



On γ -rigid regime of the Bohr–Mottelson Hamiltonian in the presence of a minimal length



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ABSTRACT

A prolate γ -rigid regime of the Bohr–Mottelson Hamiltonian within the minimal length formalism, involving an infinite square well like potential in β collective shape variable, is developed and used to describe the spectra of a variety of vibrational-like nuclei. The effect of the minimal length on the energy spectrum and the wave function is duly investigated. Numerical calculations are performed for some nuclei revealing a qualitative agreement with the available experimental data.

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1. Introduction

During the last decade, the models based on the concepts of critical point symmetries (CPS) related to shape phase transitions provide a very interesting theoretical framework for studies of nuclear structure phenomena. Actually, this interest has increased even more with the insertion of an additional critical point symmetries. Shape phase transitions have been first considered in the framework of the interacting boson model [1], which describes collective states of nuclei in terms of collective bosons of angular momentum zero (s-boson) and two (d-boson) in the context of a U(6) overall symmetry, having a dynamical U(5) (vibrational), SU(3) (prolate deformed rotational or axial rotor) and O(6) (γ -unstable) as limiting symmetries. Another important symmetries called E(5) [2] and X(5) [3], which approximate special solutions of the Bohr–Mottelson model [4] with an infinite-well potential and which were offered for the critical points of the shape phase transitions U(5) ↔ O(6) and U(5) ↔ SU(3) respectively, have been realized by Iachello. Later, a γ -rigid (with $\gamma = 0$) version of the critical symmetry X(5), called X(3) has been introduced in [5]. Other models considering the extension of X(3), such as X(3)- β^{2n} ($n = 1, 2$) [6]

and X(3)- β^6 [7] have also been developed not long ago. Besides, several additional attempts have been done to obtain solutions of the Bohr Hamiltonian with a constant mass parameter [8,9] as well as within the deformation dependent mass formalism [10,11].

Recently, a lot of attention has been attracted by the quantum mechanical problems implying a generalized modified commutation relations which include a minimal length or Generalized Uncertainty Principle (GUP). Such an important idea was motivated by noncommutative geometry [12,13] in the quantum gravity [14–16] and the string theory context [17–19]. However, the concept of minimal length can be incorporated in the study of physical systems by considering the deformed canonical commutation relation,

$$[X, P] = i\hbar \left(1 + \alpha^2 P^2\right) \quad (1)$$

here α represents the minimal length parameter (a very small positive parameter). This commutation relation leads to the uncertainty relation

$$\Delta X \Delta P \geq \frac{\hbar}{2} \left(1 + \alpha (\Delta P)^2\right) \quad (2)$$

which implies the existence of a minimal length given by $(\Delta X)_{\min} = \hbar\sqrt{\alpha}$. It should be noted that, since the elaboration of the fundamental principles of the quantum mechanics with GUP in [20–23], a much development, in this direction, has been accomplished in order to study the effect of the minimal length

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on quantum systems as well as on classical ones. Nevertheless, only few problems are shown to be solved exactly or approximately. Among them one can cite the Schrodinger equation for: the harmonic oscillator [24], the hydrogen atom [25–29], the inverse square potential [30], the scattering problem by Yukawa and Coulomb potentials [31] and square well potential [32]. In this Letter, we study a γ -rigid version of the Bohr–Mottelson Hamiltonian, within an infinite square well potential in β collective shape variable as X(3) model, in the presence of a minimal length. Particularly, we investigate the effect of a minimal length on the physical observables such as energy spectrum and eigenfunctions as well as B(E2) electromagnetic transition rates.

2. Minimal length formalism

The theoretical background of minimal length formalism (MLF) motivated by a Heisenberg algebra and implying a generalized uncertainty principle (GUP) has been considered recently in [24,32]. In the framework of this formalism, the generalization of the deformed canonical commutation relation (1) is given by [24,32]

$$[\hat{X}_i, \hat{P}_j] = i\hbar (\delta_{ij} + \alpha \hat{P}^2 \delta_{ij} + \alpha' \hat{P}_i \hat{P}_j) \quad (3)$$

where α' is an additional parameter which is of the order of α . In this case, the components of the momentum operator commute to one another

$$[\hat{P}_i, \hat{P}_j] = 0 \quad (4)$$

However, the commutator between two position operators is in general different from zero

$$[\hat{X}_i, \hat{X}_j] = i\hbar \frac{(2\alpha - \alpha') + (2\alpha + \alpha')\alpha \hat{P}^2}{1 + \alpha \hat{P}^2} (\hat{P}_i \hat{X}_j - \hat{P}_j \hat{X}_i) \quad (5)$$

