RECENT ACHIEVEMENTS OF AXIOMATIC FIELD THEORY

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INTRODUCTION

These lecture notes are an attempt to describe something of what has been achieved in so-called axiomatic field theory in the last couple of years with the emphasis on those results which are particularly neat.

Two significant projects currently under way which probably are very deep and certainly are very difficult will not be mentioned: Symanzik's structure analysis and the pursuit of the so-called "linear programme" by Källén and others. Fortunately, these are excellently summarized in [43].

The paper is divided into two parts. The results presented in the first half are characterized by the fact that, once one has had the proper insight, they can be proved with a few simple manipulations. In the second part there is a steep rise in the difficulty of the analysis.

No attempt will be made to rationalize the rather mathematical pre-occupations of these lectures; for one reason, the author has tried it before [1]. The root-mean-square deviation from the mean of opinion on what is a sensible thing to try to do in elementary particle theory seems to be one of those unrenormalizable infinities one hears about.

Of all the work reported, the most significant seems Borcher's discovery of equivalence classes of local fields and Ruelle's rigorization of Haag's collision theory. The first was totally unsuspected and represents the kind of insight which is indispensable if one is ever going to be able to get back to calculating cross-sections in relativistic quantum field theory. The second shows that in relativistic quantum field theory the collision theory (or asymptotic particle description) is already uniquely determined by the fields, a result which accords with one's physical intuition and supplies strong evidence that axiomatic field theory is on the right track.

PART ONE

This first part will describe a number of results which have simplicity and generality in common. All mathematical technicalities will be deferred to Part 2.

1.1. RECOLLECTION OF THE PCT THEOREM

The PCT theorem will be used again and again in the course of this paper so it will be presented here briefly in the form given by JOST [1].

If $A(x)$ is a charged scalar field, its transform under the PCT operation is $A(-x)^*$. The anti-unitary operator Θ on the states which generates this transformation of the fields therefore satisfies

$$\Theta A(x)\Theta^{-1} = A(-x)^* . \tag{1}$$

(A charged rather than a neutral scalar field will be considered temporarily to bring out the rôle of the Hermitian adjoint in the definition of PCT.)
In any theory of a field (or a denumerable set of fields) that has the vacuum Ψ_0 as cyclic vector (i.e. for which polynomials in the smeared fields $\mathcal{P}(A(g)\ldots)$ applied to the vacuum Ψ_0 yield a dense set in $\mathcal{H}$ the Hilbert space of states), (1) is equivalent to an identity of the vacuum expectation values:

$$(\Psi_0, A_1(x_1)\ldots A_n(x_n)\Psi_0) = [(\Psi_0, A_1(-x)^* \ldots A_n(-x_n)^* \Psi_0)]^* \tag{2}$$

or equivalently:

$$(\Psi_0, A_1(x_1)\ldots A_n(x_n)\Psi_0) = (\Psi_0, A_n(-x_n)\ldots A_1(-x_1)\Psi_0). \tag{3}$$

This reduces the problem of determining whether a theory has PCT symmetry to an examination of its vacuum expectation values. If (3) or equivalently (2) holds for all $x_1 \ldots x_n$, we say the n-fold vacuum expectation value has PCT symmetry. On the other hand, from the Lorentz invariance of the field

$$U(a,\Lambda)A(x)\,U(a,\Lambda)^{-1} = A(\Lambda x + a)\,, \tag{4}$$

the vacuum expectation values satisfy

$$(\Psi_0, A_1(\Lambda x_1 + a)\ldots A_n(\Lambda x_n + a)\Psi_0) = (\Psi_0, A_1(x_1)\ldots A_n(x_n)\Psi_0). \tag{5}$$

(Only invariance under restricted Lorentz transformations $\det \Lambda = 1$, $\operatorname{sgn} \Lambda^0_0 = 1$, is assumed.)

From this and the spectral condition it follows that

$$(\Psi_0, A_1(x_1)\ldots A_n(x_n)\Psi_0)$$

$$= \int [\exp -i \sum_{j=1}^{n-1} p_j(x_j - x_{j+1})]\, G^{A_1\ldots A_n}(p_1,\ldots p_{n-1})dp_1 \ldots dp_{n-1}\,, \tag{6}$$

where $G^{A_1\ldots A_n}$ vanishes for $p_1 \ldots p_{n-1}$ outside the physical spectrum which must be in the future light cone. From this in turn it follows that there is an analytic function $F^{A_1\ldots A_n}$ of n-1 complex from vector variables,

$$z_j = (x_j - x_{j+1}) - i\,\eta_j \text{ (where } j = 1, 2, \ldots n-1)\,,$$

$$F^{A_1\ldots A_n}(z_1,\ldots z_{n-1}) = \int [\exp(-i \sum_{j=1}^{n-1} p_j z_j)]\, G^{A_1\ldots A_n}(p_1,\ldots p_{n-1})dp_1 \ldots dp_{n-1}, \tag{7}$$

analytic in the tube, $\mathfrak{J}_{n-1}$, which is the set of $z_1, \ldots z_{n-1}$ for which $\eta_j \in V_+$, the future light cone for $j=1, \ldots n-1$ and such that

$$(\Psi_0, A_1(x_1) \ldots A_n(x_n)\Psi_0) = \lim_{\substack{\eta_1 \ldots \eta_{n-1} \to 0 \\ \text{in } V_+}} F^{A_1 \ldots A_n}(z_1, \ldots z_{n-1}). \tag{8}$$

$F^{A_1 \ldots A_n}$ is also Lorentz invariant

$$F^{A_1 \ldots A_n}(z_1, \ldots z_{n-1}) = F^{A_1 \ldots A_n}(\Lambda z_1, \ldots \Lambda z_{n-1}), \tag{9}$$

which implies that $F^{A_1 \ldots A_n}$ possesses a single-valued continuation to the extended tube $\mathfrak{J}'_{n-1}$, which consists of all points of the form $\Lambda z_1, \ldots \Lambda z_{n-1}$ with Λ a complex Lorentz transformation of determinant one and

$$z_1, \ldots z_{n-1} \in \mathfrak{J}_{n-1}.$$

In particular,

$$F^{A_1 \ldots A_n}(z_1, \ldots z_{n-1}) = F^{A_1 \ldots A_n}(-z_1, \ldots -z_{n-1}) \tag{10}$$

at each point of $\mathfrak{J}'_{n-1}$. Finally, it should be remembered that the extended tube contains real points, the so-called Jost points; $\xi_1 \ldots \xi_{n-1}$ is a Jost point if it is real and $\sum_{j=1}^{n-1} \lambda_j \xi_j$ is space-like for all $\lambda_j (j=1, \ldots n-1)$ such that

$$\lambda \geq 0 \text{ and } \sum_{j=1}^{n-1} \lambda_j > 0. \tag{11}$$

PCT Theorem

If W(eak) L(ocal) C(ommutativity)

$$(\Psi_0, A_1(x_1) \ldots A(x_n)\Psi_0) = (\Psi_0, A_n(x_n) \ldots A_1(x_1)\Psi_0) \tag{12}$$

holds for $x_1, \ldots x_n$ such that $x_1 - x_2, \ldots x_{n-1} - x_n$ fill a real neighbourhood of a Jost point, then (3) holds for all $x_1, \ldots x_n$ and the n-fold vacuum expectation value has PCT symmetry.

Conversely, if the n-fold vacuum expectation value has PCT symmetry, then WLC holds in the neighbourhood of every Jost point.

Proof

If WLC holds in the neighbourhood of the Jost point $x_1 - x_2, \ldots x_{n-1} - x_n$, then

$$F^{A_1 \ldots A_n}(z_1, \ldots z_{n-1}) = F^{A_n \ldots A_1}(-z_{n-1}, \ldots -z_1) \tag{13}$$

in an open set of real space. Therefore, the analytic functions on the left-hand side and right-hand side coincide throughout $\mathfrak{J}'_{n-1}$, using the fact that two functions analytic in an open set of complex space and coinciding on a real subset which is open in the real subspace coincide everywhere.

Using (10), this says:

$$F^{A_1 \ldots A_n}(z_1, \ldots z_{n-1}) = F^{A_n \ldots A_1}(z_{n-1}, \ldots z_1) \tag{14}$$

throughout $\mathcal{J}'_{n-1}$. (Note that if $z_1, \ldots z_{n-1}$ is a Jost point so is $-z_{n-1}, \ldots -z$ and if $z_1 \ldots z_{n-1} \in \mathcal{J}'_{n-1}$ then $z_{n-1}, \ldots z_1 \in \mathcal{J}'_{n-1}$.) Passing to the boundary values with $\eta_j \in V_+$, one gets

$$(\Psi_0, A_1(x_1) \ldots A_n(x_n)\Psi_0) = (\Psi_0, A_n(-x_n) \ldots A_1(-x_1)\Psi_0) \tag{15}$$

for all $x_1 \ldots x_n$, which is PCT symmetry.

Conversely, suppose (15) holds for all $x_1 \ldots x_n$, then it holds for a real neighbourhood of a Jost point. Then (14) and (13) follow at every real point of analyticity, and that is exactly WLC at every Jost point.

Of course, WLC is implied by LC:

$$[A_i(x), A_j(y)] = [A_i(x), A_j(y)^*] = 0 \tag{16}$$

It is important in applications that the PCT operator of an irreducible set of fields is essentially uniquely determined [3].

If $$\Theta_1 A_j(x) \Theta_1^{-1} = A_j(-x)^*$$

and $$\Theta_2 A_j(x) \Theta_2^{-1} = A_j(-x)^*,$$

then $$\Theta_2 \Theta_1 A_j(x) \Theta_1^{-1} \Theta_2^{-1} = A_j(x) \tag{17}$$

so by the irreducibility of $A_j(x)$,

$$\Theta_2 \Theta_1 = \lambda \mathbb{1} . \tag{18}$$

Now because $(PCT)^2 = 1$, $\Theta^2 = \mu \mathbb{1}$ with $|\mu| = 1$. (A priori Θ_j^2 need only be constant in each coherent subspace of states, i.e. states not separated by super selection rules. But (17) implies $[\Theta_1^2, A_j(x)] = 0$, so, by the irreducibility of A_j, $\Theta_1^2 = \mu \mathbb{1}$. If one had a more complicated transformation law, say that for appropriate two-component scalar field, it could be arranged to have $[\Theta^2, A_j(x)]_+ = 0$, then one would have $\Theta^2 = +1$ on states obtained from the vacuum by applying an even number of $A_{j\alpha}$ and -1 on those obtained by apply ing an odd number to the vacuum. In that case Θ^2 generates a super selection rule. While these applications have an interest of their own they will not be pursued here.) For anti-unitary operators $\Theta_j^2 = \mu \mathbb{1}$ with $|\mu| = 1$ implies $\mu = \pm 1$, $(\Theta(\Theta\Theta) = \Theta(\mu 1) = (\Theta\Theta)\Theta = \mu\Theta$ so μ is real and therefore $= \pm 1)$; thus (18) implies

$$\Theta_1^2 = |\lambda|^2 \Theta_2^{-2}$$

so $|\lambda|^2 = 1$ and Θ_1 and Θ_2 differ only by a phase factor. It is customary to fix this phase factor so that

$$\Theta \Psi_0 = \Psi_0 \ . \tag{19}$$

Then Θ is unique. That the left-hand side of (19) must be proportional to the right follows from a comparison of the transformation law (4) of A_j under $U(a, \Lambda)$ and (1) under Θ. One immediately deduces that $\Theta^{-1}U(a,\Lambda)^{-1}\bar{\Theta}U(-a,\Lambda)$ commutes with A_j so

$$\omega(a,\Lambda)U(a,\Lambda) = \Theta U(-a,\Lambda)\Theta^{-1}(\text{where } |\omega| = 1),$$

and since the inhomogeneous Lorentz group possesses no one-dimensional representations, $\omega = 1$.

$$U(a,\Lambda) = \Theta U(-a,\Lambda)\Theta^{-1} \ . \tag{20}$$

Thus the energy momentum operator satisfies

$$P^\mu = \Theta P^\mu \Theta^{-1} \ . \tag{21}$$

The anti-unitary character of Θ is essential here; if Θ were unitary, (21) would have a minus sign and negative energy states would exist. Finally, (21) and the convention (19) imply that $\Theta\Psi_0 = \Psi_0$. The essential point is that the algebraic structure of the set of field operators, as displayed in the symmetries of their vacuum expectation values, uniquely determines a Θ and a transformation law of the fields under Θ.

The relation of Θ to scattering theory is very simple:

$$\Theta A^{in}(x)\Theta^{-1} = A^{out}(-x)^* \ . \tag{22}$$

This is easy to see if one has a theory in which the simple form of the asymptotic condition is valid.

$$A^{in}(x) = A(x) - \int \Delta_R(x-y)\, j(y)\, dy \ ,$$

$$\begin{aligned} \Theta A^{in}(x)\Theta^{-1} &= A(-x)^* - \int \Delta_R(x-y)\, j(-y)^* dy \\ &= A(-x)^* - \int \Delta_A(x-y)\, j(y)^* dy \\ &= A^{out}(-x)^* \end{aligned}$$

because

$$\Delta_A(-x) = \Delta_R(x) \ .$$

(22) is still true in the most general scattering theory we know where the correspondence between A_j and A_j^{in} need not be one to one. This will be discussed below.

It is clear from (22) that Θ is <u>not</u> the PCT operator for $A^{out}(x)$; by the PCT theorem there must be another anti-unitary operator U satisfying

$$UA_j^{out}(x)\,U^{-1} = A_j^{out}(-x)^*\tag{23}$$

because A^{in} is local and the A_j^{in} are irreducible, which we assume for the collision states to be complete. Now we know a unitary operator, the S operator, which satisfies

$$A^{out}(x) = S^{-1}A^{in}(x)\,S\,. \tag{24}$$

By comparing (24), (23) and (22) and using the familiar argument above, we get: $\Theta^{-1}\,U = S$. The collision operator is the relative PCT transformation of the basic fields A_j and the out fields A_j^{out} . It is clear from this that one can define a "relative S operator" of two fields even when they do not satisfy the asymptotic condition $S_{AB} = \Theta_A\Theta_B$*.

1.2. THE TRANSITIVITY OF WLC AND LC; EQUIVALENCE CLASSES OF LOCAL FIELDS [3]

One of the most striking recent discoveries in quantum field theory was made by Borchers. Roughly, it says (a) that if A is an irreducible field which is LC and B is LC relative to A, i.e.

$$[A(x), B(y)] = 0 \text{ for } (x^2 - y^2) < 0\ , \tag{25}$$

then B is LC; and (b) if A is irreducible and is LC and B and C are LC relative to A then B is LC relative to C. This shows that local fields fall into equivalence classes (also called Borchers classes), two being equivalent if they are relatively local. Similar statements hold for WLC. Finally, Borchers showed [3] that if two fields lie in the same equivalence class and satisfy the LSZ asymptotic condition they have the same S operator. He also shows that if the fields are A and B, $A^{in} = \pm B^{in}$. This shows that in order to get theories with a non-trivial S operator one must use fields outside the equivalence class of any free field. It should be emphasized that each member of equivalence classes of fields acts in the same Hilbert space and has the same representation of the inhomogeneous Lorentz group. Two free fields of different mass are not comparable in this classification. It remains an open question whether there are Borchers classes, other than those of free fields, which have the same representation as free fields. Of course, there is nothing now known to prevent different Borchers classes from having the same S operator. In fact, this happens for free fields of the same mass which are not local relative to one another. Incidentally, it should also be emphasized that one can prove the required properties of the equivalence classes only by assuming that there is at least one irreducible field in the class. Thus B and C local relative to A need not imply B and C relatively local unless A is irreducible.

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* The simple but interesting remark that the S operator is a relative PCT transformation was made by SYMANZIK [4]. The relative S operator is definable even for models with a space-time containing a finite number of points [4a].

Before the proof of Borchers' result, an example of a Borchers class and an application of his theorems to prove the non-existence of solutions of certain theories will be given.

