

PAPER • OPEN ACCESS

## Recent progress in simulations of gauge theories on the lattice

To cite this article: Luigi Del Debbio 2015 *J. Phys.: Conf. Ser.* **640** 012049

View the [article online](#) for updates and enhancements.

### Related content

- [RECENT PROGRESS IN UNDERSTANDING THE ERUPTIONS OF CLASSICAL NOVAE](#)  
Michael M. Shara
- [Recent progress on detection and attribution of the water cycle](#)  
Peter Stott
- [Recent progress of SPAN towards neutrino mass spectroscopy](#)  
T Masuda, H Hara, Y Miyamoto et al.



**IOP | ebooks™**

Bringing you innovative digital publishing with leading voices to create your essential collection of books in STEM research.

Start exploring the collection - download the first chapter of every title for free.

# Recent progress in simulations of gauge theories on the lattice

**Luigi Del Debbio**

Higgs Centre for Theoretical Physics, The University of Edinburgh,  
Edinburgh, United Kingdom.

E-mail: [luigi.del.debbio@ed.ac.uk](mailto:luigi.del.debbio@ed.ac.uk)

**Abstract.** There has been considerable progress in simulating gauge theories coupled to fermions in the last decade. As a result, Monte Carlo simulations are now capable of taking the fermion determinant properly into account when generating gauge configurations even for light fermion masses. We briefly summarize the physical intuitions behind this progress, and discuss the impact on the range of problems that can currently be addressed using a computational approach to evaluate the path integral of gauge theories defined on discretized Euclidean spacetime lattices. We focus in particular on recent results for QCD at the physical point, and on new nonperturbative models for physics beyond the Standard Model.

## 1. Introduction

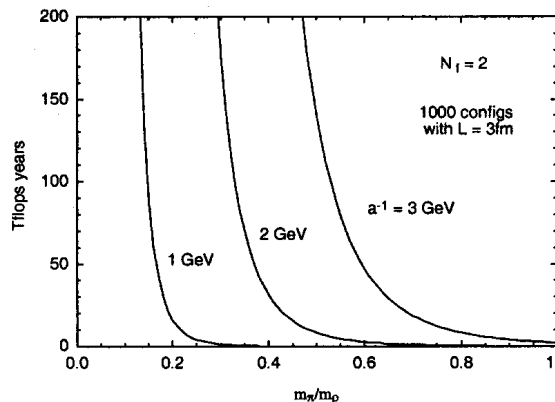
At the International Symposium on Lattice Field Theory in Berlin in 2001, a panel discussion was dedicated to the cost of quantum chromodynamics (QCD) simulations with dynamical fermions. The computational cost of simulating light dynamical fermions was identified as the main obstacle towards robust studies of quantum field theories in the nonperturbative regime by means of numerical simulations. The computing power required for such simulations was summarized by A. Ukawa in the plot taken from Ref. [1], and reproduced in Fig. 1, which became known as the *Berlin wall* plot. It is clear, even before entering into any technical discussion, that the computational effort diverges as the fermion mass is reduced. The actual scaling law proposed at the time was:

$$\text{cost} \propto m_\pi^{-6} L^5 a^{-7}, \quad (1)$$

where  $m_\pi$  is the pion mass, which in QCD is proportional to the square root of the fermion mass,  $L$  is the physical size of the lattice, and  $a$  is the lattice spacing.

In this talk, I will briefly summarize the basic principles underlying Monte Carlo simulations of gauge theories, focussing on two main points: the importance of dynamical fermions, and the difficulties that arise when trying to reach the chiral limit; the latter are clearly exposed by the scaling reported in Eq. 1. It is interesting to discuss some of the solutions that have been found to this problem, as they are based on a deeper understanding of the underlying physics. I will then briefly recall recent results in QCD that have an important impact on flavor phenomenology. The final section of this talk is devoted to the possibility of studying new models of nonperturbative dynamics beyond the Standard Model by numerical simulations. A discussion of the identification of strongly-interacting fixed points in four-dimensional gauge theories using Monte Carlo data is presented.





**Figure 1.** *Berlin wall* plot, illustrating the cost of numerical simulations as a function of the fermion mass and the lattice spacing. The curves show the cost in TFlops years of generating 1000 independent gauge configurations as a function of the fermion mass. The three curves correspond to three different values of the lattice spacing  $a$ , while the lattice linear size is kept constant. Plot from Ref. [1]

## 2. Simulating dynamical fermions

Gauge theories can be defined in a discretized Euclidean space starting from the path integral:

$$Z = \int \mathcal{D}U \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S[\psi, \bar{\psi}, U]}. \quad (2)$$

The integration variables in the path integral are the fermion fields  $\psi(x)$  and  $\bar{\psi}(x)$ , defined on the sites of the lattice, and the link variables  $U(x, \mu)$  that describe the degrees of freedom associated with the gauge field. The integration over the link variables is the Haar measure over the gauge group. In a finite discretized volume, Eq. (2) is a well-defined multidimensional integral.

