Large $\mathcal{R}$-charge operators in $\mathcal{N}=4$ super Yang-Mills and their gravity duals

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Declaration

I declare that all results presented in this thesis are original except where reference is made to the work of others. The following is the list of my original works discussed in this thesis.

1. R. de Mello Koch, N. Ives, M. Stephanou, "Correlators in Non-Trivial Backgrounds," (2008), Physical Review D, Volume 79, Issue 2, ID 026004, [arXiv:hep-th/0810.4041].
2. R. de Mello Koch, T. Dey, N. Ives, M. Stephanou, "Correlators of Operators with a Large R-Charge," (2009), Journal of High Energy Physics, Volume 2009, Number 8, [arXiv:hep-th/0905.2273].
3. R. de Mello Koch, T. Dey, N. Ives, M. Stephanou,"Hints of Integrability Beyond the Planar Limit: Nontrivial Backgrounds," (2009), Journal of High Energy Physics, Volume 2010, Number 1, [arXiv:hep-th/0911.0967].

These papers are references [46], [73], [84] in the bibliography and appear in sections 2, 3 and 4, respectively. This thesis is being submitted for the degree of Doctor of Philosophy to the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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Abstract

Operators in $\mathcal{N}=4$ super Yang-Mills theory with an $\mathcal{R}$-charge of $O(N^2)$ are dual to backgrounds which are asymtotically $\mathrm{AdS}_5\times\mathrm{S}^5$. In this thesis we develop efficient techniques that allow the computation of correlation functions in these backgrounds. We find that (i) contractions between fields in the string words and fields in the operator creating the background are the field theory accounting of the new geometry, (ii) correlation functions of probes in these backgrounds are given by the free field theory contractions but with rescaled propagators and (iii) in these backgrounds there are no open string excitations with their special end point interactions; we have only closed string excitations. Furthermore, these correlation functions are not well approximated by the planar limit. The non-planar diagrams, which in the bulk spacetime correspond to string loop corrections, are enhanced by huge combinatorial factors. We show how these loop corrections can be resummed. As a typical example of our results, in the half-BPS background of M maximal giant gravitons we find the usual $1/N$ expansion is replaced by a $1/(M+N)$ expansion. Further, we find that there is a simple exact relationship between amplitudes computed in the trivial background and amplitudes computed in the background of M maximal giant gravitons. We also find strong evidence for the BMN-type sectors suggested in arXiv:0801.4457. The problem of computing the anomalous dimensions of (nearly) half-BPS operators with a large $\mathcal{R}$-charge is reduced to the problem of diagonalizing a Cuntz oscillator chain. Due to the large dimension of the operators we consider, non-planar corrections must be summed to correctly construct the Cuntz oscillator dynamics. These non-planar corrections do not represent quantum corrections in the dual gravitational theory, but rather, they account for the backreaction from the heavy operator whose dimension we study. Non-planar corrections accounting for quantum corrections seem to spoil integrability, in general. It is interesting to ask if non-planar corrections that account for the backreaction also spoil integrability. We find a limit in which our Cuntz chain continues to admit extra conserved charges suggesting that integrability might survive.

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Contents

1 Introduction

1.1 Overview

An interesting phenomenon in string theory is the AdS/CFT correspondence [1]. This term refers to a class of dualities between superconformal field theories and superstring theories in anti-De Sitter spacetimes. Several such dualities have been proposed. Presently, we will focus on the duality between $\mathcal{N} = 4$ super Yang-Mills theory on 4-dimensional Minkowski spacetime and Type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$. The gauge theory has gauge group N and effective 't Hooft coupling λ. In the string theory, we have the string coupling g_s and radius of curvature of the spacetime, R_{AdS}. The strong form of the correspondence conjectures an exact duality, valid for all N and λ. We will make extensive use of the large-N limit, where N is much greater than any other parameter. (We will however be studying operators with dimension of order N^2). To see why the large-N limit is useful, consider the relation between gauge and string parameters.

$$g_s = \frac{\lambda}{N}, \left(\frac{R_{\mathrm{AdS}}}{l_s}\right)^4 = 4\pi\lambda$$

The effect of large N is to suppress both non-planar contributions to correlators in the gauge theory (with caveats to be addressed in this thesis) and string interactions. Only perturbative expansions in the 't Hooft coupling are typically tractable in the gauge theory, and so we must work with λ small. On the gravity side, in order to neglect curvature corrections we must have large λ. $\lambda = g_{YM}^2$, where g_{YM}^2 is the gauge theory coupling constant. Consequently, we can only compare quantities across the correspondence whose λ-dependence permits interpretation in both regimes. Dependence on λ shows up in the anomalous scaling dimension of operators in the gauge theory [4], with successive terms in the perturbation expansions having higher powers of λ. If we want to study configurations that can be interpreted on the gravity side, we must restrict attention to operators having zero anomalous dimension, or whose anomalous dimension is suppressed by other quantum numbers. These are the BPS and near-BPS operators.

Type IIB string theory has a global spacetime symmetry group $SO(2,4) \times SO(6)$, associated with the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background. The corresponding global symmetries of $\mathcal{N} = 4$ arise differently. The $SO(2,4)$ factor is a spacetime symmetry - the conformal group in 4 dimensions. The $SO(6)$ factor is an $\mathcal{R}$-symmetry generated by mixing of the six scalars in the theory. In this thesis we will be concerned with BPS and near-BPS operators built out of these scalars and their interpretation as geometric objects in the string theory. The interpretation changes as the number of scalars (the $\mathcal{R}$-charge) in the field theory operator changes. On the field theory side, the size of the operator determines the type of Wick contractions which contribute to correlators. On the string side, the geometric interpretation changes. For this reason the scalar sector of $\mathcal{N} = 4$ SYM is a useful laboratory for studying the emergence of a geometric description of field theory computations.

Operators in $\mathcal{N} = 4$ super Yang-Mills theory with a conformal dimension of $O(N^2)$ are dual to backgrounds which are asymptotically $\mathrm{AdS}_5 \times \mathrm{S}^5$. This is a consequence of the AdS/CFT map, in which conformal dimension in the gauge theory maps to energy in the string theory. The larger the dimension, the greater the energy of the dual state. States with very large energy have a non-negligible backreaction on the geometry. Their dual interpretation is not as a geometric object in $\mathrm{AdS}_5\times\mathrm{S}^5$, but rather as a new background. Appealing to the state-operator map, we can treat these large conformal dimension operators as defining new gauge theory states. Then we can compute correlation functions of smaller operators in these states. In the dual picture we have geometric objects corresponding to the smaller operators now moving in a new, asymptotically $\mathrm{AdS}_5\times\mathrm{S}^5$ background. This is an exciting situation. The geometric objects are now probing a spacetime other than the usual $\mathrm{AdS}_5\times\mathrm{S}^5$. We can see what changes in the gauge theory compared with the usual case to learn about how gravitational phenomena are encoded there. In section 2 we study this problem for string word operators and large Schur polynomial background operators dual to annular LLM geometries. We find that (i) contractions between fields in the string words and fields in the operator creating the background are the field theory accounting of the new geometry, (ii) correlation functions of probes in these backgrounds are given by the free field theory contractions but with rescaled propagators and (iii) in these backgrounds there are no open string excitations with their special end point interactions; we have only closed string excitations. Also notable are the efficient techniques for computing correlation functions of these large operators. Section 2 is the publication [46], edited slightly for inclusion in this thesis.

Correlation functions of operators with a conformal dimension of $O(N^2)$ are not well approximated by the planar limit. The non-planar diagrams, which in the bulk spacetime correspond to string loop corrections, are enhanced by huge combinatorial factors. In section 3 we show how these loop corrections can be resummed. As a typical example of our results, in the half-BPS background of M maximal giant gravitons we find the usual $1/N$ expansion is replaced by a $1/(M+N)$ expansion. Further, we find that there is a simple exact relationship between $\frac{1}{2}$-BPS and near-BPS amplitudes computed in the trivial background and the same amplitudes computed in the background of M maximal giant gravitons. We also find strong evidence for the BMN-type sectors suggested in [24]. The decoupling limit of [24] captures the decoupled low energy world volume theory of the intersecting giant graviton system and this theory is weakly coupled even when the original $\mathcal{N}=4$ super Yang-Mills theory is strongly coupled. Section 3 is based on the publication [73].

In section 4, which is based on the publication [84], we study the problem of computing the anomalous dimensions of a class of (nearly) half-BPS operators with a large $\mathcal{R}$-charge. This problem can be reduced to the problem of diagonalizing a Cuntz oscillator chain. Due to the large dimension of the operators we consider, and in accordance with results of sections 2 and 3, non-planar corrections must be summed to correctly construct the Cuntz oscillator dynamics. These non-planar corrections do not represent quantum corrections in the dual gravitational theory, but rather, they account for the geometric backreaction from the heavy operator whose dimension we study. Non-planar corrections accounting for quantum corrections seem to spoil integrability, in general. It is interesting to ask if non-planar corrections that account for the backreaction also spoil integrability. We find a limit in which the Cuntz chain continues to admit extra conserved charges, suggesting that integrability might survive.

1.2 Technical background and notation

Many of the technical tools developed in sections 2, 3 and 4 are techniques for manipulating Schur polynomials and performing computations with them. In this section we give a brief review of Schur polynomials and an important generalization - the restricted Schurs.

We will be concerned in this thesis with operators built of one or two scalar fields in $\mathcal{N} = 4$ super Yang-Mills. These fields transform in the adjoint representation of the gauge group. We will write them as $N \times N$ matrices. The matrix indices are gauge indices and so only traces or products of traces are gauge invariant. There are two points influencing the choice of basis for these operators: (i) the basis must organize the operators so that computations are mathematically tractable and (ii) in the context of AdS/CFT, the basis states should have a convenient interpretation as dual states. Several distinct bases are known, among them the Schur polynomials (for operators built from a single field) and the restricted Schur polynomials (for operators built from two or more fields). These polynomials are particularly useful for studying general bound states of giant gravitons with strings attached, and also include the operators dual to annular LLM geometries. For a review of the subject of bases in this sector and a discussion of various bases and their applications and relations, see [76].

Schur polynomials were shown to be a basis for the $\frac{1}{2}$-BPS operators in [5]. This relatively simple class of operators has been invaluable in exploring the AdS/CFT correspondence. Of particular historical and technical interest are the operators dual to giant gravitons. We give a brief description in section 1.2.3. First we take a quick look at the definition and important properties of Schur polynomials and their generalization, the restricted Schurs.

1.2.1 Schur polynomials

Schur polynomials are gauge theory operators constructed using a complex combination of two $U(N)$ adjoint fields in $\mathcal{N} = 4$ SYM, say $Z = \phi_1 + i\phi_2$. The definition is

$$\chi_R(Z) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \mathrm{Tr}\,(\sigma Z^{\otimes n}).$$

The label R is a Young diagram with n boxes and at most N rows. $\chi_R(\sigma)$ is the character of σ in S_n representation R. That is

$$\chi_R(\sigma) = \mathrm{Tr}\,_R(\Gamma_R(\sigma)),$$

a trace of the representation matrix $\Gamma_R(\sigma)$ in the vector space carrying R. On the right hand side, the tensor product has its indices mixed by σ:

$$\mathrm{Tr}\,(\sigma Z^{\otimes n}) = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \cdots Z^{i_n}_{i_{\sigma(n)}}.$$

The Schurs have an $\mathcal{R}$-charge $J = n$ and transform in $(0, n, 0)$ of the $SU(4)$ $\mathcal{R}$ symmetry group. They also have conformal dimension of $\Delta = n$ and hence preserve $\frac{1}{2}$ of the supersymmetries of the gauge theory. They provide a complete orthogonal basis in the $\frac{1}{2}$-BPS sector [5].

Properties of Schur polynomials can be computed from the Young diagram labels. The definition of the weight of a box (not the Dynkin weight) in a Young diagram is $N - i + j$ if the box is in the i^{th} row and j^{th} column. We use "factor" and "weight" of a box interchangeably. Denote the product of the weights of all boxes in a Young diagram R by f_R. For example, if

$$R = \yng(3,1)$$

then

$$f_R = N(N+1)(N+2)(N-1).$$

The two-point function for Schurs is simply

$$\langle \chi_R(Z)\chi_S(Z^\dagger)\rangle = \delta_{RS} f_R.$$

Schur polynomials are character polynomials of $U(N)$. From the product rule for $U(N)$ representations we can infer a product rule for Schurs.

$$\chi_R(Z)\chi_S(Z) = \sum_T g_{RST}\chi_T(Z),$$

where the sum on the right hand side is over all irreps T appearing in the decomposition of the product of irreps R and S and g_{RST} gives the multiplicity of T. g_{RST} is a Littlewood-Richardson number, obtained in the usual way from the rules for multiplying Young diagrams. So, for example,

$$\langle \chi_R(Z)\chi_S(Z)\chi_T(Z^\dagger)\rangle = g_{RST} f_T.$$

We will also make use of the "hook length" associated with a box. To obtain the hook length, draw a line starting from the given box towards the bottom of the page until you exit the Young diagram, and another line starting from the same box towards the right until you again exit the diagram. These two lines form an elbow - what we call the hook. The hook length for the given box is obtained by counting the number of boxes the elbow belonging to the box passes through. The notation hooks_R means the product of hook lengths of Young diagram R. Thus, for example

$$\text{hooks}_{\yng(2,1)} = 1 \cdot 1 \cdot 3\,.$$

1.2.2 Restricted Schur polynomials

In order to study more complicated systems, we need to generalize the Schur polynomials. The dual of a giant graviton is a Schur polynomial, which is labeled by a Young diagram. Operators dual to excitations of giant gravitons are obtained by inserting words $(W^{(a)})^j_i$ describing the open strings (one word for each open string) into the operator describing the system of giant gravitons. One index is used up per string, and the associated box on the Young digram is marked (or if there is just one string we can equivalently indicate the Young diagram which remains after removing the associated box). Notice that on the right hand side the S_n character has been replaced by a trace over the subspace determined by the smaller Young diagram R_1 rather than the full space carrying R. For this reason these operators are called restricted Schur polynomials.

$$\begin{aligned}\chi^{(k)}_{R,R_1}(Z,W^{(1)},...,W^{(k)}) &= \frac{1}{(n-k)!}\sum_{\sigma\in S_n}\mathrm{Tr}_{R_1}(\Gamma_R(\sigma))\mathrm{Tr}\,(\sigma Z^{\otimes n-k}W^{(1)}\cdots W^{(k)}), \\ &= \frac{1}{(n-k)!}\sum_{\sigma\in S_n}\mathrm{Tr}\,(\Pi\Gamma_R(\sigma))\mathrm{Tr}\,(\sigma Z^{\otimes n-k}W^{(1)}\cdots W^{(k)}),\end{aligned} \tag{1}$$

$$\mathrm{Tr}\,(\sigma Z^{\otimes n-k}W^{(1)}\cdots W^{(k)}) = Z^{i_1}_{i_{\sigma(1)}}Z^{i_2}_{i_{\sigma(2)}}\cdots Z^{i_{n-k}}_{i_{\sigma(n-k)}}(W^{(1)})^{i_{n-k+1}}_{i_{\sigma(n-k+1)}}\cdots(W^{(k)})^{i_n}_{i_{\sigma(n)}}.$$

Π is a product of projection operators and/or intertwiners, used to implement the restricted trace. Π is defined by the sequence of irreducible representations used to subduce R_1 from R and the chain of subgroups to which these representations belong. Since the row and column indices of the block that we trace over (denoted by R_1 in the above formula) need not coincide, we need to specify this data separately for both indices. Denote the chain of subgroups involved in the reduction by $\mathcal{G}_k \subset \mathcal{G}_{k-1} \subset \cdots \subset \mathcal{G}_2 \subset \mathcal{G}_1 \subset S_n$. $\mathcal{G}_m$ is obtained by taking all elements S_n that leave the indices of the strings $W^{(i)}$ with $i \leq m$ inert. To specify the sequence of irreducible representations employed in subducing R_1, place a pair of labels into each box, a lower label and an upper label. The representations needed to subduce the row label of R_1 are obtained by starting with R. The second representation is obtained by dropping the box with upper label equal to 1; the third representation is obtained from the second by dropping the box with upper label equal to 2 and so on until the box with label k is dropped. The representations needed to subduce the column label are obtained in exactly the same way except that instead of using the upper label, we now use the lower label. For further details see [8, 31, 32, 33].

We will denote the weight of the box that must be dropped from R to obtain R_1 as c_{RR_1}.

The restricted Schurs provide and orthonormal basis for multimatrix operators [49]. There is an analogous restricted Littlewood-Richardson number defining the product rule [53].

1.2.3 Giant gravitons

Sphere giant gravitons are expanded D3-branes in the S^5 part of the background, carrying angular momentum due to orbiting on a plane in the S^5 and coupling to the 5 form flux. The term AdS giant is used to refer to similar objects, but expanded in the AdS part. These objects were discovered in [14]. The action for such an object has two terms: the kinetic Dirac-Born-Infeld term, and the coupling to a 4-form Chern-Simons potential. The Chern-Simons potential is proportional to N, the rank of the gauge group for the dual gauge theory. To minimize the action, with increasing angular momentum the DBI term favours a smaller D3-brane, while the Chern-Simons potential favours an expanded D3-brane. For a particular angular momentum, J, the size of the giant graviton which minimizes the action is

$$R = \sqrt{\frac{J}{N}} R_{\mathrm{S}^5},$$

where R_{S^5} is the radius of the S^5 part of the geometry. In the dual gauge theory this single particle state is described by a sub-determinant of a product of J complex scalar fields:

$$\mathcal{O} = \epsilon_{i_1 \cdots i_J}^{j_1 \cdots j_J} Z_{j_1}^{i_1} \cdots Z_{j_J}^{i_J}, \tag{2}$$

where Z is a complex combination of any two of the six adjoint scalars of $\mathcal{N} = 4$ SYM. The choice of complex combination determines the plane of the S^5 in which the graviton orbits. From the AdS/CFT map we can infer that the angular momentum of the giant graviton has integer values for J. Note that J is bounded above by N. On the gauge theory side we see this because the operator above for $J > N$ is not independent of the set of operators for $J = 1, \ldots, N$. In the string picture, the radius of the D3-brane in the S^5 is bounded by the radius of S^5. This construction gives some insight into the stringy exclusion principle, which places a bound of N on the possible single-particle states which can be realized on the spherical part of the geometry. A giant graviton with $J = N$, and hence $R = R_{\mathrm{S}^5}$, is called a maximal giant graviton.

Although the gravitons expand from point-like for any value of J, D-brane tension increases and the string coupling decreases as N increases, with λ held fixed. Consequently, in the large N limit, the giant gravitons take on the nature of extended branes for $J = O(N)$. For $O(1)$ fields, the gravitons are effectively point-like. In the gauge theory, operators with different numbers of traces are no longer orthogonal when $J = O(N)$ and we need a new basis of diagonal states. The Schur polynomials are one such basis.

D-branes are surfaces on which open strings can end. The open strings are oriented, and one can associate positive and negative charges with the end-points, depending on their orientation. For a state of M D-branes, there are M^2 open string sectors. Interactions between giant gravitons are mediated by open strings stretching between them. Investigations of these interactions in the string and gauge theory pictures have given insight into the AdS/CFT correspondence. Two important questions must be answered in order to undertake such investigations: (i) what are the gauge theory operators dual to bound states of giant gravitons? and (ii) how do we define gauge theory operators dual to giants with strings attached?

A natural guess for operators dual to bound states of giant gravitons is a product of subdeterminant operators like (2),

$$\mathcal{O}_{J_1,J_2} = \epsilon^{j_1 \cdots j_{J_1}}_{i_1 \cdots i_{J_1}} Z^{i_1}_{j_1} \cdots Z^{i_{J_1}}_{j_{J_2 1}} \epsilon^{j_1 \cdots j_{J_2}}_{i_1 \cdots i_{J_2}} Z^{i_1}_{j_1} \cdots Z^{i_{J_2}}_{j_{J_2}}.$$

It turns out that this is not correct. Consider the set of 2-graviton states in $\mathrm{AdS}_5 \times \mathrm{S}^5$. These form a 2-dimensional state space. The labels correspond to the angular momenta of the giants, and states are orthogonal with respect to these labels. In the field theory then, the proposed dual states have to be orthogonal. So the operators written above would have to satisfy

$$\langle \mathcal{O}^{\dagger}_{J_1,J_2}\,\mathcal{O}_{J_3,J_4}\rangle \simeq \delta_{J1,J3}\delta_{J2,J4}+\delta_{J1,J4}\delta_{J2,J3}.$$

This only holds if at least one label in each operator is exactly N. That is, the naive product operator is only a candidate for the dual to states where all but one giant is maximal.

The subdeterminant operators like (2) are in one-to-one correspondence with the Schur polynomials labelled by a Young diagram with a single column. So it is natural to associate these Schurs with the sphere giants. The stringy exclusion principle is satisfied in this picture because the Young diagrams can have at most N rows. If one considers AdS giants, then there is no bound on the angular momentum for a particular giant, but the number of giants is bounded by N [5]. Since the length of a row is not bounded, it is tempting to identify Schurs labelled by a Young diagram with a single row as operators dual to AdS giants. Moreover, multiple AdS giant states are naturally associated with Young diagrams with one row for each AdS giant. Then the bound on the number of rows is in agreement with the bound on the number of possible AdS giants (due to screening of the 5 form flux which causes the branes to expand). Similarly, Young diagrams with k columns label the operators dual to bound states of k sphere giants, with each giant having an angular momentum according to the length of the corresponding column. The product of two Schurs each labelled by a single column is a single two-column Schur only if at least one of the columns has length N, explaining why the naive product of subdeterminants doesn't give a valid two-graviton state.

The authors of [3] computed the spectrum of small fluctuations of both sphere and AdS giants. These are expected to capture the dynamics of excited giants at low energy. An excited giant graviton has open strings attached, inducing the fluctuations. In the gauge theory, the way to attach an open string to a giant is to replace a field in the subdeterminant with a matrix operator representing the open string, see [2] and [17]. In the language of Schur polynomials one marks a box on the Young diagram associated with the open string. The rules for choosing a box reproduce the counting of the number of possible states for any giants/strings configuration. This was discussed in detail in [8]. The spectrum of anomalous dimensions for gauge theory operators dual to giants with strings attached was calculated [17] and compared with the spectrum of small vibrations. These spectra matched. Furthermore, one can build coherent states for the stringy excitations and calculate a semiclassical Hamiltonian for the system. This matches exactly the low-energy sigma model describing open string excitations of giants in the string theory. These results provide excellent evidence to support the giant graviton/Schur polynomial dictionary.

As a further test of this picture, [31] studied string interactions on systems of M bound giant gravitons. The low-energy dynamics of the open strings in this case should realise a $U(M)$ gauge theory local on a new spacetime - the 3+1 dimensional worldvolume of the giants ([8] and references therein). It was found that the amplitude for string interaction has a $\frac{1}{r}$ dependence, where r is the separation between the strings. This would be reproduced in the Born approximation by a potential which falls off like $\frac{1}{r}$, exactly as we expect for a 3+1 dimensional gauge theory. The separation between the interacting strings in this example, and in general, is given by the separation of the corresponding marked boxes on the Young diagrams labelling the gauge theory operators.

1.2.4 Summary of Schur polynomial / string theory map

Taken together, these results provide compelling support for a dictionary mapping gauge theory states labelled by Schur polynomials to string theory states. For operators of $O(N)$ gauge fields we have two main configurations. States labelled by Young diagrams with $O(1)$ rows of length $O(N)$ are dual to bound states of AdS giant gravitons. Those labelled by $O(1)$ columns of length $O(N)$ are dual to sphere giant gravitons. Open strings can be attached to the giants by replacing a particular field with a string word, and labelling the associated box on the row or column corresponding to the brane on which the string endpoints live. Both string endpoints can be in a single box - on the same brane, or endpoints can be labelled in separate boxes - a string stretched between branes. Gauss' law applies on the brane worldvolumes, as discussed in [8].

The $\mathcal{R}$-charge of gauge theory states maps into the energy of string theory states. It is possible to construct operators whose duals have enough energy that gravitational backreaction cannot be neglected and the target spacetime deforms. The dual state becomes a spacetime that is only asymptotically AdS. There are two cases of interest in this thesis. For an $\mathcal{R}$-charge of $O(N^{\frac{3}{2}})$, the dual spacetime looks like an AdS black hole [24]. See section 3.2.5 for an investigation of these operators. The main class of operator we will be studying in this thesis has $\mathcal{R}$-charge of $O(N^2)$. These operators describe LLM geometries with a radially symmetric plane at $y = 0$. See 2.2 for details.

1.2.5 Cuntz oscillators in the standard planar limit

Suppose we would like to study a system of operators constructed from matrix fields Z and Y, where operators with different numbers of Z fields, but a fixed number of Y fields can mix. This situation arises, for example, when studying open string words attached to giant gravitons. Z fields can "hop" back and forth between the string and giant graviton. Spin chain approaches to the open string word operators are not convenient when the length of the chain changes. Instead we can imagine the Y fields setting up a lattice with occupation number n_i determined by the number of Z fields between the i^{th} and $i+1^{\text{th}}$ Y field. Let Z fields be created in site i by $a_i^\dagger$ and destroyed by a_i. The excitations are neither fermions nor bosons, satisfying instead the Cuntz algebra [25]

$$a_i|0\rangle_i = 0, \;\; a_i a_j^\dagger = \delta_{ij},$$

where $|0\rangle_i$ represents the empty state for site i. $\sum_i a_i^\dagger a_i = 1 - |0\rangle_{ii}\langle 0|$ follows from completeness. We will show that the effects of very large background operators (instead of giant gravitons) can be described in terms of a modification of this algebra.

2 Correlators in Nontrivial Backgrounds

2.1 Introduction

The $\frac{1}{2}$-BPS sector of $\mathcal{N}=4$ super Yang-Mills theory is a rich laboratory [5, 6, 7, 9, 10] for the study of the gauge theory/gravity duality [1]. This is due, in part, to the fact that as the $\mathcal{R}$-charge (J) of an operator in the $\mathcal{N}=4$ super Yang-Mills theory is changed, its interpretation in the dual quantum gravity changes. This can be viewed as a consequence of the Myers effect [11]: as we increase J, the coupling to the background RR five form flux increases and the graviton expands. It puffs out to a radius

$$R=\sqrt{\frac{J}{N}}R_{\rm AdS},\qquad \text{where}\qquad R^2_{\rm AdS}=\sqrt{g^2_{YM}N}\alpha'\,.$$

We will consider the limit that N is very large with g^2_{YM} fixed and very small. For $J\sim O(1)$ the operator is dual to an object of zero size in string units, that is, a point-like graviton [1]. For $J\sim O(\sqrt{N})$ the operator is dual to an object of fixed size in string units - this is a string [12]. For $J\sim O(N)$ the operator is dual to an object whose size is of the order of $R_{\rm AdS}$ - as argued in [5, 13] these are the giant gravitons of [14]. The case that is of interest to us in this section is $J\sim O(N^2)$. Naively, the size of these objects diverge, even when measured in units with $R_{\rm AdS}=1$. This divergence is simply an indication that these operators do not have an interpretation in terms of a new object in $\mathrm{AdS}_5\times\mathrm{S}^5$: these operators correspond to new backgrounds [7, 9].

A natural way to explore the physics of these new geometries, is to compute correlation functions in the presence of the operator creating the new background. Since the operator creating the background has $O(N^2)$ fields, this task is non trivial. For the special case of operators built only from Z or from $Z^\dagger$ [72] has shown that these correlators are easily computed using the known product rule and two point function of Schur polynomials [5]. These results showed how to define operators in the super Yang-Mills theory dual to gravitons that are local in the bulk[1] of the dual quantum gravity. The definition of these local operators was in terms of a modified product rule, which is a refinement of the usual Littlewood-Richardson rule. When using the usual Littlewood-Richardson rule, to take the product $\Box \times R$, the single box would be added to all possible rows of the Young diagram R as long as R with the box added is again a legal Young diagram. In contrast to this, the local operators only add boxes to a specific location in the Young diagram. Thus, for example, we can define a local operator that would only add a box to the first row. We label these local operators by the location on the Young diagram to which they would add (in the case of acting with $\mathrm{Tr}\, Z$) or remove (in the case of $\mathrm{Tr}\,\frac{d}{dZ}$) boxes. These locations are labeled as a_i (for inward point corners) and b_i (for outward pointing corners) with i increasing as you move along the edge of the Young diagram from the upper right towards the lower left. See figure 1 for an example of our labeling. Correlators of these local operators are easily computed using the modified product rule [72]. Local operators built with $O(1)$ fields, that do not mix Z and $Z^\dagger$ are dual to gravitons; they are $\frac{1}{2}$ BPS probes.

Probing the background with an operator that is not $\frac{1}{2}$ BPS gives much richer information. In this case we have two natural possibilities: we can excite the background by attaching an open string to obtain a restricted Schur polynomial along the lines of [8, 31, 32, 33], or we could probe the new background with closed strings [15, 16, 72]. The interpretation of the open

[1]More precisely, they are local in the radial direction of the LLM plane and are located at $y = 0$ - i.e. on the LLM plane. They are s-waves on both S^3s in the geometry and are smeared along the ϕ coordinate of the LLM plane. See [72].

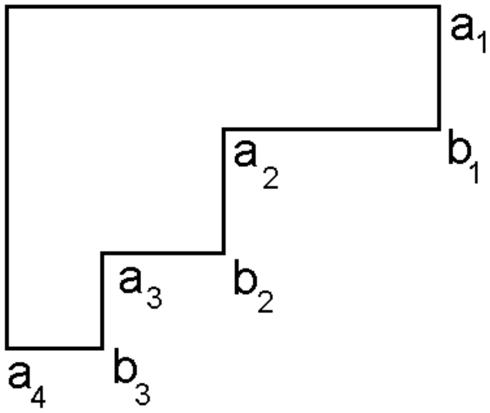


Figure 1: This figure illustrates our labeling of the corners of a Young diagram.

string excitation is not at all obvious. When the $\mathcal{R}$-charge of the operator to which the string is attached is $O(N)$, we know that the excitation indeed behaves like an open string attached to a giant graviton [8, 17, 18, 31, 32, 33]. These excited giant graviton operators are the restricted Schur polynomials. In this case the backreaction of the giant graviton can be neglected and the system is well described as a giant graviton, with open strings attached, moving in the $\mathrm{AdS}_5\times\mathrm{S}^5$ geometry. This is nothing like the situation we study in this section. When the operator to which the open string is attached has an $\mathcal{R}$-charge of $O(N^2)$ it deforms the geometry - it is not a surface on which open strings can end, it is a new classical geometry: a new metric with some background fluxes. Our results clearly show that there is nothing special about how the endpoints of the string interact; they behave just like the bulk of the string. This is a clear demonstration that there is no brane on which string endpoints end[2]: the operator which is being excited is not a membrane; its a new geometry. To arrive at this conclusion, we need to

[2]Of course, it is possible to excite giant gravitons on these geometries, in which case open strings excitations do appear. The perturbative string spectrum contains no open strings.

compute correlators of traces that mix Z and $Z^\dagger$.

To probe the geometry with a closed string, one needs to compute correlators of single trace operators of the form

$$\mathrm{Tr}\,(YZ_{a_i}^{n_1}YZ_{a_i}^{n_2}Y\cdots YZ_{a_i}^{n_L}).$$

This closed string is localized at the corner a_i in the geometry. The Z matrices decompose into a block diagonal structure with each block being associated with one of the corners a_i of the Young diagram, and the additional labels on the Z fields in the expression above indicate where the field is localized in the emergent geometry. See the discussion 2.4 and [72] for more details. Because these operators are nearly BPS their anomalous dimensions receive only a small correction and we can safely work to one loop. By studying this correction, we can obtain geometric information about the new background [15, 16, 72] indicating that this probe is indeed a valuable source of information about the geometry. The Wick contraction of the Y fields is straight forward because there are no Ys in the operator which creates the new background. After Wick contracting the Y fields, we are left with the problem of computing correlators of traces that mix Z and $Z^\dagger$.

These mixed correlators can not be computed using the modified product rule. In [72] it was conjectured that these mixed correlators can be computed using modified ribbon diagrams. The modification simply amounts to rescaling the old propagator by c/N, where c is the weight of the box added to the background Young diagram by the (local) operator. If true, this is a considerable simplification.

In this section we develop techniques that allow the direct computation of these correlation functions. Our results are in perfect agreement with the conjecture of [72]. Although we have focused on $\frac{1}{2}$ BPS backgrounds our results will certainly be applicable more generally. In situations in which backreaction can be ignored, we have already developed techniques for computing the correlation functions of restricted Schur polynomials [31, 32, 33]. In these cases contractions between fields belonging to open string words and the remaining fields in the restricted Schur, make a subleading contribution in a systematic large N expansion. We will argue that back reaction in the gauge theory is accounted for by including these contractions. Our approach to computing these extra contributions starts by noting that the two point correlator (we supress spacetime dependence which plays no role in this thesis)

$$\left\langle (Z^\dagger)^k_l Z^i_j \right\rangle = \delta^i_l \delta^k_j ,$$

is reproduced by identifying

$$(Z^\dagger)^k_l \leftrightarrow \frac{d}{dZ^l_k} .$$

In this way, the contributions to a correlation function of two restricted Schur polynomials coming from contractions between Zs that belong to the open string and Zs that belong to the brane, can be written as a differential operator acting on the restricted Schur polynomials. This differential operator will in general, contain a product of derivatives with respect to the open string words as well as derivatives with respect to Z and $Z^\dagger$. We give a rule for "cutting" any such product up into eight basic types of derivatives and then derive simple formulas for the action of these derivatives. In this way, we can compute arbitrary mixed trace correlators, in any background, to any order in a systematic large N expansion. By specializing to the annulus geometry, we find significant simplifications allowing us to prove the modified ribbon rule of [72]. We then consider LLM geometries that correspond to a set of well seperated concentric rings. The rings give a picture of the eigenvalue density of Z [72]: the eigenvalues split into well separated clumps. In the large N and large 't Hooft coupling limit the off diagonal modes connecting eigenvalues in different rings will be very heavy and decouple. Thus, Z becomes block diagonal with the number of blocks matching the number of rings. Recycling the annulus result then gives us a more general proof of the modified ribbon rule. These results are arranged as follows: In the next section, we consider "open string excitations" of the annulus background. The treatment of closed string excitations then follows, with no extra work. In section 2.3 we generalize our results to backgrounds which correspond to a set of concentric rings. In section 2.4 we discuss our results. Appendices A to G collect some relevant background and the technical details.

2.2 Backreaction: Annulus Geometry

The calculation of two point correlation functions of restricted Schur polynomials with open strings attached has been studied in [8, 31, 32, 33]. In these studies, contractions between fields in the open string and fields in the operator representing the brane were neglected. In the present section, the number of fields in the restricted Schur polynomial is $O(N^2)$. Operators with $\mathcal{R}$ charge of $O(N^2)$ are dual to new geometries, so that the back reaction of the operator must be taken into account. In section 2.2.1 we will argue that the contractions between fields in the open string word and the remaining fields in the operator can no longer be neglected. This is how the backreaction of the operator on the geometry is accounted for in the gauge theory. The open string words that we consider will use Z and Y as letters. To compute correlators in the large N limit, it is useful to treat the Ys as defining a lattice populated by Zs. The Zs themselves can be represented by

Cuntz oscillators, which simply keep track of the planar contractions. In this way the problem of computing anomalous dimensions of operators becomes the problem of computing the spectrum of a Cuntz oscillator Hamiltonian. In section 2.2.2 we will argue that the net effect of the backreaction is to produce a scaling of the Cuntz oscillators, in agreement with [72]. A special case of this result was first obtained in [16], for an LLM geometry with annulus boundary condition on the LLM plane. In section 2.2.3 we will show that the open string endpoints behave exactly like the bulk of the string. We will further argue that the "open string" excitations are best thought of as closed strings propagating on a new background. Finally, in section 2.2.4 we consider probing the new backgrounds with closed strings.

