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Testing the complexity conjecture in regular black holes geometry

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Abstract

Motived by the new complexity conjecture [1] suggesting that the fastest computer in nature are the black holes. We study the action growth rate for a variety of four-dimensional regular black holes such as Hayward, Bardeen and the new class proposed in [2]. Generally, we show that action growth rates of the Wheeler-De Witt patch are finite for such black hole configurations at the late time approach and satisfy the Lloyd bound on the rate of quantum computation. Also, the case of three dimensions space is investigated. In each regular black hole configuration, we found that the form of the Lloyd bound formula remains unaltered but the energy is modified due to the effect of the nonlinear electrodynamics where some extra-therm have appeared in the total growth action.

1. Introduction

Nowadays, the quantum information physics play a crucial role in our comprehension of the emergence of spacetime and gravity from gravitational degrees of freedom in the gauge/gravity duality framework. Various tools have been involved ranging from entanglement [3–6], quantum

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error-correcting codes [7], and even quantum state complexity [8] have all been used to put light on many key aspects of the emergent gravitational degrees of freedom.

In this context, Maldecena and Susskind have made establish a beautiful link between the Einstein-Podolsky-Rosen (EPR) correlation with the Einstein-Rosen bridge (also known as wormholes) [9]. Rigorously speaking, they have attempted to relate quantum physics with the theory of gravity by the so-called **ER=EPR conjecture**. A simple and nice interpretation of this relation is to allow the communications between Alice and Bob from opposite sides of the wormhole [8,10].

Moreover, the computational complexity regarding AdS/CFT duality can be a huge step to illuminate the mystery inside the black hole and its related geometry, in addition, it supports to build quantum computers. One can found actually two conjectures that relate complexity on the boundary to the geometry of bulk:

At the first and older one complexity is supposed to be equal to the maximal volume in the space-like slice into the bulk [11], or `complexity = volume` (CV) as

$$\mathcal{C}(t_L, t_R) \sim \frac{V}{G\ell}, \tag{1}$$

which states that to compute the complexity of the boundary state, one can evaluate the volume of a codimension-one bulk hypersurface intersecting with the asymptotic boundary on the desired time slice. The second and recent complexity formalism stipulate that the following conjecture `complexity = Action` (CA), which relates complexity to the gravitational action $\mathcal{A}_{\mathcal{WDW}}$ evaluated in a Wheeler-DeWitt patch [1,12]:

$$\mathcal{C}(t_L, t_R) \sim \frac{\mathcal{A}_{\mathcal{WDW}}}{\pi\hbar}. \tag{2}$$

Given the energy of a quantum system, as already shown by Lloyd [13], the growth rate of the bulk action or the computational rate of the boundary state should have an upper bound,

$$\text{the computational rate} \leq \frac{2\,E}{\pi\hbar}, \tag{3}$$

where E is the excited energy of the boundary state. Substituting Eq. (2) into Eq. (3), one can obtain

$$\frac{d\mathcal{A}_{\mathcal{WDW}}}{dt} \leq \frac{2\,E}{\pi\hbar}. \tag{4}$$

In the case where the CA duality holds, the action growth rates of the black holes in the Wheeler-DeWitt patch has also to be bounded. This proposal is confirmed by studying the action growth rate of some AdS black hole configuration in the WDW patch at the late time approximation [1,12]. However, Cai et al. in Ref. [14] found that the above bound will be violated for both small and large charged static AdS black holes cases, so a new and general form of the above bound has been proposed as the difference between the value of thermodynamical quantities for a rotating charged black hole as follows

$$\frac{d\mathcal{A}}{dt} \leq (M - \Omega_+ J - \mu_+ Q) - (M - \Omega_- J - \mu_- Q), \tag{5}$$

where Ω and μ denote, respectively, the angular velocity and chemical potential of the black hole and the parameters M, Q, and J stand for the black hole mass, charge, and angular momentum, respectively. The subscript $_{+/-}$ denotes the quantities are taken at the outer/inner horizon of

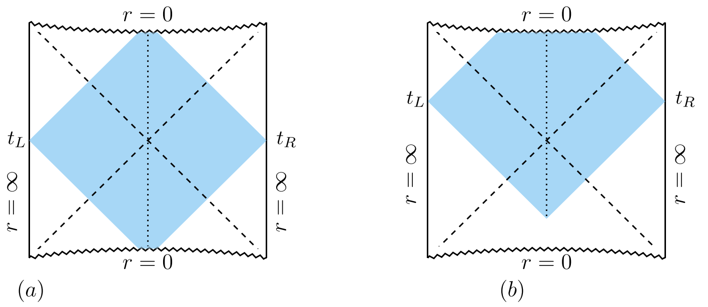

Fig. 1. Penrose diagram for a neutral two sided black hole and WDW patch in (a) initial and (b) late time regimes.

the black hole. Many papers in the literature have been devoted to checking this conjecture in different kind of black holes whether it is valid or not [14–64] and they assert that a black hole can be considered as the fastest computer in nature.

Our present understanding of some form of singularities is mainly due to pioneer works of Penrose and Hawking [65,66], then Bardeen discovered a regular black hole obeying the weak energy condition, which was pivotal in leading the direction of research in this context. A various Bardeen-like black holes solution that came after the original proposal owe the irregularity to topological changes, which offers the possibility to obtain spaces with a maximum curvature inside the black hole [67–70]. This feature, of course, involves the introduction of matter fields in a rather ad-hoc manner. Other regular black hole models were also proposed by coupling Einstein's theory with nonlinear electrodynamics [2,71]. Regular black hole interior solutions were also found in Loop Quantum Gravity [72] and they represent one of the key ingredients for our comprehension of such theory.

