

Time evolution of cascade decay

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Abstract

We study non-perturbatively the time evolution of cascade decay for generic fields $\pi \rightarrow \phi_1 \phi_2 \rightarrow \phi_2 \chi_1 \chi_2$ and obtain the time dependence of amplitudes and populations for the resonant and final states. We analyze in detail the different time scales and the manifestation of unitary time evolution in the dynamics of production and decay of resonant intermediate and final states. The probability of occupation (population) ‘flows’ as a function of time from the initial to the final states. When the decay width of the parent particle Γ_π is much larger than that of the intermediate resonant state Γ_{ϕ_1} there is a ‘bottleneck’ in the flow, the population of resonant states builds up to a maximum at $t^* = \ln[\Gamma_\pi/\Gamma_{\phi_1}]/(\Gamma_\pi - \Gamma_{\phi_1})$ nearly saturating unitarity and decays to the final state on the longer time scale $1/\Gamma_{\phi_1}$. As a consequence of the wide separation of time scales in this case the cascade decay can be interpreted as evolving sequentially $\pi \rightarrow \phi_1 \phi_2$; $\phi_1 \phi_2 \rightarrow \phi_2 \chi_1 \chi_2$. In the opposite limit the population of resonances (ϕ_1) does not build up substantially and the cascade decay proceeds almost directly from the initial parent to the final state without resulting in a large amplitude of the resonant state. An alternative but equivalent non-perturbative method useful in cosmology is presented. Possible phenomenological implications for heavy sterile neutrinos as resonant states and consequences of quantum entanglement and correlations in the final state are discussed.

Keywords: cascade decay, non-perturbative methods, time evolution



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1. Introduction

The decay of unstable or metastable states via a cascade $A \rightarrow BC \rightarrow BXY$ is of interdisciplinary importance in particle physics, quantum optics and cosmology. Early studies in particle physics proposed to use the time evolution of intermediate (resonant) states in cascade decays of heavy (B) mesons to study aspects of CP violation, mixing phenomena [1–6] and CPT violation [7, 8]. The Belle collaboration [9] has reported on remarkably precise measurements of (EPR) entanglement and correlations in cascade decays $Y(4S) \rightarrow B^0\bar{B}^0 \rightarrow (l, J/\Psi, K)$ via the analysis of the time dependence of the flavor asymmetry [10]. The BaBar collaboration [11, 12] has reported on the first direct measurement of time reversal violation in the $B^0\bar{B}^0$ system from $Y(4S)$ decay at rest by studying the time dependent correlations between the members of the entangled $B^0\bar{B}^0$ pairs also in a cascade decay.

More recently cascade decays have been proposed as possible mechanisms of CP and lepton flavor violation mediated by heavy sterile neutrinos as resonant states [13] and as possible solutions to the LSND/MiniBooNe anomalies [14]. Cascade decays may probe new particles beyond the standard model via kinematic edges associated with the new degrees of freedom in the intermediate states [15].

In quantum optics and cavity quantum electrodynamics, the cascade decay of multi-level atoms or quantum dot systems is studied as a source of correlated photons and entanglement [16–21] with potential applications in quantum information. The possibility of observation of polarization entangled photon pairs in cascade decay in quantum dots has recently been advanced, such a measurement is, fundamentally, similar to the observation of CP violating amplitudes in the time evolution of a cascade decay in B-mesons and similar meson systems addressed in [3–6].

In inflationary cosmology the rapid expansion of the Universe entails that there is no time-like Killing vector and as a consequence of the lack of kinematic thresholds quanta of a field can decay into quanta of the *same* field [22–26] leading to the recent suggestion of cascade decay of inflaton quanta during inflation [27] and the concomitant kinematic entanglement of the produced particles.

Motivation and goals: in the analysis of cascade decays in [1–5], the time evolution of the cascade is analyzed as a *sequential* series of events. Consider an initial state I decaying into an intermediate state M , which in turn decays into a final state f , the amplitude for such process is proposed to be

$$A\left[I \xrightarrow{t_I} M \xrightarrow{t_M} f\right] = e^{-iW_I t_I} e^{-iW_M t_M} \mathcal{M}_{I \rightarrow M} \mathcal{M}_{M \rightarrow f}, \quad (1.1)$$

where t_I is the time at which the initial state I decays into the intermediate resonant state M and t_M is the time at which the intermediate state M decays into the final state in their respective rest frames, $W_{I,M} = m_{I,M} - i\Gamma_{I,M}$ are the complex energies of the corresponding states and \mathcal{M} are the corresponding transition amplitudes (we neglect here the possibility of mixing in the initial, intermediate or final states).

While this sequential characterization may be phenomenologically suitable to the description of experimental situations in which there is a wide separation in time scales as a consequence of large differences in widths and masses, it clearly *assumes* that the decays occur at specific (proper) times t_I, t_M in sequence.

However, in a general case, this is at odds with the description of radiative cascades in multi-level systems in quantum optics [16–18] and with the description of decay of an unstable (or metastable) state as a continuous process with a (generally exponential) time distribution in which the amplitude for the parent particle decays exponentially and the amplitude of the daughter (resonant state) increases continuously on a similar time scale.

The interdisciplinary relevance of cascade decay motivates us to study the time evolution within a framework that is suitable to extension to the realm of cosmology and that could prove useful in other areas such as quantum optics and condensed matter physics.

The goal of this paper is to study the full time evolution of a cascade decay process directly from the quantum mechanical evolution of an initial quantum state. In particular we address the important issue of unitarity in the time evolution from the initial to the final state, focusing on the time evolution of the amplitudes and populations of the intermediate resonant and final states and how unitarity is manifest in these amplitudes.

For this purpose we provide a non-perturbative quantum field theoretical generalization of the methods ubiquitous in quantum optics [16–19] adapted to the realm of a full quantum field theory. Furthermore, one of our objectives is to provide a non-perturbative method to study the time evolution directly in real time that would be suitable for applications in cosmology where the expansion of the Universe introduces an explicit time dependence in the evolution Hamiltonian [23, 26, 27].

In this paper we consider the case of only one resonant intermediate state to introduce and develop the methods and to exhibit the main physical processes in a simpler setting, postponing the study of mixing and oscillations in the resonant state for future study.

Brief summary of results: we implement a non-perturbative method that yields the full time evolution of an initial state of a decaying parent particle in a model of generic fields that incorporates the main features of a cascade decay via a resonant state $\pi \rightarrow \phi_1 \phi_2 \rightarrow \phi_2 \chi_1 \chi_2$ although the results are general. We obtain the time evolution of the amplitudes for the intermediate and final states and show explicitly that the method is akin to a Dyson-type resummation of self-energies and fulfills unitarity. Unitary time evolution is manifest in a transfer or ‘flow’ of probability or population from the initial through the intermediate resonant to the final state. We analyze in detail the evolution of the populations on the different time scales. The population of the resonant state grows at early times reaching a maximum at $t^* = \ln \left[\Gamma_\pi / \Gamma_{\phi_1} \right] / (\Gamma_\pi - \Gamma_{\phi_1})$ and decaying to the final state.

The build-up of the population of the intermediate resonant state depends crucially on the ratio of decay widths $\Gamma_\pi / \Gamma_{\phi_1}$. For $\Gamma_\pi / \Gamma_{\phi_1} \gg 1$ there is a ‘bottleneck’ in the sense that the population of the resonant state builds up to nearly saturate unitarity on a time scale $t \simeq t^*$ and decays into the final state on much longer time scales $\simeq 1/\Gamma_{\phi_1}$. In the opposite limit, the population of the initial state is transferred almost directly to the final state with a very small build-up of the population of the intermediate resonant state.

The final asymptotic state after both the parent particle and intermediate resonant state decays is a many particle state featuring quantum entanglement and correlations among the final particles.

We also provide a quantum field theoretical generalization of the Wigner–Weisskopf method that provides a similar type of non-perturbative resummation directly in real time and

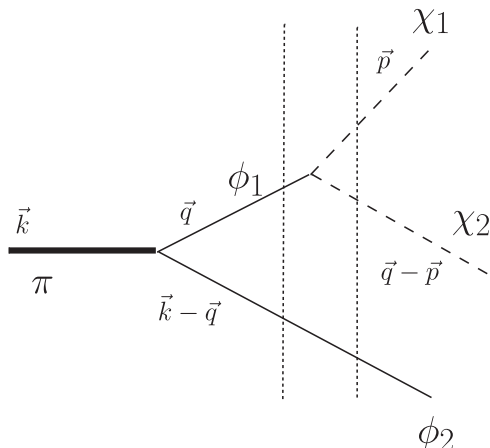


Figure 1. Cascade decay $\pi \rightarrow \phi_1\phi_2$; $\phi_1 \rightarrow \chi_1\chi_2$. The dashed lines depict the intermediate two particle state and the final three particle state.

yields the same results but is amenable of implementation in cosmology where the interaction Hamiltonian is explicitly dependent on time.

We conjecture on possible phenomenological consequences in particular for heavy sterile neutrinos as intermediate resonant states in pseudoscalar decays.

2. The model

We consider a model of generic real scalar fields π , $\phi_{1,2}$, $\chi_{1,2}$ of masses M_π ; $m_{1,2}^\phi$, $m_{1,2}^\chi$ respectively to study the relevant phenomena in the simplest setting and to focus on the main aspects of the method and physical processes, however the main results will be argued to be general.

The total Hamiltonian is $H = H_0 + H_I$ with H_0 the free field Hamiltonian and

$$H_I = \int d^3x \left\{ g_\pi \pi(x) \phi_1(x) \phi_2(x) + g_\phi \phi_1(x) \chi_1(x) \chi_2(x) \right\}. \quad (2.1)$$

Let us consider an initial state with one π particle of momentum \vec{k} and the vacuum for the other fields, namely

$$|\Psi(t=0)\rangle = |\pi_{\vec{k}}\rangle. \quad (2.2)$$

Upon time evolution this state evolves into $|\Psi(t)\rangle$ obeying

$$\frac{d}{dt} |\Psi(t)\rangle = -i(H_0 + H_I) |\Psi(t)\rangle, \quad (2.3)$$

when $m_\pi > m_{\phi_1} + m_{\phi_2}$; $m_{\phi_1} > m_{\chi_1} + m_{\chi_2}$ the interaction Hamiltonian (2.1) describes the cascade process depicted in figure 1.

At any given time the state $|\Psi(t)\rangle$ can be expanded in Fock states of the non-interacting Hamiltonian H_0

$$|\Psi(t)\rangle = \sum_n C_n(t) |n\rangle ; H_0 |n\rangle = E_n |n\rangle, \quad (2.4)$$

the time evolution of the coefficients $C_n(t)$ is obtained from (2.3), and projecting onto the states $|n\rangle$ namely

$$\dot{C}_n(t) = -iE_n C_n(t) + \sum_\kappa \langle n | H_I | \kappa \rangle C_\kappa(t). \quad (2.5)$$

This is an infinite hierarchy of equations, progress is made by truncating the hierarchy at a given order and solving the coupled equations, this method is equivalent to a *Dyson resummation of self-energy diagrams* as will be seen clearly and in detail below.