2.2.1 Brane/Sring Contractions

To simplify the presentation of our methods, we will study an operator labeled by a rectangular Young diagram with N_1 rows and M_1 columns. Denote the irreducible representation of $S_{N_1M_1}$ that this Young diagram corresponds to by R. We will consider exciting this BPS operator by attaching a single open string. The open string word has to be associated to the box in the[3] N_1th row and M_1th column, since this is the only box that can be removed to leave a valid Young diagram. Denote the irreducible representation of $S_{N_1M_1-1}$ obtained by removing the box associated to the open string by R_1. The operator we study is

$$\begin{aligned}\chi^{(1)}_{R,R_1}(Z,W) &= \frac{1}{(n-1)!}\sum_{\sigma\in S_n}\mathrm{Tr}_{R_1}\left(\Gamma_R(\sigma)\right) Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n-1}}_{i_{\sigma(n-1)}} W^{i_n}_{i_{\sigma(n)}} \\ &\equiv \mathcal{F}(R,R_1)^a_b\, W^b_a.\end{aligned} \tag{3}$$

[3]The row closest to the top is the first row; the leftmost column is the first column.

For concreteness, consider an open string with a single impurity

$$W_i^j = (Y^{n_1} Z Y^{J-n_1})_i^j .$$

We assume that J is $O(\sqrt{N})$ with $g_2 = \frac{J^2}{N} \ll 1$ so that when contracting the open string words we need only sum planar diagrams [19]. The correlation function we wish to compute is (attach the same open string word to both operators)

$$I_{R\,R_1,S\,S_1} = \left\langle \chi^{(1)}_{R,R_1} \chi^{(1)}_{S,S_1}{}^{\dagger} \right\rangle .$$

We will seperate the computation of this correlator into two pieces: $I^{(0)}_{R\,R_1,S\,S_1}$ obtained by neglecting contractions between the impurity in the open string word and fields in the $\mathcal{F}^a_b$ piece of the operator and $I^{(1)}_{R\,R_1,S\,S_1}$ obtained by contracting the impurity in the open string word with a field in the $\mathcal{F}^a_b$ piece of the operator.

First, consider $I^{(0)}_{R\,R_1,S\,S_1}$. Using the results of [31], we find

$$I^{(0)}_{R\,R_1,S\,S_1} = N_1 M_1 N^J f_R \left(1 + O(g_2^2)\right) \delta_{RS} \delta_{R_1 S_1} . \tag{4}$$

Associate a weight $N-j+i$ to the box in the ith column and jth row of R. f_R is the product of the weights of the Young diagram R. In the language of [31] only the F_0 contraction of the open strings contribute in the large N limit of the correlator (4). If $R \neq S$ but $R_1 = S_1$, then in the language of [31], the only contribution comes from the F_1 contraction of the open string words. It is straight forward to consider this case using our methods, although we do not do so in this thesis.

Next, consider $I^{(1)}_{R\,R_1,S\,S_1}$. After contracting all of the Y fields in W with the $Y^\dagger$ fields in $W^\dagger$, contract a Z in W with a $Z^\dagger$ in $\mathcal{F}^\dagger$ and a $Z^\dagger$ in $W^\dagger$ with a Z in $\mathcal{F}$. We obtain

$$
\begin{aligned}
I^{(1)}_{R\,R_1,S\,S_1} &= N^{J-2}\left\langle \frac{d\mathcal{F}(R,R_1)^a_b}{dZ^c_d}\frac{d(\mathcal{F}(S,S_1)^\dagger)^b_a}{d(Z^\dagger)^d_c}\right\rangle \\
&= N^{J-2}\left\langle \frac{d}{dZ^c_d}\frac{d}{d(Z^\dagger)^d_c}\frac{d}{dW^b_a}\frac{d}{d(W^\dagger)^a_b}\chi^{(1)}_{R,R_1}\chi^{(1)}_{S,S_1}{}^\dagger\right\rangle \\
&= N^{J-2}\left\langle \mathrm{Tr}\left(\frac{d}{dZ}\frac{d}{dZ^\dagger}\right)\mathrm{Tr}\left(\frac{d}{dW}\frac{d}{dW^\dagger}\right)\chi^{(1)}_{R,R_1}\chi^{(1)}_{S,S_1}{}^\dagger\right\rangle .
\end{aligned}
$$

We will now introduce a convenient graphical notation. The derivative operator that we need to consider is determined by the fields from the open strings that are contracted with $\mathcal{F}$ and $\mathcal{F}^\dagger$. Our notation keeps track of these fields and gives a simple picture from which we can read off the relevant derivative operator. We denote $\mathcal{F}$ and $\mathcal{F}^\dagger$ by open ellipses, with a single index line entering the ellipse and a single index line leaving the ellipse. We do not draw the fields in $\mathcal{F}$ and $\mathcal{F}^\dagger$ or their contractions. The contractions of fields in the open string words are drawn using the usual ribbon diagram (also called "fat graph" or "double line") representation. The Y contractions are given by filled ribbons. The Z contractions are empty ribbons. Fields left uncontracted in the diagram are to be contracted with the fields in $\mathcal{F}$ and $\mathcal{F}^\dagger$. The graphical representation of the two terms we have considered are given in figure 2.

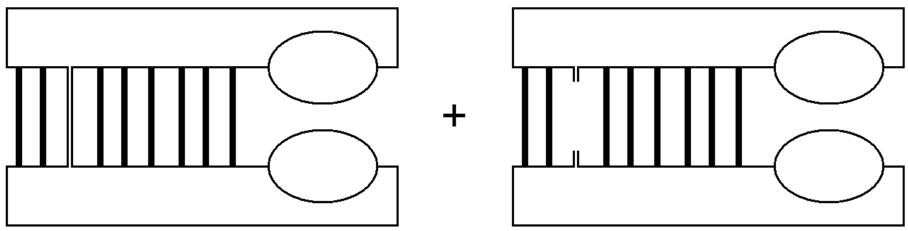


Figure 2: The graphical representation of $I^{(0)}_{R\,R_1,S\,S_1}$ and $I^{(1)}_{R\,R_1,S\,S_1}$. We have set $n_1 = 2$ and $J = 8$.

To read the derivative operator from the diagram, replace each upper "open stub" (= uncontracted Z field) by a derivative with respect to $Z^\dagger$, each lower "open stub" (= uncontracted $Z^\dagger$ field) by a derivative with respect to Z, the upper ellipse by a derivative with respect to the open string word W and the lower ellipse by a derivative with respect to the open string word $W^\dagger$. All derivatives in the same index loop are in the same trace.

In general, when we have many impurities in the open string word, we may have multiple contractions between fields belonging to the open strings and fields in $\mathcal{F}$ and $\mathcal{F}^\dagger$. In all of these cases we will be able to write these contributions as the expectation value of a derivative operator acting on $\chi^{(1)}_{R,R_1}\chi^{(1)}_{S,S_1}{}^\dagger$. The precise structure of the derivative operator will depend on the details of the specific contractions we consider. As another example, if the reader translates the diagram shown in figure 3, she should obtain

$$\left\langle \mathrm{Tr}\left(\frac{d^2}{dZ^2}\right)\mathrm{Tr}\left(\frac{d^2}{dZ^{\dagger 2}}\right)\mathrm{Tr}\left(\frac{d^3}{dZ^3}\frac{d^3}{dZ^{\dagger 3}}\right)\mathrm{Tr}\left(\frac{d}{dW}\frac{d}{dW^\dagger}\right)\chi^{(1)}_{R,R_1}\chi^{(1)}_{S,S_1}{}^\dagger\right\rangle .$$

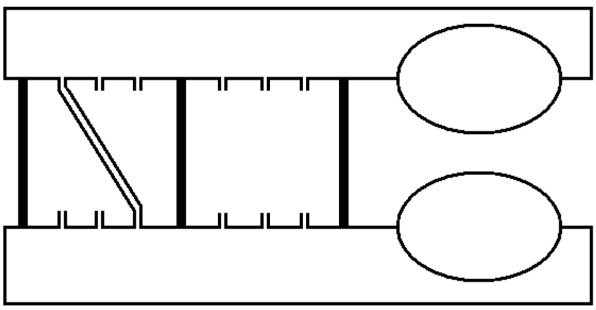


Figure 3: The graphical representation of one term contributing to the correlator $\left\langle\chi^{(1)}_{R,R_1}\chi^{(1)}_{S,S_1}{}^\dagger\right\rangle$. The open string word $W = YZ^3YZ^3Y$.

To get the full set of contributions to the correlator we need to draw all distinct diagrams allowed such that all possible connections of solid ribbons are included, and all possible combinations of connections of hollow ribbons as well as disconnected stumps are included.

The fact that we can account for contractions between fields in the open string words and fields in $\mathcal{F}$ or $\mathcal{F}^\dagger$ as a derivative operator acting on the restricted Schur polynomials is a useful observation because, in general, we can break an arbitrary derivative operator into a product of eight basic types of derivatives, as shown in appendix A. We call this process "cutting". The first cutting rule allows us to cut single derivatives out of any given trace[4] to leave a product of alternating holomorphic and antiholomorphic derivatives. The second rule allows us to cut the trace of a product of holomorphic and antiholomorphic derivatives into a product of traces of purely holomorphic or purely antiholomorphic derivative. In both cases the restricted Schur polynomial is modified by inclusion of an extra factor in the restricted character. The reader can consult appendix A for the details. The action of these basic derivatives on a general restricted Schur polynomial, is described by the simple formulas collected in appendices B and C. We call these formulas "reduction rules". After applying the cutting and then the reduction rules, it is straight forward to obtain

$$I^{(1)}_{R\,R_1,S\,S_1} = N^{J-2}(N_1M_1)^2 f_R\left(1+O(g_2^2)\right). \tag{5}$$

Comparing (4) and (5), we see that the contraction between the impurity in the open string and fields in $\mathcal{F}(R,R_1)$ and $\mathcal{F}(S,S_1)^\dagger$ need only be taken into account when N_1M_1 is $O(N^2)$. This is precisely the regime in which the operator is dual to a very heavy state whose back reaction on the original $\mathrm{AdS}_5\times\mathrm{S}^5$ space produces a new geometry, so it is natural to interpret these

[4] To cut a holomorphic (antiholomorphic) derivative out of the trace the derivative on its left must also be holomorphic (antiholomorphic).

contractions as the field theory accounting of the back reaction of the heavy state: by including these contractions, the string "interacts with the back reacted geometry". This is the key result of this section, and although we have only illustrated it in a simple example the conclusion is general.

Summary: *The contractions between fields in the open string word and the remaining fields in the operator need only be taken into account when the number of fields in the operator creating the background is $O(N^2)$. These contractions are the field theory accounting of the back reaction of this heavy state.*

2.2.2 Modified Cuntz Oscillators

In this section, we will set $N_1 = N$ and $M_1 = M$. This corresponds to taking an annulus boundary condition for the dual LLM geometry. In this case, we can have excitations of the two edges of the annulus [72]: by acting with Z_a we add boxes to the upper right corner of the Young diagram (corresponding to the outer edge of the annulus) and by acting with $\frac{d}{dZ_b}$ we erode boxes from the lower right corner (corresponding to the inner edge of the annulus). There is a huge simplification that arises for the annulus: we can simply replace the local operators Z_a and $\frac{d}{dZ_b}$ by Z and $\frac{d}{dZ}$. This is simply because Z is unable to add boxes anwehere except the first few rows and $\frac{d}{dZ}$ is unable to remove boxes from anyhwere except the last few rows. Our open string lives at the outer edge of the annulus which implies that R_1 is a rectangle with M columns and N rows and R has one extra box in the first row, giving $M + 1$ boxes in the first row. The simplest situation in which to illustrate our result is to consider open string excitations that have multiple impurities at a single site. For 9 impurities, the open string word is $W^i_j = (Y(Z)^9Y)^i_j$. We will get contributions from contracting n impurities in the open string with fields in $\mathcal{F},\mathcal{F}^\dagger$ for $n=0,1,2,...,9$. There are $9!/(n!(9-n)!)$ distinct contractions for a given n. The specific details of the contractions matters. For example, in the case that $n=4$, if none of the impurities in the open string that are contracted with $\mathcal{F},\mathcal{F}^\dagger$ are adjacent (see figure 4), we obtain the following contribution (this formula is correct to leading order at large N)

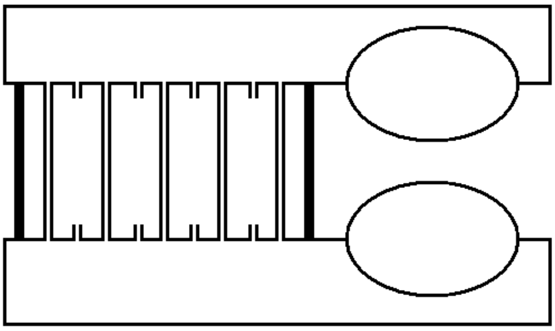


Figure 4: The diagram giving the contribution in (6)

$$N^2\left\langle \left[\mathrm{Tr}\left(\frac{d}{dZ}\frac{d}{dZ^\dagger}\right)\right]^4 \mathrm{Tr}\left(\frac{d}{dW}\frac{d}{dW^\dagger}\right)\chi^{(1)}_{R,R_1}(\chi^{(1)}_{R,R_1})^\dagger\right\rangle = (MN)^4N^2\frac{M+N}{N}f_R\left(1+O(g_2^2)\right). \tag{6}$$

Now consider the contribution coming from the term with all four impurities adjacent (see figure 5)

$$\begin{aligned} I_4 &= N^5\left\langle \mathrm{Tr}\left(\left[\frac{d}{dZ}\right]^4\left[\frac{d}{dZ^\dagger}\right]^4\right)\chi^{(1)}_{R,R_1}(\chi^{(1)}_{R,R_1})^\dagger\right\rangle \\ &= N^5\left\langle \mathrm{Tr}\left(\frac{d}{dX_4}\frac{d}{dX_4^\dagger}\right)\prod_{i=1}^{3}\mathrm{Tr}\left(\frac{d}{dX_i}\right)\mathrm{Tr}\left(\frac{d}{dX_i^\dagger}\right)\chi^{(1,4)}_{R,R_1;P}(\chi^{(1,4)}_{R,R_1;P})^\dagger\right\rangle \end{aligned} \tag{7}$$

where

$$P=(n-4,n-3,n-2,n-1).$$

To obtain this expression, we have used the methods of appendix A to decompose the derivative operator into a product of basic types. This can now be evaluated using the methods developed in appendices B, C and D. The details of some similar example calculations are summarized in appendix E. Although the details are completely different to the (6) calculation, we find *exactly the same result*

$$I_4 = N^6 M^4 \frac{M+N}{N} f_R \left(1 + O(g_2^2)\right). \tag{8}$$

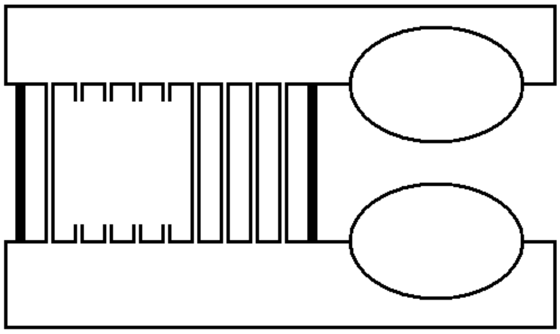


Figure 5: The diagram giving the contribution in (8)

This is general: if we have p impurities at a site, the contribution to the correlator coming from all $p!/(n!(p-n)!)$ contractions between n impurities on the open string and the fields in $F, F^\dagger$ are all the same size. Further, it is now straight forward to check that each of the terms contributing when we have p impurities in the open string words contracting with fields in $\mathcal{F}$ and $\mathcal{F}^\dagger$, gives

$$N^{10} \left(\frac{M}{N}\right)^p \frac{M+N}{N} f_R$$

and therefore that

$$\langle\chi^{(1)}_{R,R_1}\chi^{(1)}_{R,R_1}{}^{\dagger}\rangle=\sum_{p=0}^{9}N^{10}\left(\frac{M}{N}\right)^{p}\frac{M+N}{N}f_R\frac{9!}{p!(9-p)!}=N^9(M{+}N)f_R\left(1+\frac{M}{N}\right)^{9}.$$

If we had n impurities in the site, we'd have obtained

$$\langle\chi^{(1)}_{R,R_1}\chi^{(1)}_{R,R_1}{}^{\dagger}\rangle=(M+N)N^n f_R\left(1+\frac{M}{N}\right)^{n}. \tag{9}$$

Recall that adding an extra impurity in the open string word corresponds to applying another Cuntz oscillator to the state. Clearly, in view of (9), the correct way to account for the background is to rescale the Cuntz oscillators describing the impurities in the open string

$$aa^{\dagger}=\left(1+\frac{M}{N}\right),\qquad a^{\dagger}a=\left(1+\frac{M}{N}\right)(1-|0\rangle\langle 0|)\,.$$

The factor $1+\frac{M}{N}$ is $\frac{c}{N}$ with c the weight of boxes in the upper right hand region of the Young diagram.

We can give this calculation a slightly different interpretation which will allow us to state the general result: after contracting the Y fields planarly, the above correlator can be viewed as the expectation value of a product of single trace operators, in the new background

$$\begin{aligned}\left\langle\chi^{(1)}_{R,R_1}(Z,W)(\chi^{(1)}_{R,R_1}(Z,W))^{\dagger}\right\rangle &= \left\langle\mathrm{Tr}\left((Z_a)^n(Z_a^{\dagger})^n\right)\chi_{R_1}(Z)(\chi_{R_1}(Z))^{\dagger}\right\rangle\\ &\equiv \left\langle\mathrm{Tr}\left((Z_a)^n(Z_a^{\dagger})^n\right)\right\rangle_{R_1}.\end{aligned}$$

These fields only add boxes in the first few rows, i.e. in the upper right region of the Young diagram. They are thus local operators according to [72]. Thus, when computing correlators in the annulus background, we can reproduce the above result by using free field theory, after rescaling all propagators by $\frac{c}{N}$ where c is the weight of the added boxes. Below we will show how this generalizes for an LLM background comprised of concentric annuli.

In appendix F we give a rigorous derivation of this result. We also compute the expectation value of

$$O = \langle \mathrm{Tr} \left(\frac{d^n}{dZ^n} \frac{d^n}{d(Z^\dagger)^n} \right) \rangle ,$$

for the annulus background. The result is:

Summary: *In the annulus background the original matrix Z is a local operator in the sense that it only adds boxes in the first few rows of the Young diagram. The derivative $\frac{d}{dZ}$ is also a local operator in the sense that it only removes boxes from the last few rows of the Young diagram. To compute correlation functions of these local operators one uses ribbon diagrams, where each ribbon carries an extra factor of $\frac{c}{N}$ where c is the weight of the boxes added or removed by the local operator.*

2.2.3 Tying up loose ends

Since we are considering open string excitations, we need to pay some attention to the end point interactions. General methods to determine the interations for a single string [32] or for multistrings [33] are known. The strength of this interaction is given by $\sqrt{\frac{c}{N}}$. Consider a string built using $L+1$ Ys. These Ys form a lattice on which the Zs hop. The Hamiltonian for the string takes the form (this endpoint interaction assumes that the open string is attached to a single brane and not a boundstate of branes - see [32])

$$H = 2\lambda \sum_{l=1}^{L} a_l^\dagger a_l - \lambda \sum_{l=1}^{L-1} (a_l^\dagger a_{l+1} + a_l a_{l+1}^\dagger) + \sqrt{\frac{c}{N}} \lambda (a_1 + a_1^\dagger + a_L + a_L^\dagger). \quad (10)$$

Here c is the weight of the box occupied by the open string.

If the $\mathcal{R}$-charge of the background is $O(1)$ or $O(\sqrt{N})$ the operator we are studying is dual to a graviton or a string, but not a brane. In this case, the weight of the box occupied by the open string is $O(N)$, so that $\sqrt{\frac{c}{N}} = 1$. Further, the Cuntz oscillators satisfy

$$a_l a_l^\dagger = 1, \qquad a_l^\dagger a_l = 1 - |0\rangle\langle 0|.$$

This implies that hopping onto and off of the string is no different from hopping between bulk sites. This implies that the end point dynamics is not special: the end points are not "stuck to a brane". Our string is a closed string, not an open string. If we now consider the case of an operator with an $\mathcal{R}$ charge of $O(N)$ and further that the operator has $O(1)$ rows (or O(1) columns), then the open string is attached to a box with a weight of αN with $\alpha = O(1)$. The Cuntz oscillators are unchanged. This implies that the end point dynamics is special: hopping onto and off of the string has a weight $\sqrt{\alpha}$. In this case, we do indeed have an open string excitation, as has been verified in [17, 18, 32]. Finally, consider the case of interest to us here, when the operator has an $\mathcal{R}$-charge of $O(N^2)$ and all edges with a length of $O(N)$. This requirement on the length of all edges is needed if the operator is to correspond to a regular LLM geometry. [5] In this case, the Cuntz oscillators are modified to

$$a_l a_l^\dagger = \frac{c}{N}, \qquad a_l^\dagger a_l = \frac{c}{N}\left(1 - |0\rangle\langle 0|\right).$$

[5]Indeed, a rectangular Young diagram with M columns and N rows, plus one more column with αN boxes with $\alpha = O(1)$ corresponds to a finite size D3-brane on the back reacted LLM geometry. This D3 will admit open string excitations. We are considering operators dual to geometries without any D3-branes which is achieved precisely by our restriction that all edges have a length of $O(N)$.

To make all dependence on the weight of the box occupied by the open string explicit, use the rescaled oscillators $a_l = \sqrt{\frac{c}{N}}\tilde{a}_l$. In terms of these oscillators

$$
H = \frac{c}{N}\left(2\lambda\sum_{l=1}^{L}\tilde{a}_l^\dagger\tilde{a}_l - \lambda\sum_{l=1}^{L-1}(\tilde{a}_l^\dagger\tilde{a}_{l+1} + \tilde{a}_l\tilde{a}_{l+1}^\dagger) + \lambda(\tilde{a}_1 + \tilde{a}_1^\dagger + \tilde{a}_L + \tilde{a}_L^\dagger)\right),
$$

$$
\tilde{a}_l\tilde{a}_l^\dagger = 1, \qquad \tilde{a}_l^\dagger\tilde{a}_l = 1 - |0\rangle\langle 0| \,. \tag{11}
$$

Once again there is nothing special about the string endpoints which behave exactly like the bulk of the string! The astute reader might object that hopping in the bulk is between two sites of the string which is different to hopping off of and onto the string, which is what happens at the string endpoints. This is simply an artifact of how we have split the restricted Schur polynomial into a string plus background. Indeed as Zs hop off the string, extra boxes are added to the Young diagram. One could rather describe these extra boxes as impurities in an $L+1$th site of the string. For example, if there are no extra boxes in the Young digram, no Zs can hop onto the string; with the new interpretation we would say that the $L+1$th site is empty and hence nothing can hop out of this site. There are two facts that make this reinterpretation possible:

- Each time we add a Z in the open string word, we get an extra index loop giving an extra N and an extra $\frac{c}{N}$ from the extra (rescaled ribbon) propagator, giving a total extra factor of c. By adding an extra box, the factor of the product of the weights (f_R) in the restricted Schur correlation function has an extra factor of c. Thus adding a box or an impurity contributes the same factor.

- We deal with Cuntz oscillators, that is, distinguishable particles. Thus, there are no extra $n!$ type normalizations that appear for n bosons. Corresponding to this, the correlators of the restricted Schur polynomials is proportional to 1 if the Young diagrams participating have the same shape, and to 0 otherwise. (See appendix G for a detailed matching.)

This again suggests that the excitation is best thought of as a closed string and not an open string. This has an appealing interpretation: the operator we are exciting has an $\mathcal{R}$-charge of $O(N^2)$. It does not correspond to a brane, but rather to a new geometry. In this case we do not expect to see any open string excitations in the spectrum. It is satisfying that this is indeed the case.

Summary: *The dynamics of the string "endpoints" is identical to the dynamics of the bulk of the string. The excitation behaves like a closed string, not an open string. This is expected since the operator being excited is dual to a new background and not a brane.*

2.2.4 Back Reaction: Closed Strings

To consider closed strings we should probe the geometry with a single trace operator

$$\mathcal{O} = \mathrm{Tr}\,(YZ^{n_1}YZ^{n_2}Y\cdots YZ^{n_L}).$$

The leading large N contribution to this correlator is given by contracting the Y fields planarly. The above correlator then becomes the expectation value of a product of single trace operators, in the new background. This has been computed above.

2.3 Backreaction: Multi Rings

In this section we will consider LLM geometries that correspond to a set of well seperated thick rings. The background with three rings would for example, be described by a Young diagram with N rows and the same shape as the one in figure 1. The black rings can be viewed as a picture of the eigenvalue density of Z [72]. Thus, the eigenvalues will split into three well separated clumps. In the limit that we expect a classical geometry to emerge (large N and large 't Hooft coupling) the off diagonal modes connecting these three subsectors will be very heavy and decouple. We expect that, when studying almost BPS states, the effect of these modes on the dynamics can be neglected. There is no reason to neglect off diagonal modes connecting eigenvalues in the same sector. Thus, for our purposes, we can replace Z by a block diagonal matrix with the number of blocks matching the number of clumps of eigenvalues. If clump i contains N_i eigenvalues it corresponds to an $N_i \times N_i$ block.

A geometry with M rings can thus be considered as an M matrix model. The matrices Z_i are $N_i \times N_i$ dimensional matrices, where clump i contains N_i eigenvalues. Acting with $\mathrm{Tr}\,(Z_i)$ will only add boxes to the rows corresponding to ring i [72]. These boxes have weight c_i. The matrices are not interacting so that we actually have M one matrix models. Each of these matrix models has an annulus background - one described by a Young diagram with N_i rows. To make sure that the eigenvalues localize correctly into the multi-ring geometry, one needs to ensure that the weight of the boxes in the rightmost column match the weights of the corresponding boxes in the original Young diagram. This follows because the weights give the radius squared of the position of the corresponding eigenvalue on the LLM plane [72]. Note that we are not just projecting the eigenvalues. Indeed, for block i we integrate over the full set of N_i^2 matrix elements. We can now easily recycle the results of appendix F to obtain

$$\frac{\langle \chi_B \chi_B^\dagger \mathrm{Tr}\,(Z_i^n Z_i^{\dagger\, n})\rangle}{\langle \chi_B \chi_B^\dagger \rangle} = N_i c_i^n \,,$$

$$\frac{\langle \chi_B \chi_B^\dagger \mathrm{Tr}\,\left(\frac{d^n}{dZ_i^n}\frac{d^n}{dZ_i^{\dagger\, n}}\right)\rangle}{\langle \chi_B \chi_B^\dagger \rangle} = N_i c_i^n \,,$$

where c_i are the weights of the boxes added or removed, respectively. The computations of these correlators is one of the main results of this section.

Summary: *In the multi-ring LLM background the original matrix Z breaks into local blocks Z_i, which are $N_i \times N_i$ dimensional matrices, where clump i contains N_i eigenvalues. To compute correlation functions of these local operators one uses ribbon diagrams, where each ribbon carries an extra factor of $\frac{c}{N}$ where c is the weight of the boxes added or removed by the local operator and one includes a factor of $\frac{N_i}{N}$ for each trace in the local operator.*

As a nontrivial consequence of our result, note that the net affect of the background on the Cuntz Hamiltonian (11) is simply to scale the Cuntz oscillators by $\frac{c}{N}$.

2.4 Discussion

In this section we have developed techniques which allow us to compute correlation functions in the presence of an operator with an $\mathcal{R}$-charge of $O(N^2)$. The backgrounds we have considered are LLM geometries that correspond to a set of concentric rings. We have probed these backgrounds with operators corresponding to both open strings and closed strings. Contractions between fields in the string words and fields in the operator creating the background need only be taken into account when the number of fields in the operator creating the background is $O(N^2)$; these contractions are the field theory accounting of the back reaction on the geometry. From the results of [72], we know that in the new background we can break the original matrix Z into "local pieces", Z_i, which add boxes at specific locations on the Young diagram. In this section we have given a precise definition for this decomposition: the original Z matrix decomposes into a block diagonal matrix. There is a block for each ring. The dimension N_i of the blocks is equal to the number of eigenvalues in each ring. These blocks are the Z_i. To compute correlation functions of these local operators, use the usual free field theory ribbon diagrams, but each ribbon now carries an extra factor of $\frac{c}{N}$ with c the weight of the boxes added by the local operator. The complete effect of the background is the extra $\frac{c}{N}$ factor now carried by each propagator, in perfect agreement with [72]. This is a considerable simplification.

Our study of open string excitations shows that the dynamics of the string endpoints is identical to the dynamics of the bulk of the string. Open string excitations of the operators with an $\mathcal{R}$-charge of $O(N^2)$ behave like a closed string; there are no open string excitations with their special end point interactions: in the new background we have only closed string excitations. This is expected since the operator being excited is dual to a new geometry and not a brane.

Finally, the techniques we have developed here are equally applicable to the computation of correlators in the presence of the multi-matrix operators of [21, 47, 48, 49].

3 Correlators Of Operators with a Large R-charge

3.1 Introduction

According to the AdS/CFT correspondence [1], the conformal dimension of an operator in the $\mathcal{N} = 4$ super Yang-Mills theory maps into the energy of the corresponding state in IIB string theory on the $\mathrm{AdS}_5\times\mathrm{S}^5$ background. Thus, operators with a very large dimension will map into states with a very large energy. If this energy is large enough, back reaction can not be neglected and the state is best thought of as a new geometry which is only asymptotically $\mathrm{AdS}_5\times\mathrm{S}^5$. Good examples of such operators include the Schur polynomials [5, 6, 22] with $\mathcal{R}$-charge of $O(N^2)$ (dual to LLM geometries [7, 9]) and the operators obtained by distributing a gas of defects on Schur polynomials (which seem to be dual to asymptotically $\mathrm{AdS}_5\times\mathrm{S}^5$ charged black holes [23, 24]). To build a proper understanding of these operators in the gauge theory one would like, at least, to compute the anomalous dimension of these operators and to deal with their mixing.

In this section we consider the problem of computing correlators of heavy operators. The operators we have in mind are a small perturbation of a BPS operator which has $\mathcal{R}$-charge $\sim$ conformal dimension $\sim N^2$. In the language of [26] we study *almost* BPS operators. To solve this problem one must reorganize perturbation theory by resumming an infinite number of diagrams. The need for this reorganization is that for these operators non-planar diagrams can't be neglected [13]. A similar situation is provided by the BMN sector [12] of $\mathcal{N} = 4$ super Yang-Mills theory. In this case one considers operators with $\mathcal{R}$-charge $J \sim O(\sqrt{N})$. One finds the usual $\frac{1}{N}$ expansion parameter is replaced by a new expansion parameter equal to

$\frac{J^2}{N}$ [19]. Thus, one must take $\frac{J^2}{N} \ll 1$ to suppress non-planar diagrams.

The question we ask and answer in this section is:

Is there a reorganization of the $\frac{1}{N}$ expansion for almost BPS operators which have $\mathcal{R}$-charge of $O(N^2)$ and if so, what is the expansion parameter?

We focus on almost BPS operators since we want to extrapolate our computations to strong coupling, where we can compare with the dual string theory.

A particularly useful way to describe the half-BPS sector is to use the Schur polynomials [5, 6, 22]. Among the operators we consider, are small perturbations of a Schur polynomial $\chi_B(Z)$ with B a Young diagram that has M columns and N rows, and M is $O(N)$. This corresponds to an LLM geometry [7] with boundary condition that is an annulus. The inner radius of the annulus is $\propto \sqrt{M}$ and the outer radius is $\propto \sqrt{M+N}$. We demonstrate, in section 3.2.1, that there is a reorganization of the perturbation theory in the half-BPS sector and that the new expansion parameter is $\frac{1}{M+N}$. In section 3.2.2 we confirm this answer by studying the holography of the IIB supergravity in the relevant LLM geometry. We then generalize these results to multi-ring LLM geometries (in section 3.2.3), backgrounds with more than one charge (in sections 3.2.4 and 3.2.5) and beyond the BPS sector (in section 3.3). The Schwinger-Dyson equations provide a very powerful approach to the computation of correlators in the annulus background. We present these details in the appendices H to J.

3.2 Half-BPS Sector

Schur polynomials provide a very convenient reorganization of the half-BPS sector. This is due to the fact that their two point function is known to all orders in $\frac{1}{N}$ [5, 22] and that they satisfy a product rule allowing computation of exact n-point correlators using only two-point functions. In this section we will use this Schur technology to provide an answer, in the half-BPS sector, to the question posed above.

$\mathcal{N} = 4$ super Yang-Mills theory has 6 scalars ϕ_i transforming in the adjoint of the gauge group and in the **6** of the $SU(4)_{\mathcal{R}}$ symmetry. We shall use the complex combinations

$$Z = \phi_1 + i\phi_2, \qquad Y = \phi_3 + i\phi_4, \qquad X = \phi_5 + i\phi_6,$$

in what follows. We focus on the contributions coming from the color combinatorics; we drop all spacetime dependence from two point correlators[6], that is, we use the two point functions

$$\left\langle Z_{ij} Z^{\dagger}_{kl} \right\rangle = \left\langle Y_{ij} Y^{\dagger}_{kl} \right\rangle = \left\langle X_{ij} X^{\dagger}_{kl} \right\rangle = \delta_{il}\delta_{jk} \, . \tag{12}$$

Since we are not explicitly displaying the spacetime dependences, it is important to point out that all holomorphic operators are inserted at a specific spacetime event and all anti-holomorphic correlators are inserted at a second spacetime event. These correlators are called extremal correlators; there are non-renormalization theorems protecting these correlators [27, 28].

When we talk about a half-BPS operator in what follows, we mean an operator built only from Zs. These operators will not break any further supersymmetries beyond those broken by the background itself. We obtain almost BPS operators by sprinkling Ys and Xs in the operator.

[6]It is simple to reinstate the spacetime dependence in the final result.

3.2.1 Super Yang-Mills Amplitudes

The exact computation of multi-trace correlators at zero coupling is most easily carried out by expressing the multi-trace operators of interest in terms of Schur polynomials

$$\prod_i \mathrm{Tr}\,(Z^{n_i}) = \sum_R \alpha_R \chi_R(Z), \qquad \prod_j \mathrm{Tr}\,((Z^\dagger)^{m_j}) = \sum_R \beta_R \chi_R(Z^\dagger). \tag{13}$$

The coefficients α_R and β_R appearing in the above expansion have no dependence on N. It will be useful first to compute these correlators with a trivial background. A little Schur magic now gives

$$\begin{aligned} \mathcal{A}(\{n_i; m_j\}, N) \equiv \left\langle \prod_{i,j} \mathrm{Tr}\,(Z^{n_i})\mathrm{Tr}\,((Z^\dagger)^{m_j}) \right\rangle &= \sum_{R,T} \alpha_R \beta_T \left\langle \chi_R(Z)\chi_T(Z^\dagger)\right\rangle \\ &= \sum_R \alpha_R \beta_R f_R\,. \end{aligned} \tag{14}$$

In this last equation f_R is the product of the weights[7] (one for each box in the Young diagram) for Young diagram R.

