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The Supersymmetry Approaches for Kratzer Potential in Constant Positive Curvature

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abstract

In this paper, we study Schrödinger equation for the Kratzer potential in constant positive curvature. By comparing the corresponding Schrödinger equation of Kratzer potential in constant positive curvature to the Gegenbauer polynomials differential equation, we obtain the energy spectrum and wave function. These lead us to have raising and lowering operators which are first order equations. We take advantage from these first order equations and discuss the supersymmetry algebra. Also, we obtain the corresponding partner hamiltonian for Kratzer potential and investigate the commutation relation for the generators algebra.
Keywords: The Kratzer potential; Supersymmetry approaches; Gegenbauer polynomials; Raising and lowering operators

I. INTRODUCTION

The accidental degeneracy for the first time discussed by Schrodinger [1], Infeld [2] and Stivenson[3]. This subject play important role in physics, because it connect to nontrivial realization of hidden symmetry and also apply to construct many - particle wave functions [4], non relativistic models of quark systems [5] and solutions of two - center problem [6]. Also the accidental degeneracy in this system are discussed by Ref.s [7-13]. They have shown that the complete degeneracy of spectrum of the Columb problem and harmonic oscillator on the three dimensional sphere in the orbital and azimutal quantum number is caused by an additional integral of motion. In case of Kratzer potential, we have some completed problem, we see Ref[14] which discussed the Kratzer potential with algebra point of view. In that paper they considered Kratzer potential in flat space, but here we consider the Kratzer potential in positive curvature and we do same calculation as the Ref [14] mentioned. So, we consider the following kratzer potential in flat space with non-curvature.

$$v(r,\theta) = D_e\left(\frac{r-r_e}{r}\right)^2 + [\frac{\beta'\cos^2\theta}{r^2\sin^2\theta}] \tag{1.1}$$

whereD_e is the dissociation energy between two atoms in a solid, r_e is the equilibrium internuclear separation and β' is positive real constant. We find that this potential (1.1) reduces to the modified Krazter potential in the limiting case of $\beta' = 0$. The group of hidden symmetry of this systems with accidental degeneracy give us motivation to discuss the Kratzer potential with constant positive curvature. Also note that the concept of shape invariance has extended to ordinary differential equations. In that case the second order differential operator will decompose the multiplication of raising and lowering operators [15-20]. In this paper, we use the factorization method and shape invariance of the Gegenbauer differential equation with respect to parameters n and obtain the factorized Schrödinger equations for the Kratzer potential in constant positive curvature. Also we obtain the shape invariance relation for the corresponding potential. So, the paper organize as follows: Section 2 presents the general form of Kratzer potential in spherical coordinates with spaces of constant curvature. Section 3 we use the Gegenbauer equation and solve the corresponding equation. Section 4 by using the factorization method from the Gegenbauer equation we obtain the raising and lowering operators, and finally we obtain the super algebra which is important in super symmetry system.

II. KRATZER POTENTIAL ON THE CONSTANT POSITIVE CURVATURE

As we know the three - dimensional space of constant positive curvature can also be realized geometrically on the three - dimensional sphere S_3 of the radius R, imbedded into the four - dimensional Euclidean space,

$$\xi_0{}^2 + \xi_i\xi_i = R^2. \tag{2.1}$$

We note that the relation between the coordinates x_i in the tangent space and $\xi_\mu(\mu = 0, 1, 2, 3)$ is given by,

$$\xi_i = \frac{x_i}{\sqrt{1 + \frac{r^2}{R^2}}} \qquad \xi_0 = \frac{R}{\sqrt{1 + \frac{r^2}{R^2}}}, \tag{2.2}$$

where the coordinates ξ_i change in the region $\xi_i\xi_i \leq R^2$.
Now we are going to write the general form of Kratzer potential in space of constant curvature. By using the $r^2 = x_1{}^2 + x_2{}^2 + x_3{}^2$ and equation (2.2), one can obtain the following potential,

$$V(r) = V(\xi) = D_e[1 - \frac{2r_e}{\frac{\xi}{(1-\frac{\xi^2}{R^2})^{\frac{1}{2}}}} + \frac{r_e^2}{\frac{\xi^2}{1-\frac{\xi^2}{R^2}}}]. \tag{2.3}$$