In this paper, we wonder whether that conjecture under the same conditions is also valid for the variety of combined gravity and nonlinear electrodynamics family of regular solutions or not in AdS spacetime. The outline of this paper is as follows: in section 2 we obtain the evolution of complexity growth at the late time approximation and check the Lloyd bound in such black hole models. Then we investigate the effect of nonlinear electrodynamics on this quantity for each case. The last section is devoted to discussion and conclusions.

2. The complexity growth of regular black holes

The authors of the references [1,12] show that the growth rate of the action of a Wheeler-DeWitt patch of the two-sided black hole at the late time, i.e. $t_L + t_R >> \beta$, corresponds to the increasing complexity rate of the boundary state. At the late time and without any shock wave the contribution of the region behind the past horizon goes to zero exponentially. The addition of a conserved charge involves a slowing at the end of the WDW patch compared to the neutral case. Fig. 1 depicts the general Penrose diagram and WDW patch for a neutral two-sided black hole for initial times and late times approximations.

The left panel ($1.a$) shows the intersection between the patch and both future and past singularities at $r = 0$ in the initial times while the right panel ($2.a$) indicates this intersection only with the future singularity in late time.

In what follows, we will turn our attention to a variety of regular black holes class [2,73,74] inside the bulk. Indeed, we will investigate the action growth rate for the late time approximation and Lloyd bound under such gravity models effect. It was known that the action of Einstein

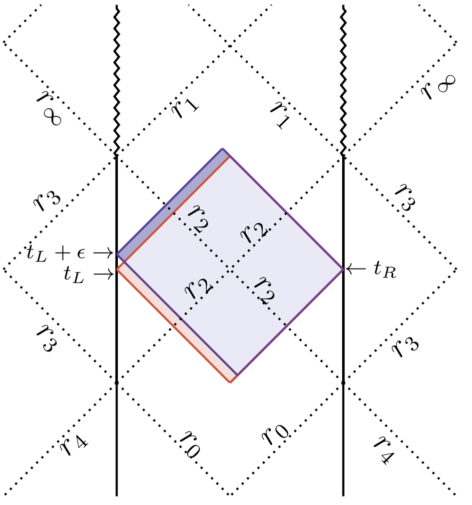

Fig. 2. Penrose diagram for a charged two sided black hole with multiple horizons and WDW patch in late time approximation. r_1 is the most internal horizon and r_0 is the null spatial infinity. The wavy lines indicate the singularities at $r=0$, and r_∞ stands for $r=-\infty$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

gravity coupled to a non-linear electromagnetic field with the York-Gibbons-Hawking terms [2] reads

$$\begin{aligned}\mathcal{A} &= \mathcal{A}_{bk} + \mathcal{A}_{bd} \\ &= \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\left(\mathcal{R}+\frac{6}{\ell^2}-\mathcal{L}(\mathcal{F})\right)+\frac{1}{8\pi G}\int\limits_{\partial\mathcal{M}} d^3x\sqrt{-h}\mathcal{K},\end{aligned} \tag{6}$$

where $F=dA$ is the field strength of the vector field, $\mathcal{F}\equiv F_{\mu\nu}F^{\mu\nu}$ and the Lagrangian density $\mathcal{L}$ is a function of $\mathcal{F}$. The first integral equation represents the action in the bulk while the second one stands for the boundary part of the WDW patch (WDW patch located in our two-sided black hole is indicated in figure Fig. 2 in which only dark blue region contributes to the complexity growth at late time approximation). The h is the induced metric of the hypersurface and $\mathcal{K}$ is the trace of the extrinsic curvature. In this paper, we consider static spherical symmetric black holes with magnetic charges. The most general ansatz is given by [2]

$$ds^2=-f(r)dt^2+\frac{dr^2}{f(r)}+r^2d\Omega_2^2, \qquad A=Q_m\cos\theta\, d\phi\,, \tag{7}$$

where $d\Omega_2^2=d\theta^2+\sin\theta^2 d\phi^2$ denotes the metric of a unit 2-sphere, Q_m is the total magnetic charge defined by

$$Q_m=\frac{1}{4\pi}\int F\,, \tag{8}$$

while the conjugate potential should be redefined as [75]

$$\Psi=\tilde{A}_t(r_h)-\tilde{A}_t(\infty), \quad \tilde{F}=d\tilde{A}=\mathcal{L}_{\mathcal{F}}{}^\star F \tag{9}$$

$$\Pi = \frac{1}{4}\int_{r_0}^{\infty} dr\sqrt{-g}\,\frac{\partial \mathcal{L}}{\partial \alpha} \tag{10}$$

Here, for our purposes, we will focus on well-known regular black hole models, namely the Hayward, Bardeen and the New class black holes proposed in [2,73,76] generalized in AdS space-time [73].

2.1. Case 1: Hayward class

The first class solution presented here is based on the Lagrangian density

$$\mathcal{L} = \frac{4\mu}{\alpha}\frac{\left(\alpha\mathcal{F}\right)^{\frac{\mu+3}{4}}}{\left(1+\left(\alpha\mathcal{F}\right)^{\frac{\mu}{4}}\right)^2}, \tag{11}$$

where $\mu > 0$ is a dimensionless constant and $\alpha > 0$ has the dimension of length squared. In the weak field limit, the vector field behaves as $\mathcal{L} \sim \alpha^{\frac{\mu-1}{4}}\mathcal{F}^{\frac{\mu+3}{4}}$. It could be either stronger ($\mu > 1$) or weaker ($0 < \mu < 1$) than a Maxwell field. A critical case occurs when $\mu = 1$ at which the nonlinear electrodynamics reduces to a Maxwell field in the weak field limit. The general static spherically symmetric solution reads