In what follows, B denotes the rectangular Young diagram with N rows and M columns. We will often refer to B as the annulus background because the corresponding LLM geometry is obtained by taking the LLM boundary condition to be a black annulus on a white plane. The expectation value of an operator O in background B is given by

$$\langle O \rangle_B \equiv \frac{\left\langle \chi_B(Z)\chi_B(Z^\dagger) O\right\rangle}{\langle \chi_B(Z)\chi_B(Z^\dagger)\rangle}\,. \tag{15}$$

Our normalization is chosen so that the expectation value of the identity is 1. Recall that B is a Young diagram with N rows and M columns. The product

[7] Recall that the box in row i and column j has a weight $N + j - i$.

rule satisfied by the Schur polynomials can be viewed as a consequence of the fact that the Schur polynomials themselves are characters of $SU(N)$ and that (i) the character of a direct product of representations is just the product of the characters and (ii) the character of a given reducible representation is equal to the sum of characters of the irreducible representations appearing in the reducible representation. In the product

$$\chi_R(Z)\chi_S(Z) = \sum_T g_{RST}\chi_T(Z),$$

the Littlewood-Richardson number g_{RST} counts the number of times irreducible representation T appears in the direct product of the irreducible representations R and S. Since we choose our background B to be a Young diagram with M columns and N rows, it is a singlet of $SU(N)$ and consequently we have the product shown in figure 6.

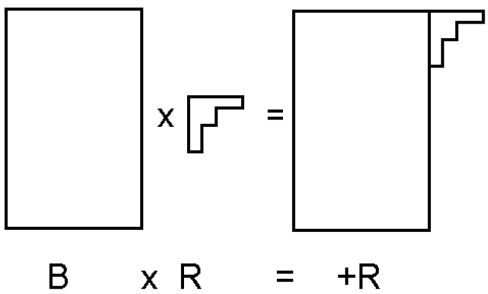


Figure 6: The product rule used to compute correlators in background B. This figure defines $+R$.

Using this product rule, the correlator in the annulus background is

$$\mathcal{A}_B(\{n_i;m_j\},N) \equiv \left\langle \prod_{i,j} \mathrm{Tr}\,(Z^{n_i})\mathrm{Tr}\,((Z^\dagger)^{m_j} \right\rangle_B \quad = \quad \sum_{R,T} \alpha_R\beta_T \frac{\left\langle \chi_B(Z)\chi_R(Z)\chi_B(Z^\dagger)\chi_T(Z^\dagger)\right\rangle}{f_B}$$

$$= \sum_{R,T} \alpha_R \beta_T \frac{\left\langle \chi_{+R}(Z)\chi_{+T}(Z^\dagger)\right\rangle}{f_B}$$

$$= \sum_{R} \alpha_R \beta_R \frac{f_{+R}}{f_B}\,. \tag{16}$$

Recall that f_{+R} is the product of the weights appearing in Young diagram $+R$ and f_B is the product of the weights appearing in Young diagram B. All of the weights appearing in the product f_B are repeated in the product f_{+R} so that after canceling common factors f_{+R}/f_B is simply equal to the product of the weights of the extra boxes stacked to the right of B to form $+R$. Thus, f_{+R}/f_B is obtained from f_R by replacing $N \to N+M$. Comparing (16) and (14) we find

$$\mathcal{A}_B(\{n_i;m_j\},N) = \mathcal{A}(\{n_i;m_j\},M+N). \tag{17}$$

We know that the correlator $\mathcal{A}(\{n_i;m_j\},N)$ admits an expansion in $\frac{1}{N}$; (17) tells us that $\mathcal{A}_B(\{n_i;m_j\},N)$ admits an expansion in $\frac{1}{N+M}$. If we assume that $\sum_i n_i \sim O(1)$, we obtain correlators of operators that are dual to gravitons. Thus, for gravitons in the background B $\frac{1}{N+M}$ clearly plays the role of a loop expansion parameter. *Thus, we learn that there is a reorganization of the $\frac{1}{N}$ expansion for these correlators in the annulus background B and further, that the new expansion parameter is $\frac{1}{N+M}$.* Our relation (17) implies that the only effect of the background is to shift $N \to N+M$.

One can easily check that this relation (17) is not a property of the full theory. Although a slight modification of (17) does allow us to relate the one point functions[8] $\langle \mathrm{Tr}\,(Z^n Z^{\dagger\,n})\rangle$ and $\langle \mathrm{Tr}\,(Z^n Z^{\dagger\,n})\rangle_B$, we have not worked out a relation between amplitudes in general. Our interest in the one point functions $\langle \mathrm{Tr}\,(Z^n Z^{\dagger\,n})\rangle_B$ is that they appear in the intermediate steps of the computation of correlators of BMN-like probes of the annulus so that we

[8]Again computed in the theory with gauge group $U(N+M)$.

manage to obtain a simple relation between the trivial background and the annulus background for both the BPS and near-BPS sectors of the theory.

The amplitudes $\mathcal{A}(\{n_i; m_j\}, M+N)$ are the amplitudes of a theory with a gauge group of rank $N+M$ and no background. In the LLM language, the boundary condition for this geometry is simply a black disk of radius $\sqrt{N+M}$. In our theory (with gauge group of rank N) the background B corresponds to an annulus with inner radius $\sqrt{M}$ and outer radius $\sqrt{M+N}$. Thus, one way to interpret the relation (17) is that *any* of the half-BPS probes considered above are unable to detect the hole in the middle of the annulus. We have not considered probes built using $\frac{d}{dZ}$ instead of Z; these will detect the hole. Indeed, acting with traces of Z on the background $\chi_B(Z)$ produces a new Schur polynomial that has extra boxes stacked adjacent to the upper right hand corner of B - this corner maps to the outer edge of the annulus defining the LLM boundary condition. Acting with $\frac{d}{dZ}$ erodes corners from the lower right corner of B - this corner maps to the inner edge of the annulus. See [72] for further details. Probes built using $\frac{d}{dZ}$ instead of Z are also half-BPS probes.

We have related amplitudes in the gauge theory with a background of M giant gravitons and gauge group $U(N)$ to amplitudes in the gauge theory with trivial background and gauge group $U(N+M)$. This is reminiscent of the infrared duality proposed in [29] which exchanges the rank of the gauge group and the number of giant gravitons. In fact, (17) tells us that half-BPS correlators computed in the $U(N)$ gauge theory in the background of M giant gravitons are exactly equal to the same half-BPS correlators computed in the $U(M)$ gauge theory in the background of N giant gravitons. Since these correlators are extremal and hence not renormalized, our computations give the value of these correlators in the deep infrared limit of the gauge theory and thus seem to provide nontrivial support for the duality proposed in [29]. Note however, that correlators of operators built using $\frac{d}{dZ}$ (which are also extremal) will not agree. This is not obviously in conflict with the proposed duality of [29]. In [29] near extremal black holes are considered. These can be understood as a condensate of giant gravitons [30] represented in $\mathcal{N} = 4$ super Yang-Mills theory by a Schur polynomial corresponding to a triangular Young diagram [9]. The LLM boundary condition for our background is a black annulus and $\frac{d}{dZ}$ correlators explore the inner edge of the annulus. The LLM boundary condition dual to the Schur polynomial with triangular Young diagram label is a grey disk; there is no inner edge. It would be interesting to see how many of our results for the black annulus boundary condition can be generalized to the grey disk boundary condition. This generalization is nontrivial.

The main evidence given in [29] for the proposed duality was an exact correspondence between the gravitational entropy formulae of small and large charge solutions. We can give arguments, in the free field theory, that suggest that the entropy of the state formed by N_g condensed giant gravitons in the theory obtained by taking the near-horizon limit of N D3-branes is equal to the entropy of the state formed by N condensed giant gravitons in the theory obtained by taking the near-horizon limit of N_g D3-branes. This provides further evidence for the duality of [29] in a completely different region of parameter space to that probed by supergravity. According to [23] the near extremal black holes in AdS space can be obtained by attaching open string excitations to the Schur polynomials with triangular Young diagram labels. If we are very close to extremality, the number of open string excitations attached is much smaller compared to the total number of fields in the operator. In this situation instead of attaching open strings by adding boxes to the original triangular label, it should be a good approximation to fix the tableau shape and replace some boxes (in arbitrary places on the Young diagram) with open strings. Assuming that the state obtained by deleting the open strings is again a typical state, i.e. again a triangle we have something like the situation given in figure 7. To compute the entropy associated with these states, we need to count the number of different ways of attaching the open string defects. This is given by the number of different ways we can pull the occupied boxes off the triangular Young diagram. We don't know exactly how to compute this number. However, it is easy to argue that this number is invariant under flipping the Young diagram so that the two states in figure 7 are swapped: this follows from two facts (i) the counting problem is constrained by the rule that we can pull boxes off in any order as long as after each box is pulled off we continue to have a valid Young diagram[9] and (ii) a shape that is (is not) a valid Young diagram before the flip is (is not) a valid Young diagram after the flip. Thus, the entropy of these two states agree.

A few comments about the above flip of the Young diagram are in order. The half-BPS sector of $\mathcal{N}=4$ super Yang-Mills theory can be mapped to the quantum Hall system with filling factor equal to one [34, 35]. In the half-BPS sector the above flip of the Young diagram is a Z_2 symmetry that exchanges particles and holes [34, 35, 36]. If one employs a continuum (field theory) description of the quantum Hall fluid, the resulting fluid is incompressible so that the corresponding continuum Lagrangian has a gauge invariance under area preserving diffeomorphisms [37]. Small fluctuations of the density, for a fluid of charged particles in a background magnetic field, are well described by a Chern-Simons theory; the $U(1)$ gauge invariance is nothing but the

[9]This is required because removing each box must define a valid subduction. See [8, 31, 32, 33].

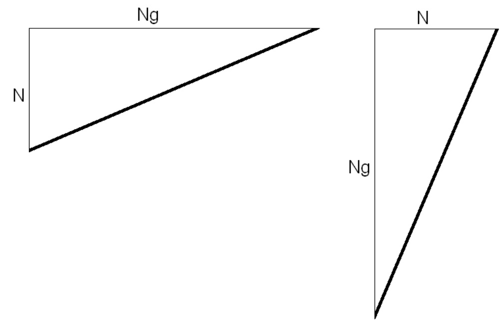


Figure 7: The typical state of N_g condensed giant gravitons in the theory with gauge group $U(N)$ (on the left) and the typical state of N condensed giant gravitons in the theory with gauge group $U(N_g)$ (on the right). The black stripe represents the boxes occupied by open string defects.

area preserving diffeomorphisms for the small fluctuations [37]. The fluid description correctly captures the long distance physics of the quantum Hall effect. It does not capture the fact that, since the fluid is described by N electrons, it has an intrinsic granular structure. One can capture this granular structure by using a Chern-Simons matrix model description [37]. In this Chern-Simons matrix model description the above Z_2 symmetry is nothing but level/rank duality of the Chern-Simons matrix model [38, 39].

One puzzling feature of the duality of [29] is the fact that it mixes giant gravitons and the branes (called background branes in [29]) whose near horizon geometry is the AdS space we work in. There is a big difference between the background branes and the giant gravitons. Indeed, in string theory the background branes carry a net RR-charge; giant gravitons carry no net charge - they are dipoles. In the dual super Yang-Mills theory, changing the number of background branes changes the rank of the gauge group i.e. the number of fields we integrate over when performing a path integral quantization. Changing the number of giant gravitons leaves the rank of the gauge group unchanged, but it does change the background i.e. the integrand we use when performing a path integral quantization. If the duality of [29] is correct, we need to understand how, in this case, changing the integrand has exactly the same effect as changing the number of variables over which we integrate. Computing $\frac{1}{2}$-BPS correlators in the annulus geometry gives a nice toy model in which to explore this issue. This is because we can reduce the whole problem to eigenvalue dynamics in zero dimensions. Consider the computation of the correlator $\left\langle \mathrm{Tr}\,(Z)\mathrm{Tr}\,(Z^\dagger)\right\rangle$ in the trivial vacuum of the $U(N+M)$ theory (Δ is usual the Van der Monde determinant)

$$\int \prod_{i=1}^{M+N} dz_i d\bar{z}_i \Delta(z)\Delta(\bar{z}) \sum_{j=1}^{N+M} z_j \sum_{k=1}^{N+M} \bar{z}_k \, e^{-\sum_{i=1}^{N+M} z_i\bar{z}_i} \,. \tag{18}$$

The same correlator in the $U(N)$ theory with a background of M giant gravitons is

$$\int \prod_{i=1}^{N} dz_i d\bar{z}_i \Delta(z)\Delta(\bar{z}) \prod_{l=1}^{N}(z_l\bar{z}_l)^M \sum_{j=1}^{N} z_j \sum_{k=1}^{N} \bar{z}_k \, e^{-\sum_{i=1}^{N} z_i\bar{z}_i} \,. \tag{19}$$

We already know that (18) and (19) give the same result. The reason why the two agree is now evident: in (18) we integrate over an extra M variables; these extra contributions add to give a larger result than that obtained for the same correlator in the $U(N)$ theory. In (19) the extra factor in the integrand implies that the integrand now peaks at larger values for the $|z_l|$; this again gives a larger result than that obtained for the same correlator in the $U(N)$ theory in the trivial vacuum. Our computation gives a simple picture of how, in this case, changing the integrand has exactly the same effect as changing the number of variables over which we integrate. It also suggests that the duality of [29] might be derived by starting with a suitable $U(N+N_g)$ theory and (i) integrating out the degrees of freedom associated with N_g colors to get a $U(N)$ gauge theory with a background given by a condensate of N_g giant gravitons or (ii) integrating out the degrees of freedom associated with N colors to get a $U(N_g)$ gauge theory with a background given by a condensate of N giant gravitons.

3.2.2 Supergravity Amplitudes

According to the AdS/CFT correspondence, correlation functions can be computed in the strong coupling limit of $\mathcal{N} = 4$ super Yang-Mills theory using the dual hologram, which is type IIB supergravity. The paper [40], has given a powerful general approach to holography in the LLM backgrounds, generalizing and extending the Coulomb branch analysis of [41, 42]. The formalism given in [40] is an application of the general method of [43] which employs the method of holographic renormalization [44]. An important result of [40] was the demonstration that the asymptotics of the LLM solutions correctly encode the vacuum expectation values of all single trace $\frac{1}{2}$ BPS operators to the leading order in the large N expansion. *Can we reproduce the results of the last section using holography in the LLM background?*

One case of interest to us in this section is that of graviton three point functions, in the background created by a heavy operator. We have computed these correlation functions in the free field theory; supergravity will reproduce these correlators in the strong coupling limit. We will now argue that it is natural to expect that these two results will agree since we can argue that there is a non-renormalization theorem protecting the three point functions of interest to us. The logic of this argument is very similar to the argument of [40] which argued that the generic one point function in the LLM background is protected. The three point functions we are interested in

$$\Big\langle \mathrm{Tr}(Z^n)\mathrm{Tr}(Z^m)\mathrm{Tr}(Z^{\dagger\, m+n})\Big\rangle_B$$

can of course, also be understood as a ratio of (higher point) correlators in the original trivial background

$$\frac{\Big\langle \chi_B(Z)\chi_B(Z^\dagger)\mathrm{Tr}(Z^n)\mathrm{Tr}(Z^m)\mathrm{Tr}(Z^{\dagger\, m+n})\Big\rangle}{\langle \chi_B(Z)\chi_B(Z^\dagger)\rangle}.$$

The correlators appearing in the above expressions are extremal correlators in the language of [45]. The computations of [45] show that at the leading order in large N, the extremal correlators take the same value at large 't Hooft coupling as in the free field theory and hence it is natural to expect that all extremal correlators are not renormalized[10].

To extract three point functions using the methods of [40], we would need to solve the equations of motion, to quadratic order, around the LLM solution. We have not managed to do this. For the case of extremal correlators, there is some room for optimism: the analysis of [45] argued that for extremal correlators, the bulk cubic supergravity coupling vanishes and the entire contribution comes from a boundary term. The LLM geometries are asymptotically AdS so that one might have hoped that there was a simple explanation of the results of the previous section. We have not found one. Note also that the fact that the bulk cubic supergravity coupling vanishes in the AdS$_5$ background need not imply that it continues to vanish in the LLM background.

Rather than pursuing the computation of three (and higher) point functions directly, we have found it simpler to relate the three point functions

[10] Note that the supergravity result suggests that the planar contribution is not renormalized. Here we are using the stronger conjecture [45] which claims the non-renormalization for any N. Our results provide further evidence for this stronger conjecture.

we'd like to compute to one point functions, since these have already been obtained in [40]. For concreteness, consider the three point function (B is again the annulus diagram)

$$\Big\langle \mathrm{Tr}(Z^2)\mathrm{Tr}(Z^2)\mathrm{Tr}(Z^{\dagger\,4})\Big\rangle_B$$

which can also be written as the one point function of $\mathrm{Tr}(Z^2)$ in the normalized state

$$|\Phi\rangle = \mathcal{N}(\mathrm{Tr}\,(Z^2) + \mathrm{Tr}\,(Z^4))\chi_B(Z)|0\rangle\,.$$

$\mathcal{N}$ is a normalization factor. Now, summing only planar diagrams

$$\Big\langle \mathrm{Tr}(Z^2)\mathrm{Tr}(Z^2)\mathrm{Tr}(Z^{\dagger\,4})\Big\rangle = 16N^3$$

so that according to our result of the previous section

$$\Big\langle \mathrm{Tr}(Z^2)\mathrm{Tr}(Z^2)\mathrm{Tr}(Z^{\dagger\,4})\Big\rangle_B = 16(N+M)^3\,.$$

The supergravity one point function is computed with a normalized state $|\Phi\rangle$; the above three point function is computed with a different normalization. Noting that (we are again using results from the previous section)[11]

$$\Big\langle (\mathrm{Tr}\,(Z^2) + \mathrm{Tr}\,(Z^4))(\mathrm{Tr}\,(Z^{\dagger\,2}) + \mathrm{Tr}\,(Z^{\dagger\,4}))\chi_B(Z)\chi_B(Z^\dagger)\Big\rangle$$

$$= \Big[4(N+M)^4 + 22(N+M)^2\Big]\,f_B = 4(N+M)^4 f_B\,\Big(1 + O((N+M)^{-2})\Big)\,,$$

we easily find

$$\langle\Phi|\mathrm{Tr}(Z^2)|\Phi\rangle = \frac{4}{N+M}\,.$$

[11]Note that since $|\Phi\rangle$ has two additive components the one-point function has additional cross terms; these however evaluate to zero leaving only the term we wish to compute.

This is the result we would like to reproduce from supergravity.

We will summarize the main steps involved in extracting correlators of local operators in the boundary gauge theory, from the ten dimensional asymptotically $\mathrm{AdS}_5\times\mathrm{S}^5$ solutions of Type IIB supergravity. For all the details and further references consult [40]. Given an asymptotically $\mathrm{AdS}_5\times\mathrm{S}^5$ solution of Type IIB supergravity, one must first systematically reduce the ten dimensional solution to a solution of five dimensional gravity coupled to an infinite number of five dimensional fields. If this reduction is performed in terms of gauge invariant variables (as it is in [40]), it does not matter in which gauge the original solution is supplied. This reduction is then supplemented with a non-linear field redefinition, performed in such a way that the equations of motion in terms of the redefined fields can be obtained by minimizing a local five dimensional action. One can then apply the usual holographic rules to compute the correlation function of boundary operators from the bulk five dimensional description. To obtain the renormalized correlation functions in the gauge theory one must appropriately renormalize the bulk gravitational action using holographic renormalization. The supergravity field that couples to the boundary operator that we are interested in ($\mathrm{Tr}(Z^2)$) is a mixture of (components of the) metric on the S^5 and the four form RR potential on the S^5. It is denoted by S^{22} in [40]. The one point function we want can now be obtained by variation of the renormalized on shell supergravity action with respect to the boundary value of S^{22}. The result is [40][12]

$$\langle \mathcal{O}_{S^{22}} \rangle = N^2 \int r^3 \rho\, e^{2i\phi} dr d\phi \,,$$

[12]This differs by a factor of $\sqrt{2}\pi^2$ from formula (3.60) of [40] because we are using a different normalization. Further, since we have dropped all spacetime dependence, we have also dropped a factor of e^{2it}.

where ρ is the boundary condition for the LLM geometry. This boundary condition is most easily determined by exploiting the free fermion description of the half-BPS sector of $\mathcal{N}=4$ super Yang-Mills theory [5, 6]. To make the transition to the free fermion description, rewrite the state $|\Phi\rangle$ in terms of Schur polynomials. This is easily accomplished using

$$\mathrm{Tr}(Z^2) = \chi_{\yng(2)}(Z) - \chi_{\yng(1,1)}(Z)\,,$$

$$\mathrm{Tr}(Z^4) = \chi_{\yng(4)}(Z) - \chi_{\yng(3,1)}(Z) + \chi_{\yng(2,1,1)}(Z) - \chi_{\yng(1,1,1,1)}(Z)\,,$$

and the product rule for Schur polynomials. The Schur polynomials correspond to energy eigenkets of the free fermions. The energies of the free fermions in the state are

$$E_i = \lambda_i + N - i + 1 \qquad i = 1, ..., N,$$

where λ_i is the number of boxes in the ith row of the Young diagram label for the Schur polynomial. Using this interpretation it is straightforward to obtain the fermion phase space density. The density we obtain in this way is

$$\rho = \rho_1 + \cos(2\phi)\rho_2\,,$$

where

$$\rho_2 = \frac{1}{2\pi(N+M)^4}\frac{e^{-Nr^2}}{(N+M-1)!}\Big[(Nr^2)^{N+M+2} + (N+M-1)(Nr^2)^{N+M+1}$$
$$-(N+M)(N+M+1)(N+M-1)(N+M-2)(Nr^2)^{N+M-2}$$
$$-(N+M)(N+M-1)^2(N+M-2)(N+M-3)(Nr^2)^{N+M-3}\Big]\,.$$

We will not need ρ_1 in what follows. As already noted in [40], the state $|\Phi\rangle$ which is a superposition of a small number (for us, 6) of Schur polynomials, does not give rise to a smooth supergravity solution. Indeed, for a regular supergravity solution, $\pi\rho$ should only take the values $\{0, 1\}$; this is not the case for the above ρ. It is now straightforward to check that

$$\langle \mathcal{O}_{S^{22}} \rangle = \frac{4}{N+M}$$

which is the result we wanted to demonstrate.

3.2.3 Multi Ring Backgrounds

Using the results of [46, 72] we can generalize our results for the annulus to a general multi ring LLM geometry. The Young diagrams of these geometries are not a rectangle, but rather are shapes with more than 4 edges and all edges have a length of order N boxes. In a geometry with m thick rings, local graviton operators were defined in [72]. When taking a product of the Young diagram of the background (B_{ring}) with the Young diagram of the probe (R), one removes boxes from the probe Young diagram and adds them, at all possible positions, to B_{ring} (respecting the usual rules for multiplying Young diagrams). The local graviton operators are defined by giving their product rule: the boxes removed from R can only be added at a specific location on B_{ring}. This definition is precise enough to allow the computation of correlators; see [72] for more details. We can also give a rough constructive definition of the local graviton operators: since we consider a boundary condition with m thick rings, the eigenvalue distribution in the dual matrix model will split into m well separated clumps. In the limit that we expect a classical geometry to emerge (large N and large 't Hooft coupling) the off diagonal modes connecting these m subsectors will be very heavy and decouple. We expect that, when studying almost BPS states, the effect of these modes on the dynamics can be neglected. There is no reason to neglect off diagonal modes connecting eigenvalues in the same sector. Z can thus be replaced by a block diagonal matrix with m blocks. If clump i contains N_i eigenvalues it corresponds to an $N_i \times N_i$ block. To construct a local graviton operator, do not use the full matrix Z, but rather use one of the blocks Z_i.

It is now straight forward to verify that, if ring i has outer radius $\sqrt{N+M_i}$ then we again get an exact relation

$$\mathcal{A}_B(\{n_i; m_j\}, N) = \mathcal{A}(\{n_i; m_j\}, M_i + N), \tag{20}$$

for correlators $\mathcal{A}_B(\{n_i; m_j\}, N)$ of gravitons localized at edge i. Thus, the gravitons at the edge of each ring have their own expansion parameter equal to $\frac{1}{N+M_i}$.

Finally, consider a background obtained by exciting k_2 sphere giants; each giant is assumed to carry a momentum $k_3 < N$. The corresponding LLM geometry will be a black disk (of area $N - k_3$) surrounded by a white annulus of area k_2 which is itself surrounded by a black annulus of area k_3 (this boundary condition is shown in figure 3 of [35]). From the results of this section we know that excitations of the central black droplet could be described using either a $U(N)$ theory with a background of k_2 sphere giants or using a $U(N - k_3)$ theory. This is closely related to remarks made in [35], arrived at using a totally different approach.

3.2.4 Backgrounds with > 1 Charges

In this section we would like to study the zero coupling limit of backgrounds made from two matrices Z and Y. There are a number of bases which we can employ for this study. The basis described in [47] builds operators with definite quantum numbers for any desired global symmetry groups acting on the matrices. The basis of [48] uses the Brauer algebra to build correlators involving Z and $Z^\dagger$; this basis seems to be the most natural for exploring brane/anti-brane systems. The basis of [49] most directly allows one to consider open string excitations [8, 17, 31, 32, 33] of the operator; this is the restricted Schur basis. These three bases do not coincide and a detailed link between them has been discussed in [50]. Casimirs that distinguish these bases have been given in [21]. Finding a spacetime interpretation of these Casimirs is a promising approach towards understanding the dual description of these bases. Although all three bases diagonalize the two point functions in the free field theory limit, computing anomalous dimensions is not yet possible[13]. In particular, we do not know how to extract the multi-matrix BPS operators from these bases.

In [53] a particularly simple set of restricted Schur polynomials were identified; these are the operators that we will focus on in this section. The operator is built using NM_1 Z fields and NM_2 Y fields. The restricted Schur polynomial has two labels $\chi_{R,(r_1,r_2)}$. R is an irreducible representation of $S_{N(M_1+M_2)}$. We take R to be a Young diagram with $M_1 + M_2$ columns and N rows. (r_1, r_2) is an irreducible representation of the $S_{NM_1} \times S_{NM_2}$ subgroup. We take r_1 to be a Young diagram with M_1 columns and N rows and r_2 to

[13]For some progress see [32, 33, 51]. In particular, these articles show that at one loop, operators can only mix if their labels differ by the location of a single box; thus, it seems that the operator mixing should have a nice simple description. This is proved in the restricted Schur basis in [32, 33] and in the basis of [47] in [51].

be a Young diagram with M_2 columns and N rows. We take both M_1 and M_2 to be $O(N)$. For this particular restricted Schur polynomial we have

$$\chi_{R,(r_1,r_2)}(Z,Y) = \frac{\text{hooks}_R}{\text{hooks}_{r_1}\,\text{hooks}_{r_2}}\chi_{r_1}(Z)\chi_{r_2}(Y)\,,$$

where $\chi_{r_1}(Z)$ and $\chi_{r_2}(Y)$ are standard Schur polynomials. The simplicity of this background is a consequence of the fact that the above operator factorizes. These operators are not BPS or even near-BPS. Indeed, in appendix J we compute the anomalous dimension of these operators, which is given by

$$\Delta = N(M_1+M_2) + 4\lambda M_1 M_2\,, \qquad \lambda = N g_{YM}^2.$$

Thus, the results of this section can not be extrapolated to strong coupling. It is still interesting to ask if, at weak coupling, we can reorganize the usual $\frac{1}{N}$ expansion.

After normalizing so that the identity has an expectation value of 1, we find

$$\langle O\rangle_{(r_1,r_2)} = \frac{\left\langle \chi_{r_1}(Z)\chi_{r_1}(Z^\dagger)\chi_{r_2}(Y)\chi_{r_2}(Y^\dagger)O\right\rangle}{\langle \chi_{r_1}(Z)\chi_{r_1}(Z^\dagger)\chi_{r_2}(Y)\chi_{r_2}(Y^\dagger)\rangle}\,.$$

Using exactly the same methods as were used in section 3.2.1, we find that the complete effect of the background on correlators of operators built using only Zs with operators built using only $Z^\dagger$s is to replace $N \to N+M_1$. In addition, we find that the complete effect of the background on correlators of operators built using only Ys with operators built using only $Y^\dagger$s is to replace $N \to N+M_2$. Thus, these two types of observables admit a different reorganization of the $1/N$ expansion; the small parameter of the two expansions is not the same.

3.2.5 A Generalization of the BMN Limit for Two Charge Backgrounds

The AdS/CFT correspondence forces us to take the limit of large 't Hooft coupling λ, if we are to suppress curvature corrections in the string theory. It is thus a strong/weak coupling duality. This naive observation is evaded by the remarkable result of BMN [12]: the expansion parameter of the super Yang-Mills theory may be suppressed by large quantum numbers for "almost BPS operators" [26]. The existence of double scaling BMN-like limits provides a very rich class of tractable problems. In this section we would like to explore the proposed BMN-like sectors discovered in [24][14]. The work [24] itself was an extension of [23]. The existence of these limits may be very relevant to understanding the gauge theory description of near-extremal charged black holes in AdS_5. Using the technology we have developed, it should be possible to study these limits.

We use the same two-charge background that we used in the previous subsection. We will however take $M_1 = O(\sqrt{N})$ and $M_2 = O(\sqrt{N})$ with the 't Hooft coupling λ large but fixed[15]. In this case, the charges $J_1 = NM_1 \sim O(N^{\frac{3}{2}})$ and $J_2 = NM_1 \sim O(N^{\frac{3}{2}})$, whilst the difference

$$\Delta - J_1 - J_2 = 4\lambda M_1 M_2 \sim N$$

so that[16]

$$\eta \equiv \frac{\Delta - J_1 - J_2}{J_1 + J_2} \sim N^{-\frac{1}{2}} \to 0\,.$$

[14]We thank Shahin Sheikh-Jabbari for extremely useful correspondence, which lead to the results of this subsection.

[15]This is not quite the same limit as that described in [24]; in [24] the limit with g_{YM}^2 fixed is considered.

[16]We kept the 't Hooft coupling fixed in order to obtain this scaling for η, which matches precisely what one obtains in the BMN limit.

Our background (plus open string excitations which do not modify this scaling) is a near BPS operator, in the language of [26]. These results are in perfect agreement with the supergravity arguments of [24] - the strong and weak coupling results match. The operator we have obtained is a "near $\frac{1}{4}$ BPS" state. The fact that the anomalous dimension is proportional to $M_1 M_2$ strongly suggests that it arises from the dynamics of the open strings stretching from the stack of M_1 giants (described by $\chi_{r_1}(Z)$) to the stack of M_2 giants (described by $\chi_{r_2}(Y)$).

Let us now recall a few well known facts. 't Hooft [52] has already made the beautiful observation that the perturbative expansion of a matrix model can be written in the form

$$\sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda)\,, \tag{21}$$

where $f_g(\lambda)$ is a polynomial in the 't Hooft coupling - there is no additional N dependence apart from the multiplying factor of N^{2-2g}. Here, $\frac{1}{N}$ plays the role of a genus counting parameter. In the BMN set up, one learnt that there is a new effective genus counting parameter $g_2 = \frac{J^2}{N}$ replacing $\frac{1}{N}$, by studying the two point functions of BMN loops (each has angular momentum J). In terms of this effective genus counting parameter, the effective 't Hooft coupling is $g_{YM}^2/g_2 = g_{YM}^2 N/J^2$. Can we repeat this analysis for the operators considered here? Towards this end, we set $M_1 = M_2 = M$ and compute the two point function

$$I_2 = \left\langle \chi_{r_1}(Z)^\dagger \chi_{r_2}(Y)^\dagger \chi_{r_1}(Z)\chi_{r_2}(Y)\right\rangle = \left(\frac{G_2(N+M+1)}{G_2(N+1)G_2(M+1)}\right)^2 ,$$

where $G_2(n+1)$ is the Barnes function defined by ($\Gamma(z)$ is the Gamma function)

$$G_2(z+1) = \Gamma(z) G_2(z)\,, \qquad G_2(n+1) = \prod_{k=1}^{n-1} k!\,.$$

We will now set $N = \alpha M^2$ where α is a number of $O(1)$ in the large N limit. The Barnes function has the asymptotic expansion

$$\log G_2(N{+}1) = \frac{N^2}{2}\log(N) - \frac{1}{12}\log(N) - \frac{3}{4}N^2 + \frac{N}{2}\log(2\pi) + \zeta'(-1) + \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)N^{2g-2}}\,,$$

where B_{2g} are the Bernoulli numbers and $\zeta'(-1)$ is the derivative of the Riemann zeta function evaluated at -1. Using this asymptotic expansion, it is clear that I_2 can be split into a product $F_{\mathrm{non-pert}}F_{\mathrm{pert}}$, where $F_{\mathrm{non-pert}}$ is a non-perturbative piece that can not be expanded in $\frac{1}{M}$ and F_{pert} is a factor that does admit an expansion in $\frac{1}{M}$[17]. This suggests that we should identify $g_2 = \frac{1}{M} \sim \frac{1}{\sqrt{N}}$. In the BMN case only even powers of g_2 appear in the genus expansion; here we can not be sure if only even powers of g_2 appear. This depends on precisely how we factorize I_2 into $F_{\mathrm{non-pert}}F_{\mathrm{pert}}$. Using this genus expansion, we would expect an effective 't Hooft coupling[18]

$$\tilde{\lambda} = g_{YM}^2 M = \frac{1}{\sqrt{\alpha}} g_{YM}^2 \sqrt{N}.$$

Keeping this effective 't Hooft coupling fixed but arbitrarily small, one finds

$$\Delta = 2NM\left(1 + 2\tilde{\lambda}\right) = 2\alpha M^3\left(1 + 2\tilde{\lambda}\right)\,.$$

This looks like a polynomial is $\tilde{\lambda}$ times some power of M. Clearly, in view of (21), $\tilde{\lambda}$ is indeed the correct 't Hooft coupling. To properly specify our state of

[17] This factorization is not unique; it will be fixed by the physics of the problem. We do not yet know how to do this.

[18] With this scaling, which is different to what we had above, η does not scale with N, but can be made arbitrarily small.

intersecting giants, we need to specify the three Young diagrams which would label the restricted Schur polynomial of the giant system. Clearly, with our simple choice it seems that we have correctly obtained the correct Yang-Mills operator to describe the decoupling limits of [23, 24] to one loop. It would be interesting to compute higher loop corrections and verify that one does indeed continue to obtain a polynomial in $\tilde{\lambda}$ times M^3. If extra M dependence does appear, it might be a sign that our simple guess for the background needs to be corrected at $O(\tilde{\lambda})$. This is not completely unexpected: these operators are not protected so one would expect corrections in general. Finally, holding $\tilde{\lambda}$ fixed we find the 't Hooft coupling $\lambda = g_{YM}^2 N = \tilde{\lambda}\sqrt{\alpha N}$ goes to infinity. Thus, even though we have large λ the one loop correction can be made arbitrarily small. We are thus optimistic that, similar to what happens in the BMN limit, it will be possible to compare perturbative field theory results to results obtained from the dual gravitational description.

The genus counting parameter $g_2 = \frac{1}{M}$ and the effective 't Hooft coupling $\tilde{\lambda} = g_{YM}^2 M$ are exactly the parameters we would expect for the description of the large M limit of a $U(M)$ gauge theory. This is rather natural: we have on the order of M giant gravitons whose worldvolume theory at low energy will be a $U(M)$ gauge theory. This provides strong field theory evidence that the sector of the theory identified in [23, 24] is indeed captured by the dynamics of open string defects distributed on a boundstate of intersecting giant gravitons. Its natural to think that the decoupling limits of [23, 24] capture the decoupled low energy world volume theory of the intersecting giant gravitons in the same way that Maldacena's limit [1] captures the decoupled low energy world volume theory of N D3 branes. Our explicit computations show that the low energy world volume theory of the giant gravitons is weakly coupled even when the original $\mathcal{N} = 4$ super Yang-Mills theory is strongly coupled.