The time evolution of the initial state depicted up to second order in the interaction in figure 1 leads to considering the following form

$$\begin{aligned} |\Psi(t)\rangle = & C_\pi(\vec{k}, t) |\pi_{\vec{k}}\rangle + \sum_{\vec{q}} C_{\phi\phi}(\vec{k}, \vec{q}; t) |\phi_{1,\vec{q}} \phi_{2,\vec{k}-\vec{q}}\rangle \\ & + \sum_{\vec{q}, \vec{p}} C_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}; t) |\phi_{2,\vec{k}-\vec{q}} \chi_{1,\vec{p}} \chi_{2,\vec{q}-\vec{p}}\rangle + \dots \end{aligned} \quad (2.6)$$

where the dots stand for many particle states that emerge in higher order in H_I . The matrix elements of the interaction Hamiltonian in (2.5) describe transitions between single and multiparticle states, and the functions $C_\pi(\vec{k}, t)$; $C_{\phi\phi}(\vec{k}, \vec{q}; t)$; $C_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}; t)$ represent the amplitudes of the initial single particle state and multiparticle states in the time evolved state. The initial conditions on these amplitudes are

$$C_\pi(\vec{k}, 0) = 1 ; C_{\phi\phi}(\vec{k}, \vec{q}; 0) = 0 ; C_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}; 0) = 0. \quad (2.7)$$

The evolution equations for the amplitudes are obtained by projection, as in (2.5). Introducing the shorthand definitions

$$M_{\pi \rightarrow \phi_1 \phi_2}(\vec{k}, \vec{q}) \equiv \langle \pi_{\vec{k}} | H_I | \phi_{1,\vec{q}} \phi_{2,\vec{k}-\vec{q}} \rangle, \quad (2.8)$$

$$M_{\phi_1 \rightarrow \chi_1 \chi_2}(\vec{q}, \vec{p}) \equiv \langle \phi_{1,\vec{q}} | H_I | \chi_{1,\vec{p}} \chi_{2,\vec{q}-\vec{p}} \rangle, \quad (2.9)$$

and

$$E_{\phi\phi}(\vec{k}, \vec{q}) \equiv E_{\vec{q}}^{\phi_1} + E_{\vec{k}-\vec{q}}^{\phi_2}, \quad (2.10)$$

$$E_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}) \equiv E_{\vec{k}-\vec{q}}^{\phi_2} + E_{\vec{p}}^{\chi_1} + E_{\vec{q}-\vec{p}}^{\chi_2}, \quad (2.11)$$

where $E_{\vec{k}}$ are the single particle energies for π , ϕ , χ respectively we find

$$\dot{C}_\pi(\vec{k}, t) = -iE_{\vec{k}}^\pi C_\pi(\vec{k}, t) - i \sum_{\vec{q}} M_{\pi \rightarrow \phi_1 \phi_2}(\vec{k}, \vec{q}) C_{\phi\phi}(\vec{k}, \vec{q}; t), \quad (2.12)$$

$$\begin{aligned} \dot{C}_{\phi\phi}(\vec{k}, \vec{q}; t) = & -iE_{\phi\phi}(\vec{k}, \vec{q}) C_{\phi\phi}(\vec{k}, \vec{q}; t) - iM_{\pi \rightarrow \phi_1 \phi_2}^*(\vec{k}, \vec{q}) C_\pi(\vec{k}, t) \\ & - i \sum_{\vec{p}} M_{\phi_1 \rightarrow \chi_1 \chi_2}(\vec{q}, \vec{p}) C_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}; t), \end{aligned} \quad (2.13)$$

$$\dot{C}_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}; t) = -iE_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p})C_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}; t) - iM_{\phi_1 \rightarrow \chi_1 \chi_2}^*(\vec{q}, \vec{p})C_{\phi\phi}(\vec{k}, \vec{q}; t). \quad (2.14)$$

We have truncated the hierarchy of equations up to second order in the interaction Hamiltonian (2.1), thereby neglecting the higher order branches with four particles etc. As it will be demonstrated below, the solution of the coupled hierarchy of equations up to this order provides Dyson resummations of *both* the propagator of the π particle *and* the propagator for the ϕ_1 particle which corresponds to the resonant intermediate state in the cascade decay process. This will become clear below.

In what follows we will suppress the arguments of the various functions introduced above to simplify notation.

It proves convenient to solve these coupled set of equations by Laplace transform as befits an initial value problem. Defining the Laplace transform of $C(t)$ as

$$\tilde{C}(s) = \int_0^\infty e^{-st} C(t) dt \quad (2.15)$$

and with the initial conditions (2.7) the set of coupled equations (2.12)–(2.14) becomes

$$\tilde{C}_\pi(\vec{k}, s) \left[s + iE_k^\pi \right] = 1 - i \sum_{\vec{q}} M_{\pi \rightarrow \phi_1 \phi_2}(\vec{k}, \vec{q}) \tilde{C}_{\phi\phi}(\vec{k}, \vec{q}; s), \quad (2.16)$$

$$\begin{aligned} \tilde{C}_{\phi\phi}(\vec{k}, \vec{q}; s) \left[s + iE_{\phi\phi} \right] &= -iM_{\pi \rightarrow \phi_1 \phi_2}^*(\vec{k}, \vec{q}) \tilde{C}_\pi(\vec{k}; s) \\ &\quad - i \sum_{\vec{p}} M_{\phi_1 \rightarrow \chi_1 \chi_2}(\vec{q}, \vec{p}) \tilde{C}_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}; s), \end{aligned} \quad (2.17)$$

$$\tilde{C}_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}; s) \left[s + iE_{\phi\chi\chi} \right] = -iM_{\phi_1 \rightarrow \chi_1 \chi_2}^*(\vec{q}, \vec{p}) \tilde{C}_{\phi\phi}(\vec{k}, \vec{q}; s). \quad (2.18)$$

We now solve this system of algebraic equations from the bottom up obtaining,

$$\tilde{C}_\pi(\vec{k}; s) = \frac{1}{s + iE_k^\pi + i\Sigma^\pi(\vec{k}; s)}, \quad (2.19)$$

$$\tilde{C}_{\phi\phi}(\vec{k}, \vec{q}; s) = -i \frac{M_{\pi \rightarrow \phi_1 \phi_2}^* \tilde{C}_\pi(\vec{k}; s)}{s + iE_{\phi\phi} + i\Sigma^\phi(\vec{k}, \vec{q}; s)}, \quad (2.20)$$

$$\tilde{C}_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}; s) = -i \frac{M_{\phi_1 \rightarrow \chi_1 \chi_2}^* \tilde{C}_{\phi\phi}(\vec{k}; s)}{s + iE_{\phi\chi\chi}}, \quad (2.21)$$

where the self-energies are given by

$$i\Sigma^\pi(\vec{k}; s) = \sum_{\vec{q}} \frac{|M_{\pi \rightarrow \phi_1 \phi_2}|^2}{s + iE_{\phi\phi} + i\Sigma^\phi(\vec{k}, \vec{q}; s)}, \quad (2.22)$$

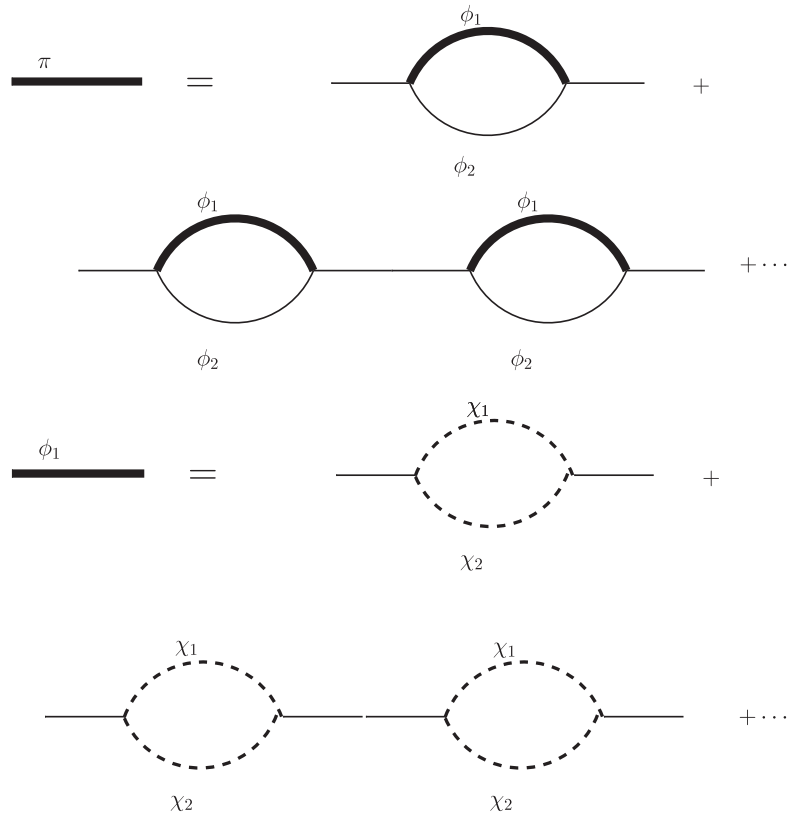


Figure 2. π propagator, the thick ϕ_1 line is the full ϕ_1 propagator with self-energy resummation.

and

$$i\Sigma^\phi(\vec{k}, \vec{q}; s) = \sum_{\vec{p}} \frac{|M_{\phi_1 \rightarrow \chi_1 \chi_2}|^2}{s + iE_{\phi_{ZZ}}}. \quad (2.23)$$

These equations have a familiar interpretation in terms of Dyson resummations of self-energy diagrams depicted in figure 2: the irreducible self-energy for the π -field is in terms of the full propagator of the (resonant) ϕ_1 particle which is itself given by a Dyson resummation of irreducible self-energy diagrams involving a loop of the final state particles. It is at this stage that the equivalence between the truncation of the hierarchy of equations and the resummation in terms of a Dyson series becomes manifest. Therefore, the solution of the truncated hierarchy up to the given (second) order, provides a non-perturbative resummation of self-energy diagrams both for the decaying parent particle *and* the intermediate resonant state. The π self-energy (2.22) includes the self-energy correction to the (intermediate) ϕ_1 state.

2.1. Time evolution

The time evolution is obtained by performing the inverse Laplace transform:

$$C(t) = \int_C \frac{ds}{2\pi i} e^{st} \tilde{C}(s), \quad (2.24)$$

where C stands for the Bromwich contour parallel to the imaginary axis in the complex s plane and to the right of all the singularities of the function $\tilde{C}(s)$.

With the purpose of using the convolution theorem

$$\int_C \frac{ds}{2\pi i} e^{st} \tilde{F}(s) \tilde{G}(s) = \int_0^t F[t-t'] G[t'] dt', \quad (2.25)$$

where $F[t]$, $G[t]$ are the inverse Laplace transforms of $\tilde{F}(s)$, $\tilde{G}(s)$ respectively, it proves convenient to rewrite (2.20), (2.21) as

$$\tilde{C}_{\phi\phi}(\vec{k}, \vec{q}; s) = -iM_{\pi \rightarrow \phi_1 \phi_2}^* \tilde{C}_\pi(\vec{k}; s) \tilde{G}_{\phi\phi}(s); \quad \tilde{G}_{\phi\phi}(s) = \frac{1}{s + iE_{\phi\phi} + i\Sigma^\phi(s)}, \quad (2.26)$$

$$\tilde{C}_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}; s) = -iM_{\phi_1 \rightarrow \chi_1 \chi_2}^* \tilde{C}_{\phi\phi}(\vec{k}; s) \tilde{G}_{\phi\chi\chi}(s); \quad \tilde{G}_{\phi\chi\chi}(s) = \frac{1}{s + iE_{\phi\chi\chi}}. \quad (2.27)$$

The amplitudes $\tilde{C}(s)$ given by equations (2.19)–(2.21) generally feature branch cuts above multiparticle thresholds and complex poles with $Re s \leq 0$ namely either along the imaginary axis or to its left as befits stable particles, multiparticle cuts (along the imaginary axis) or decaying resonances. Therefore the Bromwich contour corresponds to integration along the path parallel to the imaginary axis $s = i\omega + \varepsilon$; $-\infty \leq \omega \leq \infty$; $\varepsilon \rightarrow 0^+$ and

$$C(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} \tilde{C}(s = i\omega + \varepsilon), \quad (2.28)$$

where the contour must now be closed in the upper half ω - plane.

The first step in obtaining the time evolution of the amplitudes is to obtain $C_\pi(t)$ for which we need to identify the singularities in

$$\tilde{C}_\pi(s = i\omega + \varepsilon) = \frac{1}{i\omega + iE_k^\pi + i\Sigma^\pi(\vec{k}, i\omega + \varepsilon) + \varepsilon}. \quad (2.29)$$

The self-energy features a two-particle cut and the π particle becomes a resonance if E_k^π is embedded in the two-particle continuum. For a weakly coupled theory this resonance is described by a complex pole in the upper half plane very near the real axis, since in absence of perturbations the pole is at $\omega = -E_k^\pi$. Consistently in perturbation theory we write $\omega = -E_k^\pi$ in the argument of Σ^π , with

$$i\Sigma^\pi(\vec{k}, -iE_k^\pi + \varepsilon) = \sum_{\vec{q}} \frac{|M_{\pi \rightarrow \phi_1 \phi_2}|^2}{-iE_k^\pi + iE_{\phi\phi} + i\Sigma^\phi(\vec{k}, \vec{q}; -iE_k^\pi + \varepsilon) + \varepsilon}, \quad (2.30)$$

and from (2.23) we find

$$i\Sigma^\phi(\vec{k}, \vec{q}; s = -iE_k^\pi + \varepsilon) = i \sum_{\vec{p}} \frac{|M_{\phi_1 \rightarrow \chi_1 \chi_2}|^2}{(E_k^\pi - E_{\phi\chi\chi}) + i\varepsilon} = i\delta E + \frac{1}{2} \gamma(\vec{k}, \vec{q}), \quad (2.31)$$

$$\delta E = \sum_{\vec{p}} \mathcal{P} \frac{|M_{\phi_1 \rightarrow \chi_1 \chi_2}|^2}{(E_k^\pi - E_{\phi\chi\chi})}, \quad (2.32)$$

$$\gamma(\vec{k}, \vec{q}) = 2\pi \sum_{\vec{p}} |M_{\phi_1 \rightarrow \chi_1 \chi_2}|^2 \delta(E_k^\pi - E_{\phi\chi\chi}), \quad (2.33)$$

we note that δE is *not* the renormalization of the ϕ_1 energy (mass) and $\gamma(\vec{k}, \vec{q})$ is *not* its decay width, because in the denominator of (2.32) and the argument of the delta function in (2.33) is $E_k^\pi - E_{\vec{k}-\vec{q}}^{\phi_2} - E_{\vec{p}}^{\chi_1} - E_{\vec{q}-\vec{p}}^{\chi_2}$ instead of being $E_{\vec{q}}^{\phi_1} - E_{\vec{p}}^{\chi_1} - E_{\vec{q}-\vec{p}}^{\chi_2}$ (see below for clarification on this point).