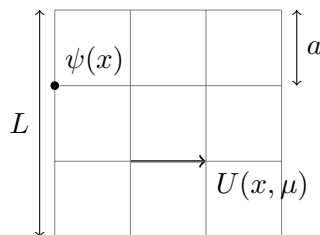
The dynamics of the theory is encoded in the action  $S$ , which determines the statistical weight of each field configuration. Different choices of the lattice action  $S$  are currently used by practitioners in the field; in general these actions differ by lattice artefacts, i.e. by operators that are suppressed by some power of the lattice spacing  $a$ . For our purposes we do not need to describe the lattice action in detail, we only assume that some discretized version of the Dirac operator  $D(x, y; U)$  exists, so that:

$$S = \sum_{x, y} \bar{\psi}(x) D(x, y; U) \psi(y) + S_g[U]. \quad (3)$$

Note that the Dirac operator depends on the gauge link variables, and encodes the interaction of the fermions with the gauge field. The fermion mass is one of the parameters that enter the definition of the Dirac operator. We also assume that we have a discretized version of the pure gauge action  $S_g$ , which we do not need to specify. For instance QCD, the theory of strong interactions, is defined as an SU(3) gauge theory coupled to three flavors of light fermions in the fundamental representation of the gauge group.

The action is quadratic in the Grassmann variables  $\bar{\psi}$  and  $\psi$ , and therefore the integration over the fermionic fields can be done analytically, yielding:

$$Z = \int \mathcal{D}U e^{-S_g[U]} (\det D[U])^{n_f}, \quad (4)$$



**Figure 2.** Field variables in a lattice formulation of gauge theories. The fermionic fields are defined on the sites of the lattice, the gauge fields are associated to the links. The linear size of the system in physical units is  $L$ , while the lattice spacing is  $a$ . The number of sites in each direction is  $N = L/a$ . The discretization of spacetime acts as a regulator for the high-energy divergences, the inverse lattice spacing  $a^{-1}$  being the UV cutoff.

where  $n_f$  is the number of fermionic species in the theory, which we assumed to be degenerate in this example. Eq. 4 shows that the probability distribution of the gauge link variables depends on the fermionic determinant, as well as on the gauge part of the action. Ignoring the contribution of the fermionic determinant defines the so-called *quenched approximation*, where the link variables are distributed according to the value of the gauge action only. Clearly such an approximation violates unitarity, and introduces systematic errors whose size is unacceptable for the precision studies in contemporary phenomenology. *Dynamical fermions* is the name used to identify those simulations that take fully into account the contribution of the Dirac determinant.

In order to understand the cost of simulating dynamical fermions as shown in Fig. 1, it is useful to recall how the fermionic determinant is usually computed. Introducing complex bosonic fields, the determinant can be written as:

$$\det D[U]^2 = \int \mathcal{D}\phi \mathcal{D}\phi^* \exp \left[ - \sum_{x,y \in \Lambda} |D^{-1}(x,y;U)\phi(y)|^2 \right]. \quad (5)$$

It is clear from the latter equation that the cost of computing the fermionic determinant is dictated by the cost of inverting the Dirac operator  $D$ , i.e. by the cost of solving:

$$\sum_y D(x,y)\psi(y) = \eta(x), \quad (6)$$

for a given source  $\eta(x)$ . The Dirac operator is a large, sparse matrix, and Eq. 6 can be solved using one of several iterative Krylov space solvers, see e.g. Ref. [2]. The inversion of  $D$  becomes more computationally expensive as the condition number of the Dirac operator increases, *i.e.* the small eigenvalues of the Dirac operator are responsible for the cost of the simulations summarised in the *Berlin wall* plot. We shall refer to the eigenstates corresponding to these low eigenvalues as *low modes* or *low eigenstates* of the Dirac operator. Clearly any IR regulator, e.g. the fermion mass or the finite volume of the box, will prevent a substantial increase of the condition number of  $D$ ; as summarized by the scaling law above, the problem arises only when trying to reach the physical regime of small masses and large volumes.

A very successful way to improve this scaling behaviour, based on the factorization of the Dirac determinant, was proposed shortly after the Berlin conference in Ref. [3]. In this talk, we will focus on a complementary approach, which relies on understanding the dynamical properties of the Dirac low eigenmodes. It is important to point out here that the number of low modes of the Dirac operator is driven by the dynamics of the system under study. In a theory where chiral symmetry is spontaneously broken, as is the case in QCD, the order parameter of the

broken symmetry, is directly related to the density of zero modes of the massless Dirac operator through the Banks-Casher relation [4]:

$$\langle \bar{\psi}\psi \rangle = \pi \lim_{\lambda \rightarrow 0} \lim_{V \rightarrow \infty} \rho(\lambda, 0), \quad (7)$$

where  $\rho(\lambda, m)$  denotes the density of modes with eigenvalue  $\lambda$ , and  $m$  is the fermion mass that appears in the Dirac operator. A non-vanishing chiral condensate in the left-hand side of Eq. 7 implies a non-vanishing density on the right-hand side. As shown in Ref. [5], integrating the Banks-Casher relation yields the number of low modes with eigenvalue  $\lambda$  such that  $|\lambda| < M$ :

$$N_{|\lambda| < M} = \frac{2}{\pi} M \langle \bar{\psi}\psi \rangle V + O(M^2). \quad (8)$$

It is clear from this equation that, as the fermion mass goes to zero, the number of low modes of  $D$  increases linearly with the physical volume of the system.