It is clear that the generalized canonical commutation relation (3) leads to the minimal observable length $(\Delta X_i)_{min} = \hbar\sqrt{3\alpha + \alpha'}$. In the same context, we have different representations for the canonical operators X_i and P_i . Among these representations, one can cite the momentum space representation [24]:

$$\hat{X}_i = i\hbar \left[(1 + \alpha p^2) \frac{\partial}{\partial p_i} + \alpha' p_i p_j \frac{\partial}{\partial p_j} \right] + \eta p_i, \quad \hat{P}_i = p_i \quad (6)$$

and the position representation given by [26,27]:

$$\hat{X}_i = \hat{x}_i + \frac{(2\alpha - \alpha')(\hat{p}^2 \hat{x}_i + \hat{x}_i \hat{p}^2)}{4}, \quad \hat{P}_i = \hat{p}_i \left(1 + \frac{\alpha'}{2} \hat{p}^2 \right) \quad (7)$$

where \hat{x}_i and \hat{p}_i are the usual position and momentum operators respectively, which obey the following relations $[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}$ and $\hat{p}^2 = \sum_i \hat{p}_i^2$. Note that in the case of $\alpha' = 2\alpha$, and for the first order on α , the following canonical commutator $[\hat{X}_i, \hat{X}_j]$ vanishes. As a consequence, Eq. (7) reduces to

$$\hat{X}_i = \hat{x}_i, \quad \hat{P}_i = \left(1 + \alpha \hat{p}^2 \right) \hat{p}_i \quad (8)$$

In addition, we can interpret p_i and P_i shown in Eq. (8) according to string theory: p_i is the momentum operator at low energies and P_i is the momentum operator at high energies. Moreover, p is the magnitude of the p_i vector.

3. Bohr–Mottelson model with a minimal length

In the context of the collective geometrical model of Bohr–Mottelson [4], the classical expression for the rigid-body kinetic

energy associated with the rotation and surface deformations of a nucleus has the form [4,33]

$$\hat{T} = \frac{1}{2} \sum_{k=1}^3 \mathcal{J}_k \omega_k^2 + \frac{B_m}{2} (\beta^2 + \beta^2 \dot{\gamma}^2), \quad (9)$$

where β and γ are the usual collective variables, B_m is the mass parameter. Also,

$$\mathcal{J}_k = 4B_m \beta^2 \sin^2(\gamma - \frac{2}{3}\pi k) \quad (10)$$

are the three principal irrotational moments of inertia, and ω'_k ($k = 1, 2, 3$) are the components of the angular velocity (angular frequencies) on the body-fixed k -axes, which can be expressed in terms of the time derivatives of the Euler angles ϕ, θ, ψ [33,34]

$$\begin{aligned} \omega'_1 &= -\sin\theta \cos\psi \dot{\phi} + \sin\psi \dot{\theta}, \\ \omega'_2 &= \sin\theta \sin\psi \dot{\phi} + \cos\psi \dot{\theta}, \\ \omega'_3 &= \cos\theta \dot{\phi} + \dot{\psi}. \end{aligned} \quad (11)$$

Going further, by assuming the nucleus to be γ -rigid (i.e. $\dot{\gamma} = 0$), as a non-adiabatic approach proposed by Davydov and Chaban in [35], and considering in particular the axially symmetric prolate case of $\gamma = 0$, we see that the third irrotational moment of inertia \mathcal{J}_3 vanishes, while the other two become equal $\mathcal{J}_1 = \mathcal{J}_2 = 3B_m \beta^2$, thus the kinetic energy of Eq. (9) is simply [33,36]

$$\begin{aligned} \hat{T} &= \frac{3}{2} B_m \beta^2 (\omega_1^2 + \omega_2^2) + \frac{B_m}{2} \dot{\beta}^2 \\ &= \frac{B_m}{2} \left[3\beta^2 (\sin^2\theta \dot{\phi}^2 + \dot{\theta}^2) + \dot{\beta}^2 \right]. \end{aligned} \quad (12)$$

Since in the case of axial symmetry the nucleus can rotate only about directions perpendicular to the symmetry axis, the collective motions in the nucleus are characterized by only three degrees of freedom: $q_1 = \phi$, $q_2 = \theta$, and $q_3 = \beta$. Having in mind the position space representation (8), the kinetic energy operator, in this case, can be expressed in terms of the Laplacian and bi-Laplacian operators as follows

$$\begin{aligned} T &= -\frac{\hbar^2}{2B_m} \left(1 - 2\alpha\hbar^2 \frac{1}{\sqrt{g}} \sum_{ij} \frac{\partial}{\partial q_i} \sqrt{g} g_{ij}^{-1} \frac{\partial}{\partial q_j} \right) \\ &\times \left[\frac{1}{\sqrt{g}} \sum_{ij} \frac{\partial}{\partial q_i} \sqrt{g} g_{ij}^{-1} \frac{\partial}{\partial q_j} \right] \end{aligned} \quad (13)$$

where the matrix g_{ij} having a diagonal form

$$g_{ij} = \begin{pmatrix} 3\beta^2 \sin^2\theta & 0 & 0 \\ 0 & 3\beta^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (14)$$

where g is the determinant of the matrix g_{ij} and g_{ij}^{-1} is the inverse matrix of g_{ij} . Using the general procedure of quantization (Pauli–Podolsky prescription) in curvilinear coordinates, we obtain, in compact form, the collective Hamiltonian operator, up to the first order of α ,

$$\hat{H} = -\frac{\hbar^2}{2B_m} \Delta + \frac{\alpha\hbar^4}{B_m} \Delta^2 + V(\beta) \quad (15)$$

with

$$\Delta = \left[\frac{1}{\beta^2} \frac{\partial}{\partial \beta} \beta^2 \frac{\partial}{\partial \beta} + \frac{1}{3\beta^2} \Delta_\Omega \right] \quad (16)$$

where Δ_Ω is the angular part of the Laplace operator

$$\Delta_{\Omega} = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}. \quad (17)$$

