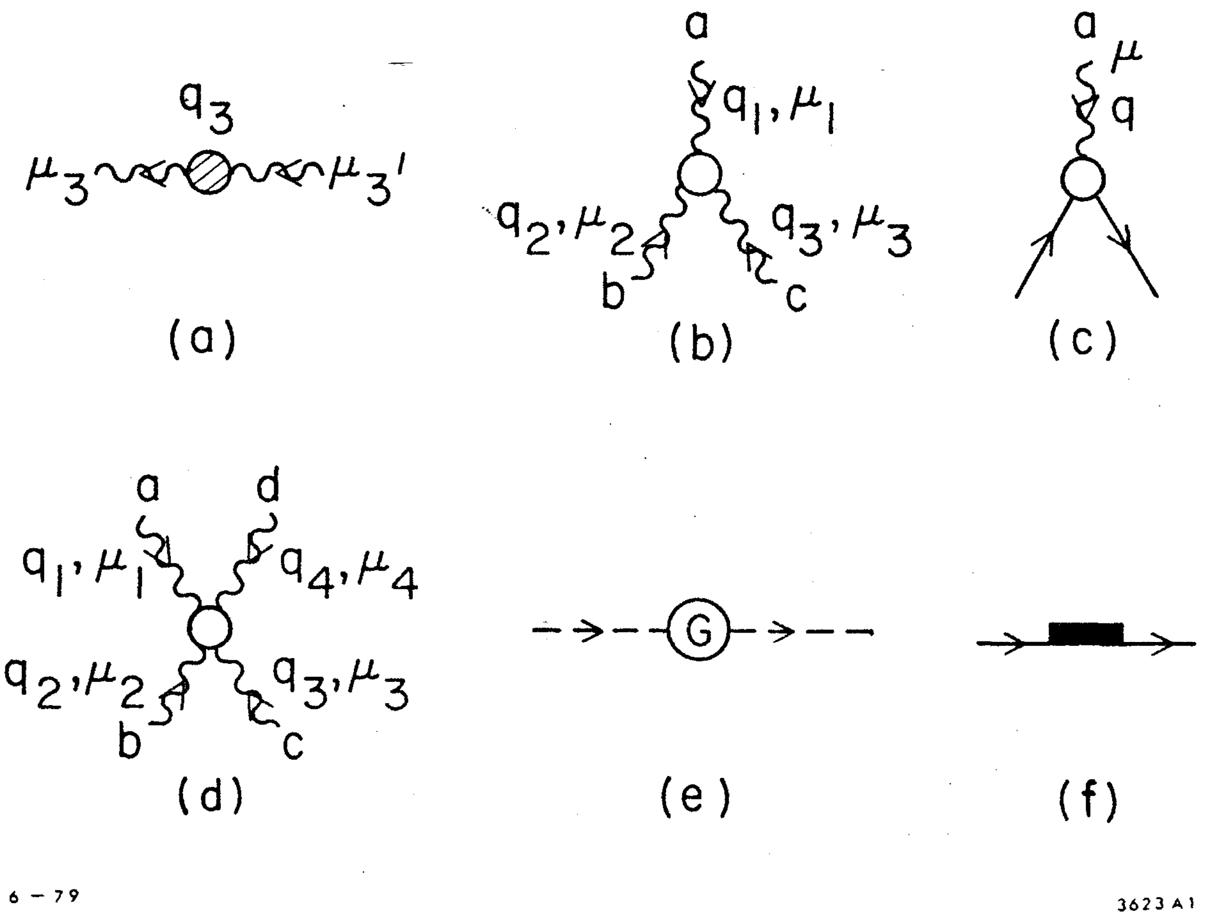
$\bar{g}^2(q^2/\mu^2, g_R)$
Long Distance
Intermed. Distance
Short Distance
$\sim\mu^2/q^2$
?
$\sim 1/\ln q^2/\mu^2$
q^2/μ^2
0
O(1)
8 79
3671A7