In the spherical system of coordinates we have,

$$\begin{aligned} &\xi_1 = R\sin\psi\sin\theta\cos\phi, \qquad \xi_2 = R\sin\psi\sin\theta\sin\phi \\ &\xi_3 = R\sin\psi\cos\theta, \qquad \xi_0 = R\cos\psi, \end{aligned} \tag{2.4}$$

where $0 \leq \psi < \pi$, $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$.
So, the kratzer potential (2.3) in spherical system is,

$$V(\psi) = D_e[1 - \frac{2r_e\cot\psi}{R} + \frac{r_e^2\cot^2\psi}{R^2}], \tag{2.5}$$

III. SOLUTION OF THE SCHRÖDINGER EQUATION FOR THE KRATZER POTENTIAL

In order to solve the Schrödinger equation, we need to write the corresponding equation (2.5) on constant curvature,

$$\left[-\frac{\hbar^2}{2\mu}\triangle + V\right]\Psi = E\Psi, \tag{3.1}$$

where $\triangle$ is the Laplace - Beltrami operator and is given by,

$$\triangle = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\sqrt{g}g^{ik}\frac{\partial}{\partial x^k} \tag{3.2}$$

so the metric is,

$$ds^2 = g_{ik}dx^i dx^k \tag{3.3}$$

where $g = det(g_{ik})$, $g^{ik} = (g_{ik})^{-1}$ and $r^2 = x_i x_i$ $(i, k = 1, 2, 3)$. and the metric can be obtained the following equation,

$$ds^2 = d\xi_1^2 + d\xi_2^2 + d\xi_3^2 + d\xi_0^2, \tag{3.4}$$

By using equation (2.4), one can obtain the equation (3.5) as following,

$$ds^2 = R^2 d^2\psi + R^2 \sin^2\psi d^2\theta + R^2 \sin^2\psi \sin^2\theta d^2\varphi, \tag{3.5}$$

and,

$$g = R^2 \sin^4\psi \sin^2\theta, \tag{3.6}$$

We put equations (2.5),(3.2),(3.5) and(3.6)into equation (3.1), and obtain the Schrödinger equation,

$$[\frac{1}{\sin^2\psi}\frac{\partial}{\partial\psi}\sin^2\psi\frac{\partial}{\partial\psi}]\Phi(\psi) + \frac{2\mu R^2}{\hbar^2}[E - \frac{\hbar^2}{2\mu R^2}\frac{m'(m'+1)}{\sin^2\psi} \tag{3.7}$$

$$-D_e(1 - \frac{2r_e\cot\psi}{R} + \frac{r_e^2\cot^2\psi}{R^2})]\Phi(\psi) = 0$$

Now we are going to choice the variables $\Phi(\psi) = \frac{W}{\sin\psi}$and $x = cos\psi$, and we have,

$$(1-x^2)W''(x) - \ xW'(x) + [1 + \frac{2\mu R^2}{\hbar^2}(E - \frac{\hbar^2}{2\mu R^2}\frac{m'(m'+1)}{1-x^2} \tag{3.8}$$

$$-D_e(1 + \frac{r_e^2}{R^2}\frac{x^2}{1-x^2})]W(x) = 0$$

where we have assume the following condition,

$$\cot\psi \geq -\frac{2R}{r_e} \tag{3.9}$$

In order to solve equation (3.8), we consider the following variable,

$$W(x) = g(x)p(x) \tag{3.10}$$

from which we obtain,

$$(1-x^2)P''(x) + [\ 2(1-x^2)\frac{g'(x)}{g(x)} - \ x]P'(x) \tag{3.11}$$

$$[1 + \frac{2\mu R^2}{\hbar^2}(E - \frac{\hbar^2}{2\mu R^2}\frac{m'(m'+1)}{1-x^2} - D_e(1 + \frac{r_e^2}{R^2}\frac{x^2}{1-x^2})]P(x) = 0$$

and compare with the following Gegenbauer equation [21],

$$(1-x^2)P''^{\lambda}_{n,m}(x) - 2(\lambda+1)xP'^{\lambda}_{n,m}(x) + [n(2\lambda+n+1) - \frac{m(2\lambda+m)}{1-x^2}]p^{\lambda}_{n,m}(x) = 0 \tag{3.12}$$