Example: the equivalence class of an irreducible free neutral scalar field

Denote the field A. Then $D^\alpha A$ is again a field (no longer scalar!) and LC with respect to A. Here

$$D^\alpha A(x) = \partial^{|\alpha|} A(x)/(\partial x^0)^{\alpha_0}\,\partial(x^1)^{\alpha_1}\ldots\partial(x^3)^{\alpha_3}, \tag{26}$$

where $|\alpha| = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$. Furthermore, the Wick ordered product $: D^\alpha A(x) D^\beta A(x) :$ is again a field and LC with respect to A. It is defined by

$$\lim_{x_1,\ldots x_\ell \to x} \; : D^{\alpha^{(1)}} A(x_1)\, D^{\alpha^{(2)}} A(x_2)\ldots D^{\alpha^{(\ell)}} A(x_\ell) :$$

$$= \sum_{r=0}^{[\ell/2]} (-1)^r \sum_{c_r} [D^{\alpha^{(j_1)}} A(x_{j_1})\ldots D^{\alpha^{(j_{2r})}} A(x_{j_{2r}})] \cdot D^{\alpha^{(k_1)}} A(x_{k_1})\ldots D^{\alpha^{(k_{\ell-2r})}} A(x_{k_{\ell-2r}}), \tag{27}$$

where $[\ell/2]$ is the largest integer less than $\ell/2$. The sum Σ_{c_r} is over all partitions of the integers ℓ into two subsets $j_1 \ldots j_{2r}$ and $k_1 \ldots k_{\ell-2r}$ satisfying $j_1 < j_2 \ldots < j_{2r}$ and $k_1 < k_2 \ldots k_{\ell-2r}$. The Hafnian $[\ldots\ldots]$ is defined by:

$$\left[D^{\alpha^{(j_1)}} A(x_{j_1})\ldots D^{\alpha^{(j_{2r})}} A(x_{j_{2r}})\right] = \sum_{\substack{k_1\ldots k_r\\ k_1'\ldots k_r'}} \prod_{s=1}^{2r} (\psi_0, D^{\alpha^{(k_s)}} A(x_{k_s}) \cdot D^{\alpha^{(k_s')}} A(x_{k_s'})\Psi_0),$$

where, here, the summation is over all partitions $(k_1, k_1') \ldots (k_r, k_r')$ of $j_r \ldots j_{2r}$ in disjoint subsets so that $k_s < k_s'$ $(s=1,\ldots r)$. Thus the equivalence class of the free field must include all invariant Wick polynomials of the form:

$$\sum_{n,\alpha,\beta} c_{n\alpha\beta} : D^\alpha A(x)\, D^\beta A(x):$$

where the indices on the derivatives are summed to give invariant combinations. For example,

$$\sum_{j=1}^{n} \alpha_j : A(x)^j : \quad , \quad : \frac{\partial}{\partial x^\mu} A(x) \frac{\partial}{\partial x_\mu} A(x) :$$

SCHROER [5] has shown recently that the invariant Wick polynomials exhaust the equivalence class of an irreducible neutral scalar free field of mass m. An obvious possibility,

$$\sum_{j=1}^{\infty} \alpha_j : A(x)^j : \tag{28}$$

with α_j decreasing very fast with j, is excluded because it describes a theory with an infinite number of subtractions*. This will be discussed in detail later.

The fact that the invariant Wick polynomial in a free field and its derivatives were then the only known examples of local fields suggested to the author some years ago [7] that one should try to use them as currents, i. e. to look for local solutions of,

$$A(x) = A^{in}(x) + \int \Delta_R(x-y)\, j(y)\, dy, \tag{29}$$

where j is an invariant Wick polynomial in a given free field $A^{(0)}(x)$. A^{in} is also a free field but a priori not in any way related to $A^{(0)}(x)$. EPSTEIN and the author have shown that there are no LC solutions in the special case $j(x) = g : A^{(0)}(x)^2$ [7]. ARAKI, HAAG and SCHROER [8] pointed out that when j is irreducible, Borchers' result enables one to give a very much more general and certainly neater discussion.

Theorem 1

If
$$(\Box + m^2)\, A(x) = j(x), \tag{30}$$

one of A and j is irreducible and A is LC, then A and j lie in the same equivalence class.

If j is an invariant Wick polynomial in a free field $A^{(0)}$ and its derivatives and is irreducible, S = 1. Furthermore, (29) has no non-trivial solutions unless j = 0 and $A = A^{in} = A^{(0)}$.

Proof

The first statement is an immediate consequence of Borchers' result. To obtain the second, note that first-degree Wick polynomials in $A^{(0)}$ are inadmissible in (29) because their retarded potentials do not exist. Because of the assumed irreducibility of j and the assumption that it is an invariant polynomial in $A^{(0)}$ and its derivatives A, $A^{(0)}$ and j lie in the same equivalence class. (A and $A^{(0)}$ are LC relative to an irreducible j. Therefore, A is LC relative to $A^{(0)}$.) Therefore, the "in" fields associated with A and $A^{(0)}$ coincide up to a sign $A^{in}(x) = \pm A^{(0)}(x)$, and the Borchers theorem implies S = 1. If A were local, this would imply

$$A^0(x) + \int \Delta_R(x-y)\, j(y)\, dy \tag{31}$$

is local which is impossible as a direct calculation shows. Sticklers for completeness can support this last step by the somewhat more general statement [9].

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* The fact that (28) does not satisfy the ordinary axioms of quantum field theory if an infinite number of α's $\neq 0$ was pointed out by GLASER [6]. It is a freak that this statement is true in three- and four-dimensional but not in two-dimensional space-time where such expressions occur in the Thirring model. That operator guage transformations give rise to such "unrenormalizable fields" was emphasized by KÄLLÉN [6].

Theorem 2

If A is a neutral scalar field of the form

$$A(x) = \sum_{n=1}^{N} \int dx_1 \ldots dx_n f_n(x, x_1 \ldots x_n) : \; A^0(x_1) \ldots A^{(0)}(x_n) : , \tag{32}$$

where $A^{(0)}$ is a free field and A is LC then A is an invariant Wick polynomial in $A^{(0)}$ and its derivatives.

Now let us consider the precise statement of the Borchers theorem and its proof. It comes in four parts, the first two relating to WLC and the second two to LC.

Theorem 3

Let A and B be neutral scalar fields but not necessarily LC. Suppose A irreducible and that A satisfies WLC. Then B satisfies WLC if the identities,

$$\begin{aligned}&(\Psi_0, A(x_1) \ldots A(x_j) B(y) A(x_{j+1}) \ldots A(x_n)\, \Psi_0)\\ &= (\Psi_0, A(x_n) \ldots A(x_{j+1}) B(y) A(x_j) \ldots A(x_1)\, \Psi_0),\end{aligned} \tag{33}$$

hold for $x_1 - x_2, \ldots x_{j-1} - x_j, x_j - y, y - x_{j+1}, \ldots x_{n-1} - x_n$ in a neighbourhood of some Jost point for each $n = 0, 1, \ldots$ and each j, $1 \leqslant j \leqslant n$. Furthermore, the PCT operator of B coincides with that of A so A and B together satisfy WLC.

Proof

Assume (33) holds. Because Θ is anti-unitary, one has:

$$(\Theta\, \Phi, \; \Theta B(z) \Theta^{-1} \Theta \Psi) = [(\Phi, B(z)\Psi)]^* = (\Psi, B(z)\Phi). \tag{34}$$

In particular, (34) holds for vectors of the form

$$\begin{aligned}\Phi &= \sum_k \int \ldots \int dx_1 \ldots dx_k f_k(x_1, \ldots x_k) A(x_1) \ldots A(x_k) \Psi_0\\ \Psi &= \sum_\ell \int \ldots \int dx_1 \ldots dx_\ell g_\ell(x_1, \ldots x_\ell) A(x_1) \ldots A(x_\ell) \Psi_0 \end{aligned} \tag{35}$$

for which

$$\begin{aligned}\Theta\Phi &= \sum_k \int \ldots \int dx_1 \ldots dx_k (f_k(x_1, \ldots x_k))^* A(-x_1) \ldots A(-x_k) \Psi_0\\ \Theta\Psi &= \sum_\ell \int \ldots \int dx_1 \ldots dx_\ell (g_\ell(x_1, \ldots x_\ell))^* A(-x_1) \ldots A(-x_\ell) \Psi_0 .\end{aligned}$$

The identities (33) in the neighbourhood of a Jost point imply the identities

$$(\Psi_0, A(x_1)\ldots A(x_j)B(y)A(x_{j+1})\ldots A(x_n)\Psi_0)$$
$$= (\Psi_0, A(-x_n)\ldots A(-x_{j+1})B(-y)A(-x_j)\ldots A(-x_1)\Psi_0) \tag{36}$$

for all $x_1 \ldots x_n$ and y. (The argument is that used in the proof of the PCT theorem.) Thus,

$$(\Psi, B(z)\Phi) = \sum_{k,\ell} \int\ldots\int dy_1\ldots dy_\ell\, dx_1\ldots dx_k \cdot [g_\ell(y_1,\ldots y_\ell)]^*$$
$$\cdot f_k(x_1\ldots x_k)(\psi_0, A(y_0)\ldots A(y_1)B(z)A(x_1)\ldots A(x_k)\psi_0)$$
$$= \sum_{k,\ell} \int\ldots\int dx_1\ldots dx_k\, dy_1\ldots dy_\ell\, f_k(x_1,\ldots x_k)$$
$$\cdot [g_\ell(y_1,\ldots y_\ell)]^*(\Psi_0, A(-x_k)\ldots A(-x_1)B(-z)A(-y_1)\ldots A(-y_\ell)\Psi_0)$$
$$= (\Theta\Phi, B(-z)\Theta\Psi). \tag{37}$$

Since by assumption states of the form (35) are dense in $\mathcal{H}$, (37) implies

$$\Theta B(z)\Theta^{-1} = B(-z); \tag{38}$$

i.e. B has a PCT operator which is the same as that of A. This implies the statements of the theorem.

It is worth noting that the last statement of the theorem is equivalent to the non-trivial result that the identities (33) linear in B imply the analogous identities with an arbitrary number of B's. When the identities (33) hold, we say, B is weakly local relative to A; or is WLC relative to A.

Theorem 4

Suppose A, B and C are WLC and A is irreducible. Let B be WLC relative to A and C be WLC relative to A, then B is WLC relative to C.

Proof

By theorem (3), A, B and C all have the same PCT operator, say Θ, which implies immediately B is WLC relative to C. In fact, it implies that A, B and C altogether are WLC.

Theorems 3 and 4 together establish a kind of weakened transitivity for WLC. Recall that a relation r is transitive if a r b and b r c implies a r c.

Theorem 5

If A is LC and irreducible and B is LC relative to A, i.e.

$$[A(x), B(y)] = 0$$

for space-like x-y, then B is LC.

This theorem is a special case of the following (take B = C).

Theorem 6

If A is LC and irreducible and B and C are each LC relative to A, then B is LC relative to C; i. e.

$$[B(x), C(y)] = 0$$

for space-like x-y.

Proof

By Theorem 4, A, B and C are together WLC. From this and the assumptions of the theorem one gets for any $x_1 \ldots x_n, y_1, y_2$ such that the set of successive difference vectors $(x_1-x_2, x_{j-1}-x_j, x_j-y_1, y_1-y_2, y_2-x_{j+1}, \ldots x_{n-1}-x_n$ is a Jost point

$$\begin{aligned}
&(\Psi_0, A(x_1)\ldots A(x_j)B(y_1)C(y_2)A(x_{j+1})\ldots A(x_n)\Psi_0)\\
&= (\Psi_0, A(x_n)\ldots A(x_{j+1})C(y_2)B(y_1)A(x_j)\ldots A(x_1)\Psi_0)\\
&= (\Psi_0, A(x_1)\ldots A(x_j)C(y_2)B(y_1)A(x_{j+1})\ldots A(x_n)\Psi_0) \qquad (39)
\end{aligned}$$

(the first step by WLC; the second by assumption).

Now the first and third expressions in (39) are boundary values of analytic functions, being

$$\lim_{\substack{\eta_1\ldots\eta_{n-1},\eta',\eta''\to 0\\ \text{in } V^+}} F^{(1)}([x_1-x_2-i\eta_1],\ldots[x_{j-1}-x_j-i\eta_{j-1}],[x_j-y_1-i\eta_j],$$

$$[y_1-y_2-i\eta'],\ [y_2-x_{j+1}-i\eta''],\ [x_{j+1}-x_{j+2}-i\eta_{j+1}]\ldots$$

$$[x_{n-1}-x_n-i\eta_{n-1}])$$

and

$$\lim_{\substack{\eta_1\ldots\eta_{n-1},\eta',\eta''\to 0\\ \text{in } V^+}} F^{(2)}([x_1-x_2-i\eta_1],\ldots[(x_j-y_1-i\eta_j)+(y_1-y_2-i\eta')],$$

$$-[y_1-y_2+i\eta''],\ [(y_1-y_2)-i\eta'+(y_2-x_{j+1}-i\eta'')]\ldots[x_{n-1}-x_n-i\eta_{n-1}]),$$

respectively. For the next step in the argument we use not the functions $F^{(1)}$ and $F^{(2)}$ but two functions derived from them by setting $\eta' = 0$ and smearing in (y_1-y_2) with a test function φ where support consists entirely of space-like vectors $f^{(j)} = \int\varphi(y_1-y_2)\,d(y_1-y_2)\,F^j(\ldots, y_1-y_2,\ldots)$. The $f^{(j)}$ are then analytic in $\mathfrak{J}_n$ in the variables $x_1-x_2-i\eta_1, \ldots [x_j-y_1-i\eta], \ldots [y_2-x_{j+1}-i\eta''] \ldots x_{n-1}-x_n-i\eta_{n-1}$, and therefore the same is true of $f = f^{(1)} - f^{(2)}$. Furthermore, the boundary value of f vanishes in an open set of real vectors, at least if the support of φ is sufficiently small. (This statement is obtained by smea ing (39) with φ in the variable y_1-y_2.)

This vanishing of f's boundary values implies that f_1-f_2 vanishes identically, so the first and third expressions in (39) are equal for all $x_1 \ldots x_n$ when y_1-y_2 is space-like; thus

$$[B(y_1), C(y_2)] = 0 \text{ for } (y_1-y_2)^2 < 0 \ .$$

The fact that f's boundary values vanishing in an open set implies f = 0 is a generalization of a large class of theorems in one complex variable of which the Theorem of the Brothers Riesz is typical: let f(z) be analytic in the unit disc $|z| = 1$ and continuous on $|z| = 1$. If f(z) = 0 for $|z| = 1$ and arg z in an open interval, then f = 0 throughout the closed unit disc [44]. If one takes the "Edge of the wedge" theorem [10] for granted, one has an easy proof. $f(\varphi, z_1 \ldots z_n)$ is analytic in $\mathfrak{J}_n$, $[f(\varphi, \bar{z}_1 \ldots \bar{z}_n]^*$ in $-\mathfrak{J}_n$, their boundary values coincide in an open set S of real space (and are zero!) and therefore f(z) is analytic there. Since the value in S is zero, f = 0. This implies tha the identity given by equating the first and third expressions in (39) is valid for all space-like y_1-y_2 and all $x_1 \ldots x_n$. Since A is irreducible, this mean that B is LC relative to C.

Now let us examine the question of the equality of the S operator for different fields. Borchers gives us the simple criterion.

Theorem 7

Let A be LC and irreducible and the same for B. Suppose

$$A^{in} = B^{in} \tag{40}$$

and the in fields are irreducible. Then the S-operator of the two theories is the same if A and B are together WLC.

Remarks

The theorem has been stated as though there were a single "in" field in each theory. This is by no means necessarily so, as will be seen from the proof. What is assumed is that the set of "in" fields for the two theories coincide and are determined by A and B in such a way that (41) and (42) below hold.

Proof

Suppose A and B are together WLC; then by the PCT theorem both have the same PCT operator Θ. Then

$$\Theta A^{in}(x)\Theta^{-1} = A^{out}(-x) \tag{41}$$

$$\Theta B^{in}(x)\Theta^{-1} = B^{out}(-x) \tag{42}$$

so $A^{in} = B^{in}$ implies $A^{out} = B^{out}$ and therefore $S_A S_B^{-1}$ commutes with A^{in}, which implies $S_A = S_B$ (since we normalize $S_A \Psi_0 = S_B \Psi_0 = \Psi_0$).

Conversely, suppose

$$B^{out} = S^{-1} B^{in} S = S^{-1} A^{in} S = A^{out} \quad . \tag{43}$$

Since A and B are LC, they have PCT operators Θ_A and Θ_B, respectively. Now Θ_A and Θ_B are uniquely determined by A^{in} and A^{out} and B^{in} and B^{out} via the relations

$$\Theta_A A^{in}(x) \Theta_A^{-1} = A^{out}(-x)$$

$$\Theta_B B^{in}(x) \Theta_B^{-1} = B^{out}(-x)$$

(the argument is always the same: assume two Θ_A and Θ_A', say; then prove $\Theta_A \Theta_A'$ commutes with A^{in}). Therefore, by (43), $\Theta_A = \Theta_B$, and A and B are together WLC.

