Flavour hierarchies and unification in String Theory

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Abstract

In this thesis we study the flavour problem in the context of String Theory. In particular, we analyse how to generate hierarchical Yukawa couplings in F-theory models of grand unification by including non-perturbative effects. We construct realistic local models that are able to reproduce the observed pattern of masses for quarks and charged leptons which justifies the initial motivation for considering F-theory models.

Furthermore, we explore the possibility of having non-abelian discrete symmetries acting on the flavour degrees of freedom in Type II D-brane models, both intersecting and magnetised. By means of simple examples we see how geometrical symmetries in higher dimensions may descend to this kind of discrete symmetries in 4d and wether these are exact or approximate. We construct explicit models based on intersecting D-branes on toroidal orbifolds that show how these symmetries constrain the Yukawa operators.

Finally, we study the kinetic mixing between different $U(1)$ gauge groups in Type II D-brane models. We explore an alternative way of computing the mixing by analysing the space of magnetic monopoles and using the Witten effect. Following this procedure we propose a supergravity formula for the open-open kinetic mixing that accounts for the one-loop correction in the open string channel. Also, using generalised complex geometry we compute the kinetic mixing of the hypercharge $U(1)$ with the closed string $U(1)$ sector in F-theory $SU(5)$ GUTs.

Resumen

En esta tesis se estudia el problema del sabor en el contexto de la Teoría de Cuerdas. En particular, se analiza cómo generar acoplos de Yukawa jerárquicos en modelos de gran unificación en Teoría F al incluir efectos no perturbativos. Se construyen modelos locales realistas que son capaces de reproducir el patrón de masas observado para quarks y leptones cargados, lo que justifica la motivación inicial para considerar modelos en Teoría F.

Además, se explora la posibilidad de tener simetrías discretas no abelianas que actúan sobre los grados de libertad de sabor en modelos de D-branas en la Tipo II, tanto intersecantes como magnetizadas. A través de ejemplos sencillos se ve cómo las simetrías geométricas en dimensiones superiores pueden descender a este tipo de simetrías en 4d y si éstas son exactas o aproximadas. Se construyen modelos explícitos basados en D-branas intersecantes en orbifolds toroidales que muestran cómo dichas simetrías afectan a los operadores de Yukawa.

Finalmente, se estudia la mezcla cinética entre diferentes grupos gauge $U(1)$ en modelos de D-branas en la Tipo II. Se explora una forma alternativa de calcular dichas mezclas al estudiar el espacio de monopolos magnéticos y el efecto Witten. Siguiendo este procedimiento se propone una fórmula en supergravedad para la mezcla abierta-abierta que incorpora las correciones a un loop en el canal de cuerda abierta. Además, utilizando geometría compleja generalizada se calcula la mezcla del $U(1)$ de hipercarga con el sector de $U(1)$s de cuerda cerrada en modelos de Teoría F de gran unificación $SU(5)$.

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1
Introduction

Our current understanding of the fundamental laws of nature is based on Quantum Mechanics. This is not a theory but rather a framework where we should formulate the theory that accounts for the experimental observations. As of today, every interaction except for gravity is well understood in terms of the Standard Model (SM) of particle physics, a quantum field theory (QFT) that has been tested experimentally with remarkable accuracy.

At the classical level, gravity is described by General Relativity (GR) which is a classical field theory whose dynamical field is the metric and can be seen to arise as the low energy limit of a massless spin two particle, the graviton. The naive way to proceed is to quantise GR but it seems, however, that gravity refuses to fit into this scheme and new ideas are needed to formulate a consistent theory of quantum gravity. This suggests that maybe the geometrical nature of gravity is just an effective description at large distances while at short distances the correct degrees of freedom could have nothing to do with the metric. It is perhaps not surprising that gravity cannot be formulated in terms of QFT since it is actually the physics of spacetime itself which makes it a completely different subject from the rest of interactions that are understood as occurring in an underlying spacetime.[1]

String theory (ST) provides the most promising candidate of a theory of quantum gravity. Quantum field theory can be thought of as a quantum theory of particles that is relativistic and similarly, ST is the relativistic quantum theory of strings. This might look like an innocent generalisation but it actually has profound consequences. For instance, there are an infinite number of different quantum field theories but only a few consistent string theories and the dimension of spacetime where they can be formulated is dictated by consistency. A remarkable feature of such string theories is that they inevitably contain a spin two massless particle so they automatically include gravity.

It was realised in the mid 90s that all the consistent STs are not independent from each other but rather fit together into a richer structure that goes by the name of M-theory. It is fair to say that M-theory is still not a well-defined theory, it is a (yet to formulate[2]) physical theory that reduces to the known string theories in certain limits. It was also noticed that these contain higher-dimensional dynamical objects so M-theory can be though of as the correct way to describe relativistic quantum extended objects.

In addition to that, these naturally include gauge interactions that, in principle, allow to incorporate the SM of particle physics into the same description. Thus, string/M-

[1]The recently discovered B-modes of the CMB give the first experimental hint that gravity must be quantised although this has been acknowledged for much longer for other reasons.

[2]Or discover, depending on one's taste.

theory provides a unified structure that can be used to model all the necessary ingredients to describe nature at the fundamental level in a logically consistent way. It is, however, highly non-trivial to go from the ST to experimental predictions that can be tested to confirm or rule out the theory.

1.1 String landscape

In order to test ST we should compute its predictions and then compare them to experiment. However, when trying to test ST *in practice* there is a previous stage that has to be addressed, one has to choose a particular vacuum.

The characteristic scale of the theory is set by the length of the string l_s which is supposed to be very small, $l_s \sim 10^{-33}$ cm. Thus, in order to test it directly one should perform experiments with a centre of mass energy of that order, namely, around 10^{19} GeV. However, due to technological limitations this seems to be out of reach and probably will be in the near future. This means that we are only able to probe the low energy limit of the theory and if we want to rule it out we have to compute all the possible low energy predictions and check wether any of them match with observations. This is a very complicated task since these are not unique as we explain in the following.

In general, a quantum theory may have many stable solutions known as vacua[3] and it is necessary to identify which is the correct one for a given physical situation. These are characterised by the vacuum expectation values (vev) of the quantum fields in the theory and, depending on the vacuum, the physics may look completely different.

String theory is formulated in ten spacetime dimensions and since our universe appears to have only four, it means that we must be living in a non-trivial vacuum of the theory. Depending on which vacuum we pick, we end up with a universe that *superficially*, i.e. at low energies, looks one way or another. This does not mean that the basic laws that govern all of them are fundamentally different, it is only at low energies that they may look different.

The space of all string/M-theory vacua is usually called the string landscape and, as of today, there is no known vacuum that precisely describes our universe. Much effort is being devoted to understand the landscape which has led to remarkable results in physics beyond the SM, string theory itself and pure mathematics. However, it is still poorly understood for various reasons, namely, our understanding of string/M-theory is incomplete, the necessary mathematical tools are not yet developed or simply for technical reasons (the calculations are too complicated).

To summarise, in order to test ST experimentally one could try to probe distances of the order of the string length l_s to obtain 'vacuum-independent' data but this is not likely to happen in the near future due to our technological limitations. Alternatively, given low energy experimental data, in order to rule out ST one should check that there is no vacuum compatible with observations which, for theoretical limitations, will probably not happen soon. So far, as our knowledge of the landscape increases, we are able to construct models that describe our universe more precisely. Also, ST has proved to be a very rich and interesting mathematical structure that provides non-trivial connections

[3] This definition of vacuum only requires it to be stable so in general it will not correspond to having 'empty' space which we refer to as the true vacuum. Actually, one should require a weaker condition, the vacuum may be metastable as long as its lifetime is much larger than the age of the universe.

between different physical theories.

At a more fundamental level, given the vast landscape (and therefore possible universes), a natural question is what is the dynamical mechanism that selects a particular vacuum. This opens the possibility of realising the anthropic principle which might be the solution to some puzzles that seem to lack a conventional explanation (e.g. cosmological constant problem). It could very well be so, however, there is still no good proposal for a vacuum selection mechanism and even if there was it is not clear that it could ever be tested experimentally.

1.2 The Standard Model within String Theory

In this thesis we study some aspects of a small corner of the landscape that looks phenomenologically promising (from our limited perspective). In particular, we focus on the embedding of the SM or its minimally supersymmetric extension (MSSM) in string theory. Ultimately one should worry about other issues such as inflation and cosmology, however, it is still soon to address everything at once.

The SM has several features (gauge group and matter content) and parameters (couplings, etc.) for which there is no deep explanation and are simply measured experimentally. From the ST perspective though, these must be related to the particular vacuum we pick and, at least in principle, given a vacuum one should be able to compute such low-energy data.

Let us briefly mention the most important aspects of the SM. It is a four-dimensional gauge theory with gauge group $SU(3) \times SU(2) \times U(1)$ with charged chiral fermions, see table 1.1. It also contains a complex scalar charged under $SU(2) \times U(1)$, the Higgs field H, which triggers electroweak symmetry breaking since it has a non-zero vev, $\langle |H|^2 \rangle = (246 \text{ GeV})^2$.

Field	# of copies	Spin	$SU(3)$	$SU(2)$	$U(1)$
Gluons	1	1	$\mathbf{8}$	$\mathbf{1}$	0
$W^\pm$, W^0	1	1	$\mathbf{1}$	$\mathbf{3}$	0
B	1	1	$\mathbf{1}$	$\mathbf{1}$	0
H	1	0	$\mathbf{1}$	$\mathbf{2}$	$-\frac{1}{2}$
l	3	$\frac{1}{2}$	$\mathbf{1}$	$\mathbf{2}$	$-\frac{1}{2}$
$\bar{e}$	3	$\frac{1}{2}$	$\mathbf{1}$	$\mathbf{1}$	1
q	3	$\frac{1}{2}$	$\mathbf{3}$	$\mathbf{2}$	$\frac{1}{6}$
$\bar{u}$	3	$\frac{1}{2}$	$\bar{\mathbf{3}}$	$\mathbf{1}$	$-\frac{2}{3}$
$\bar{d}$	3	$\frac{1}{2}$	$\bar{\mathbf{3}}$	$\mathbf{1}$	$\frac{1}{3}$

Table 1.1: Field content of the Standard Model. The fermions correspond to left-handed Weyl spinors.

The Higgs mechanism makes the $W^\pm$ bosons massive ($M_W = 80$ GeV) as well as the

Z boson ($M_Z = 91$ GeV) that corresponds to the combination $Z = -\sin\theta_W B + \cos\theta_W W^0$, where θ_W is the weak angle ($\sin^2\theta_W = 0.23$ at M_Z). The orthogonal combination remains massless and is identified with the photon γ. The interaction of the fermions with the vacuum of the Higgs field, via the Yukawa operators, generates masses for all the quarks and charged leptons, see figure 1.1. Finally, the Higgs boson acquires a mass $m_H \simeq 126$ GeV.

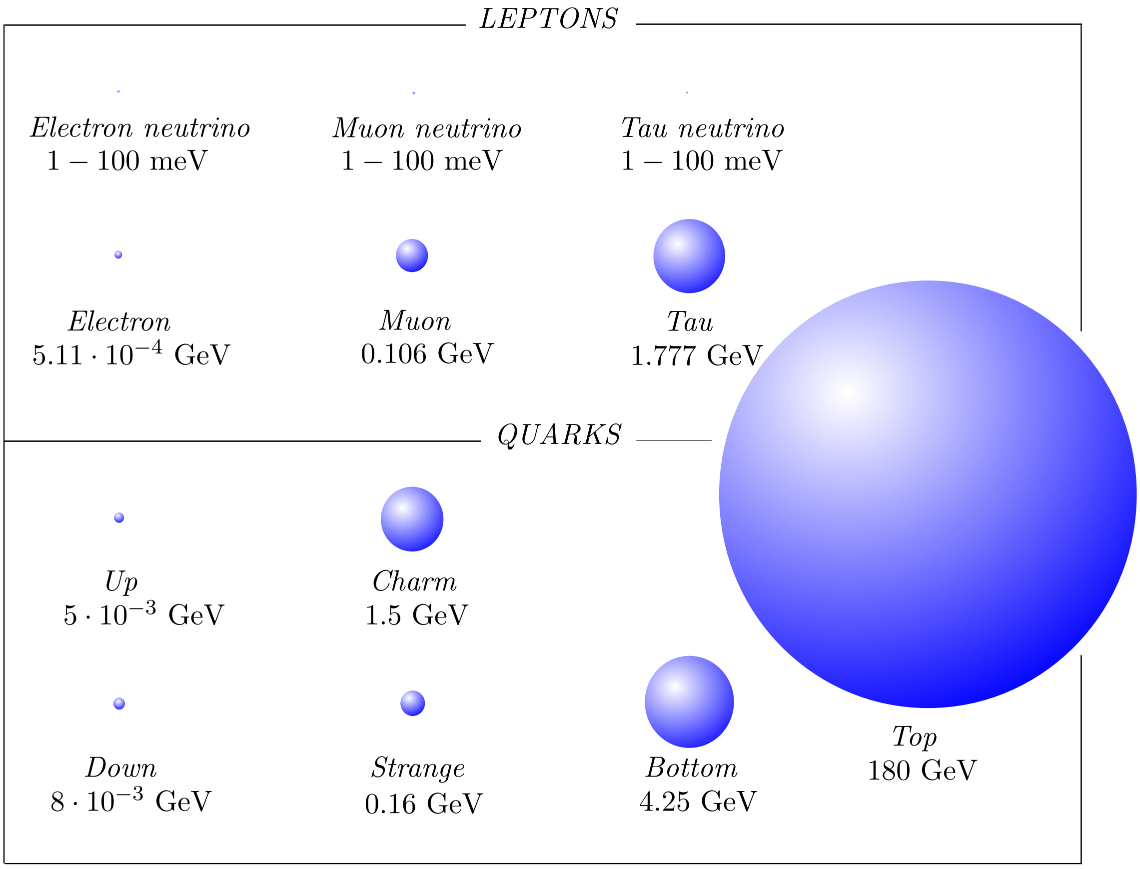

Figure 1.1: Masses of leptons and quarks.

As it stands, the SM is in very good agreement with experiments up to the TeV scale. However, even ignoring gravity, there are some experimental results that are not correctly described by the SM which include neutrino masses and dark matter. There are several extensions of the SM that allow to explain them but these have not been confirmed by experiment yet.[4] One of the extensions that has received much attention is the MSSM in which the fields are promoted to superfields that contain particles of spins differing by one half.

This means that we should find a string vacuum that reduces to the SM (or some extension) at the TeV scale. The most important step towards this is to reduce the number of 'visible' dimensions from ten to four which is usually achieved by considering that six of

[4] We emphasise that the deep reason to consider ST is gravity. There is no reason to believe that these problems cannot be solved within conventional QFT.

the spatial dimensions form a compact space. If such compact space is sufficiently small we need a large energy to resolve it and at the typical energies of the SM the theory becomes effectively four-dimensional. The scale at which the extra dimensions are visible is called the compactification scale M_c and is believed to be larger that 10^{16} GeV although models with much lower scale have also been considered. Then, the low-energy features depend on this compact space as well as in the vevs of the stringy fields in them.

For instance, in chapters 2 and 3 we construct MSSM-like vacua focusing on mechanisms that lead to a pattern of Yukawa couplings that reproduces the masses in figure 1.1.

1.3 Plan of the thesis

This thesis is organised as follows. In chapter 2 we start by considering the idea of grand unification in both field and string theory by means of simple examples. In particular, we introduce a class of models based on intersecting/magnetised D7-branes in type IIB ST and show that, for phenomenological reasons, it is convenient to reconsider them within F-theory. After, a brief overview of F-theory we rephrase these D-brane models in those terms and discuss some generalities about model building. The effective theory of exceptional 7-branes is discussed, especially the computation of Yukawa couplings in local models which is explained in detail and how to build such models. We provide a non-perturbative mechanism to generate a hierarchical pattern of Yukawa couplings in qualitative agreement with experimental data. Then, we explicitly build two different local models to describe both of the Yukawa operators in $SU(5)$ unification, the $SO(12)$ and E_6 models. In appendix A we collect all the relevant wavefunctions and normalisation factors for the $SO(12)$ and E_6 models.

In chapter 3 we explore an alternative mechanism to generate textures in Yukawa matrices based on discrete symmetries. We start with a general discussion about discrete symmetries distinguishing between global and gauge. Then we consider a compactification on a manifold with discrete isometries which yields a theory with non-abelian discrete gauge symmetries in lower dimensions. We show that it is closely related to magnetised D-branes in which the non-abelian symmetry acts on the flavour degrees of freedom which allows to constrain Yukawa couplings. Taking this example as starting point, we study non-abelian flavour symmetries in both intersecting D6- and magnetised D9-branes. In order to make the discussion concrete we consider the $\mathbb{Z}_2 \times \mathbb{Z}_2'$ toroidal orientifold. We also present a couple of Pati-Salam models which enjoy non-abelian flavour symmetries and we discuss how the Yukawa couplings get constrained. Appendix B contains a brief review of the relevant facts about the $\mathbb{Z}_2 \times \mathbb{Z}_2'$ orbifold and appendix C includes the dimensional reduction of the D6-branes at angles and magnetised D9-branes that shows the appearance of discrete gauge symmetries.

In chapter 4 we study the kinetic mixing between different $U(1)$ gauge groups in Type II D-brane models. We start computing the open-closed mixing for D6-branes in Type IIA by performing the dimensional reduction of the effective action for the D-branes. Then, we show that this mixing can be understood in terms of the Witten effect by looking at the space of magnetic monopoles of the compactification which allows to arrive at a general expression for the mixing. Also, this point of view provides a formula that resembles the closed-closed mixing and, according to this idea, we propose a supergravity formula for the mixing between open string $U(1)$s that corresponds to the one-loop corrected expression in the open string channel. Furthermore, we analyse the same problem in the context of magnetised D7-branes which is particularly interesting for F-theory model building. This case, although more involved, works very much in the same way when expressed in terms of generalised complex geometry, that seems to be a convenient tool to describe D-branes with worldvolume fluxes. Using this method we compute the kinetic mixing of the hypercharge $U(1)$ with the closed string sector $U(1)$s in F-theory $SU(5)$ GUTs. Finally, we propose a formula for the open-open kinetic mixing in 'closed string variables' that includes one-loop corrections for this case too. Appendix D reviews the concept of gerbe associated with a cycle and its role in the definition of linear equivalence. The next appendix contains the M-theory lift of parallel D6-branes and the origin of the forms ϖ that appear in the formula for the kinetic mixing from this perspective. Some basic definitions of generalised homology are given in appendix F which are useful in section 4.2. The last appendix contains some technical details about the derivation of the formula 4.90.

2

Flavour in F-theory local GUTs

In this chapter we study the pattern of Yukawa couplings within local F-theory grand unified theories (GUTs). We start with an overview of grand unification both in field and string theory and, in particular, we present a class of models based on magnetised D7-branes. We discuss why it is convenient to reconsider them in terms of F-theory which we review briefly and discuss some phenomenological aspects of these models. Moreover, we explain in detail how to build ultra-local models using the 7-brane effective action and to compute wavefunctions and Yukawa couplings, both holomorphic and physical. We also explore in this context the concept of T-brane which is useful in order to generate up-type Yukawa couplings. Since generically these Yukawa couplings have rank one, we review the necessary non-perturbative effects that allow to go beyond rank one and generate hierarchical masses for the lighter generations. In the last two sections we build two local models that correctly reproduce the observed hierarchies for down-type particles and up-type quarks which we call $SO(12)$ and E_6 models respectively.

2.1 Grand unified theories

The idea of grand unification is a vast subject both in field and string theory so we will only touch upon some aspects by means of simple examples. Hopefully this will give an idea of the similarities and differences between the two approaches as well as the successes and challenges of each of the constructions.

2.1.1 Grand unification in field theory

There are some intriguing features of the parameters of the Standard Model at low energies ($\sim M_W = 80$ GeV) that hint a remarkable structure at high energies. Indeed, consider the gauge coupling constants α_1, α_2, α_3 ($\alpha_i = g_i^2/4\pi$) and extrapolate their value to higher energies Q using the renormalisation group equations (assuming no new physics), namely

$$\frac{1}{\alpha_i(Q)} = \frac{1}{\alpha_i(M_W)} - \frac{b_i}{4\pi} \log\left(\frac{Q^2}{M_W^2}\right), \tag{2.1}$$

where b_i are the one-loop β-function coefficients which for the SM read $(b_1, b_2, b_3) = (\frac{41}{10}, -\frac{19}{6}, -7)$. As can be seen in Figure 2.1 the coupling constants tend to unify at a scale of the order 10^{15} GeV. Of course, the agreement is far from being perfect but our assumption of no new physics is very strong (and unjustified) so this plot should be taken with a grain of salt.

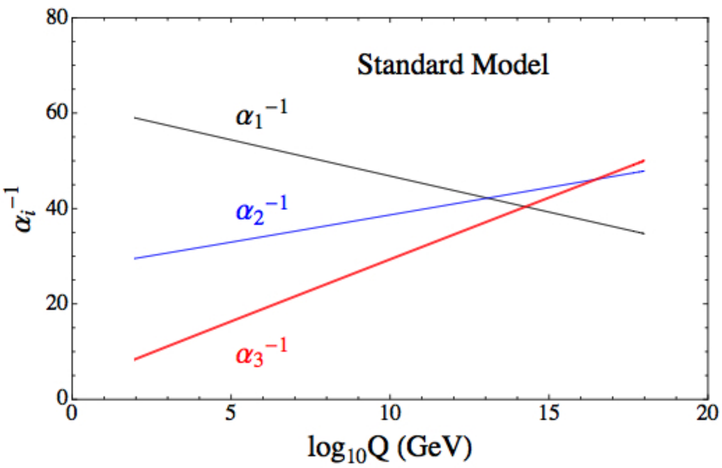

Figure 2.1: Running of the SM coupling constants at one loop assuming no new physics.

One can however consider the possibility that the coupling constants do meet at some energy M_{GUT} because beyond that scale there is a unified gauge theory with gauge group G that is broken down to the SM group by virtue of a Higgs mechanism [2] (see e.g. [3] for a review). The group G should be big enough to contain $SU(3)\times SU(2)\times U(1)$ as a subgroup and should admit complex representations to allow for chirality. One can check that the minimal possibility is $SU(5)$ although $SO(10)$ and E_6 have also been extensively studied. We will focus on the first possibility for concreteness and because it is the case we will explore in subsequent sections.

Let us see how the field content in the SM fits in representations of $SU(5)$. The gauge bosons can be accommodated in the adjoint representation, namely

$$\begin{aligned} SU(5) &\longrightarrow SU(3)\times SU(2)\times U(1) \\ \mathbf{24} &\longrightarrow (\mathbf{8},\mathbf{1})_0\oplus(\mathbf{1},\mathbf{3})_0\oplus(\mathbf{1},\mathbf{1})_0\oplus(\mathbf{3},\mathbf{2})_{-\frac{5}{6}}\oplus(\bar{\mathbf{3}},\mathbf{2})_{\frac{5}{6}}\,. \end{aligned} \tag{2.2}$$

The first two entries in each term denote the representations under $SU(3)$ and $SU(2)$ respectively while the subscript corresponds to the charge under the hypercharge $U(1)$ which is defined as

$$Q_Y=\frac{1}{6}\,\mathrm{diag}(-2,-2,-2,3,3) \tag{2.3}$$

and satisfies $Q_Y=Q-T^3$, where Q and T^3 are respectively the electric charge and third component of weak isospin. The canonical hypercharge generator $\tilde{Q}_Y$, defined such that $\mathrm{Tr}\,(\tilde{Q}_Y^2)=\frac{1}{2}$, is related to Q_Y by $\tilde{Q}_Y=\sqrt{\frac{3}{5}}Q_Y$. From the unfolding (2.2) we see that we obtain the gauge bosons for the SM group as well as some additional bosons in the representation $(\mathbf{3},\mathbf{2})_{-\frac{5}{6}}\oplus c.c.$ dubbed X, Y bosons.[1] If the $SU(5)$ group is broken at the scale $M_{GUT}\sim 10^{15}$ GeV by a Higgs mechanism the mass of the X, Y bosons is of the order of M_{GUT} which is the reason why we do not observe them at the weak scale.

The matter content of the SM also fits nicely in $SU(5)$ representations [2], as follows. A family of left-handed leptons and quarks corresponds to the representation $\bar{\mathbf{5}}_M\oplus\mathbf{10}_M$

[1]More precisely one should write $X_i^\pm$, $Y_i^\pm$ with $i=1,2,3$ a colour index, X and Y label the different components of an $SU(2)$ doublet and the $\pm$ distinguish conjugate representations.

as can be checked from the decomposition of these representations under the higgsing of the gauge group. More explicitly,

$$\begin{aligned} SU(5) &\longrightarrow SU(3)\times SU(2)\times U(1) \\ \bar{\mathbf{5}}_M &\longrightarrow (\bar{\mathbf{3}},\mathbf{1})_{\frac{1}{3}}\oplus(\mathbf{1},\mathbf{2})_{-\frac{1}{2}} \\ \mathbf{10}_M &\longrightarrow (\bar{\mathbf{3}},\mathbf{1})_{-\frac{2}{3}}\oplus(\mathbf{3},\mathbf{2})_{\frac{1}{6}}\oplus(\mathbf{1},\mathbf{1})_{1}. \end{aligned} \tag{2.4}$$

Comparing this with the table 1.1 one can check that indeed, the matter content is that of a generation of the SM. It is quite amusing to notice that $\bar{\mathbf{5}}\oplus\mathbf{10}$ is the simplest anomaly-free combination of chiral fermions that one can include in this gauge theory. However, the fact that there are exactly three families remains unexplained in this construction. It is also interesting to notice that since the electric charge generator Q belongs to $SU(5)$, it is traceless which implies that the sum of the charges of the fermions in a given representation should vanish. For instance, from the $\bar{\mathbf{5}}$ one finds that the charge of the quark d is one third of that of the electron. This kind of relations can be used to see that the charge of the proton is (minus) the charge of the electron and that all electric charges are multiples of a basic unit of charge.

The SM Higgs doublet sits in a $\mathbf{5}_H$ of $SU(5)$ which forces us to introduce additional matter in the representation $(\mathbf{3},\mathbf{1})_{-\frac{1}{3}}$, usually referred to as colour triplets. Since these particles have not been observed one should construct a model in which the triplets are much heavier that the doublets, known as doublet-triplet splitting, which typically involves fine-tuning.

There is another remarkable prediction of this simple model which is the weak angle. At the unification scale M_{GUT} the three coupling constants satisfy

$$g_3^2 = g_2^2 = \frac{5}{3}g_1^2 \tag{2.5}$$

where the factor $\frac{5}{3}$ comes from the relative normalisation of Q_Y and $\tilde{Q}_Y$ discussed above. Thus, the weak angle satisfies

$$\sin^2\theta_W = \frac{g_1^2}{g_1^2+g_2^2} = \frac{3}{8} \tag{2.6}$$

at energies where the $SU(5)$ gauge symmetry remains unbroken. However, at lower energies each coupling runs independently according to eq.(2.1) so the weak angle can be extrapolated to the weak scale (again, assuming no new physics) and compared to the measured value. This yields $\sin^2\theta_W(M_W)\approx 0.21$ in good agreement with the experimental value [4] $\sin^2\theta_W(M_W)=0.23$, given the strong assumptions we made to arrive at such number. Notice that in the SM the weak angle is a free parameter that is fixed by observation and the fact that it is so close to the value predicted by this simple grand unified theory is rather impressive and provides further evidence in favour of this proposal.