Fig. 1



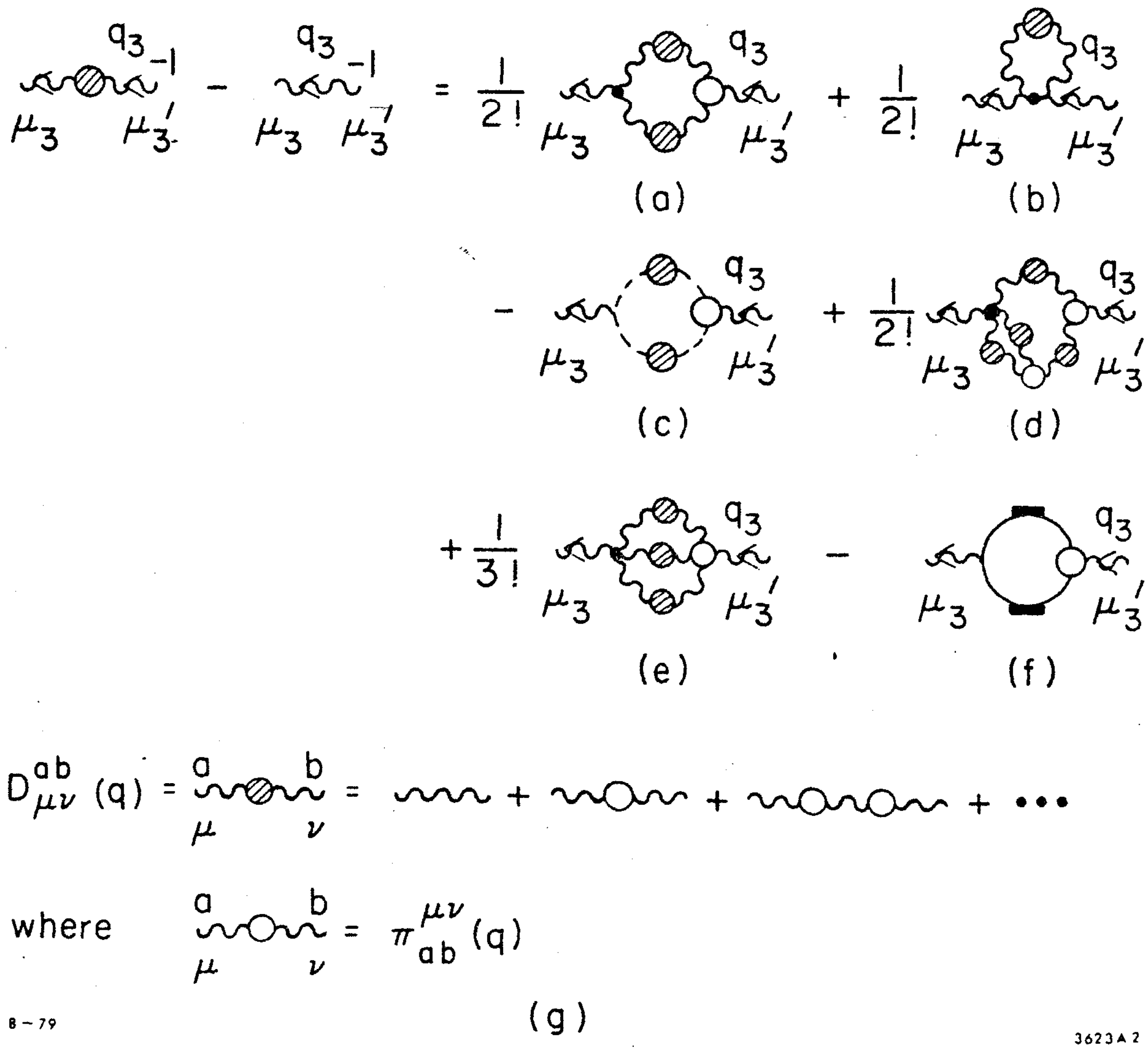
q2
q3 -ℓ
G
ℓ
q1
(a)
q2
q3 -ℓ
G
ℓ
q1
(b)
q2
q3
q1
=
q2
q3
q1
+
q2
q3 -ℓ
G
ℓ
q1
q3
8-79
(c)
3671A8

Fig. 2



q3
μ3
μ3'
(a)
a
q1, μ1
q2, μ2
q3, μ3
b
c
(b)
a
μ
q
(c)
a
d
q1, μ1
q4, μ4
q2, μ2
q3, μ3
b
c
(d)
G
(e)
(f)
6 − 79
3623 A1

Fig. 3



q3 −1
μ3 μ3′
=
1/2!
(a)
+
(b)
(c)
(d)
1/3!
(e)
(f)
$D^{ab}_{\mu\nu}(q)$ =
a b
μ ν
+ • • •
where
$= \pi^{\mu\nu}_{ab}(q)$
(g)
8 − 79
3623A2