$$f(r) = 1 - \frac{2M}{r} - \frac{2\alpha^{-1}q^3 r^{\mu-1}}{r^\mu + q^\mu} + \frac{r^2}{\ell^2}, \tag{12}$$

where M is associated with the condensate of the massless graviton, originating from its self-interactions and q is a free integration constant which is related to the magnetic charge

$$Q_m = \frac{q^2}{\sqrt{2\alpha}}. \tag{13}$$

The AMD mass is given by [77]

$$M_{AMD} = M + \alpha^{-1}q^3 \tag{14}$$

Focusing on the AMD mass, or equivalently the condensate of the massless graviton, we can easily note two kinds of contributions, one comes from the self-interactions of the graviton, giving rise to the Schwarzschild mass, and the other is originated from the non-linear interactions between the graviton and the (nonlinear) photon, leading to the charged term $\alpha^{-1}q^3$. For Simplicity, we set $\mu = 3$ in the rest of the manuscript. We can remark that for the Hayward black hole,[1] the maximal number of the horizons defined by the roots (both real and one imaginary or one real and two imaginary or three imaginary) of the equation $f(r) = 0$ is exactly equal to μ [2]. In our present study, we will focus on the two black hole horizon structure.

Taking the function f as an equipotential surface $f(r) = constant$ the first law of the black hole thermodynamics could be derived for which

$$\begin{aligned} df(S, M, P, Q_m, \alpha) &= 0 \\ &= \frac{\partial f}{\partial S}dS + \frac{\partial f}{\partial M}dM + \frac{\partial f}{\partial P}dP + \frac{\partial f}{\partial Q_m}dQ_m + \frac{\partial f}{\partial \alpha}d\alpha, \end{aligned} \tag{15}$$

[1] This result is valid for all black hole class, namely Bardeen and new class solutions.

where $S=\pi r^2$ is the entropy of the black hole and $P=\frac{3}{8\pi\ell^2}$ is the pressure of AdS spacetime. Applying the above relation we can obtain the

$$dM = TdS + VdP + \Psi dQ_m + \Pi d\alpha. \tag{16}$$

Although less straightforward, it would be interesting to derive the generalized Smarr relations as

$$M = 2TS + \Psi Q_m - 2VP + 2\Pi\alpha. \tag{17}$$

Having presented some essential features of the black hole solution and its corresponding first law of thermodynamics, we will turn our attention to calculate the complexity growth rate in such a background. Regarding the above solution Eq (12), Ricci scalar could be achieved as:

$$\mathcal{R} = -\frac{12}{\ell^2} + \frac{12\left(2q^9 - q^6r^3\right)}{\alpha\left(q^3+r^3\right)^3} \tag{18}$$

giving the bulk action growth by

$$\begin{aligned}\frac{d\mathcal{A}_{bk}}{dt} &= \frac{1}{16\pi G}\iint_{r_-}^{r_+} r^2\left[-\frac{6}{\ell^2} - \frac{12q^6}{\alpha\left(q^3+r^3\right)^2} + \frac{12\left(2q^9-q^6r^3\right)}{\alpha\left(q^3+r^3\right)^3}\right]drd\Omega_2 \\ &= -\frac{1}{2\ell^2}(r_+^3 - r_-^3) + \frac{q^6\left(r_-^3-r_+^3\right)\left(-2q^6+q^3\left(r_-^3+r_+^3\right)+4r_-^3r_+^3\right)}{2\alpha\left(q^3+r_-^3\right)^2\left(q^3+r_+^3\right)^2}.\end{aligned} \tag{19}$$

While the extrinsic curvature associated with metric Eq. (7) can be written as

$$\mathcal{K} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\sqrt{f(r)}) = \frac{2}{r}\sqrt{f(r)} + \frac{f'(r)}{2\sqrt{f(r)}}. \tag{20}$$

The growth rate of the York-Gibbons-Hawking (YGH) surface term within WDW patch at late-time approximation which is second integral in Eq. (6) reads as

$$\begin{aligned}\frac{d\mathcal{A}_{bd}}{dt} &= \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d\Omega_2(\sqrt{-h}\mathcal{K}) = \frac{1}{2}\left[r^2\sqrt{f(r)}\left(\frac{2}{r}\sqrt{f(r)} + \frac{f'(r)}{2\sqrt{f(r)}}\right)\right]_{\partial\mathcal{M}} \\ &= (r_+ - r_-) + \frac{3\left(r_+^3-r_-^3\right)}{2\ell^2} + \frac{3q^9\left(r_-^3-r_+^3\right)\left(2q^3+r_-^3+r_+^3\right)}{2\alpha\left(q^3+r_-^3\right)^2\left(q^3+r_+^3\right)^2},\end{aligned} \tag{21}$$

leading to the total growth rate of action for such black hole configuration within WDW patch at late time approximation

$$\frac{d\mathcal{A}}{dt} = (r_+ - r_+) + \frac{r_+^3 - r_-^3}{\ell^2} + \frac{2q^6\left(r_-^3-r_+^3\right)}{\alpha\left(q^3+r_-^3\right)\left(q^3+r_+^3\right)}. \tag{22}$$