There are, however, serious drawbacks to this model. For example, due to the existence of the X, Y bosons charged under both $SU(3)$ and $SU(2)$ one can convert quarks into leptons and quarks into anti-quarks. These processes are responsible for proton decay with a lifetime [3] $\tau_p\sim 10^{30}\,yr$ in contradiction with the Super-Kamiokande experimental bounds [5] $\tau_p > 5\cdot 10^{33}\,yr$.

Another serious problem with this model is the pattern of Yukawa couplings. As explained in the introduction the Yukawa couplings in the SM are completely unrelated for the leptons, down and up quarks but in GUTs the Yukawa operators for leptons and down-type quarks have a common origin. Indeed, there are two different Yukawa terms in the Lagrangian, one for the $\bar{\mathbf{5}}_M$ and another one for the $\mathbf{10}_M$, schematically

$$\begin{aligned} Y_U^{ij} \quad &: \quad \mathbf{10}_M^i \times \mathbf{10}_M^j \times \mathbf{5}_H \\ Y_{D,L}^{ij} \quad &: \quad \bar{\mathbf{5}}_M^i \times \mathbf{10}_M^j \times \bar{\mathbf{5}}_H . \end{aligned} \tag{2.7}$$

Thus, at the GUT scale we have $Y_D = Y_L$ which gives boundary conditions for the renormalisation group equations. In particular, one finds that [6] $\frac{m_b}{m_\tau} \approx 3.0$, $\frac{m_s}{m_\mu} \approx 4.0$ in reasonably good agreement with experiment, however the analogous prediction for the first generation fails by an order of magnitude.

Now one can try to fix the problems that have been pointed out, retaining the desirable properties, within field theory by constructing more involved models. In fact, this has been subject of extensive work since the paper by Georgi and Glashow in the mid-1970s which we will not discuss further.

Let us just mention that including low scale supersymmetry in the model makes some of the predictions (e.g. gauge coupling unification and weak angle) in better agreement with observations but generically still suffers from unacceptable flaws such as rapid proton decay[2] or fermion mass relations. In the minimal supersymmetric case (MSSM) one needs to include two Higgs doublets, up-Higgs $\mathbf{5}_U$ and down-Higgs $\bar{\mathbf{5}}_D$, in order to cancel anomalies and to generate masses for all the SM fermions. The Yukawa couplings read

$$\begin{aligned} Y_U^{ij} \quad &: \quad \mathbf{10}_M^i \times \mathbf{10}_M^j \times \mathbf{5}_U \\ Y_{D,L}^{ij} \quad &: \quad \bar{\mathbf{5}}_M^i \times \mathbf{10}_M^j \times \bar{\mathbf{5}}_D . \end{aligned} \tag{2.8}$$

This concludes our brief overview of GUTs in field theory and we turn to the embedding of this idea in the context of string theory in the next section.

2.1.2 Grand unification in string theory

Let us discuss qualitatively the idea of grand unification in the realm of ST. We will focus on the two main constructions in perturbative ST: heterotic and type II D-brane models by means of simple examples that illustrate some of the general features. As we will see there are a number of differences compared with the field theory models which stem from the fact that ST provides new ingredients such as extra dimensions and higher dimensional dynamical objects. We will omit many technical details in order to get to the phenomenological implications as soon as possible. The details can be found in e.g. [1] and references therein.

Heterotic models

The most straightforward route to achieve non-abelian gauge symmetries in 4d is by compactifying the HE, HO or type I string theories since these contain large gauge groups

[2]In these models the scale of unification is about ten times higher which reduces the rate of proton decay. However, due to the new particles there are additional decay channels which generically make the proton lifetime smaller than the experimental bound.

already in 10d. Remarkably enough, one of the first attempts to build realistic models in ST, known as 'standard embedding', yields an $\mathcal{N}=1$, E_6 GUT without much effort (see [7] for a nice discussion on the subject).

Consider the HO theory on $M_4 \times X$ where M_4 is 4d Minkowski and X is a 6d manifold. The conditions to have unbroken $\mathcal{N}=1$ SUSY in 4d amount to taking X a manifold with $SU(3)$-structure (to lowest order in α'). The simplest realisation is to take vanishing H-flux and constant dilation, $H = d\phi = 0$, which makes X a manifold of $SU(3)$ holonomy (i.e. Calabi Yau 3-fold). The only condition that has to be met is the Bianchi identity which reads

$$\mathrm{Tr}F^2 = \mathrm{Tr}R^2 \tag{2.9}$$

where F is the $E_8 \times E_8$ field strength and R is the Ricci form. The so-called 'standard embedding' takes the gauge field to be equal to the spin connection so (2.9) is immediately satisfied. This triggers the breaking $E_8 \times E_8 \to E_8 \times E_6$. The dynamics of E_8 and E_6 decouple so we may focus on the latter which is a good candidate to be a grand unification group (admits complex representations unlike the former). One can check that upon this breaking we get fields that are charged under E_6 in the **27** representation which has room to accommodate all the particles in the SM. Namely, the decomposition of the adjoint of E_8 is

$$\begin{aligned} E_8 \quad &\longrightarrow \quad SU(3) \times E_6 \\ \mathbf{248} \quad &\longrightarrow \quad (\mathbf{3}, \mathbf{27}) \oplus (\bar{\mathbf{3}}, \bar{\mathbf{27}}) \oplus (\mathbf{8}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{78}). \end{aligned} \tag{2.10}$$

In this construction the net number of chiral families is related to the Euler number of the internal manifold X which gives a nice geometrical interpretation to a property that is completely obscure in field theory (although it does not explain why there are three and not any other number).[3] It is not straightforward to find a model in which we have three generations but, given the number of simplifying assumptions, this still looks like a rather promising starting point to building a GUT in ST.

In order to break the GUT group down to the SM group one has to use a new mechanism, absent in conventional field theory, so as to preserve the nice properties of the previous solution. The reason for this is that the only field available (in this approximation) to break the GUT symmetry below the string scale is the E_6 gauge field, but giving a vacuum expectation value to such a field would generically generate a flux that breaks supersymmetry. However, the extra dimensions allow to give a vev to the gauge field while keeping the equations of motion unchanged, as follows. Consider an internal manifold X which is not simply connected, $\pi_1(X) \neq 0$, and imagine giving a vev to the gauge fields A such that the field strength vanishes but

$$W(\Gamma) = P \exp\left[i \oint_\Gamma A\right] \neq 1 \tag{2.11}$$

where Γ is a non-contractible loop in X. Notice that this quantity is gauge-invariant so it cannot be set to one by a gauge transformation. The kinetic term for the E_6 gauge bosons F^2, when expanded around this vacuum, will give masses to the generators that do not

[3] The net number of chiral families is counted by the index of the Dirac operator [7] in the representations **3** and $\bar{\mathbf{3}}$ since the massless fermions must satisfy the Dirac equation. In general, this depends on the gauge flux but since in this case the flux is related to the geometry through the embedding of the spin connection one finds that the number of generations is related to the geometry of the internal manifold.

commute with $W(\Gamma)$ so one can engineer models in which the Wilson lines (2.11) are such that the remaining gauge group is that of the SM. Notice that since the Wilson lines are a global property, the equations of motion remain unaffected and the previous solution is still valid.

This example illustrates an amusing difference between QFT in 4d and ST. On the one hand, in QFT one has much more freedom to add additional fields into the game while ST is more restrictive in that respect. On the other, ST is a richer structure that often provides new mechanisms that are unavailable in QFT in 4d.

Let us discuss now some phenomenological implications of these models. Calabi-Yau manifolds X have a finite fundamental group F and there exists a simply connected finite cover $\tilde{X}$ such that $X = \tilde{X}/F$ so $\tilde{X}$ admits the action of F.[4] It is convenient to formulate the physics in $\tilde{X}$ and impose certain constraints on the fields to obtain the spectrum in X. The Wilson lines $W(\Gamma)$ may be regarded as a homomorphism $f \to W(\Gamma)$ from F into E_6 which yields the breaking $E_6 \to G_{SM} \times \hat{F}$ where $\hat{F}$ is the image of such homomorphism and G_{SM} is the remaining part of the commutant (which should be the SM group). Therefore we have the following unfolding

$$\begin{aligned} E_6 \quad &\longrightarrow \quad G_{SM} \times \hat{F} \\ \mathbf{27} \quad &\longrightarrow \quad \bigoplus_i R_i \oplus T_i \end{aligned} \tag{2.12}$$

where R_i is a representation of G_{SM} and T_i of $\hat{F}$. The charges that can be measured in 4d are those corresponding to R_i however, T_i is still important indirectly as we explain in the following.

Let $\psi(x)$ be a matter field in $\tilde{X}$ transforming in the $\mathbf{27}$ under E_6. Since this is a charged field it will feel the Wilson lines introduced to break the GUT group and the field that actually survives in the quotient X must satisfy $\psi(fx) = W(\Gamma)\psi(x)$. The fact that this condition is met depends on the representation of $\psi(x)$ under $\hat{F}$ so only part of the $\mathbf{27}$ is well defined on X and is part of the theory. This has profound phenomenological consequences since, as opposed with the field theory models, different SM particles come from different $\mathbf{27}$ multiplets. For example, the down-type quarks and charged leptons will originate from different $\mathbf{27}$s in X.

At the end of the day this means that there is no relation among the Yukawa couplings of the different particles of the SM which, as we saw in the previous section, is often troublesome to fit with observations. On the other hand, there is a single gauge field in $\tilde{X}$ so the different gauge bosons in 4d have a common origin meaning that the gauge coupling unification and $\sin^2\theta_W$ GUT predictions remain true. Moreover, this provides an elegant way to achieve doublet-triplet splitting that does not introduce fine-tuning.

The computation of Yukawa couplings in heterotic models is somewhat similar to the method used in F-theory in the field theory approximation that will be explained in subsequent sections. For that reason we refrain from discussing it here and refer the reader to section 16.6 in [7] for an excellent account of the subject.

As usual, the simplest realisation does not quite work quantitatively (see e.g. [1]) so one can take this example as the starting point to building more realistic models (see [8] for an extensive review). This is, however, quite a challenging task since in the heterotic

[4] We will assume this action to be free in order to avoid orbifold singularities in X which would force us to introduce additional degrees of freedom at the singularities.

string both gauge and gravitational interactions have a common origin and therefore the GUT coupling and Newton's constant are related. It has been noticed [8] that typically the prediction for Newton's constant is about 400 times larger than what it should. In [9] it was suggested that one may overcome this problem by going into the strong coupling regime of the heterotic string, meaning M-theory compactified on $\mathbf{S}^1/\mathbb{Z}_2$ or type I string theory.

There is another issue concerning the heterotic models, namely, *moduli stabilisation*. In supersymmetric compactifications of string theories one typically finds a large number of massless scalars in 4d, dubbed moduli, which is certainly not desirable.[5] Indeed, these fields must couple at least gravitationally to matter and appear as long range forces that have not been observed. Of course, upon SUSY breaking these fields generically acquire a mass of the order of the SUSY breaking scale due to quantum corrections so it appears that it is not such a big threat. However, having a minimum in the potential that is generated for the moduli typically requires having higher-order quantum correction being more important than the lower-order ones meaning that we are in a non-perturbative regime.[6]

To sum up, GUT models seem quite natural in the context of heterotic strings. One can retain some of the nice properties of field theory unification such as gauge coupling unification and a reasonable prediction for the weak angle while avoiding the problematic fermion mass relations. On the down side, perturbative models tend to predict a too large Newton's constant and moduli stabilisation is poorly understood in this context.

D-brane models

Let us change gears and turn our attention towards another corner of the string landscape: D-brane models. We will follow the same approach as before and describe very briefly a simple realisation of a GUT model within this setup which illustrates some general properties of these constructions. There are plenty of models that one can build and we will choose the one that is closely related to the F-theory model that will be discussed in the following sections which is based on intersecting and magnetised D7-branes.

Type II string theories contain non-perturbative extended objects, known as D-branes, that are of utmost importance for various reasons. Namely, they proved to be essential tools that triggered the second superstring revolution, they were central in the formulation of Maldacena's correspondence, have provided a microscopic understanding of the entropy of certain black holes, etc. For our present purposes D-branes are interesting due to the fact that they carry gauge interactions in their worldvolumes which can be used to construct SM-like solutions of ST that are significantly different from their heterotic counterparts. A review of this vast subject is beyond the scope of this work so we refer the reader to [12] for a pedagogical introduction.

Let us start with a similar solution to the one we encountered in the heterotic theory in the context of type IIB. Consider type IIB on $M_4 \times X$ where X is a CY 3-fold which can be seen to yield an $\mathcal{N} = 2$ theory in 4d with no non-abelian gauge interactions.

In order to incorporate gauge interactions we may include a stack of five D7-branes

[5] Examples include the string coupling constant g_s and the volume of the CY 3-fold. In general there may be as much as thousands of these moduli.

[6] This is usually referred to as the Dine-Seiberg problem [10]. See e.g. chapter 2 of [11] for a clear discussion.

on $M_4 \times S_{GUT}$ with S_{GUT} a holomorphic divisor in X. For consistency reasons (tadpole cancelation) we should take an orientifold projection which amounts to finding a $\mathbb{Z}_2$ action that is well-defined on the theory and truncating it to the invariant sector.[7]

This yields a $U(5)$ gauge theory[8] on its worldvolume that descends to a gauge theory in 4d and can be arranged in such a way that preserves $\mathcal{N} = 1$ supersymmetry.[9]

A crucial difference with the previous example is that in this case the origin of the gravitational and gauge interactions is different and the respective coupling constants are not related. Roughly, these scale as [1]

$$G_N^{4d} \propto (\mathrm{Vol}(X))^{-1}, \qquad (g_{YM}^{4d})^2 \propto (\mathrm{Vol}(S))^{-1} \tag{2.13}$$

which makes the models more flexible since both quantities can be, in principle, tuned at will. This is starting to look like a good candidate to being a $SU(5)$ GUT model so let us proceed and see how far one can get. Thus far, we lack chiral matter transforming in the desired representations under $U(5)$ which means that we must include additional ingredients in our construction.

Consider adding an extra spacetime filling D7-brane on $S' \subset X$ that intersects the original stack in a complex curve $\Sigma = S_{GUT} \cap S'$, dubbed matter curve, which introduces an additional $U(1)'$ gauge group. At the intersection one finds new degrees of freedom whose microscopic description are open strings stretched between the two D-branes. These new fields have two important properties, namely, they are massless since the length of the open string is zero (so they live at the intersection $M_4 \times \Sigma$) and transform in the $\mathbf{5}_{-1} \oplus \bar{\mathbf{5}}_1$ of $U(5) \times U(1)'$. This seems a reasonable way to obtain the $\mathbf{5}$s that accommodate the down-type quarks, charged leptons and Higgs bosons, however, as it stands the model is non-chiral. Therefore we have to break parity in the internal directions somehow in order to generate a net number of chiral generations in M_4, as follows.

In principle, nothing prevents us from including a gauge flux in the $U(1)'$ of the brane at S' (as long as it preserves Poincaré invariance in 4d) usually referred to as magnetic flux, which breaks parity as desired. For consistency this flux F' has to satisfy certain quantisation conditions, meaning that its integral over any 2-cycle $\sigma \subset S'$ must be an integer, more precisely, $\int_\sigma F \in 2\pi\mathbb{Z}$. In particular, this holds for the intersection cycle Σ so $\int_\Sigma F = 2\pi n$. When compactifying to 4d [7] we have to solve Dirac's equation on Σ for the charged matter which couples to the said flux restricted to Σ. The net number of families in M_4 is given by the Dirac index which, in this case, is precisely n.[10] We may use this mechanism three times, one that generates the three copies of $\bar{\mathbf{5}}_M$ living in Σ_5, another that generates the up-Higgs $\mathbf{5}_U$ in Σ_U and finally another to get the down-Higgs $\bar{\mathbf{5}}_D$ at Σ_D.

[7] At this point we are not explicit about these technicalities since we want to arrive at the phenomenological implications of the models as soon as possible. These constructions have been extensively studied, see e.g. [1] and references therein.

[8] This group can be decomposed as $U(5) = (SU(5) \times U(1))/\mathbb{Z}_5$ where the $\mathbb{Z}_5$ is the center of $SU(5)$. One refers to this extra $U(1)$ factor as the center of mass group. As we will see in the following it has very negative phenomenological implications.

[9] The D-branes contribute to the energy-momentum tensor and necessarily modify the Calabi-Yau condition. One can see that near the D7-branes their effect on the metric can be consistently neglected. Far away, however, this is not possible but let us not worry about that for the moment, we will reconsider this in the next section.

[10] This is a different avatar of the same mechanism that appeared in the heterotic model of the previous section. See footnote 3.

This was relatively simple since generically the endpoints of an open string transform in the (anti)fundamental representation of the gauge group living in the corresponding D-brane. However, it is not obvious how one can manage to include the **10** representation of $SU(5)$ needed to incorporate the up-type quarks. In the following we briefly discuss how this representation can be generated.

As was mentioned earlier this model is obtained by taking an orientifold projection of a CY compactification which amounts to taking a $\mathbb{Z}_2$ quotient of the theory. This action acts both in the worldsheet and in spacetime and it typically leaves 8d submanifolds invariant of the form $M_4 \times S_{O7}$ with S_{O7} a holomorphic divisor in X. At such fixed loci one finds the so-called orientifold 7-planes (or O7-planes). In perturbative ST these are non-dynamical objects with negative tension and electric charge under the RR potential C_8.[11] It can be shown [1] that when a stack of N D7-branes intersects an O7-plane, the orientifold projection produces charged matter at the intersection transforming in the antisymmetric representation of $SU(N)$ with charge $+2$ under the center of mass $U(1)$. Luckily enough, the **10** of $SU(5)$ is precisely the antisymmetric representation. Notice that had we needed a different representation, say the spinorial one, this construction would be impossible in perturbative type IIB.[12] This is the reason why $SO(10)$ unification cannot be realised since matter comes from the **16**.

In order to generate chirality for the **10**s one has to introduce fluxes as in the previous case. Let Σ_{10} be the intersection of S_{GUT} with S_{O7} such that the restriction of a magnetic flux along the center of mass $U(1) \subset U(5)$ has the appropriate first Chern class, i.e. $\int_{\Sigma_{10}} F$. This produces a net number of chiral fermions in M_4 in the **10** which completes the minimal spectrum to achieve $SU(5)$ unification.

Let us discuss Yukawa couplings in this construction. The down-type Yukawa coupling comes from $\mathbf{5}_M \times \mathbf{10}_M \times \mathbf{5}_D$ which should be invariant under the full gauge group in order to be non-zero. One can convince himself that this requires that the matter curve Σ_D comes from the intersection of S_{GUT} with the orientifold image of the D7-brane that contains $U(1)'$ and produces the $\bar{\mathbf{5}}_M$ when intersecting S_{GUT}. Thus, including the relevant $U(1)$ charges the coupling reads

$$Y_{D,L}^{ij} \quad : \quad \bar{\mathbf{5}}^i_{M,-1,+1} \times \mathbf{10}^j_{M,+2,0} \times \bar{\mathbf{5}}_{D,-1,-1} \tag{2.14}$$

which indeed contains a gauge invariant combination. Here the first subscript denotes the charge under the center of mass $U(1)$ while the second is the charge under $U(1)'$. This does not mean that the Yukawa coupling is necessarily non-vanishing, it just means that it is allowed by the symmetries.

As for the up-type Yukawa coupling, one can see that it is impossible to generate a coupling that is invariant under the overall $U(1) \subset U(5)$ since the $\mathbf{10}_M$ has charge $+2$ and there are two of these fields that enter in the Yukawa operator. Thus, it seems that this is as far as we can get since a vanishing Yukawa matrix for the up quarks is completely unacceptable. Let us take a closer look at the source of the problem to see if it can be fixed somehow.

[11] Negative tension objects are unstable under perturbations in General Relativity which is why they have to be non-dynamical in perturbation theory. At strong coupling one can see that they are dynamical objects that microscopically correspond to bound states of several objects with positive mass such that the binding energy accounts for the apparent negative tension [12].

[12] This stems from the fact that open strings in perturbation theory have two endpoints which is where the gauge charge is supported. Thus, only classical Lie groups may appear as gauge symmetries and charged matter is in bifundamentals, rank-2 symmetric or antisymmetric.

This D-brane construction produces a $U(5)$ gauge theory which is more than we need and the extra $U(1)$ is in fact what ruins the model so one may wonder if it is possible to get rid of it. In string theory $U(1)$ factors are typically massive due to interactions with closed string modes, known as Stückelberg mechanism. Indeed, in this example there is a term in the D7-brane effective action of the form (we neglect the contribution coming from the image branes for simplicity)

$$S_{D7} \supset \int\limits_{M_4 \times S_{GUT}} C_4 \wedge \mathrm{tr}\langle F \rangle \wedge \mathrm{tr} F \tag{2.15}$$

where C_4 is a RR potential, $\mathrm{tr}F$ corresponds to the center of mass $U(1)$ and $\mathrm{tr}\langle F \rangle$ denotes the vev needed to generate chiral matter in the $\mathbf{10}_M$. Upon compactification this produces a term in 4d which makes this $U(1)$ massive, namely[13]

$$S_{4d} \supset \int_{M_4} B \wedge \mathrm{tr} F \tag{2.16}$$

with B a 2-form that comes from reducing C_4 on a harmonic 2-cycle in X. However, this $U(1)$ remains as a global symmetry in perturbation theory and is only violated by non-perturbative corrections to the superpotential coming from instantons [1]. This means that it is not impossible to generate a Yukawa coupling for the up-type quarks but it will be exponentially suppressed which does not look very promising since the top Yukawa is of order one.

There is an alternative way of thinking of this Yukawa which explains why it is not generated in perturbation theory. The down-type Yukawa includes a **10** and two **5**s so it corresponds to the triple intersection of an O7-plane and two D7-branes which are mapped into each other by the orientifold action. This situation is well described by open string perturbation theory. On the other hand, the up-type Yukawa includes two copies of the **10** meaning that it arises from the intersection of two orientifold planes which lacks a perturbative description.

Similarly to the field theory and heterotic cases, the simplest example does not quite work so it is necessary to improve the model in order to overcome the problems that arise.

Let us summarise what we have learned from this example. In perturbative type II models it is possible to engineer $SU(5)$ models with the correct particle content but other grand unification groups such as E_6 or $SO(10)$ are directly impossible to implement. On the other hand, the localisation of the gauge dynamics on a submanifold (S_{GUT}) of the internal space leads to the conclusion that much of the physics related to the SM is insensitive to what happens far away and one can ignore gravitational effects in a first stage. Indeed, much of what we described did not depend on the particular CY 3-fold unlike in the heterotic constructions in which everything comes from closed strings so gauge interactions and gravity necessarily have to be addressed at once. Moreover, as opposed to the heterotic models moduli stabilisation is under better control in type IIB string theory. Although we did not mention it explicitly it is worth stressing that the field theory prediction for the weak angle is still valid while one has to worry (as usual in GUTs) about keeping the proton sufficiently stable.

[13] See section 3.1.1 for a detailed explanation of why this actually gives a mass to the corresponding gauge boson.

The main issue in these models is that the top Yukawa has to be generated by non-perturbative effects. This suggests that a non-perturbative version of these models may work since the $U(1)$ that forbids such coupling is generically a global symmetry in perturbation theory. Luckily enough, F-theory provides a strong coupling description of type IIB so it opens the door to a refined version of this kind of constructions.

2.2 Some aspects of F-theory $SU(5)$ GUTs

In this section we give an overview of F-theory which allows to rephrase to D7-brane model of the last section in a different language and we discuss model building in this context. Moreover, we introduce the 7-brane effective action and explain in some detail how to build local models.

2.2.1 F-theory

F-theory [14] can be regarded as a convenient way of thinking about type IIB string theory at the non-perturbative level in the string coupling g_s (but tree level in $\alpha' = \frac{l_s^2}{4\pi^2}$). There is, however, no microscopic description of the theory, unlike for the perturbative strings, so a deep understanding of what F-theory really is remains unknown. The best way to give a concrete definition is via duality with M-theory for which we also lack a microscopic definition but before doing so, let us study type IIB supergravity. See [11, 16, 17] for reviews on constructing F-theory vacua and model building.

F-theory via type IIB supergravity

The supergravity description of a string theory is a 10-dimensional field theory where the fundamental degrees of freedom correspond to the massless modes that one obtains from quantising the string. Since it treats the strings as point-like objects this corresponds to the zeroth-order term in an expansion in powers of α' that measures the length of the string. Alternatively, it can be regarded as a low energy approximation where the typical energies involved are too small to resolve the spatial extension of the string. It is, however, non-perturbative in the string coupling so one can obtain information about the theory that is unreachable in perturbation theory.

Type IIB has $\mathcal{N} = 2$ supersymmetry in 10d and the field content consists of two sectors, NS and RR. The NS sector contains a metric $g_{\mu\nu}$, a 2-form B_2 and a scalar ϕ known as the dilation. In perturbation theory the dilaton yields the string coupling through the relation $g_s = e^{\phi}$. In the RR sector we find several p-forms, namely, C_0 (RR axion), C_2 and C_4 with the associated field strengths $F_p = dC_p$. It is convenient to arrange these fields as follows

$$\begin{aligned}\tau &= C_0 + ie^{-\phi} \\ G_3 &= F_3 - \tau H_3 \\ \tilde{F}_5 &= F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3\end{aligned} \tag{2.17}$$

where we defined $H_3 = dB_2$. It is troublesome to write down an action for this theory since the field strength $\tilde{F}_5$ is self-dual. However, one may produce an action that yields the correct equations of motion and after one imposes the self-duality condition by hand. The bosonic part of the action is

$$\begin{aligned} S_{IIB} &= \frac{2\pi}{l_s^8}\left(\int d^{10}x\sqrt{-g}R \right. \\ &\left. -\frac{1}{2}\int \frac{1}{(\mathrm{Im}\ \tau)^2}d\tau\wedge * d\bar{\tau} + \frac{1}{\mathrm{Im}\ \tau}G_3\wedge * G_3 + \frac{1}{2}\tilde{F}_5\wedge * \tilde{F}_5 + C_4\wedge H_3\wedge F_3\right). \end{aligned} \tag{2.18}$$

One can check that (2.18) enjoys a $SL(2,\mathbb{R})$ symmetry that, in order to preserve the charge quantisation condition, is broken to $SL(2,\mathbb{Z})$ acting on the fields as

$$\tau \longrightarrow \frac{a\tau+b}{c\tau+d}, \qquad \begin{pmatrix} F_3 \\ H_3 \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} F_3 \\ H_3 \end{pmatrix} \tag{2.19}$$

with $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$ and leaving the metric and $\tilde{F}_5$ invariant. This discrete symmetry is known to be exact in perturbation theory and it is believed to survive at the non-perturbative level too (see e.g. [15]).

Consider the particular case of this transformation with $a = d = 0$ and $b = -c = 1$ and take $C_0 = 0$ for simplicity. This has the following effect

$$g_s \longrightarrow 1/g_s, \qquad F_3 \longleftrightarrow -H_3 \tag{2.20}$$

so it corresponds to a weak-strong transformation meaning that type IIB at very strong coupling is described by another type IIB theory related to the original one by field redefinitions. Moreover, since the fundamental string $F1$ couples to B_2 and the D1-brane to C_2 this shows that at strong coupling the fundamental degrees of freedom correspond to D1-branes while the $F1$ becomes a non-perturbative state.