3.3 Beyond the Half-BPS Sector

In this section we would like to determine the sector of the theory in which our reorganization of perturbation theory is valid. First, the near BPS operators we are interested in are BMN-like loops

$$\mathcal{O}(\{n\}) = \mathrm{Tr}\,(YZ^{n_1}YZ^{n_2}YZ^{n_3}Y\cdots YZ^{n_L}).$$

To compute the two point correlator $\left\langle \mathcal{O}(\{n\})\mathcal{O}^\dagger(\{n\})\right\rangle_B$ we start by contracting the Y fields planarly. After performing the Y contractions we need to compute

$$\left\langle \prod_i \mathrm{Tr}\,(Z^{n_i}Z^{\dagger\, n_i})\right\rangle_B . \tag{22}$$

We need to verify that the above amplitude admits an expansion with parameter $\frac{1}{(N+M)}$ to demonstrate that our reorganization of perturbation theory does indeed apply to the half-BPS and almost BPS sectors.

We will demonstrate that a relation very similar to (17) holds for the leading contribution to the one point functions (22). We will then derive an exact relation. The first property we use is factorization (valid to leading order and if $\sum_i n_i \sim O(1)$)

$$\left\langle \prod_i \mathrm{Tr}\,(Z^{n_i}Z^{\dagger\, n_i})\right\rangle_B = \prod_i \left\langle \mathrm{Tr}\,(Z^{n_i}Z^{\dagger\, n_i})\right\rangle_B .$$

Factorization in the annulus background has been discussed in [72] and in appendix H. Thus, we need only consider

$$\left\langle \mathrm{Tr}\,(Z^{n}Z^{\dagger\, n})\right\rangle_B = \left\langle \mathrm{Tr}\,(Z^{n}Y)\mathrm{Tr}\,(Z^{\dagger\, n}Y^\dagger)\right\rangle_B .$$

This computation can be completed in exactly the same way as the computation we tackled in section 3.2.1. We start by writing the loop of interest in terms of restricted Schur polynomials

$$\mathcal{O}(n) = \mathrm{Tr}\,(Z^n Y) = \sum_{(R,R')} \alpha_{(R,R')}\chi_{(R,R')}(Z,Y). \tag{23}$$

R is a Young diagram with $n+1$ boxes and R' a Young diagram with n boxes. The expansion coefficients $\alpha_{R,R'}$ are independent of N. The two point function of restricted Schur polynomials has been computed in [31, 49]. Using these results, we find

$$\mathcal{C}(n,N) = \left\langle \mathcal{O}(n)\mathcal{O}^\dagger(n) \right\rangle = \sum_{(R,R')} \alpha^2_{R,R'} \frac{\mathrm{hooks}_R}{\mathrm{hooks}_{R'}} f_R \, .$$

The factor $\mathrm{hooks}_R/\mathrm{hooks}_{R'}$ does not depend on N. To compute these amplitudes in the annulus background, we need the analog of the product rule given in figure 6. The product rule for restricted Schur polynomials [53] was used in [46] to compute precisely the restricted Littlewood-Richardson number needed here. Again, only a single term enters in the product

$$\chi_B(Z)\chi_{(R,R')}(Z,Y) = g_{B\,(R,R')\,(+R,+R')}\;\chi_{(+R,+R')}(Z,Y).$$

$+R'$ is obtained from $+R$ by dropping a single box. This restricted Littlewood-Richardson number was computed in [46] from its definition; this involves a summation over restricted characters. There is however, a much simpler way to obtain this result. Once one has established the form $\chi_B(Z)\chi_{(R,R')}(Z,Y) =$

$g_{B\,(R,R')\,(+R,+R')}\ \chi_{(+R,+R')}(Z,Y)$, reducing both sides with respect to Y gives[19]

$$c_{R,R'}\chi_B(Z)\chi_{R'}(Z) = c_{R,R'}\chi_{+R'}(Z) = c_{+R,+R'}g_{B\,(R,R')\,(+R,+R')}\ \chi_{+R'}(Z),$$

where we have used the product rule of section 3.2.1: $\chi_B(Z)\chi_{R'}(Z) = \chi_{+R'}(Z)$ and where $c_{T,T'}$ is the weight of the box that must be dropped from T to obtain T'. Thus,

$$g_{B\,(R,R')\,(+R,+R')} = \frac{c_{R,R'}}{c_{+R,+R'}}.$$

This formula is exact. Although it is possible to compute things exactly, we will also make good use of the leading large $N+M$ version of our results. For this reason we start with a large $N+M$ analysis and then give exact results.

Large N Analysis: To leading order in large $N+M$ we have $c_{R,R'} = N$ and $c_{+R,+R'} = N+M$ which reproduces the known result [46]. It is now straight forward to find

$$\begin{aligned}
\mathcal{C}_B(n,N) &= \left\langle \mathcal{O}(n)\mathcal{O}^\dagger(n)\right\rangle_B \\
&= \sum_{(R,R')} \alpha^2_{R,R'}\left(\frac{N}{N+M}\right)^2 \frac{\text{hooks}_{+R}}{\text{hooks}_{+R'}}\frac{f_{+R}}{f_B} \\
&= \left(\frac{N}{N+M}\right)\sum_{(R,R')} \alpha^2_{R,R'}\frac{\text{hooks}_R}{\text{hooks}_{R'}}\frac{f_{+R}}{f_B}\,. \qquad (24)
\end{aligned}$$

Arguing exactly as we did in section 3.2.1 we find

$$\mathcal{C}_B(n,N) = \left(\frac{N}{N+M}\right)\mathcal{C}(n,N+M)\,. \qquad (25)$$

This is very similar to, but not quite the same as (17). Note also that (25)

[19] To reduce with respect to Y we take the derivative $\text{Tr}\,\frac{d}{dY}$. A formula for the reduction of Schur polynomials is given in [20]; a formula for the reduction of restricted Schur polynomials is given in [31].

was only derived at leading order; (17) is however exact. Some insight into these relations can be obtained by noticing that at leading order in N we have

$$\left\langle \mathrm{Tr}\,(Z^n)\mathrm{Tr}\,(Z^{\dagger\, n})\right\rangle = N^n, \qquad \left\langle \mathrm{Tr}\,(Z^n Z^{\dagger\, n})\right\rangle = N^{n+1}.$$

Thus, as long as we restrict ourselves to the leading order, (17) and (25) can be restated as follows: consider an operator $O(Z,Z^\dagger)$ which is a product of factors of the form (13) or (23). Then

$$\left\langle O(Z,Z^\dagger)\right\rangle_B = \left\langle O\left(\sqrt{\frac{N+M}{N}}Z, \sqrt{\frac{N+M}{N}}Z^\dagger\right)\right\rangle . \tag{26}$$

At leading order, the only effect of the background is to rescale Z. This observation will be useful below.

Exact Analysis: The box $c_{RR'}$ has a weight $N+p$ where p is a number of $O(1)$. Then

$$g_{B\,(R,R')\,(+R,+R')} = \frac{N+p}{N+M+p}, \qquad \frac{\mathrm{hooks}_{+R}}{\mathrm{hooks}_{+R'}} = \frac{N+M+p}{N+p}\frac{\mathrm{hooks}_R}{\mathrm{hooks}_{R'}}\,,$$

so that

$$\left\langle \mathrm{Tr}\,(Z^nY)\mathrm{Tr}\,(Z^{\dagger\, n}Y^\dagger)\right\rangle_B = \sum_{(R,R')}\frac{N+p}{N+M+p}\frac{\mathrm{hooks}_R}{\mathrm{hooks}_{R'}}\frac{1}{(n+1)^2}\frac{f_{+R}}{f_B}$$

$$= \sum_{(R,R')}\frac{\mathrm{hooks}_R}{\mathrm{hooks}_{R'}}\frac{1}{(n+1)^2}\frac{f_{+R}}{f_B} - M\sum_{(R,R')}\frac{1}{N+M+p}\frac{\mathrm{hooks}_R}{\mathrm{hooks}_{R'}}\frac{1}{(n+1)^2}\frac{f_{+R}}{f_B}$$

$$= \sum_{(R,R')}\frac{\mathrm{hooks}_R}{\mathrm{hooks}_{R'}}\frac{1}{(n+1)^2}\frac{f_{+R}}{f_B} - M\sum_{(R,R')}\frac{\mathrm{hooks}_R}{\mathrm{hooks}_{R'}}\frac{1}{(n+1)^2}\frac{f_{+R'}}{f_B}\,.$$

In the above sums, R runs over all hook Young diagrams with $n+1$ boxes (diagrams with at most one column containing more than one box; see [46]) and R' runs over all possible ways of removing a box from the hook (there are usually 2 possible ways). The first term in this last expression is nothing but $\mathcal{C}(n, N+M)$. Now, it is a simple matter to verify that

$$\sum_R \frac{\text{hooks}_R}{\text{hooks}_{R'}} \frac{1}{(n+1)^2} = \sum_R \frac{d_{R'}}{d_R} \frac{1}{n+1} = \frac{1}{n+1} \frac{n+1}{n} = \frac{1}{n}. \tag{27}$$

The first equality follows from the formula for an irreducible representation of S_n: $d_T = \frac{n!}{\text{hooks}_T}$; the second formula follows upon inserting the explicit expression for the dimensions of the hook Young diagrams R and R'. Thus, the coefficient of $f_{+R'}/f_B$ is $1/n$. Next consider

$$\mathcal{C}(n-1, N+M) = \sum_{(R',R'')} \frac{\text{hooks}_{R'}}{\text{hooks}_{R''}} \frac{1}{n^2} \frac{f_{+R'}}{f_B} = \sum_{(R',R'')} \frac{d_{R''}}{d_{R'}} \frac{1}{n} \frac{f_{+R'}}{f_B}. \tag{28}$$

In the above sum, R' runs over all hook Young diagrams with n boxes and R'' runs over all possible ways of removing a box from this hook (there are 2 possible ways). The coefficient of $f_{+R'}/f_B$ in this last sum is

$$\sum_{R''} \frac{d_{R''}}{d_{R'}} \frac{1}{n} = \frac{1}{n}.$$

This follows because we sum over all possible subductions R'' of R'. This proves that (27) and (28) are identical and hence we find the exact relation

$$\mathcal{C}_B(n, N) = \mathcal{C}(n, N+M) - M\mathcal{C}(n-1, N+M). \tag{29}$$

We have explicitly checked that it holds for $n = 1, 2, 3, 4, 5, 6$. $\mathcal{C}(n, N+M)$ admits an expansions in $\frac{1}{M+N}$; thus, we have demonstrated that the amplitudes $\mathcal{C}_B(n, N)$ admit an expansion in $\frac{1}{M+N}$, albeit with the above extra M dependence in the expansion. M is the number of maximal giant gravitons making up the background; it is not surprising that there are amplitudes that have some additional dependence on M. The point is that all N dependence in our amplitude has, according to (29), been replaced by an $N+M$ dependence which is perfectly consistent with what we found in the half-BPS sector.

Given the above success one may ask if we can consider more general correlators and to work beyond the leading order. This more general analysis would tell us the class of operators for which our reorganization works, that is, the full class of operators that can be expanded in $\frac{1}{M+N}$. Recall that the Schwinger-Dyson equations of a theory determine the correlation functions of the theory. Thus, one possible approach to our problem (now that the above analysis has suggested what to search for), is to demonstrate the replacement $N \to M+N$ at the level of the Schwinger-Dyson equations. This demonstration turns out to be straightforward. Start in the trivial background; consider first the simple Schwinger-Dyson equation[20]

$$0 = \int \left[dZdZ^\dagger\right] \frac{d}{dZ_{ij}} \left((Z^{n+1}Z^{\dagger\, n})_{ij} e^{-S}\right) .$$

Performing the derivative we find

$$\left\langle \mathrm{Tr}\,(Z^{n+1}Z^{\dagger\, n+1})\right\rangle = N\left\langle \mathrm{Tr}\,(Z^{n}Z^{\dagger\, n})\right\rangle + \sum_{r=1}^{n}\left\langle \mathrm{Tr}\,(Z^{r})\mathrm{Tr}\,(Z^{n-r}Z^{\dagger\, n})\right\rangle .$$

An easy computation (see appendix H) shows that, in the background B, the above Schwinger-Dyson equation becomes

$$\left\langle \mathrm{Tr}\,(Z^{n+1}Z^{\dagger\, n+1})\right\rangle_B = (N+M)\left\langle \mathrm{Tr}\,(Z^{n}Z^{\dagger\, n})\right\rangle_B + \sum_{r=1}^{n}\left\langle \mathrm{Tr}\,(Z^{r})\mathrm{Tr}\,(Z^{n-r}Z^{\dagger\, n})\right\rangle_B .$$

[20]Since we compute correlators using (12), to obtain Schwinger-Dyson equations that determine our correlators we must consider a zero dimensional matrix model with action $S = \mathrm{Tr}\,(ZZ^\dagger)$.

Clearly, the net effect of the background is to replace $N \to N+M$ in perfect harmony with (17). This conclusion is rather general: the net effect of the background, at the level of the Schwinger-Dyson equations, is the replacement $N \to N+M$ suggesting that (17) should hold for all correlators of operators built using only Z and $Z^\dagger$. This conclusion is too quick. To determine the correlators, we should imagine solving these equations iteratively; the value of $\left\langle \mathrm{Tr}\,(Z^{n+1}Z^{\dagger\,n+1})\right\rangle_B$ will thus depend on $\left\langle \mathrm{Tr}\,(Z^n Z^{\dagger\,n})\right\rangle_B$. We should start the process with the equation that follows for $n=0$

$$\left\langle \mathrm{Tr}\,(ZZ^\dagger)\right\rangle_B = (N+M)N\,.$$

The N multiplying $(M+N)$ on the right hand side comes from $\mathrm{Tr}\,(1)$. This introduces an N dependence, which is not replaced by $M+N$. This result is in perfect agreement with (25). By carefully analyzing the Schwinger-Dyson equations for the half-BPS loops, one can verify that a similar problem does not occur for these loops, which is consistent with (17). The above departure from a pure $M+N$ dependence is rather mild and one may hope that there is a simple generalization of our results.

We are claiming that we have a new $\frac{1}{M+N}$ expansion parameter. Why not simply replace $M+N \to (\mu+1)N$ with $\mu = \frac{M}{N}$ so that we have the usual $\frac{1}{N}$ expansion with μ dependent coefficients? There are (at least) two reasons to reject this proposal

- The loop expansion parameter in the half-BPS sector is $\frac{1}{M+N}$; the expansion coefficients have no additional M or N dependence. Since this sector includes gravitons, we should identify $\frac{1}{M+N}$ as the $\hbar$ for the graviton interactions. Indeed, a $\frac{1}{N}$ expansion with μ dependent coefficients is an expansion whose coefficients change as N is changed, indicating additional $\hbar$ dependence in these coefficients.

- Our relation (17) is exact and holds for any value of M. As the size of M changes the character of the expansion changes and it can be misleading to think that fluctuations are controlled by $\frac{1}{N}$. If $M = O(1)$, $M + N$ can be replaced by N and we have the usual $\frac{1}{N}$ expansion as expected - this is the theory in the $\mathrm{AdS}_5\times\mathrm{S}^5$ background. In this case, $\mu = O(\frac{1}{N})$ and the coefficients again become N independent. When $M = O(N)$, $M + N$ is itself of order N, and $\frac{1}{N}$ continues to control the size of fluctuations. This is the $\frac{1}{M+N}$ expansion we found above - for example the theory in the LLM annulus background. In this case, $\mu = O(1)$, and the coefficients themselves are of order 1. When $M = O(N^2)$ we can replace $M + N$ by M so that our $\frac{1}{M+N}$ expansion effectively becomes a $\frac{1}{M}$ expansion. The correlators are dominated by contractions with the background and fluctuations are much smaller than in the usual $\frac{1}{N}$ expansion: they are controlled by $\frac{1}{M} \sim \frac{1}{N^2}$. In this case, $\mu = O(N)$ so that the coefficients are unusually small - of size $\frac{1}{N}$. Particularly in the last case, it would be absurd to suggest that we have a $\frac{1}{N}$ expansion parameter.

This completes our demonstration that the BPS and near-BPS sectors of the theory admit a $\frac{1}{M+N}$ expansion.

A study of the closed strings probing the LLM annulus background has been completed in [15, 16, 46, 72]. One can describe the loops $O(\{n\})$ in terms of Cuntz oscillators (representing the Zs) populating a lattice (formed from the Ys) as in [18, 31, 32, 33]. The article [16] in particular, pointed out that the net effect of the background, on the one loop dilatation operator, was to rescale the Cuntz oscillators. This rescaling is nothing but the relation we have found in (26)! This rescaling could also be accounted for by using a new 't Hooft coupling

$$\lambda_{\rm new} = g_{YM}^2(N+M).$$

This again looks natural: we have the inverse of the effective genus counting parameter times g_{YM}^2. The net effect of the background on the one loop anomalous dimension operator (in this sector) is to replace $\lambda \to \lambda_{\rm new}$.

3.4 Discussion

One of the results of the papers [47, 48, 49] is a new basis for the local gauge invariant operators of multimatrix models. This new basis diagonalizes the two point functions and allows an exact computation of two point (and to some extent, multipoint) correlators in the $g_{YM}^2 = 0$ limit. If in addition we consider near BPS correlators, corrections in g_{YM}^2 are suppressed, so that we obtain a rather complete description of these correlators. This allows us to ask and answer a number of interesting questions, which may probe non-perturbative aspects of the dual quantum gravity.

In this section we have considered precisely such a question: the dynamics of operators whose classical dimension is $O(N^2)$. These operators are dual to states that have a large mass and consequently back reaction on the dual geometry is important. This is manifest in the fact that non-planar diagrams are no longer suppressed: although they come with the usual $\frac{1}{N^2}$ suppressions, these are overpowered by huge combinatoric factors; the usual $\frac{1}{N}$ expansion breaks down. In a number of cases we have shown that it is possible to reorganize the expansion and have identified the small parameters that control these expansions. These results were neatly captured by surprisingly simple relations between correlators in the trivial background and correlators in the background of our heavy operator.

Another interesting result that we have obtained, is that in the multi ring backgrounds and in the multi matrix backgrounds there was more than just one coupling. At first site this may seem puzzling, since the dilaton is a constant. To get some insight into what is going on, consider QCD. At high energies QCD is well described by a Lagrangian of quarks and gluons together with the coupling g_{YM}^2. At low energies, the coupling grows and one needs - somehow - to reorganize the theory of quarks and gluons into a low energy effective theory. The semi-classical objects in this low energy theory will be protons, neutrons, pions,... and there will be many possible coupling constants telling us how these semi-classical objects interact. The relevance of this story is that for the correlators we have studied, the planar approximation breaks down and one again needs to reorganize the theory. In this paper we have managed to explicitly perform this reorganization; the objects that we find that have different couplings are naturally interpreted as different semi-classical objects in the effective theory.

There will be effects that are non-perturbative in the new expansion parameter. An example of a non-perturbative amplitude is the transition from a maximal sphere giant graviton ($\chi_{[1^N]}(Z)$; $[1^N]$ denotes a Young diagram with one column containing N boxes) into KK gravitons with identical angular momentum $J << N$. The amplitude is given by [5, 10]

$$\frac{\left|\langle \chi_{[1^N]}(Z^\dagger)(\mathrm{Tr}\,(Z^J))^{N/J}\rangle\right|^2}{\langle \chi_{[1^N]}(Z^\dagger)\chi_{[1^N]}(Z)\rangle\ \langle \mathrm{Tr}\,(Z^{\dagger J})\mathrm{Tr}\,(Z^J)\rangle^{N/J}} \sim (2\pi)^{\frac{1}{2}} e^{-\frac{1}{g}-\frac{1}{2}\log(g)-(1/(gJ))\log(J)}$$

where $g = 1/N$. This is non-perturbative in $1/N$. To get the corresponding amplitude in the annulus, using Schur technology we can again argue that we simply need to replace $N \to N + M$, so that the transition amplitude is non-perturbative in $1/(M+N)$.

The techniques of this section allow us to reorganize the $1/N$ expansion for classes of observables in backgrounds described by Schur polynomials labeled by Young diagrams whose edges are all $O(N)$. This does not exhaust the operators that are dual to classical geometries. Indeed, the Schur polynomials correspond to spacetimes whose LLM boundary condition preserve a rotational symmetry on the plane in which the LLM boundary condition is specified. This is a small subset of the possible 1/2 BPS LLM geometries. Further, even amongst those LLM geometries with a rotationally invariant boundary condition, it is likely that we have only dealt with a subset of the geometries that can occur. Indeed, consider a triangular Young diagram T such that its longest column has N boxes, and each column has one box less than the column to its left. Thus our Young diagram has N columns in total. We have tested, in a number of cases, that factorization of the expectation value

$$\langle \mathcal{O} \rangle_T \equiv \frac{\left\langle \mathcal{O}\chi_T(Z)\chi_T(Z^\dagger)\right\rangle}{\langle \chi_T(Z)\chi_T(Z^\dagger)\rangle}$$

holds. This suggests that $\chi_T(Z)$ is again dual to a classical geometry. We have not yet understood how to reorganize the large N expansion in this case.

One very interesting class of excitations of an operator with a very large classical dimension, is obtained by attaching "open string defects" to the operator representing the background. Recent evidence [23, 24] suggests that these open string excited operators are dual to black hole micro states. Consequently, we expect that black hole microstate dynamics is captured in the dynamics of these operators. Our results (see in particular (26)) suggest that it may well be possible to write down spin chain descriptions for the open string excitations. Towards this end, we have explored the proposed

BMN-type sectors discovered in [24]. The existence of these limits is sure to play an important role in understanding the gauge theory description of near-extremal charged black holes in AdS_5. Taking $M_1 = O(\sqrt{N})$ and $M_2 = O(\sqrt{N})$ with the 't Hooft coupling λ large but fixed, the charges J_1 and J_2 scale as $O(N^{\frac{3}{2}})$, whilst the difference

$$\frac{\Delta - J_1 - J_2}{J_1 + J_2} \sim N^{-\frac{1}{2}} \to 0$$

in perfect agreement with the supergravity arguments of [24]. Thus, strong and weak coupling results agree - as expected for a BMN-like limit. We have further argued that if we hold $M_1 = M_2 = O(\sqrt{N})$ fixed we find an effective genus counting parameter $g_2 = M$ and an effective t́ Hooft coupling $\tilde{\lambda} = g_{YM}^2 M$. Keeping the effective 't Hooft coupling arbitrarily small, the one loop correction to the anomalous dimension can be made arbitrarily small, even though λ is sent to infinity in the limit. This is similar to what happens in the BMN limit suggesting that it will be possible to compare perturbative field theory results to results obtained from the dual gravitational description. The above effective 't Hooft coupling and genus counting parameters are naturally identified with those of the gauge theory living on the world volume of our system of intersecting giant gravitons. It would be very interesting to determine the dimensionality of this theory: is it a 2d gauge theory as the results of [23, 24] imply? Of course, the operators we have considered here are presumably too simple to describe the black hole microstates; for that one needs to consider triangular Young diagrams [9]. The triangular Young diagrams have a large number of corners which implies a large number of possible excitations of the operator. This is a significant increase in complexity as compared to the studies in this paper.

Finally, studying the rectangular Young diagrams that we have considered here will allow us to construct spin chains describing BMN loops in the LLM annulus background. One can look for signatures of integrability for these spin chains. This provides a set up in which one can test if integrability survives (a class of) non-planar corrections. This question is studied in the next section.

4 Hints of Integrability Beyond the Planar Limit: Nontrivial Backgrounds

4.1 Introduction

There is by now an impressive body of work suggesting that *planar* $\mathcal{N} = 4$ super Yang-Mills theory is *exactly integrable*. This would be very fortunate indeed, since it would mean the problem of computing the spectrum of all possible scaling dimensions of the gauge theory can be solved exactly, in the large N limit, by employing a Bethe ansatz. This has been established for the complete set of possible operators at one loop in [4, 56] and to two and three loops in the su(2) sector [57]. Given these results it is natural to guess that integrability extends to all orders in perturbation theory and perhaps even to the non-perturbative level [57]. There is now mounting evidence that this guess is correct [58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70].

The idea of spin chain parity played a central role in the discovery of the planar two and three loop integrability [57]. Acting on a single trace operator, parity simply reverses the order of fields inside the trace. The dilatation operator commutes with parity, so that as we would expect, the dilatation eigenstates are also parity eigenstates. In addition, [57] found that eigenstates with opposite parity were degenerate - this was quite unexpected. This degeneracy can be explained by the existence of a higher conserved charge (U_2 in the notation of [57]) that commutes with the dilatation operator but anticommutes with parity. Its also possible (and useful) to put this argument on its head: one can interpret the degeneracy of states with opposite parity as evidence for the existence of further conserved quantities. When non-planar corrections are taken into account, parity remains a good quantum number but the degeneracy is lifted (see [57] for a discussion of the $\mathcal{N} = 4$ super

Yang-Mills case and [71] for a very relevant discussion in the context of the ABJM theory). Although this only proves that the standard construction of conserved charges does not work away from the strict planar limit, it does suggests that integrability might not survive away from this limit. Clearly an important question is

Does integrability survive non-planar corrections?

In this section we will explicitly describe a situation in which we do sum (in fact, an infinite number of) non-planar corrections. Further, we collect some evidence that the resulting dynamics remains integrable. This suggests that, at least in certain situations, the answer to the above question is positive.

The non-planar corrections that we consider arise because we are interested in computing the anomalous dimensions of operators whose classical dimension is of order N^2. The usual $1/N$ suppression of non-planar diagrams is, in this case, overpowered by huge combinatoric factors [13]. In a series of articles [46, 72, 73], building on the earlier works [5, 22, 74], we have developed techniques to systematically study these operators. The specific operators we study are spelt out in detail in section 4.2.1. In section 4.2.2 we give the dilatation operator to two loops. To obtain this result, ribbon diagrams with arbitrarily large genus are summed. In section 4.2.3 we study the action of our dilatation operator. The action of this dilatation operator is *not easily formulated in a spin chain language because, nonplanar corrections allow the number of fields within each trace to change.* This translates into a spin chain with a variable lattice length. A convenient reformulation of the problem, in terms of a Cuntz lattice, was given in [75]. In the reformulation, particles (described by Cuntz oscillators) hop on a lattice of fixed size. The fact that the size of the spin chain lattice was dynamical now translates into the fact that the total number of particles populating the Cuntz lattice is dynamical. We give the Cuntz oscillator description of the dilatation operator for the class of operators we study in section 4.2.4. In section 4.3 we start to look for signs that our Cuntz chain is integrable. To start, we rewrite the spin chain (in the su(2) sector) of [4, 56, 57] in the Cuntz oscillator language. In particular, we write the conserved charge U_2 of [57] in terms of Cuntz oscillators. We have verified, using this expression for U_2, that U_2 does not commute with the Cuntz chain Hamiltonian corresponding to the annulus geometry. Another way explore the integrability of the original model is to study the semi-classical limit of the spin chain, which can be matched to the low energy limit of the principal chiral model. It is known that the principal chiral model is integrable. We can show that the Cuntz chain corresponding to the spin chain of [4, 56, 57] is indeed equivalent to the low energy limit of the principal chiral model - the spin chain and the Cuntz chain simply correspond to different choices of gauge. We give the explicit form of the gauge transformation relating the two in section 4.3.2. In section 4.3.3 we study the large M limit of our Hamiltonian and argue that we can indeed write down higher conserved charges. This suggests that integrability survives in this limit. In section 4.4 we discuss our results.

4.2 Two Loop Cuntz Chain of the LLM Background

$\mathcal{N} = 4$ super Yang-Mills theory has 6 scalars ϕ_i transforming in the adjoint of the gauge group and in the **6** of the $SU(4)_{\mathcal{R}}$ symmetry. We shall use the complex combinations

$$Z = \phi_1 + i\phi_2, \qquad Y = \phi_3 + i\phi_4, \qquad X = \phi_5 + i\phi_6, \tag{30}$$

in what follows. All operators that we study are built using only Z and Y; they belong to the su(2) sector of the theory. Many of the expressions that we write involve traces over Zs, Ys and derivatives of them. To avoid confusion, we will now spell out the index structure of a few expressions

$$\mathrm{Tr}\left(Z\frac{\partial}{\partial Z}\right) = Z_{ij}\frac{\partial}{\partial Z_{ij}}\,, \tag{31}$$

$$\mathrm{Tr}\left(ZY\frac{\partial}{\partial Z}\frac{\partial}{\partial Y}\right) = Z_{ij}Y_{jk}\frac{\partial}{\partial Z_{lk}}\frac{\partial}{\partial Y_{il}}\,. \tag{32}$$

4.2.1 Annulus Background

Schur polynomials provide a very convenient reorganization of the half-BPS sector of the $\mathcal{N}=4$ super Yang-Mills theory. This is due to the fact that their two point function is known to all orders in $\frac{1}{N}$ [5, 22] and that they satisfy a product rule allowing computation of exact n-point correlators using only two-point functions[21]. The half-BPS sector of the theory can be reduced to the dynamics of the eigenvalues of Z, which is the dynamics of N non interacting fermions in an external harmonic oscillator potential [5, 6]. The half-BPS sector of operators with $\mathcal{R}$ charge of order N^2, are dual to solutions of type IIB supergravity - the LLM geometries [7]. For a careful discussion of precisely what aspects of the eigenvalue dynamics the supergravity captures and vice versa, see [40]. For a recent discussion of the macroscopic description of the dual geometry see [77]. It is by now well known that the space of the LLM geometries is given by a black and white coloring of a two dimensional plane [7, 9]; this colored plane is isomorphic to the fermionic phase space of the eigenvalues of Z. The Schur polynomials correspond to geometries that

[21] There are generalizations of these results to multimatrix models [76]. These results will be very relevant for studies of backgrounds that preserve less supersymmetry.

are invariant under rotations in this plane.

The Schur polynomials are labeled by Young diagrams. In what follows, we will be interested in $\chi_B(Z)$ with B a Young diagram that has N rows and M columns. We take M to be of order N. Note that we can also express

$$\chi_B(Z) = (\det(Z))^M \; . \tag{33}$$

Since

$$\frac{\partial}{\partial Z_{ij}} \det(Z) = Z^{-1}_{ji} \det(Z),$$

we have

$$\frac{\partial}{\partial Z_{ij}} \chi_B(Z) = M(Z^{-1})_{ji}\chi_B(Z)\,, \tag{34}$$

a formula that we will make good use of below. The Z^{-1} factor which appears here will lead to terms in the dilatation operator which cause Z fields to "hop" over the Y fields, leading naturally to a Cuntz chain description. In fact, we will show that for $M > 0$ we can have negative occupation numbers for Cuntz lattice sites as a consequence of having a background operator. The colouring describing this dual LLM geometry is a black annulus. The inner white disk has an area of $\frac{M\pi}{N}$ whilst the black annulus itself has an area of π in units that assign an area of $\frac{\pi}{N}$ to each fermion state in phase space.

4.2.2 Two Loop Effective Dilatation Operator

The two loop dilatation operator, in the su(2) sector, has been computed in [57]. Using the conventions of [57], the dilation operator can be expanded as

$$D = \sum_{k=0}^{\infty} \left(\frac{g_{YM}^2}{16\pi^2}\right)^k D_{2k} = \sum_{k=0}^{\infty} g^{2k} D_{2k}\,, \tag{35}$$

where the tree level, one loop and two loop contributions are

$$D_0 = \mathrm{Tr}\left(Z\frac{\partial}{\partial Z}\right) + \mathrm{Tr}\left(Y\frac{\partial}{\partial Y}\right) , \tag{36}$$

$$D_2 = -2 : \mathrm{Tr}\left([Z,Y]\left[\frac{\partial}{\partial Z},\frac{\partial}{\partial Y}\right]\right) : , \tag{37}$$

$$\begin{aligned} D_4 = -2 : \mathrm{Tr}&\left(\left[[Y,Z],\frac{\partial}{\partial Z}\right]\left[\left[\frac{\partial}{\partial Y},\frac{\partial}{\partial Z}\right],Z\right]\right) : \\ -2 : \mathrm{Tr}&\left(\left[[Y,Z],\frac{\partial}{\partial Y}\right]\left[\left[\frac{\partial}{\partial Y},\frac{\partial}{\partial Z}\right],Y\right]\right) : \\ -2 : \mathrm{Tr}&\left([[Y,Z],T^a]\left[\left[\frac{\partial}{\partial Y},\frac{\partial}{\partial Z}\right],T^a\right]\right) : . \end{aligned} \tag{38}$$

The normal ordering symbols here indicate that derivatives within the normal ordering symbols do not act on fields inside the normal ordering symbols.

We allow this dilatation operator to act on an operator built mainly from Zs with a few "impurities" $=$ Ys added. For most of our study we explicitly display formulas for operators with two or three impurities. Our final formulas are however, completely general, covering the case that we have $O(1)$ impurities. Adapting the notation of [57] we define

$$\mathcal{O}_B(p,J_0,J_1,...,J_k) \equiv \chi_B(Z)\mathcal{O}_p^{J_0;J_1,...,J_k} \equiv \chi_B(Z)\mathrm{Tr}\,(YZ^pYZ^{J_0-p})\prod_{i=1}^{k}\mathrm{Tr}\,Z^{J_i} , \tag{39}$$

$$\mathcal{Q}_B(J_0,J_1,J_2,...,J_k) \equiv \chi_B(Z)\mathcal{Q}^{J_0,J_1;J_2,...,J_k} \equiv \chi_B(Z)\mathrm{Tr}\,(YZ^{J_0})\mathrm{Tr}\,(YZ^{J_1})\prod_{i=2}^{k}\mathrm{Tr}\,Z^{J_i} , \tag{40}$$

where $\chi_B(Z)$ is the operator creating the background, which was defined in the previous section. Our strategy is to define an effective dilatation operator D_{eff} as

$$D\left(\chi_B(Z)\mathcal{O}_p^{J_0;J_1,...,J_k}\right) = \chi_B(Z)D_{\mathrm{eff}}\mathcal{O}_p^{J_0;J_1,...,J_k} . \tag{41}$$

Diagonalizing the action of $D_{\rm eff}$ on the gauge invariant operators $\mathcal{O}_p^{J_0;J_1,...,J_k}$ is clearly equivalent to diagonalizing the action of D on $\mathcal{O}_B(p, J_0, J_1, ..., J_k)$. It is natural to interpret $D_{\rm eff}$ as the dilatation operator for the LLM background, that is, for the theory that is deformed by the insertion of $\chi_B(Z)\chi_B(Z^\dagger)$ in the path integral. Notice that we can write

$$D_{\rm eff} = \frac{1}{\chi_B(Z)} D\chi_B(Z)\,. \tag{42}$$

This formula remains correct even after replacing $\chi_B(Z)$ by any other operator creating the background, which depends only on Z. Ultimately, we will restrict ourselves to the large $M+N$ limit. To capture this limit, as explained in [73] one needs to resum an infinite number of nonplanar diagrams; ribbon diagrams with arbitrarily large genus are summed. *This limit is certainly not the planar limit of $\mathcal{N}=4$ super Yang Mills theory.*

The crucial observation needed in the computation of $D_{\rm eff}$ is that

$$\frac{\partial}{\partial Z_{ij}}\left(\chi_B(Z)\mathcal{O}_p^{J_0;J_1,...,J_k}\right) = \chi_B(Z)\left(\frac{\partial}{\partial Z_{ij}} + M(Z^{-1})_{ji}\right)\mathcal{O}_p^{J_0;J_1,...,J_k}\,. \tag{43}$$

Repeated application of this formula gives

$$D_{0\ \rm eff} = D_0 + MN\,, \tag{44}$$

$$D_{2\ \rm eff} = D_2 - 2M{\rm Tr}\left(\left(ZYZ^{-1} + Z^{-1}YZ - 2Y\right)\frac{\partial}{\partial Y}\right)\,, \tag{45}$$

$$\begin{aligned} D_{4\ \rm eff} &= D_4 + 4NM{\rm Tr}\left(\left(ZYZ^{-1} + Z^{-1}YZ - 2Y\right)\frac{\partial}{\partial Y}\right) \\ &+ 2M:{\rm Tr}\,[Z,Y]\left[Z^{-1},\left[Z,\left[\frac{\partial}{\partial Z},\frac{\partial}{\partial Y}\right]\right]\right]: +2M:{\rm Tr}\,[Z,Y]\left[\frac{\partial}{\partial Z},\left[Z,\left[Z^{-1},\frac{\partial}{\partial Y}\right]\right]\right]: \end{aligned}$$

$$
\begin{aligned}
&+ \quad 2M : \mathrm{Tr}\ [Z,Y]\left[\frac{\partial}{\partial Y},\left[Y,\left[Z^{-1},\frac{\partial}{\partial Y}\right]\right]\right]: \\
&- \quad 2M^2 \mathrm{Tr}\left(\left(Z^2YZ^{-2}-4ZYZ^{-1}+6Y-4Z^{-1}YZ+Z^{-2}YZ^2\right)\frac{\partial}{\partial Y}\right)
\end{aligned}
\tag{46}
$$

for the tree level, one loop and two loop contributions to D_{eff}. The formula for $D_{0\ \text{eff}}$ has a straight forward interpretation - the dimension of the gauge invariant operator $\mathcal{O}_p^{J_0;J_1,\ldots,J_k}$ is shifted by MN due to the presence of the background, which has dimension MN. The answer for $D_{2\ \text{eff}}$ has already been obtained and discussed in [16, 46, 72].