The corresponding deformed Schrödinger equation to the first order on α reads as

$$\left[-\frac{\hbar^2}{2B_m} \Delta + \frac{\alpha \hbar^4}{B_m} \Delta^2 + V(\beta) - E \right] \Psi(\beta, \theta, \phi) = 0 \quad (18)$$

which is a second order differential equation. In addition, we can see, here, that is difficult to obtain analytic solution of this differential equation, because of the bi-Laplacian $\Delta^2 \propto p^4$. However, we can get rid of the term Δ^2 in equation (18) by introducing an auxiliary wave function Φ as in [32], so that

$$\Psi(\beta, \theta, \phi) = \left[1 - 2\alpha \hbar^2 \Delta \right] \Phi(\beta, \theta, \phi) \quad (19)$$

Thus, we obtain the following differential equation satisfied by Φ ,

$$\left[(1 + 4B_m \alpha (E - V(\beta))) \Delta + \frac{2B_m}{\hbar^2} (E - V(\beta)) \right] \Phi(\beta, \theta, \phi) = 0 \quad (20)$$

where Δ is defined by Eq. (16). The latter equation can be solved by using the usual following factorization

$$\Phi(\beta, \theta, \phi) = F_{n_{\beta}}(\beta) Y_{LM}(\theta, \phi), \quad (21)$$

where $Y_{LM}(\theta, \phi)$ are the spherical harmonics. Then the angular part leads to the equation

$$\Delta_{\Omega} Y_{LM}(\theta, \phi) = -L(L+1) Y_{LM}(\theta, \phi), \quad (22)$$

where L is the angular momentum quantum number, while the radial part $F(\beta)$ obeys to:

$$\left[\frac{1}{\beta^2} \frac{d}{d\beta} \beta^2 \frac{d}{d\beta} - \frac{L(L+1)}{3\beta^2} + \frac{2B}{\hbar^2} \bar{K}(E, \beta) \right] F_{n_{\beta}}(\beta) = 0. \quad (23)$$

with

$$\bar{K}(E, \beta) = \left(\frac{E - V(\beta)}{1 + 4B_m \alpha (E - V(\beta))} \right) \quad (24)$$

and n_{β} is the radial quantum number. Eq. (23) is an effective Schrödinger equation including the minimal length. It should be noticed that in the limit $\alpha \rightarrow 0$, Eq. (23) reduces to the ordinary collective Schrödinger equation [5–7].

In what concerns the β degree of freedom, we will consider here an anharmonic behavior reflected into an infinite square well shape of the potential as in the case of X(3) symmetry [5]:

$$V(\beta) = \begin{cases} 0, & \text{if } \beta \leq \beta_{\omega} \\ \infty, & \text{if } \beta > \beta_{\omega} \end{cases}, \quad (25)$$

where β_{ω} indicates the width of the well. In this case the wave function $F(\beta)$ is a solution of the equation

$$\left[\frac{d^2}{d\beta^2} + \frac{2}{\beta} \frac{d}{d\beta} + \left(\bar{k} - \frac{L(L+1)}{3\beta^2} \right) \right] F_{n_{\beta}}(\beta) = 0 \quad (26)$$

in the interval $0 \leq \beta \leq \beta_{\omega}$, where we introduced the reduced energies

$$\varepsilon = \bar{k} = \frac{2B_m}{\hbar^2} \cdot \frac{E}{(1 + 4B_m \alpha E)} \quad (27)$$

while it vanishes outside. Substituting $F_{n_{\beta}}(\beta) = \beta^{-1/2} f_{n_{\beta}}(\beta)$ in Eq. (26), one obtains the Bessel equation

$$\left[\frac{d^2}{d\beta^2} + \frac{1}{\beta} \frac{d}{d\beta} + \left(\bar{k}^2 - \frac{\eta^2}{\beta^2} \right) \right] f_{n_{\beta}}(\beta) = 0, \quad (28)$$

with

$$\eta = \left(\frac{L(L+1)}{3} + \frac{1}{4} \right)^{\frac{1}{2}}, \quad (29)$$

and the boundary condition being $f_{n_{\beta}}(\beta_{\omega}) = 0$. The solution of Eq. (26), which is finite at $\beta = 0$, is then given by

$$F_{n_{\beta}}(\beta) = F_{s,L}(\beta) = N_{s,L} \beta^{-1/2} J_{\eta}(\bar{k}_{s,\eta} \beta), \quad s = n_{\beta} + 1 \quad (30)$$

with $\bar{k}_{s,\eta} = \chi_{s,\eta} / \beta_{\omega}$ and $\varepsilon_{s,\eta} = \bar{k}_{s,\eta}^2$, where $\chi_{s,\eta}$ is the s -th zero of the Bessel function of the first kind $J_{\eta}(\bar{k}_{s,\eta} \beta_{\omega})$. $N_{s,L}$ is a normalization constant to be determined later. The corresponding spectrum is then

$$E_{s,L} = \frac{\hbar^2}{2B_m} \times \frac{\bar{k}_{s,\eta}^2}{1 - 2\hbar^2 \alpha \bar{k}_{s,\eta}^2}, \quad \bar{k}_{s,\eta} = \frac{\chi_{s,\eta}}{\beta_{\omega}} \quad (31)$$

In the above equation, the term $2\hbar^2 \alpha \bar{k}_{s,\eta}^2$ is the correction due to the minimal length. Therefore, we conclude that the minimal length increases slightly the energy spectrum. In addition, the relative correction can be written as

$$\frac{\Delta E_{s,L}}{E_{s,L}^0} = \frac{2\hbar^2 \alpha \bar{k}_{s,\eta}^2}{1 - 2\hbar^2 \alpha \bar{k}_{s,\eta}^2} \quad (32)$$

where $E_{s,L}^0 = \lim_{\alpha \rightarrow 0} E_{s,L}$.