Inserting this expression into (2.30) we find

$$i\Sigma^\pi(\vec{k}, -iE_k^\pi + \varepsilon) = i\Delta E^\pi + \frac{\Gamma_\pi(k)}{2}, \quad (2.34)$$

where the energy shift

$$\Delta E^\pi = \sum_{\vec{q}} \frac{|M_{\pi \rightarrow \phi_1 \phi_2}|^2 (E_k^\pi - E_{\phi\phi} - \delta E)}{(E_k^\pi - E_{\phi\phi} - \delta E)^2 + \left(\frac{\gamma(\vec{k}, \vec{q})}{2}\right)^2}, \quad (2.35)$$

is absorbed into a renormalization of the π mass, and

$$\begin{aligned} \Gamma_\pi(k) &= \sum_{\vec{q}} \frac{|M_{\pi \rightarrow \phi_1 \phi_2}|^2 \gamma(\vec{k}, \vec{q})}{(E_k^\pi - E_{\phi\phi} - \delta E)^2 + \left(\frac{\gamma(\vec{k}, \vec{q})}{2}\right)^2} \\ &= 2\pi \sum_{\vec{q}} \sum_{\vec{p}} \frac{|M_{\pi \rightarrow \phi_1 \phi_2}|^2 |M_{\phi_1 \rightarrow \chi_1 \chi_2}|^2 \delta(E_k^\pi - E_{\phi\chi\chi})}{(E_k^\pi - E_{\phi\phi} - \delta E)^2 + \left(\frac{\gamma(\vec{k}, \vec{q})}{2}\right)^2}, \end{aligned} \quad (2.36)$$

is the decay width of the π particle.

As mentioned above $\gamma(\vec{k}, \vec{q})$ is *not* the width of the ϕ_1 resonance, and (2.32) is *not* the renormalization of the ϕ_1 energy (mass renormalization), however in perturbation theory Γ_π (2.36) becomes resonant as $E_k^\pi \rightarrow E_{\phi\phi}$, and the sum is dominated by this resonance. Near this resonance one can replace $E^\pi \rightarrow E_{\phi\phi}$ into (2.32), (2.33) and recognizing from (2.10), (2.11) that

$E_{\phi\phi} - E_{\phi\chi\chi} = E_{\vec{q}}^{\phi_1} - E_{\vec{p}}^{\chi_1} - E_{\vec{q}-\vec{p}}^{\chi_2}$ it follows that *near this resonance*

$$\delta E \rightarrow \Delta E^{\phi_1} = \sum_{\vec{p}} \mathcal{P} \frac{|M_{\phi_1 \rightarrow \chi_1 \chi_2}|^2}{\left(E_{\vec{q}}^{\phi_1} - E_{\vec{p}}^{\chi_1} - E_{\vec{q}-\vec{p}}^{\chi_2}\right)}, \quad (2.37)$$

is the energy shift absorbed into a renormalization of the ϕ_1 mass and

$$\gamma(\vec{k}, \vec{q}) \rightarrow \Gamma_{\phi_1}(\vec{q}) = 2\pi \sum_{\vec{p}} |M_{\phi_1 \rightarrow \chi_1 \chi_2}|^2 \delta\left(E_{\vec{q}}^{\phi_1} - E_{\vec{p}}^{\chi_1} - E_{\vec{q}-\vec{p}}^{\chi_2}\right), \quad (2.38)$$

is the decay width of ϕ_1 .

Therefore in a cascade decay where the intermediate ϕ_1 becomes resonant (near on-shell) and absorbing δE near this resonance into the renormalization of the ϕ_1 energy (mass), the π decay rate (2.36) becomes

$$\Gamma_{\pi}(k) \simeq 2\pi \sum_{\vec{q}} \sum_{\vec{p}} \frac{|M_{\pi \rightarrow \phi_1 \phi_2}|^2 |M_{\phi_1 \rightarrow \chi_1 \chi_2}|^2 \delta\left(E_k^{\pi} - E_{\vec{k}-\vec{q}}^{\phi_2} - E_{\vec{p}}^{\chi_1} - E_{\vec{q}-\vec{p}}^{\chi_2}\right)}{\left(E_k^{\pi} - E_{\vec{k}-\vec{q}}^{\phi_2} - E_{\vec{q}}^{\phi_1}\right)^2 + \left(\frac{\Gamma_{\phi_1}(\vec{q})}{2}\right)^2}, \quad (2.39)$$

This is the usual expression of a decay rate of the parent particle in a cascade in the Breit–Wigner approximation for the propagator of the resonant intermediate state.

Furthermore in the narrow width approximation (2.39), is dominated by the resonance in the denominator and to leading order in the width we can replace

$$\frac{1}{\left(E_k^{\pi} - E_{\vec{k}-\vec{q}}^{\phi_2} - E_{\vec{q}}^{\phi_1}\right)^2 + \left(\frac{\Gamma_{\phi_1}(\vec{q})}{2}\right)^2} \rightarrow \frac{2\pi}{\Gamma_{\phi_1}(\vec{q})} \delta\left(E_k^{\pi} - E_{\vec{k}-\vec{q}}^{\phi_2} - E_{\vec{q}}^{\phi_1}\right), \quad (2.40)$$

and write (2.39) as

$$\begin{aligned} \Gamma_{\pi}(k) &\simeq 2\pi \sum_{\vec{q}} |M_{\pi \rightarrow \phi_1 \phi_2}|^2 \delta\left(E_k^{\pi} - E_{\vec{k}-\vec{q}}^{\phi_2} - E_{\vec{q}}^{\phi_1}\right) \\ &\times \left[\frac{2\pi}{\Gamma_{\phi_1}(\vec{q})} \sum_{\vec{p}} |M_{\phi_1 \rightarrow \chi_1 \chi_2}|^2 \delta\left(E_{\vec{q}}^{\phi_1} - E_{\vec{p}}^{\chi_1} - E_{\vec{q}-\vec{p}}^{\chi_2}\right) \right]. \end{aligned} \quad (2.41)$$

From (2.38) we see that the bracket in this expression equals one, leading to

$$\Gamma_{\pi}(k) \simeq 2\pi \sum_{\vec{q}} |M_{\pi \rightarrow \phi_1 \phi_2}|^2 \delta\left(E_k^{\pi} - E_{\vec{k}-\vec{q}}^{\phi_2} - E_{\vec{q}}^{\phi_1}\right), \quad (2.42)$$

which is valid in the narrow width approximation. In this case there is no other decay channel for the resonant state and the bracket in (2.41) becomes unity, however if there are other decay channels this bracket would be replaced by the branching ratio $BR(\phi_1 \rightarrow \chi_1 \chi_2)$, which is the usual result for resonant decay in the narrow width approximation.

We are now in position to obtain the time evolution of the amplitudes. In the narrow width limit, the amplitude $\tilde{C}_{\pi}(\omega)$ given by (2.29) features a (narrow) resonance near $\omega \simeq -E_k^{\pi}$ and is

of the Breit–Wigner form

$$\tilde{C}_\pi(\omega \simeq -E_k^\pi) \simeq \frac{-i}{\omega + E_k^\pi - \frac{i}{2}\Gamma_\pi(\vec{k})}, \quad (2.43)$$

where now E_k^π is the renormalized π single particle energy and Γ_π is the decay width. This is equivalent to a Breit–Wigner approximation to the propagator in terms of a complex pole and as usual ignores the ‘background’ contribution which is perturbative. The time evolution of the amplitude follows straightforwardly, it is given by

$$C_\pi(\vec{k}, t) = e^{-iE_k^\pi t} e^{-\frac{\Gamma_\pi(k)}{2} t}. \quad (2.44)$$

In a similar manner we now obtain the time evolution of $G_{\phi\phi}(t)$, the anti Laplace transform of $\tilde{G}_{\phi\phi}(s)$ in equation (2.26), with

$$\tilde{G}_{\phi\phi}(s = i\omega + \varepsilon) = \frac{1}{i\omega + iE_{\phi\phi} + i\Sigma^\phi(\vec{k}; i\omega + \varepsilon) + \varepsilon}. \quad (2.45)$$

It features a complex pole near $\omega \simeq -E_{\phi\phi}$ with

$$i\Sigma^\phi(s = -iE_{\phi\phi} + \varepsilon) = i\Delta E^{\phi_1} + \frac{\Gamma_{\phi_1}}{2}, \quad (2.46)$$

where the energy shift ΔE^{ϕ_1} and decay width Γ_{ϕ_1} are given by (2.37), (2.38) respectively, and the energy shift ΔE^{ϕ_1} is absorbed into a renormalization of the ϕ_1 mass.

As in the case of $C_\pi(t)$ we now obtain

$$G_{\phi\phi}(t) = e^{-iE_{\phi\phi} t} e^{-\frac{\Gamma_{\phi_1}}{2} t}, \quad (2.47)$$

where E_{ϕ_1} in $E_{\phi\phi}$ is the renormalized single particle energy for ϕ_1 . We now implement the convolution theorem (2.25) with $C_\pi(t)$, $G_{\phi\phi}(t)$ given by (2.44), (2.47) respectively and find

$$C_{\phi\phi}(t) = M_{\pi \rightarrow \phi_1 \phi_2}^* \frac{\left[e^{-iE_k^\pi t} e^{-\frac{\Gamma_\pi(k)}{2} t} - e^{-iE_{\phi\phi} t} e^{-\frac{\Gamma_{\phi_1}}{2} t} \right]}{\left(E^\pi - E_{\phi\phi} \right) - \frac{i}{2} \left(\Gamma_\pi - \Gamma_{\phi_1} \right)}. \quad (2.48)$$

This is the amplitude of the two particle intermediate state with a resonant ϕ_1 .

From equation (2.21), (2.27) and with the anti Laplace transform of $\tilde{G}_{\phi\chi\chi}$ given by

$$G_{\phi\chi\chi}(t) = e^{-iE_{\phi\chi\chi} t}, \quad (2.49)$$

we find

$$C_{\phi\chi\chi}(t) = -iM_{\phi_1 \rightarrow \chi_1 \chi_2}^* \int_0^t C_{\phi\phi}(t') e^{-iE_{\phi\chi\chi}(t-t')} dt', \quad (2.50)$$

which yields (suppressing all the momenta labels)

$$C_{\phi\chi\chi}(t) = \frac{M_{\pi \rightarrow \phi_1 \phi_2}^* M_{\phi_1 \rightarrow \chi_1 \chi_2}^* e^{-iE_{\phi\chi\chi} t}}{\left(E^\pi - E_{\phi\phi}\right) - \frac{i}{2} \left(\Gamma_\pi - \Gamma_{\phi_1}\right)} \times \left\{ \frac{\left[e^{-i(E^\pi - E_{\phi\chi\chi})t} e^{-\frac{\Gamma_\pi}{2}t} - 1 \right]}{\left(E^\pi - E_{\phi\chi\chi}\right) - i\frac{\Gamma_\pi}{2}} - \frac{\left[e^{-i(E_{\phi\phi} - E_{\phi\chi\chi})t} e^{-\frac{\Gamma_{\phi_1}}{2}t} - 1 \right]}{\left(E_{\phi\phi} - E_{\phi\chi\chi}\right) - i\frac{\Gamma_{\phi_1}}{2}} \right\}. \quad (2.51)$$

This is the amplitude of the final three particle state. Although the amplitudes (2.48), (2.51) look unfamiliar, it will be proven in the next section that they satisfy unitarity.