One can, of course, make this theorem "covariant". Assume instead of (40) that

$$A^{in} = R B^{in} R^{-1}, \; R\Psi_0 = \Psi_0 \text{ (deducible as usual);} \tag{44}$$

then in order that the theory of A and B should predict the same results for collision one wants

$$S_A = R S_B R^{-1}, \tag{45}$$

because then

$$A^{out} = S_A^{-1} A^{in} S_A \text{ and } B^{out} = S_B^{-1} B^{in} S_B \tag{46}$$

are consistent with

$$A^{out} = R B^{out} R^{-1} \tag{47}$$

and the S matrix elements are the same in the two theories:

$$(A^{in}(x_1) \ldots A^{in}(x_j)\Psi_0, S_A A^{in}(x_{j+1}) \ldots A^{in}(x_n)\Psi_0)$$

$$= (B^{in}(x_1) \ldots B^{in}(x_j)\Psi_0, S_B B^{in}(x_{j+1}) \ldots B^{in}(x_n)\Psi_0), \tag{48}$$

which is what is meant by predicting the same results for collisions.

Under assumption (44) one has merely to replace B by $R^{-1}BR$ in Theorem 7 to get the appropriate criterion. The covariant form of Theorem 7 therefore reads: (45) follows if A and $R_{in}^{-1} B R_{in}$ have the same PCT operator where $A^{in} = R_{in} B^{in} R_{in}^{-1}$. This is not the situation in practice which may be described as follows: Let

$$\Theta A^{in}(x) \Theta^{-1} = A^{out}(-x); \; \Theta B^{in}(x) \Theta^{-1} = B^{out}(-x) \tag{48a}$$

$$S_A^{-1} A^{in}(x) S_A = A^{out}(x); \; S_B^{-1} B^{in}(x) S_B = B^{out}(x) \tag{48b}$$

$$R_{in}^{-1} A^{in}(x) R_{in} = B^{in}(x); \; R_{out}^{-1} A^{out}(x) R_{out} = B^{out}(x). \tag{48c}$$

From (48a) and (48b)

$$S_A \Theta A^{in}(x) \Theta^{-1} S_A^{-1} = A^{in}(-x),$$

and therefore

$$[(S_A\Theta)^2, A^{in}(x)] = 0;$$

so, by the usual argument,

$$\Theta S_A \Theta^{-1} = S_A^{-1}, \tag{48d}$$

and similarly

$$\Theta S_B \Theta^{-1} = S_B^{-1}. \tag{48e}$$

(This is the PCT symmetry of the S operator.) From (48a) and (48c),

$$\Theta R_{in}^{-1} A^{in}(x) R_{in} \Theta^{-1} = R_{out}^{-1} \Theta A^{in}(x) \Theta^{-1} R_{out}^{-1},$$

and so

$$R_{out} = \Theta R_{in} \Theta^{-1}. \tag{48f}$$

From (48b) and (48c)

$$S_B^{-1} R_{in}^{-1} A^{in}(x) R_{in} S_B = R_{out}^{-1} S_A^{-1} A^{in}(x) S_A R_{out};$$

so

$$R_{out} = S_A^{-1} R_{in} S_B. \tag{48g}$$

Thus

$$S_B = R_{in}^{-1} S_A (\Theta R_{in} \Theta^{-1}). \tag{48h}$$

The results (48d) to (48h) follow from (48a), (48b) and (48c). Conversely, if $\Theta A^{in}(x) \Theta^{-1} = A^{out}(-x)$ and S_A satisfies (48d), one can define R_{out} by (48f) and S_B by (48h); and then (48a), (48b) and (48c) will be satisfied for any unitary R_{in} that commutes with $U(a, \Lambda)$. This shows that to get $[\Theta, R_{in}] = 0$ and therefore the physical equivalence (48) of the operators S_A and S_B, one must use more details of the relationship between A, B, A_{out}^{in}, B_{out}^{in} and Θ. How this works out for the Haag-Ruelle collision theory will be discussed later.

The remaining step in Borchers' theory is as follows:

Theorem 8

Let A and B be LC and A be irreducible. Suppose B is LC relative to A. Then if A and B have asymptotic fields of the same mass, $B^{in} = \pm A^{in}$.

The proof as it stands in his paper uses the LSZ asymptotic condition and will not be reproduced here.

1.3. GENERALIZED FREE FIELDS AND THE SUPPORT PROPERTIES OF FIELDS IN MOMENTUM SPACE

In an effort to get out of the Borchers class of the free field, GREENBERG introduced the notion of generalized free field as any field A for which the commutator is a c-number [11]. The standard spectral representation then gives

$$[A(x), A(y)] = \int d\mu(a)\,(1/i)\ \Delta_a(x-y). \tag{49}$$

It turns out that all the vacuum expectation values of a generalized free field are obtained from those of a free field of mass m by replacing the free propagator $\frac{1}{2}\Delta^{(+)}_{m^2}(x)$ by $(1/i)\int d\mu(a)\,\Delta_a(x)$. Although generalized free fields are physically rather uninteresting, they illustrate a number of points of principle. For example, a generalized free field may be irreducible and its "in" and "out" fields exist according to LSZ prescriptions, but the "in" and "out" fields need not be irreducible. This makes evident a complication already mentioned before. The Borchers classes are not strictly equivalence classes unless one restricts one's attention to irreducible fields. Compare the result of Schroer alluded to just before equation (28) with that of Greenberg just quoted. One says that all elements of the equivalence classes of an irreducible free field of mass m are of the form (27); the other says that a reducible free field can have a generalized free field in its equivalence class and that generalized free field need not be of the form (27). When a generalized free field has an "in" field, it is LC relative to it so one does not get a new Borchers class except in pathological cases where no "in" fields exist.

A principal reason for discussing generalized free fields is that a number of elegant criteria have been given which guarantee that a field is a generalized free field. This gives some idea of what to avoid in trying to make a non-trivial theory.

Theorem 9 [12, 13, 14]

If A is LC and is irreducible and

$$[A(x),\ A(y)] = B(x-y)$$

(B may be an operator but must depend on x-y and not x + y), then A is a generalized free field.

Proof*

Consider

$$[B(x-y),\ A(z)] = \Big[[A(x+\xi),\ A(y+\xi)],\ A(z) \Big],$$

which holds for all ξ.

By the Jacobi identity it is

$$- \Big[[A(y+\xi),\ A(z)],\ A(x+\xi) \Big] - \Big[[A(z),\ A(x+\xi)],\ A(y+\xi) \Big].$$

For sufficiently large space-like ξ this vanishes, so

$$[B(x-y),\ A(z)] = 0,$$

and by the irreducibility of A, B must be a constant multiple of the identity operator so A is a generalized free field.

The second kind of criterion for a field to be a generalized free field relates to the support of the field in momentum space, i.e. the points of the spectrum of $\widetilde{A}(p) = \int e^{ip\cdot x} A(x)dx$. (This should not be confused with the spectrum of physical states, with which it is only indirectly connected.)

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* There is another proof of Theorem 9 by J. Katzin [13a] which is about as neat as that by Licht and Toll. It goes as follows. Because the commutator is by assumption translation invariant

$$U(a)[A(x),\ A(y)]\ U(a)^{-1} = [A(x),\ A(y)].$$

Then

$$U(a)[A(x), A(y)]\Psi_0 = [A(x),\ A(y)]\Psi_0$$

and therefore by the uniqueness of the vacuum

$$[A(x),\ A(y)]\Psi_0 = b(x-y)\Psi_0$$

where b is a c number.

But then

$$(\Psi_0, A(x_1)\dots A(x_j)([A(x), A(y)] - b(x-y)]A(x_{j+1})\dots A(x_n)\Psi_0 = 0$$

for all Jost points in the successive differences $f(x_1 \dots x_j\, x y\, x_{j+1} \dots x_n)$ and so by analytic continuation for all $x_1 \dots x_n$.

Therefore

$$[A(x), A(y)] = b(x-y) \quad .$$

Theorem 10 [12, 15, 16*]

Let A be LC and have the vacuum as cyclic vector. If the spectrum of $\tilde{A}$ omits an open set of space-like p, then A is a generalized free field. Two local fields whose Fourier transforms agree on such a set differ only by a generalized free field in their Borchers class.

The results of Robinson and Greenberg have been quoted. Other cases are considered by Greenberg and Dell'Antonio. For example, it is shown that, if the spectral weight of the 2-fold vacuum expectation value vanishes above some mass, then the field is a generalized free field. The proofs involve a systematic use either of the Dyson representation or holomorphy envelope calculations. Since these techniques will not be explained here, the proofs will also not be given.

It is worth noting that, unlike the case in Theorem 2, smeared polynomials in generalized free field operators can be LC [11].

1.4. THE CLUSTER DECOMPOSITION PROPERTY

Given a vacuum expectation value,

$$\langle A(x_1) \ldots A(x_j) A(x_{j+1} + a) \ldots A(x_n + a) \rangle_0 \quad ,$$

one would expect that, if $a \to \infty$ in a space-like direction, it should approach

$$\langle A(x_1) \ldots A(x_j) \rangle_0 \quad \langle A(x_{j+1}) \ldots A(x_n) \rangle_0 .$$

This can in fact be proved under appropriate assumptions and is an example of a <u>cluster decomposition property</u>. More refined statements can be obtained in which the $x_1 \ldots x_n$ are divided into k clusters which are then allowed to separate.

The significance of cluster decomposition properties for the theory of collisions was first emphasized by HAAG [17], and one of the most significant developments mentioned here is the work by RUELLE [18], which puts Haag's arguments on a rigorous mathematical foundation. Ruelle's results are based on a proof that a very refined form of the cluster decomposition property can holds in any theory of local fields in which the vacuum is cyclic. Before going into detail, I shall give two neat results which show the power of the method. Of course, the required cluster decomposition properties will be <u>assumed</u> here.

Theorem 11 [19]

Let A and B be two fields which satisfy

$$\begin{aligned} U(a, 1)\, A(x) U^{-1}(a, 1) &= A(x + a) , \\ U(a, 1)\, B(x)\, U^{-1}(a, 1) &= B(x + a) , \end{aligned} \tag{50}$$

* Borchers has obtained a number of the same results independently.

but not necessarily LC. (They could be components of general spinor fields.) Suppose

$$[A(x),\ B(y)]_{\pm} = 0 = [A(x),\ B^*(y)]_{\mp} \tag{51}$$

hold for all space-like x-y.

Then either $A(\varphi)\Psi_0 = 0 = A(\varphi)^*\Psi_0$ or $B(\varphi)\Psi_0 = 0 = B(\varphi)^*\Psi_0$ for all test functions φ. If A and B together have Ψ_0 as a cyclic vector and belong to some sets of operators which transform under homogeneous Lorentz transformation like spinors, then either A = 0 or B = 0.

Proof

Let φ and ψ be any two test functions of compact support whose supports are space-like with respect to one another. Taking

$$A(\varphi) = \int dx\varphi(x)A(x),\ B(\psi) = \int dy\,\psi(y)\,B(y)\,,$$

then

$$\|B(\psi)\,A(\varphi)^*\Psi_0\|^2 = (\Psi_0,\ A(\varphi)B(\psi)^*\,B(\psi)\,A(\varphi)^*\Psi_0)$$

$$= -\,(\Psi_0,\, B(\psi)^*B(\psi)\,A(\varphi)\,A(\varphi)^*\Psi_0)\,. \tag{52}$$

If we let the support of φ run off in a space-like direction, the last expression converges to

$$-\,(\Psi_0,\, B(\psi)^*\,B(\psi)\Psi_0)\ (\Psi_0,\ A(\varphi)\,A(\varphi)^*\Psi_0)$$

by the cluster decomposition property. (This proves incidentally that the left-hand side also converges.) But $(\Psi_0,\ B(\psi)B(\psi)^*\Psi_0)$ and $(\Psi_0, A(\varphi)A(\varphi)^*\Psi_0)$ are non-negative, so either

$$A(\varphi)^*\Psi_0 = 0 \text{ or } B(\psi)\Psi_0 = 0\ .$$

A precisely similar argument starting from $\|B(\psi)^*A(\varphi)^*\Psi_0\|^2$ yields

$$A(\varphi)^*\Psi_0 = 0 \text{ or } B(\psi)^*\Psi_0 = 0\ .$$

Finally, starting from the adjoint of the relations (51), one has the same statements with $A(\varphi)^*$ replaced by $A(\varphi)$. Thus either

$$A(\varphi)^*\Psi_0 = 0 = A(\varphi)\Psi_0 \text{ or } B(\psi)^*\Psi_0 = 0 = B(\psi)\Psi_0\ . \tag{53}$$

The last statement of the theorem is based on an argument which is, by now, standard. Look at an arbitrary vacuum expectation value:

$$(\Psi_0, \ldots . A(x). \ldots B(y). \ldots \Psi_0) \tag{54}$$

If all arguments are taken as space-like and the first of the alternatives (53) holds, take the farthest A or A^* to the right and move it through B's and B^*'s until it hits Ψ_0; conclude that (54) vanishes for such space-like separators. But the hypothesis on the transformation law of the A's and B's guarantees that the vacuum expectation values are analytic at Jost points, so the preceding argument shows all vacuum expectation values containing an A or an A^* are zero. Therefore A = 0.

This argument of Dell'Antonio actually first occurs in a slightly different connection in a paper by ARAKI [20]* in which he discusses the possible commutation relations of different fields and shows that a theory with anomalous commutation relations, distinct integer spin fields anti-commuting or half-odd-integer spin fields commuting or integer spin fields anti-commuting with half-odd-integer spin fields, is always physically equivalent to one with normal commutation relations (all integer spin fields commute with each other and all half-odd-integer spin fields, all half-odd-integer spin fields anti-commute). These two papers together with the original BURGOYNE [21], LÜDERS-ZUMINO [22] proof bring the theorem of the connection of spin with statistics to a dazzling polish.

As a second application of the cluster decomposition property, an example of SUDARSHAN and BARDACKI [23] in which it is violated will be discussed.

Consider two theories of a neutral scalar field labelled respectively by 1 and 2: Hilbert spaces $\mathcal{H}_i$, vacua Ψ_{0i}, representations of the Lorentz group $U_j(a, \Lambda)$, fields $A_j(x)$. Form a new theory with Hilbert space $\mathcal{H}_1 \oplus \mathcal{H}_2$ representations of the Lorentz group $U_1 \oplus U_2$ and field $A = A_1 \oplus A_2$. In this theory, the state vectors are pairs $\{\Psi_1, \Psi_2\}$ with the scalar product,

$$(\{\Psi_1, \Psi_2\}, \{\Phi_1, \Phi_2\}) = (\Psi_1, \Phi_1) + (\Psi_2, \Phi_2).$$

Clearly, there is a two-dimensional subspace of the Hilbert space $\mathcal{H}_1 \oplus \mathcal{H}_2$, each of whose vectors is left invariant by the representation of the Lorentz group:

$$(U_1(a,\Lambda) \oplus U_2(a,\Lambda)) (\alpha\{\Psi_{01}, 0\} + \beta\{0, \Psi_{02}\})$$
$$= \alpha\{\Psi_{01}, 0\} + \beta\{0, \Psi_{02}\},$$

which shows a grave defect of this theory; the vacuum ought to be unique. How does one recognize this defect in the vacuum expectation values? Pick a particular vacuum, say $\Psi^{(\alpha)} = \sqrt{\alpha}\,\{\Psi_{01}, 0\} + \sqrt{1-\alpha}\,\{0, \Psi_{02}\}$, $0 \leqslant \alpha \leqslant 1$, and compute

$$(\Psi^{(\alpha)}, A(x_1) \ldots A(x_n) \Psi^{(\alpha)}) = \alpha\, (\Psi_{01}, A_1(x_1) \ldots A_1(x_n) \Psi_0)$$
$$+ (1-\alpha)\, (\Psi_{02}, A_2(x_1) \ldots A_2(x_n) \Psi_{02}), \tag{55}$$

This just gives the proposal of Sudarshan and Bardacki: one takes two theories and forms a new one whose vacuum expectation values are convex linear

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* Note that Araki does not show that the normal case is physically equivalent to the abnormal case, but rather that the abnormal case is necessarily very restricted. By virtue of its abnormal commutation relation it must have selection rules which in turn yield the result that it is physically equivalent to a normal case with the same selection rules.

combinations of the vacuum expectation values of the two theories. But (55) does not have the cluster decomposition property even if the theories of A_1 and A_2 do because

$$\begin{aligned}
&(\Psi^{(\alpha)}, A(x_1)\ldots. A(x_j)\, A(x_{j+1}+a)\ldots. A(x_n+a)\Psi^{(\alpha)}) \\
= \;& \alpha(\Psi_{01}, A_1(x_1)\ldots. A_1(x_j)\, A_1(x_{j+1}+a)\ldots. A_1(x_n+a)\,\Psi_{01}) \\
+ \;& (1-\alpha)(\Psi_{02}, A_2(x_1)\ldots. A_2(x_j)\, A_2(x_{j+1}+a)\ldots. A_2(x_n+a)\Psi_{02}) \\
\rightarrow \;& \alpha(\Psi_{01}, A_1(x_1)\ldots. A_1(x_j)\,\Psi_{01})\,(\Psi_{01}, A_1(x_{j+1})\ldots. A_1(x_n)\Psi_{01}) \\
+ \;& (1-\alpha)\,(\Psi_{02}, A_2(x_1)\ldots. A_2(x_j)\,\Psi_{02})\,(\Psi_{02}, A_2(x_{j+1})\ldots. A_2(x_n)\,\Psi_{02}),
\end{aligned}$$

whereas it ought to approach

$$\begin{aligned}
&(\Psi^{(\alpha)}, A(x_1)\ldots. A(x_j)\,\Psi^{(\alpha)})\,(\Psi^{(\alpha)}, A(x_{j+1})\ldots. A(x_n)\,\Psi^{(\alpha)}) \\
= \;& [\alpha(\Psi_{01}, A_1(x_1)\ldots. A_1(x_j)\,\Psi_{01}) + (1-\alpha)\,(\Psi_{02}, A_2(x_1)\ldots. A_2(x_j)\,\Psi_{02})] \\
\cdot \;& [\alpha(\Psi_{01}, A_1(x_{j+1})\ldots. A_1(x_n)\,\Psi_{01}) + (1-\alpha)\,(\Psi_{02}, A_2(x_{j+1})\ldots. A_2(x_n)\,\Psi_{02})]\,.
\end{aligned}$$

Equating these two and assuming that some at least of the vacuum expectation values are non-zero, one finds $\alpha = 0$ or 1; i.e. the only theories of this kind with cluster decomposition property are the original constituents. Of course, there are other things wrong with these models but the fundamental trouble is the non-uniqueness of the vacuum as was first shown by HEPP, JOST, RUELLE and STEINMANN [24]. Actually, BORCHERS [25] has shown that the cluster decomposition property is not only necessary but sufficient for the uniqueness of the vacuum, if there is at least one cyclic vacuum. This point will be discussed further in the next section.