Essentially, the total wave function (19) can be written as

$$\begin{aligned} \Psi(\beta, \theta, \phi) &= \left[1 - 2\alpha \hbar^2 \Delta \right] \Phi(\beta, \theta, \phi) \\ &= \left(1 + 2\hbar^2 \alpha \bar{k}_{s,\eta}^2 \right) F_{n_{\beta}}(\beta) Y_{LM}(\theta, \phi) \end{aligned} \quad (33)$$

Finally, we have

$$\Psi(\beta, \theta, \phi) = N_{s,L} \left(1 + 2\hbar^2 \alpha \bar{k}_{s,\eta}^2 \right) \beta^{-1/2} J_{\eta}(\bar{k}_{s,\eta} \beta) Y_{LM}(\theta, \phi) \quad (34)$$

Using the normalization condition of this function, we easily obtain the factor $N_{s,L}$:

$$N_{s,L} = \frac{\sqrt{2}}{\beta_{\omega} J_{\eta+1}(\chi_{s,\eta}) \left(1 + 2\hbar^2 \alpha \bar{k}_{s,\eta}^2 \right)} \quad (35)$$

Having the analytical expression of the normalized wave function, one can readily compute the B(E2) transition probabilities. Nevertheless, it should be remarked that the full normalized wave function does not change by introducing the concept of minimal length. Therefore, the B(E2) transition probabilities, which are expressed as,

$$B(E2; sL \rightarrow s'L') = \frac{1}{2L+1} \left| \langle s'L' || T^{(E2)} || sL \rangle \right|^2 \quad (36)$$

also remain unchanged by this formalism and are similar to those obtained in [5] where $T_{\mu}^{(E2)} = t \beta \sqrt{\frac{4\pi}{5}} Y_{2\mu}(\theta, \phi)$ is the quadrupole operator for $\gamma = 0$ and t is a scaling factor. Here, some remarks concerning X(5) with a minimal length concept are worth to be mentioned:

- (1): notice that, in this case, the same Eq. (28) occurs, but with $\eta = \left(\frac{L(L+1)}{3} + \frac{9}{4} \right)^{1/2}$.

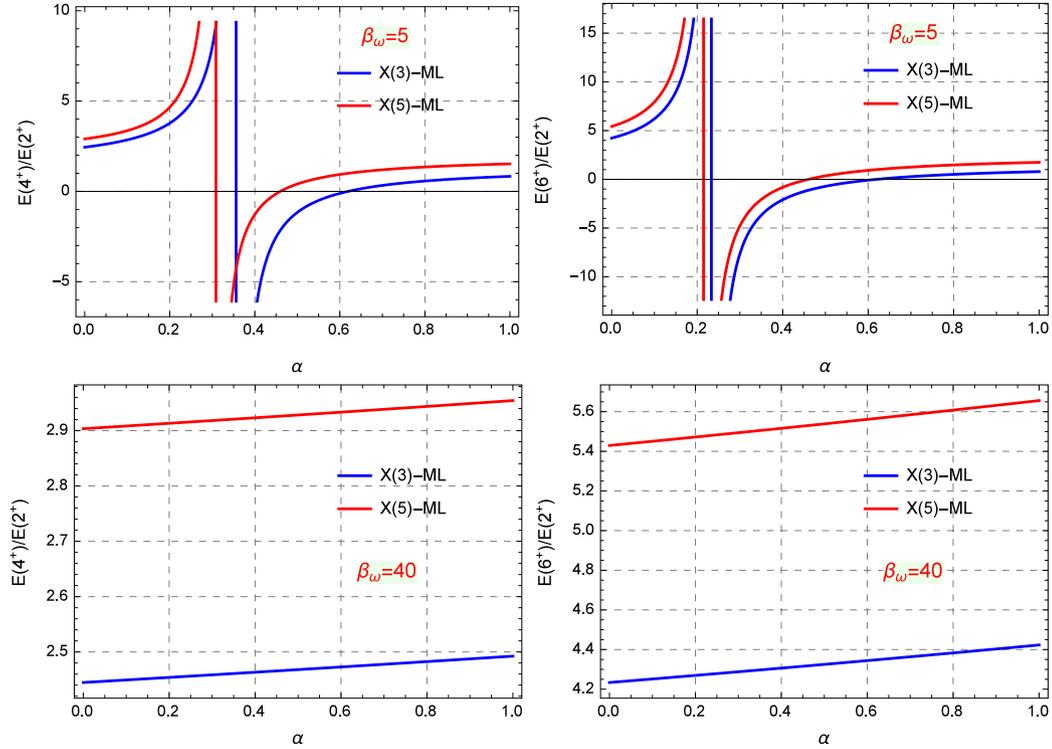


Fig. 1. The energy of the first 4^+ and 6^+ levels are plotted as function of the minimal length parameter α . (For interpretation of the colors in this figure, the reader is referred to the web version of this article.)

- (2): As in the case of X(3) model, the concept of minimal length has no effect on the $B(E2)$ transition probabilities of X(5).