More generally, this symmetry implies that the spectrum of BPS states must arrange in $SL(2,\mathbb{Z})$ multiplets. For instance, the F1 and D1 are particular examples of a more general object charged under both B_2 and C_2, namely (p,q)-strings. Here p and q denote the charges under B_2 and C_2 respectively. The magnetic duals of these are (p,q)-5-branes, bound states of D5- and NS5-branes. Similarly, one finds that the D7 is a particular instance of a more general kind of object, dubbed (p,q)-7-branes, magnetically charged under the axiodilation τ, and where (p,q)-strings can end. Finally, since $\tilde{F}_5$ is self-dual the D3-brane is not in a $SL(2,\mathbb{Z})$ orbit, rather, this group acts on it realising the Montonen-Olive duality.[14]

Notice that a single (p,q)-brane can always be brought to a $(1,0)$- or $(0,1)$-brane by choosing an appropriate $SL(2,\mathbb{Z})$ frame. However, for a state with a (p,q)- and a (p',q')-brane such that $pq' - p'q \neq 0$ (mutually non-local) this is not possible so there is no description in open string perturbation theory. Ultimately, this is what will allow us to construct a model very similar to the one introduced in the last section that allows a large top Yukawa. It is, however, not at all obvious how to describe these mutually non-local branes. Before moving into that let us take a closer look at the supergravity description of a Dp-brane.

The BPS solution of a Dp-brane for $p < 7$ along $x^0, \ldots, x^p$ is

$$\begin{aligned} ds^2 &= H_p^{-\frac{1}{2}}\eta_{\mu\nu}dx^\mu dx^\nu + H_p^{\frac{1}{2}}\delta_{ij}\,dx^i dx^j, \qquad e^{2\phi} = e^{2\phi_0}H_p^{\frac{3-p}{2}} \\ C_{p+1} &= e^{-\phi_0}(H_p^{-1}-1)\,dx^0\wedge\ldots\wedge dx^p, \qquad H_p = 1+\left(\frac{r_p}{r}\right)^{7-p} \end{aligned} \tag{2.21}$$

[14]The theory describing the dynamics of the D3-brane is $\mathcal{N} = 4$ SYM in 4d which is known to have a $SL(2,\mathbb{Z})$ weak-strong duality.

with $r_p^{7-p} \propto e^{\phi_0}$ and ϕ_0 the value of the dilaton asymptotically far away from the brane. This solution shows that the backreaction of the Dp-brane decays more rapidly the bigger the normal space is, as one might have expected. In particular, this means that we can analyse this configuration as follows. Far away from the D-brane we can choose the dilaton ϕ_0 such that the string coupling is small and do closed string perturbation theory on such background[15] and near the D-brane one can use open string perturbation theory.

On the contrary, for a D7-brane the normal space is two-dimensional and, intuitively, it is not sufficiently large to account for the backreaction. More explicitly, the axiodilaton in the vicinity of a D7-brane behaves logarithmically since it satisfies Poisson's equation in 2d, namely

$$\tau = \frac{1}{2\pi i} \log\left(\frac{z}{z_0}\right) + \ldots \tag{2.22}$$

where $z = x^8 + ix^9$ is the complex coordinate in the normal space and the dots include regular terms in z. The first thing one learns from this solution is that when encircling the D7-brane ($z \to e^{2\pi i}\, z$) the field τ undergoes a monodromy

$$\tau \longrightarrow \tau + 1, \tag{2.23}$$

consistent with the fact that the D7-brane has magnetic charge under C_0, i.e. $\oint_{\mathbf{S}^1} dC_0 = 1$. Notice that this monodromy corresponds to a $SL(2,\mathbb{Z})$ transformation, namely

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{2.24}$$

making the solution univalued. For general (p,q)-7-branes the monodromy can be computed and one can check that the associated $SL(2,\mathbb{Z})$ elements for two different 7-branes commute if and only if these are mutually local.

Near the D7-brane ($|z| < |z_0|$) this solution makes perfect sense, however, far away from it ($|z| > |z_0|$) the imaginary part becomes negative (making the string coupling negative). Moreover, by looking at Einstein's equations [12] one can see that this solution becomes locally flat at infinity but produces a deficit angle. All this means that having a single D7-brane sitting in infinite space is inconsistent so it is not at all clear that the model we presented in the last section is in fact consistent. More generally, it is non-trivial to know if a particular set of 7-branes defines a consistent vacuum.

A consistent configuration entails finding a solution to the equations of motion such that the metric is well-defined (except at the location of the 7-branes) and is modular invariant. The axiodilaton has to be a holomorphic function (to preserve $\mathcal{N} = 1$ SUSY) satisfying Im $\tau > 0$ everywhere and covariant with respect to $SL(2,\mathbb{Z})$. By analysing the monodromies of τ one can give an interpretation in terms of 7-branes which, in general, are mutually non-local. This implies that it is not possible to tune the value of the dilaton to have a small string coupling all over the space unlike what happens with lower dimensional branes.

F-theory is a clever way of formulating type IIB *supergravity* that allows to construct and study such solutions with 7-branes which is why it is often referred to as a

[15] Notice that as far as the background is concerned we need not take the string coupling small. It is only when we want to compute, say, string amplitudes on it that we have to restrict ourselves to the perturbative regime.

non-perturbative approach to type IIB. However, a truly non-perturbative microscopic description of type IIB string theory is still lacking.[16]

The key idea of F-theory is to regard the axiodilaton as the complex structure of an elliptic curve[17] fibered over the ten-dimensional spacetime. Roughly speaking, this means that we attach a 2-torus at every point in spacetime such that its shape (i.e. complex structure) varies as we move along the base. For a globally well-defined elliptic fibration the (position-dependent) shape of the fibre is given by a complex function τ such that $\operatorname{Im}\tau > 0$ and is covariant under the modular group of the fibre which are precisely the consistency condition that the axiodilaton must satisfy in type IIB.[18] We should stress that the elliptic fibre is *not* part of the physical spacetime, it is introduced just for convenience to encode the information about the axiodilaton and hence the 7-branes.

Let us be more concrete. An elliptic curve can be defined by the following equation

$$W = y^2 - x^3 - fx - g = 0 \tag{2.25}$$

with $x, y \in \mathbb{C}$ and f, g complex constants, usually referred to as Weierstaß equation.[19] The condition to have a smooth elliptic curve, meaning that $W = dW = 0$ has no solution, is equivalent to asking that the discriminant $\Delta = 27g^2 + 4f^3 \neq 0$.[20]

One can show that the shape τ of this curve is given in terms of f and g via the j-function

$$j(\tau) \equiv \frac{(\vartheta_3^8(\tau) + \vartheta_4^8(\tau) + \vartheta_2^8(\tau))^3}{8\eta^{24}(\tau)} = \frac{4(24f)^3}{\Delta} \tag{2.26}$$

where $\vartheta_n(\tau)$ are Jacobi theta functions and $\eta(\tau)$ is the Dedekind eta function.[21]

An elliptic fibration over spacetime (with coordinates z) is simply given by taking the Weierstraß form (2.25) and making the constants f, g depend on z. In this case the equation (2.26) defines τ as a function of z which should be interpreted as the axiodilaton.

A natural question is how the presence of 7-branes is encoded in the fibration. In order to see how this comes about let us look again at the solution associated to a D7-brane (2.22). One can check that at the location of the D7, $z = 0$, the axiodilaton diverges $\tau \to i\infty$. From the geometrical perspective, an elliptic curve such that $\tau = i\infty$ is *singular* so we are led to considering singular elliptic fibres as the geometrisation of 7-branes. From the type IIB point of view one can have different kinds of (p, q)-7-branes intersecting at angles in many possible ways. This is mapped to the structure of singularities associated to the corresponding fibration which is just as intricate.

The analysis of singular elliptic fibres started in the mathematical context by Kodaira around thirty years before the birth of string theory. In recent years this subject

[16] We should say that F-theory is not just type IIB supergravity since it includes the contributions coming from small instantons.

[17] In this context an elliptic curve is a complex curve of genus one.

[18] The condition of having a holomorphic axiodilaton translates into having a special kind of elliptic fibration to be discussed in the next section.

[19] One should define the Weierstraß model as a hypersurface in a weighted projective space if we want to have a compact curve. However, these technicalities are not essential at this level of discussion so we will avoid them for the sake of clarity.

[20] When $\Delta = 0$ the elliptic curve is singular and the corresponding singularity can be studied using techniques of algebraic geometry.

[21] In fact we should also choose two independent cycles α and β in the fibre and the axiodilaton is given by $\tau = \int_\beta \lambda / \int_\alpha \lambda$ where λ is the only holomorphic 1-form in the fibre.

has been revisited precisely because of its applications to F-theory and much progress has been made although there is still work to do in that direction. This is a very technical subject that we will not discuss in any detail since we will not really need it in subsequent sections. Let us just mention that from looking at the singular fibres one can read off both the gauge group associated with the system of 7-branes as well as the light matter. We will come back to this in the next section when we discuss F-theory in the light of M-theory.

F-theory via M-theory

We have seen that F-theory is formulated not in the physical spacetime but rather in an elliptic fibration over it. Thus far this is simply a mathematical trick that is convenient to encode the relevant information about the background, however, one can gain a deeper understanding of why this trick works by defining type IIB via duality with M-theory.

This duality consists of two steps: first one relates M-theory to type IIA by compactifying on a circle and then performs a T-duality transformation on another circle to get to type IIB on a dual circle. Thus, let us consider M-theory compactified on a two-torus $\mathbf{S}^1_A \times \mathbf{S}^1_B$ with radii R_A and R_B respectively and follow the dualities.

- Upon dimensional reduction along $\mathbf{S}^1_A$ one obtains type IIA on $\mathbf{S}^1_B$ with $g_s^{IIA} = \frac{R_A}{\sqrt{\alpha'}}$.
- T-duality along $\mathbf{S}^1_B$ yields type IIB on $\tilde{\mathbf{S}}^1_B$ with $g_s^{IIB} = \frac{\sqrt{\alpha'}}{R_B} g_s^{IIA} = \frac{R_A}{R_B}$.

Here $\tilde{\mathbf{S}}^1_B$ is a circle of radius $\frac{\alpha'}{R_B}$. In order to get type IIB on flat space one must take the so-called F-theory limit that corresponds to $R_B \to 0$ keeping the ratio $\frac{R_A}{R_B}$ fixed. This shows that the string coupling in type IIB is related to the complex structure of the M-theory torus as $1/g_s^{IIB} = \text{Im }\tau$. This is precisely the same relation with the elliptic fibre we introduced above. A detailed derivation can be found in [11] showing that indeed all the properties of the elliptic fibre match those of the M-theory torus.

This construction explains the appearance of an elliptic curve in type IIB supergravity and why it is not part of the physical space. Indeed, from the M-theory side the elliptic curve is part of spacetime and is no different from the other dimensions. However, when taking the F-theory limit only the complex structure of the fibre survives so there is no metric defined on it which is why it cannot possibly be a part of spacetime. In addition to that, it shows that the $SL(2, \mathbb{Z})$ invariance has a geometric origin since it corresponds to the large diffeomorphisms of $\mathbf{T}^2$ in M-theory.

One can now perform this chain of dualities fibrewise to obtain 4d vacua as follows. Consider M-theory compactified on an elliptically fibered 8-dimensional space Y with base X which yields a 3d vacuum. Dimensionally reducing along one of the circles in the fibre and T-dualising along the other one we obtain type IIB on $M_3 \times \mathbf{S}^1 \times X$. Taking the F-theory limit we decompactify the circle and obtain a solution with 4d Poincaré invariance. This allows, among other things, to obtain a low energy effective action for F-theory compactifications starting with the M-theory action in 3d [18]. From this perspective one can show that in order to obtain a supersymmetric configuration the space Y must preserve the Calabi-Yau condition meaning that from the type IIB point of view the axiodilaton is holomorphic. Notice that in general the base which corresponds to the physical space is not Calabi-Yau. This is because the derivatives of the axiodilaton contribute to the energy-momentum tensor making X a manifold with positive curvature.

Let us revisit the issue of gauge symmetries using this M-theory perspective. From our previous discussion we know that gauge symmetries are related to singularities in the elliptic fibrations so we should consider singular CY 4-folds Y. In order to study the physics on such singular spaces one often takes a continuous family of CYs such that Y belongs to it and the nearby spaces are smooth. One then studies the physics in the smooth spaces and eventually takes the limit to the original singular CY.

In the following we discuss qualitatively one of such methods that gives a physical interpretation of the appearance of gauge symmetries at singularities. Consider a singular point in Y such that replacing it by a 2-sphere (blow-up) defines a family of smooth CY parametrised by the volume of the sphere E (so Y belongs to it for the special case of vanishing volume). On the modified space $\tilde{Y}$ one can reduce the M-theory 3-form C_3 along the new 2-cycle which gives a gauge boson in 3d. In the limit $\tilde{Y} \to Y$ the gauge boson is still there so one finds a $U(1)$ gauge symmetry in the singular space too.[22]

More generally, one must introduce several spheres E_i to make the resulting space smooth so by the same argument we find a $U(1)^r$ gauge group where r is the number of 2-cycles needed. However, there is more to it than that, since the spheres intersect among themselves in a very particular way. This is characterised by a matrix of pairwise intersections which coincides with the Cartan matrix of a Lie algebra $\mathfrak{g}$ of rank r, something that was noticed by mathematicians before string theory existed and had no apparent explanation. M-theory provides a beautiful explanation of this mathematical fact showing that the actual gauge algebra associated with the singularity is $\mathfrak{g}$. Indeed, an M2-brane wrapped along the chains $E_i \cup E_{i+1} \cup \cdots \cup E_j$ gives additional massive bosons in 3d charged under the original $U(1)^r$ and which become massless in the limit of vanishing volumes. Thus, these states correspond to the charged bosons that are needed to account for a non-abelian gauge symmetry $\mathfrak{g}$.

An important feature about this mechanism is that it does not always give classical gauge groups as it happens in the perturbative regime. In particular, one can get *exceptional* groups, whose microscopic description is given in terms of (p, q)-strings associated to mutually non-local 7-branes [19]. These are especially important for phenomenological reasons as we will see in subsequent sections. For practical purposes the important point is that there is an algebraic algorithm that tells you the gauge groups given the Weierstraß form [20].

One might wonder why take the trouble of compactifying M-theory on a 4-fold Y and taking the F-theory limit to produce a 4d vacuum instead of directly considering it on a G_2 manifold with singularities. This may yield an $\mathcal{N} = 1$ 4d solution with non-abelian gauge symmetry which corresponds to intersecting D6-branes in the perturbative limit. The reason is that real manifolds, unlike complex ones, are poorly understood so for technical reasons it is possible to go much further using this F-theory route.[23]

This concludes our brief overview of F-theory which hopefully gives a flavour of the

[22]This method corresponds to studying the supersymmetric theory by moving in the Coulomb branch which is identified with performing a small resolution of the space. On the other hand, one can also study it in the Higgs branch by deforming the space which yields a complementary description that is often useful and insightful.

[23]The M in M-theory is supposed to stand for mother implying that it is the mother of all theories. In the early days of F-theory it was believed to be a 12d theory that could be thought of as the father of all theories, hence the F. However, given the current understanding it would probably be more accurate to think of F-theory as a (holomorphic) son of M-theory.

subject. In the following we rephrase the D-brane model we introduced in the last section in terms of F-theory and discuss some further phenomenological issues.

2.2.2 Model building with 7-branes

In F-theory, the D7-brane model that was explained in section 2.1.2 can be regarded as a compactification on an elliptically fibered space Y with base X and the appropriate singularities. See [22–24, 155] for the original references.

For instance, the D7-branes wrapped along the divisor $S_{GUT} \subset X$ correspond to a codimension one[24] singularity such that it yields a $SU(5)$ gauge group. Notice that in this setup it is natural to have $SU(5)$ rather than $U(5)$ as a gauge group for the reasons already explained. Along some curves in S_{GUT} the type of singularity may get worse which means that another D7-brane is intersecting the GUT branes. In particular, if there is a $U(1)$ 7-brane intersecting the GUT branes the enhancement of the gauge group associated to the singular fibre should be of rank one. At such codimension one loci Σ in S_{GUT} the extra degrees of freedom that account for the enhancement of the gauge group correspond to charged matter living at the intersection of the branes.

Thus, one should construct an elliptically fibered CY 4-fold that has a $SU(5)$ singularity at S_{GUT} and within S_{GUT} the singularity should enhance so as to produce matter in the **10** and **5**.[25] Let us see this point in more detail. Consider, for example the enhancement $SU(5) \to SO(6)$ along Σ or, by regarding it the other way around one gets

$$\begin{aligned} SU(6) &\longrightarrow SU(5) \times U(1) \\ \mathbf{35} &\longrightarrow \mathbf{24}_0 \oplus \mathbf{1}_0 \oplus \mathbf{5}_{-1} \oplus \bar{\mathbf{5}}_1 \end{aligned} \tag{2.27}$$

which gives matter localised along Σ in the **5** of $SU(5)$. Moreover, the enhancement to $SO(10)$

$$\begin{aligned} SO(10) &\longrightarrow SU(5) \times U(1) \\ \mathbf{45} &\longrightarrow \mathbf{24}_0 \oplus \mathbf{1}_0 \oplus \mathbf{10}_2 \oplus \overline{\mathbf{10}}_{-2} \end{aligned} \tag{2.28}$$

gives matter in the **10** of $SU(5)$. Notice that these two are the only possibilities that enhance $SU(5)$ by rank one and yield exactly the required representations for a minimal GUT.[26]

This construction gives hypermultiplets in $M_4 \times \Sigma$ and one has to dimensionally reduce along Σ to obtain the matter content in 4d. As explained in section 2.1.2 we have to introduce magnetic fluxes over the branes to get chirality in M_4 and these have to be such that there are three generations, etc. In F-theory all the fluxes have a common origin, namely, they come from the G_4-flux in M-theory. In particular, brane fluxes are related to G_4-fluxes along the collapsed cycles E_i that were introduced earlier. Thus, from this point of view one needs to consider a G_4-flux such that its restriction to the matter curves accounts for the correct chirality and number of families.

[24] We mean complex codimension in the base X.

[25] There can also be matter that is not localised along a codimension one loci in the GUT divisor. However, in the minimal scenario one can show [23] that the MSSM matter is indeed localised along matter curves.

[26] From the analysis of the singularity one can check that the $U(1)$ factors are massive and do not play any role.

In order to include Yukawa couplings among these modes the associated matter curves should meet at a point meaning that the gauge group is further enhanced since we find several colliding singularities.[27] One can show [25] that the down-type Yukawa is generated in a $SO(12)$ enhancement point which as we move away from it we find that generically it breaks down to the GUT group but along two complex curves it breaks only to $SU(6)$ which support the $\mathbf{5}_M$ and $\mathbf{5}_D$. Along another curve one finds the higgsing to $SO(10)$ that carries the $\mathbf{10}_M$.

Thus far only classical groups have appeared in our discussion which matches the fact that we have not achieved nothing we could not do in perturbation theory. The reason to use F-theory is manifest when we try to engineer a non-vanishing Yukawa for the up-type quarks which was troublesome in perturbative type IIB. Indeed, one can see that in order to find a phenomenologically viable Yukawa it is necessary to include a codimension three singularity with associated gauge group E_6. Away from such a point the gauge group is $SU(5)$ except at the curve supporting the $\mathbf{10}_M$ that has $SO(10)$ and the $\mathbf{5}_U$ curve with $SU(6)$. Both of these Yukawa coupling will be computed in sections 2.4 and 2.5 respectively where there are more details on how this construction works.

There is yet another possible rank two enhancement $SU(5) \to SU(7)$ that can be seen to correspond to the coupling $\bar{\mathbf{5}} \times \mathbf{5} \times \mathbf{1}$. This term generates a supersymmetric mass to the Higgs bosons so when building a model it should be arranged such that this coupling is of the order of the weak and not the GUT scale.

Before going into the details of Yukawa couplings let us discuss some more generalities of these kinds of models such as the breaking of the GUT group, gauge coupling unification and proton decay.

GUT breaking and doublet-triplet splitting

Up to now we have not discussed the mechanism responsible for the breaking of the GUT group down to the SM group in F-theory models so let us explore the different possibilities [17, 23, 24].

We have seen that in standard field theories one implements this breaking via a Higgs mechanism as in the model introduced in section 2.1.1 for which we need to have a field in the adjoint of $SU(5)$. In this case the fact that we have or not such a field depends on the geometrical properties of the divisor wrapped by the GUT branes. Indeed, normal deformations[28] of such divisor are parametrised by a complex scalar in the adjoint representation which in 4d yields precisely the desired field. In principle one can consider S_{GUT} such that this mechanism is available, however, one should worry about producing a potential to the adjoint Higgs in order to trigger this breaking dynamically. This is quite challenging and one finds difficulties in achieving doublet-triplet splitting naturally. There are more convenient ways of breaking $SU(5)$ to which we turn our attention.

The breaking induced by Wilson lines that was discussed in the heterotic context in section 2.1.2 is also a possibility in F-theory models. In this case the availability of this mechanism is also linked to geometrical properties of the divisor S_{GUT}. Indeed, the Wilson lines require to have a non-simply connected divisor as was explained for the heterotic case.

[27] We will explain this in greater detail in the following sections by means of the effective action that governs the dynamics of the 7-branes.

[28] Actually, it is not the normal sections that matter as it happens with lower dimensional D-branes but sections of the canonical bundle. We will come back to this point in section 2.2.3.

There may well be models in which this is the reason behind the breaking of the GUT symmetry. In order to avoid having matter in the adjoint representation one typically wraps the GUT branes on a divisor that has at most discrete fundamental group. Turning on discrete Wilson lines in these models leads to exotic matter in minimal models and is not necessarily the best mechanism available in general.

F-theory models allow for yet another way of breaking $SU(5) \to SU(3) \times SU(2) \times U(1)$ that is not possible in heterotic models. Consider turning on a flux along the hypercharge generator (2.3) over some 2-cycles in S_{GUT}. Similarly to Wilson lines this breaks the gauge group to the commutant of such flux which in this case is precisely the SM group. Naively, this can also be done in the heterotic model of section 2.1.2, however, the hypercharge generator then gets massive by a Stückelberg mechanism which is not phenomenologically desirable. For D-brane models this can be avoided because of the localisation of the gauge dynamics to a submanifold. More explicitly, the potential source of a mass to the hypercharge generator comes from

$$\int_{M_4 \times S_{GUT}} C_4 \wedge \mathrm{Tr}(F \wedge F) \longrightarrow \int_{S_{GUT}} \omega_2^i \wedge \langle F_Y \rangle \int_{M_4} B_2^i \wedge F_Y \tag{2.29}$$

which is part of the Chern-Simons action of the D-branes. Here $\langle F_Y \rangle \in H^2(S_{GUT}, \mathbb{Z})$ is the hypercharge flux over the 7-branes and we have expanded the 4-form $C_4 = B_2^i \wedge \omega_2^i$ in harmonic 2-forms ω_2^i of the base. When this term is non-zero the hypercharge generator is massive so for this to work we must find a way to make it vanish while keeping the flux non-zero.

This can be done by taking the hypercharge flux to be different from zero in $H^2(S_{GUT}, \mathbb{Z})$ but to correspond to the zero class in the cohomology of the base $H^2(X, \mathbb{Z})$. Notice that this condition depends on the global geometry and not only on the information in the vicinity of the GUT branes.

As usual, GUT breaking is related to the doublet-triplet splitting and this mechanism provides a natural explanation for this. As was mentioned earlier, chirality is linked to the Dirac index over the matter curves and since the hypercharge flux distinguishes between different SM particles within the same multiplet we can arrange the flux to achieve doublet-triplet splitting. Indeed, the hypercharge flux should be such that the Higgs doublets are chiral while the triplets are non-chiral and therefore have a mass of the order of the GUT scale generically. This is because unification takes place in 8d rather than in 4d so there is no need to have full GUT multiplets in M_4 similarly to the Wilson line breaking in heterotic models.

There is, however, an important difference between breaking the GUT group with Wilson lines and hypercharge flux which is the unification of gauge constants.

Gauge coupling unification

Let us discuss briefly the issue of gauge coupling unification and the effect of the hypercharge flux [24, 26, 54]. The gauge kinetic function $f_{SU(5)} = \frac{4\pi}{g_{GUT}^2} + i\theta$ can be computed by dimensionally reducing the action for the 7-branes wrapped over S_{GUT}. If we only care about the GUT branes we may take a frame in which these are D7-branes and use the usual DBI+CS action which leads to

$$f^0_{SU(5)} = \frac{1}{g_s\, l_s^4}\mathrm{Vol}(S_{GUT}) + i \int\limits_{S_{GUT}} C_4. \tag{2.30}$$

Including the contributions coming from the hypercharge flux in $\int C_0 \wedge \mathrm{tr}(F^4)$ one finds that the usual relation among the SM gauge couplings (2.5) is modified [26], namely

$$f_{SU(3)} = f_{SU(2)} + \frac{\tau}{2} \int\limits_{S_{GUT}} \langle F_Y \rangle^2 = \frac{3}{5}\left(f_{U(1)} + \frac{\tau}{2} \int\limits_{S_{GUT}} \langle F_Y \rangle^2 \right). \tag{2.31}$$

At this level of approximation it looks like we do not recover gauge coupling unification in 4d at the GUT scale which goes against the main motivation to consider GUTs in the first place. However, one expects that this result receives several corrections coming from KK modes, winding modes, etc. Also, this derivation has been done in perturbative type IIB and a fully F-theoretic analysis has not been developed yet.

In general it is difficult to have control over all these effects and as of today there is no final agreement in the literature. A concrete model that is in quantitative agreement with both the gauge coupling constants and weak angle has not appeared yet. However, precisely because of the several different contributions it seems plausible that there exists a region in the parameter space where these conditions are met.

Proton decay

As we mentioned in section 2.1.1 in generic $SU(5)$ GUTs the predicted lifetime of the proton is extremely small, in contradiction with experiment. There are dangerous dimension 4 operators such as $\bar{\mathbf{5}}_M \times \bar{\mathbf{5}}_M \times \mathbf{10}_M$ that induce rapid proton decay and should be absent from the low energy action. In conventional field theories this can be achieved by declaring that there is a discrete $\mathbb{Z}_2$ symmetry, dubbed R-parity, such that it forbids the unwanted couplings (see chapter 3 for a discussion about discrete symmetries).

In F-theory models these are generated in a very similar fashion as Yukawa couplings so, as we will see, they can be avoided by demanding there is no enhancement point which depends on geometrical properties of the model.

The lifetime of the proton is so large that getting rid of dimension 4 operators is not enough and one should also build the model such that dimension 5 operators that induce proton decay are absent. These typically induce the decay $p \to K^+\bar{\nu}$ and, even though they are naturally suppressed by a factor $1/M_{GUT}$, one finds that the prediction for the lifetime of the proton is too small. As we will see in the next section, dimensional reduction of the 7-brane effective theory only yields cubic terms in the superpotential so any dimension 5 operator must be generated by integrating out massive modes. For example, the couplings $d\theta^2 QQQL$ and $d\theta^2 UDUE$ are generated by integrating out massive higgsino triplets. Notice that, using the hypercharge flux we may eliminate the triplets in the $\mathbf{5}s$ that contain the Higgses, however, massive states with those quantum numbers are still present in the theory. The existence of a down-type Yukawa implies that the couplings

$$Q\, L\, T_D, \qquad U\, D\, T_D \tag{2.32}$$

are present. Here T_D are fields with the quantum numbers of the down-type coloured Higgs. Similarly, if the up-type Yukawa is non-vanishing we may have the operators

$$Q\,Q\,T_U, \qquad U\,E\,T_U \tag{2.33}$$

with T_U the up-type triplets. Now we see that, since T_D and T_U are localised on different matter curves the coupling $T_U T_D$ vanishes whenever these do not intersect and the offending dimension 5 operators are not generated.[29]

Finally, dimension 6 operators open a window into GUT physics since the typical prediction for the lifetime of the proton associated to them is of the order of the current experimental bound. It is interesting to notice that measuring proton decay would give us information about physics at a scale 10^{16} GeV, many orders of magnitude above the LHC energy. See e.g. [23, 24, 29] for more details on proton decay in F-theory.