Fig. 4

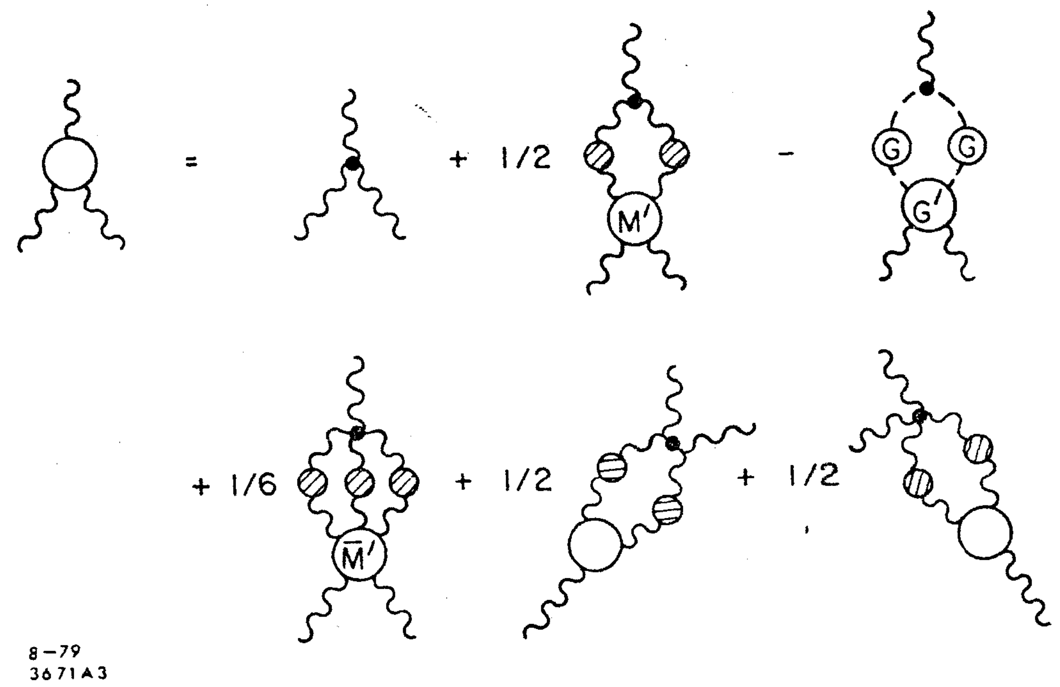


Fig. 5

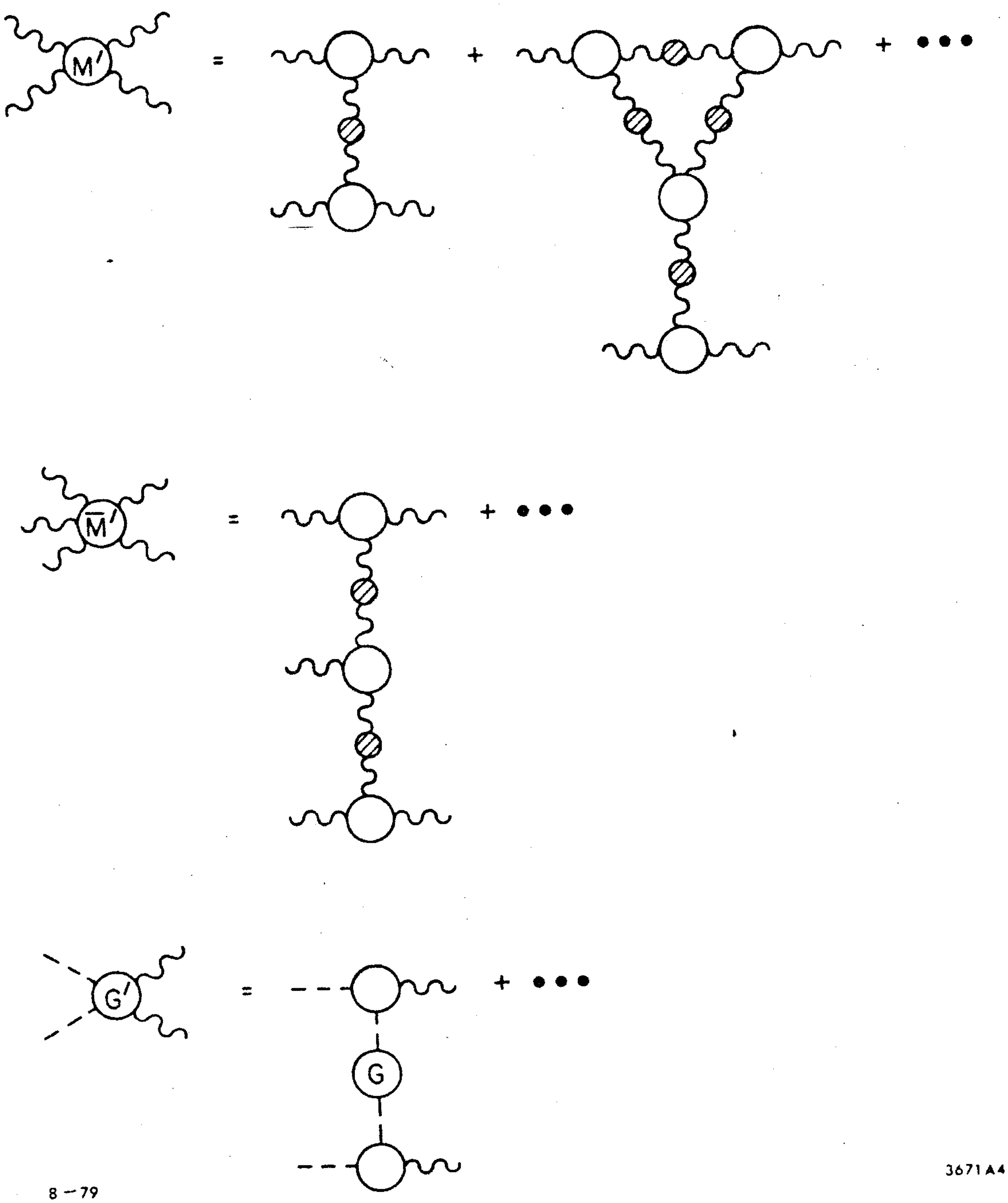


Fig. 6

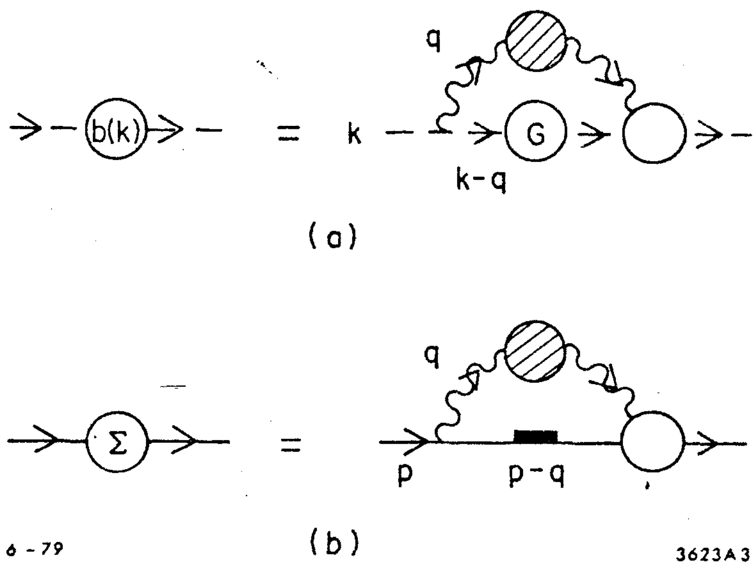

b(k)
k
q
G
k-q
(a)
Σ
q
p
p-q
(b)
6-79
3623A3

Fig. 7

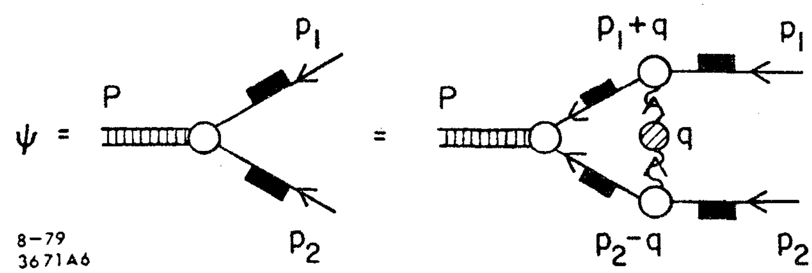

ψ =
P
p1
p2
=
P
p1+q
p1
q
p2-q
p2
8-79
3671A6

Fig. 8