Besides, from the requirement that the wave function be symmetric with respect to the perpendicular plan to the symmetry axis of the nucleus and passing through its center, it follows that only even values of the angular momentum L are allowed. Therefore no γ bands appear in the present models as expected, because the γ degree of freedom has been initially frozen to $\gamma = 0$.

4. Model applicability and numerical results

Because the γ degree of freedom has been frozen to $\gamma = 0$, the bands in the present models, like in X(3) model, are only classified by the principal quantum number n_β or $s = n_\beta + 1$. A few interesting low-lying bands are given as

- i) The energy levels of the ground state band with $s = 1$,
- ii) The β -vibrational bands with $s > 1$.

In order to avoid any ambiguity of the nomenclature between our models and the existing phenomenological models, namely: X(3) and X(5), we denote X(3)-ML and X(5)-ML in connection with X(3) and X(5) respectively. The proposed models have two free parameters, namely: the minimal length parameter α and the width of the infinite square well potential β_ω . Obviously, we do not count the mass parameter B_m since it disappears when calculating the energy ratios. However, according to the general form of the obtained energy spectrum, these parameters could be dependent from each other and check a constraint. Indeed, the energy spectrum corresponding to our models, where the effect of the minimal length is considered, is always positive $E_{s,L} \geq 0$ (this is also valid in the ordinary case i.e.: without a minimal length scenario). Due to this fact, we can write:

Table 1

Typical energy levels (ground state) of the X(3)-ML and X(5)-ML models, normalized to the 2^+_{g} excited state energy for different values of the parameter α with $\hbar = 1$.

L	X(3)-ML			L	X(5)-ML		
0^+	0.000	0.000	0.000	0^+	0.000	0.000	0.000
2^+	1.000	1.000	1.000	2^+	1.000	1.000	1.000
4^+	2.445	2.455	2.465	4^+	2.904	2.914	2.925
6^+	4.234	4.274	4.315	6^+	5.430	5.477	5.526
8^+	6.348	6.448	6.551	8^+	8.483	8.610	8.741
10^+	8.779	8.980	9.194	10^+	12.03	12.290	12.570
12^+	11.520	11.880	12.270	12^+	16.04	16.530	17.050
β_ω	-	60.0	60.0	β_ω	-	60.0	60.0
α	0	0.5	1	α	0	0.5	1

$$1 - 2\hbar^2 \alpha \bar{k}_{s,\eta}^2 > 0, \quad \bar{k}_{s,\eta} = \frac{\chi_{s,\eta}}{\beta_\omega} \quad (37)$$

which is a constraint between α and β_ω . From practical point of view, it is important to note that the value of α must be very small compared to the width of the well β_ω in order to preserve the mentioned above constraint. In Fig. 1, the energy of the first 4^+ and 6^+ levels, of X(3)-ML and X(5)-ML models, for two values of the width $\beta_\omega = 5$ and $\beta_\omega = 40$, are displayed as function of the minimal length parameter α in the interval $[0, 1]$. In the case of small value of $\beta_\omega = 5$, we see that, the energy ratios of the first 4^+ and 6^+ levels, for X(3)-ML, present a singularity nearby $\alpha = 0.3561$ and $\alpha = 0.2333$ respectively, because the condition (37), in this case, is not fulfilled. Likewise in the case of X(5)-ML, but in this time, the singularity occurs around the following values $\alpha = 0.3087$ and $\alpha = 0.2149$. While in the case of a large value of $\beta_\omega = 40$, where the above relationship is very well checked, the energy of the first 4^+ and 6^+ levels is very much influenced by α . In addition, Table 1 shows a typical energy levels of ground state of the X(3)-ML and X(5)-ML models, normalized to the 2^+_{g} excited state energy for $\beta_\omega = 60$. From this table, one can see the effect of the minimal length becomes manifest for higher values of the angular momentum. Indeed, such a fact, which re-