By re-expressing the mass M:

$$M = \frac{1}{4}\left((r_+ + r_-) + \frac{r_+^3 + r_-^3}{\ell^2} + \frac{1}{\ell^2}\sqrt{\zeta\left(\zeta(r_- - r_+)^2 + \frac{8r_-^3r_+^3\ell^2}{\alpha\xi}\right)}\right) \tag{23}$$

and the charge Q_m

$$Q_m = \left(\frac{\alpha\zeta\left(r_-^3 + r_+^3\right) - \sqrt{\alpha\zeta\xi\left(8r_-^3 r_+^3 \ell^2 + \alpha\zeta\xi(r_- - r_+)^2\right)}}{2\alpha\zeta - 4\xi\ell^2} \right)^{\frac{1}{3}} \tag{24}$$

in term of inner and outer horizon radius $r_\pm$ and ξ and ζ are given by the following formula

$$\xi = r_-^2 + r_- r_+ + r_+^2, \tag{25}$$

$$\zeta = r_-^2 + r_- r_+ + r_+^2 + \ell^2. \tag{26}$$

At this level and by attention to all above definitions, the action growth at the late time approximation can be summarized in more compact form by recalling the conjugated potentials as

$$\frac{d\mathcal{A}}{dt} = \left(M - Q_m\Psi_+ - 2\alpha\Pi_+\right) - \left(M - Q_m\Psi_- - 2\alpha\Pi_-\right), \tag{27}$$

where the chemical potentials associated with charge Q_m and nonlinearity parameter α at the inner and outer horizon $\pm$. Both quantities are respectively given with the help of Eq. (9) and Eq. (10)

$$\Psi = \frac{3q^4(2r^3 + q^3)}{\sqrt{2\alpha}(r^3 + q^3)^2}, \tag{28}$$

$$\Pi = \frac{q^6(2r^3 - q^3)}{4\alpha^2(r^3 + q^3)^2}. \tag{29}$$

Comparing with the pioneer works presented by Brown et al. [1,12], one can infer there are some extra terms due to the presence of the nonlinear electrodynamics coupling.

Now, it's obvious, that the action growth rate of the Hayward black hole can also saturate Eq. (5) but in its novel form Eq. (27) which includes the coupling constant of nonlinear electrodynamics and its conjugate potential. In particular, the quantum complexity growth rate is bounded by Eq. (4). Also in the limit where $\alpha \to \infty$, the Eq. (27) reduces to result in [12,14].

2.2. *Case 2: Bardeen class*

The second class solution we present is valid for a Lagrangian density

$$\mathcal{L} = \frac{4\mu}{\alpha} \frac{\left(\alpha\mathcal{F}\right)^{5/4}}{\left(1 + \sqrt{\alpha\mathcal{F}}\right)^{1+\mu/2}}. \tag{30}$$

In the weak field limit, the vector field behaves as $\mathcal{L} \sim \alpha^{1/4}\mathcal{F}^{5/4}$ which is slightly stronger than a Maxwell field. The black hole solution in this background is [2]

$$f(r) = 1 - \frac{2M}{r} - \frac{2\alpha^{-1}q^3 r^{\mu-1}}{\left(r^2 + q^2\right)^{\mu/2}} + \frac{r^2}{\ell^2}, \tag{31}$$

where the magnetic charge is still given by Eq. (13). In this context, the Ricci scalar curvature becomes

$$\mathcal{R} = -\frac{12}{\ell^2} + \frac{6q^5 r^2 - 24q^7}{\alpha\left(q^2 + r^2\right)^{7/2}} \tag{32}$$

Following [8], we can calculate the growth rate of the bulk action at late time approximation given by first integral equation in Eq. (6) and using the Bardeen Lagrangian Eq. (30) as follows:

$$\begin{aligned}\frac{d\mathcal{A}_{bk}}{dt} &= \frac{1}{16\pi G}\iint_{r_-}^{r_+} r^2\Big[-\frac{6}{\ell^2}-\frac{12q^5}{\alpha\left(q^2+r^2\right)^{5/2}}+\frac{6\left(4q^7-q^5r^2\right)}{\alpha\left(q^2+r^2\right)^{7/2}}\Big]drd\Omega_2 \\ &= -\frac{1}{2\ell^2}(r_+^3-r_-^3)+\frac{q^9\left(\tilde{r}_-r_+^3-r_-^3\tilde{r}_+\right)}{\alpha\tilde{r}_-^5\tilde{r}_+^5} \\ &\quad +\frac{q^7\left(r_-^5\tilde{r}_+-4r_-^3r_+^2\tilde{r}_++4r_-^2\tilde{r}_-r_+^3-\tilde{r}_-r_+^5\right)}{2\alpha\tilde{r}_-^5\tilde{r}_+^5} \\ &\quad +\frac{q^5r_-^2r_+^2(r_--r_+)(r_-+r_+)(r_-\tilde{r}_++\tilde{r}_-r_+)}{\alpha\tilde{r}_-^5\tilde{r}_+^5}+\frac{q^3r_-^4r_+^4(r_-\tilde{r}_+-\tilde{r}_-r_+)}{2\alpha\tilde{r}_-^5\tilde{r}_+^5}. \end{aligned} \tag{33}$$

Here, we have introduced the following simplified notation $\tilde{r}_\pm=\sqrt{q^2+r_\pm^2}$. Besides, the boundary action is completely given by

$$\frac{d\mathcal{A}_{bd}}{dt}=(r_+-r_-)+\frac{3\left(r_+^3-r_-^3\right)}{2\ell^2}+\frac{3q^3}{2\alpha}\left(\frac{r_-^3\left(2q^2+r_-^2\right)}{\tilde{r}_-^5}-\frac{r_+^3\left(2q^2+r_+^2\right)}{\tilde{r}_+^5}\right). \tag{34}$$

Therefore, the total growth rate of action for such black hole configuration within WDW patch at late time approximation can be calculated straightforwardly

$$\frac{d\mathcal{A}}{dt}=(r_+-r_+)+\frac{r_+^3-r_-^3}{\ell^2}+\frac{2q^3}{\alpha}\left(\frac{r_-^3}{\tilde{r}_-^3}-\frac{r_+^3}{\tilde{r}_+^3}\right). \tag{35}$$

As in the Hayward class, it's easy to check the validity of the Eq. (27) within the Bardeen background by recalling the form of mass and charge in term of the inner/outer horizon and definition of potentials associated with the charge Q_m and the parameter of the nonlinearity α Eq. (9) and Eq. (10) respectively. Given the same form of the total action growth in Hayward framework Eq. (27).