4.2.3 Action of the Two Loop Effective Dilatation Operator

A useful observation made in [57], is that, when acting with $g^2D_2+g^4D_4$ on the generic two impurity gauge invariant operators $\mathcal{O}_B(p,J_0,J_1,\ldots,J_k)$ and $\mathcal{Q}_B(J_0,J_1;J_2,\ldots,J_k)$ operators of type $\mathcal{Q}_B(J_0,J_1;J_2,\ldots,J_k)$ are never produced. This is easy to understand: acting with $g^2D_2+g^4D_4$ always inserts a commutator $[Y,Z]$ into a trace; this trace vanishes unless it contains another Y. This observation generalizes: when acting with $g^2D_{2\ \text{eff}}+g^4D_{4\ \text{eff}}$ on the generic two impurity gauge invariant operators $\mathcal{O}_p^{J_0;J_1,\ldots,J_k}$ and $\mathcal{Q}^{J_0,J_1;J_2,\ldots,J_k}$ operators of type $\mathcal{Q}^{J_0,J_1;J_2,\ldots,J_k}$ are never produced. This follows because both

$$
\mathrm{Tr}\left(\left(ZYZ^{-1}+Z^{-1}YZ-2Y\right)\frac{\partial}{\partial Y}\right) \tag{47}
$$

and

$$
\mathrm{Tr}\left(\left(Z^2YZ^{-2}-4ZYZ^{-1}+6Y-4Z^{-1}YZ+Z^{-2}YZ^2\right)\frac{\partial}{\partial Y}\right) \tag{48}
$$

annihilate gauge invariant operators containing a single Y and because the remaining terms in $g^2D_{2\ \text{eff}}+g^4D_{4\ \text{eff}}$ always insert a $[Y,Z]$ into a trace.

In what follows we will study the anomalous dimensions of the operators $\mathcal{O}_B(p, J_0, J_1, ..., J_k)$. Clearly, from the observation we just made, we do not need to consider the operators $\mathcal{Q}_B(J_0, J_1, J_2, ..., J_k)$ to do this.

Consider the action of $D_{2\ \mathrm{eff}}$ on $\mathcal{O}_p^{J_0;J_1,...,J_k}$. Following [57] we can break $D_{2\ \mathrm{eff}}$ into three pieces

$$D_{2\ \mathrm{eff}} = D_{2,0\ \mathrm{eff}} + D_{2,+\ \mathrm{eff}} + D_{2,-\ \mathrm{eff}}\,, \tag{49}$$

where $D_{2,0\ \mathrm{eff}}$ preserves the number of traces in $\mathcal{O}_p^{J_0;J_1,...,J_k}$, $D_{2,+\ \mathrm{eff}}$ increases (by 1) the number of traces and $D_{2,-\ \mathrm{eff}}$ decreases (by 1) the number of traces. From (45) it is clear that the additional term proportional to M can only contribute to $D_{2,0\ \mathrm{eff}}$. The terms $D_{2,+\ \mathrm{eff}}$ and $D_{2,-\ \mathrm{eff}}$ which involve gauge invariant operator splitting and joining will not contribute at the leading order; they will be important when computing, for example, subleading corrections to the leading $M+N$ limit. The additional contributions proportional to M in (45) have an important effect: they imply that p and $J_0 - p$ can be negative. Thus, we need to consider gauge invariant operators in which we populate the two "gaps between the Ys" with both positive and negative powers of Z. This implies that p in $\mathcal{O}_p^{J_0;J_1,...,J_k}$ is completely unrestricted.

It is easy to write down an exact expression for the action of $D_{2\ \mathrm{eff}}$

$$\begin{aligned} D_{2,0\ \mathrm{eff}}\mathcal{O}_p^{J_0;J_1,...,J_k} &= -4A_1\left(\mathcal{O}_{p+1}^{J_0;J_1,...,J_k}\right.\\ &\left.-\mathcal{O}_p^{J_0;J_1,...,J_k}\right) - 4A_2\left(\mathcal{O}_{p-1}^{J_0;J_1,...,J_k}\right.\\ &\left.-\mathcal{O}_p^{J_0;J_1,...,J_k}\right), \end{aligned} \tag{50}$$

$$D_{2,+\ \mathrm{eff}}\mathcal{O}_p^{J_0;J_1,...,J_k} = 4\sum_{J_{k+1}=1}^{p-1}\left(\mathcal{O}_{p-J_{k+1}}^{J_0-J_{k+1};J_1,...,J_k,J_{k+1}}\right.$$

$$-\mathcal{O}_{p-J_{k+1}-1}^{J_0-J_{k+1};J_1,\ldots,J_k,J_{k+1}}\Big) - 4\sum_{J_{k+1}=1}^{J_0-p-1}\Big(\mathcal{O}_{p+1}^{J_0-J_{k+1};J_1,\ldots,J_k,J_{k+1}}$$

$$-\mathcal{O}_{p}^{J_0-J_{k+1};J_1,\ldots,J_k,J_{k+1}}\Big)\,, \tag{51}$$

$$D_{2,-\ \mathrm{eff}}\mathcal{O}_p^{J_0;J_1,\ldots,J_k} = 4\sum_{i=1}^{k} J_i\Big(\mathcal{O}_{p+J_i}^{J_0+J_i;J_1,..,\hat{J}_i,..,J_k}$$

$$-\mathcal{O}_{p+J_i-1}^{J_0+J_i;J_1,..,\hat{J}_i,..,J_k}\Big) - 4\sum_{i=1}^{k} J_i\Big(\mathcal{O}_{p+1}^{J_0+J_i;J_1,..,\hat{J}_i,..,J_k}$$

$$-\mathcal{O}_{p}^{J_0+J_i;J_1,..,\hat{J}_i,..,J_k}\Big)\,, \tag{52}$$

where in the last expression hatted variables are removed from the argument of $\mathcal{O}$ and

$$\begin{aligned} A_1 &= M+N \qquad J_0 > p \\ &= M \qquad \text{otherwise} \end{aligned} \tag{53}$$

$$\begin{aligned} A_2 &= M+N \qquad p > 0 \\ &= M \qquad \text{otherwise}\,. \end{aligned} \tag{54}$$

Again motivated by [57] we can write

$$D_{4\ \mathrm{eff}} = -\frac{1}{4}(D_{2\ \mathrm{eff}})^2 + \delta D_{4\ \mathrm{eff}} \tag{55}$$

where

$$\delta D_{4\ \mathrm{eff}} = 2 : \mathrm{Tr}\ [Z,Y]\left[\frac{d}{dY},\left[Y,\left[\frac{d}{dZ},\frac{d}{dY}\right]\right]\right] : +2M : \mathrm{Tr}\ [Z,Y]\left[\frac{d}{dY},\left[Y,\left[Z^{-1},\frac{d}{dY}\right]\right]\right] : \,.$$

We can decompose $\delta D_{4\text{ eff}}$ as

$$\delta D_{4\text{ eff}} = \delta D_{4,0\text{ eff}} + \delta D_{4,+\text{ eff}} + \delta D_{4,-\text{ eff}} + \delta D_{4,+-\text{ eff}} + \delta D_{4,--\text{ eff}} + \delta D_{4,++\text{ eff}} \, . \tag{56}$$

The number of pluses/minuses in the subscripts on the right hand side of this last expression tell us how many traces are added/removed by the action of that particular term. $\delta D_{4,+-\text{ eff}}$ adds and removes a trace and hence it comes from summing higher genus ribbon diagrams which contribute to the trace conserving piece of $\delta D_{4\text{ eff}}$. It is again easy to write down exact expressions for the action of these terms. These expressions are given in appendix M.

4.2.4 Leading $M + N$ Limit

To extract the leading terms at large N (recall that we take M to be of order N) we need to rewrite the action of $D_{2\text{ eff}}$ obtained in the last subsection in terms of normalized gauge invariant operators, that is, gauge invariant operators that have a suitably normalized two point function. The relevant two point correlators have been computed in appendix K. The result is most easily written in terms of a Cuntz oscillator chain. We will now focus on gauge invariant operators that have $k = 0$. If we relax the restriction to two impurities (which we do from now on), the gauge invariant operators we study have the form

$$\mathcal{O}_B(\{p\}, J_0) \equiv \chi_B(Z)\mathrm{Tr}\,(YZ^{p_1}YZ^{p_2}\cdots YZ^{p_n}) \, , \qquad \sum_{i=1}^{n} p_i = J_0 \, . \tag{57}$$

To translate this gauge invariant operator into a Cuntz chain state, we associate a site of the Cuntz chain with each of the gaps between the Ys. The above gauge invariant operator defines a state in a Cuntz chain with n sites. Further, the number of Zs in each site gives the occupation number of that site. Finally, we require that operators with a normalized (free field) two point function map into normalized Cuntz chain states. This last point deserves a few comments. The spacetime dependence of the free field correlators we compute is trivially determined by the bare dimension of the operator. Thus we can compute all correlators in zero dimensions. In this zero dimensional model, each two point function is just a number. An operator with a normalized two point function is one for which this number is one. This map between two point functions and the norm of states is of course nothing but the usual state operator correspondence.

The dilatation operator allows the Zs to hop between sites of the Cuntz chain. At leading order in large $M+N$, the action of $D_{2\,\mathrm{eff}}$ is given entirely by $D_{2,0\,\mathrm{eff}}$. Using the correspondence given in the last equation of appendix K, the operator equation (50) can be translated into an equation for the action of $D_{2\,\mathrm{eff}}$ on the Cuntz lattice. If all Cuntz lattice occupation numbers are positive then, since $D_{2\,\mathrm{eff}}$ lowers each occupation number by at most 1, acting with $D_{2\,\mathrm{eff}}$ can't change the value $p_- = 0$ where p_- is negative the sum of all the negative occupation numbers. This implies that all gauge invariant operators have the same leading two point function and hence, when acting on a Cuntz lattice with, for example, two sites ($\lambda = g^2 N$)

$$g^2 D_{2,0\ \mathrm{eff}}|\{p_1, p_2\}\rangle = -4\lambda \frac{N+M}{N}\left(|\{p_1+1, p_2-1\}\rangle - 2|\{p_1, p_2\}\rangle + |\{p_1-1, p_2+1\}\rangle\right) . \tag{58}$$

Similarly, if all Cuntz lattice occupation numbers are negative then, since $D_{2\,\mathrm{eff}}$ raises each occupation number by at most 1, acting with $D_{2\,\mathrm{eff}}$ again can't change the value $p_- = \sum_i p_i$. Again all gauge invariant operators have the same leading two point function and hence, when acting on a Cuntz lattice with, for example, two sites

$$g^2 D_{2,0\ \text{eff}} |\{p_1, p_2\}\rangle = -4\lambda \frac{M}{N} \left(|\{p_1+1, p_2-1\}\rangle - 2|\{p_1, p_2\}\rangle + |\{p_1-1, p_2+1\}\rangle \right) . \tag{59}$$

Finally, consider for example the case that $D_{2,0\ \text{eff}}$ acts on a lattice with two sites and occupation numbers $p_1 = 0$, $p_2 = 2$. In this case, the normalization of the gauge invariant operators is not the same: three terms have $p_- = 0$ and one has $p_- = 1$. Taking this into account gives

$$\begin{aligned} g^2 D_{2,0\ \text{eff}} |\{0,2\}\rangle &= -4\lambda \frac{M+N}{N} \left(|\{1,1\}\rangle - |\{0,2\}\rangle \right) \\ &\quad -4\lambda \frac{M}{N} \left(\frac{\sqrt{M+N}}{\sqrt{M}} |\{-1,3\}\rangle - |\{0,2\}\rangle \right) . \end{aligned} \tag{60}$$

There is a nice convenient way to summarize these results [16, 46, 72]. We will introduce Cuntz oscillators which satisfy the algebra (we associate one of these oscillators to each site of the chain)

$$a^\dagger a = \frac{M}{N} + \theta(\hat{n}+1) - |0\rangle\langle 0| , \qquad a a^\dagger = \frac{M}{N} + \theta(\hat{n}+1) , \tag{61}$$

with $\hat{n}$ the number operator. Notice that when $M = 0$ we have only positive occupation numbers so that the above relations reduce to the usual ones

$$a^\dagger a = 1 - |0\rangle\langle 0| , \qquad a a^\dagger = 1 . \tag{62}$$

We can also define these oscillators by giving their action on states of good particle number

$$a|n\rangle = \sqrt{1 + \frac{M}{N}} |n-1\rangle \qquad n > 0 \tag{63}$$

$$a|n\rangle = \sqrt{\frac{M}{N}}|n-1\rangle \qquad n \le 0\,. \tag{64}$$

In terms of these Cuntz oscillators we have

$$g^2 D_{2\ \mathrm{eff}} = 2\lambda \sum_{l=1}^{L} (a_l^\dagger - a_{l+1}^\dagger)(a_l - a_{l+1})\,. \tag{65}$$

There is a rather direct way to extract (part of the) geometry of the dual LLM solution from this Cuntz oscillator description [16, 46, 72]. To see this, consider the coherent state

$$|z\rangle = \sum_{n=-\infty}^{0} \left(\frac{N}{M}\right)^{\frac{n}{2}} z^n |n\rangle + \sum_{n=1}^{\infty} \left(\frac{N}{M+N}\right)^{\frac{n}{2}} z^n |n\rangle\,. \tag{66}$$

The norm of this state

$$\langle z|z\rangle = \sum_{n=0}^{\infty} \frac{M^n}{N^n |z|^{2n}} + \sum_{n=1}^{\infty} \frac{N^n |z|^{2n}}{(M+N)^n} \tag{67}$$

is only finite if $\frac{M}{N} \le |z|^2 \le \frac{M+N}{N}$, that is, $|z|^2$ must lie within the annulus. Clearly z is a complex coordinate for the LLM plane.

This one loop result is intriguing: the effect of the background has been completely accounted for by simply modifying the Cuntz oscillator algebra. This is a remarkably simple change. It is natural to ask

Is the net effect of the background (even at higher loops) completely accounted for, by simply modifying the Cuntz oscillator algebra?

We do not have a complete answer to this question. Computing the form of the two loop answer ($D_{4\ \mathrm{eff}}$) is already a rather involved task. We have however studied this question for two sites. In this case

$$g^4 D_{4\ \mathrm{eff}} = -\frac{1}{4}(g^2 D_{2\ \mathrm{eff}})^2 + g^4 \delta D_{4\ \mathrm{eff}} \tag{68}$$

where

$$g^4\delta D_{4\text{ eff}} = 4\lambda^2\sum_{l=1}^{2}\Big(a_l^\dagger[a_{l+1},a_{l+1}^\dagger]a_{l+1} + a_l^\dagger a_{l+1}[a_l,a_l^\dagger]$$
$$-a_l^\dagger a_l[a_{l+1},a_{l+1}^\dagger] - a_l^\dagger[a_l,a_l^\dagger]a_l\Big)\ . \tag{69}$$

Thus, at two loops for a Cuntz chain with two sites we find that, again, the net effect of the background is to modify the Cuntz oscillator algebra.

4.3 Conservation Laws

In the last section we have explained how to extract an effective dilatation operator. Diagonalizing this effective dilatation operator will give the spectrum of anomalous dimensions for a class of operators whose classical dimension is of order N^2. Non-planar diagrams have to be included to obtain this dilatation operator. One consequence of this is that a spin chain description is no longer useful and we have instead passed to a Cuntz lattice description. In this section we want to answer two questions:

- The Cuntz lattice description can be employed even before a background is introduced. What is the translation of the conserved quantities of the original spin chain into the Cuntz lattice language?
- Is there any evidence that the effective dilatation operator obtained in the presence of the LLM annulus background is integrable?

4.3.1 U_2 rewritten in the Cuntz Oscillator Language

Before we obtain the conserved charge U_2 in the Cuntz oscillator language, its useful to first write it as a differential operator acting on gauge invariant operators. We will use this expression when we search for a corresponding conserved charge in the nontrivial background. It is a straight forward exercise to find that the planar action of

$$U_2 = \mathrm{Tr}\left((YZZ - ZYZ)\frac{d}{dY}\frac{d}{dZ}\frac{d}{dZ}\right) + \mathrm{Tr}\left((ZZY - YZZ)\frac{d}{dZ}\frac{d}{dY}\frac{d}{dZ}\right)$$

$$+\mathrm{Tr}\left((ZYZ - ZZY)\frac{d}{dZ}\frac{d}{dZ}\frac{d}{dY}\right) + \mathrm{Tr}\left((ZYY - YZY)\frac{d}{dZ}\frac{d}{dY}\frac{d}{dY}\right)$$

$$+\mathrm{Tr}\left((YYZ - ZYY)\frac{d}{dY}\frac{d}{dZ}\frac{d}{dY}\right)+\mathrm{Tr}\left((YZY - YYZ)\frac{d}{dY}\frac{d}{dY}\frac{d}{dZ}\right), \quad (70)$$

on single trace gauge invariant operators matches the action of U_2 on the spin chain. To illustrate how to obtain a Cuntz oscillator representation, consider the first term above: it acts as

$$ZZY \to YZZ - ZYZ\,. \quad (71)$$

This term can be represented in terms of Cuntz oscillators as

$$\sum_{l=1}^{L}(a_{l+1}^\dagger a_{l+1}^\dagger - a_l^\dagger a_{l+1}^\dagger)a_l a_l\,. \quad (72)$$

The second term in the sum should not be truncated to $a_{l+1}^\dagger a_l$ since we have to make sure that there are at least 2 Zs in lattice site l. The result for U_2 is

$$U_2 = \sum_{l=1}^{L}\Big((a_{l+1}^\dagger a_{l+1}^\dagger - a_l^\dagger a_{l+1}^\dagger)a_l a_l + (a_l^\dagger a_l^\dagger - a_{l+1}^\dagger a_{l+1}^\dagger)a_{l+1}a_l$$

$$+(a_l^\dagger a_{l-1}^\dagger - a_{l-1}^\dagger a_{l-1}^\dagger)a_l a_l + (a_{l-1}^\dagger - a_l^\dagger)a_{l+1}[a_l, a_l^\dagger]$$

$$+(a_{l+1}^\dagger - a_{l-1}^\dagger)[a_l, a_l^\dagger]a_l + (a_l^\dagger - a_{l+1}^\dagger)a_{l-1}[a_l, a_l^\dagger]\Big)\,. \quad (73)$$

Using this procedure it is straight forward to write down the Cuntz oscillator representation for any of the conserved charges of the spin chain.

It is tedious but straight forward to compute the commutator of U_2 given above and $D_{2\,\mathrm{eff}}$ - they do not commute. It seems natural to consider the operator

$$U_{2\,\mathrm{eff}} = \frac{1}{\chi_B(Z)} U_2 \chi_B(Z) \,. \tag{74}$$

It is a simple matter to compute $U_{2\,\mathrm{eff}}$ using the above expression for U_2 as a differential operator acting on gauge invariant operators. Again, $D_{2\,\mathrm{eff}}$ and $U_{2\,\mathrm{eff}}$ do not commute. However, in the leading large M limit the two do commute suggesting that this may be an interesting limit of $D_{2\,\mathrm{eff}}$. This is explored in detail in section 4.3.3 below.

4.3.2 Classical Limit

The original spin chain description of the dilatation operator can be replaced by a sigma model in the limit of a large number of sites. This sigma model precisely matches the Polyakov action describing the propagation of closed strings in $\mathrm{AdS}_5 \times \mathrm{S}^5$, in a particular limit [78, 79, 80]. This has also been extended to other examples of the AdS/CFT correspondence [18, 75, 81, 82, 83]. The Cuntz oscillator description of the dilatation operator is simply an alternative language in the undeformed background; when we consider the deformed background the spin chain description is not convenient, so that in this case it is best to use the Cuntz oscillator description. We can obtain a semiclassical description of the Cuntz chain by again considering a large number of sites [18]. In this subsection we will provide further insight into the relation between the spin chain and Cuntz oscillator descriptions by showing that in the dual string theory the two descriptions are simply related by a change of worldsheet gauge choice.

The semi-classical limit of the Cuntz chain is obtained by taking $L \sim \sqrt{N} \to \infty$, $\lambda \to \infty$ holding $\frac{\lambda}{L^2}$ fixed and by putting each lattice site into a coherent state (we are discussing the undeformed theory so there are no negatively occupied sites)

$$|z\rangle = \sqrt{1-|z|^2}\sum_{n=0}^{\infty} z^n|n\rangle\,, \qquad |n\rangle = (a^\dagger)^n|0\rangle\,. \tag{75}$$

The coherent state parameter of the l^{th} site is traded for a radius and an angle $z_l = r_l e^{i\phi_l}$. The action is given, as usual, by

$$S = \int dt\left(i\langle Z|\frac{d}{dt}|Z\rangle - \langle Z|D|Z\rangle\right) \qquad |Z\rangle = \prod_l |z_l\rangle\,. \tag{76}$$

After trading the sum over l for an integral over σ, the action is

$$S = -L\int dt\int_0^1 d\sigma\left(\frac{r^2\dot{\phi}}{1-r^2} + \frac{\lambda}{L^2}(r'^2 + r^2\phi'^2)\right)\,. \tag{77}$$

For a detailed derivation the reader could consult [18]. We would like to show how this Cuntz chain sigma model can be recovered from the standard string sigma model action on $R\times S^3$.

A string moving on $R\times S^3$ can be described by the principal chiral model with su(2) valued currents

$$j_\tau = g^{-1}\frac{\partial g}{\partial\tau}\,, \qquad j_\sigma = g^{-1}\frac{\partial g}{\partial\sigma}\,, \tag{78}$$

where

$$g = \begin{bmatrix} Z & iY \\ i\bar{Y} & \bar{Z} \end{bmatrix} \in \mathrm{SU}(2)\,, \tag{79}$$

and Z and Y are the coordinates of a sphere

$$|Z|^2 + |Y|^2 = 1\,. \tag{80}$$

We are choosing to employ a principal chiral model description since this description manifests the integrability of the model. Parametrize the sphere coordinates as follows

$$Z = re^{i(\kappa\tau - \phi)}\,, \qquad Y = \sqrt{1-r^2}e^{i(\varphi+\kappa\tau)}\,. \tag{81}$$

The equations of motion

$$\partial_\tau j_\tau - \partial_\sigma j_\sigma = 0 \tag{82}$$

can be obtained from the Polyakov action in conformal gauge and after fixing the residual conformal diffeomorphism freedom by choosing $t = \kappa\tau$. In this gauge, energy is homogeneously distributed along σ. To obtain the low energy limit, we take $\kappa \to \infty$ holding $\kappa\dot{r}$, $\kappa\dot{\phi}$ and $\kappa\dot{\varphi}$ fixed. It is precisely in this gauge and in this limit that [78] matched the semiclassical limit of the one loop spin chain to the string sigma model. The Lagrangian in this limit becomes

$$\mathcal{L} = -\frac{1}{4}(j_\tau^2 - j_\sigma^2) = r^2\kappa\dot{\phi} - (1-r^2)\kappa\dot{\varphi} + \frac{1}{2}r^2\phi'^2 + \frac{1}{2}(1-r^2)\varphi'^2 + \frac{1}{2}\frac{r'^2}{1-r^2}\,.$$

The equations of motion following from this action needs to be supplemented by the usual Virasoro constraints, which in this limit are $j_\tau^2 + j_\sigma^2 = 0$ and

$$\kappa\varphi'(1-r^2)^2 + \kappa\phi' r^2(r^2-1) + O(1) = 0\,. \tag{83}$$

The Cuntz sigma model (77) does not contain the field φ. Thus, it should be eliminated before we can expect to obtain agreement. Integrate by parts to obtain

$$\mathcal{L} = r^2\kappa\dot{\phi} - \varphi\frac{\partial}{\partial\tau}(1-r^2)\kappa + \frac{1}{2}r^2\phi'^2 + \frac{1}{2}(1-r^2)\varphi'^2 + \frac{1}{2}\frac{r'^2}{1-r^2}\,. \tag{84}$$

Now, using the φ equation of motion (to rewrite the coefficients of φ in the action) and the (square of the) Virasoro constraint (83) we find

$$\mathcal{L} = r^2\kappa\dot{\phi} + \frac{1}{2}r^2\phi'^2 + \frac{1}{2}\frac{r'^2}{1-r^2} - \frac{1}{2}\frac{r^4}{1-r^2}\phi'^2\,. \tag{85}$$

This does not agree with the result (77).

The disagreement is not surprising: the sigma model (85) is written in a gauge in which energy is distributed homogeneously along the string; the sigma model (77) corresponds to a gauge in which p_φ (= angular momentum conjugate to φ) is distributed homogeneously along the string. In the Cuntz chain, only Ys mark lattice sites; in the usual spin chain both Ys and Zs mark lattice sites. Consequently to go from the σ_{cc} coordinate of the Cuntz chain to the σ_{sc} coordinate of the spin chain, we need to "add the Zs back in"

$$\sigma_{sc} = \sigma_{cc} + \int_0^{\sigma_{cc}} n_z(\sigma')d\sigma' = \sigma_{cc} + \int_0^{\sigma_{cc}} \frac{r^2}{1-r^2}d\sigma'\,, \qquad \tau_{sc} = \tau_{cc}\,. \tag{86}$$

In this last equation, $n_z(\sigma') = r^2/(1-r^2)$ is the expected number of Cuntz particles (= number of Zs) at σ'. It is now straight forward to compute

$$\frac{\partial\sigma_{sc}}{\partial\sigma_{cc}} = \frac{1}{1-r^2}, \qquad \frac{\partial\tau_{sc}}{\partial\tau_{cc}} = 1, \qquad \frac{\partial\tau_{sc}}{\partial\sigma_{cc}} = 0\,, \tag{87}$$

$$\frac{\partial\sigma_{sc}}{\partial\tau_{cc}} = -2r^2\frac{\partial\phi}{\partial\sigma_{cc}} = -2\frac{r^2}{1-r^2}\frac{\partial\phi}{\partial\sigma_{sc}}\,. \tag{88}$$

The integrability condition

$$\frac{\partial}{\partial\tau_{cc}}\frac{\partial}{\partial\sigma_{cc}}\sigma_{sc} = \frac{\partial}{\partial\tau_{cc}}\frac{1}{1-r^2} = \frac{\partial}{\partial\sigma_{cc}}\frac{\partial}{\partial\tau_{cc}}\sigma_{sc} = \frac{\partial}{\partial\sigma_{cc}}\left(-2r^2\frac{\partial\phi}{\partial\sigma_{cc}}\right) \tag{89}$$

is nothing but the ϕ equation of motion derived from (77). It is now a straight forward exercise to verify that after this change of coordinates (77) and (85) match perfectly.

The Cuntz oscillator Hamiltonian (77) can be written as

$$S = -L \int dt \int_0^1 d\sigma \left(n_z(\sigma)\dot{\phi} + \frac{\lambda}{L^2}(r'^2 + r^2\phi'^2) \right) . \tag{90}$$

The advantage of this rewriting is that it holds in general - that is, both for the undeformed and deformed backgrounds. Inserting the explicit expression for the expected number of Cuntz particles in the deformed background we find

$$S = -L \int dt \int_0^1 d\sigma \left(\left[\frac{r^2}{1 + \frac{M}{N} - r^2} - \frac{\frac{M}{N}}{\frac{M}{N} - r^2} \right] \dot{\phi} + \frac{\lambda}{L^2}(r'^2 + r^2\phi'^2) \right) . \tag{91}$$

If one drops either of the two terms in square braces one obtains a model which can be related to the low energy limit of the principal chiral model and hence is the low energy limit of an integrable model. The physical interpretation of such a truncation is clear: keeping only the first term corresponds to focusing on fluctuations localized at the outer edge of the annulus; keeping only the second term corresponds to focusing on fluctuations localized at the inner edge of the annulus. For these classes of fluctuations, it seems that the dynamics is integrable. It would be interesting to establish if the full model is integrable or not.

There is a very natural generalization to multi ring LLM geometries, corresponding to backgrounds created by Schur polynomials labeled by a Young diagram with more than 4 edges and each edge with a length of $O(N)$. In this case $n_z(\sigma)$ is a sum of terms, one for each edge. Restricting to fluctuations localized about a particular edge again gives a model which can be related to the low energy limit of the principal chiral model and hence is the low energy limit of an integrable model. These localized excitations have been constructed in [72].

The superstar geometry [30] has been related to an LLM geometry with boundary condition given by a sequence of concentric alternating black and white rings [9]. Rings of the same color have the same area and the total area of the black rings is π. As mentioned above, to construct $n_z(\sigma)$ we need to sum a term for each edge of the multi-ring geometry. We will consider a geometry which corresponds to the Young diagram shown in figure 8 with $n_1, n_2 << N$. In this case, we sum a very large number of terms and hence may use the Euler-Maclaurin formula to rewrite the sum as an integral. Carrying this integral out we find (we dropped an additive constant that will not contribute to the equations of motion)

$$n_z(\sigma) = \frac{\alpha}{\alpha+\beta}\frac{r^2}{1+\frac{M}{N}-r^2} \tag{92}$$

where

$$\alpha = \frac{n_1}{N}, \qquad \beta = \frac{n_2}{N}\,. \tag{93}$$

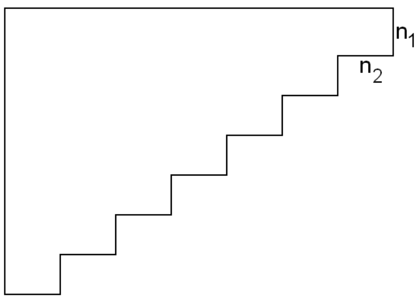


Figure 8: The Young diagram corresponding to the superstar geometry.

Thus, the semiclassical limit of the Cuntz chain model is

$$S = -L \int dt \int_0^1 d\sigma \left(\frac{\alpha}{\alpha+\beta} \frac{r^2}{1+\frac{M}{N}-r^2} \dot{\phi} + \frac{\lambda}{L^2} (r'^2 + r^2 \phi'^2) \right) . \tag{94}$$

After rescaling t we can recover the action (77) up to an overall multiplicative constant. Thus, the model can again be related to the low energy limit of an integrable model.

4.3.3 Integrability in the Large M Limit

In this subsection we will consider the large M limit, that is, we take $M, N \to \infty$ and in addition, the ratio $\frac{N}{M} \to 0$. In this limit we suppress all $\frac{N}{M}$ dependence. The dilatation operator for the sector we consider, after subtracting the classical dimension out, can be written as

$$D = D\left(Z, Y, \frac{d}{dZ}, \frac{d}{dY} \right) . \tag{95}$$

To get the large M limit of D_{eff} (we denote this operator by $\tilde{D}_{\text{eff}}$) we should simply replace $\frac{d}{dZ}$ by MZ^{-1} in the above expression to obtain

$$\tilde{D}_{\text{eff}} = \tilde{D}_{\text{eff}} \left(Z, Y, MZ^{-1}, \frac{d}{dY} \right) . \tag{96}$$

Expand this operator as

$$\tilde{D}_{\text{eff}} = \sum_n \tilde{D}_{\text{eff n}} \tag{97}$$

where $\tilde{D}_{\text{eff n}}$ has a total of n derivatives with respect to Y. From the general structure of a connected planar l-loop vertex we know that D will act on $l+1$ adjacent sites; thus, it will contain $l+1$ derivatives. The leading contribution in the large M limit would come from terms which have all $l+1$

derivatives acting on Z and these are replaced to give an $M^{l+1}\mathrm{Tr}\,(Z^{-l-1})$. This leading term is captured in $\tilde{D}_{\mathrm{eff}\,0}$, which up to an arbitrary coefficient, is now determined. Since the dimension of $\mathrm{Tr}\,(Z^J)$ is not corrected, it must be annihilated by D and hence, in the large M limit it must be annihilated by $\tilde{D}_{\mathrm{eff}\,0}$. This implies that $\tilde{D}_{\mathrm{eff}\,0}$ vanishes. Thus, the leading contribution will in fact come from the $\tilde{D}_{\mathrm{eff}\,1}$. The l-loop term will thus have a $(g^2M)^l$ dependence - this is the dependence that the present argument captures[22]. Since D is dimensionless, and preserves the total number of Zs and the total number of Ys, it is clear that, to leading order at large M

$$\tilde{D}_{\mathrm{eff}} = \tilde{D}_{\mathrm{eff}\,1} = \sum_n c_n \mathrm{Tr}\left(Z^n Y Z^{-n} \frac{d}{dY} \right) \tag{98}$$

where the c_n depend on g^2M. It is trivial to see that

$$\left[\mathrm{Tr}\left(Z^{-n} Y Z^{n} \frac{d}{dY} \right), \mathrm{Tr}\left(Z^{-m} Y Z^{m} \frac{d}{dY} \right) \right] = 0\,, \tag{99}$$

so that

$$\left[\tilde{D}_{\mathrm{eff}}, \mathrm{Tr}\left(Z^{-m} Y Z^{m} \frac{d}{dY} \right) \right] = 0\,, \tag{100}$$

which clearly demonstrates an infinite number of conserved quantities to all loops.