the wave function and energy spectrum are respectively,

$$\Phi(x) = a_{n,m}(\lambda)(1-x^2)^{\frac{2\lambda-1}{4}}p^{\lambda}_{n,m}(x) \tag{3.13}$$

$$E = D_e + \frac{\hbar^2}{2\mu R^2}\left[m'(m'+1) + 2\lambda(n-m) + n(n+1) - m^2 + (\lambda - \frac{1}{2})\right], \tag{3.14}$$

where $a_{n,m}(\lambda)$ is a real normalization coefficient.

IV. LADDER OPERATORS

By using the factorization method we construct the creation and annihilation operators. In order to obtain $A_{\pm}$ operators, we use following equation,

$$A_{\pm}\Phi_n(x) = a_{\pm}\Phi_{n\pm1}(x) \tag{4.1}$$

and,

$$A_{\pm} = \pm\alpha(x)\frac{d}{dx} \pm \beta(x) \tag{4.2}$$

In that case we apply the operators on the wave function (3.13),

$$\frac{d}{dx}\Phi(x) = (\frac{2\lambda-1}{2})(\frac{-x}{1-x^2})\Phi(x) + a_{n,m}(\lambda)(1-x^2)^{\frac{2\lambda-1}{4}}\frac{d}{dx}p^{\lambda}_{n,m}(x) \tag{4.3}$$

One possible relation for the first derivative of the Gegenbauer polynomials [21] is,

$$(1-x^2)\frac{d}{dx}p^{\lambda}_n(x) = (n+2\lambda-1)p^{\lambda}_{n-1}(x) - nxp^{\lambda}_n(x) \tag{4.4}$$

We put equation (4.4)into equation (4.3) and we obtain the lowering operator with respect to n,

$$A_-(n,x) = (1-x^2)\frac{d}{dx} + (n+\lambda-\frac{1}{2})x \tag{4.5}$$

We now proceed same the raising and obtain operator.In that case,we need to consider the Gegenbauer polynomials [21],as following,

$$(1-x^2)\frac{d}{dx}p^{\lambda}_n(x) = (n+2\lambda)xp^{\lambda}_n(x) - (n+1)p^{\lambda}_{n+1}(x) \tag{4.6}$$

We put again this equation into equation (4.2), one can obtain raising operator with respect to n,

$$A_+(n,x) = -(1-x^2)\frac{d}{dx} + (n+\lambda+\frac{1}{2})x \tag{4.7}$$

V. THE SUPERSYMMETRY APPROACHES FOR KRATZER POTENTIAL WITH CONSTANT POSITIVE CURVATURE

In order to discuss the supersymmetry for this model. we have to consider the ground state wave function. By using the hamiltonian and ground state wave function. we can obtain $V_1(r)$ as,

$$H_1\Psi_0(r) = -\frac{\hbar^2}{2m}\frac{d^2\Psi_0}{dr^2} + V_1(r)\Psi_0(r) = 0, \tag{5.1}$$

$$V_1(r) = \frac{\hbar^2}{2m}\frac{\Psi_0''(r)}{\Psi_0(r)} \tag{5.2}$$

Now, we factorize the corresponding Hamiltonian in terms of first order equation, which are called A,A^+,

$$H_1 = A^+A \tag{5.3}$$

where hamiltonian $H_1$is,

$$H_1 = -(1-x^2)^2\frac{d^2}{dx^2} + (\beta+2)x(1-x^2)\frac{d}{dx} - \alpha(1-x^2) + \beta\alpha x^2 \tag{5.4}$$

Now, we are going to obtain the hamiltonian H_2, as $H_2 = AA^+$, which is partner H_1, so we have,

$$H_2 = -(1-x^2)^2\frac{d^2}{dx^2} + (2-\alpha)x(1-x^2)\frac{d}{dx} + \beta(1-x^2) + \alpha\beta x^2 \tag{5.5}$$

The relation between H_1 and H_2, lead us to have a following equation,

$$n = -\lambda \tag{5.6}$$

By using the equation(5.6) in equation (4.5) and equation (4.7) we obtain the following relations.