2.3. Case 3: a new class

Perhaps the most interesting theories which admit regular black hole solutions are such that the vector field approaches a Maxwell field in the weak field limit. We find that such theories indeed exist within the Lagrangian density

$$\mathcal{L}=\frac{4\mu}{\alpha}\frac{\alpha\mathcal{F}}{\left(1+\left(\alpha\mathcal{F}\right)^{1/4}\right)^{\mu+1}}. \tag{36}$$

Therefore, the black hole solution is totally expressed by the following metric function

$$f(r)=1-\frac{2M}{r}-\frac{2\alpha^{-1}q^3r^{\mu-1}}{\left(r+q\right)^{\mu}}+\frac{r^2}{\ell^2}, \tag{37}$$

where the magnetic charge also still given by Eq. (13). One can easily calculate the Ricci scalar in this background as

$$\mathcal{R} = -\frac{12}{\ell^2} - \frac{24q^5}{\alpha(q+r)^5} \tag{38}$$

Meanwhile, we also have the bulk action growth is given by

$$\begin{aligned}\frac{d\mathcal{A}_{bk}}{dt} &= \frac{1}{16\pi G}\iint_{r_-}^{r_+} r^2\left[-\frac{6}{\ell^2} - \frac{12q^4}{\alpha(q+r)^4} + \frac{24q^5}{\alpha(q+r)^5}\right]drd\Omega_2 \\ &= \frac{r_-^3 - r_+^3}{2\ell^2} - \frac{q^4(r_- - r_+)}{2\alpha(q+r_-)^4(q+r_+)^4} \\ &\quad \times \left(2q^4\left(r_-^2 + r_-r_+ + r_+^2\right) - q^3(r_- + r_+)\left(r_-^2 - 8r_-r_+ + r_+^2\right)\right. \\ &\quad \left. - 4q^2r_-r_+(r_- - r_+)^2 - 6qr_-^2r_+^2(r_- + r_+) - 6r_-^3r_+^3\right)\end{aligned} \tag{39}$$

and the contribution from the boundary term within the WDW patch reads as

$$\begin{aligned}\frac{d\mathcal{A}_{bd}}{dt} &= (r_+ - r_-) + \frac{3\left(r_+^3 - r_-^3\right)}{2\ell^2} - \frac{3q^4}{\alpha}\left(\frac{r_+^3}{(q+r_+)^4} - \frac{r_-^3}{(q+r_-)^4}\right) \\ &\quad - \frac{3q^3}{2\alpha}\left(\frac{r_+^4}{(q+r_+)^4} - \frac{r_-^4}{(q+r_-)^4}\right).\end{aligned} \tag{40}$$

Hence, these two equations together give the total growth rate of action for such black hole kind of solutions within WDW patch at late time approximation as

$$\begin{aligned}\frac{d\mathcal{A}}{dt} &= (r_+ - r_+) + \frac{r_+^3 - r_-^3}{\ell^2} \\ &\quad + \frac{2q^4\left(q^2\left(r_-^3 - r_+^3\right) + 3qr_-r_+\left(r_-^2 - r_+^2\right) + 3r_-^2r_+^2(r_- - r_+)\right)}{\alpha(q+r_-)^3(q+r_+)^3}.\end{aligned} \tag{41}$$

By the same way of the previous sections, we can easily check that the New class of black hole is not the exception and it verifies the Eq. (27).

2.4. Three dimension regular solution

Having investigated a variety of regular black hole in four dimensions, it's crucial to check the validity of the complexity conjecture in three dimensional spacetimes. Staring with the $(2+1)$ Einstein theory coupled to a non linear electrodynamics given by the following action [78]

$$\begin{aligned}\mathcal{A} &= \mathcal{A}_{bk} + \mathcal{A}_{bd} \\ &= \int \mathrm{d}^3x\sqrt{-g}\left(\frac{\mathcal{R} - 2\Lambda}{16\pi G} + G\mathcal{L}(\mathcal{F})\right) + \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^2x\sqrt{-h}\mathcal{K},\end{aligned} \tag{42}$$

where, G denotes Newton's constant, g is the determinant of the metric tensor, $\Lambda = -\frac{1}{\ell^2}$ is the cosmological constant, and $\mathcal{L}(\mathcal{F})$ is the Lagrangian of the nonlinear electrodynamics with $\mathcal{F} = F^{\mu\nu}F_{\mu\nu}$. For the static, circularly symmetric spacetime we take the simplest metric ansatz [78]

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\phi^2 \tag{43}$$

where

$$f(r) = -m + \frac{r^2}{\ell^2} - 2q^2\left(\log\left(\frac{q}{\alpha\ell} + \frac{r}{\ell}\right) + \frac{q}{\alpha r + q}\right) \tag{44}$$

and

$$\mathcal{L}(r) = \frac{\frac{r}{q} - \frac{1}{\alpha}}{8\pi\left(\frac{r}{q} + \frac{1}{\alpha}\right)^3}, \quad E(r) = \frac{r^2/q^2}{16\pi\left(\frac{r}{q} + \frac{1}{\alpha}\right)^3} \tag{45}$$

where, $\mathcal{F} = -2E^2$.[2] Clearly, the regular black hole solution Eq. (44) will reduce to the static, charged BTZ black hole when the parameter $\alpha \to \infty$.