3. Unitarity: population flow

Unitary time evolution of the state $|\Psi(t)\rangle$ (2.4) with the initial condition (2.2) implies $\langle\Psi(t)|\Psi(t)\rangle = 1$, namely

$$\sum_n \left| C_n(t) \right|^2 = 1, \quad (3.1)$$

which for the state (2.6) implies

$$\left| C_\pi(\vec{k}, t) \right|^2 + \sum_{\vec{q}} \left| C_{\phi\phi}(\vec{k}, \vec{q}; t) \right|^2 + \sum_{\vec{q}, \vec{p}} \left| C_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}; t) \right|^2 + \dots = 1. \quad (3.2)$$

The number of π particles with momentum \vec{k} is given by

$$n_{\vec{k}}^\pi(t) = \langle\Psi(t) | a_{\pi, \vec{k}}^\dagger a_{\pi, \vec{k}} | \Psi(t) \rangle = \left| C_\pi(\vec{k}; t) \right|^2, \quad (3.3)$$

similarly the number of ϕ_1 particles with particular momentum \vec{q} in the time evolved state $|\Psi(t)\rangle$ is given by

$$n_{\vec{q}}^{\phi_1}(t) = \langle\Psi(t) | a_{\phi_1, \vec{q}}^\dagger a_{\phi_1, \vec{q}} | \Psi(t) \rangle = \left| C_{\phi\phi}(\vec{k}, \vec{q}; t) \right|^2, \quad (3.4)$$

and the number of *pairs* of $\chi_{1,2}$ particles with momenta $\vec{p}, \vec{q} - \vec{p}$ respectively is

$$n_{\vec{p}}^{\chi_1}(t) n_{\vec{q}-\vec{p}}^{\chi_2}(t) = \langle\Psi(t) | \left(a_{\chi_1, \vec{p}}^\dagger a_{\chi_1, \vec{p}} \right) \left(a_{\chi_2, \vec{q}-\vec{p}}^\dagger a_{\chi_2, \vec{q}-\vec{p}} \right) | \Psi(t) \rangle = \left| C_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}; t) \right|^2, \quad (3.5)$$

where $a_\alpha^\dagger; a_\alpha$ are the annihilation and creation operators for the quanta of the respective fields. Therefore the probabilities $|C(t)|^2$ are also a measure of the *population* of the many particle states upon decay of the initial state.

Since $\left| C_\pi(\vec{k}, 0) \right| = 1; \left| C_{\phi\phi}(\vec{k}, \vec{q}; 0) \right| = 0; \left| C_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}; 0) \right| = 0$, as time evolves the probabilities $\left| C_{\phi\phi}(\vec{k}, \vec{q}; t) \right|^2$ of the intermediate (resonant) state $|\phi_{1, \vec{q}} \phi_{2, \vec{k}-\vec{q}}\rangle$ will grow as the $|\pi\rangle$ state decays therefore producing ϕ_1 particles, however these particles eventually decay into final

state particles $|\phi_{2,\vec{k}-\vec{q}}\chi_{1,\vec{p}}\chi_{2,\vec{q}-\vec{p}}\rangle$ whose population $|C_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}; t)|^2$ will grow in time.

Therefore we expect the physical picture: the amplitude of the initial state decays, while the amplitude of the intermediate resonant state grows at early times eventually decays into $\chi_{1,2}$ particles, and the amplitude of the final state grows more slowly than the intermediate state at early times, but it reaches an asymptotic final value to fulfill the unitarity relation (3.2) with

$$|C_{\pi}(\vec{k}, \infty)| = 0; \quad |C_{\phi\phi}(\vec{k}, \vec{q}; \infty)| = 0; \quad |C_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}; \infty)| \neq 0 \quad \text{with} \\ \sum_{\vec{q}} \sum_{\vec{p}} |C_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}; \infty)|^2 = 1. \quad (3.6)$$

This physical picture describes probability or *population flow*, namely that the probabilities of the various states evolve in time in such a way that the $\langle \Psi(t) | \Psi(t) \rangle$ is constant but population and probability flows among multiparticle states and at asymptotically long times only the stable final particle states feature non-vanishing amplitudes.

The main goal of this section is to understand how unitarity is manifest in the time evolution of the probabilities. At first this notion seems puzzling: at $t = 0$ the initial state has unit probability and the intermediate and final states vanishing probability. The matrix elements connecting the initial, intermediate and final states are all perturbatively small in the couplings, yet at asymptotically late time when the initial state has decayed, the total probability of the final state, related to the initial state by perturbative matrix elements must be unity, highlighting the non-perturbative aspects of the dynamics.

We now study this process in detail to analyze the various time scales associated with this ‘population flow’. Let us introduce

$$\mathcal{E} = E_{\phi\phi} - E_k^\pi; \quad \Delta\Gamma = \Gamma_\pi - \Gamma_{\phi_1}, \quad (3.7)$$

in terms of which

$$|C_{\phi\phi}(\vec{k}, \vec{q}; t)|^2 = |M_{\pi \rightarrow \phi_1 \phi_2}|^2 e^{-\Gamma_{\phi_1} t} \frac{|e^{i\mathcal{E}t} e^{-\frac{\Delta\Gamma}{2}t} - 1|^2}{\left[\mathcal{E}^2 + \left(\frac{\Delta\Gamma}{2}\right)^2\right]}. \quad (3.8)$$

In the narrow width limit this expression becomes proportional to $\delta(\mathcal{E})/|\Delta\Gamma|$, in order to find the proportionality factor we integrate (3.8) in $-\infty \leq \mathcal{E} \leq \infty$ and obtain to leading order in the narrow width limit

$$|C_{\phi\phi}(\vec{k}, \vec{q}; t)|^2 = 2\pi |M_{\pi \rightarrow \phi_1 \phi_2}|^2 \frac{(e^{-\Gamma_{\phi_1} t} - e^{-\Gamma_\pi t})}{\Gamma_\pi - \Gamma_{\phi_1}} \delta(E_k^\pi - E_{\vec{q}}^{\phi_1} - E_{\vec{k}-\vec{q}}^{\phi_2}). \quad (3.9)$$

This result is obviously non-perturbative: while in the numerator is $|M_{\pi \rightarrow \phi_1 \phi_2}|^2 \propto g_\pi^2$ the denominator is *also* $\propto g_\pi^2, g_\phi^2$ exhibiting the non-perturbative nature of the result. At very early times $t \ll 1/\Gamma_\pi; 1/\Gamma_{\phi_1}$ the total number of ϕ_1 particles grows linearly with time with a rate

$$\Gamma(\pi \rightarrow \phi_1 \phi_2) = 2\pi \sum_{\vec{q}} |M_{\pi \rightarrow \phi_1 \phi_2}|^2 \delta(E_k^\pi - E_{\vec{q}}^{\phi_1} - E_{\vec{k}-\vec{q}}^{\phi_2}), \quad (3.10)$$

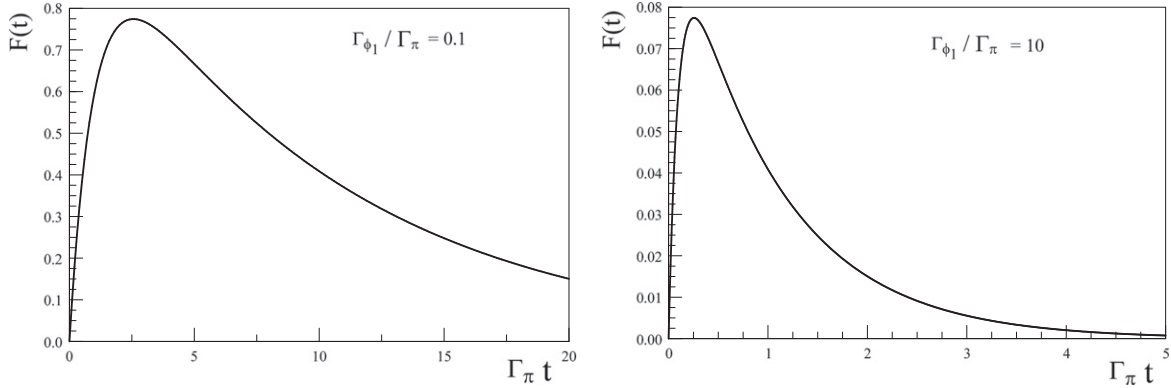


Figure 3. The function $F(t) = (e^{-\Gamma_{\phi_1} t} - e^{-\Gamma_{\pi} t}) / (1 - \Gamma_{\phi_1} / \Gamma_{\pi})$ for $\Gamma_{\phi_1} / \Gamma_{\pi} = 0.1, 10$.

however it reaches a maximum at $t = t^*$ given by

$$t^* = \frac{\ln \left[\frac{\Gamma_{\pi}}{\Gamma_{\phi_1}} \right]}{\Gamma_{\pi} - \Gamma_{\phi_1}}, \quad (3.11)$$

and falls off exponentially on a time scale determined by the smaller of $\Gamma_{\pi}, \Gamma_{\phi_1}$.

In particular if the π particle decays at rest the total number of ϕ_1 particles is given by

$$N_{\phi_1}(t) = \sum_{\vec{q}} \left| C_{\phi\phi}(\vec{k}, \vec{q}; t) \right|^2 = \frac{\Gamma(\pi \rightarrow \phi_1 \phi_2)}{\Gamma_{\pi}} \left[\frac{e^{-\Gamma_{\phi_1}^* t} - e^{-\Gamma_{\pi} t}}{1 - \frac{\Gamma_{\phi_1}^*}{\Gamma_{\pi}}} \right], \quad (3.12)$$

where

$$\Gamma_{\phi_1}^* = \Gamma_{\phi_1}(q^*) ; \quad q^* = \frac{1}{2m_{\pi}} \left[m_{\pi}^4 + m_{\phi_1}^4 + m_{\phi_2}^4 - 2m_{\pi}^2 m_{\phi_1}^2 - 2m_{\pi}^2 m_{\phi_2}^2 - 2m_{\phi_1}^2 m_{\phi_2}^2 \right]^{\frac{1}{2}}. \quad (3.13)$$

The time dependent function in the bracket in (3.12) that determines the population of the resonant particles is shown in figure 3 as a function of $\Gamma_{\pi} t$ for $\Gamma_{\phi_1} / \Gamma_{\pi} = 0.1, 10$, displaying the behavior discussed above: an early linear growth from the production of resonant states from the decay of the parent particle for $t \ll \Gamma_{\pi}, \Gamma_{\phi_1}$ growing to a maximum and falling off exponentially on the longer time scale determined by the smaller of the decay widths.

We now turn to $|C_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}; t)|^2$ focusing first on the asymptotic $t \rightarrow \infty$ limit, we find

$$\sum_{\vec{q}, \vec{p}} \left| C_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}; \infty) \right|^2 = \sum_{\vec{q}, \vec{p}} \frac{\left| M_{\pi \rightarrow \phi_1 \phi_2} \right|^2 \left| M_{\phi_1 \rightarrow \chi_1 \chi_2} \right|^2}{\left[(E_{\phi\chi\chi} - E_k)^2 + \left(\frac{\Gamma_{\pi}}{2} \right)^2 \right] \left[(E_{\phi\chi\chi} - E_{\phi\phi})^2 + \left(\frac{\Gamma_{\phi_1}}{2} \right)^2 \right]}. \quad (3.14)$$

Remarkably this expression is very similar to the asymptotic limit of the probability of a two photon state in a two-level radiative cascade in quantum optics [16, 32].

In the narrow width limit this expression is dominated by the resonances and as usual we approximate

$$\frac{1}{\left[(E_{\phi\chi\chi} - E_k^\pi)^2 + \left(\frac{\Gamma_\pi}{2}\right)^2 \right]} \rightarrow \frac{2\pi}{\Gamma_\pi} \delta(E_{\phi\chi\chi} - E_k^\pi), \quad (3.15)$$

therefore

$$\sum_{\vec{q}, \vec{p}} \left| C_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}; \infty) \right|^2 = \frac{2\pi}{\Gamma_\pi} \sum_{\vec{q}, \vec{p}} \frac{|M_{\pi \rightarrow \phi_1 \phi_2}|^2 |M_{\phi_1 \rightarrow \chi_1 \chi_2}|^2 \delta(E_{\phi\chi\chi} - E_k^\pi)}{\left[(E_k^\pi - E_{\phi\phi})^2 + \left(\frac{\Gamma_{\phi_1}}{2}\right)^2 \right]}, \quad (3.16)$$

where in the denominator we used the delta function to set $E_{\phi\chi\chi} = E_k^\pi$.