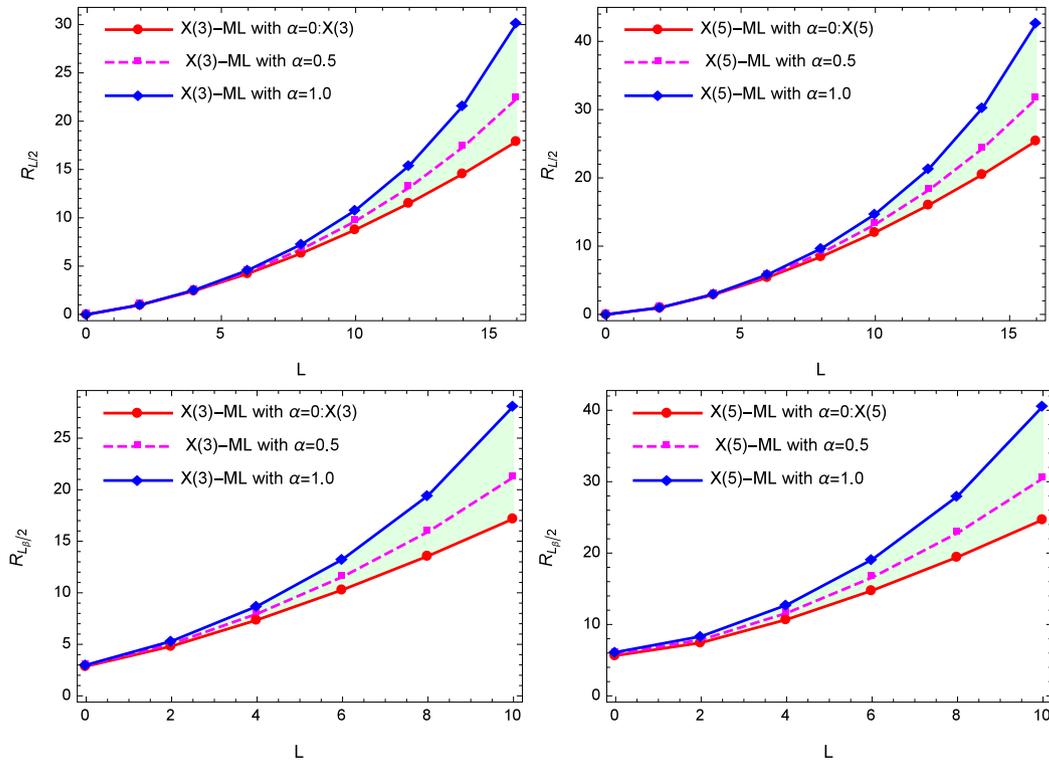


Fig. 2. The energy of the ground state and the β_1 band, normalized to the energy of the first excited state in the X(3)-ML and X(5)-ML models are plotted as function of angular momentum L for different values of the minimal length parameter α . The ground state is labeled by $R_{L/2}$, while β_1 band is labeled by $R_{L_{\beta_1}/2}$. The X(3) and X(5) predictions are also shown for comparison.

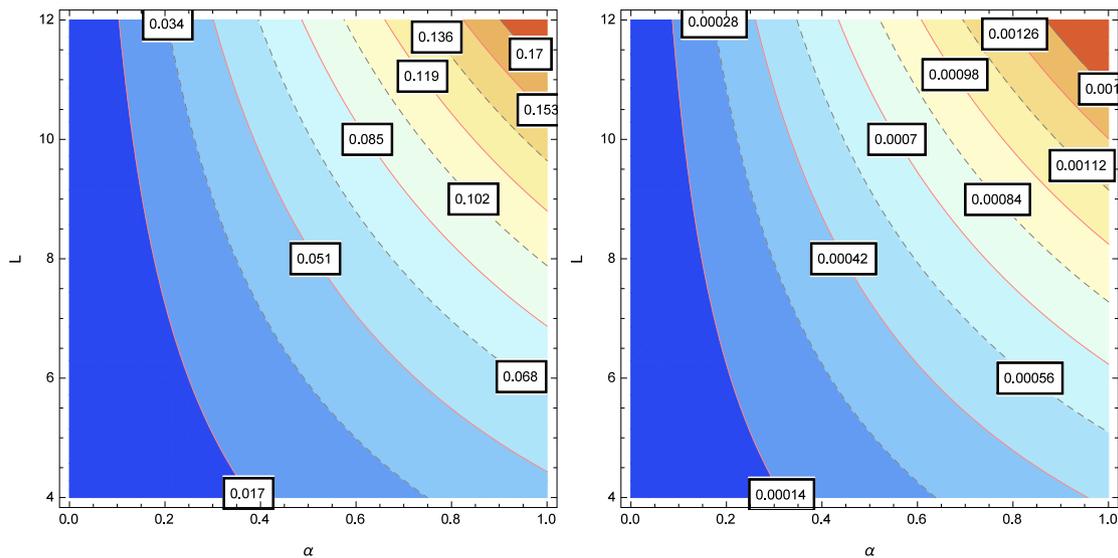


Fig. 3. Map contour lines of the relative correction (32) for the X(3)-ML model drawn as a function of the angular momentum L and the minimal length parameter α for $\beta_\omega = 40$ (left) and $\beta_\omega = 400$ (right).

sults from the uncertainty principle Eq. (2) as expected from string theory, is well illustrated schematically in Fig. 2 where the evolution of the energy spectrum of the ground state and the β_1 bands, normalized to the first 2^+ excited state, is presented. Furthermore, one can see that the effect of the minimal length is more important for the X(3) symmetry than for the X(5) one. Such an effect could be beneficial when trying to reproduce the experimental data for concrete nuclei in comparison, particularly, with the pure X(3) model as it can be seen subsequently. Moreover, from this figure one can see that the ground state band as well

as β_1 band are very much influenced by α for higher angular momentum. Besides, as is mentioned above, the minimal length effect increases slightly the energy spectrum. In Fig. 3, we present the variations of the relative correction of our model to the X(3) symmetry given by Eq. (32), as a function of the angular momentum L and the minimal length as well as the width β_ω . The map contour lines are lines with a constant relative correction. The area delimited by two successive contour lines represents the recovery rate of the X(3) symmetry by our model. From Fig. 3, one can see that in the vicinity of $\alpha \rightarrow 0$ and for lower values of the angu-