### 3. Deflated inverters

The problem of the large condition number for the Dirac operator can be alleviated using deflation. Deflation techniques are well known to practitioners in lattice QCD, see e.g. Refs. [6, 7, 8, 9, 10]; they are based on the idea of projecting out the low modes of the Dirac operator. The Dirac operator restricted to the orthogonal complement of the space spanned by the low modes has a better condition number, and hence can be inverted at a lower cost. More specifically, let us denote by  $\mathcal{S}$  the subspace spanned by  $N$  orthogonal vectors:

$$\mathcal{S} = \text{span}\{\phi_1, \dots, \phi_N\}, \quad (9)$$

then the projector on  $\mathcal{S}$  can be defined as:

$$P\psi = \sum_{k=1}^N \phi_k (\phi_k, \psi). \quad (10)$$

The little Dirac operator is defined to be the restriction of the Dirac operator to the subspace  $\mathcal{S}$ :

$$PDP\psi = \sum_{k,l} \phi_k A_{kl} (\phi_l, \psi), \quad A_{kl} = (\phi_k, D\phi_l). \quad (11)$$

If  $P$  is effective at projecting on the low eigenmodes of the Dirac operator, then the condition number of the Dirac operator in the orthogonal subspace is clearly improved. As the number of low modes scales with the physical volume, a naive deflation procedure has a cost  $O(V^2)$ , and becomes rapidly unaffordable as large volumes are used.

Multi-level algorithms have been proposed a long time ago in order to improve the scaling of the cost with the size of the system [12]. These ideas were developed and applied to QCD with Wilson fermions in Refs. [14, 15, 16, 17]. Unfortunately it is not possible to present here a full account of the work done in this area. A recent account of this work was presented at the Lattice conference this year. Instead here we shall summarize briefly one particular approach, proposed by Lüscher in Ref. [5].

Lüscher's idea is based on an empirical property of low eigenmodes called *local coherence*; it refers to the fact that the low eigenmodes of the Dirac operator, when restricted to a small physical volume, are well approximated by a relatively small number of independent vectors. Local coherence suggests a new recipe for constructing deflation subspaces. Here we follow closely the notation introduced in Ref. [5].

The lattice is divided into non-overlapping rectangular blocks, denoted by  $\Lambda$ . In each block, a local deflation subspace can be defined by choosing a set of orthonormal fields  $\phi_l^\Lambda$ , where  $l = 1, \dots, N_s$ . The total number of fields introduced in this way is  $N_b N_s$ , where  $N_b$  denotes the number of blocks. The block fields are defined starting from  $N_s$  global fields  $\psi$ , projecting them to the blocks:

$$\phi_l^\Lambda(x) = \begin{cases} \psi_l(x), & \text{if } x \in \Lambda, \\ 0 & \text{otherwise,} \end{cases} \quad (12)$$

and orthonormalizing the resulting fields. They can be used to build a projector as explained above, and for any given normalized field  $\psi$ , the accuracy of the approximation provided by the deflation subspace can be monitored by introducing the deficit:

$$\epsilon = \|(1 - P)\psi\|^2. \quad (13)$$

Given a set of low eigenmodes of  $D^\dagger D$ , a domain-decomposed subspace can be constructed from a smaller subset of these modes chosen randomly. Local coherence implies that the deficit for *all* the low modes is small, independently of whether they were included in the subset used to define the projector. In the numerical experiments presented in Ref. [5], 12 eigenmodes out of 48 were chosen at random to construct the subspace; the remaining 36 modes were well approximated within this subspace with deficits of the order of 3% to 6%, using blocks of size  $\approx 0.3$  fm.

In practice an efficient deflation subspace can be obtained starting from global fields that are only approximate eigenmodes of the Dirac operator. These can be computed at small cost using some relaxation procedure, which yields a set of  $N_s$  global fields such that:

$$\|D\psi_l\| \leq M\|\psi_l\|, \quad (14)$$

where  $M$  is the size of the largest eigenvalue that we want to project out using deflation. The important point here is that the number of modes computed does not depend on the volume of the lattice.

Recent numerical investigations using Wilson fermions show that this approach overcomes the problem of the proliferation of low modes with volume, and delivers a significant improvement in the inversion of the Dirac operator for small fermion masses. An eloquent example of the improved performance, as reported in Ref. [18], is shown in Fig. 3.

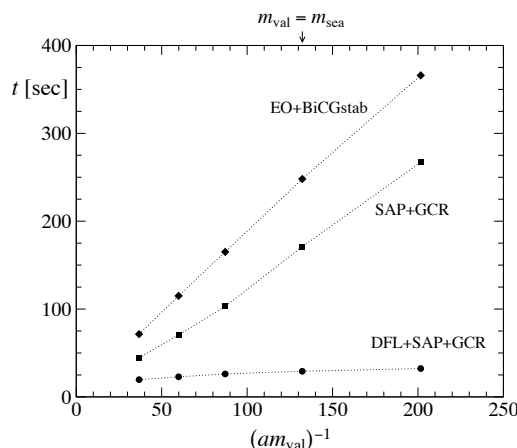
A similar procedure has been recently implemented for domain-wall fermions in Ref. [11]. Domain wall fermions are a five-dimensional formulation of lattice fermions which preserves some kind of chiral symmetry at finite lattice spacing [19]. The five-dimensional formulation requires to solve a number of new problems; the implementation in Ref. [11] has successfully addressed these issues, and has delivered a significant gain in the inversion of the Dirac operator for chiral fermions.

In all these cases, it is interesting to note that the algorithmic progress has been driven by the understanding of the underlying dynamics, and its relation to the cost of numerical simulations.

#### 4. QCD at the physical point

As a result of the recent algorithmic improvements, simulations of lattice QCD are now performed on large volumes and physical light quark masses. These simulations allow for an unprecedented control of systematic errors, and high statistical accuracy. Current lattice results provide numerous nonperturbative inputs for precision studies in the flavor sector of the Standard Model.