What is the physical meaning of this limit? Recall that the dilatation operator can be read from the two point functions of the theory. Restricting to operators constructed from scalar fields only is, in general, not possible due to operator mixing. However, it is possible to show [57] that it is consistent to restrict to operators built using only the two complex fields Y and Z. In this

[22]Of course, our large M limit is a double scaling limit in which we take $M \to \infty$, $g^2 \to 0$ holding g^2M fixed. This limit is the natural one: our effective genus counting parameter is $\frac{1}{M}$ so that $\lambda = g^2M$ is the obvious definition of the 't Hooft coupling. See [73] for further details.

case, we can compute the two point functions in a reduced model comprising of only the Y and Z matrices. Interpreted in this way, the dilatation operator can be understood as implementing the Wick contractions associated with the F-term vertex. For example, consider the combination

$$\frac{d}{dZ} + MZ^{-1} \tag{101}$$

which replaces $\frac{d}{dZ}$ in transforming the undeformed into the deformed Cuntz chain. The $\frac{d}{dZ}$ term represents a contraction between the vertex and a $Z^\dagger$ in one of the fields whose two point function we are computing; to see this connection it is useful to remember that

$$\langle Z^\dagger_{ij} Z_{kl} \rangle = \delta_{jk}\delta_{il} = \frac{d}{dZ_{ji}} Z_{kl} \,. \tag{102}$$

In contrast to this, the MZ^{-1} term represents a contraction between the vertex and a $Z^\dagger$ in the operator representing the background. In the large M limit, the contractions with the background completely dominate as compared to contractions with fields belonging to the operators of the two point functions we are computing. One can think that the matrices entering into the operators are "bits of a string". In the limit that we consider, the different bits in the string do not interact with each other - they interact only with the background. We would indeed expect the dynamics to simplify in this limit.

In the large M limit, the action of the Cuntz chain (91) becomes

$$S = -L \int dt \int_0^1 d\sigma \left(-\dot{\phi} + \frac{\lambda}{L^2}(r'^2 + r^2\phi'^2) \right) \,. \tag{103}$$

Since $\dot{\phi}$ is a total derivative, all time derivatives drop out of the equations of motion. This implies that the dynamics becomes trivial which is indeed consistent with integrability. It is rather interesting that there is a class of operators in $\mathcal{N} = 4$ super Yang-Mills theory that have such a simple description.

In terms of the dual LLM boundary condition, the large M limit corresponds to an annulus with a large radius and fixed area, so that the annulus is becoming very thin.

4.4 Discussion

The problem of computing the anomalous dimensions of operators with a large ($O(N^2)$) R-charge corresponds to a generalization, in the dual gravitational description, to string dynamics in spacetimes that are only asymptotically $\mathrm{AdS}_5\times\mathrm{S}^5$. The problem can again be reduced to diagonalizing a Hamiltonian. In the $\mathrm{AdS}_5\times\mathrm{S}^5$ spacetime, this Hamiltonian was an integrable spin chain. As a consequence of the fact that our strings can exchange angular momentum with the background, our Hamiltonian describes Cuntz particles hopping on a lattice. In the gauge theory description, the terms in the dilatation operator that allow the strings to exchange angular momentum with the background arise from summing (an infinite number of) non-planar corrections. It is surprisingly straight forward to write down very explicit expressions for the relevant Cuntz chain Hamiltonians.

A natural question to ask is if our Cuntz chain Hamiltonians correspond to integrable systems. We don't know. However, we have given some evidence that the large M limit of our Hamiltonian does admit higher conserved charges and that certain localized semiclassical excitations are described by the low energy limit of a principal chiral model, so an optimist would indeed conjecture that our Cuntz chain Hamiltonian is integrable. We hope that we have managed to convince the reader (even if she is pessimistic) that these are interesting limits of the original $\mathcal{N} = 4$ super Yang-Mills theory that warrant further study.

5 Discussion

One of the major contributions of the work presented in this thesis is the computational framework for calculating multipoint correlators of operators with an $\mathcal{R}$-charge of $O(N^2)$. These previously intractable regimes have revealed some interesting features of how supergravity geometry is encoded in $\mathcal{N} = 4$ SYM. A general theme seems to be that different geometric regimes in the supergravity correspond to situations where different types of contractions can be neglected when computing gauge theory correlators in the large-N limit. For example, in calculating correlators of operators dual to open strings attached to giant gravitons, contractions between the string words and giant graviton operators can be neglected. In the case of strings attached to larger operators, with $\mathcal{R}$-charge of $O(N^2)$, these contractions can no longer be neglected and it is their contributions which lead to, for example, the string endpoints no longer having a special role. This mirrors the dual picture in which there no longer is an open string attached to a D-brane, but rather a closed string in a new spacetime. It is noteworthy that in all cases the data required for correlator calculations can be read off of the Young diagram labelling the operators. At least in the $\frac{1}{2}$-BPS sector then, it appears to be the case that the supergravity description is encoded in symmetric group data associated with field theory operators.

A particularly interesting way to extend this work would be to learn how to calculate correlators for triangular Young diagrams, with horizontal and vertical edges of length $O(N)$, with a small number of attached open strings. Recent results in [23, 24] suggest that such states may be the microstates of near-extremal black holes in AdS_5. It would be fascinating, for example, to understand the emergence of an horizon in terms of the gauge theory operators.

It would also be useful to complete the analysis of integrability of strings in the presence of large background operators. The evidence acquired in section 4 strongly suggests that the system *is* integrable. The result necessitates summing an infinite number of non-planar diagrams. It would be valuable to compare this case in detail to situations where including non-planar corrections is known to break integrability. It would also be interesting to fully understand the system in the $M \gg N$ limit.

Recall the definition of the restricted Schur, (1). For some terms in the sum, the restricted trace $\mathrm{Tr}_{R_1}(\Gamma_R(\sigma))$ is over an element not in the subgroup of S_n labelled by R_1. Such elements pose a computational difficulty. However, they are central in the general expression for the dilatation operator acting on a restricted Schur built from two matrices. Being able to calculate such traces in closed form would lead to the identification of all $\frac{1}{4}$-BPS states in terms of the shape of their Young diagram labels.

Recent work making use of restricted Schur polynomials includes [85], wherein the authors computed the anomalous dimension of a class of operators built from two complex $\mathcal{N} = 4$ scalars with bare scaling dimension of $O(N)$. In a particular limit the dilatation operators takes the form of a lattice second derivative, with the lattice emerging from the Young diagram labels on the operators. This lattice is naturally interpreted in the supergravity as the worldvolume of the dual D3-brane.

There is also work under way [86, 87] exploring number theoretic properties of restricted Littlewood-Richardson numbers. They are related to a generalization of Grothendieck's Dessins d'Enfants with coloured edges and are proving to be a useful tool for finding invariants in Galois theory.

Appendix

A Decomposing Derivative Operators

As argued in section 2.2.1, the contributions to a correlation function of two restricted Schur polynomials, coming from contractions between Zs that belong to the open string and Zs that belong to the brane, can be written as a differential operator acting on the restricted Schur polynomials. In this appendix we will show that any such string of derivatives can be written in terms of eight basic types of derivatives, acting on modified restricted Schur polynomials. This result is a useful one because it is possible to work out general formulas for the action of these eight basic derivative types on the modified restricted Schur polynomials. We will illustrate the basic procedure with an example, leaving a statement of the general result for the next section. In section A.3 we show some examples of the use of the cutting rules.

A.1 Warm Up

The example we study is

$$\begin{aligned} I_2 &= \left(\frac{d}{dZ^c_c}\right)\left(\frac{d}{dZ^d_e}\frac{d}{dZ^e_f}\frac{d}{d(Z^\dagger)^f_d}\right)\left(\frac{d}{dZ^g_h}\frac{d}{d(Z^\dagger)^h_g}\right)\left(\frac{d}{d(Z^\dagger)^k_l}\frac{d}{d(Z^\dagger)^l_k}\right)\times \\ &\times \left(\frac{d}{dZ^a_b}\frac{d}{dW^b_a}\right)\left(\frac{d}{d(Z^\dagger)^m_n}\frac{d}{d(W^\dagger)^n_m}\right)\chi^{(1)}_{R,R_1}(Z,W)\left(\chi^{(1)}_{R,R_1}(Z,W)\right)^\dagger . \end{aligned}$$

Using the notations of (3), computing the derivatives with respect to the open string words gives

$$I_2 = \left(\frac{d}{dZ^c_c}\right)\left(\frac{d}{dZ^d_e}\frac{d}{dZ^e_f}\frac{d}{d(Z^\dagger)^f_d}\right)\left(\frac{d}{dZ^g_h}\frac{d}{d(Z^\dagger)^h_g}\right)\left(\frac{d}{d(Z^\dagger)^k_l}\frac{d}{d(Z^\dagger)^l_k}\right)\frac{d\mathcal{F}^a_b}{dZ^a_b}\frac{d(\mathcal{F}^\dagger)^m_n}{d(Z^\dagger)^m_n}.$$

Computing the remaining derivatives and summing over repeated indices, we easily obtain

$$\begin{aligned}
I_2 &= \left[\frac{1}{(n-6)!}\right]^2 \sum_{\sigma\in S_n}\sum_{\tau\in S_n} \mathrm{Tr}_{R_1}\left(\Gamma_R(\sigma)\right)\mathrm{Tr}_{R_1}\left(\Gamma_R(\tau)\right)^* \times \\
&\times\ Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n-6}}_{i_{\sigma(n-6)}}(Z^\dagger)^{j_1}_{j_{\tau(1)}}\cdots(Z^\dagger)^{j_{n-6}}_{j_{\tau(n-6)}}\delta^{i_n}_{i_{\sigma(n-1)}}\delta^{i_{n-1}}_{i_{\sigma(n)}}\delta^{i_{n-2}}_{j_{\tau(n-4)}} \times \\
&\times\ \delta^{j_{n-4}}_{i_{\sigma(n-2)}}\delta^{i_{n-3}}_{i_{\sigma(n-4)}}\delta^{j_{n-5}}_{i_{\sigma(n-3)}}\delta^{i_{n-4}}_{j_{\tau(n-5)}}\delta^{i_{n-5}}_{i_{\sigma(n-5)}}\delta^{j_n}_{j_{\tau(n-1)}}\delta^{j_{n-1}}_{j_{\tau(n)}}\delta^{j_{n-2}}_{j_{\tau(n-3)}}\delta^{j_{n-3}}_{j_{\tau(n-2)}}. \qquad (104)
\end{aligned}$$

Now, define the permutations

$$P=(n,n-1)(n-3,n-4), \qquad Q=(n,n-1)(n-2,n-3).$$

Further, set

$$\sigma=\psi P, \qquad \tau=\lambda Q.$$

Changing variables in the above sums (104) from σ to ψ and from τ to λ we find

$$\begin{aligned}
I_2 = &\left[\frac{1}{(n-6)!}\right]^2 \sum_{\psi\in S_n}\sum_{\lambda\in S_n} \mathrm{Tr}_{R_1}\left(\Gamma_R(\psi P)\right)\mathrm{Tr}_{R_1}\left(\Gamma_R(\lambda Q)\right)^* \times \\
\times\ &Z^{i_1}_{i_{\psi(1)}}\cdots Z^{i_{n-6}}_{i_{\psi(n-6)}}(Z^\dagger)^{j_1}_{j_{\lambda(1)}}\cdots(Z^\dagger)^{j_{n-6}}_{j_{\lambda(n-6)}}\delta^{i_n}_{i_{\psi(n)}}\delta^{i_{n-1}}_{i_{\psi(n-1)}}\delta^{i_{n-3}}_{i_{\psi(n-3)}} \times \\
\times\ &\delta^{i_{n-5}}_{i_{\psi(n-5)}}\delta^{j_n}_{j_{\lambda(n)}}\delta^{j_{n-1}}_{j_{\lambda(n-1)}}\delta^{j_{n-2}}_{j_{\lambda(n-2)}}\delta^{j_{n-3}}_{j_{\lambda(n-3)}}\delta^{i_{n-2}}_{j_{\lambda(n-4)}}\delta^{j_{n-4}}_{i_{\psi(n-2)}}\delta^{j_{n-5}}_{i_{\psi(n-4)}}\delta^{i_{n-4}}_{j_{\lambda(n-5)}}.
\end{aligned}$$

The reason why we made the change of variables from σ and τ to λ and ψ is now clear: in (104) Kronecker deltas with two i indices or two j indices did not have the property that the upper index was related to the lower index by permutation; after the change of variables, all such Kronecker deltas do have this property. This is useful, because a Kronecker delta with this property is produced by acting on the restricted Schur polynomial with the trace of a derivative. One is tempted to replace all such Kronecker deltas with indices

i_j $j < n$ by the trace of a derivative with respect to Z; this is not quite correct. As an example, $\delta^{i_{n-1}}_{i_{\psi(n-1)}}$ in the last expression above is obtained by differentiating only $Z^{i_{n-1}}_{i_{\psi(n-1)}}$ - the trace of a derivative with respect to Z will generate this term as well as terms that come from acting on every single other Z in the polynomial. Further, due to the prescence of P and Q it really does make a difference which Z is differentiated. This is, however, easily overcome: we can replace $Z^{i_{n-1}}_{i_{\psi(n-1)}}$ by a new matrix $X^{i_{n-1}}_{i_{\psi(n-1)}}$ so that $\delta^{i_{n-1}}_{i_{\psi(n-1)}}$ can safely be replaced by the trace of a derivative with respect to X. We call these new matrices "open string place holders". It is easy to see that I_2 now takes the form

$$I_2 = \mathrm{Tr}\,\frac{d}{dX_1}\mathrm{Tr}\,\frac{d}{dX_3}\mathrm{Tr}\,\frac{d}{dX_5}\mathrm{Tr}\,\frac{d}{dW}\mathrm{Tr}\,\frac{d}{dX_1^\dagger}\mathrm{Tr}\,\frac{d}{dX_2^\dagger}\mathrm{Tr}\,\frac{d}{dX_3^\dagger}\mathrm{Tr}\,\frac{d}{dW^\dagger}\mathrm{Tr}\,\frac{d}{dX_2}\frac{d}{dX_4^\dagger}\times$$

$$\times\mathrm{Tr}\,\frac{d}{dX_4}\frac{d}{dX_5^\dagger}\chi^{(1,5)}_{R,R_1;P}(Z,W)(\chi^{(1,5)}_{R,R_1;Q}(Z,W))^\dagger,$$

where we have introduced the new notation

$$\chi^{(1,m)}_{R,R_1;\Lambda}(Z,W) \equiv \frac{1}{(n-1)!}\sum_{\sigma\in S_n}\mathrm{Tr}_{\,R_1}\left(\Gamma_R(\sigma\Lambda)\right) Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n-m-1}}_{i_{\sigma(n-m-1)}}\prod_{k=1}^{m} X^{i_{n-k}}_{i_{\sigma(n-k)}} W^{i_n}_{i_{\sigma(n)}}\,.$$

In this formula Λ is any element of the symmetric group. Thus, the original derivative operator has been decomposed into a product of basic operations as advertised. The Schur polynomial has been modified by the inclusion of a new factor (Λ in the last equation) inside the trace; we call this factor the *trace insertion*. Since the trace insertion is a new factor in the trace, our notation includes the trace insertion after the existing trace labels.

A.2 General Rule

In this section we give general rules for decomposing a differential operator into a product of basic operations. The full set of basic operations is

$$\mathrm{Tr}\left(\frac{d}{dZ}\right),\quad \mathrm{Tr}\left(\frac{d}{dW}\right),\quad \mathrm{Tr}\left(\frac{d}{dZ^\dagger}\right),\quad \mathrm{Tr}\left(\frac{d}{dW^\dagger}\right),\quad \mathrm{Tr}\left(\frac{d}{dZ}\frac{d}{dZ^\dagger}\right),$$

$$\mathrm{Tr}\left(\frac{d}{dW}\frac{d}{dZ^\dagger}\right),\quad \mathrm{Tr}\left(\frac{d}{dZ}\frac{d}{dW^\dagger}\right)\quad \text{and}\quad \mathrm{Tr}\left(\frac{d}{dW}\frac{d}{dW^\dagger}\right).$$

We call the last four operators "mixed derivatives".

A general rule must give a recipe for reading off the trace insertion and product of basic operations (the new derivative operator) from any differential operator to be disected. Of course, it is just a summary of what happens when one performs the analog of the $\sigma,\tau\to\psi,\lambda$ change of variables of the last section.

In this section, we assume that the open string word is associated with the nth index i_n as in (3). In what follows we will switch to an obvious matrix notation, illustrated in the following example

$$\frac{d}{dZ^a_b}\frac{d}{dW^b_c}\frac{d}{d(Z^\dagger)^c_d}\frac{d}{d(W^\dagger)^d_a}\to(DD_WD^\dagger D_W^\dagger)\,.$$

Terms within a single bracket are traced. We start by giving each of the derivatives with respect to W or Z a label, counting down from n. D_W is given the label n. We then give each of the derivatives with respect to $W^\dagger$ or $Z^\dagger$ a label, again counting down from n. $D_W^\dagger$ is given the label n. As an example, the operator

$$(D)(DDD^\dagger)(DD^\dagger)(D^\dagger D^\dagger)(DD_W)(D^\dagger D_W^\dagger)$$

is labelled as follows (the labels for D, D_W appear above the operator; the labels for $D^\dagger, D^\dagger_W$ appear below the operator)

$$\begin{array}{cccccccccccc}
n-5 & n-4 & n-3 & . & n-2 & . & . & . & n-1 & n & . & . \\
(D) & (D & D & D^\dagger) & (D & D^\dagger) & (D^\dagger & D^\dagger) & (D & D_W) & (D^\dagger & D^\dagger_W)\,. \\
. & . & . & n-5 & . & n-4 & n-3 & n-2 & . & . & n-1 & n
\end{array}$$

The Z derivatives with labels will be replaced with open string place holders. There are two cutting rules:

First cutting rule: If, within any given trace, D (or any other holomorphic derivative) has another holomorphic derivative to its left, it can be removed from the trace and placed into its own trace. The two cycle which swaps the label of D and the label of its neighbour on the left is added, on the left, to the trace insertion of the holomorphic Schur polynomial. If, within any given trace, $D^\dagger$ (or any other antiholomorphic derivative) has another antiholomorphic derivative to its left, it can be removed from the trace and placed into its own trace. The two cycle which swaps the label of $D^\dagger$ and the label of its neighbour on the left is added, on the left, to the trace insertion of the antiholomorphic Schur polynomial.

Second cutting rule: If within any given trace $DD^\dagger$ (or any other product of a holomorphic with an antiholomorphic derivative) has a second $DD^\dagger$ (or any other product of a holomorphic with an antiholomorphic derivative) to its right, then the "middle two" derivatives can be removed from the existing trace and placed into their own trace. The two cycle which swaps the labels of the two holomorphic derivatives is added, on the left, to the trace insertion of the holomorphic Schur polynomial. If within any given trace $D^\dagger D$ (or any other product of an antiholomorphic with a holomorphic derivative) has a second $D^\dagger D$ (or any other product of an antiholomorphic with a holomorphic derivative) to its right, then the "middle two" derivatives can be removed from the existing trace and placed into their own trace. The two cycle which swaps the labels of the two antiholomorphic derivatives is added, on the left, to the trace insertion of the antiholomorphic Schur polynomial.

We have stated the rules using the terms "holomorphic/antiholomorphic" derivative. Stated in this way, the rule are valid even if there is more than one open string attached to the restricted Schur polynomial. Any derivatives cut out of the product, with respect to Z or $Z^\dagger$ are replaced by derivatives with respect to open string place holders.

A.3 Examples

In this appendix we give some examples of how the cutting rules are used. This is done so that the reader can test that she understands how to correctly apply the rules. The operator

$$\mathrm{Tr}\left(\frac{d}{dZ}\frac{d}{dZ}\frac{d}{dZ}\frac{d}{dW}\right)$$

becomes

$$\mathrm{Tr}\left(\frac{d}{dX_3}\right)\mathrm{Tr}\left(\frac{d}{dX_2}\right)\mathrm{Tr}\left(\frac{d}{dX_1}\right)\mathrm{Tr}\left(\frac{d}{dW}\right).$$

The antiholomorphic trace insertion is 1; the holomorphic trace insertion is $(n-3, n-2, n-1, n)$. The operator

$$\mathrm{Tr}\left(\frac{d}{dZ}\frac{d}{dZ^\dagger}\frac{d}{dZ}\frac{d}{dZ^\dagger}\frac{d}{dZ}\frac{d}{dZ^\dagger}\right)$$

becomes

$$\mathrm{Tr}\left(\frac{d}{dX_3}\frac{d}{dX_2^\dagger}\right)\mathrm{Tr}\left(\frac{d}{dX_2}\frac{d}{dX_1^\dagger}\right)\mathrm{Tr}\left(\frac{d}{dX_1}\frac{d}{dX_3^\dagger}\right).$$

The antiholomorphic trace insertion is 1; the holomorphic trace insertion is $(n-3,n-2)(n-1,n-2)$. By cycling a derivative around the operator we have dissected can be written as

$$\mathrm{Tr}\left(\frac{d}{dZ^\dagger}\frac{d}{dZ}\frac{d}{dZ^\dagger}\frac{d}{dZ}\frac{d}{dZ^\dagger}\frac{d}{dZ}\right)$$

Cutting this operator up gives a holomorphic trace insertion of 1 and a non-trivial antiholomorphic trace insertion. Clearly the result of cutting is not unique. Of course, these different dissections all lead to the same value for the correlation function.

B Mixed Derivative Rules

In this appendix we will explain how to evaluate

$$\left\langle \left[\hat{O} \chi^{(1)}_{R,R_1}(Z,W)(\chi^{(1)}_{S,S_1}(Z,W'))^\dagger \right] \right\rangle$$

in free field theory, in the case that $\hat{O}$ is one of the mixed derivative operators. All the arguments in this appendix are unchanged if a trace insertion factor is included.

B.1 $\hat{O} = \mathbf{Tr}\left(\frac{d}{dZ}\frac{d}{dZ^\dagger}\right)$

Consider the Schwinger-Dyson equation

$$0 = \int dZ dZ^\dagger dY dY^\dagger \frac{d}{dZ^a_b}\left(\chi^{(1)}_{R,R_1}(Z,W) \left[\frac{d}{d(Z^\dagger)^b_a} (\chi^{(1)}_{S,S_1}(Z,W'))^\dagger \right] e^{-S} \right),$$

where

$$S = \mathrm{Tr}\,(ZZ^\dagger + YY^\dagger).$$

The Schwinger-Dyson equation implies

$$\begin{aligned} \left\langle \left[\mathrm{Tr}\left(\frac{d}{dZ}\frac{d}{dZ^\dagger}\right) \chi^{(1)}_{R,R_1}(Z,W)(\chi^{(1)}_{S,S_1}(Z,W'))^\dagger \right] \right\rangle &= \left\langle \chi^{(1)}_{R,R_1}(Z,W)(Z^\dagger)^b_a \left[\frac{d}{d(Z^\dagger)^b_a}(\chi^{(1)}_{S,S_1}(Z,W'))^\dagger \right] \right\rangle \\ &= n_{Z^\dagger} \left\langle \chi^{(1)}_{R,R_1}(Z,W)(\chi^{(1)}_{S,S_1}(Z,W'))^\dagger \right\rangle \end{aligned}$$

where $n_{Z^\dagger}$ is the number of $Z^\dagger$ matrices appearing in $(\chi^{(1)}_{S,S_1}(Z,W'))^\dagger$. The correlator $\langle \chi^{(1)}_{R,R_1}(Z,W)(\chi^{(1)}_{S,S_1}(Z,W'))^\dagger \rangle$ is now easily evaluated using the results of [31].

B.2 $\hat{O} = \mathbf{Tr}\left(\frac{d}{dW'}\frac{d}{dW^\dagger}\right)$

This operator simply "contracts" the two open string words. The most general form that the two point function of open string words can take is

$$\left\langle (W)^i_j (W^\dagger)^k_l \right\rangle = F_0 \delta^i_l \delta^k_j + F_1 \delta^i_j \delta^k_l .$$

$$\left\langle \mathrm{Tr}\left(\frac{d}{dW'}\frac{d}{dW^\dagger}\right) \chi^{(1)}_{R,R_1}(Z,W')(\chi^{(1)}_{S,S_1}(Z,W))^\dagger \right\rangle$$

is simply equal to the F_0 contribution to the correlator above.

B.3 $\hat{O} = \mathbf{Tr}\left(\frac{d}{dZ}\frac{d}{dW^\dagger}\right)$

Explicitely performing the derivative with respect to Z in

$$I = \frac{d}{dZ^e_d}\chi^{(1)}_{R,R_1}(Z,W)\frac{d}{d(W^\dagger)^d_e}(\chi^{(1)}(Z,W))^\dagger$$

we obtain

$$\begin{aligned} I &= \frac{1}{(n-2)!}\sum_{\sigma\in S_n} \mathrm{Tr}_{R_1}\left(\Gamma_R(\sigma)\right) Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_{n-2}}_{i_{\sigma(n-2)}} \delta^{i_{n-1}}_e \delta^d_{i_{\sigma(n-1)}} W^{i_n}_{i_{\sigma(n)}} \frac{d}{d(W^\dagger)^d_e}(\chi^{(1)}(Z,W))^\dagger \\ &= \frac{1}{(n-2)!}\frac{d}{dX^e_d}\sum_{\sigma\in S_n} \mathrm{Tr}_{R_1}\left(\Gamma_R(\sigma)\right) Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_{n-2}}_{i_{\sigma(n-2)}} X^{i_{n-1}}_{i_{\sigma(n-1)}} W^{i_n}_{i_{\sigma(n)}} \frac{d}{d(W^\dagger)^d_e}(\chi^{(1)}(Z,W))^\dagger . \end{aligned}$$

If we now introduce the representations T_α defined by removing a single box from R_1, so that

$$R_1 = \oplus_\alpha T_\alpha,$$

we obtain

$$I = \sum_\alpha \frac{d}{dX^e_d}\chi^{(2)}_{R,T_\alpha}(Z,X,W)\frac{d}{d(W^\dagger)^d_e}(\chi^{(1)}(Z,W))^\dagger,$$

where in the restricted Schur polynomial $\tilde{\chi}^{(2)}_{R,T_\alpha}(Z,X,W)$, W is associated with the box that must be removed from R to obtain R_1 and X is associated with the box that must be removed from R_1 to obtain T_α. After using the subgroup swap rule of [31] to swap X and W, this correlator can be evaluated exactly as in the previous subsection.

B.4 $\hat{O} = \mathbf{Tr}\left(\frac{d}{dW}\frac{d}{dZ^\dagger}\right)$

The evaluation of this term is essentially the same as the term treated in the last subsection.

C Reduction Rules

In this section we will consider the action of

$$\mathrm{Tr}\left(\frac{d}{dZ}\right)\equiv D_Z, \quad \text{and} \quad \mathrm{Tr}\left(\frac{d}{dW}\right)\equiv D_W,$$

on restricted Schur polynomials. By D_W we mean either a reduction with respect to the open string attached to the restricted Schur polynomial or with respect to any of the open string place holders. We call these "reductions" of the restricted Schur polynomial because the action of the operators removes boxes from the Young diagram label of the polynomial. The action of D_W on a restricted Schur polynomial has been worked out in [31]. D_W removes the box associated with W, thereby producing a Schur polynomial and multiplies this polynomial by the weight of the removed box.

Now, consider the action of D_Z. If D_Z acts after D_W has acted, we need the action of D_Z on a Schur polynomial. This action has been worked out in [20] and [31]. D_Z when acting on a Schur polynomial produces all Schur polynomials that can be obtained by removing a single box from the Schur polynomial it acts on. Each of the polynomials produced are multiplied by the weight of the removed box.

Finally, we will evaluate the action of D_Z on a restricted Schur polynomial. By explicitely evaluating the derivative, we have

$$\begin{aligned}
\frac{d}{dZ^a_a}\chi^{(1)}_{R,R_1}(Z,W) &= \frac{1}{(n-2)!}\sum_{\sigma\in S_n}\mathrm{Tr}\left(\Gamma_R(\sigma)\right) Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n-2}}_{i_{\sigma(n-2)}}\delta^{i_{n-1}}_{i_{\sigma(n-1)}}W^{i_n}_{i_{\sigma(n)}} \\
&= D_X\sum_\alpha \chi^{(2)}_{R,T_\alpha}(Z,X,W), \qquad (105)
\end{aligned}$$

where in the restricted Schur polynomial $\chi^{(2)}_{R,T_\alpha}(Z,X,W)$, W is associated with the box that must be removed from R to obtain R_1 and X is associated with the box that must be removed from R_1 to obtain T_α. In this last formula, the representations T_α are all representations that can be obtained by removing a single box from R_1, so that

$$R_1 = \oplus_\alpha T_\alpha .$$

The reduction with respect to X in (105) is now easily computed using the subgroup swap rule of [31]. Clearly, the arguments in this appendix are unchanged if a trace insertion factor is included.

C.1 Example

For this subsection we will use a graphical notation for the labels of the restricted Schur polynomial. We draw R as a Young diagram and write the open string word w in the box which must be removed to obtain R_1. Similarly, we write x into the box that must be removed to obtain T_α. In this notation, an explicit example of (105) is

$$D_Z \chi_{\ytableaushort{{}{}w,{}}} = D_x \left(\chi_{\ytableaushort{{}{}w,x}} + \chi_{\ytableaushort{{}xw,{}}} \right) .$$

We can simply evaluate the action of D_x because when the polynomial is constructed we first reduce with to the w box and then with respect to the x box; we need to swap these two using the subgroup swap rule. To apply the subgroup swap rule, we do not need to worry about twisted string states because we are reducing with respect to x (see [31]). Thus, after swapping we obtain

$$D_x \left(\frac{1}{9} \chi_{\ytableaushort{{}{}x,w}} + \frac{8}{9} \chi_{\ytableaushort{{}{}w,x}} + \chi_{\ytableaushort{{}wx,{}}} \right) .$$

To reduce with respect to x we now simply remove the box populated by x and multiply by its weight so that we finally obtain

$$D_Z \chi_{\ytableaushort{{}{}w,{}}} = \frac{N+2}{9}\chi_{\ytableaushort{{}{},w}} + \frac{8(N-1)}{9}\chi_{\ytableaushort{{}{}w}} + (N+2)\chi_{\ytableaushort{{}w,{}}}.$$

D Formulas for Restricted Characters

The cutting rules introduce an insertion factor for each restricted Schur polynomial in the correlator. Evaluating this extra factor is most easily done using restricted characters. In [33] general formulas for restricted characters were obtained. In this appendix we will review these methods. In the next appendix we illustrate our methods with a nontrivial example.

A restricted character is given by taking a restricted trace of a group element. By a restricted trace, we mean that we don't trace over the whole carrier space on which the group acts; we trace only over a subspace

$$\chi_{R,R_1}(\sigma) = \mathrm{Tr}_{R_1}\left(\Gamma_R(\sigma)\right).$$

R is an irrep of S_n; we can think of R as a Young diagram with n boxes. The subspace R_1 is the carrier space of a subgroup of S_n. Consequently, a convenient way to specify which subspace of the full space we consider, is by knocking boxes off the Young diagram R; the smaller Young diagram is R_1. Finally, we also need to consider restricted characters in which the row and column indices are traced over different subspaces. In this case, we compute

$$\chi_{R,R_1R_2}(\sigma) = \mathrm{Tr}_{R_1R_2}\left(\Gamma_R(\sigma)\right)$$

by summing the row index over R_1 and the column index over R_2. This requires that we have an isomorphism between R_1 and R_2 because we need to correlate the row and column indices in the sum. This ismorphism amounts to a choice of basis and is specified by requiring for σ in the subgroup of which R_1 and R_2 are irreducible representations, we have $\Gamma_{R_1}(\sigma) = \Gamma_{R_2}(\sigma)$. We represent these subspaces graphically by drawing R as a Young diagram and placing two labels in each box to be dropped. If a total of m boxes are to be dropped the labels run from 1 to m. To get the row (column) subspace R_1 (R_2) drop boxes from R according to the upper (lower) index in each box.

Looking back at the cutting rules, it is clear that we only need to compute restricted characters of cycles $(i_1 i_2 \cdots i_k)$ for the case that the indices $i_1, i_2, \cdots i_k$ are associated to dropped boxes, i.e. they are left inert by the subgroup whose carrier space we trace over. We have this in mind for the remainder of this appendix. The general algorithm used to compute these restricted characters has three steps:

- Decompose the group element whose trace is to be computed into a product of two cycles of the form $\Gamma_R\left((i, i+1)\right)$. Insert a complete set of states between each factor.
- The only non-zero matrix elements of each $\Gamma_R\left((i, i+1)\right)$ factor, are obtained when the order of boxes dropped to obtain the carrier space of the bra matches the order of boxes dropped to obtain the carrier space of the ket, except for the $(n-i+1)^{\text{th}}$ and $(n-i+2)^{\text{th}}$ boxes, whose order can be swapped.
- The known value of the matrix elements for precisely the two cases arising in the previous point are plugged in to get the value of the restricted character.

A very convenient way to implement this algorithm is by using *strand diagrams* [33]. If, after factorizing the group element as described in the first point above, n indices are involved, we draw a picture with n columns. The columns are populated by labeled strands - each strand represents one of the boxes that are to be dropped. Label the strands by the upper index in the box. The box that appears in the first column is to be dropped first; the box in the second column is to be dropped second and so on. The strands are ordered at the top of the diagram, according to the order in which they must be dropped to get the row index. The strands are ordered at the bottom of the diagram according to the column index. The strands move from the top of the diagram to the bottom of the diagram, without breaking, so that strand ends at the top connect to the corresponding strand ends at the bottom. To connect the strands (which in general are in a different order at the top and bottom of the diagram) we need to weave the strands, thereby allowing them to swap columns. The allowed swaps depends on the specific group element whose trace we are computing. To determine the allowed swaps, write the group element as a product of cycles of the form $(i, i+1)$. Each cycle $(i, i+1)$ is drawn as a box which straddles the columns i and $i+1$. Boxes on the right are drawn above boxes on the left. When the strands pass through a box, they may do so without swapping or by swapping columns. Each box is associated with a factor. Imagine that the strands passing through the box, reading from left to right, are labeled n and m. The weights associated with these boxes are c_n and c_m respectively. If the strands do not swap inside the box the factor for the box is

$$f_{\text{no swap}} = \frac{1}{c_n - c_m}.$$

If the strands do swap inside the box, the factor is

$$f_{\text{swap}} = \sqrt{1 - \frac{1}{(c_n - c_m)^2}}.$$

Denote the product of the factors, one from each box, by F. We have

$$\mathrm{Tr}_{R_1,R_2}\Big(\Gamma_R(\sigma)\Big) = \sum_i F_i \mathrm{dim}_{R_1},$$

where the index i runs over all possible paths consistent with the boundary conditions.

With a little thought, the astute reader should be able to convince herself that this graphical rule is nothing but a convenient representation of the algorithm given above. We end with an example. The character

$$\chi_1 = \mathrm{Tr}_{\begin{ytableau} \; & \; & {}^{1}_{3} \\ \; & {}^{2}_{1} \\ {}^{3}_{2} \end{ytableau}} \Big(\Gamma_{\begin{ytableau} \; & \; & \; \\ \; & \; \\ \; \end{ytableau}}((6,4))\Big).$$

is represented by the strand diagram of figure 9. To obtain this strand dia-

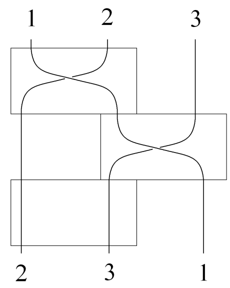


Figure 9: The strand diagram used in the computation of χ_1.

gram write $(6,4) = (6,5)(4,5)(6,5)$. The factors for the upper most, middle and lower most boxes are $\sqrt{1-\frac{1}{(c_1-c_2)^2}}$, $\sqrt{1-\frac{1}{(c_1-c_3)^2}}$, and $\frac{1}{c_2-c_3}$ respectively. Thus,

$$\begin{aligned}\chi_1 &= \sqrt{1-\frac{1}{(c_1-c_2)^2}}\sqrt{1-\frac{1}{(c_1-c_3)^2}}\frac{1}{c_2-c_3}\dim_{\begin{ytableau} \; & \; \\ \; \end{ytableau}} \\ &= 2\sqrt{1-\frac{1}{(c_1-c_2)^2}}\sqrt{1-\frac{1}{(c_1-c_3)^2}}\frac{1}{c_2-c_3}.\end{aligned}$$

For further details and more examples, see [33].