Upon comparing this result with the result for Γ_π given by (2.39) (see the definitions (2.10), (2.11)) we find

$$\sum_{\vec{q}, \vec{p}} \left| C_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}; \infty) \right|^2 = 1, \quad (3.17)$$

which is the manifestation of unitarity in the asymptotic limit.

The analysis for finite time is lengthy and relegated to (appendix A), here we present the main results.

In the narrow width limit we replace

$$\frac{1}{\left[(E_{\phi\phi} - E_k^\pi)^2 + \left(\frac{\Gamma_{\phi_1}}{2}\right)^2 \right]} \rightarrow \frac{2\pi}{\Gamma_{\phi_1}} \delta(E_{\phi\phi} - E_k^\pi), \quad (3.18)$$

and to leading order in the widths it follows that

$$\sum_{\vec{q}, \vec{p}} \left| C_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}; \infty) \right|^2 = (2\pi)^2 \sum_{\vec{q}, \vec{p}} \frac{|M_{\pi \rightarrow \phi_1 \phi_2}|^2 |M_{\phi_1 \rightarrow \chi_1 \chi_2}|^2}{\Gamma_\pi \Gamma_{\phi_1}} \delta(E_{\phi\chi\chi} - E_k^\pi) \delta(E_{\phi\phi} - E_k^\pi), \quad (3.19)$$

combining this expression with (A.3) and (A.11) we finally find

$$\begin{aligned} \sum_{\vec{q}, \vec{p}} \left| C_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}; t) \right|^2 &= (2\pi)^2 \sum_{\vec{q}, \vec{p}} \frac{|M_{\pi \rightarrow \phi_1 \phi_2}|^2 |M_{\phi_1 \rightarrow \chi_1 \chi_2}|^2}{\Gamma_\pi \Gamma_{\phi_1} (\Gamma_\pi - \Gamma_{\phi_1})} \\ &\quad \times \left\{ \Gamma_\pi (1 - e^{-\Gamma_{\phi_1} t}) - \Gamma_{\phi_1} (1 - e^{-\Gamma_\pi t}) \right\} \\ &\quad \times \delta(E_{\phi\chi\chi} - E_k^\pi) \delta(E_{\phi\phi} - E_k^\pi). \end{aligned} \quad (3.20)$$

This expression may be simplified by realizing that $\Gamma_{\phi_1}(\vec{q})$ depends on \vec{q} but not on \vec{p} and because of the second delta function in (3.20) we can set $\delta(E_{\phi\chi\chi} - E_k^\pi) \rightarrow \delta(E_{\phi\chi\chi} - E_{\phi\phi}) = \delta(E_{\vec{q}}^{\phi_1} - E_{\vec{p}}^{\chi_1} - E_{\vec{q}-\vec{p}}^{\chi_2})$ in the first delta function, using the result (2.38) we finally find

$$\sum_{\vec{q}, \vec{p}} \left| C_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}; t) \right|^2 = \frac{2\pi}{\Gamma_\pi} \sum_{\vec{q}} \left| M_{\pi \rightarrow \phi_1 \phi_2} \right|^2 \times \left[\frac{\Gamma_\pi (1 - e^{-\Gamma_{\phi_1} t}) - \Gamma_{\phi_1} (1 - e^{-\Gamma_\pi t})}{(\Gamma_\pi - \Gamma_{\phi_1})} \right] \delta(E_k^\pi - E_{\vec{k}-\vec{q}}^{\phi_2} - E_{\vec{q}}^{\phi_1}). \quad (3.21)$$

Combining this result with (3.9) and using (2.41) we find

$$\sum_{\vec{q}} \left| C_{\phi\phi}(\vec{k}, \vec{q}; t) \right|^2 + \sum_{\vec{q}, \vec{p}} \left| C_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}; t) \right|^2 = 1 - e^{-\Gamma_\pi t} = 1 - |C_\pi(t)|^2, \quad (3.22)$$

this is the final result confirming unitary time evolution and the flow of population from the initial through the intermediate to the final state.

Therefore an important conclusion that follows from this analysis is that the amplitudes (2.48), (2.50) are the correct ones insofar as they manifestly satisfy unitarity.

In particular when the π particle decays at rest, from the result (3.21) and the identification (3.5) the total number of pairs of χ_1, χ_2 particles is given by

$$N_{\chi\chi}(t) = \sum_{\vec{q}, \vec{p}} \left| C_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}; t) \right|^2 = \left[\frac{\Gamma_\pi (1 - e^{-\Gamma_{\phi_1}^* t}) - \Gamma_{\phi_1}^* (1 - e^{-\Gamma_\pi t})}{(\Gamma_\pi - \Gamma_{\phi_1}^*)} \right], \quad (3.23)$$

where $\Gamma_{\phi_1}^*$ is given by (3.13). In summary:

$$N_\pi(t) = e^{-\Gamma_\pi t}, \quad (3.24)$$

$$N_{\phi_1}(t) = \frac{e^{-\Gamma_{\phi_1}^* t} - e^{-\Gamma_\pi t}}{1 - \frac{\Gamma_{\phi_1}^*}{\Gamma_\pi}}, \quad (3.25)$$

$$N_{\chi\chi}(t) = \frac{\Gamma_\pi (1 - e^{-\Gamma_{\phi_1}^* t}) - \Gamma_{\phi_1}^* (1 - e^{-\Gamma_\pi t})}{(\Gamma_\pi - \Gamma_{\phi_1}^*)}, \quad (3.26)$$

with

$$N_\pi(t) + N_{\phi_1}(t) + N_{\chi\chi}(t) = 1. \quad (3.27)$$

The populations are shown in figure 4 as a function of time for $\Gamma_{\phi_1}/\Gamma_\pi = 0.1, 10$.

Discussion: the expressions (3.24)–(3.26) and the (3), (4) display the behavior of the populations of resonant and final states and describe the main time dependent physical phenomena of the cascade decay. As discussed above at early time $t \ll 1/\Gamma_\pi, 1/\Gamma_{\phi_1}$ the number of resonances grows linearly in t as $N_{\phi_1}(t) \simeq \Gamma_\pi t$, as the decay the the parent particle increases the population of resonances, reaches a maximum at t^* given by (3.11) and decays on the longer time scale. However the population of the final state $\chi\chi\phi_2$ at early time grows as $N_{\chi\chi}(t) \simeq \Gamma_\pi \Gamma_{\phi_1} t^2/2$, namely much slower than the population of the intermediate resonant state. This is a result of a second order process as the build-up of the final state requires first to build up the population of the resonant state from the decay of the parent state as $\simeq \Gamma_\pi t$ and the build-up of the final state from the decay of the populated intermediate resonant state as $\Gamma_{\phi_1} t$.

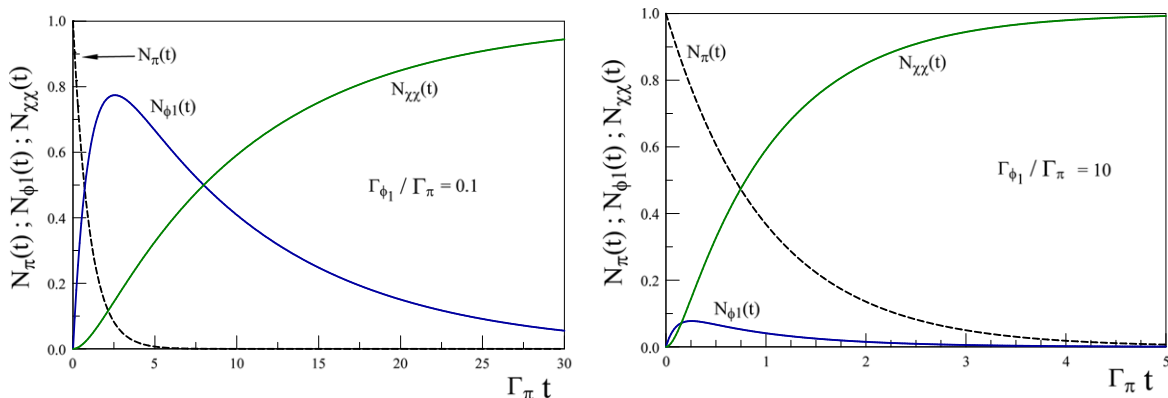


Figure 4. The number of π -particles $N_\pi(t)$, resonant states $N_{\phi_1}(t)$ and χ -pairs $N_{\chi\chi}(t)$ versus $\Gamma_\pi t$ for $\Gamma_{\phi_1}/\Gamma_\pi = 0.1, 10$.

The behavior of the populations at intermediate and long times depends on the particular cases $\Gamma_\pi \gg \Gamma_{\phi_1}$ and $\Gamma_\pi \ll \Gamma_{\phi_1}$.

- $\Gamma_\pi \gg \Gamma_{\phi_1}$: in this case the slowly decaying intermediate resonant state acts as a ‘bottleneck’, the rapid decay of the parent particle builds up the population of the resonant states which grows to a maximum at t^* with an amplitude $N_{\phi_1}(t^*)$. For $\Gamma_\pi \gg \Gamma_{\phi_1}$ it follows that $e^{-\Gamma_\pi t^*} \ll 1$; $e^{-\Gamma_{\phi_1} t^*} \sim 1$ and $N_{\phi_1}(t^*) \sim 1$. The decay of the resonant state into the final state particles occurs on a *much slower* time scale of the order of $1/\Gamma_{\phi_1}$. In this limit the cascade decay can be described *sequentially* as $\pi \rightarrow \phi_2\phi_1$; $\phi_2\phi_1 \rightarrow \chi_1\chi_2\phi_2$ where the intermediate resonant state attains an amplitude $\simeq 1$, nearly saturating unitarity on a short time scale $t \simeq t^* = \ln[\Gamma_\pi/\Gamma_{\phi_1}]/(\Gamma_\pi - \Gamma_{\phi_1})$ and decays slowly on the time scale $1/\Gamma_{\phi_1}$. For $t > 1/\Gamma_\pi$ it follows that the population of the final state $N_{\chi\chi}(t) \simeq (1 - e^{-\Gamma_{\phi_1}^* t})$ consistently with the decay of an initial resonant state of nearly unit amplitude.
- $\Gamma_\pi \ll \Gamma_{\phi_1}$: In this case the intermediate resonant state decays on time scales shorter than that of the parent particle, as a result there is very little population build-up of the resonant state and the initial population of the parent particle is ‘transferred’ directly to the final state on a time scale $\simeq 1/\Gamma_\pi$. In this limit $\Gamma_{\phi_1} t^* \gg 1$ and $N_{\phi_1}(t^*) \simeq e^{-\Gamma_\pi t^*} \Gamma_\pi/\Gamma_{\phi_1} \ll 1$, and for $t \gtrsim t^*$ it follows that $N_{\chi\chi}(t) \simeq (1 - e^{-\Gamma_\pi t})$ describing the build up of the population of the final states directly from the decay of the parent particle. In this limit the decay of the parent particle can be described as a direct decay into the final states as the intermediate resonant state is so short-lived that the population of resonances does not build up substantially.
- $\Gamma_\pi = \Gamma_{\phi_1}$: in this (unlikely) case it is straightforward to find

$$N_{\phi_1}(t) = \Gamma_\pi t e^{-\Gamma_\pi t}; N_{\chi\chi}(t) = 1 - e^{-\Gamma_\pi t} [1 + \Gamma_\pi t]; t^* = 1/\Gamma_\pi. \quad (3.28)$$

The ‘bottleneck’ in the case when $\Gamma_\pi \gg \Gamma_{\phi_1}$ is reminiscent of a similar phenomenon in the production of Helium during Big Bang Nucleosynthesis (BBN) via the formation of a deuteron bound state in $n - p$ collisions. The deuteron is photo-dissociated by the high energy photons in the blackbody tail and does not form until the ambient temperature falls to about 80 keV resulting in a delayed transition from the initial $n - p$ to the final state. However, although there is a similarity in that the decay to the final state is delayed, the physical reasons are different: in the case under consideration in this article, the delayed decay to the final state is a consequence of the on-shell formation of a long-lived resonant state, whereas in BBN the delay is a consequence of photodissociation of the deuteron intermediate state which does not form until the temperature falls well below the binding energy of the bound state ≈ 2 MeV. Once the deuteron is formed the fusion reactions end in ${}^4\text{He}$ very fast.