It is not possible to cover the whole spectrum of QCD results in this talk; instead we will have to make a selection, and refer the reader to the proceedings of the annual Lattice conference for



**Figure 3.** Cost of the inversion of the Dirac operator using deflation as discussed in the text. Figure from Ref. [18].

a more complete survey of current results. Several new results have been reviewed in parallel sessions at this conference, they illustrate well the breadth of the topics that are currently under study:

- nonperturbative QCD contributions to the anomalous magnetic moment of the muon;
- isospin breaking and QED effects;
- strongly-interacting matter;
- structure of nucleons;
- nonperturbative inputs for flavor physics phenomenology;
- theories of composite dark matter;
- supersymmetry.

The calculation of the hadronic contributions to the anomalous magnetic moment of the muon were reviewed by T. Izubuchi. This computation is important in view of the planned experiments aiming at measuring  $g - 2$ . Several results have appeared this year, see e.g. Refs. [20, 21, 22, 23, 24, 25], and further references therein.

As a consequence of the precision of current simulations, isospin and hence QED effects have become a significant source of systematic error; first studies have appeared that address the formulation of lattice QCD in the presence of isospin breaking, and coupled to a U(1) electromagnetic field. Interesting results have appeared recently [26, 27].

The Equation of State of QCD describes the phase transition that strongly-interacting matter undergoes as the temperature of the system is increased. Recent results have been reported this year in Refs. [28, 29].

Current lattice simulations also allow nucleons to be studied from first principles. Computations of the nucleon vector and axial form factors, of the nucleon axial charge, and quark momentum fraction can now be computed with quarks at the physical point, with an accuracy at around 10% [30]. See also Ref. [31] for an interesting new approach.

An up-to-date summary of lattice results for flavor physics is provided by the FLAG collaboration in Ref. [32], and was summarized at this conference in a parallel session by A. El-Khadra. The combination of new lattice gauge configurations generated with dynamical light quarks, and several complementary formulations for the heavy quark sector, yield a variety of results with high accuracy. A review of results was presented at this conference, see e.g. [33, 34].

Novel work on the possibility of constructing models of composite dark matter was reported in one of the parallel sessions. The idea behind these models is that Dark Matter is made of the hadrons of a new strongly-interacting sector. Investigating numerically the nuclear spectroscopy of these theories allows the computation of the nonperturbative parameters that describe the dynamics of these systems. Ultimately these inputs are crucial for the construction of a predictive and constrained theory. The results of these studies for an  $SU(2)$  gauge theory with two Dirac fermions in the fundamental representation, using Wilson fermions, have appeared in Refs. [35, 36]. Similar directions have been investigated in Refs. [37, 38].

Finally, recent progress in formulating supersymmetric theories on a discretized space-time have been reviewed by S. Catterall. An excellent review of the theoretical issues can be found in Ref. [39].

Hopefully this partial report shall help the reader to get a snapshot of the recent directions in which the field is moving. The list of references should also provide an entry point to the contemporary literature on these topics.

## 5. IR fixed points for composite Higgs models

The discovery of the Higgs boson at the LHC is an impressive verification of the Standard Model. The first measurements of the couplings of the Higgs boson show that the Brout-Englert-Higgs (BEH) mechanism provides an excellent description of electroweak symmetry breaking. However the Higgs sector of the Standard Model lagrangian does not provide any insight on the dynamical origin of the symmetry breaking, which is simply parametrized by the Higgs potential. The measured value of the mass of the Higgs requires new physics beyond the Standard Model to explain the stabilization of the electroweak scale. Traditional solutions like minimal supersymmetry need to be finely tuned in order to be compatible with current observations.

Assuming that the Higgs boson is a composite particle provides an appealing explanation of the mechanism of electroweak symmetry breaking; indeed it would be very much in line with the BCS explanation of superconductivity, which in turn inspired the BEH mechanism. In composite scenarios the nonperturbative dynamics of some new sector beyond the Standard Model is advocated in order to explain the origin of the electroweak scale, in analogy to the appearance of an hadronic scale in QCD. The problem with these kind of models is that computations in the nonperturbative regime of quantum field theories are difficult, and hence quantitative predictions do not come easy. The naive idea that the new sector could be a rescaled version of QCD is already ruled out by LHC data, and therefore some new dynamics needs to be found if these scenarios have any chance of describing experimental results.

Strongly-interacting dynamics that is qualitatively different from QCD could be obtained by deforming theories that possess an infrared (IR) fixed point [40, 41, 42]. This kind of fixed point is usually thought to be generated by the quantum effects of a sufficiently large number of massless fermions, hence these problems could only be addressed as a result of the algorithmic progresses discussed above. Numerical simulations of gauge theories regulated on a space-time lattice provide a tool to study the nonperturbative regime of these theories from first principles. Hence they have a great potential to become a valuable instrument for addressing the questions discussed above. In the rest of this article, we will discuss some of the methods that have been developed to identify the existence of fixed points, and to compute the associated anomalous dimensions, using lattice simulations.