Other phenomenological issues

We conclude this section with some general remarks about the phenomenology of these constructions. We should stress that, although the technical part related to F-theory model building has received a lot of attention, at the phenomenological level there are still many challenges that have not been addressed in detail.

One of the first things one has to worry about when constructing a model is moduli stabilisation. In F-theory this is believed to work very much like in perturbative type IIB. The complex structure moduli of the CY four-fold (which include the position of the 7-branes) are, in principle, stabilised at tree level by a suitable G_4 flux. The Kähler moduli, on the other hand, cannot be stabilised at tree level which is troublesome because of the Dine-Seiberg problem that was mentioned earlier. There is, however, a scenario where this can be achieved in a controlled way known as KKLT [30,31]. In this framework the Kähler moduli are stabilised by including the non-perturbative effect of euclidean D3-branes or strong gauge dynamics on the worldvolume of a D7-brane wrapping a holomorphic divisor in the base. This produces a fully stabilised AdS solution that has to be uplifted to a de Sitter space which is done by including additional sources of tension in the compactification. There is a generalisation of this procedure that is more flexible known as Large Volume Scenario (LVS) in which the leading α' corrections to the Kähler potential are included [32–34]. These compete with the non-perturbative terms in the superpotential and lead generically to large volume moduli stabilisation. These mechanisms should be regarded as a road map towards moduli stabilisation but their implementation in detail is highly non-trivial.

Another important point that has not been mentioned at all is supersymmetry breaking. It is clear that our universe is not supersymmetric at low energies (below the weak scale) so in these models it must be broken at some scale between M_{EW} and M_{GUT}. Having supersymmetry breaking near the weak scale has the advantage that it gives a natural explanation as to why the Higgs mass is so low which is why is it has received so much attention in the past. However, the fact that the LHC has not found any evidence for low scale SUSY so far and the measured Higgs mass has motivated considering an intermediate SUSY breaking scale in the context of F-theory [35–37].

[29] It is not necessary that the curves Σ_D and Σ_U do not intersect but it is sufficient. It could happen that they do intersect but still the coupling $T_D T_U$ is not generated. Also, one could engineer additional $U(1)$ symmetries to forbid the couplings [24].

There are mainly two scenarios for SUSY breaking: moduli dominated and gauge mediation. In the former one considers the coupling between the Kähler modulus associated with the overall volume of the GUT divisor with the MSSM fields. Giving a vev to the F-term of such modulus generates soft SUSY breaking masses which have been studied in detail in [38, 39]. The latter assumes that the volume is stabilised due to high energy dynamics and breaks SUSY by giving a vev to a gauge singlet that couples to vector-like pair messengers [23, 40]. Then gauge and messenger loops generate soft masses for the MSSM particles.

It is fair to say that GUTs in F-theory are still in a rather early stage and, although it is a field that is developing quickly, there is not yet a clear global picture that takes into account all the phenomenology at once.

2.2.3 Local models

A powerful tool to analyse these models is the low energy effective theory of the 7-branes. This approach is fundamentally local since it focuses only on the physics in the vicinity of the GUT divisor S_{GUT} so one can proceed from a bottom-up perspective [41]. In a first stage one tries to use this local approach to match the low energy data as much as possible and then embeds the model in a global setup. This is particularly fruitful for studying Yukawa couplings since these are essentially insensitive to what happens far away from the GUT branes.

In perturbative type II the effective theory that governs the dynamics of the D-branes is the DBI + CS action. However, for our purposes we need a theory capable of describing exceptional 7-branes which was derived in [22] and we briefly summarise in the following.

7-brane effective action

The effective worldvolume description of a seven-brane wrapping $M_4 \times S$, with S a holomorphic divisor of the base X, can be determined starting with a maximally supersymmetric Yang-Mills theory in ten dimensions. The field content consists of a gauge field and an adjoint-valued Majorana-Weyl fermion.

Dimensionally reducing to $M_4 \times S$ leads to a theory with a gauge boson and a complex scalar that describes the transverse position of the 7-brane plus fermionic partners, all of them transforming in the adjoint representation of the gauge group. In order to preserve $\mathcal{N} = 1$ SUSY in 4d we need to embed S in X in a supersymmetric way. This means that the field theory that describes the massless modes must be topologically twisted and since S is Kähler the twist turns out to be unique. Due to this twist the scalars and fermions transform as forms on S. Table 2.1 shows the field content together with their transformation properties.[30] The details of the computation of the action can be found in [22].

For our purposes, the most interesting part of the 4d effective action is the super-

[30] Notice that the embedding of the 7-brane is given by a field in the canonical bundle rather than in the normal bundle. This can be seen from another point of view by considering deformations of the complex structure $\Omega^{(4,0)}$ of the four-fold. Namely, along a singular point $\delta\Omega^{(4,0)} = \Phi_i^{(2,0)} \wedge \omega^i$ where ω^i are the Poincaré duals of the the 2-cycles needed to resolve the singularity.

Field	$\mathcal{N}=1$ multiplet	Forms on S
(A_μ, η_α)	Gauge	$(0,0)$-form
$(A_{\bar{m}}, \psi_{\bar{m}})$	Chiral	$(0,1)$-form
$(\Phi_{mn}, \chi_{\alpha mn})$	Chiral	$(2,0)$-form

Table 2.1: Field content and their transformation properties. m,n take values in S so if we take complex coordinates x,y we have $m,n=x,y$. μ is a vector index and α a left-handed spinor index in 4d.

potential

$$W_S = m_*^4 \int_{M_4\times S} d^2\theta\, \mathrm{Tr}\,(\mathbf{F}\wedge\mathbf{\Phi}) \tag{2.34}$$

where m_* is the F-theory characteristic mass scale[31] and $\mathbf{F}=d\mathbf{A}-i\mathbf{A}\wedge\mathbf{A}$ is the field strength of the gauge field $\mathbf{A}_{\bar{m}}$. Also, the superfields decompose as

$$\begin{array}{rcl} \mathbf{A}_{\bar{m}} & = & A_{\bar{m}} + \sqrt{2}\theta\psi_{\bar{m}} + \dots \\ \mathbf{\Phi}_{xy} & = & \Phi_{xy} + \sqrt{2}\theta\chi_{xy} + \dots \\ \mathbf{V} & = & -\theta\sigma^\mu\bar{\theta}A_\mu + i\theta\theta\bar{\theta}\bar{\eta} - i\bar{\theta}\theta\theta\eta + \dots \end{array} \tag{2.35}$$

where the dots include the auxiliary fields and $\bar{m}=\bar{x},\bar{y}$. The superpotential (2.34) yields the following equations of motion (F-term equations) for the bosonic fields

$$\bar{\partial}_A\Phi = 0 \tag{2.36a}$$

$$F^{0,2} = 0 \tag{2.36b}$$

with $\bar{\partial}_A=\bar{\partial}-i[A,\cdot]\wedge$ and the superscript denotes the Hodge type. From the 4d viewpoint, these equations follow from the requirement that the potential is F-flat. The D-flatness condition is given by the vanishing of the following D-term

$$D = \int_{M_4\times S} d^2\theta\, d^2\bar{\theta}\, \left(\omega\wedge\mathbf{F} + \frac{1}{2}[\mathbf{\Phi},\mathbf{\Phi}^\dagger]\right) \tag{2.37}$$

from where we obtain the so-called D-term equation

$$\omega\wedge F + \frac{1}{2}[\Phi,\Phi^\dagger] = 0 \tag{2.38}$$

where ω stands for the fundamental form on S. Equations (2.36-2.38) determine the bosonic internal wavefunctions and are usually referred to as the Hitchin system [42].

The action also determines the equations of motion for the massless fermionic degrees of freedom. Namely, the part of the action bilinear in fermions without kinetic terms is

$$S \supset m_*^4 \int_{M_4\times S} \mathrm{Tr}\left(\chi\wedge\partial_A\psi + 2i\sqrt{2}\,\omega\wedge\partial_A\eta\wedge\psi + \frac{1}{2}\psi\wedge[\Phi,\psi] + \sqrt{2}\,\eta\,[\Phi^\dagger,\chi] + h.c.\right) \tag{2.39}$$

[31] This is the scale at which the effective theory is no longer valid which from the type IIB perspective corresponds to the string scale.

which gives the following equations of motion

$$\begin{aligned} \bar{\partial}_A\chi + i[\Phi,\psi] - 2i\sqrt{2}\,\omega\wedge\partial_A\eta &= 0 \\ \bar{\partial}_A\psi - i\sqrt{2}\,[\Phi^\dagger,\eta] &= 0 \\ \omega\wedge\partial_A\psi - \frac{1}{2}[\Phi^\dagger,\chi] &= 0. \end{aligned} \tag{2.40}$$

For supersymmetric configurations one can simply focus on the bosonic equations since the fermionic wavefunctions are the same as the bosonic ones.

The Hitchin system describes, in principle, the dynamics of an isolated 7-brane which from the geometrical point of view corresponds to the degrees of freedom associated to an irreducible singularity over a smooth, compact, complex surface. However, we would like to handle more general situations in which several singularities collide to model the intersection of additional 7-branes with the GUT 7-branes. As explained earlier this is necessary to include charged matter in the model. Let us see how this can be arranged.

Intersecting and magnetised branes

We want to include additional branes intersecting the GUT divisor S_{GUT} by means of the Hitchin system. In the spirit of our local approach we consider these other branes to be wrapping a non-compact divisor S' which effectively decouples their dynamics.[32]

There are basically two ways to proceed at first order in the scale m_* depending on whether the angles between S_{GUT} and S' are large or small. At the intersection between the branes Σ there are charged hypermultiplets that are microscopically open strings or, more generally, (p,q)-strings. The fact that these degrees of freedom are confined at the intersection is quite intuitive. Indeed, such an open string has an endpoint in each of the 7-branes so when it tries to go away from the intersection its length necessarily increases meaning that there is a potential driving the string towards Σ. Quantum mechanically the open strings are not exactly localised along Σ, one expects rather that they are in a tubular neighbourhood of Σ.

When the angles are large, the region around Σ where the open strings localise is of the order of the string length $l_s \sim 1/m_*$. This means that we cannot resolve its spatial extension using our effective theory since it does not include higher order terms in m_*. Thus, one can describe the interaction between the brane and the charged matter by treating it as a topological defect. This is encoded in the following form of the Hitchin system [22]

$$\begin{aligned} \bar{\partial}_A\Phi &= \delta_\Sigma\langle\langle\sigma^c,\sigma\rangle\rangle \\ F^{0,2} &= 0 \\ \omega\wedge F + \frac{1}{2}[\Phi,\Phi^\dagger] &= -\frac{i}{2}\omega\wedge\delta_\Sigma\left[\mu(\bar{\sigma},\sigma) - \mu(\bar{\sigma}^c,\sigma^c)\right]. \end{aligned} \tag{2.41}$$

Here (σ,σ^c) is the hypermultiplet living at the intersection and δ_Σ is a two-form with delta-function support along Σ. Locally, the outer product looks like $\langle\langle\sigma^c,\sigma\rangle\rangle = \sigma^c_a(T^I)^a_b\sigma^b$ with $(T^I)^a_b$ the generators of the gauge group on S_{GUT}. Moreover, the moment map is given locally by $\mu(\bar{\sigma},\sigma) = \bar{\sigma}^{\bar{a}}(T^I)_{\bar{a}b}\sigma^b$.

[32] Recall that the gauge coupling constant is given by (2.13) so it is zero for infinite volume.

Equations (2.41) describe the wavefunction of the charged matter using a Dirac delta function in agreement with the fact that the effective theory is unable to resolve it. Although useful in some cases[33] this approach is not particularly satisfactory for our purposes since we want to study the properties of the hypermultiplets themselves that ultimately yield the SM particles.

This brings us to the case in which the angles are small. To see what happens in this situation imagine that we start with both branes parallel and wrapping $S_{GUT} = S'$. In such situation the gauge group enhances to G_Σ and the charged matter corresponds to the extra degrees of freedom of the vector multiplet whose wavefunction is spread all over S_{GUT}. As we start deforming the branes such that they intersect the gauge group breaks $G_\Sigma \to SU(5) \times U(1)$ and these fields start localising along Σ. As long as the angles are small enough this process can be described in field theory as we explain in the following.

Consider the Hitchin system with gauge group G_Σ. Recall that the field Φ is valued in the adjoint of G_Σ and describes the embedding of the 7-branes in the base. Thus, by giving a vev to Φ we can describe intersecting branes which is known in mathematics as a Higgs bundle [43]. This vev breaks the gauge symmetry G_Σ down to the commutant of Φ. So Φ has to be valued in the $U(1) \subset G_\Sigma$ such that the commutant is $SU(5) \times U(1)$, the gauge symmetry that the intersecting branes preserve.

From the elliptic fibration point of view what we are doing is taking a Weierstraß model that has a G_Σ singularity and we are deforming it such that it describes two colliding singularities. Indeed, one can see that the deformation parameters are the gauge invariant polynomials of the Higgs field Φ in the Hitchin system [43].[34] These polynomials are the spectral data associated to the Higgs bundle, i.e. eigenvalues and eigenvectors, which (for unitary groups) are the coefficients of the polynomial

$$\det(sI - \Phi) = 0 \tag{2.42}$$

where s is a coordinate on the canonical bundle of S_{GUT}. This equation can be regarded as a multiple cover of the GUT brane known as the spectral cover. From the geometric perspective this equation is obtained by restricting the discriminant locus Δ, defined in section 2.2.1, to the vicinity of the the $SU(5)$ singularity S_{GUT}. The coordinate s corresponds then to the normal coordinate to the GUT branes.

In order to include more 7-branes and matter curves into the game the idea is basically the same. Consider a Yukawa point, meaning a codimension 3 singularity where the gauge group has a rank two enhancement to G_p. Then, the idea is to take the Hitchin system with gauge group G_p such that an appropriate Higgs field breaks it down to $SU(5)$ generically. At the matter curves there is a partial enhancement to $G_\Sigma \subset G_p$. If we want to include another Yukawa point p' then the gauge group G of our Hitchin system should be such that it contains G_p and $G_{p'}$ with the corresponding matter curves, etc.[35]

As already explained, we have in to include gauge fluxes that restrict non-trivially to the matter curves in order to generate chirality. In the Hitchin system it is clear how to include these fluxes, simply giving a vev to the abelian generators in G that are not in $SU(5)$. These correspond to fluxes turned along the $U(1)$ branes that intersect the GUT

[33] This equations can be used to describe the recombination of the branes on S_{GUT} and S' when the fields σ, σ^c have a non-vanishing expectation value [22].

[34] This is only true for bundles satisfying $[\Phi, \bar{\Phi}] = 0$. We will relax this condition in section 2.3.3.

[35] We should say that these enhanced gauge groups are always broken by the intersections so they do not lead to symmetries in 4d.

7-branes. Also, there must be a flux in the hypercharge direction to break the GUT group and achieve doublet-triplet splitting. From the spectral cover point of view this is done by specifying a line bundle on the spectral curve (2.42). At this level in m_* we should require that the flux densities are small compared with the cut-off scale.

The spectral cover construction has been widely used in this context to build local models (see e.g. [44–46]). However, we will not need it in the following and since it is fairly technical we will not explain it in any detail. Let us just mention that this construction appears naturally in heterotic models where the internal space is elliptically fibered since it allows to construct gauge bundles in a neat way [47, 48]. Both spectral covers are mapped to each other in models which have both an F-theoretic and heterotic description.

Since we are ultimately interested in computing Yukawa couplings we will take an even more local and simple approach. Indeed, Yukawa couplings are localised at triple intersections of matter curves and are not sensitive to what happens far away from such a point. Thus, under certain assumptions that we will explain in the next section, we may compute such Yukawa couplings by taking our gauge theory to live in an open flat patch that contains such point. We illustrate this idea with a simple toy model in the following.

Toy model: $U(3)$ point

Let us construct a toy model that will clarify many of the statements that were made in the last section. We want to describe three matter curves meeting at a point $p \in X$ where the singularity type is $U(3)$ [49].

Let us take an open neighbourhood of $p \in X$ with local coordinates $x, y, z \in \mathbb{C}^3$. If the GUT brane is located at $z = 0$ then the fields in the Hitchin system depend on x, y and Φ is identified with the normal coordinate z through $\Phi = m_*^2 z$. Consider the following vev for the Higgs field

$$\Phi = \frac{m^2}{3} \begin{pmatrix} -2x+y & 0 & 0 \\ 0 & x+y & 0 \\ 0 & 0 & x-2y \end{pmatrix} dx \wedge dy \tag{2.43}$$

that solves the F- and D-terms (2.36-2.38) since the gauge field is zero and Φ is diagonal and holomorphic.

The surviving gauge group is given by the commutant of Φ in $U(3)$ which generically reduces to $U(1)^3$ locally, the group of three intersecting branes. Notice that globally the surviving group may be trivial. The embedding of the branes in $\mathbb{C}^3$ can be read off from eq.(2.42), namely

$$\begin{aligned} S_1 &: \quad \left\{ z = \frac{\sigma}{3}(y-2x) \right\} \\ S_2 &: \quad \left\{ z = \frac{\sigma}{3}(x+y) \right\} \\ S_3 &: \quad \left\{ z = \frac{\sigma}{3}(x-2y) \right\} \end{aligned} \tag{2.44}$$

where we have defined $\sigma = \frac{m^2}{m_*^2}$ that is related to the intersection angle between the branes.

This intersection yields the matter curves

$$\begin{aligned} \Sigma_a = S_1 \cap S_2 \quad &: \quad \left\{ z = \frac{\sigma}{3} y, \;\; x = 0 \right\} \\ \Sigma_b = S_2 \cap S_3 \quad &: \quad \left\{ z = \frac{\sigma}{3} x, \;\; y = 0 \right\} \\ \Sigma_c = S_3 \cap S_1 \quad &: \quad \left\{ z = \frac{\sigma}{3} x, \;\; x = y \right\}. \end{aligned} \tag{2.45}$$

These are the loci at which there is a rank one enhancement and therefore where charged matter lives. For instance, at $x = 0$ we find locally $U(2) \times U(1)$. The charged matter comes from the 12 and 21 entries in the 3×3 matrix representation of $U(3)$.

The three matter curves meet at $x = y = z = 0$ where the gauge group enhances to $U(3)$ locally so there is a rank two enhancement. At such a point there is a Yukawa coupling between the three different modes a, b and c.

One can check that the equations of motion for the modes are obtained from the Hitchin system by expanding in fluctuations $(a_{\bar{x}}, a_{\bar{y}}, \varphi_{xy})$ around the background $(A_{\bar{x}}, A_{\bar{y}}, \Phi_{xy})$. For the curve a we find the following solution for the charged modes[36]

$$\vec{\Psi}_{a^{\pm}} = \begin{pmatrix} a_{\bar{x}} \\ a_{\bar{y}} \\ \varphi_{xy} \end{pmatrix} = \begin{pmatrix} \mp i \\ 0 \\ 1 \end{pmatrix} e^{-m^2 |x|^2} f(y) \tag{2.46}$$

where the $\pm$ denotes the two possible charges for a given chirality. The first thing we notice is that the solution is peaked along the matter curve $x = 0$ as expected. Moreover, we see that it depends on an arbitrary holomorphic function $f(y)$ of the coordinate along the curve. In a model that includes the whole matter curve and not just a patch this function is determined by the boundary conditions on Σ_a. Actually, for a model without fluxes we expect to obtain a solution for both charges while including a flux should generate net chirality. Let us see how this works.

Consider adding a gauge flux of the form

$$F = i\, \frac{M}{3} \left(dy \wedge d\bar{y} - dx \wedge d\bar{x} \right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.47}$$

with M a real positive constant. This flux satisfies the equations of motion and modifies the previous wavefunctions, namely

$$\vec{\Psi}_{a^{\pm}} = \begin{pmatrix} -i \frac{\lambda_{\pm}}{m^2} \\ 0 \\ 1 \end{pmatrix} e^{-\sqrt{\left(\frac{M}{2}\right)^2 + m^4} |x|^2} e^{\mp \frac{M}{2} |y|^2} f(y). \tag{2.48}$$

Here $\lambda_{\pm}$ is a constant that depends on M and m^2 (see [49] for more details on this model). The important point about this solution is that it only converges for one of the charges depending on the sign of the flux density M. For $M < 0$ we have that Ψ_{a^+} converges while Ψ_{a^-} is not normalisable so it should be discarded. Again, the boundary conditions on Σ_a determine the function $f(y)$ as well as the number of distinct solutions (see e.g. [50, 51]).

[36] We will be more explicit in the next chapter where we explain in detail how to solve for the wavefunctions.

In this ultra-local approach this information is lost so we should choose the functions $f(y)$ by hand.

Two possible choices are [52]

$$\begin{aligned} f_j(y) &= m_*^j y^j \\ f_j(y) &= e^{a_j y} \end{aligned} \tag{2.49}$$

for $j = 0, 1, 2$ and a_j complex numbers. In the first case we are just expanding the functions $f_j(y)$ in Taylor series around the Yukawa point and choosing a basis in which each family is associated to a monomial. In the second possibility the different values of a_j produce a displacement of the peak along the matter curve. Since we are going to work in a patch around $x = y = z = 0$ it is more convenient to use the first choice since it keeps the wavefunctions localised within that patch.

Regarding the curve c, one can readily check that the gauge flux restricts to zero. This means that at this level the modes that come from reducing along Σ_c are non-chiral.

Hopefully this example clarifies some of what was said in the previous sections. In the following we explain how to obtain Yukawa couplings in this construction and their features. We will keep using this example to illustrate some general statements in subsequent sections.

2.3 Yukawa couplings in $SU(5)$ F-theory GUTs

In this section we explain in detail how to compute the Yukawa couplings using this ultra-local approach. We start by reviewing how to obtain supersymmetric backgrounds as well as the associated matter curves and wavefunctions, both holomorphic and real. We also introduce the concept of T-brane, necessary to generate a Yukawa coupling for the top quark. Finally, we analyse the effect of some non-perturbative contributions to the superpotential that are needed to obtain phenomenologically viable Yukawa patterns. See [49, 53–56] for some of the original references.

2.3.1 Generalities

Let us start by considering the effective field theory introduced in the last section with gauge group G and living in $M_4 \times S$ with S a holomorphic divisor of the base X. The superpotential in 8d reads

$$W_S = m_*^4 \int_{M_4 \times S} d^2\theta \, \mathrm{Tr}\,(\mathbf{F} \wedge \mathbf{\Phi}) \tag{2.50}$$

which contains a triple field interaction

$$W_S \supset -i m_*^4 \int_{M_4 \times S} d^2\theta \, \mathrm{Tr}\,(\mathbf{A} \wedge \mathbf{A} \wedge \mathbf{\Phi}) \, . \tag{2.51}$$

After expanding in fluctuations around the background this generates Yukawa couplings in 8d among the modes living on the matter curves. Then, upon dimensional reduction over S we get Yukawa couplings in 4d. This will be the basic strategy that we will follow in the remainder of this chapter. Namely, we solve for the internal wavefunctions of the chiral matter and plug them in (2.51) to obtain Yukawa couplings after integrating over S.

Let us be more explicit. Consider expanding the superpotential in fluctuations in 8d which yields[37]

$$W_S \supset -im_*^4 \int_{M_4\times S} d^2\theta \,\mathrm{Tr}\,(\mathbf{a}\wedge\mathbf{a}\wedge\phi) \tag{2.52}$$

where the lowercase superfields denote the fluctuations. We want to compute the coupling between three modes which we label a, b and c so we may expand the fields as follows,

$$\begin{aligned} \mathbf{a} &= \left(\mathbf{a}_{\bar{x}}^{\alpha}\, d\bar{x} + \mathbf{a}_{\bar{y}}^{\alpha}\, d\bar{y}\right) E_\alpha \\ \phi &= \phi_{xy}^{\alpha}\, dx\wedge dy\, E_\alpha \end{aligned} \tag{2.53}$$

where $\alpha = a, b, c$, E_α is the element in the gauge algebra associated to each mode and x, y are local coordinates in S.[38] We only write down the antiholomorphic piece of the Wilson lines $\mathbf{a}$ since this is the only part that contributes. Substituting this in eq.(2.52) we find

$$W_S \supset -im_*^4 \int_{M_4\times S} d^2\theta\,\mathrm{Tr}\,([E_a,E_b]E_c)\,\det\begin{pmatrix} \mathbf{a}_{\bar{x}}^{a} & \mathbf{a}_{\bar{x}}^{b} & \mathbf{a}_{\bar{x}}^{c} \\ \mathbf{a}_{\bar{y}}^{a} & \mathbf{a}_{\bar{y}}^{b} & \mathbf{a}_{\bar{y}}^{c} \\ \phi_{\bar{x}}^{a} & \phi_{\bar{x}}^{b} & \phi_{\bar{x}}^{c} \end{pmatrix} dx\wedge dy\wedge d\bar{x}\wedge d\bar{y} \tag{2.54}$$

Notice that the internal wavefunction associated to a given mode is in general a combination of $\mathbf{a}_{\bar{x}}, \mathbf{a}_{\bar{y}}$ and ϕ_{xy} as it happened in the $SU(3)$ toy model discussed above. Thus, we may decompose the 8d superfields as

$$\begin{pmatrix} \mathbf{a}_{\bar{x}} \\ \mathbf{a}_{\bar{y}} \\ \phi_{xy} \end{pmatrix} = \mathbf{K}\vec{\Psi}, \qquad \vec{\Psi} = \begin{pmatrix} a_{\bar{x}} \\ a_{\bar{y}} \\ \varphi_{xy} \end{pmatrix} \tag{2.55}$$

where $\mathbf{K}$ is the chiral superfield in Minkowski space and $\vec{\Psi}$ is a vector of scalar functions in the internal space S that satisfies the equations of motion, c.f.(2.48). This allows to write the superpotential as

$$W_S = \int_{M_4} d^2\theta \left[m_*^4\, f_{abc} \int_S \det(\vec{\Psi}_a, \vec{\Psi}_b, \vec{\Psi}_c)\,\mathrm{dvol}_S \right] \mathbf{K}^a\mathbf{K}^b\mathbf{K}^c \tag{2.56}$$

with $f_{abc} = -i\mathrm{Tr}\,([E_a,E_b]E_c)$ the structure constants of G and $\mathrm{dvol}_S = \frac{1}{(2i)^2} dx\wedge dy\wedge d\bar{x}\wedge d\bar{y}$. We can now read off the Yukawa coupling, namely

$$Y_{abc} = m_*^4\, f_{abc} \int_S \det(\vec{\Psi}_a, \vec{\Psi}_b, \vec{\Psi}_c)\,\mathrm{dvol}_S. \tag{2.57}$$

It is also useful to write it as

$$Y = -im_*^4 \int_S \mathrm{Tr}\,(a\wedge a\wedge\varphi)\,. \tag{2.58}$$

[37] The terms quadratic in fluctuations vanish on shell.

[38] The three modes a, b, c need not be different. When, say, $a = b$ then E_a should be a linear combination of different elements in the algebra to get a non-zero contribution. This is what happens for example in the E_6 model in section 2.5.

Notice that the Yukawa couplings in 4d are the numbers that multiply the fermion-fermion-Higgs terms in the Lagrangian once the kinetic terms are canonical. In our case this means that the internal wavefunctions should be appropriately normalised since this is what gives the kinetic term in 4d. Thus, one often distinguishes between holomorphic and physical Yukawas. The former are the quantities that one gets when the normalisation is ignored and the latter are those that correspond to canonically normalised wavefunctions.

In the subsequent sections we explain in detail how to obtain the wavefunctions $\vec{\Psi}$ and elaborate on the properties of the couplings.