Generality of the results: although we focused on the specific example given by the interaction Hamiltonian (2.1) the procedure leading to the equations for the amplitudes is general. In particular we have left the matrix elements (2.8), (2.9) indicated without using their explicit form and the final results for the amplitudes depend solely on the transition matrix elements, decay widths of the parent and resonant states and single particle energies. Therefore the extension of the above results to any other theory in which a cascade decay proceeds via an intermediate resonant state is straightforward and can be described by the above results by replacing the proper matrix elements and decay widths.

4. Field theoretic generalization of Wigner–Weisskopf

The Wigner–Weisskopf theory of spontaneous emission [28] plays an important role in quantum optics [16–18] and in particle physics it is one of the main approaches to study the dynamics of the $K^0 - \bar{K}^0$ system [29, 30] (also $B^0 - \bar{B}^0$). A quantum field theoretical generalization of this important method was provided in [31] with an extension in cosmology in [24, 27]. However, to the best of our knowledge the method has not been extended to the case of radiative cascades. This is the goal of this section.

Again, writing the Hamiltonian as $H = H_0 + H_I$, and passing to the interaction picture instead of using the Schrodinger picture, as in (2.3), the time evolution of the quantum state is given by

$$i \frac{d}{dt} | \Psi(t) \rangle_I = H_I(t) | \Psi(t) \rangle_I, \quad (4.1)$$

where $H_I(t)$ is the interaction Hamiltonian in the interaction picture.

The state $| \Psi(t) \rangle_I$ is then expanded in the basis of free particle Fock states $| m \rangle$ eigenstates of H_0 , namely

$$| \Psi(t) \rangle_I = \sum_m A_m(t) | m \rangle. \quad (4.2)$$

Using the orthogonality relation on any state expanded in this fashion leads to the following relation

$$\dot{A}_n(t) = -i \sum_m \langle n | H_I(t) | m \rangle A_m(t). \quad (4.3)$$

This allows for a *time dependent* interaction Hamiltonian, a situation common to field theories in curved spacetime [24, 27]. Specifically, the method described below, when generalized to the case of an expanding cosmology is amenable to be implemented to study the cascade decay of inflationary quanta discussed in [27]. We intend to apply the method developed here to the case of the cosmological cascade decay in future studies.

For the purposes of this work, ensuring that the Wigner–Weisskopf procedure reproduces the results obtained via Laplace transform will confirm that this real time method provides a non-perturbative resummation that yields the correct time evolution at least in the cases where it can be compared to known results.

In the interaction picture the quantum state that describes the cascade decay is given by

$$\begin{aligned} |\Psi(t)\rangle_I = & A_\pi(\vec{k}, t) |\pi_{\vec{k}}\rangle + \sum_{\vec{q}} A_{\phi\phi}(\vec{k}, \vec{q}; t) |\phi_{1,\vec{q}} \phi_{2,\vec{k}-\vec{q}}\rangle \\ & + \sum_{\vec{q}, \vec{p}} A_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}; t) |\phi_{2,\vec{k}-\vec{q}} \chi_{1,\vec{p}} \chi_{2,\vec{q}-\vec{p}}\rangle + \dots \end{aligned} \quad (4.4)$$

When the interaction Hamiltonian in the Schrodinger picture is time independent as is the case in Minkowski space time in absence of external sources, the coefficients in this expression differ from those in section 2 by the relation $C = Ae^{-iEt}$, where E is the energy (eigenvalue of H_0) of the particular state in the expansion, this is the difference between the two pictures.

Restricting attention to Minkowski spacetime in absence of explicit time dependent sources, the orthogonality relations lead to matrix elements of the interaction Hamiltonian given by

$$M_{mn}(t) = \langle m | H_I(t) | n \rangle = e^{i(E_m - E_n)t} \langle m | H_I | n \rangle; H_I(t) = e^{iH_0 t} H_I e^{-iH_0 t}. \quad (4.5)$$

Again, consider the interaction Hamiltonian given by (2.1) and the initial conditions given by (2.2). To simplify notation, the definitions introduced in equations (2.8)–(2.11) will be used leading to the following equations for the coefficients

$$\dot{A}_\pi(\vec{k}, t) = -i \sum_{\vec{q}} M_{\pi \rightarrow \phi_1 \phi_2} A_{\phi\phi}(\vec{k}, \vec{q}, t) e^{i(E_{\vec{k}} - E_{\phi\phi})t}; A_\pi(\vec{k}, 0) = 1, \quad (4.6)$$

$$\begin{aligned} \dot{A}_{\phi\phi}(\vec{k}, \vec{q}, t) = & -i M_{\pi \rightarrow \phi_1 \phi_2}^* A_\pi(\vec{k}, t) e^{i(E_{\phi\phi} - E_{\vec{k}})t} \\ & -i \sum_{\vec{p}} M_{\phi_1 \rightarrow \chi_1 \chi_2} A_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}, t) e^{i(E_{\phi\phi} - E_{\phi\chi\chi})t}; A_{\phi\phi}(\vec{k}, \vec{p}, 0) = 0, \end{aligned} \quad (4.7)$$

$$\dot{A}_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}, t) = -i M_{\phi_1 \rightarrow \chi_1 \chi_2}^* e^{-i(E_{\phi\phi} - E_{\phi\chi\chi})t} A_{\phi\phi}(\vec{k}, \vec{q}, t); A_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}, 0) = 0. \quad (4.8)$$

Again we solve the hierarchy from the bottom up. The solution of (4.8) is

$$A_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}, t) = -i M_{\phi_1 \rightarrow \chi_1 \chi_2}^* \int_0^t e^{-i(E_{\phi\phi} - E_{\phi\chi\chi})t'} A_{\phi\phi}(\vec{k}, \vec{q}, t') dt' \quad (4.9)$$

and combining (4.7) and (4.9) yields (now suppressing the momenta in the arguments)

$$\dot{A}_{\phi\phi}(t) + \sum_{\vec{p}} \left| M_{\phi_1 \rightarrow \chi_1 \chi_2} \right|^2 \int_0^t dt' e^{-i(E_{\phi\phi} - E_{\phi\chi\chi})(t-t')} A_{\phi\phi}(t') = -i M_{\pi \rightarrow \phi_1 \phi_2}^* e^{-i(E_\pi - E_{\phi\phi})t} A_\pi(t). \quad (4.10)$$

A perturbative solution to this integro-differential equation in terms of A_π is straightforward, however, it leads to resonant denominators and its eventual breakdown. Instead we implement a non-perturbative approach that provides a resummation that incorporates consistently the width of the resonant state.

In order to implement this method to solve (4.10), let us first focus on the homogeneous case neglecting the right hand side. Consider

$$\dot{A}_{\phi\phi}^H(t) + \sum_{\bar{p}} \left| M_{\phi_1 \rightarrow \chi_1 \chi_2} \right|^2 \int_0^t dt' e^{-i(E_{\phi\phi} - E_{\phi\chi\chi})(t'-t)} A_{\phi\phi}^H(t') = 0; A_{\phi\phi}^H(0) = 1. \quad (4.11)$$

This equation simplifies by implementing a Markovian approximation which is justified in weak coupling. Since the term inside the integrand $\left| M_{\phi_1 \rightarrow \chi_1 \chi_2} \right|^2 \sim g_\phi^2 \ll 1$ it follows that $\dot{A}_{\phi\phi}^H \propto g_\phi^2 \ll 1$, namely the amplitudes vary *slowly* in time. To clearly see the nature of the approximation, the following is introduced

$$W_0^\phi(t, t') = \sum_{\bar{p}} \left| M_{\phi_1 \rightarrow \chi_1 \chi_2} \right|^2 \int_0^{t'} dt'' e^{-i(E_{\phi\phi} - E_{\phi\chi\chi})(t''-t)}, \quad (4.12)$$

which has the properties

$$\frac{d}{dt'} W_0^\phi(t, t') = \sum_{\bar{p}} \left| M_{\phi_1 \rightarrow \chi_1 \chi_2} \right|^2 e^{-i(E_{\phi\phi} - E_{\phi\chi\chi})(t'-t)} \sim \mathcal{O}(g_\phi^2); W_0^\phi(t, 0) = 0, \quad (4.13)$$

An integration by parts produces

$$\int_0^t dt' \frac{d}{dt'} W_0^\phi(t, t') A_{\phi\phi}(t') = W_0^\phi(t, t) A_{\phi\phi}(t) - \int_0^t dt' \dot{A}_{\phi\phi}(t') W_0^\phi(t, t'). \quad (4.14)$$

This can be repeated systematically, producing higher order derivatives by using the natural definition

$$W_N^\phi(t, t') = \int_0^{t'} dt'' W_{N-1}^\phi(t, t''); W_N^\phi(t, 0) = 0. \quad (4.15)$$

Repeated integration by parts produces the series

$$\int_0^t dt' \frac{d}{dt'} W_0^\phi(t, t') A_{\phi\phi}(t') = W_0^\phi(t, t) A_{\phi\phi}(t) - W_1^\phi(t, t) \dot{A}_{\phi\phi}(t) + W_2^\phi(t, t) \ddot{A}_{\phi\phi}(t) + \dots \quad (4.16)$$

where each term has a multiplicative factor $W_N^\phi \sim g_\phi^2$ and $\dot{A} \propto g_\phi^2$; $\ddot{A} \propto g_\phi^4$ etc. Truncating this series to leading order, namely keeping only W_0^ϕ gives

$$\dot{A}_{\phi\phi}^H + W_0^\phi(t, t) A_{\phi\phi}^H(t) = 0 ; A_{\phi\phi}^H(0) = 1. \quad (4.17)$$

This makes it apparent that the lowest order solution results from keeping only W_0 , this is the Markovian approximation. So to lowest order, the homogenous solution is written

$$A_{\phi\phi}^H(t) = e^{-\int_0^t dt' W_0^\phi(t', t')}, \quad (4.18)$$

where

$$\int_0^t dt' W_0^\phi(t', t) = it \sum_{\bar{p}} \frac{|M_{\phi_1 \rightarrow \chi_1 \chi_2}|^2}{(E_{\phi\phi} - E_{\phi\chi\chi})} \left\{ \left(1 - \frac{\sin(E_{\phi\phi} - E_{\phi\chi\chi})t}{(E_{\phi\phi} - E_{\phi\chi\chi})t} \right) - i \left(\frac{1 - \cos(E_{\phi\phi} - E_{\phi\chi\chi})t}{(E_{\phi\phi} - E_{\phi\chi\chi})t} \right) \right\}. \quad (4.19)$$

We prove in (appendix B) (see also [24]) that in the long time limit for time scales that are much larger than the energy uncertainty ($t \gg 1/(E_{\phi\phi} - E_{\phi\chi\chi})$),

$$\int_0^t dt' W_0^\phi(t', t) \rightarrow t \left(i\mathcal{P} \sum_{\bar{p}} \frac{|M_{\phi_1 \rightarrow \chi_1 \chi_2}|^2}{E_{\phi\phi} - E_{\phi\chi\chi}} + \pi \sum_{\bar{q}} |M_{\phi_1 \rightarrow \chi_1 \chi_2}(p, q)|^2 \delta(E_{\phi\phi} - E_{\phi\chi\chi}) \right). \quad (4.20)$$

However, the same result can be obtained from replacing $E_{\phi\phi} - E_{\phi\chi\chi} \rightarrow E_{\phi\phi} - E_{\phi\chi\chi} + i\varepsilon$ with $\varepsilon \rightarrow 0^+$ and $t = t' \rightarrow \infty$ in (4.12) which yields

$$\lim_{t \rightarrow \infty} \int_0^t dt' W_0^\phi(t', t) = it \sum_{\bar{p}} \frac{|M_{\phi_1 \rightarrow \chi_1 \chi_2}|^2}{(E_{\phi\phi} - E_{\phi\chi\chi} + i\varepsilon)} = \left(i\Delta E^{\phi_1} + \frac{\Gamma_{\phi_1}}{2} \right) t, \quad (4.21)$$

where ΔE^{ϕ_1} and Γ_{ϕ_1} are given by (2.37), (2.38) respectively, and the homogenous solution becomes

$$A_{\phi\phi}^H(t) = e^{-i\Delta E^{\phi_1}t} e^{-\frac{\Gamma_{\phi_1}}{2}t}. \quad (4.22)$$