The behaviour of couplings in the neighbourhood of a fixed point is described by the so-called linearized renormalization group (RG) equations. The RG equations describe the change of the couplings in a theory as the energy scale is varied: the couplings are said to *flow* in parameter

space. This flow is described by the beta functions:

$$\mu \frac{d}{d\mu} \hat{g}_k = \beta_k(\hat{g}),$$

where  $\hat{g}$  are dimensionless couplings, and  $\mu$  is the energy scale. The fixed points are identified by the zeroes of the beta functions, and the equations in the neighbourhood of a fixed point can be linearized:

$$\beta_k(\hat{g}^*) = 0 \implies \beta_k(\hat{g}) = \left. \frac{\partial \beta_k}{\partial \hat{g}_j} \right|_* (\hat{g}_j - \hat{g}_j^*).$$

Working in a basis in the space of couplings where the matrix  $\left. \frac{\partial \beta_k}{\partial \hat{g}_j} \right|_*$  is diagonal, the linearized evolution is easily solved:

$$\hat{g}_k(\mu) = \hat{g}_{k,0} \left( \frac{\mu}{\Lambda_{UV}} \right)^{-y_k} \implies \boxed{\Lambda_{IR} = \hat{g}_0^{1/y_k} \Lambda_{UV}}, \quad (15)$$

where  $y_k$  are the eigenvalues of the linearized evolution matrix, and are related to the anomalous dimensions of operators at the fixed point. In Eq. 15  $\hat{g}_{k,0}$  is the value of the coupling at the UV scale  $\Lambda_{UV}$ , while  $g_k(\mu)$  is the value at the scale  $\mu$ . The IR scale  $\Lambda_{IR}$  is defined as the value of the energy scale where the coupling becomes of order 1. It is clear that a small value of the exponent  $y_k$  leads naturally to a large separation between the UV and the IR scale, which otherwise would require a fine-tuning of the coupling at the UV scale  $g_{k0}$ . Hence we see explicitly that the existence of a fixed point with small exponents  $y_k$  can be used to generate a hierarchy of scales without fine-tuning.

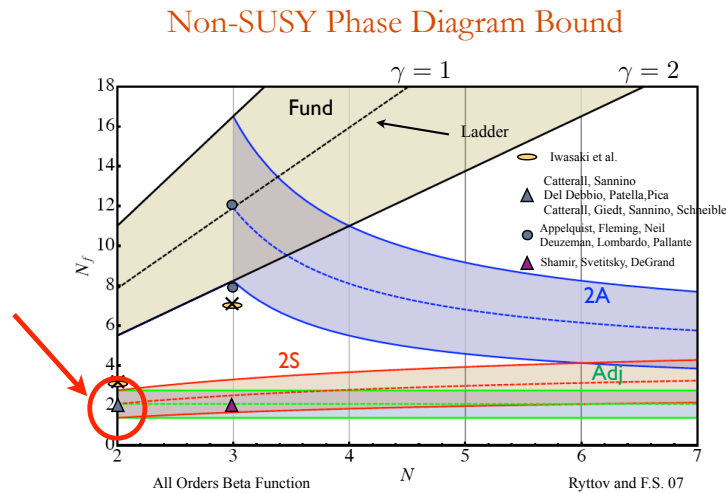
Theories that have an IR fixed point are said to be inside the *conformal window*. Analytical results about the conformal window are difficult to derive in the absence of further symmetries, like e.g. supersymmetry. The expectation for gauge theories coupled to fermions, usually based on some kind of approximations, is that these theories develop a fixed point as the number of fermions  $n_f$  is increased, and hence there is a range in  $n_f$  for which there is an IR fixed point before asymptotic freedom is lost. The conjectured conformal window for SU(N) theories coupled to fermions in different representations of the gauge group are shown in Fig. 4.

Phenomenologically viable models can be built by deformations of theories inside the conformal window, or by finding theories close to the edge of the conformal window. Since the existence of an IR fixed point has some unambiguous signatures, we focus on the possibility of finding robust evidence that a given theory is indeed inside the conformal window. This entails a characterization of the fixed point obtained from a determination of the anomalous dimensions.

The existence of a IR fixed point has its theoretical signatures. Here we shall concentrate on the scaling of the spectrum of the theory. Indeed if the theory becomes scale invariant at large distances, it cannot develop a mass gap, and therefore *all* the masses of the states in the spectrum are expected to scale to zero as the fermion mass is sent to zero, i.e. as the explicit source of breaking of scale invariance is removed. More precisely, the masses scale to zero as a power of the fermion mass, and the scaling exponent is determined by the critical dimension of the mass operator in the theory [44, 45, 46]:

$$M_H \propto m^{1/(1+\gamma^*)}, \quad (16)$$

where  $M_H$  is the mass of the physical states,  $m$  is the fermion mass, and  $\gamma^*$  is the anomalous dimension of the mass at the fixed point. Clearly this behaviour is at odds with the one expected



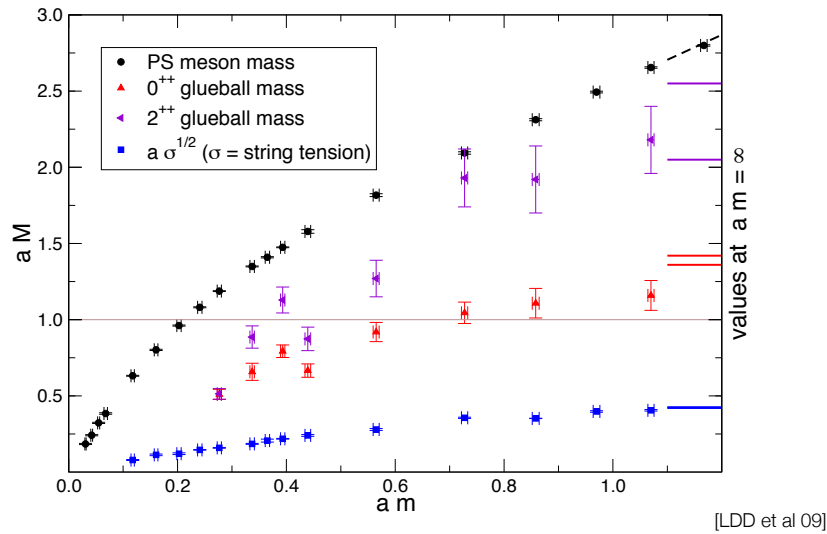
**Figure 4.** Conformal windows for  $SU(N)$  theories coupled to fermions. The coordinate on the horizontal axis labels the gauge group, while the one on the vertical axis represents the number of fermion families. Above the upper bound of each band, the theory is no longer asymptotically free. Different colours correspond to different representations of the fermions. The arrow indicates the  $SU(2)$  theory with two adjoint fermions for which we report results below. The phase diagram sketched here is taken from Ref. [43].

in a theory outside the conformal window, where chiral symmetry breaking leads to massless pions, but other states remain massive even when the fermion mass goes to zero. The latter is the behaviour that is expected e.g. in QCD.