A third application of the cluster decomposition property comes about as follows. The author considers that finding non-trivial examples of internally consistent field theories is one of the most important problems of the subject at the present moment. One approach to this problem which might be attempted is to simplify it mathematically without losing its essential nature. For example, suppose one assumes that U contains only the vacuum and one irreducible representation. Can one find local fields which transform according to (4)? The answer is no, if Ψ_0 is cyclic:

Theorem 12 [26]

In a theory of a neutral scalar field with cyclic vacuum, the physical spectrum must be additive.

Remark

A point p lies in the physical spectrum if for each open set W of four momenta containing p there is a non-zero vector whose energy momentum spectrum lies in W. That the spectrum is additive means p_1 in the spectrum, and p_2 in the spectrum implies $p_1 + p_2$ in the spectrum.

Proof

Let S_1 be an open neighbourhood of p_1 and S_2 an open neighbourhood of p_2. The first step in the proof consists in choosing field operators $B_1(x)$ and $B_2(x)$ satisfying

$$U(a, 1)\, B_i(x)\, U^{-1}(a, 1) = B_i(x+a) \tag{56}$$

and test functions φ_j which have Fourier transforms with supports in S_1 and S_2 such that

$$B_1(\varphi_1)\Psi_0 \neq 0 \text{ and } B_2(\varphi_2)\Psi_0 \neq 0\,. \tag{57}$$

It follows from (56) that the energy momentum spectra of these vectors are in S_1 and S_2, respectively. (Note that $U(a, 1)B_j(\varphi_j)\Psi_0 = B_j(\{a, 1\}\varphi_j)\Psi_0$ where $(\{a, 1\}\varphi_j)(x) = \varphi_j(x-a)$, so a momentum analysis of the vector is equivalent to a momentum analysis of φ_j.)

To get the required B's, choose open neighbourhoods T_1 and T_2 of p_1 and p_2, respectively, such that the closures $\overline{T}_1$ and $\overline{T}_2$ satisfy $\overline{T}_1 \subset S_1, \overline{T}_2 \subset S_2$. Let $\mathcal{H}_{\overline{T}_j}$ be the closed subspace of $\mathcal{H}$ consisting of all vectors whose spectrum lies in $\overline{T}_j$. Then because Ψ_0 is cyclic there exist vectors of the form

$$\sum_{n=1}^{N} \int \ldots \int f_{nj}(x_1, \ldots x_n)\, A(x_1) \ldots A(x_n)\, dx_1 \ldots dx_n\, \Psi_0 \tag{58}$$

which are respectively not orthogonal to $\mathcal{H}_{\overline{T}_j}$.
Define

$$B_j(x) = \sum_{n=1}^{N} \int \ldots \int f_{nj}(x-x_1, \ldots x-x_n)\, A(x_1) \ldots A(x_n)\, dx_1 \ldots dx_n\,. \tag{59}$$

Then clearly (56) holds. (Quantities of the form (59) are called <u>almost local fields</u> by Haag.) Let $\widetilde{\varphi}_j$ have a support in S_j that includes $\overline{T}_j$. Then $B_j(\varphi_j)\Psi_0 \neq 0$ for some such φ_j; otherwise (58) would be orthogonal to $\mathcal{H}_{\overline{T}_j}$. Thus the required $B_j(\varphi_j)\Psi_0 \neq 0$ can be constructed.

Now consider the vectors

$$B_1(\varphi_1)\, U(a, 1)\, B_2(\varphi_2)\, \Psi_0\,.$$

Their support must lie in $S_1 + S_2$ by the same argument as before. Can they vanish for all a? To prove not, assume the contrary:

$$\begin{aligned} 0 &= \left\| B_1(\varphi_1)\, U(a, 1)\, B_2(\varphi_2)\, \Psi_0 \right\|^2 \\ &= \langle B_2(\varphi_2)^*\, U(a, 1)^*\, B_1(\varphi_1)^*\, B_1(\varphi_1)\, U(a, 1)\, B_2(\varphi_2) \rangle_0 \end{aligned} \tag{60}$$

Now apply the cluster decomposition property in a stronger form than before. It is asserted and will be discussed in detail later that, as $a \to \infty$ in a space-like direction, (60) converges to

$$\langle B_2(\varphi_2)^* B_2(\varphi_2)\rangle_0 \langle B_1(\varphi_1)^* B_1(\varphi_1)\rangle_0 ,$$

so either $B_2(\varphi_2)\Psi_0 = 0$ or $B_1(\varphi_1)\Psi_0 = 0$ is a contradiction. Therefore, $p_1 + p_2$ lies in the spectrum.

To get a neat statement of the required cluster decomposition property it is advisible to introduce the notion of the truncated part of a vacuum expectation value [27]. This is defined by induction:

$$\langle A(x)\rangle_0 = \langle A(x)\rangle_{0T} ,$$

$$\langle A(x_1)A(x_2)\rangle_0 = \langle A(x_1)A(x_2)\rangle_{0T} + \langle A(x_1)\rangle_{0T} \langle A(x_2)\rangle_{0T} ,$$

$$\begin{aligned}\langle A(x_1)A(x_2)A(x_3)\rangle_0 &= \langle A(x_1)A(x_2)A(x_3)\rangle_{0T} + \langle A(x_1)A(x_2)\rangle_{0T} \langle A(x_3)\rangle_{0T} \\ &+ \langle A(x_1)A(x_3)\rangle_{0T} \langle A(x_2)\rangle_{0T} + \langle A(x_2)A(x_3)\rangle_{0T} \langle A(x_1)\rangle_{0T} \\ &+ \langle A(x_1)\rangle_{0T} \langle A(x_2)\rangle_{0T} \langle A(x_3)\rangle_{0T} ,\end{aligned} \tag{61}$$

or generally

$$\langle A(x_1)\dots A(x_n)\rangle_0 = \Sigma \prod \langle A(x_j)\rangle_{0T} , \tag{62}$$

where the sum is overall partitions of $1 \dots n$ into non-empty subsets and the product is over the truncated vacuum expectation values of the subsets, all x's occurring in the subsets in the order they occur in $1 \dots n$. The definition works both for the almost local fields defined by (59) and for the field A.

The truncated part calculated in perturbation theory is just the sum of all connected diagrams. The various cluster decomposition properties can be stated thus: the truncated parts go to zero as their arguments separate (under various conditions).

The actual calculation for (60) is the following:

$$\begin{aligned}&\langle B_2(\varphi_2)^* B_1(\{-a,1\}\varphi_1)^* B_1(\{-a,1\}\varphi_1) B_2(\varphi_2)\rangle_0 \\ &= \langle B_2(\varphi_2)^* B_1(\{-a,1\}\varphi_1)^* B_1(\{-a,1\}\varphi_1) B_2(\varphi_2)\rangle_{0T} \\ &+ \langle B_2(\varphi_2)^*\rangle_{0T} \langle B_1(\{-a,1\}\varphi_1)^* B_1(\{-a,1\}\varphi_1) B_2(\varphi_2)\rangle_{0T} \\ &+ \langle B_1(\{-a,1\}\varphi_1)^*\rangle_{0T} \langle B_2(\varphi_2)^* B_1(\{-a,1\}\varphi_1) B_2(\varphi_2)\rangle_{0T} \\ &+ \langle B_1(\{-a,1\}\varphi_1)\rangle_{0T} \langle B_2(\varphi_2)^* B_1(\{-a,1\}\varphi_1)^* B_2(\varphi_2)\rangle_{0T} \\ &+ \langle B_2(\varphi_2)\rangle_{0T} \langle B_2(\varphi_2)^* B_1(\{-a,1\}\varphi_1)^* B_1(\{-a,1\}\varphi_1)\rangle_{0T} \\ &+ \langle B_2(\varphi_2)^* B_1(\{-a,1\}\varphi_1)^*\rangle_{0T} \langle B_1(\{-a,1\}\varphi_1) B_2(\varphi_2)\rangle_{0T}\end{aligned}$$

$$+ \langle B_2(\varphi_2)^* B_1(\{-a,1\}\varphi_1)\rangle_{0T} \langle B_1(\{-a,1\}\varphi_1)^* B_2(\varphi_2)\rangle_{0T}$$

$$+ \langle B_2(\varphi_2)^* B_2(\varphi_2)\rangle_{0T} \langle B_1(\{-a,1\}\varphi_1)^* B_1(\{-a,1\}\varphi_1)\rangle_{0T} \text{ (i)}$$

$$+ \langle B_2(\varphi_2)^*\rangle_{0T} \langle B_2(\varphi_2)\rangle_{0T} \langle B_1(\{-a,1)\}\varphi_1)^* B_1(\{-a,1\}\varphi_1)\rangle_{0T} \text{ (ii)}$$

$$+ \langle B_2(\varphi_2)^*\rangle_{0T} \langle B_1(\{-a,1\}\varphi_1)\rangle_{0T} \langle B_1(\{-a,1\}\varphi_1)^* B_2(\varphi_2)\rangle_{0T}$$

$$+ \langle B_2(\varphi_2)^*\rangle_{0T} \langle B_1(\{-a,1\}\varphi_1)^*\rangle_{0T} \langle B_1(\{-a,1\}\varphi_1) B_2(\varphi_2)\rangle_{0T}$$

$$+ \langle B_1(\{-a,1\}\varphi_1)^*\rangle_{0T} \langle B_1(\{-a,1\}\varphi_1)\rangle_{0T} \langle B_2(\varphi_2)^* B_2(\varphi_2)\rangle_{0T} \text{ (iii)}$$

$$+ \langle B_1(\{-a,1\}\varphi_1)^*\rangle_{0T} \langle B_2(\varphi_2)\rangle_{0T} \langle B_2(\varphi_2)^* B_1(\{-a,1\}\varphi_1)\rangle_{0T}$$

$$+ \langle B_1(\{-a,1\}\varphi_1)\rangle_{0T} \langle B_2(\varphi_2)\rangle_{0T} \langle B_2(\varphi_2)^* B_1(\{-a,1\}\varphi_1)^*\rangle_{0T}$$

$$+ \langle B_1(\{-a,1\}\varphi_1)^*\rangle_{0T} \langle B_1(\{-a,1\}\varphi_1)\rangle_{0T} \langle B_2(\varphi_2)^*\rangle_{0T} \langle B_2(\varphi_2)\rangle_{0T} \text{ (iiii)}$$

Of all these terms only the numbered ones (i), (ii), (iii), (iiii) are constant in a; the rest go to zero as $a \to \infty$ in a space-like direction, because the expressions separate into two clusters.

Clearly, here one needs the cluster decomposition property for almost local fields rather than the local fields of which the almost local fields are constructed. This will be developed later.

A much stronger result than Theorem 12 can be derived from the work of Ruelle described below. It can be shown that U necessarily contains as sub-representation the representation belonging to the theory of free fields, one for each irreducible representation contained in U. This shows that there are no non-trivial mathematical idealizations of local field theory which simplify U. U must be as complicated as physics tells us it is in a theory of particles.

PART TWO

This part will be quite precise mathematically and will begin with axioms for a theory of scalar fields.

2.1. AXIOMS AND THE RECONSTRUCTION THEOREM

Such a theory has a continuous unitary representation of the restricted inhomogeneous Lorentz group $\{a, \Lambda\} \to U(a, \Lambda)$ and a unique vacuum, Ψ_0, in a separable Hilbert space $\mathcal{H}$. A field is a linear function A with domain $\mathcal{D}$, and values linear operators in $\mathcal{H}$. It is assumed:

I. As φ runs over $\mathcal{D}$, $A(\varphi)$ and $A(\varphi)^*$ possess a common linear dense domain D such that

$$A(\varphi)\,D \subset D \qquad A(\varphi)^*\,D \subset D$$
$$\Psi \in D \qquad U(a,\Lambda)\,D \subset D \tag{63}$$

A is an operator valued distribution in the sense that for each $\Phi, \Psi \in D$, $(\Phi, A(\varphi)\Psi)$ is a distribution in $\mathcal{D}$, i.e. a continuous linear functional on $\mathcal{D}$.
II. On D

$$U(a,\Lambda)\,A(\varphi)\,U(a,\Lambda)^{-1} = A(\{a,\Lambda\}\varphi) \tag{64}$$

III. On D

$$[A(\varphi),\ A(\psi)] = 0 = [A(\varphi),\ A(\psi)^*] \tag{65}$$

for $\varphi, \psi \in \mathcal{D}$ such that

$$\varphi(x)\psi(y) = 0 \text{ for } (x-y)^2 \geqslant 0 \tag{66}$$

If

$$A(\varphi)^* = A(\bar{\varphi}) \text{ on } D, \tag{67}$$

A is called neutral or Hermitian.

It follows directly from I that the vacuum expectation values

$$(\Psi_0,\ A_{j_1}(\varphi_1)\ldots.\ A_{j_n}(\varphi_n)\,\Psi_0)$$

are multilinear functionals in $\varphi_1\ldots.\varphi_n$ separately continuous in their arguments. The Schwartz Nuclear Theorem asserts that these functionals can be uniquely extended by continuity to be distributions in the n variables [28].

$$\int dx_1\ldots.\,dx_n\,\varphi(x_1\ldots.\,x_n)\,(\Psi_0,\ A_{j_1}(x_1)\ldots.\ A_{j_n}(x_n)\,\Psi_0).$$

Conversely, as was shown some time ago, one can take a set of distributions satisfying certain conditions and construct a theory having just those for vacuum expectation values [29]. The only reason for talking about this now is that these have significant recent improvements in the sharpness of this reconstruction theorem.