E Example Correlator

In this appendix we give the details of the computation of a correlator of the type considered in section 2.2.2

$$I_{RR_1,RR_1} = \left\langle \chi^{(1)}_{R,R_1} (\chi^{(1)}_{R,R_1})^\dagger \right\rangle .$$

We deal with three impurities in the open string $W^i_j = (YZ^3Y)^i_j$ and take R_1 to be the rectangular Young diagram with N rows and M columns with $M = O(N)$. R is given by adding a box in the upper right hand corner, i.e. in the first row. This example is already involved enough to nicely illustrate the use of our technology.

No Brane/String Contractions: This contribution comes from the diagram given below.

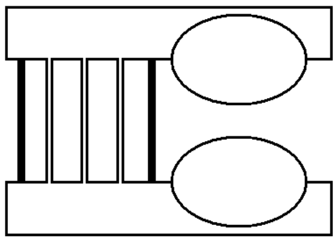


Figure 10: The contribution with no brane/string contractions.

Using the rules of [31] we easily obtain, at leading order in a large N expansion

$$I^{(0)}_{RR_1,RR_1} = N^4 \frac{\text{hooks}_R}{\text{hooks}_{R'}} f_R = N^3(M+N) f_R .$$

One Brane/String Contraction: This contribution comes from the three diagrams given below.

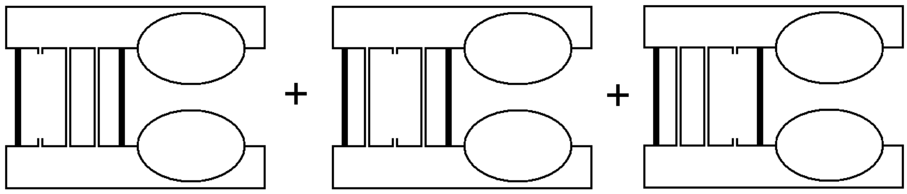


Figure 11: The contribution with one brane/string contraction.

All three diagrams give the same contibution. We do not need to use our cutting rules yet; we do use the results of appendices B.1 and B.2. The result is

$$
\begin{aligned}
I^{(1)}_{RR_1,RR_1} &= 3N^2 \left\langle \mathrm{Tr}\left(\frac{d}{dZ}\frac{d}{dZ^\dagger}\right)\mathrm{Tr}\left(\frac{d}{dW}\frac{d}{dW^\dagger}\right)\chi^{(1)}_{R,R_1}(\chi^{(1)}_{R,R_1})^\dagger\right\rangle \\
&= 3N^2(MN)\frac{\mathrm{hooks}_R}{\mathrm{hooks}_{R'}}f_R = 3MN^2(M+N)f_R\,.
\end{aligned}
$$

Two Brane/String Contractions: This contribution comes from the three diagrams given below.

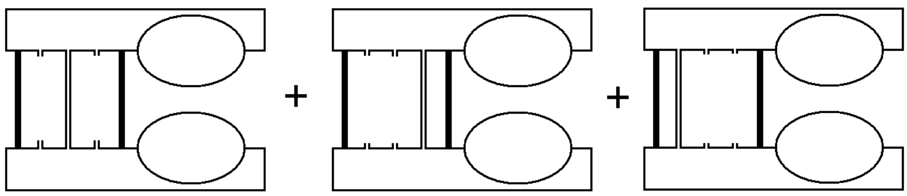


Figure 12: The contribution with two brane/string contractions.

The first diagram is the simplest to evaluate. We can again do it without using the cutting rules. The result is

$$
\left\langle \mathrm{Tr}\left(\frac{d}{dZ}\frac{d}{dZ^\dagger}\right)^2\mathrm{Tr}\left(\frac{d}{dW}\frac{d}{dW^\dagger}\right)\chi^{(1)}_{R,R_1}(\chi^{(1)}_{R,R_1})^\dagger\right\rangle
$$

$$
= (MN)^2\frac{\mathrm{hooks}_R}{\mathrm{hooks}_{R'}}f_R = M^2N(M+N)f_R\,.
$$

The evaluation of the second and third diagrams are exactly the same. Consider the second diagram. We need to evaluate

$$N\left\langle \mathrm{Tr}\left(\frac{d}{dZ}\frac{d}{dZ}\frac{d}{dZ^\dagger}\frac{d}{dZ^\dagger}\right)\mathrm{Tr}\left(\frac{d}{dW}\frac{d}{dW^\dagger}\right)\chi^{(1)}_{R,R_1}(\chi^{(1)}_{R,R_1})^\dagger\right\rangle .$$

We now need to use our cutting rules and the associated open string holders. We start to use the graphical notation that draws the Young diagram, with the open string word (w) and the open string place holders (1 and 2) on R (see appendices C.1 and G). We draw R_1 with 5 rows and 5 columns, but our results hold for general M and N. After cutting we have to evaluate

$$\chi_A = \mathrm{Tr}\left(\frac{d}{dX_1}\right)\left[\tilde{\chi}\begin{ytableau} \; & \; & \; & \; & \; & w \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & 2 \\ \; & \; & \; & \; & 1 \end{ytableau} + \tilde{\chi}\begin{ytableau} \; & \; & \; & \; & \; & w \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \\ \; & \; & \; & 2 & 1 \end{ytableau}\right].$$

The tilde on χ is to denote the fact that there is a trace insertion factor of $(n-1, n-2)$ arising from the cutting. Using strand diagrams we can eliminate the trace insertion factors for each term

$$\tilde{\chi}\begin{ytableau} \; & \; & \; & \; & \; & w \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & 2 \\ \; & \; & \; & \; & 1 \end{ytableau} = -\chi\begin{ytableau} \; & \; & \; & \; & \; & w \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & 2 \\ \; & \; & \; & \; & 1 \end{ytableau}, \qquad \tilde{\chi}\begin{ytableau} \; & \; & \; & \; & \; & w \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \\ \; & \; & \; & 2 & 1 \end{ytableau} = \chi\begin{ytableau} \; & \; & \; & \; & \; & w \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \\ \; & \; & \; & 2 & 1 \end{ytableau}.$$

After using the subgroup swap rule to swap w and X_1, we can compute the reduction to obtain (there are some terms that arise from swapping w and

1; these are however $O\left(\frac{1}{N^2}\right)$ so they can be dropped to leading order in N)

$$\chi_A = -M\chi_{\begin{ytableau} \ & \ & \ & \ & \ & w \\ \ & \ & \ & \ & \ \\ \ & \ & \ & \ & \ \\ \ & \ & \ & \ & 2 \\ \ & \ & \ & \ \end{ytableau}} + M\chi_{\begin{ytableau} \ & \ & \ & \ & \ & w \\ \ & \ & \ & \ & \ \\ \ & \ & \ & \ & \ \\ \ & \ & \ & \ & \ \\ \ & \ & \ & 2 \end{ytableau}}$$

To get the contribution from the second diagram, we now simply need to compute

$$N\left\langle \mathrm{Tr}\left(\frac{d}{dX_2}\frac{d}{dX_2^\dagger}\right)\mathrm{Tr}\left(\frac{d}{dW}\frac{d}{dW^\dagger}\right)\chi_A\chi_A^\dagger\right\rangle .$$

To obtain this, we need to use

$$h_1 = \frac{\mathrm{hooks}_{\ydiagram{6,5,5,5,4}}}{\mathrm{hooks}_{\ydiagram{5,5,5,4,4}}} = \frac{M(M+N)}{2}, \qquad h_2 = \frac{\mathrm{hooks}_{\ydiagram{6,5,5,5,4}}}{\mathrm{hooks}_{\ydiagram{5,5,5,5,3}}} = \frac{M(M+N)}{2}.$$

It is now straight forward to obtain

$$\begin{aligned} N\left\langle \mathrm{Tr}\left(\frac{d}{dX_2}\frac{d}{dX_2^\dagger}\right)\mathrm{Tr}\left(\frac{d}{dW}\frac{d}{dW^\dagger}\right)\chi_A\chi_A^\dagger\right\rangle &= NMf_R\,(h_1+h_2) \\ &= NM^2(M+N)f_R\,. \end{aligned}$$

Notice that although the computation for diagram 2 was completely different to the computation for diagram 1, they give exactly the same result. As already mentioned, the third diagram gives exactly the same contribution as the second so that

$$I^{(2)}_{RR_1,RR_1} = 3M^2N(M+N)f_R\,.$$

Three Brane/String Contractions: This contribution comes from the diagram given below.

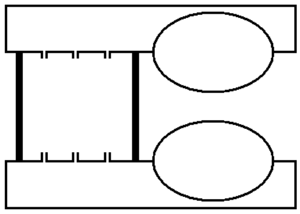


Figure 13: The contribution with three brane/string contractions.

For this contribution we need to evaluate

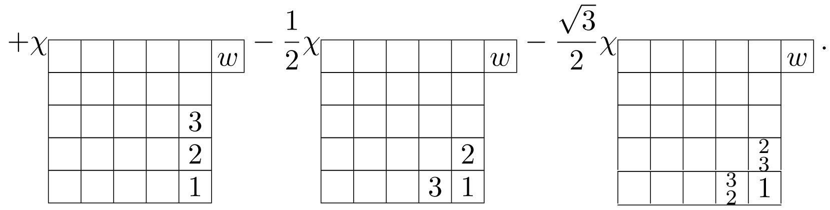
$$\left\langle \mathrm{Tr}\left(\frac{d}{dZ}\frac{d}{dZ}\frac{d}{dZ}\frac{d}{dZ^\dagger}\frac{d}{dZ^\dagger}\frac{d}{dZ^\dagger}\right)\mathrm{Tr}\left(\frac{d}{dW}\frac{d}{dW^\dagger}\right)\chi^{(1)}_{R,R_1}(\chi^{(1)}_{R,R_1})^\dagger\right\rangle .$$

We cut two holomorphic derivatives and two antiholomorphic derivatives out of the trace. Thus, we will need a total of three open string place holders; the trace insertion factor is $(n-3,n-2)(n-1,n-2)$. To recover the trace over R_1 we again need to sum over all ways of distributing the open string place holders. The result is

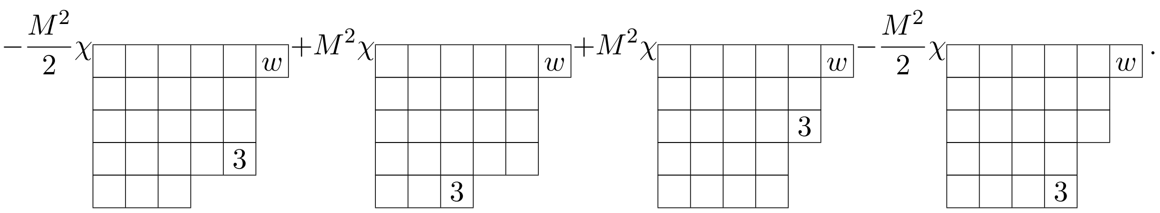
$$\tilde{\chi}\begin{ytableau} \; & \; & \; & \; & \; & w \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & 3 \\ \; & \; & \; & 2 & 1 \end{ytableau} + \tilde{\chi}\begin{ytableau} \; & \; & \; & \; & \; & w \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \\ \; & \; & 3 & 2 & 1 \end{ytableau} + \tilde{\chi}\begin{ytableau} \; & \; & \; & \; & \; & w \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & 3 \\ \; & \; & \; & \; & 2 \\ \; & \; & \; & \; & 1 \end{ytableau} + \tilde{\chi}\begin{ytableau} \; & \; & \; & \; & \; & w \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & 2 \\ \; & \; & \; & 3 & 1 \end{ytableau} .$$

After accounting for the trace insertion factor, we obtain

$$-\frac{1}{2}\chi\begin{ytableau} \; & \; & \; & \; & \; & w \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & 3 \\ \; & \; & \; & 2 & 1 \end{ytableau} + \frac{\sqrt{3}}{2}\chi\begin{ytableau} \; & \; & \; & \; & \; & w \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & {}^{3}_{2} \\ \; & \; & \; & {}^{2}_{3} & 1 \end{ytableau} + \chi\begin{ytableau} \; & \; & \; & \; & \; & w \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \\ \; & \; & 3 & 2 & 1 \end{ytableau}$$



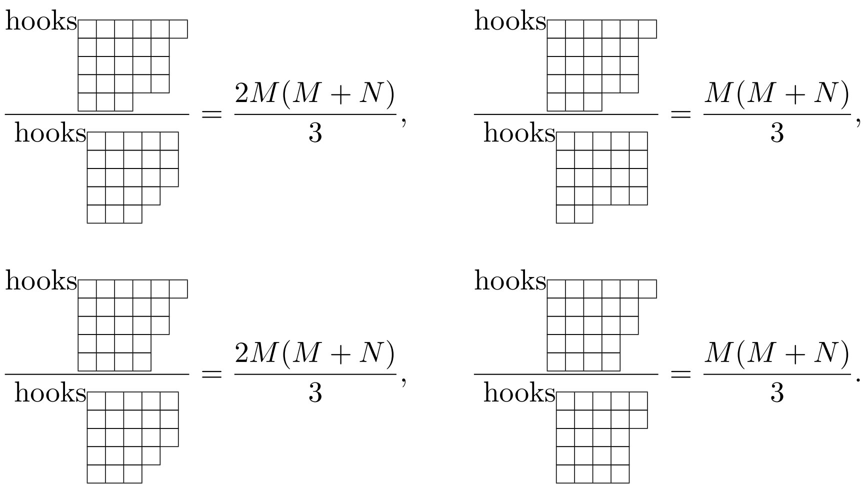
We now need to use the subgroup swap rule so that we can reduce with respect to X_1 and X_2. There is again a dramatic simplification because the terms in which the location of w changes are suppressed at large N. The result after reducing is



To get the contribution from the three brane/string contractions, we now need to compute

$$\left\langle \chi_A \chi_A^\dagger \right\rangle = M^3(M+N) f_R.$$

To get this we used



Putting things together, we have

$$I_{RR_1,RR_1} = (N^3 + 3MN^2 + 3M^2N + M^3)(M+N) f_R = \left(1 + \frac{M}{N}\right)^3 N^3 (M+N) f_R \,.$$

F Exact Results for the Annulus

In this appendix we consider a background $\chi_B(Z)$ where B is a Young diagram with M columns and N rows. We will compute the two correlators

$$I_1 = \frac{\langle \chi_B \chi_B^\dagger \mathrm{Tr}\,(Z^n Z^{\dagger\, n})\rangle}{\langle \chi_B \chi_B^\dagger \rangle},$$

and

$$I_2 = \frac{\langle \mathrm{Tr}\,\left(\frac{d^n}{dZ^n}\frac{d^n}{dZ^{\dagger\, n}}\right)\chi_B \chi_B^\dagger\rangle}{\langle \chi_B \chi_B^\dagger \rangle},$$

in the large N limit.

F.1 Computation of I_1

We will make use of a dummy field D which does not interact with Z and has a two point function

$$\left\langle (D^\dagger)^k_l D^i_j \right\rangle = \delta^i_l \delta^k_j,$$

Including D does not change the value of any normalized correlation functions of operators built only out of Z and $Z^\dagger$. In particular, it does not change the value of I_1. Using D, we can rewrite

$$I_1 = \frac{\langle \chi_B \chi_B^\dagger \mathrm{Tr}\,(Z^n D) \mathrm{Tr}\,(D^\dagger Z^{\dagger\, n})\rangle}{\langle \chi_B \chi_B^\dagger \rangle}.$$

This is a useful step, because after using the identities

$$\mathrm{Tr}\,(Z^n D) = \frac{1}{n+1} D_{ij} \frac{d}{dZ_{ij}} \mathrm{Tr}\,(Z^{n+1}),$$

and (this identity was proved in appendix 6 of [22])

$$\mathrm{Tr}\,(Z^{n+1}) = \sum_{s=0}^{n} (-1)^s \chi_{(n+1-s,1^s)}(Z),$$

where $(n+1-s, 1^s)$ denotes a Young diagram with $s+1$ rows; the first row has $n+1-s$ boxes and all remaining rows have one box, we can write $\mathrm{Tr}\,(Z^n D)$ as a sum of restricted Schur polynomials

$$\mathrm{Tr}\,(Z^n D) = \frac{1}{n+1}\sum_{s=0}^{n}\sum_{h_s}(-1)^s \chi_{(n+1-s,1^s),h_s}(Z,D),$$

where h_s is an irreducible representation of S_n. The sum over h_s is a sum over all possible S_n irreducible representations that can be suduced from the S_{n+1} representation $(n+1-s, 1^s)$. To proceed, we would like to evaluate the product

$$\chi_B(Z)\chi_{(n+1-s,1^s),h_s}(Z,D) \tag{106}$$

for any s and h_s. This product can be computed using the restricted Littlewood-Richardson rule derived in the second of [49]. The difficult part of this computation entails evaluating the restricted Littlewood-Richardson numbers, which include the sum

$$\sum_{\sigma_1\in S_{NM}}\sum_{\sigma_2\in S_{n+1}}\chi_B(\sigma_1)\chi_{(n+1-s,1^s),h_s}(\sigma_2)\chi_{R,R'}(\sigma_1\circ\sigma_2).$$

To evaluate this sum, note that both

$$\frac{d_B}{NM!}\sum_{\sigma_1\in S_{NM}}\chi_B(\sigma_1)\sigma_1 \qquad \text{and} \qquad \frac{d_{(n+1-s,1^s)}}{(n+1)!}\sum_{\sigma_2\in S_{n+1}}\chi_{(n+1-s,1^s),h_s}(\sigma_2)\sigma_2$$

are projection operators. Thus, the sum we need to compute is simply the partial trace (over (R,R')) of the direct product of two projectors. In general this is not a very useful observation because one can't choose a basis which is both simultaneously a basis of B and $((n+1-s,1^s),h_s)$ on the one hand and (R,R') on the other. However, for the case we consider here a simultaneous basis can indeed be chosen as we now explain.

The above sum is needed to compute the coefficient of the term $\chi_{R,R'}(Z,D)$ appearing in the product (106). Since B has N rows, we can only stack $((n+$

$1-s,1^s),h_s)$ as a complete Young diagram, to the right of B; denote this new Young diagram by $(+(n+1-s,1^s),+h_s)$. To see that this is the case, note that we could start with (R,R') which is an irreducible representation of S_{NM+n+1} and keep restricting to smaller and smaller subgroups, by freezing the indices that S_{n+1} acts on. Doing $n+1$ restrictions we have the subgroup S_{NM} and we must have reduced (R,R') to B. This forces (R,R') to be $(+(n+1-s,1^s),+h_s)$ and it provides a simultaneous basis for B and $((n+1-s,1^s),h_s)$ and for $(+(n+1-s,1^s),+h_s)$.

It is now straight forward to see that

$$\frac{d_B}{NM!}\frac{d_{(n+1-s,1^s)}}{(n+1)!}\sum_{\sigma_1\in S_{NM}}\sum_{\sigma_2\in S_{n+1}}\chi_B(\sigma_1)\chi_{(n+1-s,1^s),h_s}(\sigma_2)\chi_{+(n+1-s,1^s),+h_s}(\sigma_1\circ\sigma_2)=d_{h_s}d_B,$$

where the right hand side is nothing but the dimension of the space that we traced over. Consequently,

$$\sum_{\sigma_1\in S_{NM}}\sum_{\sigma_2\in S_{n+1}}\chi_B(\sigma_1)\chi_{(n+1-s,1^s),h_s}(\sigma_2)\chi_{R,R'}(\sigma_1\circ\sigma_2)=(n+1)!NM!\frac{d_{h_s}}{d_{(n+1-s,1^s)}}.$$

Some straightforward manipulations now give

$$\chi_B(Z)\mathrm{Tr}\,(Z^nD)=\frac{1}{n+1}\frac{N}{N+M}\sum_{s=0}^{n}\sum_{h_s}(-1)^s\chi_{+(n+1-s,1^s),+h_s}(Z,D).$$

Thus, we have reduced the computation of I_1 to the computation of a two point function which is easily performed (we keep only the leading term at large N)

$$\begin{aligned}
I_1 &= \frac{\langle\chi_B\chi_B^\dagger\mathrm{Tr}\,(Z^nZ^{\dagger\,n})\rangle}{\langle\chi_B\chi_B^\dagger\rangle}\\
&= \frac{1}{f_B}\frac{1}{(n+1)^2}\frac{N^2}{(N+M)^2}\langle\sum_{s=0}^{n}\sum_{h_s}(-1)^s\chi_{+(n+1-s,1^s),+h_s}(Z,D)\sum_{t=0}^{n}\sum_{h_t}(-1)^t\chi_{+(n+1-t,1^t),+h_s}(Z,D)^\dagger\rangle\\
&= \frac{1}{(n+1)^2}N(M+N)^n\sum_{s=0}^{n}\sum_{h_s}\frac{(\mathrm{hooks})_{(n+1-s,1^s)}}{(\mathrm{hooks})_{h_s}}\\
&= N(M+N)^n\,.
\end{aligned}$$

F.2 Computation of I_2

It is clear that we can write

$$I_2 = \frac{\langle \chi_B \chi_B^\dagger : \mathrm{Tr}\,(Z^n Z^{\dagger\, n}) :\rangle}{\langle \chi_B \chi_B^\dagger \rangle} \equiv \langle : \mathrm{Tr}\,(Z^n Z^{\dagger\, n}) : \rangle_B ,$$

where $: O :$ denotes the normal ordering of O. Thus, we can obtain I_2 from I_1 by subtracting all terms with an odd number of self contractions (contractions between two fields in $\mathrm{Tr}\,(Z^n Z^{\dagger\, n})$) from I_1 and adding back all the terms with an even number of self contractions. The term with one self contraction, for example, gives

$$\sum_{r=1}^{n} \langle \mathrm{Tr}\,(Z^{n-r}(Z^\dagger)^{n-r}) \mathrm{Tr}\,(Z^{r-1}(Z^\dagger)^{r-1}) \rangle_B = \sum_{r=1}^{n} \langle \mathrm{Tr}\,(Z^{n-r}(Z^\dagger)^{n-r}) \rangle_B \langle \mathrm{Tr}\,(Z^{r-1}(Z^\dagger)^{r-1}) \rangle_B$$

$$= \sum_{r=1}^{n} N(M+N)^{n-r} N(M+N)^{r-1} = nN^2(M+N)^{n-1}.$$

To obtain this result we made use of large N factorization and the result of the previous subsection. A very similar argument gives

$$\frac{n!}{c!(n-c)!} N^{1+c}(N+M)^{n-c}$$

for the term with c self contractions. Thus

$$\begin{aligned} I_2 &= N(N+M)^n - \sum_{c=1}^{n} \frac{n!}{c!(n-c)!} (-N)^{1+c}(N+M)^{n-c} \\ &= N \sum_{c=0}^{n} \frac{n!}{c!(n-c)!} (-N)^{c}(N+M)^{n-c} \\ &= N(N+M-N)^n \\ &= NM^n \,. \end{aligned}$$

G Last Site Dictionary

In this section we will explain how to translate between a "closed string" description of the operator

$$w = \mathrm{Tr}\,(YZ^{n_1}YZ^{n_2}Y\cdots YZ^{n_L})$$

and an "open string" description

$$\sum_{R,R'}\alpha_{R,R'}\chi^{(1)}_{R,R'}(Z,w) \qquad w^i_j = (YZ^{n_1}YZ^{n_2}Y\cdots YZ^{n_{L-1}}Y)^i_j,$$

where in this second description the last site is described by the Young diagrams R, R'. One simply makes repeated use of the identity

$$\chi^{(1)}_{R,R'}(Z,w) - \chi_{R'}(Z)\mathrm{Tr}\,(w) = \sum_\alpha \frac{1}{d_{R''_\alpha}}\mathrm{Tr}_{\,R''_\alpha}(\Gamma_R\,[(n,n-1)])\chi^{(1)}_{R',R''_\alpha}(Z,Zw).$$

which was derived in [32]. The second term on the LHS in the above identity does not contribute at large N. Start from

$$\chi^{(1)}_{\square,\cdot}(Z,Z^{n_L}w) \equiv \mathrm{Tr}\,(Z^{n_L}w),$$

and use the identity to pull Z's off Z^{n_L} and onto the Young diagram R. For example, for $n_L = 1,2,3$ we have

$$\mathrm{Tr}\,(Zw) = \frac{1}{2}\left(\chi_{\begin{ytableau} \ & w \end{ytableau}} - \chi_{\begin{ytableau} \ \\ w \end{ytableau}}\right),$$

$$\mathrm{Tr}\,(Z^2w) = \frac{1}{3}\left(\chi_{\begin{ytableau} \ & \ & w \end{ytableau}} - \chi_{\begin{ytableau} \ & \ \\ w \end{ytableau}} - \chi_{\begin{ytableau} \ & w \\ \ \end{ytableau}} + \chi_{\begin{ytableau} \ \\ \ \\ w \end{ytableau}}\right),$$

$$\mathrm{Tr}\,(Z^3w) = \frac{1}{4}\left(\chi_{\begin{ytableau} \ & \ & \ & w \end{ytableau}} - \chi_{\begin{ytableau} \ & \ & \ \\ w \end{ytableau}} - \chi_{\begin{ytableau} \ & \ & w \\ \ \end{ytableau}} + \chi_{\begin{ytableau} \ & \ \\ \ \\ w \end{ytableau}} + \chi_{\begin{ytableau} \ & w \\ \ \\ \ \end{ytableau}} - \chi_{\begin{ytableau} \ \\ \ \\ \ \\ w \end{ytableau}}\right).$$

These formulas are exact.

H Schwinger-Dyson Equations in the Annulus Background

The Schwinger-Dyson equations provide a powerful approach to computing correlators in the annulus background. They are far more computationally efficient that the approach based on cutting rules developed in [46]. The advantage of the cutting rules are their generality: the cutting rules work in any background. In deriving the Schwinger-Dyson equation, a crucial observation is that for the background we consider (recall that B is a Young diagram with M columns and N rows)

$$\chi_B(Z) = \det(Z)^M \,.$$

We make repeated use of this fact and consequently, the results of this appendix apply only to the annulus background.

H.1 Schwinger-Dyson Equations

Start by considering

$$0 = \int \left[dZ dZ^\dagger\right] \frac{d}{dZ_{ij}} \left((Z^{n+1} Z^{\dagger\, n})_{ij} \chi_B(Z) \chi_B(Z^\dagger) e^{-S} \right) \,.$$

Carrying out the derivative is straightforward, except perhaps for the term obtained when the derivative acts on the background. To evaluate this term, note that

$$\frac{d}{dZ_{ij}} \chi_B(Z) = \frac{d}{dZ_{ij}} \det(Z)^M = M (Z^{-1})_{ji} \det(Z)^M \,.$$

Thus, this term contributes

$$M \left\langle \mathrm{Tr}\,(Z^n Z^{\dagger\, n}) \right\rangle_B$$

to the Schwinger-Dyson equation. Next, focus on the term obtained by acting on the first Z in $(Z^{n+1}Z^{\dagger\, n})_{ij}$ which gives

$$N\left\langle \mathrm{Tr}\,(Z^n Z^{\dagger\, n})\right\rangle_B \;;$$

these two terms combine to give the claimed $N \to M + N$ replacement in the Schwinger-Dyson equations. Writing out all of the terms we have

$$\left\langle \mathrm{Tr}\,(Z^{n+1}Z^{\dagger\, n+1})\right\rangle_B = (N+M)\left\langle \mathrm{Tr}\,(Z^n Z^{\dagger\, n})\right\rangle_B + \sum_{r=1}^{n}\left\langle \mathrm{Tr}\,(Z^r)\mathrm{Tr}\,(Z^{n-r}Z^{\dagger\, n})\right\rangle_B \,.$$

A slightly more general Schwinger-Dyson equation which we found useful in the computation of correlators reads

$$0 = \int \left[dZdZ^\dagger\right]\frac{d}{dZ_{ij}}\left((Z^m Z^{\dagger\, n})_{ij}\mathrm{Tr}\,(Z^{n+1-m})\chi_B(Z)\chi_B(Z^\dagger)e^{-S}\right)$$

which is easily seen to give

$$\left\langle \mathrm{Tr}\,(Z^{\dagger\, n+1}Z^m)\mathrm{Tr}\,(Z^{n+1-m})\right\rangle_B = \sum_{r=1}^{m-1}\left\langle \mathrm{Tr}\,(Z^{\dagger\, n}Z^{m-r-1})\mathrm{Tr}\,(Z^r)\mathrm{Tr}\,(Z^{n+1-m})\right\rangle_B$$

$$+(N+M)\left\langle \mathrm{Tr}\,(Z^{\dagger\, n}Z^{m-1})\mathrm{Tr}\,(Z^{n+1-m})\right\rangle_B + (n+1-m)\left\langle \mathrm{Tr}\,(Z^n Z^{\dagger\, n})\right\rangle_B \,.$$

This starting point could easily be generalized to

$$0 = \int \left[dZdZ^\dagger\right]\frac{d}{dZ_{ij}}\left((Z^m Z^{\dagger\, n})_{ij}\mathrm{Tr}\,(Z^{n+1-m})O(Z,Z^\dagger)\chi_B(Z)\chi_B(Z^\dagger)e^{-S}\right)$$

where $O(Z,Z^\dagger)$ is any gauge invariant operator.

Computing correlators is now straightforward. We can obtain all correlators of the form $\left\langle \prod_{i,j}\mathrm{Tr}\,(Z^{n_i})\mathrm{Tr}\,((Z^\dagger)^{m_j})\right\rangle_B$ using (17). Using these in the above Schwinger-Dyson equations, we can easily determine the correlators $\left\langle \prod_i \mathrm{Tr}\,(Z^{n_i}Z^{\dagger\, n_i})\right\rangle_B$ which are of relevance for the near-BPS sector of the theory.

H.2 Testing Factorization

Now that we can use the Schwinger-Dyson equations to compute correlators exactly, we can answer some interesting questions. One obvious question is if the annulus geometry provides a good background. For this to be the case, we need to have a factorization of the background expectation values of gauge invariant observables. This implies that a single saddle point is dominating the gauge theory path integral. By the gauge theory/gravity correspondence, this saddle point represents a particular space-time geometry in gravity, that is, a classical spacetime has emerged. One nice general result follows from

$$0 = \int \left[dZ dZ^\dagger\right] \frac{d}{dZ_{ij}} \left(Z_{ij}[\mathrm{Tr}\,(ZZ^\dagger)]^n \chi_B(Z)\chi_B(Z^\dagger)e^{-S}\right)$$

which implies that

$$\left\langle [\mathrm{Tr}\,(ZZ^\dagger)]^{n+1}\right\rangle_B = (N^2+MN+n)\left\langle [\mathrm{Tr}\,(ZZ^\dagger)]^{n}\right\rangle_B \,.$$

This recursion relation is easily solved to give

$$\left\langle [\mathrm{Tr}\,(ZZ^\dagger)]^{n+1}\right\rangle_B = \prod_{i=0}^{n}(N^2+MN+i)\,.$$

Keeping only the leading order[23], we have

$$\left\langle [\mathrm{Tr}\,(ZZ^\dagger)]^{n+1}\right\rangle_B = (N^2+MN)^{n+1} = \left\langle \mathrm{Tr}\,(ZZ^\dagger)\right\rangle_B^{n+1} \,,$$

demonstrating factorization for these amplitudes. We can easily generalize this by considering

$$0 = \int \left[dZ dZ^\dagger\right] \frac{d}{dZ_{ij}} \left(Z_{ij}\mathrm{Tr}\,(Z^p Z^{\dagger p})[\mathrm{Tr}\,(ZZ^\dagger)]^n \chi_B(Z)\chi_B(Z^\dagger)e^{-S}\right)$$

[23] Of course, n and p (used below) are $O(1)$.

which implies

$$\left\langle [\mathrm{Tr}\,(ZZ^\dagger)]^{n+1}\mathrm{Tr}\,(Z^pZ^{\dagger p})\right\rangle_B = (N^2+MN+n+p)\left\langle [\mathrm{Tr}\,(ZZ^\dagger)]^{n}\mathrm{Tr}\,(Z^pZ^{\dagger p})\right\rangle_B \,.$$

Once again this is easy to solve, giving

$$\left\langle [\mathrm{Tr}\,(ZZ^\dagger)]^{n+1}\mathrm{Tr}\,(Z^pZ^{\dagger p})\right\rangle_B = \prod_{i=0}^{n}(N^2+MN+i+p)\left\langle \mathrm{Tr}\,(Z^pZ^{\dagger p})\right\rangle_B \,,$$

which becomes, at the leading order,

$$\begin{aligned}\left\langle [\mathrm{Tr}\,(ZZ^\dagger)]^{n+1}\mathrm{Tr}\,(Z^pZ^{\dagger p})\right\rangle_B &= (N^2+MN)^{n+1}\left\langle \mathrm{Tr}\,(Z^pZ^{\dagger p})\right\rangle_B \\ &= \left\langle \mathrm{Tr}\,(Z^pZ^{\dagger p})\right\rangle_B \left\langle \mathrm{Tr}\,(ZZ^{\dagger})\right\rangle_B^{n+1} \,,\end{aligned}$$

again demonstrating factorization.

Above we have been careful to compute things to all orders. If we simply assume factorization and keep only the leading order, we get additional information about the leading behavior of various loops. For example,

$$\begin{aligned}\left\langle \mathrm{Tr}\,(Z^{p+1}Z^{\dagger\,p+1})\prod_i \mathrm{Tr}\,(Z^{n_i}Z^{\dagger\,n_i})\right\rangle_B &= (N+M)\left\langle \mathrm{Tr}\,(Z^{p}Z^{\dagger\,p})\prod_i \mathrm{Tr}\,(Z^{n_i}Z^{\dagger\,n_i})\right\rangle_B \\ &+\sum_{r=1}^{p}\left\langle \mathrm{Tr}\,(Z^{p-r}Z^{\dagger\,p})\mathrm{Tr}\,(Z^{r})\prod_i \mathrm{Tr}\,(Z^{n_i}Z^{\dagger\,n_i})\right\rangle_B \\ &+\sum_j\sum_{r=0}^{n_j-1}\left\langle \mathrm{Tr}\,(Z^{r+p+1}Z^{\dagger\,p}Z^{n_j-r-1}Z^{\dagger\,n_j})\prod_{i\neq j} \mathrm{Tr}\,(Z^{n_i}Z^{\dagger\,n_i})\right\rangle_B \,,\end{aligned}$$

becomes, after assuming factorization

$$\begin{aligned}\left\langle \mathrm{Tr}\,(Z^{p+1}Z^{\dagger\,p+1})\right\rangle_B \prod_i \left\langle \mathrm{Tr}\,(Z^{n_i}Z^{\dagger\,n_i})\right\rangle_B &= (N+M)\left\langle \mathrm{Tr}\,(Z^{p}Z^{\dagger\,p})\right\rangle_B \prod_i \left\langle \mathrm{Tr}\,(Z^{n_i}Z^{\dagger\,n_i})\right\rangle_B \\ &+\sum_j\sum_{r=0}^{n_j-1}\left\langle \mathrm{Tr}\,(Z^{r+p+1}Z^{\dagger\,p}Z^{n_j-r-1}Z^{\dagger\,n_j})\right\rangle_B \prod_{i\neq j} \left\langle \mathrm{Tr}\,(Z^{n_i}Z^{\dagger\,n_i})\right\rangle_B \,,\end{aligned}$$

The second term of the right hand side is subleading compared to the first term because (i) the first term is multiplied by $(N+M)$ and (ii) the second term has one less trace in it. Thus it may be dropped to give

$$\left\langle \mathrm{Tr}\,(Z^{p+1}Z^{\dagger\,p+1})\right\rangle_B \prod_i \left\langle \mathrm{Tr}\,(Z^{n_i}Z^{\dagger\,n_i})\right\rangle_B = (N{+}M)\left\langle \mathrm{Tr}\,(Z^{p}Z^{\dagger\,p})\right\rangle_B \prod_i \left\langle \mathrm{Tr}\,(Z^{n_i}Z^{\dagger\,n_i})\right\rangle_B \,.$$

Iterating this relation, the above result is clearly equivalent to

$$\left\langle \mathrm{Tr}\,(Z^{q}Z^{\dagger\,q})\right\rangle_B = N(N+M)^q \,.$$

which we know is correct.