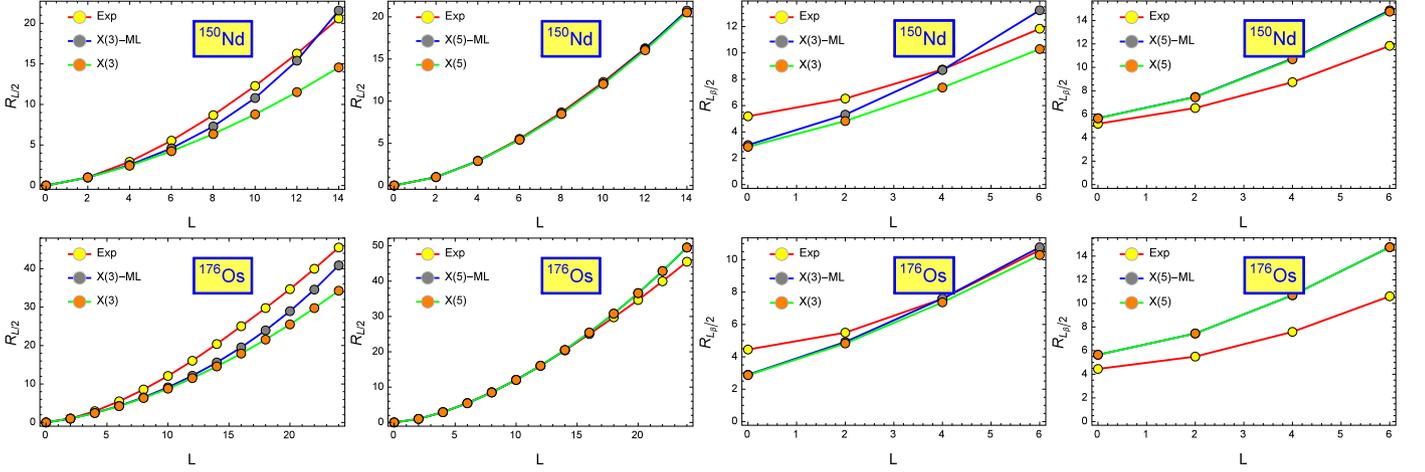


Fig. 4. Theoretical results for energy levels of the ground state and the β_1 -bands of the X(3)-ML and X(5)-ML models, compared with the available experimental data [37] for ^{150}Nd and ^{176}Os . The levels of each band are normalized to the 2_1^+ state. The ground state is labeled by $R_{L/2}$, while β_1 band is labeled by $R_{L\beta/2}$.

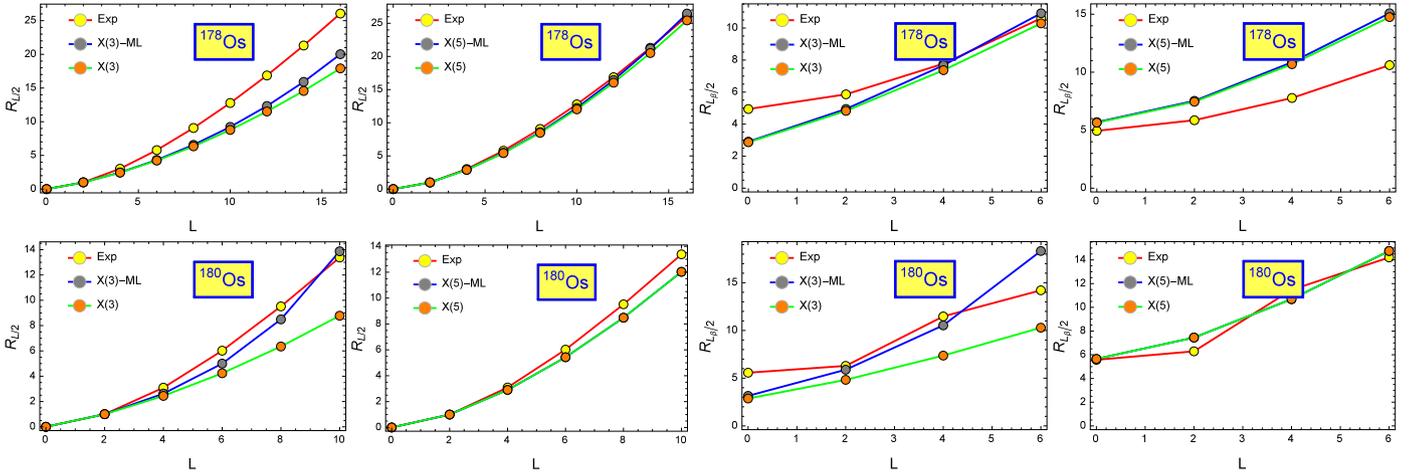


Fig. 5. Theoretical results for energy levels of the ground state and the β_1 -bands of the X(3)-ML and X(5)-ML models, compared with the available experimental data [37] for ^{178}Os and ^{180}Os . The levels of each band are normalized to the 2_1^+ state. The ground state is labeled by $R_{L/2}$, while β_1 band is labeled by $R_{L\beta/2}$.