Let us briefly recapitulate the conditions for a single neutral scalar field. Then the vacuum expectation values may be labelled

$$F^{(n)}(x_1-x_2,\ldots.\,x_{n-1}-x_n) = (\Psi_0,\ A(x_1)\ldots.\ A(x_n)\,\Psi_0),$$

where $n = 0, 1\ldots.$ From (67) and hermiticity:

$$(\Psi_0,\ A(\varphi_1)\ldots.\ A(\varphi_n)\,\Psi_0) = [(\Psi_0,\ A(\varphi_n)^*\ldots.\ A(\varphi_1)^*\Psi_0)]^*;$$

$$\text{and}\quad F^{(n)}(\xi_1,\ldots.\,\xi_{n-1}) = [F^{(n)}(\xi_{n-1},\ldots.\,\xi_1)]^*. \tag{68}$$

<u>The hermiticity conditions</u>

From Schwartz's inequality

$$|\Psi_0, \int \varphi_1(x_1, \ldots. x_j)\, dx_1 \ldots. dx_j\, A(x_1) \ldots. A(x_j)\, U(a,1)$$

$$\cdot \int \varphi_2 (x_{j+1}, \ldots. x_n)\, A(x_{j+1}) \ldots. A(x_n)\, \Psi_0)|$$

$$\cdot \leqslant || \int [\varphi_1(x_1, \ldots. x_j)]^* \, dx_1 \ldots. dx_j\, A(x_j)^* \ldots. A(x_1)^* \, ||$$

$$\cdot \, || \int \varphi_2 (x_{j+1}, \ldots. x_n)\, dx_{j+1} \ldots. dx_n\, A(x_{j+1}) \ldots. A(x_n)\, \Psi_0 \, || ; \qquad (69)$$

this shows that

$$(\Psi_0, \int \varphi_1(x_1, \ldots. x_j)\, dx_1 \ldots. dx_j\, A(x_1) \ldots. A(x_j)\, U(a, 1)$$

$$\cdot \int \varphi_2 (x_{j+1}, \ldots. x_n)\, dx_{j+1} \ldots. dx_n\, A(x_{j+1}) \ldots. A(x_n) \Psi_0) \qquad (70)$$

is bounded in a. Since it is also infinitely differentiable in a (moving U(a,1) to the right, it can be expressed as a translation of φ_2 which is infinitely differentiable) we can Fourier transform it and find that the Fourier transform is zero except for p in the physical spectrum [30]. (These are the spectral conditions.) The boundedness of (70) in a also has the consequence that $F^{(n)}(\varphi_1, \ldots. \varphi_{n-1})$ can be extended to a continuous linear functional on $\mathcal{S}$, the space of infinitely differentiable functions which, together with their derivatives, vanish at infinity faster than any power of the distance. (Continuity is then defined in the standard manner of SCHWARTZ [31].) Finally,

$$\lim_{a\to\infty} (70) = (\Psi_0, \int \varphi_1(x_1, \ldots. x_j)\, dx_1 \ldots. dx_j\, A(x_1) \ldots. A(x_j)\, \Psi_0)$$

$$\cdot \, (\Psi_0, \int \varphi_2 (x_{j+1}, \ldots. x_n)\, dx_{j+1} \ldots. dx_n\, A(x_{j+1}) \ldots. A(x_n) \Psi_0),$$

which expressed as a property of F^n is

$$\lim_{a\to\infty} F^{(n)}(\varphi_1, \ldots. \{a,1\}\, \varphi_j, \ldots. \varphi_{n-1}) = F^{(j)}(\varphi_1, \ldots. \varphi_{j-1})$$

$$\cdot \, F^{(n-j)} (\varphi_j, \ldots. \varphi_{n-1}) , \qquad (71)$$

as $a\to\infty$ in a space-like direction. This is the cluster decomposition property [27, 30, 33].

Lastly, because

$$|| \Sigma\, \alpha_k\, A(\varphi_{k1}) \ldots. A(\varphi_{kk}) \Psi_0 \, ||^2 \geqslant 0 ,$$

for any finite set of complex numbers α_k,

$$\sum_{k,\ell} \bar{\alpha}_k \alpha_\ell \int \dots \int [\varphi_{kk}(x_k), \dots \varphi_{k1}(x_1)]^* [\varphi_{\ell 1}(y_1) \dots \varphi_{\ell\ell}(y_\ell)]$$

$$\cdot F^{(k+\ell)}(x_k - x_{k+1}, \dots x_2 - x_1, x_1 - y_1, y_1 - y_2, \dots (y_{\ell-1} - y_\ell)$$

$$\cdot dx_k \dots dx_1 dy_1 \dots dy_\ell \geqslant 0 \tag{72}$$

these are usually referred to as the positive definiteness conditions.

Now the reconstruction theorem can be stated precisely.

Theorem 13

For each n = 0, 1, 2, ... let $F^{(n)}$ be a distribution in $\mathscr{D}'$ depending on (n-1) four-vector variables and invariant under the transformations

$$\xi_1, \dots \xi_{n-1} \rightarrow \Lambda \xi_1, \dots \Lambda \xi_{n-1} .$$

Suppose the $F^{(n)}$ are extendable to $\mathscr{S}$ in each of their arguments, the others being held fixed. If the $F^{(n)}$ satisfy the hermiticity conditions, the spectral conditions, the positive definiteness conditions and the cluster decomposition property, then there exists a Hilbert space $\mathscr{H}$, a continuous unitary representation of the Lorentz group $\{a,\Lambda\} \rightarrow U(a,\Lambda)$ with energy-momentum spectrum in or on the future light cone and unique vacuum Ψ_0, and a Hermitian scalar field $A(\varphi)$ satisfying Axioms I and II with $D = D_0$ and such that

$$(\Psi_0, A(x_1) \dots A(x_n)\Psi_0) = F^{(n)}(x_1 - x_2, \dots x_{n-1} - x_n).$$

This realization is unique up to unitary equivalence.

Axiom III is also satisfied if in addition the $F^{(n)}$ satisfy the local commutativity conditions.

The proof will not be given here; it is the same as in [29] or [25], except for the uniqueness of the vacuum which is obtained from [25].

2.2. $\mathscr{D}$ VERSUS $\mathscr{S}$ AS DEFINITION DOMAIN FOR $A(\varphi)$; DISCUSSION OF D; SELF ADJOINTNESS FOR HERMITIAN FIELDS

Those things which could be proved by assuming test functions in $\mathscr{D}$ and those which also required assuming the fields defined for test functions in $\mathscr{S}$ were not very carefully distinguished in Part One. Clearly some of the constructions required the latter, for example, that in the proof of Theorem 12. Physically, it is very natural to assume fields defined for test functions in $\mathscr{D}$: then $A(\varphi)$, φ real would describe a field measurement in a bounded region of space time. It would be very satisfactory if one could prove from this that $A(\varphi)$ could be extended to $\mathscr{S}$. Fields defined for test functions in $\mathscr{S}$ are desirable for a very practical reason. They permit one to use Fourier transforms freely and to derive dispersion relations for scattering amplitudes. It should be borne in mind that what one is excluding in such a proof that fields can be extended to $\mathscr{S}$ is worse than polynomial growth in x-space. The argument in connection with the spectral conditions (just before Theorem 13) shows that the vacuum expectation values are bounded in any one difference variable with the others held fixed. So the worse-than-exponential growth to be excluded appears only when two or more difference vectors go to infinity simultaneously. Such a growth is wildly implausible behaviour for a quantity which measures correlations between field measurements in the vacuum.

On the other hand, field quantities do behave in a way which would lead one to use test functions with compact support in p-space rather than x-space.

One finds in the perturbation theory of unrenormalized field theories evidence that one must expect momentum space vacuum expectation values which would grow faster than any power of the momentum. To make sense of these one needs test functions of compact support in p-space and therefore entire functions of exponential growth in x-space. The idea that one should adapt the axioms to such possibilities has been urged particularly by GÜTTINGER [34]. It provides a natural way of making the distinction between renormalizable and unrenormalizable theories independent of any detailed classification of Lagrangians.

Let us now discuss the domain D, again a subject which was glossed over in Part One. The first natural question is: Why not simplify the problem by assuming the field operators are everywhere defined, i.e. $D = \mathcal{H}$? The answer is that for φ real (and therefore $A(\varphi)$ Hermitian) this would imply that $A(\varphi)$ is a bounded and therefore continuous operator, i.e. $\sup_{||\Phi||=1} ||A\Phi|| < \infty$. This happens to be false for the free field, and there is every reason to believe that interesting theories should be worse rather than better than the free field. Thus D must not be all of $\mathcal{H}$. The best we can hope for is that the Hermitian unbounded $A(\varphi)$ are self-adjoint, $A(\varphi)^* = A(\varphi)$. But it is known that such operators are everywhere discontinuous on their domain of definition, so it appears that one must face up to unbounded discontinuous operators.

Recall that the adjoint of an operator T with dense domain $D(T) \subset \mathcal{H}$, and range $R(T) \subset \mathcal{H}_2$ and graph Γ_T, consisting of all pairs $\{\Phi, T\Phi\}$ with $\Phi \in D(T)$ is the uniquely defined linear operator T^* from $\mathcal{H}_2$ to $\mathcal{H}_1$ whose graph Γ_{T^*} is $\{-\Psi^*, \Psi\}$ where $\{\Psi^*, \Psi\}$ runs over the orthogonal complement of Γ_T in $\mathcal{H}_1 \oplus \mathcal{H}_2$. That means that Ψ lies in $D(T^*)$ and $T^*\Psi = \Psi^*$, if for all $\Phi \in D(T)$

$$(\Psi^*, \Phi) = (\Psi, T\Phi).$$

An operator T is Hermitian if $T \in T^*$, i.e. if $D(T) \in D(T^*)$ and $T = T^*$ on $D(T)$. An operator T is self-adjoint if $T = T^*$. It is essentially self-adjoint if $T^{**} = T^*$. A self-adjoint operator cannot be extended to any other vector without losing the property $T = T^*$. A useful criterion for the essential self-adjointness of an Hermitian operator is that there are no solutions of the equations:

$$T^*\Phi = \pm i \Phi.$$

In general, when T is Hermitian the number of linearly independent solutions of these two equations are respectively the defect indices of T. If the defect indices of T are equal, then T possesses at least one self-adjoint extension. Evidently, in the first half of these notes the precise distinctions made in this paragraph were not noted, but they will be from here on [35].

The very best we can presume for the operators $A(\varphi)$, φ real, is that they are essentially self-adjoint on the domain D_0, whose vectors are of the form $P(A(\psi)\ldots)\Psi_0$ where P is a polynomial in the smeared operators for $\psi \in \mathcal{D}$. Clearly, $D_0 \in D$, so I write $A(\varphi)|_{D_0}$ for the restriction of $A(\varphi)$ to D_0. Written out, the required essential self-adjointness is

$$[A(\varphi)|_{D_0}]^{**} = [A(\varphi)|_{D_0}]^{*}.$$

It is possible to prove this for the free field.

Theorem 14

If A is a free field and φ is real and $\in \mathcal{D}$ then $A(\varphi)|_{D_0}$ is essentially self-adjoint.

The proof is not long but makes very explicit use of a configuration space realization of the free field [36].

For a general field satisfying I, II or I, II and III, there is no such result proved at present. However, one can prove that the defect indices of $A(\varphi)|_{D_0}$ are equal. In outline, the proof is as follows: From the discussion just before Theorem 13, it follows that $F^{(n)}$ is the boundary value of an analytic function in each of its variables, the others being held fixed and smeared with test functions in $\mathcal{D}$. The analyticity in question is in the tube $\mathcal{J}$. It then follows from a theorem of ZERNER [37]* that there exists a unique function analytic in $\mathcal{J}_{n-1}$ which reduces to $F^{(n)}$. This function is invariant under the homogeneous Lorentz group so that one can use the theorem of Hall to prove the PCT theorem as at the beginning of Part One Thus the PCT theorem is valid for an irreducible field satisfying I, II and III. The PCT operator Θ leaves D_0 invariant.

Now suppose φ is not only real but even under $x \to -x$. Then Θ satisfies

$$\Theta A(\varphi)|_{D_0}\Theta^{-1} = A(\varphi)|_{D_0}.$$

But then if Φ satisfies

$$(A(\varphi)|_{D_0})^{*}\Phi = \pm i\Phi,$$

$\Theta\Phi$ will satisfy

$$(A(\varphi)|_{D_0})^{*}\Theta\Phi = \mp i\Theta\Phi.$$

(If Θ commutes with $A(\varphi)$ and leaves D_0 invariant, it maps D_0 one to one onto itself and commutes with $A(\varphi)|_{D_0}^{*}$ as can easily be verified directly from

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* In the simplest case of two complex variables Zerner's result is as follows: if $f(x_1, z_2)$ is analytic for $z_2 > 0$ for each real value of x_1 and $f(z_1, x_2)$ is analytic in $z_1 > 0$ for each real x_2 and $f(x_1, x_2)$ is continuous, then there exists a unique function f analytic for $z_1 > 0$ and $z_2 > 0$ which reduces to the given data on $z_1 = 0$, $z_2 \geqslant$ and $z_2 = 0$, $z_1 \geqslant 0$.

the definitions.) Thus, there are as many solutions with the plus sign as with the minus sign and the defect indices of $A(\varphi)$ are equal when φ is real and even. The general case of φ real is easily reduced to this.

There does not appear to be any evidence against the conjecture that $A(\varphi)|_{D_0}$ is essentially self-adjoint in the general case. At the moment, however, the best we have is the following:

Theorem 15

If φ is real and $\varphi \in \mathscr{D}$ and A is an irreducible field satisfying I, II and III, then $A(\varphi)|_{D_0}$ has equal defect indices and therefore possesses at least one self-adjoint extension.

The importance of self-adjointness is that it makes available one of the most powerful tools for the study of operators in Hilbert space, the spectral theorem. If $\hat{A}(\varphi)$ is a self-adjoint extension of $A(\varphi)|_{D_0}$, then

$$\hat{A}(\varphi) = \int_{-\infty}^{\infty} \lambda \, dE(\lambda, \varphi),$$

where $E(\lambda, \varphi)$ is a spectral resolution.

There may be physical requirements which single out a particular self-adjoint extension (for example, LC for the extended operators). If it turns out that even after these additional requirements have been applied the $A(\varphi)|_{D_0}$ do not possess unique self-adjoint extensions, one will have to say that the theory is not completely given by its vacuum expectation values. This would not be a catastrophe.

There is one additional simple remark about domains: The extension of the vacuum expectation values from multilinear functionals $(\Psi_0, A(\varphi_1) \ldots A(\varphi_n)\Psi_0)$ to distributions in all the variables,

$$\int dx_1 \ldots dx_n \, \varphi(x_1, \ldots x_n) \, (\Psi_0, A(x_1) \ldots A(x_n)\Psi_0),$$

permits an analogous extension for vectors:

$$A(\varphi_1) \ldots A(\varphi_n)\Psi_0 \to \int dx_1 \ldots dx_n \, \varphi(x_1, \ldots x_n) A(x_1) \ldots A(x_n)\Psi_0. \tag{73}$$

This last expression is then a vector valued distribution where continuity for the vectors is in the norm topology of Hilbert space [18, 30]. This permits an extension of the operators $A(\varphi)$ to the domain D of all vectors such as (73).

2.3. VON NEUMANN ALGEBRAS ASSOCIATED WITH A DOMAIN OF SPACE-TIME AND A FIELD

It is natural to try to associate an algebra of bounded operators with the field. (This is the reverse situation from that customary in mathematics where one is given an algebra of bounded operators and associates unbounded operators with it.) HAAG has particularly emphasized the significance of associating an algebra of bounded operators $R(\theta)$ with the set of field operators $A(\varphi)$ where the supports of the φ lie in a fixed domain θ of space-time [38].

There would be a straightforward way to define $\mathcal{R}(\theta)$ if we knew that the $A(\varphi)|_{D_0}$ were essentially self-adjoint: take the von Neumann algebra generated by the spectral projections of the self-adjoint operators $(A(\varphi)|_{D_0})^*$. (Recall that a von Neumann algebra is a set $\mathcal{R}$ of bounded operators with the properties: $1\in\mathcal{R}$; if $A\in\mathcal{R}$, then $A^*\in\mathcal{R}$; if A and $B\in\mathcal{R}$, then AB and $aA+bB\in\mathcal{R}$; if A_n (where $n = 1, 2, \ldots$) is a weakly convergent sequence of operators $\in\mathcal{R}$, then $\lim A_n\in\mathcal{R}$.) This definition would still work with our present knowledge but might give different $\mathcal{R}(\theta)$ depending on which self-adjoint extension of $A(\varphi)|_{D_0}$ is used. Alternatively one can proceed as follows [18]. Define: C, a bounded operator, commutes with $A(\varphi)$ if

$$(A(\varphi)^* \Phi, C\Psi) = (\Phi, CA(\varphi)\Psi) \tag{74}$$

for all Φ, Ψ in D. Then define $X\in\mathcal{R}(\theta)$ if X commutes with all C that satisfy (14) for every $A(\varphi)$ and $A(\varphi)$ with support of φ in θ. The relations among the various possible definitions are well worth exploring. The first steps in this direction are in [39]. One particular result is so simple and important that it must be given here [40].

Theorem 16

Let A be a neutral field satisfying I and II, but with test functions in $\mathcal{S}$ (including as usual the requirement that the vacuum be unique). Suppose Ψ_0 is cyclic. Then A is irreducible in the sense that any operator C satisfying

$$(A(\varphi)^* \Phi, C\Psi) = (\Phi, CA(\varphi)\Psi) \tag{75}$$

for all $\varphi\in\mathcal{D}$ and all $\Phi, \Psi\in D_0$ is a constant multiple of the identity.