H.3 Testing the Cutting Rules

In [46] a method to compute general correlators in any arbitrary LLM background was given. Now that we have an efficient way to compute correlators in the annulus background we can ask: Do the cutting rule methods of [46] really work? In this appendix we will compute a specific correlator, first using the Schwinger-Dyson equations and then using the cutting rules. We have found complete agreement between the cutting rule result and the result from the Schwinger-Dyson equations for any correlator we have computed. The cutting rules work.

Starting from

$$0 = \int \left[dZdZ^\dagger\right] \frac{d}{dZ_{ij}} \left((Z^{\dagger\,n} Z^m Z^{\dagger\,p} Z^{n+p-m+1})_{ij} \chi_B(Z)\chi_B(Z^\dagger) e^{-S}\right)$$

we obtain

$$\left\langle \mathrm{Tr}\,(Z^{\dagger n+1} Z^m Z^{\dagger p} Z^{n+p-m+1})\right\rangle_B = \sum_{r=0}^{m-1} \left\langle \mathrm{Tr}\,(Z^{\dagger n} Z^r)\mathrm{Tr}\,(Z^{\dagger p} Z^{n+p-r})\right\rangle_B$$

$$+ \sum_{r=0}^{n+p-m-1} \left\langle \mathrm{Tr}\,(Z^{\dagger n} Z^m Z^{\dagger p} Z^r)\mathrm{Tr}\,(Z^{n+p-m-r})\right\rangle_B + (N{+}M)\left\langle \mathrm{Tr}\,(Z^{\dagger n} Z^m Z^{\dagger p} Z^{n+p-m})\right\rangle_B \,.$$

Setting $n = 0$, $m = 1$ and $p = 1$ we have

$$\left\langle \mathrm{Tr}\,(Z^\dagger Z Z^\dagger Z)\right\rangle_B = (2N+M)\left\langle \mathrm{Tr}\,(Z^\dagger Z)\right\rangle_B = (2N+M)N(N+M)\,.$$

We will summarize the cutting rule computation; for more details the reader should consult [46]. Evaluating this correlator using cutting rules, there are four contributions: the term with no contractions with the background gives $2N^3$. The terms coming from contracting one Z in the loop with a $Z^\dagger$ in $\chi_B(Z^\dagger)$ give

$$4N\left\langle \mathrm{Tr}\,\left(\frac{d}{dZ}\frac{d}{dZ^\dagger}\right)\chi_B(Z)\chi_B(Z^\dagger)\right\rangle = 4N^2M.$$

The terms coming from contracting both Zs in the loop with $Z^\dagger$s in $\chi_B(Z^\dagger)$ give

$$\left\langle \mathrm{Tr}\,\left(\frac{d}{dZ}\frac{d}{dZ^\dagger}\frac{d}{dZ}\frac{d}{dZ^\dagger}\right)\chi_B(Z)\chi_B(Z^\dagger)\right\rangle = M^2N - N^2M.$$

To evaluate this last contribution we had to cut the trace of four derivatives into a product of two traces, each containing two derivatives. This is accompanied by a nontrivial trace insertion factor that we evaluated in representation B. Summing these terms we have

$$2N^3 + 4N^2M + M^2N - N^2M = (2N+M)N(N+M)\,.$$

I Schwinger-Dyson Equations for > 1 Charge Background

The background of interest in this appendix is (recall that r_1 is a rectangular Young diagram with N rows and M_1 columns and r_2 is a rectangular Young diagram with N rows and M_2 columns)

$$\chi_{r_1}(Z)\chi_{r_2}(Y) = (\det(Z))^{M_1}(\det(Y))^{M_2}\,.$$

The Schwinger-Dyson equations continue to provide a powerful approach to correlator computations, when we consider this background built using more than one matrix. In this appendix we will give a few example computations. Consider the identity[24]

$$0 = \int [dZdZ^\dagger dYdY^\dagger]\frac{d}{dZ_{ij}}\left((Z^nY^mY^{\dagger\,m}Z^{\dagger\,n-1})_{ij}\chi_{r_1}(Z)\chi_{r_1}(Z^\dagger)\chi_{r_2}(Y)\chi_{r_2}(Y^\dagger)e^{-S}\right)$$

which leads to the following Schwinger-Dyson equation

$$\left\langle \mathrm{Tr}\,(Z^nY^mY^{\dagger\,m}Z^{\dagger\,n})\right\rangle_{(r_1,r_2)} = (N+M_1)\left\langle \mathrm{Tr}\,(Z^{n-1}Y^mY^{\dagger\,m}Z^{\dagger\,n-1})\right\rangle_{(r_1,r_2)}$$
$$+\sum_{r=1}^{n-1}\left\langle \mathrm{Tr}\,(Z^r)\mathrm{Tr}\,(Z^{n-1-r}Y^mY^{\dagger\,m}Z^{\dagger\,n-1})\right\rangle_{(r_1,r_2)}\,.$$

If we had been working in the trivial vacuum, the only difference would have been to replace $N+M_1$ in the above equation by N. One way to think about the above Schwinger-Dyson equation is that to go from the left hand side to the right hand side, we perform one of the Z Wick contractions. To obtain the equation that follows when we perform a Y Wick contraction, start with the identity

$$0 = \int [dZdZ^\dagger dYdY^\dagger]\frac{d}{dY_{ij}}\left((Y^nZ^mZ^{\dagger\,m}Y^{\dagger\,n-1})_{ij}\chi_{r_1}(Z)\chi_{r_1}(Z^\dagger)\chi_{r_2}(Y)\chi_{r_2}(Y^\dagger)e^{-S}\right)$$

[24]The action $S = \mathrm{Tr}\,(ZZ^\dagger) + \mathrm{Tr}\,(YY^\dagger)$.

which leads to the following Schwinger-Dyson equation

$$\left\langle \mathrm{Tr}\,(Y^n Z^m Z^{\dagger\, m} Y^{\dagger\, n})\right\rangle_{(r_1,r_2)} = (N+M_2)\left\langle \mathrm{Tr}\,(Y^{n-1} Z^m Z^{\dagger\, m} Y^{\dagger\, n-1})\right\rangle_{(r_1,r_2)}$$

$$+\sum_{r=1}^{n-1}\left\langle \mathrm{Tr}\,(Y^r)\mathrm{Tr}\,(Y^{n-1-r} Z^m Z^{\dagger\, m} Y^{\dagger\, n-1})\right\rangle_{(r_1,r_2)} .$$

If we had been working in the trivial vacuum, the only difference would have been to replace $N+M_2$ in the above equation by N. This structure is parallel to the structure we found for backgrounds constructed using a single matrix: in this case we have found that to reproduce correlators of operators built only using Zs or $Z^\dagger$s (Ys or $Y^\dagger$s) we simply replace $N\to N+M_1$ ($N\to N+M_2$). This structure is again emerging at the level of the Schwinger-Dyson equations.

Consider the last Schwinger-Dyson equation given above. In the large N limit the first term on the right hand side gives the leading contribution. The second term has one more trace in it whilst the first term is multiplied by $(N+M_2)$. Naive counting of powers of N would suggest that these two terms are of the same order. However, the leading contribution to the second term $\langle \mathrm{Tr}\,(Y^r)\rangle_{(r_1,r_2)}\left\langle \mathrm{Tr}\,(Y^{n-1-r} Z^m Z^{\dagger\, m} Y^{\dagger\, n-1})\right\rangle_{(r_1,r_2)}$ vanishes. Dropping the second term and iterating we find

$$\left\langle \mathrm{Tr}\,(Y^n Z^m Z^{\dagger\, m} Y^{\dagger\, n})\right\rangle_{(r_1,r_2)} = (N{+}M_2)^n\left\langle \mathrm{Tr}\,(Z^m Z^{\dagger\, m})\right\rangle_{(r_1,r_2)} = N(N{+}M_1)^m(N{+}M_2)^n\,. \tag{107}$$

This looks very similar to the relation (26); indeed, we can write

$$\left\langle O(Z,Z^\dagger,Y,Y^\dagger)\right\rangle_{(r_1,r_2)} = \left\langle O\left(\sqrt{\frac{N+M_1}{N}}Z,\sqrt{\frac{N+M_1}{N}}Z^\dagger,\sqrt{\frac{N+M_2}{N}}Y,\sqrt{\frac{N+M_2}{N}}Y^\dagger\right)\right\rangle .$$

Relations of this type would again be very useful in deriving spin chains for loops in this two matrix background.

Finally, although we have already argued that there are large 't Hooft coupling corrections to the background, it is still interesting to ask if factorization holds. Such backgrounds are naturally interpreted as classical backgrounds that receive curvature corrections. For operators that do not mix Zs and Ys in the same trace, correlators factorize into a Z correlator times a Y correlator. Using the results of Appendix H we clearly have factorization in this case. For operators with mixed traces, a bit more work is needed. We will give some examples in which factorization is clear. Consider

$$0=\int [dZdZ^\dagger dYdY^\dagger]\frac{d}{dY_{ij}}\Big((Y^{n+1}Z^mZ^{\dagger\,m}Y^{\dagger\,n})_{ij}\times$$

$$\times\prod_a \mathrm{Tr}\,(Z^{n_a}Z^{\dagger n_a})\chi_{r_1}(Z)\chi_{r_1}(Z^\dagger)\chi_{r_2}(Y)\chi_{r_2}(Y^\dagger)e^{-S}\Big)$$

which implies

$$\Big\langle \mathrm{Tr}\,(Y^{\dagger\,n}Y^nZ^mZ^{\dagger m})\prod_a \mathrm{Tr}\,(Z^{n_a}Z^{\dagger n_a})\Big\rangle_{(r_1,r_2)}$$

$$=(N+M_2)\Big\langle \mathrm{Tr}\,(Y^{\dagger\,n-1}Y^{n-1}Z^mZ^{\dagger m})\prod_a \mathrm{Tr}\,(Z^{n_a}Z^{\dagger n_a})\Big\rangle_{(r_1,r_2)}$$

$$+\sum_{r=1}^{n-1}\Big\langle \mathrm{Tr}\,(Y^r)\mathrm{Tr}\,(Y^{n-1-r}Z^mZ^{\dagger\,m}Y^{\dagger\,n-1})\prod_a \mathrm{Tr}\,(Z^{n_a}Z^{\dagger n_a})\Big\rangle_{(r_1,r_2)}\,.$$

In the large N limit the first term on the right hand side gives the leading contribution so that we can drop the second term. Iterating, we find

$$\Big\langle \mathrm{Tr}\,(Y^{\dagger\,n}Y^nZ^mZ^{\dagger m})\prod_a \mathrm{Tr}\,(Z^{n_a}Z^{\dagger n_a})\Big\rangle_{(r_1,r_2)}=(N+M_2)^n\Big\langle \mathrm{Tr}\,(Z^mZ^{\dagger m})\prod_a \mathrm{Tr}\,(Z^{n_a}Z^{\dagger n_a})\Big\rangle_{(r_1,r_2)}$$

$$=(N+M_2)^n\Big\langle \mathrm{Tr}\,(Z^mZ^{\dagger m})\Big\rangle_{(r_1,r_2)}\prod_a\Big\langle \mathrm{Tr}\,(Z^{n_a}Z^{\dagger n_a})\Big\rangle_{(r_1,r_2)}\,,$$

where to get the last equality we used factorization of the $Z,Z^\dagger$ correlators. Now, use (107) to identify $(N+M_2)^n\Big\langle \mathrm{Tr}\,(Z^mZ^{\dagger m})\Big\rangle$ as $\Big\langle \mathrm{Tr}\,(Y^{\dagger n}Y^nZ^mZ^{\dagger m})\Big\rangle$ in the last line above so that

$$\Big\langle \mathrm{Tr}\,(Y^{\dagger n}Y^nZ^mZ^{\dagger m})\prod_a \mathrm{Tr}\,(Z^{n_a}Z^{\dagger n_a})\Big\rangle_{(r_1,r_2)}$$

$$= \left\langle \mathrm{Tr}\,(Y^{\dagger n} Y^n Z^m Z^{\dagger m}) \right\rangle_{(r_1,r_2)} \prod_a \left\langle \mathrm{Tr}\,(Z^{n_a} Z^{\dagger n_a}) \right\rangle_{(r_1,r_2)} .$$

We can give a rather general argument for factorization: consider

$$0 = \int [dZdZ^\dagger dYdY^\dagger] \frac{d}{dY_{ij}} \left((Y^{p+1} Z^n Z^{\dagger\, n} Y^{\dagger\, p})_{ij} \mathcal{O} \chi_{r_1}(Z) \chi_{r_1}(Z^\dagger) \chi_{r_2}(Y) \chi_{r_2}(Y^\dagger) e^{-S} \right)$$

where $\mathcal{O}$ is any gauge invariant operator. This implies

$$\left\langle \mathrm{Tr}\,(Y^{\dagger\, p+1} Y^{p+1} Z^n Z^{\dagger n}) \mathcal{O} \right\rangle_{(r_1,r_2)} = (N + M_2) \left\langle \mathrm{Tr}\,(Y^{\dagger\, p} Y^p Z^n Z^{\dagger n}) \mathcal{O} \right\rangle_{(r_1,r_2)}$$

$$+ \sum_{r=1}^{p} \left\langle \mathrm{Tr}\,(Y^r) \mathrm{Tr}\,(Y^{p-r} Z^m Z^{\dagger\, m} Y^{\dagger\, p}) \mathcal{O} \right\rangle_{(r_1,r_2)} + \left\langle (Y^{p+1} Z^n Z^{\dagger\, n} Y^{\dagger\, p})_{ij} \frac{d}{dY_{ij}} \mathcal{O} \right\rangle_{(r_1,r_2)} .$$

The second term on the right hand side can be dropped - it vanishes at leading order. The third term on the right hand side can also be dropped - it represents a loop joining term, so that it has one less trace than the first term. If we now rescale $Y \to \frac{N}{N+M_2} Y$ and $Z \to \frac{N}{N+M_1} Z$ we recover the Schwinger-Dyson equations of the theory in the trivial vacuum (after dropping the same two terms justified with the same two reasons). We know that factorization was a property of the old Schwinger-Dyson equations so that we have just learnt that it is a property of the new Schwinger-Dyson equations too.

J Anomalous Dimension for > 1 Charge Background

In this section we will explain how to compute the one loop anomalous dimension of the backgrounds considered in section 3.2.4.

J.1 An Identity: Excited Giant Correlators

Correlation functions of restricted Schur polynomials have been computed in [31] (see also [32, 33]). The logic in these computations is first to contract the open string words and then to compute the remaining contractions. In this section we will obtain a formula that describes the result of contracting all fields *except* the open string words. Although we derive our formula for the case of one string attached, it is simple to extend it to the general case. The formula we are after says

$$\left\langle \chi^{(1)}_{R,R'}(Z,W)\chi^{(1)}_{R,R'}(Z^\dagger,W^\dagger)\right\rangle = A\left\langle \mathrm{Tr}\,(WW^\dagger)\right\rangle + B\left\langle \mathrm{Tr}\,(W)\mathrm{Tr}\,(W^\dagger)\right\rangle .$$

Recall[31] that the allowed index structure for open string word two point functions is

$$\left\langle W^i_j (W^\dagger)^k_l \right\rangle = \delta^i_l\delta^k_j F_0 + \delta^i_j\delta^k_l F_1 .$$

Thus,

$$\left\langle \chi^{(1)}_{R,R'}(Z,W)\chi^{(1)}_{R,R'}(Z^\dagger,W^\dagger)\right\rangle = A(N^2F_0+NF_1)+B(N^2F_1+NF_0) .$$

From the technology developed in [31], we also know that

$$\left\langle \chi^{(1)}_{R,R'}(Z,W)\chi^{(1)}_{R,R'}(Z^\dagger,W^\dagger)\right\rangle = \frac{\mathrm{hooks}_R}{\mathrm{hooks}_{R'}} f_R F_0 + c_{RR'} f_R F_1 .$$

It is now trivial to find

$$A = \left(\frac{\text{hooks}_R}{\text{hooks}_{R'}} N^2 - c_{RR'} N\right) \frac{f_R}{N^4 - N^2} \, ,$$

$$B = \left(N^2 c_{RR'} - N \frac{\text{hooks}_R}{\text{hooks}_{R'}}\right) \frac{f_R}{N^4 - N^2} \, .$$

J.2 Leading Contribution to Background Correlator

The formulas we write in this section will not be general; we are considering the backgrounds r_1 and r_2 of section3.2.4. With a little extra effort one could be general. We are interested in computing the normalized correlation function

$$\left\langle \chi_{r_1}(Z^\dagger)\chi_{r_2}(Y^\dagger)\chi_{r_1}(Z)\chi_{r_2}(Y) \right\rangle \, ,$$

to one loop. According to Appendix B of [54], the D-term, self energy and gluon exchange cancel at one loop order (using techniques of [55]), so to this order we only need to consider the contributions from the F-term. Towards this end we will now evaluate

$$I_1 = \left\langle \chi_{r_1}(Z^\dagger)\chi_{r_2}(Y^\dagger)\chi_{r_1}(Z)\chi_{r_2}(Y) \mathrm{Tr}\left([Z,Y][Z^\dagger,Y^\dagger]\right) \right\rangle \, .$$

The tricky part of this computation is the evaluation of the color combinatoric factor. To do this evaluation we can work in zero dimensions. Since we drop self energy corrections, the F-term is normal ordered and hence the above correlator can be written as

$$\left\langle \mathrm{Tr}\,([\frac{\partial}{\partial Z}, \frac{\partial}{\partial Y}][\frac{\partial}{\partial Z^\dagger}, \frac{\partial}{\partial Y^\dagger}]) \chi_{r_1}(Z^\dagger)\chi_{r_2}(Y^\dagger)\chi_{r_1}(Z)\chi_{r_2}(Y) \right\rangle \, .$$

This can be rewritten, using dummy open string variables, as

$$\left\langle \mathrm{Tr}\,([\frac{\partial}{\partial W}, \frac{\partial}{\partial V}][\frac{\partial}{\partial W^\dagger}, \frac{\partial}{\partial V^\dagger}]) \chi^{(1)}_{r_1,r_1'}(Z^\dagger, W^\dagger)\chi^{(1)}_{r_2,r_2'}(Y^\dagger, V^\dagger)\chi^{(1)}_{r_1,r_1'}(Z,W)\chi^{(1)}_{r_2,r_2'}(Y,V) \right\rangle \, .$$

For both r_1 and r_2 there is only one way possible to attach the open string. Using the results of the previous subsection we now obtain

$$\mathrm{Tr}\,([\frac{\partial}{\partial W},\frac{\partial}{\partial V}][\frac{\partial}{\partial W^\dagger},\frac{\partial}{\partial V^\dagger}])\left(A_1\mathrm{Tr}\,(WW^\dagger)+B_1\mathrm{Tr}\,(W)\mathrm{Tr}\,(W^\dagger)\right)\left(A_2\mathrm{Tr}\,(VV^\dagger)+B_2\mathrm{Tr}\,(V)\mathrm{Tr}\,(V^\dagger)\right),$$

where

$$A_1=\frac{M_1}{N}f_{r_1},\quad A_2=\frac{M_2}{N}f_{r_2},\quad B_i=0\,.$$

It is a simple matter to find

$$\frac{I_1}{f_{r_1}f_{r_2}}=-2NM_1M_2+2\frac{M_1M_2}{N}\,.$$

The leading contribution to this correlator comes from the terms

$$\mathrm{Tr}\,(ZYY^\dagger Z^\dagger)+\mathrm{Tr}\,(YZZ^\dagger Y^\dagger)\,.$$

This computation will allow us, in the next subsection, to identify and evaluate the leading contribution to the one loop anomalous dimension.

J.3 Leading Contribution to the One Loop Dilatation Operator

The $O(g_{YM}^0)$ contribution to the anomalous dimension

$$D_0=\mathrm{Tr}\left(Z\frac{\partial}{\partial Z}\right)+\mathrm{Tr}\left(Y\frac{\partial}{\partial Y}\right)$$

gives $\Delta_0=NM_1+NM_2$. To obtain the leading piece of the $O(g_{YM}^2)$ contribution to the anomalous dimension, which we have identified in the previous subsection, replace

$$D_1=-2g_{YM}^2\mathrm{Tr}\,:\Big[\frac{\partial}{\partial Y},\frac{\partial}{\partial Z}\Big]\Big[Y,Z\Big]:\to 2g_{YM}^2\mathrm{Tr}\left(ZY\frac{\partial}{\partial Y}\frac{\partial}{\partial Z}\right)+2g_{YM}^2\mathrm{Tr}\left(YZ\frac{\partial}{\partial Z}\frac{\partial}{\partial Y}\right)\,.$$

The normal ordering symbols here indicate that derivatives within the normal ordering symbols do not act on fields inside the normal ordering symbols. It is now straightforward to argue that

$$\left(2g_{YM}^2 \mathrm{Tr}\left(ZY\frac{\partial}{\partial Y}\frac{\partial}{\partial Z}\right) + 2g_{YM}^2 \mathrm{Tr}\left(YZ\frac{\partial}{\partial Z}\frac{\partial}{\partial Y}\right)\right)\chi_{r_1}(Z)\chi_{r_2}(Y)$$

$$= 4g_{YM}^2 N M_1 M_2 \chi_{r_1}(Z)\chi_{r_2}(Y)\,.$$

We are interested in two cases:

- Both M_1 and M_2 are $O(N)$. Holding $g_{YM}^2 N = \lambda$ fixed and large (which is the regime in which we expect that we can trust the dual geometry), we find a one loop correction of $O(N^2)$ times λ to the tree level value which is itself $O(N^2)$. At large N the tree level and one loop results are of the same order.
- Both M_1 and M_2 are $O(\sqrt{N})$. Holding $g_{YM}^2 N = \lambda$ fixed and large, we find a one loop correction of $O(N)$ times λ to the tree level value which is itself $O(N^{3/2})$. At large N the tree level result dominates the one loop correction.

Notice that if we take M_1 to be $O(N)$ and keep M_2 to be $O(1)$ or if we take M_2 to be $O(N)$ and keep M_1 to be $O(1)$, the one loop correction becomes negligible as compared to the tree level value, as we would expect. It is also interesting to note that our operator is an eigenoperator of D_1, so that at one loop and at large N it does not mix with other operators.

K Two Point Functions

In this appendix we will compute the two point functions used in section 4.2. We only want these two point functions to the leading order in an $M+N$ expansion. We will use f_B to denote the product of the weights of Young diagram B. It is straight forward to obtain

$$f_B = \frac{G_2(N+M+1)}{G_2(N+1)G_2(M+1)}\,, \tag{108}$$

where $G_2(n+1)$ is the Barnes function defined by ($\Gamma(z)$ is the Gamma function)

$$G_2(z+1) = \Gamma(z)G_2(z)\,. \tag{109}$$

In particular, for an integer $z=n$ we have

$$G_2(n+1) = \prod_{k=1}^{n-1} k!\,. \tag{110}$$

First consider the free field theory two point function

$$\left\langle \chi_B(Z)\chi_B(Z^\dagger)\mathrm{Tr}\,(Y^2 Z^{J_0})\mathrm{Tr}\,(Y^2 Z^{J_0})^\dagger \right\rangle . \tag{111}$$

According to [46], this two point function is given, at the leading order in a large $M+N$ expansion, by f_B times $\left(\frac{M+N}{N}\right)^{J_0}$ times the free field theory two point function in the trivial background

$$\left\langle \mathrm{Tr}\,(Y^2 Z^{J_0})\mathrm{Tr}\,(Y^2 Z^{J_0})^\dagger \right\rangle = N^{J_0+2} + O(N^{J_0})\,. \tag{112}$$

Thus,

$$\left\langle \chi_B(Z)\chi_B(Z^\dagger)\mathrm{Tr}\,(Y^2 Z^{J_0})\mathrm{Tr}\,(Y^2 Z^{J_0})^\dagger \right\rangle = f_B(N+M)^{J_0} N^2\,. \tag{113}$$

Arguing in exactly the same way, we find

$$\left\langle \chi_B(Z)\chi_B(Z^\dagger)\mathrm{Tr}\,(YZ^nYZ^{J_0-n})\mathrm{Tr}\,(YZ^mYZ^{J_0-m})^\dagger \right\rangle = f_B(N+M)^{J_0}N^2\delta_{mn}\,. \tag{114}$$

Next, consider

$$\left\langle \chi^{(1)}_{B,B'}(Z,YZ^{J_0+1}Y)\chi^{(1)}_{B,B'}(Z,YZ^{J_0+1}Y)^\dagger \right\rangle \tag{115}$$

where

$$\chi^{(1)}_{B,B'}(Z,YZ^{J_0+1}Y) = \left(YZ^{J_0+1}Y\right)_{ij}\frac{\partial}{\partial Z_{ij}}\chi_B(Z)\,. \tag{116}$$

These correlators have been computed in [31]. The result is

$$\left\langle \chi^{(1)}_{B,B'}(Z,W)\chi^{(1)}_{B,B'}(Z^\dagger,W^\dagger) \right\rangle = \frac{\mathrm{hooks}_B}{\mathrm{hooks}_{B'}}f_BF_0 + c_{BB'}f_BF_1\,. \tag{117}$$

We find $c_{BB'} = M$ and $\frac{\mathrm{hooks}_B}{\mathrm{hooks}_{B'}} = MN$. Further, for the open string word $W = YZ^{J_0+1}Y$ we find that at the leading order

$$F_0 = N^{J_0+2}\left(\frac{M+N}{N}\right)^{J_0+1}\,, \qquad F_1 \sim N^{J_0-1}\left(\frac{M+N}{N}\right)^{J_0+1}\,, \tag{118}$$

so that, to the leading order we have

$$\left\langle \chi^{(1)}_{B,B'}(Z,YZ^{J_0+1}Y)\chi^{(1)}_{B,B'}(Z,YZ^{J_0+1}Y)^\dagger \right\rangle = f_BMN^2(M+N)^{J_0+1}\,. \tag{119}$$

This correlator result can also be written as

$$\left\langle \chi_B(Z)\chi_B(Z^\dagger)\mathrm{Tr}\,(YZ^{-1}YZ^{J_0+1})\mathrm{Tr}\,(YZ^{-1}YZ^{J_0+1})^\dagger \right\rangle = f_B\frac{1}{M}(N+M)^{J_0+1}N^2\,. \tag{120}$$

Again arguing in exactly the same way, we find

$$\left\langle \chi_B(Z)\chi_B(Z^\dagger)\mathrm{Tr}\,(YZ^{-n}YZ^{J_0+n})\mathrm{Tr}\,(YZ^{-m}YZ^{J_0+m})^\dagger \right\rangle = f_B\frac{1}{M^n}(N+M)^{J_0+n}N^2\delta_{mn}\,. \tag{121}$$

Notice that at large enough M that $\frac{N}{M}$ can be neglected, all the gauge invariant operators considered have exactly the same two point function.

Finally, by using the methods of this appendix, we can obtain the general result

$$\langle \mathcal{O}_B(\{p\}, J_0) \mathcal{O}_B(\{p\}, J_0)^\dagger \rangle = N^2 f_B \frac{(M+N)^{J_0+p_-}}{M^{p_-}} \,, \tag{122}$$

where $\{p\}$ denotes the occupation numbers of the Cuntz chain and p_- is negative the sum of all the negative occupation numbers. Thus, we have the correspondence

$$\mathcal{O}_B(\{p\}, J_0) \leftrightarrow \sqrt{N^2 f_B \frac{(M+N)^{J_0+p_-}}{M^{p_-}}} |\{p\}\rangle \,, \tag{123}$$

between operators and normalized Cuntz lattice states.

L More on the Cuntz Chain

To specify the general Cuntz chain model (90) one needs to specify the expected number of Cuntz particles $n_z(\sigma)$. Given $n_z(\sigma)$, what is the corresponding supergravity background? Using the results of the first of [16] as well as (90), the metric on the $y = 0$ plane and the circle along which the string moves (parametrized by[25] φ) can be written as

$$ds^2 = -h^{-2}(Dt)^2 + h^2 dz d\bar{z} + h^{-2} d\varphi^2 \,, \qquad Dt = dt - \frac{1}{2} i\bar{V} dz + \frac{1}{2} iV d\bar{z} \,, \tag{124}$$

$$V = \frac{n_z}{\bar{z}}, \qquad h^4 = \frac{\partial V}{\partial z}, \qquad z = re^{i\phi} \,. \tag{125}$$

[25]The angular momentum along this circle is due to the Y fields appearing in the gauge invariant operator dual to the string.

M Explicit Expressions for the Two Loop Dilatation Operator

In these expressions hatted indices are again to be dropped, $\theta(x) = 1$ if $x > 0$ and vanishes otherwise. To obtain this result we have assumed $J_0 > 0$, $J_0 - p \geq 0$ and $p \geq 0$ - assumptions which can easily be relaxed if need be.

$$\delta D_{4,0\ \text{eff}} \mathcal{O}_p^{J_0;J_1,\ldots,J_k} = 4(\mathcal{O}_1^{J_0;J_1,\ldots,J_k} - \mathcal{O}_0^{J_0;J_1,\ldots,J_k})\times$$

$$\times N(N+M)(\delta_{p=0} + \delta_{p=J_0} - \delta_{p=1} - \delta_{p=J_0-1})\,, \tag{126}$$

$$\delta D_{4,+\ \text{eff}} \mathcal{O}_p^{J_0;J_1,\ldots,J_k} = 4(M+2N)\theta(p-1)\times$$

$$\times(\mathcal{O}_0^{J_0-p+1;J_1,\ldots,J_k,p-1} - \mathcal{O}_1^{J_0-p+1;J_1,\ldots,J_k,p-1})$$

$$+4(M+2N)\theta(J_0-p-1)(\mathcal{O}_0^{p+1;J_1,\ldots,J_k,J_0-p-1} - \mathcal{O}_1^{p+1;J_1,\ldots,J_k,J_0-p-1})$$

$$+4(N+M)\theta(p)(\mathcal{O}_1^{J_0-p;J_1,\ldots,J_k,p} - \mathcal{O}_0^{J_0-p;J_1,\ldots,J_k,p})$$

$$+4(N+M)\theta(J_0-p)(\mathcal{O}_1^{p;J_1,\ldots,J_k,J_0-p} - \mathcal{O}_0^{p;J_1,\ldots,J_k,J_0-p})$$

$$+\sum_{s=1}^{J_0-1} 4N(\delta_{p=0} + \delta_{p=J_0})(\mathcal{O}_1^{J_0-s;J_1,\ldots,J_k,s} - \mathcal{O}_0^{J_0-s;J_1,\ldots,J_k,s})\,, \tag{127}$$

$$\delta D_{4,-\ \text{eff}} \mathcal{O}_p^{J_0;J_1,\ldots,J_k} = 4N\sum_{j=1}^{k} J_j\,(\delta_{p=0} + \delta_{p=J_0})\times$$

$$\times(\mathcal{O}_1^{J_0+J_j;J_1,\ldots,\hat{J}_j,\ldots,J_k} - \mathcal{O}_0^{J_0+J_j;J_1,\ldots,\hat{J}_j,\ldots,J_k})$$

$$-4N(\delta_{p=0} + \delta_{p=J_0})\sum_{i=1}^{k} \delta_{J_i=1}(\mathcal{O}_1^{J_0+1;J_1,\ldots,\hat{J}_i,\ldots,J_k} - \mathcal{O}_0^{J_0+1;J_1,\ldots,\hat{J}_i,\ldots,J_k})\,, \tag{128}$$

$$\delta D_{4,+-\ \text{eff}} \mathcal{O}_p^{J_0;J_1,\ldots,J_k} = 4\theta(p)\sum_{i=1}^{k} J_i(\mathcal{O}_1^{J_0+J_i-p;J_1,\ldots,\hat{J}_i,\ldots,J_k,p}$$

$$-\mathcal{O}_0^{J_0+J_i-p;J_1,\ldots,\hat{J}_i,\ldots,J_k,p}) + 4\theta(J_0-p)\sum_{i=1}^{k} J_i(\mathcal{O}_1^{J_i+p;J_1,\ldots,\hat{J}_i,\ldots,J_k,J_0-p}$$

$$-\mathcal{O}_0^{J_i+p;J_1,\ldots,\hat{J}_i,\ldots,J_k,J_0-p}) - 4\theta(J_i-1)\sum_{i=1}^{k} J_i(\mathcal{O}_1^{p+1;J_1,\ldots,\hat{J}_i,\ldots,J_k,J_0+J_i-p-1}$$

$$-\mathcal{O}_0^{p+1;J_1,\ldots,\hat{J}_i,\ldots,J_k,J_0+J_i-p-1}) - 4\theta(J_i-1)\sum_{i=1}^{k} J_i(\mathcal{O}_1^{J_0-p+1;J_1,\ldots,\hat{J}_i,\ldots,J_k,J_i+p-1}$$

$$-\mathcal{O}_0^{J_0-p+1;J_1,\ldots,\hat{J}_i,\ldots,J_k,J_i+p-1})\,, \tag{129}$$

$$\delta D_{4,++\ \mathrm{eff}}\mathcal{O}_p^{J_0;J_1,\ldots,J_k} = 4\sum_{r=1}^{p-2}(\mathcal{O}_0^{J_0-p+1;J_1,\ldots,J_k,r,p-r-1}$$

$$-\mathcal{O}_1^{J_0-p+1;J_1,\ldots,J_k,r,p-r-1}) + 4\sum_{r=1}^{J_0-p-2}(\mathcal{O}_0^{p+1;J_1,\ldots,J_k,r,J_0-p-r-1}$$

$$-\mathcal{O}_1^{p+1;J_1,\ldots,J_k,r,J_0-p-r-1}) + 4\theta(p)\sum_{s=1}^{J_0-p-1}(\mathcal{O}_1^{J_0-p-s;J_1,\ldots,J_k,s,p}$$

$$-\mathcal{O}_0^{J_0-p-s;J_1,\ldots,J_k,s,p}) + 4\theta(J_0-p)\sum_{s=1}^{p-1}(\mathcal{O}_1^{p-s;J_1,\ldots,J_k,s,J_0-p}$$

$$-\mathcal{O}_0^{p-s;J_1,\ldots,J_k,s,J_0-p})\,, \tag{130}$$

$$\delta D_{4,--\ \mathrm{eff}}\mathcal{O}_p^{J_0;J_1,\ldots,J_k} = 0\,. \tag{131}$$

Setting $M=0$ in the above expressions gives exact agreement with appendix E of [57] except for the last term in our expression for $\delta D_{4,-}$. The extra term that we have ensures that no joinings between the trace with the Ys and a trace without Ys and a single Z can occur. In this case, the $\frac{d}{dZ_{ij}}$ in $\delta D_{4\ \mathrm{eff}}$ acts on $\mathrm{Tr}\,(Z)$ to produce δ_{ij}. This vanishes because $\frac{d}{dZ_{ij}}$ appears inside a commutator; the extra term is needed.

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