2.3.2 Holomorphic gauge and holomorphic Yukawas

Consider the supersymmetry equations which we repeat here for convenience,

$$\begin{aligned} \bar{\partial}_A \Phi &= 0 \\ F^{0,2} &= 0 \\ \omega \wedge F + \frac{1}{2}[\Phi, \Phi^\dagger] &= 0. \end{aligned} \tag{2.59}$$

The second equation means that the gauge bundle is holomorphic and the first that the embedding of the 7-brane is holomorphic with respect to the said bundle. As we will see, the D-term can be interpreted as an equation for the metric in the fibre.

The gauge group acts on the fields as

$$\begin{aligned} A' &= g\,A\,g^{-1} + ig\,d\,g^{-1} \\ \Phi' &= g\,\Phi\,g^{-1} \end{aligned} \tag{2.60}$$

where $g \in G$ so we have that $g^{-1} = g^\dagger$. For the first two equations, however, we may take g to be in the complexified gauge group $G_{\mathbb{C}}$, i.e. $g^{-1} \neq g^\dagger$ in general, and they remain invariant. The D-term involves the field $\Phi^\dagger$ so it is only invariant under the real group G. This can be seen directly from the superpotential and D-term respectively. This fact has important consequences for the following reason. One can show that the space of solutions to the F- and D-terms modulo the action of G is isomorphic to the space of solutions to the F-term equations modulo the action of $G_{\mathbb{C}}$.

Let us expand the fields in fluctuations around the background, namely

$$\begin{aligned} A &= \langle A \rangle + a \\ \Phi &= \langle \Phi \rangle + \varphi. \end{aligned} \tag{2.61}$$

Here A denotes the part of the gauge field of Hodge type $(0,1)$ and since the total gauge field is real we have that $A^\dagger = \langle A^\dagger \rangle + a^\dagger$ is the part of type $(1,0)$. The F-terms linear in fluctuations are

$$\begin{aligned} \bar{\partial}_{\langle A \rangle}\varphi - i[a, \langle \Phi \rangle] &= 0 \\ \bar{\partial}_{\langle A \rangle} a &= 0 \end{aligned} \tag{2.62}$$

while the D-term reads

$$\omega \wedge \left(\partial_{\langle A \rangle} a + \bar{\partial}_{\langle A \rangle} a^\dagger \right) - \frac{1}{2}\left([\langle \Phi^\dagger \rangle, \varphi] + [\varphi^\dagger, \langle \Phi \rangle] \right) = 0. \tag{2.63}$$

The infinitesimal gauge transformations acting on the fluctuations are

$$\begin{aligned} a' &= a + \bar{\partial}_{\langle A\rangle}\chi \\ \varphi' &= \varphi - i[\langle\Phi\rangle, \chi] \end{aligned} \tag{2.64}$$

with χ an arbitrary function such that $\chi = \chi^\dagger$. However, under a complexified gauge transformation we may take $\chi \neq \chi^\dagger$.

Let us solve the equations of motion ignoring the D-term for the moment which means that at this level we may use the full complexified gauge group to our advantage. The first thing one can do is take an appropriate class of gauges dubbed holomorphic. This amounts to choosing a gauge within $G_{\mathbb{C}}$ such that $A = 0$ while $A^\dagger \neq 0$. This deserves some explanation. When working with the complexified gauge group the gauge field is *not* necessarily real so $A^{0,1\dagger} \neq A^{1,0}$. Thus, saying that $A = 0$ but $A^\dagger \neq 0$ is clearly an abuse of notation and by $A^\dagger$ we actually mean the $(1,0)$ part of the gauge field, even though it is not the conjugate of the $(0,1)$ piece. In addition to that, notice that the reason that we can in fact achieve such a gauge is the equation of motion $F^{0,2} = 0$.

In holomorphic gauge the F-terms simplify,

$$\begin{aligned} \bar{\partial}\varphi - i[a, \langle\Phi\rangle] &= 0 \\ \bar{\partial}a &= 0. \end{aligned} \tag{2.65}$$

Since we are ultimately interested in Yukawa couplings we restrict ourselves to a patch $U \subset S$ that contains the rank two enhancement point as already explained. Therefore, in U we may apply Poincaré's lemma and solve the second equation, namely

$$a = \bar{\partial}\xi \tag{2.66}$$

with ξ an arbitrary function in U valued in the adjoint of $G_{\mathbb{C}}$. Using this we readily solve for φ and find

$$\varphi = h - i[\langle\Phi\rangle, \xi] \tag{2.67}$$

where h is a holomorphic $(2,0)$-form also in the adjoint. Now we can take a holomorphic gauge for the fluctuations too by setting ξ to be a holomorphic function. Therefore, the space of solutions is given by the possible fields φ. These are holomorphic $(2,0)$-forms modulo commutators $[\langle\Phi\rangle, \xi]$ with ξ holomorphic. This means that if we are only interested in the number of solutions the answer is given by solving a simple algebraic problem as opposed to solving differential equations.

Let us illustrate this with the $U(3)$ toy model that was introduced in the last section. Take, for instance, the modes along the Cartan of $U(3)$. Since these commute with $\langle\Phi\rangle$ we find that the solutions are simply $\varphi = h(x, y)$ where x, y are local complex coordinates on $U \subset S$. Thus, it appears that there are an infinite number of solutions, however, this is just an artifact of taking a neighbourhood of S. As we will see in section 2.3.4 the physical wavefunctions, the ones that solve the D-term, are harmonic and there is only one harmonic function in a compact space. This solution corresponds then to the gauge multiplet associated with the $U(1)^3$ that is preserved by the background. The important point about this computation is that φ depends on both x, y which means that this mode in spread all over S.

The situation is different when we look at the charged modes. Consider now the mode along the element E_{12} of $\mathfrak{su}(3)_{\mathbb{C}}$. We have that $[\langle\Phi\rangle, \chi] = -m^2 x$ so the space of solutions is given by $\varphi_{12} = h(y)$ since any dependence of h on x can be gauged away within the class of holomorphic gauges. Again, one finds an infinite number of solutions due to the local analysis. Typically, we take a basis of solutions given by monomials, i.e. $h_j(y) = y^j$ for $j = 0, 1, \dots,$ but other choices are also possible as mentioned above. The fact that φ only depends on y signals the appearance of a matter curve $\Sigma_a = \{x = 0\}$ which we already saw in the last section.

In general we have that the Higgs field takes values in a subgroup J of G which triggers the breaking $G \to H \times J$ where $H \times J$ is the commutant of J in G. The adjoint of G decomposes as

$$\begin{aligned} G &\to H \times J \\ \mathrm{Adj} &\to \bigoplus_i (\mathcal{T}_i, \mathcal{R}_i). \end{aligned} \tag{2.68}$$

Given these definitions we can characterise algebraically the existence of charged matter in a given curve as follows. We find charged matter in the representation $\mathcal{R}$ under J localised in a curve defined by $f = 0$ when there exists a holomorphic function μ in the representation $\mathcal{R}$ and a positive integer n such that

$$\varphi = \Psi \left(\frac{\mu}{f^n} \right). \tag{2.69}$$

Here the matrix Ψ is the Higgs field acting in the representation $\mathcal{R}$ (not to be confused with the vector $\vec{\Psi}$). The intuition behind this equation is that away from $f = 0$ the field φ is gauge equivalent to zero since it is the commutator with the background. On the other hand, at $f = 0$ this is not a valid gauge transformation so φ is non zero and the gauge-invariant information is contained in the matter curve.

Going back to our previous example we have that $G = U(3)$, $J = U(1)^3$ and $H = 1$ so for the element E_{12} the corresponding representation $\mathcal{R}$ is $(1, -1, 0)$ which yields $\Psi = -m^2 x$. Thus, $f = m^2 x$, $n = 1$ and $\mu = -\varphi$ so we recover the same result.

Let us consider now the holomorphic Yukawa couplings, i.e. those that are obtained ignoring the normalisation. In this case the couplings arise just from the superpotential and nothing else which means that are also invariant under the complexified gauge group. As we will see this has important phenomenological consequences.

Using the solution to the F-terms we may substitute it in eq.(2.58) and find

$$Y = -i m_*^4 \int_U \mathrm{Tr} \left(\bar{\partial}\xi \wedge \bar{\partial}\xi \wedge (h - i[\langle \Phi \rangle, \xi]) \right). \tag{2.70}$$

Notice that if we take a holomorphic gauge in the fluctuations as before, the Yukawa coupling vanishes. This can only be done in an open patch within S but not globally. What makes sense when computing Yukawas in a patch $U \subset S$ is to restrict to a class of gauges that preserve the localisation of the wavefunction (see [56] for more details). After some manipulations the coupling can be written as

$$Y = -i \frac{m_*^4}{3} \int_U \mathrm{Tr} \left(\bar{\partial}\xi \wedge \bar{\partial}\xi \wedge h \right) \tag{2.71}$$

where we used the fact that a and φ are localised to neglect a boundary term. By writing

ξ in terms of φ and h, namely[39]

$$\xi = i\,\Psi^{-1}(\varphi_{xy} - h_{xy}) \tag{2.72}$$

we arrive at the following expression for the Yukawa

$$Y = m_*^4\, f_{abc} \int_{\mathcal{C}} \eta^a \eta^b h^c_{xy}\, dx \wedge dy \tag{2.73}$$

where $\mathcal{C}$ is diffeomorphic to the product of two circles surrounding the Yukawa point and we defined the quantities

$$\eta = -i\Psi^{-1} h_{xy}. \tag{2.74}$$

Finally, this allows to write a residue formula for the Yukawa[40]

$$Y_{abc} = m_*^4 \pi^2 f_{abc}\, \mathrm{Res}(\eta^a \eta^b h^c_{xy}). \tag{2.75}$$

This formula shows explicitly that the holomorphic Yukawas are independent of the fluxes and of the metric, they only depend on the complex structure so they are usually said to be quasi-topological.

Let us compute the Yukawa coupling that arises in the $U(3)$ toy model using the residue formula. We readily find that

$$\eta_a^i = \frac{i}{m^2 x} m_*^i y^i, \quad \eta_b^j = -\frac{i}{m^2 y} m_*^j x^j, \quad h_c^k = m_*^k (x+y)^k \tag{2.76}$$

where the indices i, j, k label the different generations for each mode. If we wanted to do a toy model of the SM we should take $i, j = 0, 1, 2$ since the curves a, b support the chiral matter and $k = 0$ to have a Higgs boson. This yields

$$Y_{abc}^{ijk} = \frac{m_*^4}{m^4} \pi^2\, \mathrm{Res}_{(0,0)} \left(\frac{m_*^{i+j+k} y^i x^j (x+y)^k}{xy} \right) = \frac{m_*^4}{m^4}\, \pi^2\, \delta_{i0}\, \delta_{j0}\, \delta_{k0}. \tag{2.77}$$

The absolute value of the Yukawa computed in this way has no physical meaning since, as we said, we are ignoring the kinetic terms. However, the important fact is that only a single generation ($i = j = 0$) gets a non-vanishing coupling. Moreover, given that the holomorphic couplings are independent of the fluxes and the metric, these will not induce any corrections that allow to generate a higher rank matrix.

This is a particular instance of a general theorem dubbed *rank theorem* in [53]. The theorem states that, given the superpotential (2.50), the rank of the Yukawa matrix Y^{ij}_{abc} (we take only one Higgs boson) associated with a given intersection point p is equal to or less than the intersection degree $(\Sigma_a \cdot \Sigma_b)_p$ of the matter curves at p. From the phenomenological point of view this means that we may give a mass to the third generation generically.

[39] In this expression ξ is a vector that transforms in the representation $\mathcal{R}$ and in (2.71) it is an element of $\mathfrak{g}_{\mathbb{C}}$ but we use the same symbol since the difference should be clear from the context. Notice that in order to have a well defined ξ we must impose that at the matter curve (i.e. $\det \Psi = 0$) $\varphi_{|\det\Psi=0} = h$.

[40] We divided by $(2i)^2$ since we want to use a canonical volume form $d\mathrm{vol}_S = \frac{\omega^2}{2}$. Even though the final expression does not look symmetric it is in fact symmetric under the permutation of the three modes. See [53, 56–58] for more details on the residue formula.

In order to enhance the rank of the Yukawa matrix to account for the masses of the lighter families we have basically three choices. First, we may take the different generations to live in different matter curves, each of which has intersection number one with the rest. However, this goes somewhat against the spirit of unification and the hierarchical pattern of the masses lacks a deep explanation. Keeping all the generations in the same matter curve such that the intersection number is three does not look very good either. If the different points are separated from each other it becomes difficult to compute the physical couplings and, again, the hierarchical structure of the Yukawas remains unexplained. Finally, we may modify the superpotential such that the theorem no longer holds. This is the approach we will take in section 2.3.6 where we include non-perturbative corrections to the superpotential that allow to go beyond rank one and naturally lead to a hierarchy for the fermion masses.

Before discussing extra contributions to the superpotential and physical wavefunctions we make a short digression into the concept of T-brane [56] that will be used to generate a Yukawa coupling for the top quarks in section 2.5.

2.3.3 T-branes

The $U(3)$ toy model that has been discussed up to now describes qualitatively how the down-type quarks and charged leptons get Yukawa couplings. The reason for this is that this coupling comes from the interaction between three different kinds of fields, namely, the $\mathbf{5}_M$, $\mathbf{10}_M$ and $\mathbf{5}_D$. Thus, one can model this by taking three different curves meeting at a point, just like in the $U(3)$ example.

For the up-type Yukawa the situation is rather different. Indeed, in this case the coupling arises from the interaction of only two different modes, i.e. $\mathbf{10}_M$ and $\mathbf{5}_U$, with the former appearing twice. This means that the model that describes this situation has to be qualitatively different if we want to keep all the families in the $\mathbf{10}_M$ reside in the same matter curve. We could take them to be located at different curves but this, as explained earlier, does not give any satisfactory explanation to the hierarchy of masses for the up-type quarks. Actually, in such situation one gets two massive generations and another very light [23], in contradiction with the data.

It was suggested in [23] that one can keep all the $\mathbf{10}$s in the same curve and engineer the model such that the corresponding matter curve self-intersects at the $\mathbf{5}_U$ curve in order to obtain a rank one Yukawa matrix. This proposal seems intuitive since after all we want the $\mathbf{10}_M$ to contribute twice to the coupling. It was realised in [25] that this may be achieved locally by taking two intersecting $\mathbf{10}_M$ curves that look different but that are identified due to 7-brane monodromy.

To illustrate this idea consider the following Higgs field in $SU(2)$

$$\langle\Phi\rangle = \begin{pmatrix} m\sqrt{mx} & 0 \\ 0 & -m\sqrt{mx} \end{pmatrix} dx\wedge dy. \tag{2.78}$$

with m a mass scale, which we will refer to as 'branched' intersecting branes. This seems to correspond to two different 7-branes with embeddings $S_1 = \{m_*^2 z = m\sqrt{mx}\}$ and $S_2 = \{m_*^2 z = -m\sqrt{mx}\}$ that intersect each other at $x = 0$. However, when encircling the intersection point $x \to xe^{2\pi i}$ we find that

$$\langle\Phi(xe^{2\pi i})\rangle = \begin{pmatrix} -m\sqrt{mx} & 0 \\ 0 & m\sqrt{mx} \end{pmatrix} dx\wedge dy, \tag{2.79}$$

so it looks like we have interchanged the branes. This corresponds to a gauge transformation that lives in the Weyl group of $SU(3)$ and tells us that both branes are really the same. Intuitively this is what we want but this description is not the most suited one. Actually, the embedding (2.78) does not satisfy the equations of motion at $x = 0$ since the square root is not holomorphic there.

It was realised in [56] that there is an alternative way to describe this situation that, although less intuitive, is well defined and satisfactory. The key idea is to allow for Higgs fields that are not diagonalisable, called *T-branes*. Thus, we distinguish between two kinds of Higgs bundles, those that satisfy $[\Phi, \Phi^\dagger] = 0$ which we call *intersecting branes* and those with $[\Phi, \Phi^\dagger] \neq 0$.[41] For instance, the $U(3)$ toy model is an example of intersecting branes.

When a matrix is diagonalisable all the information it contains is encoded in its eigenvalues or, alternatively, in its symmetric invariant polynomials. This means that for intersecting branes all the relevant information is in the spectral polynomial

$$P_\Phi(z) = \det(zI - \Phi). \tag{2.80}$$

However, a non-diagonalisable matrix is not characterised by its eigenvalues since the correspondence is not one-to-one. Thus, for T-branes there is gauge invariant information that is not contained in the spectral polynomial.

Let us introduce an example to make the discussion more concrete. Consider the following $SU(2)$ T-brane,

$$\langle\Phi\rangle = \begin{pmatrix} 0 & m \\ m^2 x & 0 \end{pmatrix} dx \wedge dy. \tag{2.81}$$

Clearly, at $x = 0$ this matrix is nilpotent so it is not diagonalisable. What is not so clear is that this configuration is actually a solution to the Hitchin system. It is homolorphic and we have not introduced any gauge bundle so the F-terms are trivially satisfied. However, the D-term is not since

$$[\langle\Phi\rangle, \langle\Phi^\dagger\rangle] = m^2(1 - m^2|x|^2) P \, dx \wedge dy \wedge d\bar{x} \wedge d\bar{y}, \qquad P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.82}$$

In order to find a supersymmetric solution we follow the following strategy. We know that the F-terms are satisfied and these are invariant under complex gauge transformations. On the other hand, the D-term is not invariant under such transformations so we may take a configuration that is related to the original one (that solves the F-terms) by a complex transformation and impose the D-term to find a solution. This procedure is guaranteed to produce a unique solution up to real gauge transformations. More explicitly, consider $g = e^{\frac{f}{2}P} \in SU(2)_{\mathbb{C}}$ and, according to (2.60)

$$\begin{aligned} \langle A\rangle &= \frac{i}{2}(\partial - \bar{\partial}) f \, P \\ \langle\Phi\rangle &= m(e^f E^+ + m x E^-) \, dx \wedge dy. \end{aligned} \tag{2.83}$$

Here A denotes the full gauge field and $E^\pm$ are the step generators of $\mathfrak{su}(2)$ with $[P, E^\pm] = \pm 2E^\pm$. Also, we may take f to be a real function by means of a real gauge transformation. Plugging this into the D-term we find an equation for f, namely

$$(\partial_x \partial_{\bar{x}} + \partial_y \partial_{\bar{y}}) \, f = m^2 \left(e^{2f} - m^2|x|^2 e^{-2f}\right) \tag{2.84}$$

[41] The fact that a complex matrix M is not diagonalisable is equivalent to having $[M, M^\dagger] \neq 0$ as can be seen by taking it to its Jordan canonical form.

where we used the Kähler form

$$\omega = \frac{i}{2}(dx \wedge d\bar{x} + dy \wedge d\bar{y}). \tag{2.85}$$

Taking the ansatz $f = f(r)$ with r the radial coordinate in the $x, \bar{x}$ plane we may write the equation as

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) f = 4m^2 \left(e^{2f} - m^2 r 2 e^{-2f}\right) \tag{2.86}$$

which is a particular case of the Painlevé III differential equation. There is no analytic solution to this equation but, in the spirit of the local approach, we may solve it near the origin $r = 0$ which yields

$$f(r) = \log c + m^2 c^2 r^2 + m^4 r^4 \left(\frac{c^4}{2} - \frac{1}{4c^2}\right) + \mathcal{O}(r^6) \tag{2.87}$$

where c is a number that is fixed by demanding the solution to be well-defined everywhere (and gives $c \sim 0.73$). However, from this local point of view it is not possible to fix it.

This shows that the supersymmetric solution of a T-brane *must* include a non-primitive flux to satisfy the D-term, as opposed to the intersecting branes where the configuration is much simpler. By looking at the behaviour of $f(r)$ for large r one can check that this non-primitive flux is concentrated in a neighbourhood of $r = 0$ which decays as $\sim e^{-\frac{8}{3}r^{3/2}}$ [56].

Let us intersect this T-brane with another brane to see if there is charged matter living at the intersection. Thus, consider the following holomorphic background

$$\langle\Phi\rangle = \begin{pmatrix} 0 & m & 0 \\ m^2 x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} dx \wedge dy. \tag{2.88}$$

The Higgs field is valued in $SU(2)$ and we want to study the fluctuations that arise in the 13 and 23 entries that correspond to the representation $\mathcal{R} = \mathbf{2}$ of $\mathfrak{su}(2)$. In order to do so we may apply the recipe that was given in the last section around eq.(2.69). We have that

$$\varphi = \begin{pmatrix} h_+ \\ h_- \end{pmatrix}, \qquad \Psi = \begin{pmatrix} 0 & m \\ m^2 x & 0 \end{pmatrix} \tag{2.89}$$

with $h_\pm$ arbitrary holomorphic functions of x, y and Ψ the Higgs field acting in the $\mathbf{2}$. We see that there is indeed a matter curve that hosts charged modes since the equation (2.69) can be met by taking

$$\mu = \begin{pmatrix} h_- \\ m x h_+ \end{pmatrix} \quad \text{and} \quad f = m^2 x. \tag{2.90}$$

Moreover, by means of complex gauge transformations we readily check that the fluctuations can be brought to the form

$$\varphi = \begin{pmatrix} 0 \\ h_-(y) \end{pmatrix}. \tag{2.91}$$

In section 2.5 we will use this kind of construction to generate a Yukawa coupling in which this doublet appears twice in the coupling as needed to generate the $\mathbf{10}_M \times \mathbf{10}_M \times \mathbf{5}_U$ in $SU(5)$ GUTs.

Let us look at the spectral polynomial associated to this T-brane. We have that

$$P_\Phi(z) = z^2 - m^3 x \tag{2.92}$$

which is precisely the same as the one of the 'branched' Higgs discussed earlier. Actually, one can check [56] that by taking a singular gauge transformation we may take the $SU(2)$ T-brane to the 'branched' intersecting branes which means that the correct way to describe the situation we had in mind at the beginning is precisely using this non-commutative background. Also, the pathologies that the 'branched' branes suffered from are simply due to a singular gauge transformation starting from a consistent configuration. Both descriptions are very different at the point $x = 0$ since from the 'branched' perspective the Higgs field vanishes and we get an enhancement to $SU(2)$. On the other hand, at $x = 0$ the background (2.83) does not vanish and there is no enhancement at all.

Let us conclude this short section with some comments about the embedding of the T-brane in spacetime. In perturbative string theory, the embedding of the branes is given by diagonalising the matrices Φ^n with n an index that labels the normal directions to the brane and then reading it off from the eigenvalues. However, it may happen that Φ^n and Φ^m do not commute for $n \neq m$ as in the Myers effect [59]. This is precisely what happens with the T-branes, the embedding is non-commutative. One may be tempted to identify the equation $P_\Phi(z) = 0$ as the embedding since for intersecting branes this is indeed the case. However, the spectral polynomial is invariant under the complexified gauge group while the physical embedding of the branes should care about more than just holomorphic data. Thus, one may think of $P_\Phi(z) = 0$ as containing part of the information about the embedding but not all.

This kind of nilpotent Higgs vevs were first considered in a topological context in [60] and further analysed in [61]. An interpretation of these configurations in terms of the global geometry in F-theory has been recently studied in [62–64].

2.3.4 Physical wavefunctions and physical Yukawas

After this short digression let us go back to the analysis of the fluctuations for a given background. Thus far we have only worried about fluctuations that satisfy the F-terms equations so the discussion has been mainly algebraic. However, the *physical* fields have to solve also the D-term constraint.

In order to find a solution to the D-term we will use the same strategy that was used in the last section to compute the T-brane background. Namely, we start with a solution to the F-terms and perform a complex gauge transformation that is determined upon imposing the D-term. Recall that the solution to the F-terms in holomorphic gauge for the background is given by

$$\begin{aligned} a &= \bar{\partial}\xi \\ \varphi &= h - i\Psi\xi. \end{aligned} \tag{2.93}$$

where we are taking a given representation $\mathcal{R}$ under the subgroup where $\langle\Phi\rangle$ is valued.[42] The next step is to go into real gauge for the background. This can be done by simply multiplying these wavefunctions by the corresponding gauge transformation $g \in G_{\mathbb{C}}$ in the

[42]Recall that Ψ is the Higgs field in the representation $\mathcal{R}$ in *holomorphic* gauge.

representation $\mathcal{R}$, namely

$$\begin{aligned} a &= g\,\bar{\partial}\xi \\ \varphi &= g(h - i\Psi\xi). \end{aligned} \tag{2.94}$$

According to (2.64) the gauge transformations act by changing ξ so in order to find a physical solution we may substitute these fields in the D-term which yields an equation for ξ whose solution is unique (up to real gauge transformations). If we define the quantity

$$U = h - i\Psi\xi \tag{2.95}$$

then the physical fluctuations can be written as

$$\begin{aligned} a &= ig\Psi^{-1}\bar{\partial}U \\ \varphi &= gU. \end{aligned} \tag{2.96}$$

Now, we may substitute in the D-term (2.63), where $\langle A\rangle$ and $\langle\Phi\rangle$ correspond to the background in real gauge,

$$\begin{aligned} \left[\langle A^{1,0}\rangle, \cdot\,\right] &= -i\partial g^{-1} g \\ [\langle\Phi\rangle, \cdot\,] &= g\,\Psi\,g^{-1}, \end{aligned} \tag{2.97}$$

and solve for U. The equation for U is

$$\partial_x(g\Psi^{-1}\partial_{\bar{x}}U) + g^{-1}\partial_x g\,g\,\Psi^{-1}\partial_{\bar{x}}U + \partial_y(g\Psi^{-1}\partial_{\bar{y}}U) + g^{-1}\partial_y g\,g\,\Psi^{-1}\partial_{\bar{y}}U = g^{-1}\Psi^\dagger g^2 U \tag{2.98}$$

where we used the fact that we can always take $g = g^\dagger$.

Examples

Let us work out some examples to see how eq.(2.98) behaves in simple cases. For instance, take the sector a^+ (i.e. $\mathcal{R} = (1, -1, 0)$) of the $U(3)$ model without fluxes. In that case we have that $\Psi = -m^2 x$ and $g = 1$ so the equation reduces to

$$\partial_x\partial_{\bar{x}}U + \partial_y\partial_{\bar{y}}U - \frac{1}{x}\partial_{\bar{x}}U - m^4|x|^2 U = 0 \tag{2.99}$$

whose normalisable solution is $U = f(y)e^{-m^2|x|^2}$ with $f(y)$ an arbitrary holomorphic function that corresponds to the family function. Using the relations (2.96) we find that this indeed yields the result (2.46).

When we include fluxes as in (2.47) the only change is that $g = \exp\left[\frac{M}{2}(|x|^2 - |y|^2)\right]$ so the equation for U is modified to

$$\partial_x\partial_{\bar{x}}U - \frac{1}{x}\partial_{\bar{x}}U - M\bar{x}\,\partial_{\bar{x}}U - m^4|x|^2 U = 0 \tag{2.100}$$

where we assumed U does not depend on $\bar{y}$. The solution is given by

$$U = f(y)\,e^{-\sqrt{\left(\frac{M}{2}\right)^2 + m^4}|x|^2} e^{-\frac{M}{2}|x|^2} \tag{2.101}$$

in agreement with (2.48) with $\lambda_+ = -\frac{M}{2} - \sqrt{\frac{M^2}{4} + m^4}$.