Proof

If (75) holds for the $A(\varphi)$, it also holds with $A(\varphi)$ replaced by

$$\sum_n \int \ldots \int dx_1 \ldots dx_n \varphi_n(x_1 \ldots x_n) A(x_1) \ldots A(x_n),$$

a fact that will be used in a moment.

Now it may be assumed that $C\Psi_0 \neq 0$ because, if $C\Psi_0 = 0$, $C\Psi = 0$ for any $\Psi\in\mathcal{D}_0$ and therefore $C = 0$.

Write $\|C\Psi_0\| = \rho > 0$, $(\Psi_0, C\Psi_0) = \alpha$. Schwartz's inequality then implies $|\alpha| \leqslant \rho$. To prove the required result it suffices to show $|\alpha| = \rho$, because then $C\Psi_0 = \alpha\Psi_0$ and this implies $C\Phi = \alpha\Phi$ for all $\Phi\in D_0$, because C commutes with the $A(\varphi)$ according to (75).

Because Ψ_0 is cyclic a polynomial exists in the smeared fields, say $\mathcal{P}$, such that $\|(C-\mathcal{P})\Psi_0\| < \epsilon$. Then

$$|(\Psi_0, C^* C\Psi_0) - (\Psi_0, \mathcal{P}^* C\Psi_0)| = |((C-\mathcal{P})\Psi_0, C\Psi_0)| < \rho\epsilon . \tag{76}$$

So far the commutation relation (75) has not been used.

Now analyse the form of $\mathcal{P}\Psi_0$ in momentum space. $\mathcal{P}$ may have a p-space support which runs over all of p-space; but when it is applied to Ψ_0, all of the contribution save that from the physical spectrum is annihilated. By multiplying the Fourier transform of the test function occurring in $\mathcal{P}$ by a function which is 1 on the physical spectrum and zero for points which are in the negative of the continuous spectrum, one can get a new operator, $\hat{\mathcal{P}}$, of the same form as $\mathcal{P}$, which satisfies

$$\hat{\mathcal{P}}\Psi_0 = \mathcal{P}\Psi, \quad \hat{\mathcal{P}}^*\Psi_0 = (\mathcal{P}\Psi_0, \Psi_0)\Psi_0. \tag{77}$$

(Crudely, what is being done is this: Replace

$$\langle p|\mathcal{P}|q\rangle \text{ by } \langle p|\hat{\mathcal{P}}|q\rangle = \theta(p^0-q^0)\theta((p-q)^2)\langle p|\mathcal{P}|q\rangle;$$

then

$$\langle p|\hat{\mathcal{P}}^*|q\rangle = \theta(q^0-p^0)\,\theta((q-p)^2)\,\overline{\langle q'|\mathcal{P}|p\rangle},$$

so

$$\langle p|\mathcal{P}|0\rangle = \langle p|\hat{\mathcal{P}}|0\rangle \text{ but } \langle p|\hat{\mathcal{P}}^*|0\rangle = \theta(-p^0)\,\theta(p^2)\,\overline{\langle 0|\mathcal{P}|p\rangle},$$

which can only be different from zero when $p = 0$ because $\langle 0|\mathcal{P}|p\rangle = 0$ unless p is in the physical spectrum. Actually, θ has to be replaced by an infinitely differentiable function, so we need the hypothesis that $p = 0$ is an isolated point of the spectrum in order to get enough room for the smoothed θ to fall to zero from the value 1 it has at 0.) Then, using (75),

$$\rho\epsilon > |\rho^2 - (\mathcal{P}\Psi_0, C\Psi_0)| = |\rho^2 - (\hat{\mathcal{P}}\Psi_0, C\Psi_0)| = |\rho^2 - (\Psi_0, C\hat{\mathcal{P}}^*\Psi_0)|$$

$$= |\rho^2 - \alpha\,(\mathcal{P}\Psi_0, \Psi_0)| . \tag{78}$$

But ϵ can be chosen arbitrarily small; and when it is, $(\mathcal{P}\Psi_0, \Psi_0)$ is arbitrarily close to $\bar{\alpha}$. Therefore $|\alpha| = \rho$.

A second remarkable result of this type has been produced by REEH and SCHLIEDER [39].

Theorem 17

Suppose A is a field satisfying I and II with D_0 dense in $\mathcal{H}$ (test functions in $\mathcal{D}$). $D_0(\theta)$ is also dense for any open set of space-time θ. $D_0(\theta)$ is the set of all vectors of the form $\mathcal{P}(A(\varphi)\ldots.)\Psi_0$ where P is a polynomial in the fields smeared with test functions whose supports lie in θ.

Proof

A matrix element of the form

$$(\chi, A(x_1)\ldots.A(x_n)\Psi_0)$$

is the (distribution!) boundary value of an analytic function G of the variables $-x_1 - i\eta_0, x_1 - x_2 - i\eta_1, \ldots x_{n-1} - x_n - i\eta_{n-1}$ defined in $\mathcal{J}_n$. This follows immediately from the arguments described above in connection with the proof of the PCT theorem under the weakened hypothesis that test functions are in $\mathcal{D}$. But then the hypothesis of the theorem implies that the boundary value of G is zero in an open set of real space. Thus by the argument given in the proof to Theorem 6, G vanishes everywhere in $\mathcal{J}_n$ and therefore so do its boundary values $(\chi, A(x_1) \ldots A(x_n)\Psi_0)$. Since D_0 has been assumed dense, we see that χ orthogonal to $D_0(\theta)$ implies $\chi = 0$, so the theorem is proved.

One might think that, by combining the arguments of the preceding theorem with the present one, one could prove the irreducibility of the set of operators $\mathcal{P}(A(\varphi) \ldots)$ with φ restricted to have support in any fixed open set of space-time. However, this is not and cannot be so because the result is false. As was first shown by HAAG and SCHROER [41], there are generalized free fields such that the set of $\mathcal{P}(A(\varphi))$ is irreducible when φ ranges over all $\mathcal{S}$ but the set of $\mathcal{P}(A(\varphi) \ldots)$ is not irreducible when the supports of the φ are restricted to lie in any time slice $-\infty < a < x^0 < b < \infty$. The reason the proof does not go through is that the construction of the $\hat{\mathcal{P}}$ used in (78) requires test functions φ which cannot be of compact support in x space.

2.4. HAAG-RUELLE COLLISION THEORY; GENERAL ACCOUNT

The first step in Haag's theory is the construction of what he calls almost local fields. These are quantities of the form

$$B(x) = \sum_n \int \ldots \int f_n(x-x_1, \ldots x-x_n)\, A(x_1) \ldots A(x_n)\, dx_1 \ldots dx_n \quad (79)$$

which satisfy

$$U(a,\Lambda)\, B(x)\, U(a,\Lambda)^{-1} = B(\Lambda x + a)$$

$$(\Psi_0, B(x)\Psi_0) = 0.$$

where $f_n \in \mathcal{S}$. We assume finite sums in (79). At one time or another Haag has considered using some kind of limit of finite sums but that does not appear to be necessary and has not been possible till now. Furthermore, it is desirable that for each irreducible representation contained in U, say of mass m_i, there exists an almost local field such that $B_i(x)\Psi_0$ lies in the subspace of that irreducible representation. (This actually implies $(\Psi_0, B(x)\Psi_0) = 0$.) Haag refers to the construction of almost local operators satisfying these requirements as the "solution of the one-body problem". It would seem that neither Haag nor Ruelle tells one in print how to "solve the one-body problem". It is clear that under some circumstances it can always be done. Suppose, for example, that the discrete mass state in question is isolated in the mass spectrum. Then the construction used in the proof of Theorem 12 will yield the required B_i. The same holds true even if the discrete mass value is not isolated, provided that conserved quantum numbers exist which label the fields and the mass value is isolated in the subspace of states with definite values of the quantum numbers. The sort of thing meant here is, say, the case of the deuteron which lies in the middle of the mass continuum if all states are considered, but which is isolated if one confines one's attention to states of baryon number 2. It should always be possible to "solve the one-body problem" with sufficient accuracy, so that the following calculations would work, but the author has not carried out the details. (The idea is that although $B_i(x)\Psi_0$ is not a pure one-particle state the left-over piece can be made sufficiently small not to matter.) For the purpose of the present exposition it is assumed that one can "solve the one-body problem" exactly.

Now define

$$B_i^f(x_i^0) = i\int dx_i \left[f_i(x)^* \frac{\partial}{\partial x_i^0} B_i(x_i) - \frac{\partial}{\partial x_i^0} f_i(x_i)^* B_i(x_i) \right], \tag{80}$$

where the Fourier transform of f_i is of the form

$$\theta(p^0)\,\delta(p^2 - m_i^2)\,\hat{f}(\vec{p}) \text{ with } \hat{f} \in \mathcal{S}.$$

Then Haag's assertion is as follows:

Theorem 18

Let B_i be an almost local field such that $B_i(x_i)$ lies in the subspace of $\mathcal{H}$ belonging to the irreducible representation $[m_i, s_i]$ of mass m_i and spin s_i. Form the states,

$$\Phi(t) = \prod B_i^f(t)\Psi_0 ;$$

then $\lim_{t\to\pm\infty} \Phi(t)$ exists in norm.

Proof

Note first that $\frac{d\Phi}{dt} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}[\Phi(t+\Delta t) - \Phi(t)]$ exists where the limit is to be understood in the norm. This is an immediate consequence of the continuity properties discussed earlier in connection with the domain D. Furthermore, in order to verify the strong convergence of $\Phi(t)$ it is sufficient to prove that $|t|^{3/2}\|d\Phi/dt\| \to 0$ as $t \to \pm\infty$, because then

$$\|\Phi(t') - \Phi(t'')\| = \left\| \int_{t'}^{t''} d\tau \frac{d\Phi(\tau)}{d\tau} \right\| \leqslant \left| \int_{t'}^{t''} d\tau \left\| \frac{d\Phi(\tau)}{d\tau} \right\| \right| \leqslant C \left| \int_{t'}^{t''} \frac{d\tau}{\tau^{3/2}} \right|,$$

and this can be made arbitrarily small for sufficiently large t' and t''. Thus, to prove the theorem it is sufficient to prove

$$|t|^{3/2}\|d\Phi/dt\| \to 0 .$$

Now $\|d\Phi/dt\|$ can be written out as a sum of terms of the form

$$\int d\vec{x}_1\, d\vec{x}_2 \ldots . d\vec{x}_k f_1(\vec{x}_1, t) f_2(\vec{x}_2, t) \ldots . f_k(\vec{x}_k, t) F(\vec{x}_1 - \vec{x}_2, \ldots . \vec{x}_{k-1} - \vec{x}_k), \quad (81)$$

where two of the f_j are actually time derivatives of the f's appearing in the theorem and F is the vacuum expectation value of the B_i fields. Note that F is time independent because $x_i^0 = x_j^0 = t$. F can now be expanded in terms of truncated vacuum expectation values. Then (81) appears as a sum of products of integrals which are again of the form (81); however, now F stands for a truncated vacuum expectation value.

There are now two steps in the proof. First, one must establish that $\sup_{\vec{x}} |f_j(\vec{x}, t)| < C/|t|^{3/2}$ for large (t) and

$$\int d\vec{x}\, |f(\vec{x}, t)| < C_1\, |t|^{3/2}.$$

Secondly, it must be shown that the (truncated) F's fall off faster than any power of

$$\sum_{j=1}^{k-1} |x_j - x_{j+1}|^2 \text{ for } k > 2 .$$

If both these things have been established, then (81) will decrease as $|t|^{(-3/2)(k-2)}$. It remains to show that no terms with k = 2 contribute. This is a result of the hypothesis that the B's "solve the one-body problem". The two steps in the proof will be returned to in the two following sections.

Some remarks about the relativistic invariance of the procedure are necessary here. What has to be shown at this point is that the same limiting state is arrived at if one carries out the same procedures along another time-like direction. For this it suffices to show that $(1 + i\epsilon\, \vec{n}.\vec{N})\, \Phi(t)$ yields the same result as $\Phi(t)$, where $\vec{n}.\vec{N}\epsilon$ is an infinitesimal pure Lorentz transformation along the direction $\vec{n}$. The term $\vec{n}.\vec{N}\Phi(t)$ will give rise to no contribution in the limit because it will involve one extra derivative of the term which approached a constant in the preceding calculation.

The next step is to define "in" and "out" operators on the "in" and "out" states which have just been defined. One writes

$$\begin{aligned} B^f_{\substack{in\\out}}\, \Phi_{\substack{in\\out}} &= \lim_{t\to\pm\infty} B^f(t)\,\Phi(t) , \\ \left(B^f_{\substack{in\\out}}\right)^* \Phi_{\substack{in\\out}} &= \lim_{t\to\pm\infty} (B^f(t))^*\, \Phi(t). \end{aligned} \quad (82)$$

To be sure that these equations actually define linear operators one has only to check the single valuedness; i.e. suppose $\Psi(t) = \sum_{j=0}^{\ell} \Phi_j(t)$ and $\Psi_{in} = 0$ or $\Psi_{out} = 0$, then one must have $\lim_{t\to\pm\infty} B^f(t)\,\Psi(t) = 0$ for the appropriate case.

But the families of vectors $\Psi(t)$ and $(B^f(t))^* B^f(t)\,\Psi(t)$ both have a strong limit, that of the first family being zero. Therefore $\lim_{t\to\pm\infty} (\Psi(t), (B^f(t))^* B^f(t)\Psi(t)) = 0$, so $B^f_{in}\Phi_{in} = 0$ or $B^f_{out}\Phi_{out} = 0$, whichever is appropriate.

The B^f_{in} and B^f_{out} and their adjoints are respectively defined on the "in" and "out" states which span two subspaces of Hilbert space $\mathcal{H}_{in}$ and $\mathcal{H}_{out}$ respectively.

We have no assurance that $\mathcal{H}_{in} = \mathcal{H}_{out}$ nor that $\mathcal{H}_{in} = \mathcal{H} = \mathcal{H}_{out}$ at the present stage, and in fact examples show that the asymptotic states need not be complete. (There are generalized free fields such that $\mathcal{H}_{in} \neq \mathcal{H}$ and $\mathcal{H} \neq \mathcal{H}_{out}$.) That is Axiom IV (Ruelle):

IV. $\mathcal{H}_{in} = \mathcal{H} = \mathcal{H}_{out}$

Notice that $\Theta\Phi_{in}$ is an "out" state; thus if χ is orthogonal to $\mathcal{H}_{in}$, then $\Theta\chi$ is orthogonal to $\mathcal{H}_{out}$.. Thus it suffices to assume $\mathcal{H}_{in} = \mathcal{H}$ to get $\mathcal{H}_{out} = \mathcal{H}$.

The B_{in} and B_{out} which have been defined are associated with the correct discrete masses m but do not have any simple transformation law under Lorentz transformation. Ruelle's next step is to extract from the B-free spinor fields with the appropriate transformation law under Lorentz transformations to describe particles of spin s_i. The construction will not be described here, but the author believes that this is the first place where the collision theory of particles of arbitrary spin has been treated systematically in so-called axiomatic field theory.

There is one subject not explored in Ruelle's paper where further investigation would seem very valuable. That is the relation between the domains of the operators B_{in}, B_{out} and the domain of the original operators A. A typical problem here would be whether one can show that all these operators can be extended to the subspace of $\mathcal{H}$ consisting of all states whose energy is less than $E < \infty$.

2.5. ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF THE KLEIN GORDON EQUATION [18]

An important role was played in Haag's original argument for the asymptotic condition by an estimate of the asymptotic behaviour for large times of the solutions of the Klein Gordon equation:

$$\frac{1}{(2\pi)^{3/2}}\int e^{-ik.x}\,\tilde{f}(k)\,d\Omega(k)$$

$$\sim \sqrt{m}\, i^{-3/2}\,(1-\vec{v}^{\,2})^{-3/4}\,\exp\left[-i\,m\,t\,(1-\vec{v}^{\,2})^{1/2}\right] t^{-3/2}\,\tilde{f}\,(m\vec{v}(1-\vec{v}^{\,2})^{-1/2}),$$

$$\text{where } \vec{v} = \vec{x}/t. \tag{83}$$

This was one of the weak points of Haag's argument, because the class of functions for which it is valid was not determined. Ruelle replaces this by the following:

Lemma

Let f be the solution of the Klein Gordon equation $(\Box + m^2)f(x) = 0$ given by

$$f(x) = (2\pi)^{-2} \int dp\,\theta(p^0)\,\delta(p^2 - m^2)\,\tilde{f}(\vec{p})\,e^{-ip\cdot x} \tag{84}$$

where $\tilde{f}(\vec{p})$ is infinitely differentiable and of compact support. Then f is infinitely differentiable and $f(\lambda u)$ goes to zero as $\lambda \to +\infty$ in two different ways depending on whether the vectors λu (where $0 < \lambda < \infty$) intersect the support of $\delta(p^2 - m^2)\tilde{f}(\vec{p})$ or not; such vectors determine a cone C.