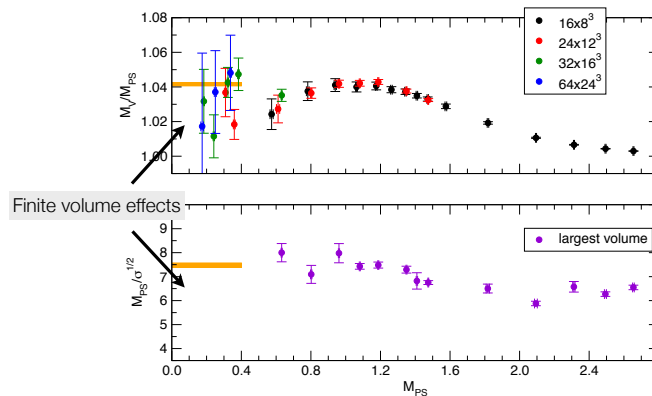
Figure 5 shows the spectrum for a number of states computed from Monte Carlo simulations of an  $SU(2)$  gauge theory with two Dirac fermions in the adjoint representation [47]. Although the scaling of the masses to zero as a function of the fermion mass is encouraging, a more quantitative analysis is needed in order to conclude that the theory possesses a fixed point. Systematic errors due to the finite value of the fermion mass, and the finite volume of the system, could obscure the physics that instead we would like to highlight.

A more quantitative description is obtained by looking at ratios of masses. The upper part of Fig. 6 shows the ratio of the mass of the vector particle  $M_V$  to the mass of the pseudoscalar one  $M_{PS}$ . This ratio remains constant as the mass of the fermions is decreased, as expected from the scaling argument presented above, and in sharp contrast with the expectation in a theory where chiral symmetry is spontaneously broken. In the latter case, the pseudoscalar particle becomes massless in the chiral limit because it is a Goldstone boson, while the rest of the spectrum remains finite, and hence the ratio in the plot should diverge. The lower part of the figure shows the ratio of the pseudoscalar mass to the square root of the string tension. Once again the qualitative behaviour is consistent with the existence of an IR fixed point with a single relevant operator which we identify as the fermion mass. A careful analysis shows that finite volume effects are sizeable for the smallest fermion masses that have been simulated so far [48]. Finite-volume effects are clearly visible when comparing the results obtained on lattices of increasing volume in the upper plot. As expected these effects are only relevant for light fermion masses: this is the regime where signatures of IR-conformality can be found, and it is crucial that all sources of systematic error are studied in detail.

Evidence for the existence of a fixed point can be reinforced by combining multiple analyses,



**Figure 5.** Scaling of the spectrum as a function of the fermion mass. The fermion mass is defined as the PCAC mass, and hence the point at  $m = 0$  is the point where explicit symmetry breaking is absent. The masses of all the states scale to zero at the same rate as predicted for theories with an IR fixed point.



**Figure 6.** Ratios of masses as a function of the fermion mass. Different colours indicate results from lattices with different volumes. All results are obtained at a single value of the lattice spacing  $a$ .

e.g. by looking simultaneously at the scaling of several quantities. In Ref. [49] scaling relations for dimensionful quantities including matrix elements of operators were established, which can be tested in numerical studies. Also the study of the density of eigenvalues of the Dirac operators in Ref. [50] has provided a determination of the mass anomalous dimension for SU(2) with adjoint fermions. The simulations for this model are all consistent with the existence of an IR fixed point, and mass anomalous dimension  $\gamma^* = 0.37(2)$ , which is related to the critical exponent by  $y_m = 1 + \gamma^*$ . This is so far the strongest numerical evidence in favour of the existence of an IR fixed point.

Recent investigations have suggested novel theories as candidates for building BSM scenarios. All these theories are at the edge of the conformal window, and could be characterized by multiple

scales, with large separations between them. Some theories, like the the  $SU(3)$  gauge theory with sextet fermions [60] have a light scalar state in the spectrum, which could be a candidate for a composite Higgs boson. On the other hand, the recent studies in Ref. [57] have found for the first time a theory with a large mass anomalous dimension. The interested reader can find a compilation of recent results in Refs. [51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61]. While this list is not exhaustive, it should provide a suitable overview of the current problems in this field.

## 6. Outlook

There has been a remarkable progress in numerical simulations of lattice field theories in the last decade, which has led to the possibility of simulating light dynamical fermions efficiently on current hardware. From the algorithmic point of view, we have witnessed a very impressive improvement, partly driven by a better understanding of the physical origin of the critical slowing down of the simulation codes for light fermion masses. The performance of the current algorithms has opened up two major research directions.