These examples correspond to intersecting branes so the representation $\mathcal{R}$ is abelian. However, when dealing with T-branes we typically find non-abelian representations as we saw in the last section. Let us reconsider the modes (2.91) and find the corresponding physical fluctuations. In this case we have

$$g = \begin{pmatrix} e^{\frac{f}{2}} & 0 \\ 0 & e^{-\frac{f}{2}} \end{pmatrix}, \qquad \Psi = \begin{pmatrix} 0 & m \\ m^2 x & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} U_+ \\ U_- \end{pmatrix} \tag{2.102}$$

so the equations for $U_\pm$ are

$$\begin{aligned} \partial_x \partial_{\bar{x}} U_+ - \partial_x f\, \partial_{\bar{x}} U_+ - e^{2f} U_+ &= 0 \\ \partial_x \partial_{\bar{x}} U_- - \frac{1}{x} \partial_{\bar{x}} U_- + \partial_x f\, \partial_{\bar{x}} U_- - |x|^2 e^{-2f} U_- &= 0 \end{aligned} \tag{2.103}$$

where we assumed once again that $U_\pm$ do not depend on $\bar{y}$. These equations cannot be solved analytically since we do not even know the function f. However, following (2.87) we may take the approximate solution for f near the Yukawa point, i.e.

$$f(r) = \log c + m^2 c^2 r^2 + \mathcal{O}(r^4) \tag{2.104}$$

which allows to find a solution for U. These are

$$U_+ = 0, \qquad U_- = f(y) e^{\lambda |x|^2} \tag{2.105}$$

with $\lambda = -\frac{m^2 c^2}{2} - \sqrt{\left(\frac{m^2 c^2}{2}\right)^2 + \frac{m^4}{c^2}}$. We take $U_+ = 0$ since we do not find localised solutions (see [58] for more details). We see that the equations for U_+ and U_- are decoupled which is by no means generic. In fact, in the E_6 model of section 2.5 we find a similar set of equations in which this is not the case.

In all of these examples the behaviour of the physical wavefunctions is given by an exponential damping of the same kind. However, this is no longer true when we have non-constant fluxes or matter curves that are not straight. In such cases, as one might guess, an analytic solution is often not possible.

Normalisation and mixing

Now that we have an idea of what the physical fluctuations look like we may compute the normalisation factors. Given our previous results we may write the wavefunction for a mode living in a matter curve as

$$\vec{\Psi}^i_\rho = \gamma^i_\rho \begin{pmatrix} i g_\rho \Psi_\rho^{-1} \partial_{\bar{x}} U^i_\rho \\ i g_\rho \Psi_\rho^{-1} \partial_{\bar{x}} U^i_\rho \\ g_\rho U^i_\rho \end{pmatrix} \tag{2.106}$$

where ρ labels the representation under the gauge group and i is a family index. Also, we have included a constant factor γ^i_ρ that makes sure that the wavefunction is normalised. The natural scalar product among such modes that corresponds to the kinetic term for the fields in M_4 is

$$K^{ij}_\rho \equiv \langle \vec{\Psi}^i_\rho, \vec{\Psi}^j_\rho \rangle = m_*^4 \gamma^i_\rho \gamma^j_\rho T(\rho) \int_S \vec{\Psi}^{i\dagger}_\rho \vec{\Psi}^j_\rho \, \mathrm{dvol}_S \tag{2.107}$$

where $T(\rho)$ is defined via $\mathrm{Tr}(E_\rho E_\kappa^\dagger) = T(\rho)\delta_{\rho\kappa}$. The dagger means complex transpose in the three dimensional vector $\overrightarrow{\Psi}$ as well as in the components, which are themselves vectors, and $d\mathrm{vol}_S = \frac{\omega^2}{2}$. We take both wavefunctions to correspond to the same curve since the scalar product of fields living in different curves is automatically zero due to gauge invariance. Asking for canonical kinetic terms translates into having $K_\rho^{ij} = \delta_{ij}$

Let us compute the kinetic mixing for the sector a^+ of the $U(3)$ model with fluxes which yields

$$\langle \overrightarrow{\Psi}^i_{a^+}, \overrightarrow{\Psi}^j_{a^+} \rangle = m_*^4 \gamma^i_{a^+} \gamma^j_{a^+} \left(1 + \frac{\lambda_+^2}{m^4}\right) \int_{\mathbb{C}^2} e^{-2\sqrt{\left(\frac{M}{2}\right)^2 + m^4}|x|^2} e^{-M|y|^2} m_*^i \bar{y}^i m_*^j y^j \, d\mathrm{vol}_{\mathbb{C}^2}$$

where we took $S = \mathbb{C}^2$ according to our local analysis. The integral in $y, \bar{y}$ immediately shows that it vanishes for $i \neq j$ so there is no mixing between the different families. Evaluation of the integrals then gives the normalisation factors.

These depend explicitly on the fluxes and metric since the normalisation condition involves taking the complex conjugate and is therefore not holomorphic. It is important to notice that for the fluxless case, $M = 0$, the integral in $y, \bar{y}$ diverges which signals no chirality in that sector. For a negative flux density $M < 0$ the integral also diverges but the mode with opposite charge has a finite normalisation factor. This is closely related to the notion of local chirality that will be introduced in the next section.

Physical Yukawa couplings

Finally, we have all the ingredients we need to compute the physical Yukawa couplings. We can arrive at the result by taking the physical zero modes, plugging them in eq.(2.57), namely,

$$Y_{abc} = m_*^4 f_{abc} \int_S \det(\overrightarrow{\Psi}_a, \overrightarrow{\Psi}_b, \overrightarrow{\Psi}_c) \, d\mathrm{vol}_S \tag{2.108}$$

and performing the integral. Alternatively, we can take the holomorphic couplings that we found earlier by means of the residue formula (2.109) and simply add the normalisation factors,

$$Y_{abc} = m_*^4 \pi^2 \gamma_a \gamma_b \gamma_c f_{abc} \, \mathrm{Res}(\eta^a \eta^b h^c_{xy}). \tag{2.109}$$

Both methods will give the same result. The nice thing about the second approach is that it shows manifestly that the only dependence of the couplings on the Kähler data is through the normalisation factors.

In the following we briefly discuss how to engineer ultra-local models with the appropriate chirality and doublet-triplet splitting.

2.3.5 Chirality and doublet-triplet splitting

When building a local model we have to include fluxes to generate chirality and achieve doublet-triplet splitting. As discussed earlier this amounts to a topological condition on S_{GUT} so it seems that this will not be visible in an ultra-local model.

More explicitly, to have a chiral spectrum in the sector ρ living at Σ_ρ we must ensure that

$$\int_{\Sigma_\rho} \mathrm{Tr}_{\mathcal{R}} \langle F \rangle \neq 0 \tag{2.110}$$

where $\mathcal{R}$ is the representation of the subgroup generated by $\langle\Phi\rangle$ and $\langle F\rangle$ under which fermions in the sector ρ transform. This condition requires a global knowledge of the matter curve Σ_ρ along S_{GUT} and the flux along it. However, we cannot impose (2.110) in practice since we do not know the geometry or fluxes away from the Yukawa point so we need an alternative characterisation of chirality suited to our local approach. The notion of local chirality was discussed in [52] and it boils down to demanding that matter wavefunctions of a certain 4d chirality are localised near the Yukawa point p. Indeed, when gauge fluxes are included such that the wavefunction for a given sector ρ is localised in the region around p, its conjugate sector $\bar{\rho}$ will not contain any localised mode in that same region, and this signals a net local chirality.

For instance, take the physical modes of the sector a of the $U(3)$ toy model, namely,

$$\vec{\Psi}_{a^\pm} = \begin{pmatrix} -i\frac{\lambda_\pm}{m^2} \\ 0 \\ 1 \end{pmatrix} e^{-\sqrt{\left(\frac{M}{2}\right)^2+m^4}|x|^2} e^{\mp\frac{M}{2}|y|^2} f(y). \tag{2.111}$$

The first thing we notice is that, regardless of the flux, the wavefunction is localised in the curve $x=0$ due to the intersection. However, the localisation along the matter curve is determined by the exponential factor $e^{\mp\frac{M}{2}|y|^2}$ which depends only on the flux density M. For $M=0$ there is no localised mode so, in particular, the normalisation factor vanishes. On the other hand, for $\pm M>0$ we have that $a^\pm$ is localised while $a^\mp$ is not.

If this local analysis is to be trusted we must make sure that the wavefunctions have support in the vicinity of the Yukawa point. If this is not the case it means that it is not possible to describe such mode in this way. Thus, we say that we have a local chiral mode whenever the corresponding physical wavefunction is localised around the Yukawa point. As the example shows this is directly related to the flux density on the matter curve. We can sharpen this connection by studying a local version of the chiral index as we show in the following.

A simple way to obtain a condition on the fluxes to have local chirality is to first T-dualise in the z, $\bar{z}$ directions to a system of magnetised D9 branes and look at the index theorem in 6d. As explained in appendix A of [49], under T-duality we get a gauge field along $\bar{z}$, $\langle A_{\bar{z}}\rangle = \langle\Phi_{xy}\rangle$, so we end up with magnetic fluxes $F_{x\bar{z}} = D_x\Phi_{xy}$ and $F_{y\bar{z}} = D_y\Phi_{xy}$. For T-brane backgrounds we also have a flux along the $z\bar{z}$ direction, $F_{z\bar{z}} = i[\Phi_{xy}, \Phi_{xy}^\dagger]$.

The Dirac index for a given representation $\mathcal{R}$ in 6d reads

$$\text{index}_{\mathcal{R}} \not{D} = \frac{1}{48(2\pi)^2}\int \left(\text{Tr}_{\mathcal{R}}\, F\wedge F\wedge F - \frac{1}{8}\text{Tr}_{\mathcal{R}}\, F\wedge \text{Tr}R\wedge R\right) \tag{2.112}$$

where F is the gauge flux and R is the Riemann tensor. The notion of local chirality in this setup translates into asking that the integrand is different from zero at the Yukawa point. Since the second term in (2.112) vanishes for flat spaces we should look at the quantity[43]

$$\begin{aligned}\mathcal{I}_{\mathcal{R}} \equiv \frac{i}{6}\text{Tr}_{\mathcal{R}}\, (F\wedge F\wedge F)_{x\bar{x}y\bar{y}z\bar{z}} = i\,\text{Tr}_{\mathcal{R}}\big(&F_{x\bar{x}}\{F_{y\bar{y}}, F_{z\bar{z}}\} + F_{x\bar{z}}\{F_{y\bar{x}}, F_{z\bar{y}}\} + \\ &F_{x\bar{y}}\{F_{y\bar{z}}, F_{z\bar{x}}\} - \{F_{x\bar{x}}, F_{y\bar{z}}\}F_{z\bar{y}} - \{F_{x\bar{y}}, F_{y\bar{x}}\}F_{z\bar{z}} - \{F_{x\bar{z}}, F_{y\bar{y}}\}F_{z\bar{x}}\big).\end{aligned} \tag{2.113}$$

[43]We take $F = F_{\alpha\bar{\beta}}\,dx^\alpha\wedge d\bar{x}^{\bar{\beta}}$ so we include a factor of i to make $\text{Tr}_{\mathcal{R}} F^3_{x\bar{x}y\bar{y}z\bar{z}}$ a real number. In our conventions $\{A,B\} = \frac{1}{2}(AB+BA)$.

The condition to have local chirality in a given sector ρ is that $\mathcal{I}_{\mathcal{R}} < 0$ for such sector in a given region. For the case of intersecting branes the flux $F_{z\bar{z}}$ vanishes and the representation $\mathcal{R}$ is abelian so $\mathcal{I}_{\mathcal{R}}$ reduces to the criterion for local chirality discussed in [52] and [57].

This shows the connection between the flux density and localisation of the wavefunction without having to actually compute the physical wavefunction which becomes useful when building a concrete model. Let us reconsider the $a^{\pm}$ case in which $\mathcal{R} = \pm(1,-1,0)$ of $U(1)^3$ and the only non-vanishing fluxes are $F_{x\bar{x}} = -F_{y\bar{y}} = -iM$ and $F_{x\bar{z}} = -m^2$ so

$$\mathcal{I}_{\pm(1,-1,0)} = \mp m^4 M, \tag{2.114}$$

in agreement with our previous result.

Thus far we have not discussed much the hypercharge flux or doublet-triplet splitting in this ultra-local analysis so let us comment on this briefly. As explained earlier, the standard way of breaking the GUT group down to the SM group is via a hypercharge flux. From this perspective this amounts to including a flux that has the following form

$$\langle F_Y \rangle = i\left[R(dx \wedge d\bar{x} - dy \wedge d\bar{y}) + N(dx \wedge d\bar{y} + dy \wedge d\bar{x})\right] Q_Y \tag{2.115}$$

with R, M real constants that parametrise the flux density and Q_Y the hypercharge generator in $SU(5)$. The different SM particles within the same GUT multiplet have different hypercharge so they feel this flux differently. In particular, the local index (2.113) is different for each of them so the condition to have doublet-triplet splitting can be rephrased as a condition for the flux densities. More specifically, a realistic model should be such that $\mathcal{I}_{\text{doublet}} \neq 0$ while $\mathcal{I}_{\text{triplet}} = 0$.

As we will see in sections 2.4 and 2.5 the hypercharge flux is an important ingredient from the model-building perspective. Indeed, this flux appears in the normalisation factors and hence in the physical Yukawa couplings. Thus, this allows to explain, for instance, the difference between the masses of the down-type quarks and charged leptons.

Up to now we have explained how to construct ultra-local models and how to compute physical Yukawa couplings. However, as we saw in section 2.3.2 these are typically rank one which means that only the third generation gets a mass. In the following we introduce extra ingredients to the game that allow to go beyond rank one.

2.3.6 Non-perturbative effects on Yukawa couplings

Non-perturbative superpotential

As shown in section 2.3.2 the holomorphic Yukawa couplings have rank one if we insist on generating them from a single intersection point. This looks like a good starting point to describe the observed fermion masses of the SM since the third family is much heavier than the rest. However, in a realistic model one should be able to generate masses for all the generations and, given the assumptions we made, the only way to modify the structure of Yukawas is to deform the superpotential. In a supersymmetric theory the superpotential is not modified in perturbation theory so we should consider the effect of non-perturbative corrections [65].

Typically, in a specific model there will be additional branes to those that are responsible for generating the SM gauge group and matter content. For instance, there may be euclidean D3-branes wrapping a holomorphic divisor $S_{np} \subset X$ or extra 7-branes that develop a gaugino condensate. These will induce a non-perturbative superpotential that may modify the Yukawa couplings as we briefly explain in the following (see [65] for more details).

Consider a perturbative type IIB vacuum with $O3/O7$-planes and a D7-brane wrapping S_{np} such that it undergoes gaugino condensation. If there is a spacetime-filling D3-brane at z_0 in the internal space then it feels a superpotential of the form [66, 67, 152]

$$W_{D3} = \mu_3 \, \mathcal{A} \, e^{-T} f(z_0). \tag{2.116}$$

Here μ_3 is the tension of the D3-brane, $\mathcal{A}$ is a holomorphic function of the closed string complex structure moduli, T is the complexified Kähler modulus of the D7-brane and $f(z_0) = 0$ is the equation of the divisor S_{np}. If there is a complex structure moduli-stabilising G-flux the function $\mathcal{A}$ can be taken to be a constant. This superpotential is the same as the one generated by an euclidean D3-brane wrapping S_{np} with the appropriate number of zero modes.

This shows the non-perturbative contributions to a D3-brane but we are interested in what happens to a D7-brane. Since the seven-branes we are using to construct the model are magnetised they will have, in general, induced D3-brane charge, i.e. $N_{D3} = \frac{1}{8\pi^2}\int_{S_{GUT}} \mathrm{Tr} F \wedge F$. Thus, for our present purposes one can think of the D7-branes as smeared D3-branes. This means that euclidean D3-branes will induce a similar non-perturbative superpotential to our model. More specifically, we have that [65]

$$W_{D7} = \mu_3 \, \mathcal{A} \, e^{-T} \exp\left(\frac{1}{8\pi^2}\int_{S_{GUT}} \mathrm{Str}(\log f F \wedge F)\right) \tag{2.117}$$

where Str denotes the symmetric trace. The key point about his superpotential is that it depends on the D7-brane moduli so it will necessarily modify their equations of motion. In order to see this one can Taylor expand (2.117) near the GUT branes which yields [57, 65]

$$W_{D7} = m_*^4 \left[\int\limits_{S_{GUT}} \mathrm{Tr}\,(\Phi_{xy} F) \wedge dx \wedge dy + \frac{\epsilon}{2} \sum_{n \geq 0} \int\limits_{S_{GUT}} \theta_n \mathrm{Str}(\Phi_{xy}^n F \wedge F) \right]. \tag{2.118}$$

We have defined

$$\epsilon = \mathcal{A} \, e^{-T} h_0^{N_{D3}}, \quad \theta_n = \frac{g_s^{-\frac{n}{2}} \mu^3}{(2\pi)^{2+\frac{3n}{2}} m_*^{4+2n}} [\partial_z^n \log(h/h_0)]_{z=0} \tag{2.119}$$

and $h_0 = \int_{S_{GUT}} h$ is the mean value of h in S_{GUT} and $\mu \sim m_*$ is the fundamental scale of the non-perturbative effect.

The first term in (2.118) is the tree-level contribution that was found previously which gets corrected by a sum of terms of decreasing importance as n increases. Also, the overall factor ϵ is small when the volume of the instanton is big which allows to treat this deformation in perturbation theory.

Let us take a closer look at this new superpotential to see what are the relevant terms for our purposes. The first term in the expansion is proportional to $\int_{S_{GUT}} \theta_0 \mathrm{Tr} F \wedge F$ which is topological in the case where θ_0 is constant but non-trivial otherwise. There is a geometric interpretation for these two possibilities, as follows. In order to avoid instanton zero modes charged under the SM group the instanton should not intersect the GUT branes so we assume $S_{GUT} \cap S_{np} = 0$. However, following the analysis of [69] the instanton must intersect some other 7-brane on $S \subset X$ to have the right structure of zero modes to generate a superpotential.[44] One should then distinguish two different cases:

- The D3-instanton intersects a 7-brane that does not intersect the GUT brane, namely, $S_{np} \cap S \neq 0$ and $S_{GUT} \cap S = 0$.
- The D3-instanton intersects a 7-brane that in turn intersects the GUT brane so $S_{np} \cap S \neq 0$ but $S_{GUT} \cap S \neq 0$.

In the first scenario the function h restricted to the GUT brane is constant so θ_0 vanishes.[45] Thus, we should consider the next term in the expansion (2.118). However, for many interesting gauge groups this terms also vanishes since it is proportional to the symmetric trace $\mathrm{Str}(T_a T_b T_c)$. Indeed, for the two groups that will be considered in the following sections, $SO(12)$ and E_6, this contribution vanishes so it is the term proportional to θ_2 that gives the leading correction to the superpotential. As shown in [57] this term produces a hierarchical structure of the form $(1, \mathcal{O}(\epsilon^2), \mathcal{O}(\epsilon^2))$ for the $SO(12)$ model which is rather unsatisfactory from the phenomenological point of view.

The second scenario corresponds to a non-constant holomorphic function θ_0 so its effect in the equations of motion is non-trivial. As we will show in the next sections this yields a hierarchical pattern $(1, \mathcal{O}(\epsilon), \mathcal{O}(\epsilon^2))$ which looks better. Thus, in order to keep the discussion as clear as possible we will restrict to this case in the following, namely

$$W_{np} = m_*^4 \left[\int_{S_{GUT}} \mathrm{Tr}\,(\Phi \wedge F) + \frac{\epsilon}{2} \int_{S_{GUT}} \theta_0 \mathrm{Tr}(F \wedge F) \right]. \tag{2.120}$$

See [49, 57] for a more complete discussion.

The strategy now is basically the same as the one we have been following but with the new superpotential (2.120). Namely, we may compute the holomorphic Yukawa couplings by means of a residue formula and the normalisation factors by solving the corresponding D-terms which, as shown in appendix C of [57], are not modified by the D3-instanton.

In what follows we compute the correction to the conditions for supersymmetry which allows to generalise the residue formula.

Generalised residue formula

The equations of motion that follow from the superpotential (2.120) are

$$\begin{aligned} \bar{\partial}_A \Phi + \epsilon \partial \theta_0 \wedge F &= 0 \\ F^{0,2} &= 0 \end{aligned} \tag{2.121}$$

which together with the D-term, $\omega \wedge F + \frac{1}{2}[\Phi, \Phi^\dagger] = 0$, give a deformed Hitchin system. Expanding in fluctuations we obtain the F-term equations of motion for the zero modes,

[44] In a perturbative limit this corresponds to the intersection with an orientifold plane to have an $O(1)$ D3-instanton that indeed generates a superpotential.

[45] If the intersection of two divisors $S_1 \cap S_2$ is homologically trivial, then the restriction of the line bundle $\mathcal{L}_{S_1}$ to S_2 is trivial and vice versa. Hence, since the divisor function h_1 of S_1 is a section of $\mathcal{L}_1$ we can always take $h_1|_{S_2}$ as a constant section of the trivial bundle $\mathcal{L}_1|_{S_2}$.

namely

$$\begin{array}{rcl} \bar{\partial}_{\langle A\rangle}\varphi - i[\alpha, \langle\Phi\rangle] + \epsilon\partial\theta_0 \wedge (\partial_{\langle A\rangle}a + \bar{\partial}_{\langle A\rangle}a^\dagger) & = & 0 \\ \bar{\partial}_{\langle A\rangle}a & = & 0 \end{array} \tag{2.122}$$

Notice that the gauge bundle is still holomorphic so we may take a holomorphic gauge for the background and solve the F-terms similarly to the original equations which yields

$$\begin{array}{rcl} a & = & \partial\xi \\ \varphi & = & h - i[\langle\Phi\rangle, \xi] + \epsilon\partial\theta_0 \wedge (a^\dagger - \partial\xi). \end{array} \tag{2.123}$$

Notice there is a dependence of φ on the conjugate field $a^\dagger$, however, as shown in appendix D of [57] the terms proportional to it in the Yukawa coupling arrange in total derivatives and vanish upon integration. Thus, we may simply set $a^\dagger = 0$ in the expression above when computing holomorphic Yukawa couplings.

We may now substitute these solutions in the formula for the Yukawa coupling (2.58) and find

$$Y = -i\frac{m_*^4}{3}\int_{S_{GUT}} \mathrm{Tr}(h \wedge \bar{\partial}\xi \wedge \bar{\partial}\xi). \tag{2.124}$$

This is the same expression we found for the tree-level superpotential, however, the function ξ receives ϵ corrections so the Yukawas get corrected too. This means that the correction to the Yukawa coupling comes simply from the correction to the wavefunction, the extra term in the superpotential does not yield any contribution. Setting $a^\dagger = 0$ in (2.123) we have that ξ satisfies

$$\Psi\xi dx \wedge dy = i(\varphi - h + \epsilon\partial\theta_0 \wedge \partial\xi). \tag{2.125}$$

This equation can be solved iteratively and, to first order in ϵ, one finds

$$\begin{array}{rcl} \xi & = & \xi^{(0)} + i\epsilon\Psi^{-1}(\partial_x\theta_0\partial_y\xi^{(0)} - \partial_y\theta_0\partial_x\xi^{(0)}) + \mathcal{O}(\epsilon^2) \\ \xi^{(0)} & = & i\Psi(\varphi_{xy} - h_{xy}). \end{array} \tag{2.126}$$

Plugging this expression in (2.124) we find, similarly to the tree-level case, a residue formula for the holomorphic Yukawa coupling that reads

$$Y_{abc} = m_*^4\pi^2 f_{abc}\,\mathrm{Res}(\eta^a\eta^b h^c_{xy}) \tag{2.127}$$

with

$$\eta = -i\Psi^{-1}h_{xy} + \epsilon\Psi^{-1}(\partial_x\theta_0\partial_y(\Psi^{-1}h_{xy}) - \partial_y\theta_0\partial_x(\Psi^{-1}h_{xy})) + \mathcal{O}(\epsilon^2). \tag{2.128}$$

This concludes the introduction to F-theory local GUTs. In the next two sections we build partial local models that describe the $\mathbf{5}_M \times \mathbf{10}_M \times \mathbf{5}_D$ and $\mathbf{10}_M \times \mathbf{10}_M \times \mathbf{5}_U$ Yukawa couplings in $SU(5)$ GUTs which shows that this approach to generating higher rank couplings provides the correct hierarchical structure in agreement with experimental data.

2.4 Down-type Yukawas

2.4.1 The $SO(12)$ model

In this section we describe in detail the $SO(12)$ local model which we will analyse in the following. According our previous discussion we will first specify the structure of 7-brane intersections and matter curves that breaks the $SO(12)$ symmetry down to $SU(5) \times U(1)^2$, and then add the worldvolume flux that induces 4d chirality and breaks the $SU(5)$ GUT spectrum down to the MSSM.

Matter curves

Following the general framework described earlier, let us consider a local model where the symmetry group at the intersection point of three matter curves is $G_p = SO(12)$. Away from this point, this group is broken to a subgroup because $\langle \Phi \rangle \neq 0$. One can then engineer a $\langle \Phi \rangle$ such that generically $SO(12)$ is broken to $SU(5) \times U(1)^2$, except for some complex curves where there is an enhancement to either $SO(10) \times U(1)$ or $SU(6) \times U(1)$. In this way, we can identify $G_{S_{GUT}} = SU(5)$ as the GUT gauge group and the enhancement curves as matter curves where chiral matter wavefunctions are localised.

In order to make the above picture more precise let us consider the generators of $SO(12)$, in terms of which we can express the particle spectrum of our local GUT model. These generators can be decomposed as $\{H_i, E_\rho\}$, where the H_i, $i = 1, \cdots, 6$, belong to the Cartan subalgebra of $SO(12)$ and the E_ρ are step generators.[46] Recall that

$$[H_i, E_\rho] = \rho_i E_\rho \tag{2.129}$$

where ρ_i is the i-th component of the root ρ. The 60 non-trivial roots are given by

$$(\underline{\pm 1, \pm 1, 0, 0, 0, 0}) \tag{2.130}$$

where the underlying means all possible permutations of the vector entries.

Let us now choose the vev of the transverse position field $\Phi = \Phi_{xy} dx \wedge dy$ to be

$$\langle \Phi_{xy} \rangle = m^2 \left(x Q_x + y Q_y \right) \tag{2.131}$$

where m^2 is related to the intersection slope between 7-branes as explained earlier, and the charge operators Q_x and Q_y are the following combinations of generators of elements of the $SO(12)$ Cartan subalgebra

$$Q_x = -H_1 \quad ; \quad Q_y = \frac{1}{2} \left(H_1 + H_2 + H_3 + H_4 + H_5 + H_6 \right) \tag{2.132}$$

This choice of $\langle \Phi \rangle$ describes a $SO(12)$ local model that is similar to the $U(3)$ toy model in several aspects. This will allow us to apply several useful results already encountered to the more realistic case at hand.

Given (2.131) one can understand the $SO(12)$ symmetry breaking pattern described above as follows. In general the step generators E_ρ satisfy

$$[\langle \Phi_{xy} \rangle, E_\rho] = m^2 q_\Phi(\rho) E_\rho \tag{2.133}$$

[46] Throughout this work we use the standard form of the $SO(2N)$ generators in the fundamental representation.

with q_Φ a holomorphic function of the complex coordinates x, y of the 4-cycle S. The subgroup of $SO(12)$ not broken by the presence of this vev corresponds to those generators that commute with $\langle\Phi\rangle$ at any point in S_{GUT}. This set is given by the Cartan subalgebra of $SO(12)$ and to those step generators E_ρ such that $q_\Phi(\rho) = 0$ for all x, y. It is easy to see that such unbroken roots are given by

$$(0, \underline{1, -1, 0, 0, 0}) \tag{2.134}$$

together with the Cartan generators. Therefore, from the symmetry group $SO(12)$ only the subgroup $SU(5)\times U(1)^2$ remains as a gauge symmetry, and we can identify $G_{S_{GUT}} = SU(5)$ as our GUT gauge group.

On the other hand, the broken generators of $SO(12)$, that have $q_\Phi \neq 0$ for generic x, y, allow us to understand the pattern of matter curves and to classify the charged matter localised therein. Such broken roots and their charges q_Φ are displayed in table 2.2.