lar momentum L , the recovery area is large. So, in this region our model is identical to the X(3) one. But, as one goes in the same given region to higher values of L , such an area narrows. Also, as α increases, the recovery area starts to contract. So, the gap between our model and the X(3) one increases, as it was mentioned above in the comment on Table 1. However, this gap between both models is worthwhile for ours insofar as it allows reproducing the experimental data, by our model, with a good precision in comparison with the pure X(3) model as it can be seen from Fig. 4, Fig. 5 and Fig. 6. In the right panel of Fig. 3, given for $\beta_\omega = 400$, we observe a similar behavior as in the left one for which $\beta_\omega = 40$ but with a bit more contracted recovery areas and lower values of the relative correction corresponding to the contour lines. This is due to the fact that for a deeper square well, the minimal length becomes smaller in concordance with the constraint (37). As a result, the models, developed here, allow to describe properties of nuclei having the signature $R_{4/2} = E(4_2^+)/E(2_2^+) \geq 2.44$, unlike the model developed in Ref. [6,7] which studies a few properties of nuclei having in this case the ratio $R_{4/2} < 2.44$. The experimental realization of the models was found to occur in some nuclei ^{150}Nd , $^{176-180}\text{Os}$, ^{156}Dy and ^{154}Gd , where the values of the used free parameters in the calculations are listed in Table 2. In Fig. 4, Fig. 5 and Fig. 6, we present the numerical results for the energy of the

ground state and the β_1 bands, normalized to the energy of the first excited level, obtained within X(3)-ML and X(5)-ML for these nuclei. From these figures, one can see the X(3)-ML model reproduces well the experimental data in comparison with the pure X(3) one in both bands. While, the X(5)-ML model is generally identical to the X(5) one and slightly better in the case of ^{150}Nd , ^{178}Os and ^{154}Gd nuclei. Such a difference, in the precision of predictions of both models: X(3)-ML and X(5)-ML, is due to the parameter η which enters in the zeros of the Bessel function $\chi_{s,\eta}$ and so defining the energy through the quantity $\bar{k}_{s,\eta}$ Eq. (31). Indeed, for a given L , the value of η is lower in the X(3)-ML than in the X(5)-ML. Hence, the minimal length formalism seems to be more suitable for studying γ -rigid nuclei in the frame of the X(3) symmetry.

5. Conclusion

In this work, we have derived new solutions of the Bohr-Mottelson Hamiltonian in the γ -rigid regime within the minimal length formalism which emerges in many higher dimension theories of quantum physics. The recall potential of the collective β -vibrations is assumed to be equal to an infinite square well as in the standard X(3) and X(5) models. So, improved versions of the X(3) and X(5) symmetries being called X(3)-ML and X(5)-ML are

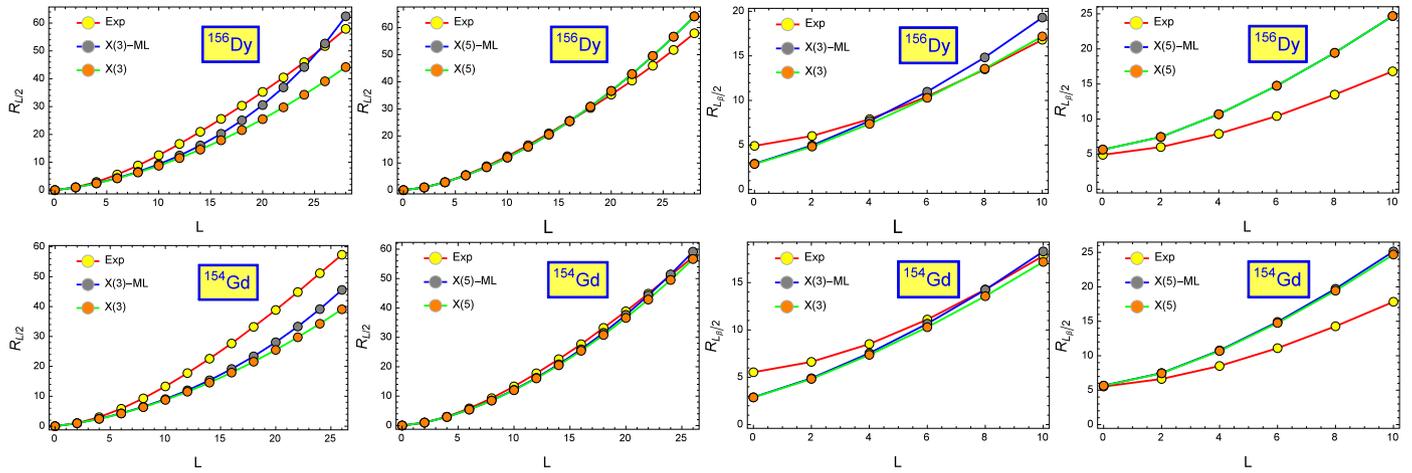


Fig. 6. Theoretical results for energy levels of the ground state and the β_1 -bands of the X(3)-ML and X(5)-ML models, compared with the available experimental data [37] for ^{156}Dy and ^{154}Gd . The levels of each band are normalized to the 2_1^+ state. The ground state is labeled by $R_{L/2}$, while β_1 band is labeled by $R_{L\beta_1/2}$.

Table 2

The values of the free parameters used in the calculations.

Models	X(3)-ML		X(5)-ML	
Nucleus	α	β_ω	α	β_ω
^{150}Nd	0.961	29.446	0.184	67.308
^{176}Os	0.421	42.517	0.000	64.670
^{178}Os	0.444	38.575	0.649	75.614
^{180}Os	0.999	21.858	0.000	56.102
^{156}Dy	0.833	50.763	0.000	95.399
^{154}Gd	0.654	60.299	0.233	65.648

elaborated. Indeed, we have shown, through this work, that the introduction of the minimal length formalism allows one to enhance the numerical calculation precision of physical observables, particularly the energy spectrum of nuclei in comparison with the X(3) and X(5) models. These later could be easily recovered by taking a null minimal length solutions of our models: X(3)-ML and X(5)-ML.

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