Now with the left hand side of eqn. (4.10) replaced by the Markovian approximation (4.17) and using the above result, the full solution of (4.10) is given by

$$A_{\phi\phi}(t) = -iM_{\pi \rightarrow \phi_1 \phi_2}^* e^{-i\left(\Delta E_{\phi_1} - i\frac{\Gamma_{\phi_1}}{2}\right)t} \int_0^t dt' A_\pi(t') e^{-i\left(E^\pi - E_{\phi\phi} - \Delta E_{\phi_1} + i\frac{\Gamma_{\phi_1}}{2}\right)t'}. \quad (4.23)$$

Inserting this solution into equation (4.6) we obtain

$$\dot{A}_\pi(t) + \sum_{\bar{q}} |M_{\pi \rightarrow \phi_1 \phi_2}|^2 \int_0^t dt' e^{i\left(E^\pi - E_{\phi\phi} - \Delta E_{\phi_1} + i\frac{\Gamma_{\phi_1}}{2}\right)(t-t')} A_\pi(t') = 0; A_\pi(0) = 1. \quad (4.24)$$

At this stage, we implement again a Markovian approximation as described above, which is justified in this case by the weak coupling $g_\pi \ll 1$ so that the integrand can be treated as slowly varying and carrying out the same expansion as above and keeping the lowest order we obtain

$$\dot{A}_\pi(t) + W_0^\pi(t, t)A_\pi(t) = 0; A_\pi(0) = 1, \quad (4.25)$$

where

$$W_0^\pi(t, t') = \sum_{\bar{q}} \left| M_{\pi \rightarrow \phi_1 \phi_2} \right|^2 \int_0^{t'} dt'' e^{-i \left(E^\pi - E_{\phi\phi} - \Delta E_{\phi_1} + i \frac{\Gamma_{\phi_1}}{2} \right) (t'' - t)}, \quad (4.26)$$

and in the same manner as before the solution is now given by

$$A_\pi(t) = e^{-\int_0^t dt' W_0^\pi(t', t)}, \quad (4.27)$$

where again taking the long time limit

$$\int_0^t dt' W_0^\pi(t', t) = i t \sum_{\bar{q}} \frac{\left| M_{\pi \rightarrow \phi_1 \phi_2} \right|^2}{E^\pi - E_{\phi\phi} - \Delta E_{\phi_1} + i \frac{\Gamma_{\phi_1}}{2} + i\varepsilon}, \quad (4.28)$$

where for $\Gamma_{\phi_1} \neq 0$ one can neglect ε . The solution now becomes

$$A_\pi(t) = e^{-i\Delta E^\pi t} e^{-\frac{\Gamma_\pi}{2} t}, \quad (4.29)$$

where

$$\Delta E^\pi = \sum_{\bar{q}} \frac{\left| M_{\pi \rightarrow \phi_1 \phi_2} \right|^2 \left(E^\pi - E_{\phi\phi} - \Delta E_{\phi_1} \right)}{\left(E^\pi - E_{\phi\phi} - \Delta E_{\phi_1} \right)^2 + \left(\frac{\Gamma_{\phi_1}}{2} \right)^2}, \quad (4.30)$$

$$\Gamma_\pi(k) = \sum_{\bar{q}} \frac{\left| M_{\pi \rightarrow \phi_1 \phi_2} \right|^2 \Gamma_{\phi_1}}{\left(E^\pi - E_{\phi\phi} - \Delta E_{\phi_1} \right)^2 + \left(\frac{\Gamma_{\phi_1}}{2} \right)^2}. \quad (4.31)$$

This is now used to solve equation (4.7) resulting in the following expression

$$\begin{aligned} A_{\phi\phi}(t) &= -i M_{\pi \rightarrow \phi_1 \phi_2}^* e^{-i \left(\Delta E_{\phi_1} - i \frac{\Gamma_{\phi_1}}{2} \right) t} \int_0^t dt' e^{-i \left(E^\pi - E_{\phi\phi} + \Delta E^\pi - \Delta E_{\phi_1} - \frac{i}{2} (\Gamma_\pi - \Gamma_{\phi_1}) \right) t'} \\ &= \frac{M_{\pi \rightarrow \phi_1 \phi_2}^* e^{-i \left(\Delta E_{\phi_1} - i \frac{\Gamma_{\phi_1}}{2} \right) t}}{E^\pi - E_{\phi\phi} + \Delta E^\pi - \Delta E_{\phi_1} - \frac{i}{2} (\Gamma_\pi - \Gamma_{\phi_1})} \\ &\quad \times \left[e^{-i \left(E^\pi - E_{\phi\phi} + \Delta E^\pi - \Delta E_{\phi_1} - \frac{i}{2} (\Gamma_\pi - \Gamma_{\phi_1}) \right) t} - 1 \right]. \end{aligned} \quad (4.32)$$

Absorbing the energy shifts into renormalizations of the mass as before, namely

$$E^\pi = E_0^\pi + \Delta E^\pi; E_{\phi\phi} = E_{\phi_1}^0 + \Delta E_{\phi_1} + E_{\phi_2}, \quad (4.33)$$

and inserting the (renormalized) solution above into (4.9) we finally find

$$\begin{aligned}
A_{\phi\chi\chi}(t) &= \frac{-iM_{\pi\rightarrow\phi_1\phi_2}^* M_{\phi_1\rightarrow\chi_1\chi_2}^*}{E^\pi - E_{\phi\phi} - \frac{i}{2}(\Gamma_\pi - \Gamma_{\phi_1})} \int_0^t dt' \\
&\times \left[e^{-i(E^\pi - E_{\phi\chi\chi} - i\frac{\Gamma_\pi}{2})t'} - e^{-i(E_{\phi\phi} - E_{\phi\chi\chi} - i\frac{\Gamma_{\phi_1}}{2})t'} \right] \\
&= \frac{M_{\pi\rightarrow\phi_1\phi_2}^* M_{\phi_1\rightarrow\chi_1\chi_2}^*}{E^\pi - E_{\phi\phi} - \frac{i}{2}(\Gamma_\pi - \Gamma_{\phi_1})} \\
&\times \left[\frac{e^{-i(E^\pi - E_{\phi\chi\chi} - i\frac{\Gamma_\pi}{2})t} - 1}{E^\pi - E_{\phi\chi\chi} - i\frac{\Gamma_\pi}{2}} - \frac{e^{-i(E_{\phi\phi} - E_{\phi\chi\chi} - i\frac{\Gamma_{\phi_1}}{2})t} - 1}{E_{\phi\phi} - E_{\phi\chi\chi} - i\frac{\Gamma_{\phi_1}}{2}} \right]. \tag{4.34}
\end{aligned}$$

Passing back to the Schrödinger picture and after renormalization of the single particle energies it follows that

$$A_\pi(t) \rightarrow e^{-iE^\pi t} A_\pi(t) = C_\pi(t), \tag{4.35}$$

$$A_{\phi\phi}(t) \rightarrow e^{-iE_{\phi\phi} t} A_{\phi\phi}(t) = C_{\phi\phi}(t), \tag{4.36}$$

$$A_{\phi\chi\chi}(t) \rightarrow e^{-iE_{\phi\chi\chi} t} A_{\phi\chi\chi}(t) = C_{\phi\chi\chi}(t), \tag{4.37}$$

finally matching the results (2.44), (2.48), (2.51) that were obtained via Laplace transform.

Discussion: the analysis above shows that the field theoretical generalization of the Wigner–Weisskopf method provides a real time realization of the non-perturbative resummation akin to the Dyson resummation of self-energies in the propagators and yields a resummation of secular terms that grow in time. The Markovian approximation based on a derivative expansion that relies on a separation of time scales valid in the weak coupling regime in the long time limit is akin to the Breit–Wigner or narrow width approximation in that it captures reliably the decay of resonances. Furthermore, the equivalence with the results of the previous section, clearly proves that the Wigner Weisskopf method is manifestly unitary.

While both methods, are equivalent in *Minkowski space time*, the quantum field theoretical Wigner–Weisskopf method features the distinct advantage of direct applicability in the cosmological context wherein the expansion implies an explicitly time dependent interaction Hamiltonian *in the Schrödinger picture* as a consequence of the cosmological expansion. To lowest order this method has been applied in cosmology in [24, 31], however the purpose of this work is to extend it to the hitherto unexplored case of *radiative cascade decay*. The results of this section indicate the reliability of the method thereby bolstering the case for its implementation in cosmology, which will be the subject of future study.

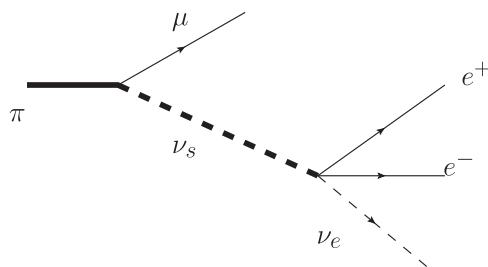


Figure 5. A possible process: $\pi \rightarrow \mu \nu_s \rightarrow e^+ e^- \nu_e$.

5. Possible phenomenological consequences and correlations

5.1. Possible phenomenological consequences

A possible phenomenological consequence may emerge if heavy sterile neutrinos ν_s exist and mix with active neutrinos. For example, consider the case of $\nu_\mu - \nu_s$; $\nu_e - \nu_s$ mixing in π -decay, if ν_s features a mass¹ $2m_e < m_s < m_\pi - m_\mu$ (here we neglect the mass of the ‘active’ neutrino mass eigenstates) then the intermediate state with a ν_s becomes resonant and can decay either via charged or neutral current interactions into $e^+ e^- \nu_e$. This process is depicted in the Fermi limit in figure 5 and would correspond to an *appearance* contribution.

If the lifetime of the heavy sterile neutrino is very long for example of the order of the baseline in long baseline neutrino experiments, the process above will yield a contribution to the appearance probability. If on the other hand the lifetime of the putative sterile neutrino is short, then it can decay on distances shorter than the oscillation length of active neutrinos and this process would contribute to the appearance probability with oscillations and the concomitant distortion in the energy spectrum, but on length scales shorter than the oscillation length. Clearly such process will be suppressed by a product $U_{es} U_{\mu s}$. Furthermore unlike oscillations of active neutrinos, the time dependence in this case would be damped exponentially on a scale of the order of the decay length of the sterile neutrino. For example if the heavy sterile neutrino decays via a charged current vertex, and neglecting the electron mass

$$\Gamma_{\nu_s} \simeq \frac{G_F^2 m_{\nu_s}^5}{192\pi^3} |U_{es}|^2, \quad (5.1)$$

with a decay length

$$c\tau_{\nu_s} \simeq \frac{860}{|U_{es}|^2} \left(\frac{100\text{MeV}}{m_{\nu_s}} \right)^5 \text{mts}. \quad (5.2)$$

For $|U_{es}| \ll 1$ the maximum in the probability of the intermediate state with the sterile neutrino resonance occurs at a distance

$$ct^* \sim -10 \ln |U_{es}| \text{mts} \quad (5.3)$$

this is approximately the distance away from the decay region of the parent pion at which the final state leptons are produced, namely the ‘decay vertex’. A detection of the charge lepton a

¹ We here simply refer to ν_s as the sterile-like *mass eigenstate*.

distance L away from the production region *may* mis-identify these charged leptons as being produced by neutrinos resulting from the decay of the parent meson or from oscillations between active neutrinos, the displaced decay vertex of the heavy sterile neutrino would then imply a *shortening* the effective baseline by a large factor if $|U_{es}| \ll 1$.

Similar ‘rare pion decay’ processes have been discussed within the context of lepton flavor violation in [13] but without addressing the full time evolution.

5.2. Entanglement and correlations

The asymptotic state after the parent particle and the intermediate resonant state have decayed is given by

$$|\Psi(\infty)\rangle = \sum_{\vec{q}, \vec{p}} C_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}; \infty) \left| \phi_{2, \vec{k}-\vec{q}} \chi_{1, \vec{p}} \chi_{2, \vec{q}-\vec{p}} \right\rangle, \quad (5.4)$$

where

$$C_{\phi\chi\chi}(\vec{k}, \vec{q}, \vec{p}; \infty) = \frac{M_{\pi \rightarrow \phi_1 \phi_2}^* M_{\phi_1 \rightarrow \chi_1 \chi_2}^* e^{-iE_{\phi\chi\chi} t}}{\left[(E_{\phi\chi\chi} - E^\pi) + i\frac{\Gamma_\pi}{2} \right] \left[(E_{\phi\chi\chi} - E_{\phi\phi}) + i\frac{\Gamma_{\phi_1}}{2} \right]} \quad (5.5)$$

this is entangled state of three particles, which apart from $\left| \phi_{2, \vec{k}-\vec{q}} \right\rangle$ is very similar to the asymptotic entangled two-photon state from a radiative cascade of two-level atomic systems in quantum optics [16, 32].