Before carrying out an explicit calculation of the complexity growth rate, let us first describe some of the geometric characteristics of the solution. First, let us calculate the Ricci scalar

$$\mathcal{R} = -\frac{6}{\ell^2} + \frac{2\alpha^2 q^2(3q + \alpha r)}{(q + \alpha r)^3}, \tag{46}$$

then, applying the on-shell condition $8G \equiv 1$ we can express the integrand in the bulk action as a function of radius r, we have

$$\frac{d\mathcal{A}_{bk}}{dt} = -\frac{2(r_+^2 - r_-^2)}{\ell^2} + 4q^2 \ln\left(\frac{q + \alpha\, r_+}{q + \alpha\, r_+}\right) + \frac{4\alpha q^3(r_- - r_+)}{(q + \alpha r_-)(q + \alpha r_+)} \tag{47}$$

and it turns out that the boundary action can be obtained to be

$$\frac{d\mathcal{A}_{bd}}{dt} = \frac{4(r_+^2 - r_-^2)}{\ell^2} - 4q^2 \ln\left(\frac{q + \alpha\, r_+}{q + \alpha\, r_+}\right) - 2q^4\left(\frac{1}{(q + \alpha\, r_+)^2} - \frac{1}{(q + \alpha\, r_-)^2}\right) \tag{48}$$

To actually get the total action growth rate, we perform the addition of the two previous equations to get

$$\frac{d\mathcal{A}}{dt} = \frac{2(r_+^2 - r_-^2)}{\ell^2} + \frac{2\alpha^2 q^3(r_- - r_+)(2\alpha r_- r_+ + q(r_- + r_+))}{(\alpha r_- + q)^2(\alpha r_+ + q)^2} \tag{49}$$

By analogy with previously treated cases, one can express the mass and the charge in terms of the inner and outer horizons $r_\pm$, then we define the chemical potentials associated with q as [78]

$$\Psi = \frac{2q^2(4\alpha r + 3q)}{32\pi(\alpha r + q)^2} + \frac{q}{16\pi}\log\left(\frac{\alpha r + q}{\alpha\ell}\right) \tag{50}$$

To rewrite Eq. (49) as

$$\frac{dA}{dt} = (M - \Psi_+ Q) - (M - \Psi_- Q), \tag{51}$$

which shares exactly the same form of the result in [14] and where we have set $Q = 8\pi q$. In particular, the quantum complexity growth rate is bounded by Eq. (4).

[2] We obtain $\mathcal{L}(r)$ and $E(r)$ as functions of r. In principle, one can always combine them to give $\mathcal{L}(\mathcal{F})$. One can easily check that the electromagnetic fields in this case satisfies the weak energy condition [78].

3. Conclusion

We have investigated the boundary term of the action and the bulk action to calculate the action growth rate. Afterward, we obtain the result of the total action growth which is related to the results of the seminal paper of the Brown et al. [12] but in more general form and we carry out the main calculations of this paper to obtain the Lloyd bound, we have found some extra terms related to the nonlinear electrodynamics are added to the total action growth. Also, it is proved that by attention to the conjugated potential for the nonlinearity parameter the Lloyd bound [13] is satisfied for all parameter states defined the regime of the nonlinear electrodynamics. Further, we have explored the differences between the calculations of the complexity of each regular black hole model in four and three-dimension background when the two-horizons structure is considered.

More importantly, the CA conjectures provide interesting results to shed light on quantum computation. Furthermore, complexity grows linearly in time if one proposes the late time approximation where the main contributions come from the WdW patch. Hence the linear rates of the active growth support the link between quantum states and it saturates the Lloyd bound on the growth of the complexity.

This is another important evidence for the idea that black holes are the fastest computers and scramblers in nature. It would also be very interesting to investigate the complexity growth rate, which is a link between spacetime geometry and quantum entanglements [8], in different gravity theories and different geometries to understand deeply its nature. We will leave it to our future projects. Also, the one horizon structure can be one of the key directions in these upcoming works.

References

[1] Adam R. Brown, Daniel A. Roberts, Leonard Susskind, Brian Swingle, Ying Zhao, Holographic complexity equals bulk action?, Phys. Rev. Lett. 116 (19) (2016) 191301.
[2] Zhong-Ying Fan, Xiaobao Wang, Construction of regular black holes in general relativity, Phys. Rev. D 94 (12) (2016) 124027.
[3] A.A. Deriglazov, Poincare covariant mechanics on noncommutative space, J. High Energy Phys. 03 (2003) 021.
[4] Shinsei Ryu, Tadashi Takayanagi, Holographic derivation of entanglement entropy from AdS/CFT, Phys. Rev. Lett. 96 (2006) 181602.
[5] Mark Van Raamsdonk, Building up spacetime with quantum entanglement, Gen. Relativ. Gravit. 42 (2010) 2323–2329, Int. J. Mod. Phys. D 19 (2010) 2429.
[6] Brian Swingle, Entanglement renormalization and holography, Phys. Rev. D 86 (2012) 065007.
[7] Ahmed Almheiri, Xi Dong, Daniel Harlow, Bulk locality and quantum error correction in AdS/CFT, J. High Energy Phys. 04 (2015) 163.
[8] Leonard Susskind, Computational complexity and black hole horizons, Fortschr. Phys. 64 (2016) 44–48, Fortschr. Phys. 64 (2016) 24.
[9] Juan Maldacena, Leonard Susskind, Cool horizons for entangled black holes, Fortschr. Phys. 61 (2013) 781–811.
[10] Leonard Susskind, Ying Zhao, Teleportation through the wormhole, Phys. Rev. D 98 (4) (2018) 046016.
[11] Douglas Stanford, Leonard Susskind, Complexity and shock wave geometries, Phys. Rev. D 90 (12) (2014) 126007.
[12] Adam R. Brown, Daniel A. Roberts, Leonard Susskind, Brian Swingle, Ying Zhao, Complexity, action, and black holes, Phys. Rev. D 93 (8) (2016) 086006.
[13] Seth Lloyd, Ultimate physical limits to computation, Nature 406 (2000) 1047.
[14] Rong-Gen Cai, Shan-Ming Ruan, Shao-Jiang Wang, Run-Qiu Yang, Rong-Hui Peng, Action growth for AdS black holes, J. High Energy Phys. 09 (2016) 161.
[15] Mohsen Alishahiha, Holographic complexity, Phys. Rev. D 92 (12) (2015) 126009.
[16] Luis Lehner, Robert C. Myers, Eric Poisson, Rafael D. Sorkin, Gravitational action with null boundaries, Phys. Rev. D 94 (8) (2016) 084046.