On the one hand simulations of QCD are now performed at physical quark masses, and hence some major sources of systematic errors have been removed from lattice inputs in flavor physics. As a consequence of this increase in accuracy, new questions need to be addressed, like e.g. the impact of electrodynamics and isospin breaking. There has been theoretical work in this direction, and first numerical results have started to appear.

On the other hand, simulations with dynamical fermions have made possible the numerical study of theories beyond QCD. Some tantalizing results have emerged from Monte Carlo simulations that have identified strongly-interacting theories as candidates to describe new sectors beyond the Standard Model. The challenge in the near future will be to scrutinize these theories even more closely, in order to draw robust conclusions that may be useful for model building. It will be interesting to follow the developments in this area as the LHC restarts its experiments in 2015.

## References

- [1] A. Ukawa [CP-PACS and JLQCD Collaborations], Nucl. Phys. Proc. Suppl. **106** (2002) 195.
- [2] Y. Saad. *Iterative Methods for Sparse Linear Systems*. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2nd edition, 2003.
- [3] M. Hasenbusch, Phys. Lett. B **519** (2001) 177 [hep-lat/0107019].
- [4] T. Banks and A. Casher, Nucl. Phys. B **169** (1980) 103.
- [5] M. Luscher, JHEP **0707** (2007) 081 [arXiv:0706.2298 [hep-lat]].
- [6] R. G. Edwards, U. M. Heller and R. Narayanan, Phys. Rev. D **59** (1999) 094510 [hep-lat/9811030].
- [7] H. Neff, N. Eicker, T. Lippert, J. W. Negele and K. Schilling, Phys. Rev. D **64** (2001) 114509 [hep-lat/0106016].
- [8] R. B. Morgan and W. Wilcox, Nucl. Phys. Proc. Suppl. **106** (2002) 1067 [hep-lat/0109009].
- [9] D. Darnell, R. B. Morgan and W. Wilcox, Nucl. Phys. Proc. Suppl. **129** (2004) 856 [hep-lat/0309068].
- [10] Dean Darnell, Ronald B. Morgan, and Walter Wilcox. *Linear Algebra and its Applications*, 429(10):2415 – 2434, 2008.
- [11] P. A. Boyle, arXiv:1402.2585 [hep-lat].
- [12] A. Brandt, *Math. Comp.* **31**:330 – 390, 1977.
- [13] M. Brezina, R. Falgout, S. MacLachlan, T. Manteuffel, S. McCormick, J. Ruge, *SIAM Journal on Scientific Computing* **25**(6):1896 – 1920, 2004.
- [14] J. Brannick, R. C. Brower, M. A. Clark, J. C. Osborn and C. Rebbi, Phys. Rev. Lett. **100** (2008) 041601 [arXiv:0707.4018 [hep-lat]].
- [15] R. Babich, J. Brannick, R. C. Brower, M. A. Clark, S. D. Cohen, J. C. Osborn and C. Rebbi, PoS LAT **2009** (2009) 031 [arXiv:0912.2186 [hep-lat]].
- [16] R. Babich, J. Brannick, R. C. Brower, M. A. Clark, T. A. Manteuffel, S. F. McCormick, J. C. Osborn and C. Rebbi, Phys. Rev. Lett. **105** (2010) 201602 [arXiv:1005.3043 [hep-lat]].
- [17] A. Frommer, K. Kahl, S. Krieg, B. Leder and M. Rottmann, SIAM J. Sci. Comput. **36** (2014) A1581 [arXiv:1303.1377 [hep-lat]].
- [18] M. Luscher, arXiv:1002.4232 [hep-lat].