ρ	root	q_Φ	$SU(5)$
a^+	$(1, \underline{-1, 0, 0, 0, 0})$	$-x$	$\overline{\mathbf{5}}$
a^-	$(-1, \underline{1, 0, 0, 0, 0})$	x	$\mathbf{5}$
b^+	$(0, \underline{1, 1, 0, 0, 0})$	y	$\mathbf{10}$
b^-	$(0, \underline{-1, -1, 0, 0, 0})$	$-y$	$\overline{\mathbf{10}}$
c^+	$(-1, \underline{-1, 0, 0, 0, 0})$	$x - y$	$\overline{\mathbf{5}}$
c^-	$(1, \underline{1, 0, 0, 0, 0})$	$-(x - y)$	$\mathbf{5}$

Table 2.2: Data of broken generators

From this table we see that there are three complex curves within S_{GUT} where the bulk symmetry $SU(5)\times U(1)^2$ is enhanced, in the sense that there $q_\Phi = 0$ for an additional set of roots. Concretely, for $x = 0$ there are 10 additional roots that together with those in (2.134) complete the $SU(6)$ root system. We have labeled such matter curve as Σ_a, so we would have that $G_{\Sigma_a} = SU(6) \times U(1)$. These extra set of roots whose q_Φ vanishes at Σ_a can be split into subsets that have different q_Φ away from Σ_a. It is easy to convince oneself that each of these subsectors must fall into complete weight representations of $SU(5)$, which in turn correspond to the matter localised at the curve. In the case of Σ_a, there are two sectors a^+ and a^- that correspond to the representations $\mathbf{5}$ and $\overline{\mathbf{5}}$ of $SU(5)$, respectively, as shown in table 2.2.

Similarly to Σ_a, at the curve $\Sigma_b = \{y = 0\}$ there are 20 extra unbroken roots and $SU(5) \times U(1)^2$ is enhanced to $SO(10) \times U(1)$, giving rise to the representations $\mathbf{10}$ and $\overline{\mathbf{10}}$. The third matter curve is given by $\Sigma_c = \{x = y\}$, where there is also an enhancement to $SU(6) \times U(1)$.

Worldvolume flux

To obtain a 4d chiral model the above pattern of matter curves is not enough, and it is necessary to add a non-trivial background worldvolume flux $\langle F\rangle$ to our local F-theory model. Just like the position field, such flux is usually chosen along the Cartan subalgebra of $SO(12)$, so that it commutes with $\langle\Phi_{xy}\rangle$ and the equations of motion of our system are simplified. Moreover, considering a component of $\langle F \rangle$ along the hypercharge generator Q_Y allows to break the GUT gauge group $SU(5)$ down to $SU(3) \times SU(2) \times U(1)_Y$, as explained earlier.

In order to construct a worldvolume flux with the desired properties we proceed in three steps. First we add a flux $\langle F_1 \rangle$ that creates chirality on the curves Σ_a and Σ_b, selecting the sectors a^+ and b^+ as the ones that contain the chiral matter of the model, as opposed to a^- and b^-. Then we add an extra piece $\langle F_2 \rangle$ such that the matter curve Σ_c also contains a chiral spectrum, a typical requirement to achieve an acceptable Higgs sector. None of these previous fluxes further break the gauge group $SU(5) \times U(1)^2$ so, finally, we include a flux $\langle F_Y \rangle$ along the hypercharge generator Q_Y that breaks $SU(5)$ down to $SU(3) \times SU(2) \times U(1)_Y$.

To proceed we then consider the flux

$$\langle F_1 \rangle \;=\; i\,(M_x\, dx \wedge d\bar{x} + M_y\, dy \wedge d\bar{y})\; Q_F \tag{2.135}$$

where

$$Q_F = \frac{1}{2}(H_1 - H_2 - H_3 - H_4 - H_5 - H_6) = -Q_x - Q_y. \tag{2.136}$$

To analyse the effect of this flux it is convenient to define the Q_F-charge of the roots E_ρ according to

$$[Q_F, E_\rho] = q_F(\rho) E_\rho \tag{2.137}$$

The roots in (2.134) are clearly neutral under this flux component $\langle F_1 \rangle$, and so the gauge symmetry $SU(5)$ is not broken further by its presence. The roots in the sectors a and b are however not neutral. Hence, if the integral of (2.135) over each of these curves does not vanish, they will each host a chiral sector of the theory. In the following we will assume that this is the case and that $\langle F_1 \rangle$ induces a net chiral spectrum of three $\bar{\mathbf{5}}$'s in the curve Σ_a and three $\mathbf{10}$'s in the curve Σ_b. If this chiral spectrum can be understood in terms of local zero modes in the sense of [52], then such chiral modes should arise in the sectors a^+ and b^+ of table 2.2, respectively, and using the local chiral index (2.113) one should choose $M_x < 0 < M_y$ to describe them locally (see also appendix B of [57]).

Notice that the roots belonging to the c sector are neutral under (2.136), and so the spectrum arising from the curve Σ_c is unaffected by the presence of $\langle F_1 \rangle$. As the $SO(12)$ triple intersection point is where down-like Yukawa couplings arise from, we do need one $\bar{\mathbf{5}}$ in such curve, but however no $\mathbf{5}$ so that no undesired $\bar{\mathbf{5}}\mathbf{5}$ mass terms appear. This chiral spectrum on the sector c can be achieved by adding the following extra piece of worldvolume flux

$$\langle F_2 \rangle \;=\; i\,(dx \wedge d\bar{y} + dy \wedge d\bar{x})\,(N_a Q_x + N_b Q_y) \tag{2.138}$$

It is easy to check that the particles localised at the matter curve Σ_c are now non-trivially charged under the flux background, and that a local chiral spectrum can be achieved if we choose $N_a \neq N_b$. In particular, for $N_a > N_b$ one obtains net local chirality in the sector c^+, yielding the desired $\bar{\mathbf{5}}$ which is the $SU(5)$ down Higgs. Notice that those particles at the curves Σ_a and Σ_b are also charged under (2.138). However, by construction the number of (local) families in such curves is independent of the flux $\langle F_2 \rangle$, as can be seen from (2.113).

Let us finally add a third piece of worldvolume flux which, unlike (2.135) and (2.138), will break the $SU(5)$ gauge group down to the MSSM. As usual, such flux should be turned along the hypercharge generator Q_Y, and a rather general choice is given by

$$\langle F_Y \rangle \;=\; i\left[(dx\wedge d\bar{y} + dy\wedge d\bar{x})\, N_Y + (dy\wedge d\bar{y} - dx\wedge d\bar{x})\, \tilde{N}_Y\right] Q_Y \tag{2.139}$$

where

$$Q_Y \;=\; \frac{1}{3}\,(H_2 + H_3 + H_4) - \frac{1}{2}\,(H_5 + H_6) \tag{2.140}$$

We have chosen the hypercharge flux to be a primitive $(1,1)$-form, so that it satisfies automatically the equations of motion for the background. Note that (2.139) has two components that are easily comparable with the previous flux components (2.135) and (2.138). The first component, proportional to the flux density N_Y, is quite similar to $\langle F_2\rangle$. Indeed, as it happens for (2.138) its pullback vanishes over the matter curves Σ_a and Σ_b, and so it does not contribute to the (local) index that computes the number of chiral families in the sectors a and b. The second component, proportional to $\tilde{N}_Y$, may in principle affect the chiral index over the curves Σ_a and Σ_b but, following the common practice in the GUT F-theory literature, we will assume that this is not the case. Globally one requires that

$$\int_{\Sigma_a} \langle F_Y \rangle \;=\; \int_{\Sigma_b} \langle F_Y \rangle \;=\; 0 \tag{2.141}$$

so that three complete families of quarks and leptons remain at the curves Σ_a and Σ_b after introducing the hypercharge flux. Locally, we demand that the local zero modes still arise from the sectors a^+ and b^+, and this amounts to requiring flux densities such that

$$M_x + q_Y \tilde{N}_Y < 0 < M_y + q_Y \tilde{N}_Y \tag{2.142}$$

for every possible hypercharge value q_Y in the sectors a^+ and b^+, see table 2.3 below.

While innocuous for the matter spectrum at the curves Σ_a, Σ_b, the hypercharge flux is supposed to modify the chiral spectrum of curve Σ_c, in order to avoid the doublet-triplet splitting problem of $SU(5)$ GUT models [23]. Indeed, one typically assumes that $\int_c \langle F_Y\rangle \neq 0$, and since (2.139) couples differently to particles with different hypercharge, this implies a different chiral index for the doublet and for the triplet of $\bar{\mathbf{5}}$. Locally, we have that the total flux seen near the Yukawa point by the doublets on the sector c^+ is

$$F_{\text{tot},\mathbf{2}} \;=\; N_Y + 2(N_a - N_b) \tag{2.143}$$

while the flux seen by the triplets is

$$F_{\text{tot},\mathbf{3}} \;=\; -\frac{2}{3}N_Y + 2(N_a - N_b) \tag{2.144}$$

Hence, in order to have a vector-like sector of triplets in the local model we can set

$$N_Y \;=\; 3(N_a - N_b) \tag{2.145}$$

and then assume that such vector-like spectrum is massive. Notice that this condition still yields a chiral sector for the doublets and so forbids a μ-term for them. Indeed, imposing (2.145) we have that

$$F_{\text{tot},\mathbf{2}} \;=\; \frac{5}{3}N_Y \tag{2.146}$$

which in general will induce a net chiral spectrum of doublets in the curve Σ_c. Hence, imposing (2.145) the combined effect of $\langle F_2\rangle$ and $\langle F_Y\rangle$ is such that doublets of $\bar{\mathbf{5}}$ in the sector c feel a net flux, while triplets do not. One may then choose the flux density N_Y such that it yields a single pair of MSSM down Higgses at the curve Σ_c.

To summarise, the total worldvolume flux on this local $SO(12)$ model is given by

$$\begin{aligned}\langle F\rangle &= i(dy\wedge d\bar{y} - dx\wedge d\bar{x})Q_P + i(dx\wedge d\bar{y} + dy\wedge d\bar{x})Q_S \\ &\quad + i(dy\wedge d\bar{y} + dx\wedge d\bar{x})M_{xy}Q_F\end{aligned} \tag{2.147}$$

where we have defined

$$Q_P = MQ_F + \tilde{N}_Y Q_Y \tag{2.148}$$

$$Q_S = N_a Q_x + N_b Q_y + N_Y Q_Y \tag{2.149}$$

and

$$M \equiv \frac{1}{2}(M_y - M_x) \qquad M_{xy} \equiv \frac{1}{2}(M_y + M_x). \tag{2.150}$$

Note that the combination of flux densities M_{xy} corresponds to an FI-term, which will be set to vanish whenever supersymmetry is imposed.

The combined effect of the background $\langle\Phi\rangle$ and $\langle A\rangle$ breaks $SO(12)$ to $SU(3)\times SU(2)\times U(1)^3$, and as a result the sectors a, b and c split into further subsectors compared to table 2.2. The content of charged particles under the surviving gauge group is shown in table 2.3, where we have also displayed the charges of each sector under the operators Q_F, Q_x, Q_y and Q_Y. We have also included the values of q_S and q_P, which are defined as

$$[Q_S, E_\rho] = q_S(\rho)E_\rho \qquad [Q_P, E_\rho] = q_P(\rho)E_\rho \tag{2.151}$$

and which, unlike the other charges, depend on the flux densities of the model. As discussed in section 2.3.2, each of these sectors obeys a different zero mode equation, and so it is described by a different wavefunction.

Sector	Root	$SU(3)$	$SU(2)$	q_Y	q_F	q_x	q_y	q_S	q_P
a_1^+	$(1,\underline{-1,0,0},0,0)$	$\bar{\mathbf{3}}$	$\mathbf{1}$	$-\frac{1}{3}$	1	-1	0	$-N_a-\frac{1}{3}N_Y$	$M-\frac{1}{3}\tilde{N}_Y$
a_1^-	$(-1,\underline{1,0,0},0,0)$	$\mathbf{3}$	$\mathbf{1}$	$\frac{1}{3}$	-1	1	0	$N_a+\frac{1}{3}N_Y$	$-M+\frac{1}{3}\tilde{N}_Y$
a_2^+	$(1,0,0,0,\underline{-1,0})$	$\mathbf{1}$	$\mathbf{2}$	$\frac{1}{2}$	1	-1	0	$-N_a+\frac{1}{2}N_Y$	$M+\frac{1}{2}\tilde{N}_Y$
a_2^-	$(-1,0,0,0,\underline{1,0})$	$\mathbf{1}$	$\mathbf{2}$	$-\frac{1}{2}$	-1	1	0	$N_a-\frac{1}{2}N_Y$	$-M-\frac{1}{3}\tilde{N}_Y$
b_1^+	$(0,\underline{1,1,0},0,0)$	$\bar{\mathbf{3}}$	$\mathbf{1}$	$\frac{2}{3}$	-1	0	1	$N_b+\frac{2}{3}N_Y$	$-M+\frac{2}{3}\tilde{N}_Y$
b_1^-	$(0,\underline{-1,-1,0},0,0)$	$\mathbf{3}$	$\mathbf{1}$	$-\frac{2}{3}$	1	0	-1	$-N_b-\frac{2}{3}N_Y$	$M-\frac{2}{3}\tilde{N}_Y$
b_2^+	$(0,\underline{1,0,0},\underline{1,0})$	$\mathbf{3}$	$\mathbf{2}$	$-\frac{1}{6}$	-1	0	1	$N_b-\frac{1}{6}N_Y$	$-M-\frac{1}{6}\tilde{N}_Y$
b_2^-	$(0,\underline{-1,0,0},\underline{-1,0})$	$\bar{\mathbf{3}}$	$\mathbf{2}$	$\frac{1}{6}$	1	0	-1	$-N_b-\frac{1}{6}N_Y$	$M+\frac{1}{6}\tilde{N}_Y$
b_3^+	$(0,0,0,0,1,1)$	$\mathbf{1}$	$\mathbf{1}$	-1	-1	0	1	N_b-N_Y	$-M-\tilde{N}_Y$
b_3^-	$(0,0,0,0,-1,-1)$	$\mathbf{1}$	$\mathbf{1}$	1	1	0	-1	$-N_b+N_Y$	$M+\tilde{N}_Y$
c_1^+	$(-1,\underline{-1,0,0},0,0)$	$\bar{\mathbf{3}}$	$\mathbf{1}$	$-\frac{1}{3}$	0	1	-1	$N_a-N_b-\frac{1}{3}N_Y$	$-\frac{1}{3}\tilde{N}_Y$
c_1^-	$(1,\underline{1,0,0},0,0)$	$\mathbf{3}$	$\mathbf{1}$	$\frac{1}{3}$	0	-1	1	$-N_a+N_b+\frac{1}{3}N_Y$	$\frac{1}{3}\tilde{N}_Y$
c_2^+	$(-1,0,0,0,\underline{-1,0})$	$\mathbf{1}$	$\mathbf{2}$	$\frac{1}{2}$	0	1	-1	$N_a-N_b+\frac{1}{2}N_Y$	$\frac{1}{2}\tilde{N}_Y$
c_2^-	$(1,0,0,0,\underline{1,0})$	$\mathbf{1}$	$\mathbf{2}$	$-\frac{1}{2}$	0	-1	1	$-N_a+N_b-\frac{1}{2}N_Y$	$-\frac{1}{2}\tilde{N}_Y$
X^+,Y^+	$(0,\underline{1,0,0},\underline{-1,0})$	$\mathbf{3}$	$\mathbf{2}$	$\frac{5}{6}$	0	0	0	$\frac{5}{6}N_Y$	$\frac{5}{6}\tilde{N}_Y$
X^-,Y^-	$(0,\underline{-1,0,0},\underline{1,0})$	$\bar{\mathbf{3}}$	$\mathbf{2}$	$-\frac{5}{6}$	0	0	0	$-\frac{5}{6}N_Y$	$-\frac{5}{6}\tilde{N}_Y$

Table 2.3: Different sectors and charges for the $SO(12)$ model.

Perturbative zero modes

Given the above background, and ignoring for the time being non-perturbative effects, one may solve for the zero mode wavefunctions on each of the sectors of table 2.3. One obtains in this way the internal profile for each of the 4d $\mathcal{N}=1$ chiral multiplets that arise from the matter curves Σ_a, Σ_b and Σ_c, and in particular for the 4d chiral fermions of the MSSM.

Following our discussion on physical zero modes in section 2.3.4 one may find all the wavefunctions which boil down to solving essentially the same equation as the one in the $SU(3)$ toy model (2.100). One finds that these are given by

$$\vec{\Psi}_\rho = \begin{pmatrix} -\dfrac{i\lambda_{\bar{x}}}{m^2} \\ \dfrac{i\lambda_{\bar{y}}}{m^2} \\ 1 \end{pmatrix} \chi_\rho^i E_\rho, \qquad \chi_\rho^i = e^{-q_\Phi(\lambda_{\bar{x}}\bar{x}-\lambda_{\bar{y}}\bar{y})} e^{i\Omega_\rho} f_i(\lambda_{\bar{x}}y+\lambda_{\bar{y}}x) \tag{2.152}$$

where we have defined

$$\Omega_\rho = \frac{i}{2}\left[(|y|^2-|x|^2)q_P + (x\bar{y}+\bar{x}y)q_S\right] \tag{2.153}$$

assuming the BPS condition $M_{xy}=0$. Also f_i are holomorphic functions of the variable $\lambda_{\bar{x}}y+\lambda_{\bar{y}}x$, with $\lambda_{\bar{x}}$, $\lambda_{\bar{y}}$ constants that depend on the flux densities q_P, q_S and the mass scale m^2, and which are different for each sector ρ. The index i runs over the different holomorphic functions that are present on each sector, or in other words over the families of zero modes localised in the same curve. Finally, recall that q_Φ is a holomorphic function of the 4-cycle coordinates x, y. We have summarised the values of these quantities for each sector of our model in table 2.4. For the sectors a_p^+, b_q^+ and c_r^+, λ_+ is defined as the lowest

ρ	q_Φ	$\lambda_{\bar{x}}$	$\lambda_{\bar{y}}$	$SU(5)$
a_p^+	$-x$	λ_+	$-q_S\frac{\lambda_+}{\lambda_+-q_P}$	$\bar{\mathbf{5}}$
a_p^-	x	λ_-	$q_S\frac{\lambda_-}{\lambda_-+q_P}$	$\mathbf{5}$
b_q^+	y	$-q_S\frac{\lambda_+}{\lambda_++q_P}$	λ_+	$\mathbf{10}$
b_q^-	$-y$	$q_S\frac{\lambda_-}{\lambda_--q_P}$	λ_-	$\overline{\mathbf{10}}$
c_r^+	$x-y$	$\frac{q_S\lambda_+-m^4}{\lambda_++q_P-q_S}$	$-\lambda_+ - \frac{q_S\lambda_+-m^4}{\lambda_++q_P-q_S}$	$\bar{\mathbf{5}}$
c_r^-	$-(x-y)$	$-\frac{q_S\lambda_-+m^4}{\lambda_--q_P+q_S}$	$-\lambda_- + \frac{q_S\lambda_-+m^4}{\lambda_--q_P+q_S}$	$\mathbf{5}$

Table 2.4: Wavefunction parameters.

(negative) eigenvalue of the flux matrix

$$\mathbf{m}_\rho = \begin{pmatrix} -q_P & q_S & im^2q_x \\ q_S & q_P & im^2q_y \\ -im^2q_x & -im^2q_y & 0 \end{pmatrix} \tag{2.154}$$

and one can check that the three lower entries of the vector in (2.152) are the corresponding eigenvector of this matrix. The same definition applies to λ_- for the sectors a_p^-, b_q^- and c_r^-.[47] In general, $\pm\lambda_\pm$ satisfy a complicated cubic equation which depends on the flux densities q_S and q_P. Since these two quantities contain the hypercharge flux, $\lambda_\pm$ will be different for each of the subsectors $a_p^\pm$, $b_q^\pm$ and $c_r^\pm$. Indeed, it is precisely in the value of the flux densities q_P and q_S that the wavefunctions within the same $SU(5)$ multiplet but with different hypercharge differ. See appendix A for a list of all the relevant wavefunctions.

Sector	Chiral mult.	$SU(3)\times SU(2)$	q_Y	q_x	q_y
a_1^+	D_R	$3(\bar{\mathbf{3}},\mathbf{1})$	$-\frac{1}{3}$	-1	0
a_2^+	L	$3(\mathbf{1},\mathbf{2})$	$\frac{1}{2}$	-1	0
b_1^+	U_R	$3(\bar{\mathbf{3}},\mathbf{1})$	$\frac{2}{3}$	0	1
b_2^+	Q_L	$3(\mathbf{3},\mathbf{2})$	$-\frac{1}{6}$	0	1
b_3^+	E_R	$3(\mathbf{1},\mathbf{1})$	-1	0	1
c_1^+	D_d	$(\bar{\mathbf{3}},\mathbf{1})$	$-\frac{1}{3}$	1	-1
c_2^+	H_d	$(\mathbf{1},\mathbf{2})$	$\frac{1}{2}$	1	-1

Table 2.5: Dictionary $SO(12)$-MSSM

[47] Although they have a similar definition, $\lambda_\pm$ have different values. Indeed, since $\mathbf{m}_{\alpha^+} = -\mathbf{m}_{\alpha^-}$ we have that $-\lambda_-(\alpha^-)$ is the highest positive eigenvalue of $\mathbf{m}_{\alpha^+}$.

Finally, one may solve the wavefunctions for the bulk sector $(X,Y)^{\pm}$, which is only sensitive to the presence of the hypercharge flux. Although the global properties of $\langle F_Y \rangle$ can be chosen so that no chiral matter arises from this sector [23], there will always be massive modes which we can be identified with the $X^{\pm}$, $Y^{\pm}$ bosons of 4d $SU(5)$ GUTs. As shown in appendix A such massive modes will have a Gaussian profile, a fact that can be used to suppress operators mediating proton decay [36].

Yukawa couplings

Let us compute the Yukawa couplings between the chiral zero modes of this model, before any non-perturbative effect is taken into account. As explained in section 2.3.1 such couplings can be written in terms of the wavefunctions above as

$$Y^{ijk}_{\rho\sigma\tau} = m_*^4 f_{\rho\sigma\tau} \int_S \det\,(\vec{\Psi}^i_\rho, \vec{\Psi}^j_\sigma, \vec{\Psi}^k_\tau)\,\mathrm{dvol}_S \tag{2.155}$$

where $f_{\rho\sigma\tau} = -i\mathrm{Tr}\,([E_\rho, E_\sigma]E_\tau)$, $\mathrm{dvol}_S = \omega^2/2$ and the vectors $\vec{\psi}^i_\rho$ are given by the three lower entries of Ψ. From the last subsection and the results of appendix A we have that these vectors read

$$\vec{\Psi}^i_{a_p^+} = \begin{pmatrix} -\dfrac{i\lambda_{a_p}}{m^2} \\ \zeta_{a_p}\dfrac{i\lambda_{a_p}}{m^2} \\ 1 \end{pmatrix} \chi^i_{a_p} \qquad \vec{\Psi}^j_{b_q^+} = \begin{pmatrix} -\zeta_{b_q}\dfrac{i\lambda_{b_q}}{m^2} \\ \dfrac{i\lambda_{b_q}}{m^2} \\ 1 \end{pmatrix} \chi^j_{b_q} \qquad \vec{\Psi}_{c_r^+} = \begin{pmatrix} \dfrac{i\zeta_{c_r}}{m^2} \\ \dfrac{i(\zeta_{c_r}-\lambda_{c_r})}{m^2} \\ 1 \end{pmatrix} \chi_{c_r} \tag{2.156}$$

where the λ's and ζ's are real constants defined in appendix A. The scalar wavefunctions χ are given by

$$\begin{aligned} \chi^i_{a_p} &= e^{\lambda_{a_p} x(\bar{x}-\zeta_{a_p}\bar{y})} e^{i\Omega_{a_p}} f_i(y+\zeta_{a_p}x) \qquad & \chi^j_{b_q} &= e^{\lambda_{b_q} y(\bar{y}-\zeta_{b_q}\bar{x})} e^{i\Omega_{b_q}} g_j(x+\zeta_{b_q}y) \\ \chi_{c_r} &= \gamma_{c_r} e^{(x-y)(\zeta_{c_r}\bar{x}-(\lambda_{c_r}-\zeta_{c_r})\bar{y})} e^{i\Omega_{c_r}} \end{aligned} \tag{2.157}$$

and the holomorphic factor can be chosen as

$$f_i = \gamma^i_{a_p} m_*^{3-i}(y+\zeta_{a_p}x)^{3-i} \qquad g_j = \gamma^j_{b_q} m_*^{3-j}(x+\zeta_{b_q}y)^{3-j} \qquad i,j=1,2,3 \tag{2.158}$$

with the normalisation factors $\gamma^i_{a_p}$, $\gamma^j_{b_q}$ and γ_{c_r} to be fixed later.

We then see that the structure of wavefunctions and Yukawas is quite similar to the one in the $U(3)$ toy model analysed in [49]. One difference is the more involved sector structure, which as illustrated in table 2.5 is necessary to accommodate the MSSM chiral spectrum. Notice also that, due to the extra components of $\langle F \rangle$ that we have introduced, the holomorphic factors in the wavefunctions not only depend on the complex coordinate along the matter curve, but also on the transverse one. This however does not affect the general result of [53], in the sense that the Yukawa matrices are of rank one. Indeed, substituting in (2.155) shows that the integrand is the product of $f_i g_j$ times an exponential whose argument is invariant under a diagonal $U(1)$ rotation of x and y. Since dvol_S is also invariant under such rotation, the integral can be non-vanishing only when f_i and g_j are constants, which happens for $i=j=3$. Computing the integral yields the only non-zero coupling

$$Y^{33}_{a_p^+ b_q^+ c_r^+} = \pi^2 \left(\frac{m_*}{m}\right)^4 \gamma^3_{a_p}\gamma^3_{b_q}\gamma_{c_r} f_{a_p^+ b_q^+ c_r^+} \tag{2.159}$$

where $f_{a_p^+ b_q^+ c_r^+}$ are the structure constants of $SO(12)$ in the fundamental representation. Except for the normalisation factors this coupling does not depend on the fluxes.

2.4.2 Non-perturbative zero modes at the SO(12) point

Let us now apply the non-perturbative scheme of section 2.3.6 to the Yukawa point of $SO(12)$ enhancement. More precisely, we would like to consider the superpotential (2.120) for the local $SO(12)$ model described in the last section and see how this new superpotential affects the wavefunction profile for the matter fields and the Yukawa couplings. The aim of this section is to solve for the zero mode wavefunctions in the presence of the non-perturbative piece of the superpotential, leaving the computation of Yukawa couplings for the next section.

As shown in section 2.3.6, the new zero mode wavefunctions can be written as a perturbative expansion in the small parameter ϵ that measures the strength of the non-perturbative effect. More precisely we have that

$$\overrightarrow{\Psi}_\rho = \overrightarrow{\Psi}_\rho^{(0)} + \epsilon \overrightarrow{\Psi}_{\rho,\theta_0}^{(1)} + \mathcal{O}(\epsilon^2) \tag{2.160}$$

where $\overrightarrow{\Psi}_\rho^{(0)}$ are the tree-level wavefunctions (2.156)-(2.158) for the sector $\rho = a_p^+, b_q^+, c_r^+$ and $\overrightarrow{\Psi}_{\rho,\theta_0}^{(1)}$ is the $\mathcal{O}(\epsilon)$ correction to this sector when all θ_{2n} vanish except θ_0. We refer the reader to [57] for a detailed discussion on the corrections due to θ_2 which we will not consider here.