[17] Brian Swingle, Gregory Bentsen, Monika Schleier-Smith, Patrick Hayden, Measuring the scrambling of quantum information, Phys. Rev. A 94 (4) (2016) 040302.
[18] Dean Carmi, Robert C. Myers, Pratik Rath, Comments on holographic complexity, J. High Energy Phys. 03 (2017) 118.
[19] Dean Carmi, Shira Chapman, Hugo Marrochio, Robert C. Myers, Sotaro Sugishita, On the time dependence of holographic complexity, J. High Energy Phys. 11 (2017) 188.
[20] Josiah Couch, Willy Fischler, Phuc H. Nguyen, Noether charge, black hole volume, and complexity, J. High Energy Phys. 03 (2017) 119.
[21] Shira Chapman, Hugo Marrochio, Robert C. Myers, Complexity of formation in holography, J. High Energy Phys. 01 (2017) 062.
[22] Daniel A. Roberts, Beni Yoshida, Chaos and complexity by design, J. High Energy Phys. 04 (2017) 121.
[23] Adam R. Brown, Leonard Susskind, Second law of quantum complexity, Phys. Rev. D 97 (8) (2018) 086015.
[24] Ben-Ami Omer, Dean Carmi, On volumes of subregions in holography and complexity, J. High Energy Phys. 11 (2016) 129.
[25] Run-Qiu Yang, Strong energy condition and complexity growth bound in holography, Phys. Rev. D 95 (8) (2017) 086017.
[26] Alan Reynolds, Simon F. Ross, Divergences in holographic complexity, Class. Quantum Gravity 34 (10) (2017) 105004.
[27] Wen-Jian Pan, Yong-Chang Huang, Holographic complexity and action growth in massive gravities, Phys. Rev. D 95 (12) (2017) 126013.
[28] Jose L.F. Barbon, Eliezer Rabinovici, Holographic complexity and spacetime singularities, J. High Energy Phys. 01 (2016) 084.
[29] Rong-Gen Cai, Misao Sasaki, Shao-Jiang Wang, Action growth of charged black holes with a single horizon, Phys. Rev. D 95 (12) (2017) 124002.
[30] Shira Chapman, Michal P. Heller, Hugo Marrochio, Fernando Pastawski, Toward a definition of complexity for quantum field theory states, Phys. Rev. Lett. 120 (12) (2018) 121602.
[31] Mohsen Alishahiha, Amin Faraji Astaneh, Ali Naseh, Mohammad Hassan Vahidinia, On complexity for F(R) and critical gravity, J. High Energy Phys. 05 (2017) 009.
[32] Run-Qiu Yang, Chao Niu, Keun-Young Kim, Surface counterterms and regularized holographic complexity, J. High Energy Phys. 09 (2017) 042.
[33] Jose L.F. Barbon, Javier Martin-Garcia, Holographic complexity of cold hyperbolic black holes, J. High Energy Phys. 11 (2015) 181.
[34] Hyat Huang, Xing-Hui Feng, H. Lu, Holographic complexity and two identities of action growth, Phys. Lett. B 769 (2017) 357–361.
[35] Bartlomiej Czech, Einstein equations from varying complexity, Phys. Rev. Lett. 120 (3) (2018) 031601.
[36] Mahdi Kord Zangeneh, Yen Chin Ong, Bin Wang, Entanglement entropy and complexity for one-dimensional holographic superconductors, Phys. Lett. B 771 (2017) 235–241.
[37] Wen-Di Guo, Shao-Wen Wei, Yan-Yan Li, Yu-Xiao Liu, Complexity growth rates for AdS black holes in massive gravity and $f(R)$ gravity, Eur. Phys. J. C 77 (12) (2017) 904.
[38] Francisco Jose Garcia Abad, Manuela Kulaxizi, Andrei Parnachev, On complexity of holographic flavors, J. High Energy Phys. 01 (2018) 127.
[39] William Cottrell, Miguel Montero, Complexity is simple!, J. High Energy Phys. 02 (2018) 039.
[40] Souvik Banerjee, Johanna Erdmenger, Debajyoti Sarkar, Connecting Fisher information to bulk entanglement in holography, J. High Energy Phys. 08 (2018) 001.
[41] Wissam Chemissany, Tobias J. Osborne, Holographic fluctuations and the principle of minimal complexity, J. High Energy Phys. 12 (2016) 055.
[42] Alan Reynolds, Simon F. Ross, Complexity in de Sitter space, Class. Quantum Gravity 34 (17) (2017) 175013.
[43] Peng Wang, Haitang Yang, Shuxuan Ying, Action growth in $f(R)$ gravity, Phys. Rev. D 96 (4) (2017) 046007.
[44] Gia Dvali, Cesar Gomez, Dieter Lust, Yasser Omar, Benedikt Richter, Universality of black hole quantum computing, Fortschr. Phys. 65 (1) (2017) 1600111.
[45] Vitaly Vanchurin, Dual field theories of quantum computation, J. High Energy Phys. 06 (2016) 001.
[46] Mohammad M. Qaemmaqami, Complexity growth in minimal massive 3D gravity, Phys. Rev. D 97 (2) (2018) 026006.
[47] Seyed Ali Hosseini Mansoori, Mohammad M. Qaemmaqami, Complexity Growth, Butterfly Velocity and Black Hole Thermodynamics, 2017.
[48] Mahdis Ghodrati, Complexity growth in massive gravity theories, the effects of chirality, and more, Phys. Rev. D 96 (10) (2017) 106020.