$$A_-(n,x) = (1-x^2)\frac{d}{dx} - \frac{1}{2}x \tag{5.7}$$

$$A_+(n,x) = -(1-x^2)\frac{d}{dx} + \frac{1}{2}x$$

In that case hamiltonian H_1 will be as,

$$H_1 = -(1-x^2)^2\frac{d^2}{dx^2} + \frac{5}{2}x(1-x^2)\frac{d}{dx} - \frac{3}{4}x^2 + \frac{1}{2} \tag{5.8}$$

and this lead us to have a following equation,

$$V_1(r) = W^2(r) - \frac{\hbar}{\sqrt{2m}}W'(r) \tag{5.9}$$

This equation is known as Riccit equation, where W_r is superpotential, so we obtain,

$$W(r) = \frac{1}{2}x \tag{5.10}$$

Finally the corresponding potential $V_1(r)$ is,

$$V_1(r) = \frac{1}{4}x^2 - \frac{1}{2}\frac{\hbar}{\sqrt{2m}} \tag{5.11}$$

Now, we are going to obtain the Hamiltonian H_2,

$$H_2 = -(1-x^2)^2\frac{d^2}{dx^2} + \frac{5}{2}x(1-x^2)\frac{d}{dx} - \frac{3}{4}x^2 + \frac{1}{2} \tag{5.12}$$

where,

$$V_2(r) = W_2^2(r) + \frac{\hbar}{\sqrt{2m}}W_2'(r) \tag{5.13}$$

The $W_2(r)$ is superpotential is,

$$W_2(r) = -\frac{1}{2}x \tag{5.14}$$

Finally the corresponding potential $V_2(r)$ is,

$$V_2(r) = \frac{1}{4}x^2 - \frac{1}{2}\frac{\hbar}{\sqrt{2m}} \tag{5.15}$$

The shape invariance condition will be as $V_2(r) = V_1(r)$. The shape invariance condition and hamiltonian partner lead us to study the aspect of supersemmetry. So, the supercharges are,

$$Q = \begin{pmatrix} 0 & 0 \\ A_n & 0 \end{pmatrix}, \quad Q^+ = \begin{pmatrix} 0 & A_n^+ \\ 0 & 0 \end{pmatrix}, \tag{5.16}$$

Also, we have the following commutation relation,

$$[\ H, Q\ ] = [\ H, Q^{+}\ ] = 0 \tag{5.17}$$

$$\{\ Q, Q^{+}\ \} = H, \{\ Q, Q\ \} = \{\ Q^{+}, Q^{+}\ \} = 0$$

and,

$$[\ H, Q\ ] = [\ \begin{matrix} 0 & 0 \\ H_2A - AH_1 & 0 \end{matrix}\ ] \tag{5.18}$$

In order to satisfy the equations (5.18), we need following equation,

$$H_2A = AH_1 \tag{5.19}$$

This completely satisfy the following anti-commutation relations ,

$$\{\ Q, Q^{+}\ \} = H, \{\ Q, Q\ \} = \{\ Q^{+}, Q^{+}\ \} = 0$$

$$\{\ Q^{+}, Q^{+}\ \} = 0 \Rightarrow \begin{pmatrix} 0 & A^{+} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & A^{+} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & A^{+} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & A^{+} \\ 0 & 0 \end{pmatrix} = 0 \tag{5.20}$$

$$\{\ Q, Q\ \} = 0 \Rightarrow \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} = 0$$

and,

$$\{\ Q, Q^{+}\ \} = H \Rightarrow \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}\begin{pmatrix} 0 & A^{+} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & A^{+} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} = \tag{5.21}$$

$$\begin{pmatrix} A^{+}A & 0 \\ 0 & AA^{+} \end{pmatrix} = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} = H$$

VI. CONCLUSION

In this paper we discuss the Kratzer potential in constant Positive curvature. We have obtained the energy spectrum and wave function. Using the factorization method we derive some raising and lowering operators. These lead us to introduce some supercharge operators. With the help of shape invariance condition it may be interesting to discuss the representation of super algebra for the Kratzer potential in constant curvature.

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