- [19] D. B. Kaplan, Phys. Lett. B **288** (1992) 342 [hep-lat/9206013].
- [20] T. Blum, S. Chowdhury, M. Hayakawa and T. Izubuchi, arXiv:1407.2923 [hep-lat].
- [21] B. Chakraborty, C. T. H. Davies, G. C. Donald, R. J. Dowdall, J. Koponen, G. P. Lepage and T. Teubner, arXiv:1403.1778 [hep-lat].
- [22] M. Benayoun, J. Bijnens, T. Blum, I. Caprini, G. Colangelo, H. Czy, A. Denig and C. A. Dominguez *et al.*, arXiv:1407.4021 [hep-ph].
- [23] A. Francis, V. Guelpers, G. Herdoiza, H. Horch, B. Jaeger, H. B. Meyer and H. Wittig, arXiv:1410.7491 [hep-lat].
- [24] M. Della Morte, A. Francis, G. Herdoiza, H. Horch, B. Jger, A. Jttner, H. Meyer and H. Wittig, PoS LATTICE **2014** (2014) 162 [arXiv:1411.1206 [hep-lat]].
- [25] A. Francis, V. Glpers, B. Jger, H. Meyer, G. von Hippel and H. Wittig, PoS LATTICE **2014** (2014) 128 [arXiv:1411.7592 [hep-lat]].
- [26] S. Borsanyi, S. Durr, Z. Fodor, C. Hoelbling, S. D. Katz, S. Krieg, L. Lellouch and T. Lippert *et al.*, arXiv:1406.4088 [hep-lat].
- [27] R. Zhou and S. Gottlieb, PoS LATTICE **2014** (2014) 024 [arXiv:1411.4115 [hep-lat]].
- [28] S. Borsanyi, Z. Fodor, C. Hoelbling, S. D. Katz, S. Krieg, C. Ratti and K. K. Szabo, arXiv:1410.7917 [hep-lat].
- [29] A. Bazavov *et al.* [HotQCD Collaboration], Phys. Rev. D **90** (2014) 9, 094503 [arXiv:1407.6387 [hep-lat]].
- [30] S. Syritsyn, T. Blum, M. Engelhardt, J. Green, T. Izubuchi, C. Jung, S. Krieg and M. Lin *et al.*, arXiv:1412.3175 [hep-lat].
- [31] A. J. Chambers, R. Horsley, Y. Nakamura, H. Perlt, D. Pleiter, P. E. L. Rakow, G. Schierholz and A. Schiller *et al.*, Phys. Rev. D **90** (2014) 014510 [arXiv:1405.3019 [hep-lat]].
- [32] S. Aoki, Y. Aoki, C. Bernard, T. Blum, G. Colangelo, M. Della Morte, S. Drr and A. X. El Khadra *et al.*, Eur. Phys. J. C **74** (2014) 9, 2890 [arXiv:1310.8555 [hep-lat]].
- [33] A. X. El-Khadra, PoS LATTICE **2013** (2014) 001 [arXiv:1403.5252 [hep-lat]].
- [34] N. H. Christ, J. M. Flynn, T. Izubuchi, T. Kawanai, C. Lehner, A. Soni, R. S. Van de Water and O. Witzel, arXiv:1404.4670 [hep-lat].
- [35] W. Detmold, M. McCullough and A. Pochinsky, arXiv:1406.2276 [hep-ph].
- [36] W. Detmold, M. McCullough and A. Pochinsky, arXiv:1406.4116 [hep-lat].
- [37] A. Hietanen, R. Lewis, C. Pica and F. Sannino, JHEP **1407** (2014) 116 [arXiv:1404.2794 [hep-lat]].
- [38] T. Appelquist *et al.* [Lattice Strong Dynamics (LSD) Collaboration], Phys. Rev. D **89** (2014) 094508 [arXiv:1402.6656 [hep-lat]].
- [39] S. Catterall, D. B. Kaplan and M. Unsal, Phys. Rept. **484**, 71 (2009) [arXiv:0903.4881 [hep-lat]].
- [40] M. J. Strassler, hep-th/0309122.
- [41] D. K. Hong, S. D. H. Hsu and F. Sannino, Phys. Lett. B **597**, 89 (2004) [hep-ph/0406200].
- [42] M. A. Luty and T. Okui, JHEP **0609**, 070 (2006) [hep-ph/0409274].
- [43] D. D. Dietrich and F. Sannino, Phys. Rev. D **75** (2007) 085018 [hep-ph/0611341].
- [44] T. DeGrand and A. Hasenfratz, Phys. Rev. D **80** (2009) 034506 [arXiv:0906.1976 [hep-lat]].
- [45] L. Del Debbio and R. Zwicky, Phys. Rev. D **82** (2010) 014502 [arXiv:1005.2371 [hep-ph]].
- [46] L. Del Debbio and R. Zwicky, Phys. Rev. D **89** (2014) 014503 [arXiv:1306.4038 [hep-ph]].
- [47] L. Del Debbio, B. Lucini, A. Patella, C. Pica and A. Rago, Phys. Rev. D **80** (2009) 074507 [arXiv:0907.3896 [hep-lat]].
- [48] F. Bursa, L. Del Debbio, D. Henty, E. Kerrane, B. Lucini, A. Patella, C. Pica and T. Pickup *et al.*, Phys. Rev. D **84** (2011) 034506 [arXiv:1104.4301 [hep-lat]].
- [49] L. Del Debbio and R. Zwicky, Phys. Lett. B **700** (2011) 217 [arXiv:1009.2894 [hep-ph]].
- [50] A. Patella, Phys. Rev. D **86** (2012) 025006 [arXiv:1204.4432 [hep-lat]].
- [51] J. Rantaharju, T. Karavirta, V. Leino, T. Rantalaiho, K. Rummukainen and K. Tuominen, arXiv:1411.4879 [hep-lat].
- [52] M. P. Lombardo, K. Miura, T. J. N. da Silva and E. Pallante, arXiv:1410.0298 [hep-lat].
- [53] T. DeGrand, Y. Shamir and B. Svetitsky, Phys. Rev. D **88** (2013) 5, 054505 [arXiv:1307.2425].
- [54] T. DeGrand, Y. Liu, E. T. Neil, Y. Shamir and B. Svetitsky, arXiv:1412.4851 [hep-lat].
- [55] Y. Aoki *et al.* [the LatKMI Collaboration], Phys. Rev. D **89** (2014) 111502 [arXiv:1403.5000 [hep-lat]].
- [56] Y. Aoki *et al.* [LatKMI Collaboration], Phys. Rev. Lett. **111** (2013) 16, 162001 [arXiv:1305.6006 [hep-lat]].
- [57] A. Athenodorou, E. Bennett, G. Bergner and B. Lucini, arXiv:1412.5994 [hep-lat].
- [58] T. Appelquist *et al.* [LSD Collaboration], Phys. Rev. D **90** (2014) 114502 [arXiv:1405.4752 [hep-lat]].
- [59] T. Appelquist, R. C. Brower, M. I. Buchoff, M. Cheng, G. T. Fleming, J. Kiskis, M. F. Lin and E. T. Neil *et al.*, Phys. Rev. Lett. **112** (2014) 111601 [arXiv:1311.4889 [hep-ph]].
- [60] Z. Fodor, K. Holland, J. Kuti, D. Nogradi, C. Schroeder and C. H. Wong, Phys. Lett. B **718** (2012) 657 [arXiv:1209.0391 [hep-lat]].
- [61] J. Kuti, PoS LATTICE **2013** (2014) 004.