[49] Koichi Nagasaki, Complexity of AdS_5 black holes with a rotating string, Phys. Rev. D 96 (12) (2017) 126018.
[50] Lorenzo Sebastiani, Luciano Vanzo, Sergio Zerbini, Action growth for black holes in modified gravity, Phys. Rev. D 97 (4) (2018) 044009.
[51] Keiju Murata, Fast scrambling in holographic Einstein-Podolsky-Rosen pair, J. High Energy Phys. 11 (2017) 049.
[52] Leonard Susskind, Dear Qubitzers GR=QM, 2017.
[53] Yan-Gang Miao, Long Zhao, Complexity-action duality of the shock wave geometry in a massive gravity theory, Phys. Rev. D 97 (2) (2018) 024035.
[54] Brian Swingle, Yixu Wang, Holographic complexity of Einstein-Maxwell-Dilaton gravity, J. High Energy Phys. 09 (2018) 106.
[55] Ning Bao, ChunJun Cao, Sean M. Carroll, Aidan Chatwin-Davies, De Sitter space as a tensor network: cosmic no-hair, complementarity, and complexity, Phys. Rev. D 96 (12) (2017) 123536.
[56] Zicao Fu, Donald Marolf, Eric Mefford, Time-independent wormholes, J. High Energy Phys. 12 (2016) 021.
[57] Yu-Sen An, Rong-Hui Peng, Effect of the dilaton on holographic complexity growth, Phys. Rev. D 97 (6) (2018) 066022.
[58] Koichi Nagasaki, Complexity growth of rotating black holes with a probe string, Phys. Rev. D 98 (12) (2018) 126014.
[59] Mahdis Ghodrati, Complexity growth rate during phase transitions, Phys. Rev. D 98 (10) (2018) 106011.
[60] Hossein Ghaffarnejad, Mohammad Farsam, Emad Yaraie, Effects of Quintessence Dark Energy on the Action Growth and Butterfly Velocity, 2018.
[61] Mohsen Alishahiha, Komeil Babaei Velni, Mohammad Reza Tanhayi, Complexity and Near Extremal Charged Black Branes, 2019.
[62] Koichi Nagasaki, Interface in AdS Black Hole Spacetime, 2018.
[63] Yi Ling, Yuxuan Liu, Cheng-Yong Zhang, Holographic subregion complexity in Einstein-Born-Infeld theory, Eur. Phys. J. C 79 (3) (2019) 194.
[64] Jun Tao, Peng Wang, Haitang Yang, Testing holographic conjectures of complexity with Born–Infeld black holes, Eur. Phys. J. C 77 (12) (2017) 817.
[65] Roger Penrose, Gravitational collapse and space-time singularities, Phys. Rev. Lett. 14 (1965) 57–59.
[66] S.W. Hawking, R. Penrose, The singularities of gravitational collapse and cosmology, Proc. R. Soc. Lond. Ser. A 314 (1970) 529–548.
[67] Valeri P. Frolov, M.A. Markov, Viatcheslav F. Mukhanov, Through a black hole into a new universe?, Phys. Lett. B 216 (1989) 272–276, 52 (1990).
[68] Valeri P. Frolov, M.A. Markov, Viatcheslav F. Mukhanov, Black holes as possible sources of closed and semiclosed worlds, Phys. Rev. D 41 (1990) 383.
[69] Valeri P. Frolov, Notes on nonsingular models of black holes, Phys. Rev. D 94 (10) (2016) 104056.
[70] Viatcheslav F. Mukhanov, Robert H. Brandenberger, A nonsingular universe, Phys. Rev. Lett. 68 (1992) 1969–1972.
[71] Ayon-Beato Eloy, Alberto Garcia, Regular black hole in general relativity coupled to nonlinear electrodynamics, Phys. Rev. Lett. 80 (1998) 5056–5059.
[72] Leonardo Modesto, Black hole interior from loop quantum gravity, Adv. High Energy Phys. 2008 (2008) 459290.
[73] Bobir Toshmatov, Zdeněk Stuchlík, Bobomurat Ahmedov, Comment on "Construction of regular black holes in general relativity", Phys. Rev. D 98 (2) (2018) 028501.
[74] Zhong-Ying Fan, Critical phenomena of regular black holes in Anti-de Sitter space-time, Eur. Phys. J. C 77 (4) (2017) 266.
[75] D.A. Rasheed, Nonlinear Electrodynamics: Zeroth and First Laws of Black Hole Mechanics, 1997.
[76] Sean A. Hayward, Formation and evaporation of regular black holes, Phys. Rev. Lett. 96 (2006) 031103.
[77] Abhay Ashtekar, Saurya Das, Asymptotically Anti-de Sitter space-times: conserved quantities, Class. Quantum Gravity 17 (2000) L17–L30.
[78] Yun He, Meng-Sen Ma, $(2+1)$-dimensional regular black holes with nonlinear electrodynamics sources, Phys. Lett. B 774 (2017) 229–234.