(a) If $u \in C$,

$$|f(\lambda u)| < A(u)\,\lambda^{-3/2} \qquad 0 < \lambda < \infty \tag{85}$$

where $A(u)$ is continuous;

(b) If $u \notin C$,

$$\lim_{\lambda\to+\infty} \lambda^n \, |f(\lambda u)| = 0 \qquad \text{for all } n = 0, 1, 2, \ldots \tag{86}$$

and uniformly for u in compact subsets of $(u^0)^2 + \vec{u}^2 = 1$.

Remark

It is helpful to recall the Riemann Lebesgue Lemma and one of its proofs in order to see why the cone C appears. Consider

$$f(x) = \int e^{ikx}\, dk\,\tilde{f}(k)$$

and suppose $\tilde{f}$ is integrable and has an integrable derivative. Then

$$f(x) = \int \tilde{f}(k)\,dk\,(1/ix)\frac{d}{dk}(e^{ikx}) = \frac{i}{x}\int \frac{d\tilde{f}(k)}{dk}\,dk\; e^{ikx}$$

$$\text{so } |f(x)| \leqslant \left(\int |d\hat{f}(k)/dk|\,dk\right)\Big/|x|\,.$$

This procedure can be repeated if $\tilde{f}$ has more integrable derivatives; each yields one more power of $|x|$ in the denominator.

For an integral of the form

$$\int e^{i\sqrt{k^2+m^2}\,x}\,\tilde{f}(k)\,dk$$

the situation is different because

$$\frac{1}{ix}\left[\frac{\sqrt{k^2+m^2}}{k}\right]\frac{d}{dk}\left(e^{i\sqrt{k^2+m^2}\,x}\right) = e^{i\sqrt{k^2+m^2}\,x}$$

and the square bracket is singular at 0. Thus the previous argument cannot be repeated indefinitely.

Proof

(84) can be written

$$f(x) = [2(2\pi)^2]^{-1}\int d\Omega_m(p)\, e^{-ipx}\, \tilde{f}(\vec{p}) , \tag{87}$$

where the integral runs over $p^2 = m^2$, $p^0 > 0$ and $d\Omega_m(p) = dp^3/\sqrt{\vec{p}^2 + m^2}$. Because the integral runs over a compact subset of $\vec{p}$ space, one can differen-

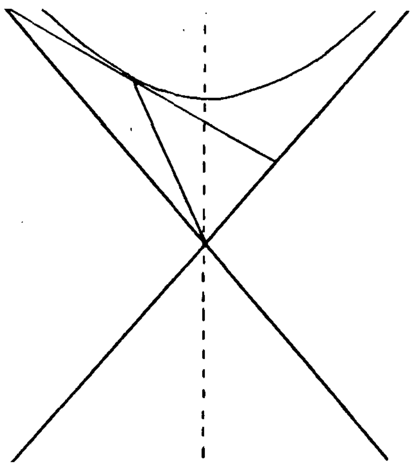

Fig. 1

tiate with respect to x^μ under the integral sign and always get convergent integrals. Therefore $f(x)$ is infinitely differentiable.

To study the asymptotic behaviour in λ when $x = \lambda u$, rewrite (87) as

$$f(\lambda u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-is\lambda}\, \tilde{f}_u(s)\, ds , \tag{88}$$

where

$$\tilde{f}_u(s) = \frac{1}{2(2\pi)^{3/2}}\int d\Omega(p)\, \delta(s - p\cdot u)\, \tilde{f}(\vec{p}) . \tag{89}$$

Now $s = p\cdot u$ is a 3 plane with normal u. It intersects the hyperboloid in a two-dimensional surface, which is the Lorentz transform of a sphere if u is plus time-like and s is sufficiently large (Fig. 1). They do not intersect for sufficiently small s and in the transition case the plane is tangent to the hyperboloid. For light-like u the plane intersects in a two-dimensional surface which runs to infinity; the same is true for space-like u. When the δ function is eliminated, there appears in the remaining integral over the curve a Jacobian which is analytic in s as long as s does not take the value for which the plane becomes tangent. If the support of $\tilde{f}$ does not contain the $\vec{p}$ of the point of tangency, $f_u(s)$ is infinitely differentiable. Since whatever u is, $\tilde{f}_u(s)$ is of compact support because the integrand will get too singular at $k = 0$. If the support of $\tilde{f}$ does not include zero, however, the preceding argument is valid. The analogue of $\vec{k} = 0$ in the integral is $p \alpha \lambda u$, which shows that one expects different behaviour for $u \in C$ and for $u \notin C$. (88) shows that $f(\lambda u)$ vanishes faster than any power of the distance. Furthermore, it will be uniformly continuous in u as long as u stays away from C. This establishes (b).

To prove (a) note that under the assumption $u \in C$, u is plus time-like, so by a Lorentz transformation it can be brought into the time axis. Then choosing for convenience $u = (1, 0, 0, 0)$, we get for (89)

$$\tilde{f}_u(s) = \frac{1}{2(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{\vec{p}^2 + m^2}} \, \delta(s - \sqrt{\vec{p}^2 + m^2}) \tilde{f}(\vec{p})$$

$$= \frac{1}{4(2\pi)^{3/2}} \sqrt{s^2 - m^2} \; \theta(s-m) \int_{|\vec{p}| = \sqrt{s^2 - m^2}} d\omega(\vec{p}) \tilde{f}(\vec{p})$$

$$= \sqrt{s-m} \; \tilde{g}(s-m), \tag{90}$$

where $\tilde{g}(s-m)$ is infinitely differentiable and of compact support on the closed half axis $0 \leqslant s < \infty$. Then

$$f(\lambda u) = (2\pi)^{-1/2} \int_m^\infty ds \, e^{-is\lambda} \sqrt{s-m} \; \tilde{g}(s-m)$$

$$= (2\pi)^{-1/2} \, e^{-im\lambda} \int_0^\infty ds \, e^{-is\lambda} \sqrt{s} \; \tilde{g}(s) \, .$$

Write

$$\sqrt{s} \; \tilde{g}(s) = \sqrt{s} \; \tilde{g}(0) \, e^{-s} + \sqrt{s} \, (\tilde{g}(s) - \tilde{g}(0) \, e^{-s}) \, .$$

The contribution from the first can be done exactly because

$$\int_0^\infty e^{-ist - s} \sqrt{s} \; ds = \sqrt{\pi} \; i \; [1 + it]^{-3/2} \, , \tag{91}$$

while the second has two integrable derivatives, so that its Fourier transform is bounded in absolute value by $a(u) |\lambda|^{-2}$. Thus

$$|f(\lambda u)| < A(u) \; |\lambda|^{-3/2} \tag{92}$$

Here A(u) can be taken to be continuous because the integral varies continuously under Lorentz transformations.

This Lemma has as an immediate consequence the following:

Lemma

If f satisfies the hypotheses of the preceding Lemma, then

$$\sup_{\vec{x}} |f(x^0, \vec{x})| \text{ decreases as } |x^0|^{-3/2} \text{ when } x^0 \to +\infty$$

and

$$\int d\vec{x}\, |f(x^0, \vec{x})| \text{ does not increase faster than } (x^0)^{3/2}$$

$$\text{when } x^0 \to +\infty .$$

Proof

Because of the uniformity of the estimates in u one has that $\sup_{x \in C} |f(x^0, \vec{x})|$ decreases as $|x^0|^{-3/2}$.

The intersection of the plane x^0 = const. with C is a compact set C_1 of three-space which lie inside a sphere of radius $< x^0$. The integral $\int d\vec{x}\, |f(x^0, \vec{x})|$ can be split into an integral over C_1 and over the rest of space. The contribution from the rest goes to zero faster than any power of x^0, while $\left|\int_{C_1}\right| \leqslant$ const. $|x_0|^{-3/2}\, |x^0|^3$.

2.6. THE REFINED CLUSTER DECOMPOSITION THEOREM [18]

First, a notation must be introduced to describe the n + 1 clusters:

$$\underline{A}_i(\underline{x}_i) = A(x_{i0})\, A(x_{i1}) \ldots . A(x_{ir(i)}) \, . \tag{93}$$

(The second index labels the points within a cluster; $\underline{x}_i$ stands for the set of vector variables $x_{i0} \ldots . x_{ir(i)}$, i = 0, ... n.) Define also

$$\underline{A}_i(\underline{x}_i + a_i) = U(a_i, 1)\, \underline{A}_i(\underline{x}_i)\, U(a_i, 1)^{-1} \, . \tag{94}$$

(If we had to deal with a set of fields A, an analogous definition could be made by putting an extra index on $\underline{A}_i$ to indicate what the constituent fields are in the i^{th} cluster. $\underline{A}_i$ would then be called a Bose or Fermi field respectively if the product contained an even or odd number of anti-commuting fields.) (93) will be referred to as a cluster product and (94) as a translated cluster product.

The translated cluster products will appear in vacuum expectation values in different orders, and the next bit of notation labels these vacuum expectation values. Let π be the element (permutation) of the symmetric group on n + 1 objects such that $\pi(0, 1, \ldots . n) = (i_0, \ldots . i_n)$ (and $\alpha_\pi = \pm 1$ according to whether, when acting on $\underline{A}_1 \ldots \underline{A}_n$ the permutation of the Fermi fields is even or odd). Then define

$$T^\pi(\underline{x}+a) = T^\pi(\underline{x}_0+a_0, \underline{x}_1+a_1, \ldots \underline{x}_n+a_n)$$
$$= \sigma_\pi < \underline{A}_{i_0}(\underline{x}_{i_0}+a_{i_0})\underline{A}(\underline{x}_{i_1}+a_{i_1}) \ldots \underline{A}_{i_n}(\underline{x}_{i_n}+a_{i_n})>_0, \tag{95}$$

$$F^\pi_\varphi(a) = \int d\underline{x}\ \underline{\varphi}(\underline{x})\, T^\pi(\underline{x}+a) \tag{96}$$

where $\underline{\varphi} \in \mathcal{S}$ in the $\sum_{k=0}^{n}[r(i_k)+1]$ vector variables,

$$x_{00}x_{01} \ldots x_{0r(0)} \ldots x_{n0} \ldots x_{nr(n)}.$$

Note that in (95) and (96) $\underline{x}$ stands for the set $\underline{x}_i$, $i = 0, \ldots n$, and a for the set a_i, $i = 0, \ldots n$.

The a_i that will be under discussion here are purely space-like, so $a = (0, \vec{a}_i)$. The diameter λ of the set $\vec{a}_0 \ldots \vec{a}_n$ is given by $\lambda^2 = \sup_{i,i'} (\vec{a}_i - \vec{a}_{i'})^2$. Let this maximum be obtained for $i = j$ and $i' = j'$. Then $\lambda^2 = (\vec{a}_j - \vec{a}_{j'})^2$. Now consider the family of all partitions of $\{0, 1, \ldots n\}$ into two subsets X and X′ such that $j \in X$ and $j' \in X'$. The maximum of the distance of the set $\{\vec{a}_i; i \in X\}$ from the set $\{\vec{a}_{i'}; i' \in X'\}$ as X varies over the family is given by

$$\mu^2 = \sup_X \Big[\inf_{i \in X,\, i' \in X'} (\vec{a}_i - \vec{a}_{i'})^2 \Big].$$

In the following discussion it will be assumed that this maximum is obtained for the partition $X = Y$ and $X' = Y'$ and that $\mu^2 = (\vec{a}_\ell - \vec{a}_{\ell'})^2$, $\ell \in Y$ and $\ell' \in Y'$.

There is an elementary but basic inequality connecting μ with the diameter λ:

$$n\mu \geqslant \lambda. \tag{97}$$

Proof

We divide the points $\vec{a}_i$, into two classes: those which can be joined to $\vec{a}_j$ by a chain of points such that 1) no point repeats, 2) the distance between successive points is $\leqslant \mu$, and those which cannot. We claim $\vec{a}_j$ lies in the former class, because every point of the latter class lies a distance $> \mu$ from every point of the former and if $\vec{a}_j$ belonged to it we would have a partition violating the definition of μ. Therefore, there is a chain of points $\vec{a}_j, \vec{a}_p, \ldots \vec{a}_j$, such that

$$\lambda = |\vec{a}_j - \vec{a}_j| \leqslant |\vec{a}_j - \vec{a}_p| + |\vec{a}_p - \ldots| + \ldots |\ldots - a_j| \leqslant n\mu.$$

Notice that $n\mu = \lambda$ when the $\vec{a}_i$ are equally spaced along a line.

A final bit of notation: the truncated vacuum expectation values corresponding to (95) will be denoted T^π_T and

$$F^{\pi}_{T\underline{\varphi}}(a) = \int d\underline{x}\ \underline{\varphi}(x)\ T^{\pi}_{T}(\underline{x}+a)$$

If

$$Y = \{i_0, i_1, \ldots . i_k\},\ Y'\ \{i'_0, i'_1, \ldots . i'_{k'}\} \text{ with } k+k' = n-1,$$

where the elements i_r within each of the subsets are written in their natural order as integers. Define permutations I and J by

$$I(0, 1, \ldots . n) = (0, 1, \ldots . n);\ J(0, 1, \ldots . n) = (i_0, i_1 \ldots . i_k, i'_0, i'_1, \ldots . i'_{k'}).$$

I is the identity permutation.

Now we are ready for the second step in the proof. Let A be a field satisfying I, II and III but with test functions in $\mathscr{S}$ rather than $\mathscr{D}$.

Theorem

Let λ be the diameter of the set $\{\vec{a}_0 \ldots . \vec{a}_n\}$. Then, for any positive integer N,

$$\lim_{\lambda\to\infty} \lambda^N\,[F^I_{T_{\underline{\varphi}}}(\vec{a}) - F^J_{T_{\underline{\varphi}}}(\vec{a})] = 0 \tag{97. a}$$

provided that the configuration of the $\vec{a}$'s remains such that the above defined j, j', Y, Y', and ℓ' stay the same.

Remarks

(1) This theorem already has been stated by HAAG [17]. He gave a plausible but somewhat hand-waving-type proof.
(2) It is this theorem which enables the commutation relations for the "in" and "out" fields to be proved.

Proof

Note first that $T^I_T(\underline{x}) - T^J_T(\underline{x})$ vanishes when all $x_{i\alpha}$, $(i \in Y)$, are spacelike to all $x_{i'\alpha'}$, $(i' \in Y')$, because of III (LC). Therefore, $\underline{\varphi}(\underline{x})$ does not contribute to the integral:

$$F^I_{T_{\underline{\varphi}}}(\vec{a}) - F^J_{T_{\underline{\varphi}}}(\vec{a}) = \int d\underline{x}\ \underline{\varphi}(\underline{x})\,[T^I_T(\underline{x}+\vec{a}) - T^J_T(\underline{x}+\vec{a})]\ ,$$

$$(x^0_{i\alpha} - x^0_{i'\alpha'})^2 < [(\vec{x}_{i\alpha} - \vec{x}_{i'\alpha'}) + (\vec{a}_i - \vec{a}_{i'})]^2\ ,$$

$$||x_{i\alpha} - x_{i'\alpha'}||^2 < (\vec{x}_{i\alpha} - \vec{x}_{i'\alpha'})^2 + [(\vec{x}_{i\alpha} - \vec{x}_{i'\alpha'}) + (\vec{a}_i - \vec{a}_{i'})]. \tag{98}$$

Now the square bracket is always greater than $||\vec{x}_{i\alpha} - \vec{x}_{i'\alpha'}| - |\vec{a}_i - \vec{a}_{i'}||$; and if, when $[(x_{i\alpha} - x_{i'\alpha'}) + (a_i - a_{i'})]^2 < 0$ for all $\alpha = 0, \ldots . r(i)$,

$$\alpha' = 0, \ldots . r(i') \text{ and all } i \in Y, \ i' \in Y'.$$

Introducing the Euclidean distance,

$$||x_{i\alpha} - x_{i'\alpha'}||^2 = (x^0_{i\alpha} - x^0_{i'\alpha'})^2 + (\vec{x}_{i\alpha} - \vec{x}_{i'\alpha'})^2, \tag{99}$$

one can get a sufficient condition for (99) to be satisfied as follows: Note (99) can be rewritten as

$$||x_{i\alpha} - x_{i'\alpha'}||^2 < (\vec{x}_{i\alpha} - \vec{x}_{i'\alpha'})^2 + [(\vec{x}_{i\alpha} - \vec{x}_{i'\alpha'}) + (\vec{a}_i - \vec{a}_{i'})]^2. \tag{100}$$

The second term on the right-hand side is always bigger than

$$[|\vec{x}_{i\alpha} - \vec{x}_{i'\alpha'}| - |\vec{a}_i - \vec{a}_{i'}|]^2,$$

so the right-hand side is bigger than

$$2|\vec{x}_{i\alpha} - \vec{x}_{i'\alpha'}|^2 + |\vec{a}_i - \vec{a}_{i'}|^2 - 2|\vec{x}_{i\alpha} - \vec{x}_{i'\alpha'}|\,|\vec{a}_i - \vec{a}_{i'}|.$$

This takes its minimum as $|\vec{x}_{i\alpha} - \vec{x}_{i'\alpha'}|$ varies when

$$|\vec{x}_{i\alpha} - \vec{x}_{i'\alpha'}| = (1/2)\,|\vec{a}_i - \vec{a}_{i'}|;$$

then it is $(1/2)\,|\vec{a}_i - \vec{a}_{i'}|^2$; thus (99) is guaranteed if

$$||x_{i\alpha} - x_{i'\alpha'}||^2 < \mu^2/2 ;$$

or, using (97);

$$||x_{i\alpha} - x_{i'\alpha'}||^2 < \lambda^2/2n^2.$$

Because $||x_{i\alpha} - x_{i'\alpha'}||^2 \leqslant (||x_{i\alpha}|| + ||x_{i'\alpha'}||)^2$, if one makes

$$||x||^2 \equiv \sum_{i=0}^{r(i)} ||x_{i\alpha}||^2 < \lambda^2/8n^2,$$

one has each $||x_{i\alpha}|| < \lambda/2\sqrt{2}n$ so $||x_{i\alpha} - x_{i'\alpha'}||^2 < (\lambda/\sqrt{2}n)^2 = \lambda^2/2n^2$. Thus there is a sphere in $\underline{x}$ space whose radius is $\lambda\sqrt{2}n$ such that $\underline{\varphi}(\underline{x})$ does not contribute to the integral (98) for $\underline{x}$ in the sphere.