Extrapolating the results to the case of phenomenological relevance discussed above, the asymptotic state $|\Psi(\infty)\rangle$ is an entangled state of four particles. Quantum entanglement entails correlations, these are completely determined by the amplitudes $C(\infty)$, which depend on the mass and width of the resonant state and the matrix elements. These correlations will be manifest as intensity Hanbury-Brown–Twiss correlations among the charged lepton pairs produced by the decay of the resonant state that *could* reveal important information on the properties of heavy sterile neutrinos in the intermediate state.

We postpone a deeper study of the time dependence of these phenomenological consequences including mixing and oscillations to a future article.

6. Conclusions and further questions

Cascade decays via resonant intermediate states are of interdisciplinary interest, being ubiquitous in particle physics and quantum optics, and more recently are discussed within the context of inflationary cosmology. In this article we generalize methods of quantum optics to the realm of quantum field theory to study the real time dynamics of cascade decay in a model quantum field theory of generic fields, however the conclusions are general.

The method is based on a hierarchical solution of the coupled equations for the amplitudes of the initial, intermediate and final multiparticle states. We show that a solution of the equations up to a given order in the interaction yields a non-perturbative resummation *a la Dyson* in terms of the self-energies of the initial and intermediate resonant states. We analyze the time evolution of the amplitudes and probabilities and show that unitary time evolution is manifest as a probability ‘flow’ from the initial through the intermediate resonant and to the

final state. When the decay width of the initial parent particle Γ_π is much larger than that of the intermediate resonant state Γ_{ϕ_1} , there is a ‘bottleneck’ in the evolution, the probability of the intermediate resonant state grows to a maximum nearly saturating unitarity on a time scale given

$$t^* = \frac{\ln \left[\frac{\Gamma_\pi}{\Gamma_{\phi_1}} \right]}{\Gamma_\pi - \Gamma_{\phi_1}}, \quad (6.1)$$

and decays on a longer time scale $\simeq 1/\Gamma_{\phi_1}$, whereas in the opposite limit the population of the resonant state does not build up substantially and the probability flows almost directly from the initial to the final state on a time scale $1/\Gamma_\pi$.

We provided an alternative formulation in terms of a quantum field theoretical generalization of the Wigner–Weisskopf method in quantum optics. This method provides a non-perturbative resummation of secular terms in time and reliably describes the time evolution of intermediate resonant states. While both methods are equivalent, the quantum field theoretical generalization of Wigner–Weisskopf is directly applicable in the cosmological setting where the interaction Hamiltonian is explicitly dependent on time.

We *conjecture* on potential phenomenological implications, in particular in the case of pseudoscalar meson decay via possible heavy sterile neutrinos as intermediate resonant states. Their production and decay into ‘active’ neutrinos and charged leptons may have experimental relevance, we argue that cascade decay in this case may lead to ‘displaced decay vertices’ which may result in important corrections to the baseline dependence of the appearance (and disappearance) probabilities. Asymptotically at long times after both the parent and resonant intermediate particles decay, the final state is a quantum entangled many particle state that features quantum correlations which are completely determined by the asymptotic amplitudes. We also conjecture that quantum entanglement of the final state may translate in intensity Hanbury-Brown–Twiss correlations that may reveal information on the mass and width of the intermediate resonant state.

The analysis and results presented above may be important in resonant leptogenesis [33–36], which is typically studied by implementing powerful non-equilibrium methods such as the Kadanoff–Baym and Keldysh formulations to obtain the kinetic description of the distribution functions that include resonant cross-sections. However, the application of the results obtained in this article to the important case of resonant leptogenesis is not direct: we obtained the evolution equations for the amplitudes of the many particle state resulting from the decay of an initial *single particle state*. Instead the Kadanoff–Baym and Keldysh approaches to resonant leptogenesis focus on the time evolution of *distribution functions* which are ensemble averages in a non-equilibrium *density matrix*. In order to apply the methods and results obtained above to this important case, first we must understand how to implement the non-perturbative methods described above to the case of a non-equilibrium density matrix rather than a single particle initial state. The importance of resonant leptogenesis motivates further study to generalize the results obtained here to the case of non-equilibrium ensembles, correlation and distribution functions on which we expect to report in the future.

Further questions: in this paper we focused on the description of the time evolution of cascade decay in a simple scenario with only one intermediate resonant state. An important case that remains to be studied is that of several intermediate resonant states that may result from

mixing. In this case there will emerge interference phenomena manifest as oscillations, and if the intermediate states are nearly degenerate these oscillations and interference may lead to important dynamics at long times. This is the case for meson mixing relevant for CP and CPT violations. As discussed in the introduction cascade ‘mixing’ is typically studied as sequential events in (proper) time, however the analysis presented in the previous sections suggests that there may be important corrections from the time dependence of the amplitudes that *may* prove to be relevant to the experimental analysis. These questions along with an assessment of potential impact on neutrino oscillation experiments from a resonant heavy sterile neutrino, and the study of correlations in the final state as potential indicators of properties of the intermediate resonant state merit further study, which is postponed to a future article.

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Appendix A. Analysis of $C_{\phi\chi\chi}(\mathbf{k}, \mathbf{q}, \mathbf{p}; t)$

To make the notation in this appendix more compact we introduce the following variables

$$E_{\phi\chi\chi} - E_k^\pi = \eta; E_{\phi\phi} - E_k^\pi = \sigma; \Delta = \Gamma_\pi - \Gamma_{\phi_1}; \Sigma = \Gamma_\pi + \Gamma_{\phi_1}, \quad (\text{A.1})$$

and suppressing the momenta labels, we write

$$C_{\phi\chi\chi}(t) \equiv D(t) + C_{\phi\chi\chi}(\infty). \quad (\text{A.2})$$

The contribution from the term $|C_{\phi\chi\chi}(\infty)|^2$ has been analyzed above, leading to equation (3.14), for the remaining terms we find

$$|D(t)|^2 + D(t)C_{\phi\chi\chi}^*(\infty) + D^*(t)C_{\phi\chi\chi}(\infty) = \left| M_{\pi \rightarrow \phi_1 \phi_2} \right|^2 \left| M_{\phi_1 \rightarrow \chi_1 \chi_2} \right|^2 \times [(a) + (b) + (c) + (d) + (e)] \quad (\text{A.3})$$

$$(a) = \frac{1}{\sigma^2 + \left(\frac{\Delta}{2}\right)^2} \left[\frac{e^{-\Gamma_\pi t} - e^{-\frac{\Gamma_\pi}{2} t} (e^{i\eta t} + e^{-i\eta t})}{\eta^2 + \left(\frac{\Gamma_\pi}{2}\right)^2} \right], \quad (\text{A.4})$$

$$(b) = \frac{1}{\sigma^2 + \left(\frac{\Delta}{2}\right)^2} \left[\frac{e^{-\Gamma_{\phi_1} t} - e^{-\frac{\Gamma_{\phi_1}}{2} t} (e^{i(\eta-\sigma)t} + e^{-i(\eta-\sigma)t})}{(\eta - \sigma)^2 + \left(\frac{\Gamma_{\phi_1}}{2}\right)^2} \right], \quad (\text{A.5})$$

$$(c) = \frac{1}{\sigma^2 + \left(\frac{\Delta}{2}\right)^2} \left[\frac{e^{i\eta t} e^{-\frac{\Gamma_\pi}{2} t} - e^{i\sigma t} e^{-\frac{\Sigma}{2} t}}{\left(\eta + i\frac{\Gamma_\pi}{2}\right)\left(\eta - \sigma - i\frac{\Gamma_{\phi_1}}{2}\right)} \right], \quad (\text{A.6})$$

$$(d) = \frac{1}{\sigma^2 + \left(\frac{\Delta}{2}\right)^2} \left[\frac{e^{i(\eta-\sigma)t} e^{-\frac{\Gamma_{\phi_1}}{2}t} - e^{-i\sigma t} e^{-\frac{\Sigma}{2}t}}{\left(\eta - i\frac{\Gamma_{\pi}}{2}\right)\left(\eta - \sigma + i\frac{\Gamma_{\phi_1}}{2}\right)} \right], \quad (\text{A.7})$$

$$(e) = \frac{1}{\sigma^2 + \left(\frac{\Delta}{2}\right)^2} \left[\frac{e^{-i\eta t} e^{-\frac{\Gamma_{\pi}}{2}t}}{\left(\eta - i\frac{\Gamma_{\pi}}{2}\right)\left(\eta - \sigma + i\frac{\Gamma_{\phi_1}}{2}\right)} + \frac{e^{-i(\eta-\sigma)t} e^{-\frac{\Gamma_{\phi_1}}{2}t}}{\left(\eta + i\frac{\Gamma_{\pi}}{2}\right)\left(\eta - \sigma - i\frac{\Gamma_{\phi_1}}{2}\right)} \right]. \quad (\text{A.8})$$

The resonant denominators result in that the dominant contribution in the narrow widths limit are proportional to $\delta(\sigma)\delta(\eta)$, in order to extract the proportionality factors we integrate the above expressions in the complex σ, η planes where the resonant denominators yield complex poles. We find

$$(a) + (b) = -\frac{(2\pi)^2}{|\Delta|} \left\{ \frac{e^{-\Gamma_{\pi}t}}{\Gamma_{\pi}} \delta(\eta) + \frac{e^{-\Gamma_{\phi_1}t}}{\Gamma_{\phi_1}} \delta(\eta - \sigma) \right\} \delta(\sigma). \quad (\text{A.9})$$

The integrals in the complex planes of (c), (d) feature vanishing residues at the complex poles in $\eta, \eta - \sigma$, therefore these integrals yield subleading contributions in the narrow width limit. Finally by the same procedure we find

$$(e) = 2\frac{(2\pi)^2}{|\Delta|} \frac{e^{-\frac{|\Delta|}{2}t} e^{-\frac{\Sigma}{2}t}}{\frac{1}{2}(|\Delta| + \Sigma)} \delta(\eta)\delta(\sigma). \quad (\text{A.10})$$

The final result is given by

$$(a) + (b) + (c) + (d) + (e) = \frac{(2\pi)^2}{\Gamma_{\pi} - \Gamma_{\phi_1}} \left\{ \frac{e^{-\Gamma_{\pi}t}}{\Gamma_{\pi}} - \frac{e^{-\Gamma_{\phi_1}t}}{\Gamma_{\phi_1}} \right\} \delta(E_k^{\pi} - E_{\phi\chi\chi}) \delta(E_k^{\pi} - E_{\phi\phi}). \quad (\text{A.11})$$

Appendix B. The long time limit of equation (4.19)

Introducing

$$\rho(\omega) = \sum_{\vec{q}} \left| M_{\phi_1 \rightarrow \chi_1 \chi_2} \right|^2 \delta(\omega - E_{\phi\chi\chi}), \quad (\text{B.1})$$

we can write

$$\begin{aligned} \int_0^t dt' W_0^{\phi}(t', t') &= i t \int_{-\infty}^{\infty} d\omega' \frac{\rho(\omega')}{(E_{\phi\phi} - \omega')} \left[1 - \frac{\sin(\omega' - E_{\phi\phi})t}{(\omega' - E_{\phi\phi})t} \right] \\ &+ \int_{-\infty}^{\infty} d\omega' \frac{\rho(\omega')}{(E_{\phi\phi} - \omega')^2} \left[1 - \cos[(\omega' - E_{\phi\phi})t] \right]. \quad (\text{B.2}) \end{aligned}$$

Asymptotically as $t \rightarrow \infty$, these integrals approach:

$$\int_{-\infty}^{\infty} d\omega' \frac{\rho(\omega')}{(E_{\phi\phi} - \omega')} \left[1 - \frac{\sin(\omega' - E_{\phi\phi})t}{(\omega' - E_{\phi\phi})t} \right] \xrightarrow{t \rightarrow \infty} \mathcal{P} \int_{-\infty}^{\infty} d\omega' \frac{\rho(\omega')}{(E_{\phi\phi} - \omega')} \quad (\text{B.3})$$

$$\int_{-\infty}^{\infty} d\omega' \frac{\rho(\omega')}{(E_{\phi\phi} - \omega')^2} \left[1 - \cos[(\omega' - E_{\phi\phi})t] \right] \xrightarrow{t \rightarrow \infty} \pi t \rho(E_{\phi\phi}). \quad (\text{B.4})$$

The second integral above can be easily recognized as the usual Fermi's golden rule by taking the time derivative .

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