Next note that the transformation $\underline{x} \to \underline{x} + a$, where all a are identical, leaves T^π_T invariant, so one can assume without loss of generality that the cluster labelled zero has its first $\vec{a}$ at the origin. Then

$$||\underline{a}||^2 = \sum_{i=0}^{n} \sum_{\alpha=0}^{r(i)} ||a_i||^2 < \sum_{i=1}^{n} (r(i)+1)\lambda^2 = L\lambda^2, \text{ where } L = n + \sum_{i=1}^{n} r(i);$$

i. e.

$$||a|| \leqslant \lambda\sqrt{L}. \tag{101}$$

To complete the proof, Ruelle introduces an important technical device: a partition of unity adapted to the problem. Partitions of unity are a standard device of distribution theory [42], but the one used here has some special features.

What is wanted is a family of non-negative functions $f_\nu(\underline{x}) \in \mathcal{S}$; $\nu = 1, 2..$ such that

(1) $\sup_{\underline{x}} f_\nu(\underline{x})$ is bounded in ν and the same holds true for each derivative of f_ν;

(2) $f_\nu(\underline{x}) \equiv f_\nu(||x||) = 0$ both if $||x|| > \nu + 1$ and $||x|| < \nu - 1$;

(3) $\sum_\nu f_\nu(\underline{x}) = 1$.

Recall that for an arbitrary open covering of space-time $\{O_i ; i \in I\}$ (where, according to the definition of open covering, I is some index set, O_i is open for all i and every x lies in some O_i) a partition of the identity is a family of φ_i ; $i \in I$ of infinitely differentiable non-negative functions with support of $\varphi_i \subset O_i$ and such that if C is any compact set of space-time, C intersects the support of almost a finite number of φ_i. In the present case, the sets may be taken as O_i, the interiors of spherical shells of thickness $(2 + \epsilon)$ and integer radius, and one has to look into the details of the proof, for example that of SCHWARTZ [42], to see that the property 1, which is usually not required for a partition of unity, can be secured. It is true but will not be proved here.

Taking the f_ν for granted, then one gets

$$F^I_{T\underline{\varphi}}(\vec{a}) - F^J_{T\underline{\varphi}}(\vec{a}) = \sum_{\nu > \lambda/2n\sqrt{2}-1} [F^I_{T\underline{\varphi}_\nu}(\vec{a}) - F^J_{T\underline{\varphi}_\nu}(\vec{a})] ,$$

where $\underline{\varphi}_\nu = f_\nu(\underline{x})\,\varphi(\underline{x})$. (The series $\sum_{r=1}^{\infty} \varphi_\nu$ converges to φ in $\mathcal{S}$. There is no contribution from the terms with $\nu + 1 < \lambda/2n\sqrt{2}$ because support of φ_2 is then entirely in the sphere $||x|| < \lambda/2n\sqrt{2}$).

Since $T^I_T - T^J_T$ is a temperate distribution, it may be written as $T^I_T - T^J_T = D^\alpha g$, where g is a continuous function of $\underline{x}$ of at most polynomial growth. D^α is the differentiation operator defined in Eq. (26). Thus

$$F^I_{T\underline{\varphi}_\nu}(\vec{a}) - F^J_{T\underline{\varphi}_\nu}(\vec{a}) = \int d\underline{x}\, \underline{\varphi}_\nu(\underline{x})\, D^\alpha g(\underline{x} + a)$$

$$= \int d\underline{x}\, [D^\alpha \underline{\varphi}_\nu(\underline{x})]\, g(\underline{x} + a). \qquad (102)$$

Now the numbers $\sup_{\underline{x}} |D^\alpha \underline{\varphi}_\nu(\underline{x})|$ decrease with ν faster than any power of ν^{-1}. (The reason for this is that $\varphi \in \mathcal{S}$ so $\sup_x |x^\beta D^\gamma \varphi(x)| < \infty$. But the derivatives of φ_ν are uniformly bounded in ν. This supplies $\sup |x^\beta \varphi_\nu(x)| < C$ independent of ν so

$$\sup_{\underline{x}} |D^\alpha \varphi_\nu(x)| < C(\alpha, \beta)/\nu^\beta$$

for all integer ν and each β.)

Thus, since

$$|g(x)| \leqslant C(1+||x||^2)^{k/2}$$

$$|F^I_{T\underline{\varphi}_\nu}(\vec{a}) - F^J_{T\underline{\varphi}_\nu}(\vec{a})| \leqslant S(\nu+1) \sup_x |D^\alpha \underline{\varphi}_\nu(x)| C,$$

$$\sup_{||x||=\nu+1} (1+||x||^2)^{k/2} \leqslant S(\nu+1) \sup_x |D^\alpha \underline{\varphi}_\nu(x)| C(1+2(\nu+1)^2)^{k/2} (1+2\lambda^2 L)^{k/2},$$

where $S(\nu+1)$ is the volume of the sphere in $\underline{x}$ space of radius $\nu+1$ and the inequality $1+||x+a||^2 < (1+2||x||^2) \times (1+2||a||^2)$ has been used.

Now the numbers $C_\nu = \max_{\underline{x}} |D^\alpha \underline{\varphi}_\nu(\underline{x})| [C S(\nu+1) \times (1+2(\nu+1)^2)^{k/2}]$ decrease faster than any power of ν^{-1}; therefore, in the inequality

$$|F^I_{T\underline{\varphi}}(\vec{a}) - F^J_{T\underline{\varphi}}(\vec{a})| < (\Sigma_{\nu > \lambda/2\sqrt{2}\, n-1} C_\nu)(1+2L\lambda^2)^{k/2}$$

the first factor decreases faster than any power of λ^{-1}.

(ΣC_ν decreases as $N^{-(\ell-1)}$ for $\ell \geqslant 2$; as proof of this compare with an integral which can be integrated explicitly.) Therefore,

$$\lim_{\ell \to \infty} \lambda^N [F^I_{T\underline{\varphi}}(\vec{a}) - F^J_{T\underline{\varphi}}(\vec{a})] = 0$$

for all N, as was to be proved.

It is well to look over the proof to see why it works. Evidently, it uses the sphere in $\underline{x}$, within which there is no contribution to the integral. Furthermore, it uses the assumption that the T^π_T are temperate in order to conclude that they can be written in terms of a derivative of a continuous polynomial bounded g.

The next theorem is the one which gives the title to this section.

Theorem

With the same hypotheses as in the previous theorem but, in addition, the requirement that $p = 0$ be an isolated point of the physical momentum spectrum, $F^\pi_{T\varphi}(\vec{a})$ as well as $D_0 F^\pi_{T\varphi}(\vec{a})$ where D_0 is any derivative with respect to the a^0_i are functions in $\mathcal{S}$.

Proof

Introduce now in x-space the new variables,

$$x = x_{i_0 0};\ \xi = x_{i'_0 0} - x_{i_0 0};\ \xi_i = x_{i0} - x_{i_0 0} \quad (i \neq i_0);$$

$$\xi_{i'} = x_{i'0} - x_{i'_0 0},\ (i' \neq i'_0);\ \xi_{i\alpha} = x_{i\alpha} - x_{i0} \quad (\alpha \neq 0);$$

$$\xi_{i'\alpha'} = x_{i'\alpha'} - x_{i'0} \quad (\alpha' \neq 0).$$

That is, single out one point with index in Y, $x_{i_0 0}$ and one with index in Y′, $x_{i'_0 0}$. Introduce the first as x and their difference as ξ. Then introduce the differences of the first points of the clusters in Y relative to $x_{i_0 0}$ and call them ξ_i; introduce the differences of the first points of the clusters in Y′ relative to $x_{i'_0 0}$ and call them $\xi_{i'}$. Finally, introduce the differences between the $x_{i\alpha}$ and the first point of their clusters $\xi_{i\alpha} = x_{i\alpha} - x_{i0}$, and the corresponding differences between the $x_{i'\alpha'}$ and the first point of their clusters $\xi_{i'\alpha'} = x_{i'\alpha'} - x_{i'0'}$.

Denote by $\underline{\xi}$ the family of all $\xi_i\, \xi_{i'} \xi_{i\alpha}\, \xi_{i'\alpha'}$. Then T_T^π is a function of ξ and $\underline{\xi}$, and φ a function of $x, \xi, \underline{\xi}$. Define Fourier transforms by

$$(\mathfrak{F}\, T_T^\pi)(P, \underline{P}) = (2\pi)^{-2L} \int \ldots \int d\xi d\underline{\xi}\, e^{-i(P\xi + \underline{P}\underline{\xi})}\, T_T^\pi(\xi, \underline{\xi}) ,$$

$$(\mathfrak{F}\, \underline{\varphi})(p, P, \underline{P}) = (2\pi)^{-2(L+1)} \int \ldots \int dx d\xi d\underline{\xi}\, e^{+i(px + P\xi + \underline{P}\underline{\xi})} \underline{\varphi}(x, \xi, \underline{\xi}).$$

Here the P's are labelled in the same way as the $\underline{\xi}$'s. Incidentally, this formula displays what was already clear from first principles: $F^\pi_{T\underline{\varphi}}$ is an infinitely differentiable function of at most polynomial growth. Then

$$F^\pi_{T\underline{\varphi}}(\underline{a}) = (2\pi)^2 \int dP\, d\underline{P}\, (\mathfrak{F}\, \underline{\varphi})(0, P, \underline{P})\, (\mathfrak{F}\, T_T^\pi)\, (\mathfrak{F}\, T_T^\pi)(P, \underline{P})$$

$$\times \exp + i\,[P(a_{i'_0} - a_{i_0}) + \sum_{i=i_1}^{i_k} P_i(a_i - a_{i_0}) + \sum_{i'=i'_1}^{i'_{k'}} P_{i'}(a_{i'} - a_{i'_0})] .$$

Up to this point in the proof there is essentially nothing but notation for Fourier-transforms. Now comes the idea. Notice that $(\mathfrak{F}\, T_T^J)(P, \underline{P}) = 0$ unless $P \in \bar{V}_+^M$ (where V_+^M stands for all vectors P with $Q^2 > M^2$, $\varphi^0 > 0$ and the bar denotes closure). This is true because P is conjugate to the difference $\xi = x_{i'_0 0} - x_{i_0 0}$. (Insert U(a, 1) just after $A(x_{i_0 0})$ in the vacuum expectation value, multiply by $e^{-iQ\cdot\alpha}$ and integrate. The result has to be zero except when Q is in the physical spectrum but has the effect $\xi \to \xi + a$ so that P must be in the physical spectrum.) M is the assumed lower limit on the mass of the system. The vacuum does not appear as an intermediate state because the vacuum expectation values have been truncated. A full formal proof of this last intuitively obvious statement is contained in [27]. Furthermore if K is the permutation $K(0,1,\ldots n) \to (i'_0 \ldots i'_k, i_0 \ldots i_k)$, K changes ξ into $\underline{\xi}$ without changing $\underline{\xi}$ so $(\mathfrak{F}\, T_T^K)(P, P) = 0$ unless $P \in V_-^M$. Now define $(\mathfrak{F}\, \Psi)(p, P, \underline{P}) = h(P)\tilde{\varphi}(p, P, \underline{P}) \in \mathcal{S}$ where h is infinitely differentiable on $\bar{V}_+^M$ and vanishes outside of V_+. Then, clearly

$$F^J_{T\underline{\psi}}(a) = F^J_{T\underline{\varphi}}(a), \quad F^J_{T\underline{\psi}}(a) = 0 . \tag{103}$$

Now the argument of the preceding theorem was made for two permutations, I and J, but it would differ only in notation if carried out for J and K. Thus

$$\lim_{\lambda \to \infty} \lambda^N F^J_{T\underline{\varphi}}(\vec{a}) = 0 \tag{104}$$

under the same conditions described in the preceding theorem. Those conditions involve the points j, j′, ℓ, ℓ' and the sets Y and Y′. But if the $\vec{a}$ have a configuration such that j, j′, etc. are different, the conclusion is the same and there are only a finite number of possible choices for the j, j′... Thus, whatever the configuration of the $\vec{a}$, (104) holds. Applying D_0 to F is equivalent to changing $\underline{\varphi}$, so the theorem is proved.

For the application to Haag collision theory one needs the preceding conclusion but for almost local fields. Actually, this case is covered by the preceding argument if a change in notation is made. Write

$$B_i(x_i) = U(x_i, 1)\underline{A}_i(\underline{\varphi}_i)U(x_i, 1)^{-1}$$

and call x_i the former variables a_i, and replace a by x. Then

$$F(x) = (\Psi_0, B_0(x_0)B_1(x_1)\ldots B_n(x_n)\Psi_0)$$

is a special case of the (untruncated) F's considered before with $\underline{\varphi} = \varphi_0 \otimes \varphi_1 \otimes \ldots \otimes \varphi_n$. The truncated vacuum expectation values are defined with respect to the B's as in Part One, not as above with respect to the A's, but one sees immediately from the above proof that the vacuum will be eliminated equally well in the intermediate states by this procedure.

Corollary

The preceding theorem is also true for truncated vacuum expectation values of almost local fields built out of local fields (test functions again in $\mathcal{S}$) provided the vacuum is an isolated point of the spectrum.

2.7. FINAL REMARKS ON THE HAAG-RUELLE COLLISION THEORY

The preceding sections have explained how one can construct collision states of all the elementary systems associated with irreducible representations of the Poincaré group contained in U. A natural question is then: Are the collision states unique? The answer is yes. Suppose that by choosing two different sets of B's, say B and $\hat{B}$, and carrying out the preceding constructions, one was led to two states $\Phi(t)$ and $\hat{\Phi}(t)$. The argument which follows Eq. (81) shows that they actually converge to the same "in" or "out" state. The argument goes just as before, except that instead of the terms with two operators not contributing because their time derivatives are zero, here it is because the contributions of Φ and $\hat{\Phi}$ cancel. Both cases are covered by the statement that there is no contribution because the one-body problem has been solved, assuming the one particle states $B_f\Psi_0$ and $\hat{B}_f\Psi_0$ are normalized in the same way. Thus, the Haag-Ruelle Collision Theory will give a unique set of "in" and "out" fields and consequently a unique collision matrix.

These statements hold even if Axiom IV does not hold. Then, however, the S operator is a unitary mapping of $\mathcal{H}_{out}$ onto $\mathcal{H}_{in}$ which is undefined on those vectors of $\mathcal{H}$ which are not in $\mathcal{H}_{out}$. There might be some point in investigating (in the spirit of Heisenberg's elementary particle theory) theories for which Axiom IV does not hold.

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