In the remaining of this section we will solve for the first order corrections $\overrightarrow{\Psi}_{\rho,\theta_0}^{(1)}$ to the wavefunctions (2.156)-(2.158).

Zero modes for $\theta_0 \neq 0$

Let us then consider the wavefunction corrections for the case where $\theta_0 \neq 0$ but $\theta_n = 0$ for all $n > 0$. Notice that in [49] θ_0 was also present but assumed to be constant, and shown that Yukawa couplings are independent of it. As discussed earlier we may now relax this condition and take θ_0 to be non-constant and holomorphic on x, y. For simplicity let us take it to be

$$\theta_0 = im^2 \left(\theta_{00} + x\, \theta_{0x} + y\, \theta_{0y}\right) \tag{2.161}$$

where θ_{00}, θ_{0x}, θ_{0y} are complex constants, and the factor im^2 has been added for later convenience. We will now see that the corrected wavefunctions do depend on θ_{0x} and θ_{0y}, and in the next section that they also enter into the corrected Yukawa couplings.

The corrected equations of motion for the fluctuations were derived in section 2.3.6, and they can be conveniently rewritten by using the notation

$$\overrightarrow{\Psi}_{(\rho)} = \overrightarrow{\Psi}_\rho E_\rho = \begin{pmatrix} a_{\rho\bar{x}} \\ a_{\rho\bar{y}} \\ \varphi_{\rho xy} \end{pmatrix} E_\rho. \tag{2.162}$$

Recall that because of supersymmetry the bosonic fluctuations $(a_{\rho\bar{m}}, \varphi_{\rho xy})$ pair up with fermionic fluctuations $(\psi_{\rho\bar{m}}, \chi_\rho)$ analysed in section 2.4.1, and so in the absence of non-perturbative effects the components of $\overrightarrow{\psi}_\rho$ match those of the vectors in eq. (2.156). Using the above notation we can express the corrected F-term equations as

$$\begin{aligned} \partial_{\bar{x}} a_{\rho\bar{y}} - \partial_{\bar{y}} a_{\rho\bar{x}} &= 0 \\ \partial_{\bar{m}} \varphi_{\rho xy} + im^2 q_\Phi(\rho) a_{\rho\bar{m}} &= i\epsilon m^2 \left[\theta_{0y}\partial_x a_{\rho\bar{m}} - \theta_{0x}\partial_y a_{\rho\bar{m}}\right] + \mathcal{O}(\epsilon^2) \end{aligned} \tag{2.163}$$

which, in the holomorphic gauge that we are considering, do not depend on the world-volume flux densities M_{xy}, q_P and q_S. By the results of [53, 55] this implies that the holomorphic Yukawa couplings cannot depend on these quantities either.

On the other hand, the D-term translates into[48]

$$\begin{aligned} &\{\partial_x + \bar{x}[q_P(\rho) - M_{xy}q_F(\rho)] - \bar{y}q_S(\rho)\}\, a_{\rho\bar{x}} \\ &\qquad + \quad \{\partial_y - \bar{y}[q_P(\rho) + M_{xy}q_F(\rho)] - \bar{x}q_S(\rho)\}\, a_{\rho\bar{y}} - im^2\bar{q}_\Phi(\rho)\varphi_{\rho xy} \\ = &\qquad i\epsilon m^2\bar{\theta}_{0x}\,\{y[M_{xy}q_F(\rho) + q_P(\rho)] + xq_S(\rho)\}\,\varphi_{\rho xy} \\ &\qquad - \quad i\epsilon m^2\bar{\theta}_{0y}\,\{x[M_{xy}q_F(\rho) - q_P(\rho)] + yq_S(\rho)\}\,\varphi_{\rho xy} \end{aligned} \tag{2.164}$$

where the specific values of the charges q_Φ, q_F, q_P, and q_S, for each sector ρ are given in tables 2.2 and 2.3. Notice that as usual the D-term equation depends on the flux densities, and in particular on the hypercharge fluxes contained in q_P and q_S. Hence, just like in [49], the holomorphic Yukawa couplings will not depend on the hypercharge but the physical Yukawas will, as we show in the next section.

As already mentioned, the zero modes to zeroth order in ϵ are given by eq.(2.156). To find the corrected zero modes the strategy is to start with an ansatz motivated by the zeroth order solutions and then proceed perturbatively in ϵ. Notice that the zeroth order solutions consist of a fixed vector $\vec{v}_\rho$ multiplying a scalar wavefunction $\chi_\rho = \varphi_{\rho xy}$, that has a simple dependence on the complex variables $\lambda_{\bar{x}}y + \lambda_{\bar{y}}x$ and $\lambda_{\bar{x}}x - \lambda_{\bar{y}}y$. We find that the first order solutions are also of this form, but now with a corrected scalar wavefunction, namely

$$\varphi_{\rho xy} = \varphi^{(0)}_{\rho xy} + \epsilon\varphi^{(1)}_{\rho xy} + \mathcal{O}(\epsilon^2) \tag{2.165}$$

where $\varphi^{(0)}_{\rho xy}$ are given by the scalar wavefunctions χ_ρ in eq. (2.157). In the following we report the results for the correction $\varphi^{(1)}_{\rho xy}$ in the different sectors, dropping the subscripts xy to avoid cluttering the equations. While we will keep the worldvolume flux dependence in the zero mode equations, for simplicity we will set $M_{xy} = 0$. Our solutions are however easily generalised for non-vanishing M_{xy}.

- *Sector* a^+

Let us first define the complex variables

$$u_a = x - \zeta_a y \qquad\qquad v_a = y + \zeta_a x \tag{2.166}$$

that come from a rescaling of $\lambda_{\bar{x}}y+\lambda_{\bar{y}}x$ and $\lambda_{\bar{x}}x-\lambda_{\bar{y}}y$ for this sector. To simplify notation we have suppressed the subindex $p = 1, 2$, for each sector $\rho = a_p^+$, that labels elements of the $\bar{\mathbf{5}}$ representation with different hypercharge, and therefore with different values of λ_a and ζ_a. Notice that the variables u_a and v_a are different for each of these subsectors.

[48] This D-term equation is written in holomorphic gauge which, in principle, does not make sense since the D-term in not invariant under complexified gauge transformations. However, for abelian backgrounds one can safely go to holomorphic gauge even at the level of D-terms and switch to real gauge at the end of the computation by multiplying by $e^{i\Omega}$.

The correction to the wavefunctions $\varphi^i_{a^+}$ can be conveniently written in terms of (u_a, v_a). Concretely,

$$\varphi^{i(1)}_{a^+} = m_* e^{-q_\Phi \lambda_a \bar{u}_a} \left(A_i(v_a) + \Upsilon^i_{a^+}\right) \tag{2.167}$$

where $q_\Phi(a^+) = -x$ for these sectors, A_i is a holomorphic function of v_a and

$$\Upsilon^i_{a^+} = \frac{1}{2}\theta_{0y}\lambda_a^2\bar{u}_a^2 f_i(v_a) + \lambda_a\bar{u}_a(\zeta_a\theta_{0y} - \theta_{0x})f_i'(v_a) + \left[\frac{\alpha_1}{2}x^2 + \alpha_2 x v_a\right] f_i(v_a) \tag{2.168}$$

The last two terms are in fact only necessary to fulfill the D-term equation (2.164). In particular, we obtain that the coefficients α_1 and α_2 must take the following values

$$\begin{aligned} \alpha_1 &= -\frac{m^4}{\lambda_a}\left\{\bar{\theta}_{0x}\left(q^a_S - \zeta_a q^a_P\right) + \bar{\theta}_{0y}\left(q^a_P + \zeta_a q^a_S\right)\right\} \\ \alpha_2 &= -\frac{m^4}{\lambda_a}\left\{\bar{\theta}_{0x} q^a_P - \bar{\theta}_{0y} q^a_S\right\} \end{aligned} \tag{2.169}$$

To sum up, taking into account the zeroth order solution in eq.(2.157) the final result for $\varphi^i_{a^+}$ can be cast as

$$\varphi^i_{a^+} = m_* e^{-q_\Phi \lambda_a \bar{u}_a} \left(f_i(v_a) + \epsilon A_i(v_a) + \epsilon \Upsilon^i_{a^+}\right) + \mathcal{O}(\epsilon^2) \tag{2.170}$$

Naively the functions A_i remain unfixed by the above equations of motion. This is because we could think of them as a $\mathcal{O}(\epsilon)$ correction to the holomorphic functions f_i, which are also not fixed. However, given the choice (2.158) of f_i, the A_i are fixed as follows. Notice that the F-term equation (2.163) implies $a_{\rho\bar{m}} = \partial_{\bar{m}}\xi_\rho$, where ξ_ρ is a regular function [53]. As shown in appendix D of [57], this function can be found by integrating (2.163). Imposing that ξ_ρ is regular at the loci $q_\Phi(\rho) = 0$ where the zero modes are localised implies nontrivial constraints for the wavefunctions. In particular, for the a^+ sector requiring $\xi^i_{a^+}$ to be free of poles at $x = 0$ fixes the A_i, which read

$$A_2 = A_3 = 0 \quad ; \quad A_1 = \gamma^1_a a_0 \quad ; \quad a_0 = m_*^2\zeta_a(\zeta_a\theta_{0y} - 2\theta_{0x}) \tag{2.171}$$

Interestingly, this form of A_1 guarantees that the Yukawa couplings computed via overlap of zero modes will be flux independent up to normalisation factors, as we comment on the next section.

- *Sector b^+*

The sectors b^+ and a^+ are quite similar, so let us first define the variables

$$u_b = y - \zeta_b x \qquad\qquad v_b = x + \zeta_b y \tag{2.172}$$

that again are different for each subsector $\rho = b^+_q$, $q = 1, 2, 3$. As in (2.167) we obtain

$$\chi^{j(1)}_{b^+} = m_* e^{q_\Phi \lambda_b \bar{u}_b} \left(B_j(v_b) + \Upsilon^j_b\right) \tag{2.173}$$

where now $q_\Phi(b^+) = y$, B_j is a holomorphic function of v_b and

$$\Upsilon^i_b = \frac{1}{2}\theta_{0x}\lambda_b^2\bar{u}_b^2 g_j(v_b) + \lambda_b\bar{u}_b(\zeta_b\theta_{0x} - \theta_{0y})g_j'(v_b) + \left[\frac{\beta_1}{2}y^2 + \beta_2 y v_b\right] g_j(v_b) \tag{2.174}$$

Again, the terms proportional to β_1, β_2 are required to satisfy the D-term equation. These coefficients can be obtained from (2.169) by replacing $\bar{\theta}_{0x} \leftrightarrow \bar{\theta}_{0y}$, $q_P^a \to -q_P^b$, $q_S^a \to q_S^b$, and consistently $\lambda_a \to \lambda_b$, $\zeta_a \to \zeta_b$.

Including the zeroth order solution from eq.(2.157) yields the full result

$$\varphi^j_{b^+} = m_* e^{q_\Phi \lambda_b \bar{u}_b} \left(g_j(v_b) + \epsilon B_j(v_b) + \epsilon \Upsilon^j_b \right) + \mathcal{O}(\epsilon^2) \tag{2.175}$$

The functions B_i are determined demanding that $\xi^j_{b^+}$ be free of singularities. In this way we find

$$B_2 = B_3 = 0 \quad ; \quad B_1 = \gamma^1_b b_0 \quad ; \quad b_0 = m_*^2 \zeta_b(\zeta_b \theta_{0x} - 2\theta_{0y}) \tag{2.176}$$

- *Sector c^+*

In this case it is helpful to introduce the variables

$$u_c \;=\; x - \tau_c y \qquad\qquad v_c \;=\; y + \tau_c x \tag{2.177}$$

with $\tau_c = (\lambda_c - \zeta_c)/\zeta_c$. The quantities λ_c and ζ_c, defined in appendix A, actually depend on the subsector c_r^+, $r = 1, 2$. The correction to the wavefunction is found to be

$$\varphi^{(1)}_{c^+} = m_* \gamma_c e^{q_\Phi \zeta_c \bar{u}_c} \left(C(v_c) + \Upsilon^{(1)}_{c^+} \right) \tag{2.178}$$

where now $q_\Phi = (x - y)$, C is a function of v_c, and

$$\Upsilon^{(1)}_{c^+} = -\frac{1}{2}\zeta_c^2 \bar{u}_c^2(\theta_{0x} + \theta_{0y}) + \frac{\delta_1}{2}(x-y)^2 + \delta_2(x-y)v_c \tag{2.179}$$

The constants δ_1 and δ_2 are given by

$$\begin{aligned}
\delta_1 &= \frac{m^4}{\zeta_c(1+\tau_c)^2}\left\{\bar{\theta}_{0x}\,(q_S^c - \tau_c q_P^c) + \bar{\theta}_{0y}\,(q_P^c + \tau_c q_S^c)\right\} \\
\delta_2 &= \frac{m^4}{\zeta_c(1+\tau_c)^2}\left\{\bar{\theta}_{0x}\,(q_P^c + q_S^c) + \bar{\theta}_{0y}\,(q_P^c - q_S^c)\right\}
\end{aligned} \tag{2.180}$$

The holomorphic terms in $\Upsilon^{(1)}_{c^+}$, which depend on $\bar{\theta}_{0x}$ and $\bar{\theta}_{0y}$ through δ_1 and δ_2, are needed to satisfy the corrected D-term equation. In appendix D of [57] it is shown that $C = 0$.

2.4.3 Normalisation and mixings of corrected zero modes

Given the corrected zero mode wavefunction one must compute their normalisation factors, since it is through these factors that the physical Yukawa couplings depend on worldvolume fluxes. The computation of normalisation factors for perturbative zero modes is explained in section 2.3.4 and the results can be found in appendix A, where the following norms are computed explicitly

$$K^{ij}_\rho \;=\; \langle \overrightarrow{\Psi}_{\rho i} | \overrightarrow{\Psi}_{\rho j} \rangle \;=\; m_*^4 \int_S \mathrm{Tr}\,(\overrightarrow{\Psi}^\dagger_{\rho i} \cdot \overrightarrow{\Psi}_{\rho j})\, \mathrm{dvol}_S \;=\; 2\,||\overrightarrow{v}_\rho||^2 X^{ij}_\rho \tag{2.181}$$

with $\vec{v}_\rho$ the constant vector in $\vec{\Psi}$ in eq.(2.152) and the scalar wavefunction metrics

$$X_\rho^{ij} = m_*^4 \left(\chi_\rho^{i\,\mathrm{real}}, \chi_\rho^{j\,\mathrm{real}}\right) = m_*^4 \int_S \left(\chi_\rho^{i\,\mathrm{real}}\right)^* \chi_\rho^{j\,\mathrm{real}}\, d\mathrm{vol}_S \tag{2.182}$$

are calculated by extending the local patch to $\mathbb{C}^2$. Here the superscript 'real' stands for the zero mode wavefunction expressed in a real gauge rather than in the holomorphic gauge that we have used in the previous section. One may switch from wavefunctions in the holomorphic to the real gauge by multiplying them by an appropriate sector dependent prefactor. For instance, in the sector a^+ we have

$$\chi_{a^+}^{i\,\mathrm{real}} = e^{\frac{q_P^a}{2}(|x|^2-|y|^2)-q_S^a\,\mathrm{Re}(x\bar{y})}\chi_{a^+}^i \tag{2.183}$$

where $\chi_{a^+}^i$ is the zero mode computed in the holomorphic gauge. As we are now dealing with non-perturbative zero modes, we should take $\chi_{a^+}^i$ to be the corrected scalar wavefunction $\varphi_{a^+}^i$ given in (2.170).

For the perturbative zero modes, $X_\rho^{ij} = 0$ for $i \neq j$, since the integrand needs to be invariant under the $U(1)$ diagonal rotation $(x, y) \to e^{i\alpha}(x, y)$. However, this no longer needs to be true at order ϵ. Indeed, let us consider the sector a^+ and the corrected zero modes in this sector when θ_0 is present. Due to the correction $\Upsilon_{a^+}^i$ given in (2.168), one can in principle have non-diagonal metrics. In fact, to order ϵ one finds that only $X_{a^+}^{31}$ and its conjugate are different from zero.

Substituting the corrected zero modes and computing the Gaussian integrals we find

$$K_{a^+}^{ij} = \frac{2\pi^2 m_*^4}{\Delta_a q_P^a}||\vec{v}_a||^2 \gamma_a^{i*}\gamma_a^j\, \mathbf{x}_a^{ij} + \mathcal{O}(\epsilon^2) \tag{2.184}$$

where $\Delta_a = -(2\lambda_a + q_P^a(1+\zeta_a^2))$ and the matrix $\mathbf{x}_a$ is

$$\mathbf{x}_a = \begin{pmatrix} 2\frac{m_*^4}{(q_P^a)^2} & 0 & \epsilon\left[m_*^2(2r_a\bar{\theta}_{0x} + r_a^2\bar{\theta}_{0y})\right] \\ 0 & \frac{m_*^2}{q_P^a} & 0 \\ \epsilon\left[m_*^2(2r_a\theta_{0x} + r_a^2\theta_{0y})\right] & 0 & 1 \end{pmatrix}, \tag{2.185}$$

$$r_a = -\frac{q_S^a}{q_P^a}. \tag{2.186}$$

The quantity r_a is the quotient between the off-diagonal and diagonal worldvolume fluxes felt by this sector. Hence, when off-diagonal fluxes are turned on $K_{a^+}^{31}$ is non-zero. Finally, notice that the diagonal terms $K_{a^+}^{ii}$ do not get corrections to order ϵ.

As the metric for the zero-modes is non-trivial, the Yukawas computed from them do not yet correspond to the physical couplings. From the 4d effective theory viewpoint, to obtain physical Yukawas one performs a field redefinition such that the 4d chiral fields have canonical kinetic terms. The higher dimensional counterpart of this field redefinition is to take a linear combination of zero modes such that the matrix K_{a^+} becomes the identity. In the absence of non-perturbative effects $\epsilon = 0$ and such redefinitions are rather easy to perform, since K_{a^+} is diagonal and one only needs to choose the normalisation factors to have $K_{a^+} = I_3$. The same applies to order $\mathcal{O}(\epsilon)$ whenever $q_S^a = 0$.

In general we will have 4d fields Φ_ρ^m whose metric K_ρ has off-diagonal terms, and so a wavefunction rescaling is not enough to have canonically normalised fields. However, as the field metric K_ρ must be hermitian and definite positive, we can always write it as

$$K_\rho = P_\rho^\dagger P_\rho \tag{2.187}$$

and describe the fields with canonical kinetic terms and the physical Yukawa couplings as

$$\Phi^i_{\text{phys}} = (P_\rho)^i{}_m \Phi^m \qquad Y^{\text{phys}}_{ijk} = (P^{-1}_\rho)_i^m (P^{-1}_{\rho'})_j^n (P^{-1}_{\rho''})_k^p Y_{mnp} \tag{2.188}$$

where Y_{mnp} stand for the initial set of Yukawa couplings. Finally, the set of internal wavefunctions Ψ_m associated to these fields will transform under this change of basis as

$$\Psi_i^{\text{phys}} = (P^{-1\,t})_i^m \Psi_m. \tag{2.189}$$

In the case of K_{a^+} one can easily find a matrix P_{a^+} such that (2.187) is satisfied

$$P_{a^+} = \frac{\sqrt{2}\pi m_*^2 ||\vec{v}_a||}{\sqrt{\Delta_a q_P^a}} \begin{pmatrix} \sqrt{2}\frac{m_*^2}{q_P^a} & & \epsilon\bar{\mu}_a \\ & \frac{m_*}{(q_P^a)^{1/2}} & \\ & & 1 \end{pmatrix} \begin{pmatrix} \gamma_a^1 & & \\ & \gamma_a^2 & \\ & & \gamma_a^3 \end{pmatrix} + \mathcal{O}(\epsilon^2) \tag{2.190}$$

$$\mu_a = \frac{q_P^a}{\sqrt{2}} (2 r_a \theta_{0x} + r_a^2 \theta_{0y}) \tag{2.191}$$

Upon choosing the normalisation factors γ_a^i as in appendix A this matrix simplifies to

$$P_{a^+} \to \begin{pmatrix} 1 & & \epsilon\bar{\mu}_a \\ & 1 & \\ & & 1 \end{pmatrix} + \mathcal{O}(\epsilon^2) \tag{2.192}$$

Hence, in order to have canonically normalised fields we not only need to choose appropriate normalisation factors, but also perform a rotation among the families of this sector. One can express such rotation in terms of the holomorphic representatives f_i that describe each of our families as

$$\tilde{f}_1 = f_1 \qquad \tilde{f}_2 = f_2 \qquad \tilde{f}_3 = f_3 - \epsilon\bar{\mu}_a f_1, \tag{2.193}$$

$\tilde{f}_i$ being the holomorphic representatives that describe canonically normalised 4d fields.

The analysis of the metrics in the b^+ sector proceeds along the same lines. Again considering the case where only $\theta_0 \neq 0$, the non-zero off-diagonal entries of the mixing matrix amount to $X^{31}_{b^+}$ and its conjugate $X^{13}_{b^+}$. Similarly to the sector a^+ there are no $\mathcal{O}(\epsilon)$ corrections to the diagonal terms $X^{ii}_{b^+}$, and obtaining canonically normalised fields amounts to appropriately choosing the normalisation factors γ_b^j and performing a redefinition of chiral representatives. More precisely, we must choose γ_b^j as in appendix A and perform the redefinition

$$\tilde{g}_1 = g_1 \qquad \tilde{g}_2 = g_2 \qquad \tilde{g}_3 = g_3 - \epsilon\bar{\mu}_b g_1 \tag{2.194}$$

where now

$$\mu_b = \frac{q_P^b}{\sqrt{2}} (2 r_b \theta_{0y} + r_b^2 \theta_{0x}) \qquad r_b = \frac{q_S^b}{q_P^b}. \tag{2.195}$$

To summarise, in the presence of non-diagonal fluxes of the form (2.138) non-trivial corrections appear at first order in ϵ for the wavefunction metrics. The corrections appear in the off-diagonal metrics K_ρ^{ij}, $i \neq j$ that vanish at zeroth order in ϵ. One can perform a change of basis of the families of zero modes in order to set the off-diagonal entries of K_ρ to vanish, and choose the appropriate wavefunction factors γ_ρ^i to have canonically normalised 4d kinetic terms. Notice that such factors are the ones found in appendix A, namely the normalisation factors at the perturbative level, and that the only effect of $\mathcal{O}(\epsilon)$ corrections to the field metrics is encoded in the redefinitions (2.193) and (2.195). These redefinitions have however no effect for the physical Yukawa couplings, at least at the level of approximation that we are working.

Indeed, the above redefinitions are only nontrivial in the cases $f_3 \to \tilde{f}_3$ and $g_3 \to \tilde{g}_3$, and amount to saying that in this new basis the third families of both sectors a^+ and b^+ have a contamination of $\mathcal{O}(\epsilon)$ from the first family. In principle this contamination will modify the Yukawa couplings Y_{3j} and Y_{i3}. However, as the Yukawa couplings involving the first family are already of order ϵ, the modification will be $\mathcal{O}(\epsilon^2)$. The same result is obtained by directly applying (2.188) to the Yukawa couplings computed in the next section.

We then find that, although non-perturbative effects modify the metrics for the 4d matter fields, this modification can be neglected at the level of approximation that we are working, at least for the purpose of computing fermion masses. The whole effect of wavefunction normalisation is already captured by tree-level wavefunctions. This fact is quite relevant in the present scheme because holomorphic Yukawas do not depend on worldvolume fluxes. Hence, the only place where the hypercharge flux will enter into the expression for the physical Yukawa couplings will be via the normalisation factors of perturbative zero modes.

2.4.4 Yukawa couplings

As already explained in section 2.3.6, for $\theta_0 \neq 0$ the non-perturbative contribution to the Yukawa couplings comes only from the correction to the wavefunctions but the extra term in the superpotential does not give any further contribution.

The corrected Yukawas implied by θ_0 are easy to compute by substituting the zero modes of section 2.4.2 in (2.155) and evaluating the integrals or by applying the generalised residue formula. The only non-vanishing couplings turn out to be

$$Y^{22}_{a^+b^+c^+} = \frac{\epsilon\pi^2 m_*^6}{m^4} f_{a^+b^+c^+}\gamma_a^2\gamma_b^2\gamma_c(\theta_{0x}+\theta_{0y}) \tag{2.196a}$$

$$Y^{31}_{a^+b^+c^+} = \frac{\epsilon\pi^2 m_*^6}{m^4} f_{a^+b^+c^+}\gamma_a^3\gamma_b^1\gamma_c\theta_{0y} \tag{2.196b}$$

$$Y^{13}_{a^+b^+c^+} = \frac{\epsilon\pi^2 m_*^6}{m^4} f_{a^+b^+c^+}\gamma_a^1\gamma_b^3\gamma_c\theta_{0x}. \tag{2.196c}$$

To obtain these couplings we have used the functions A_i and B_j given in (2.171) and (2.176) respectively. In particular, the constants a_0 and b_0 are such that the couplings Y^{13} and Y^{31} do not depend on the fluxes, up to normalisation factors. For instance, before substituting the value of a_0 we find that

$$Y^{13}_{a^+b^+c^+} = \frac{\epsilon\pi^2 m_*^4}{m^4} f_{a^+b^+c^+}\gamma_a^1\gamma_b^3\gamma_c\left[(1+2\zeta_a)m_*^2\theta_{0x} - \zeta_a^2 m_*^2\theta_{0y} + a_0\right]. \tag{2.197}$$

We see that taking $a_0 = m_*^2(\zeta_a^2\theta_{0y} - 2\zeta_a\theta_{0x})$ indeed leads to (2.196c). In appendix D of [57] it is shown in detail how the couplings (2.196) can also be deduced from the residue formula.

To conclude, we have derived a set of Yukawa couplings that are flux independent up to the $\mathcal{O}(\epsilon^0)$ normalisation factors γ_ρ^i, similarly to the structure obtained in [49].

2.4.5 Quark-lepton mass hierarchies and hypercharge flux

Gathering the results of eqs.(2.159-2.196) one obtains that the physical Yukawa mass matrix has a structure of the form

$$Y_{D/L} \;=\; 2\pi^2\varrho^{-2}\gamma_c \begin{pmatrix} \mathcal{O}(\epsilon^2) & \mathcal{O}(\epsilon^2) & \epsilon\varrho^{-1}\gamma_a^1\gamma_b^3\tilde{\theta}_{0x} \\ \mathcal{O}(\epsilon^2) & \epsilon\varrho^{-1}\gamma_a^2\gamma_b^2(\tilde{\theta}_{0x}+\tilde{\theta}_{0y}) & \mathcal{O}(\epsilon^2) \\ \epsilon\varrho^{-1}\gamma_b^1\gamma_a^3\tilde{\theta}_{0y} & \mathcal{O}(\epsilon^2) & \gamma_a^3\gamma_b^3 \end{pmatrix} \tag{2.198}$$

where the coefficients

$$\tilde{\theta}_{0x} = m^2\theta_{0x} \qquad \tilde{\theta}_{0y} = m^2\theta_{0y} \tag{2.199}$$

are adimensional, and we have defined the quotient of scales

$$\varrho \;=\; \left(\frac{m}{m_*}\right)^2 \;=\; (2\pi)^{3/2}g_s^{1/2}\sigma. \tag{2.200}$$

The second expression for ϱ can be derived in the type IIB orientifold limit of the $SO(12)$ model. There, $\sigma = (m/m_{st})^2$ describes the intersection slope of the 7-branes in units of the string scale $m_{st}^{-2} = 2\pi\alpha'$. In this limit we also obtain the relation $m_{st}^4 = g_s(2\pi)^3 m_*^4$, and combining both results the second equality of (2.200) follows.

As already explained, the holomorphic Yukawa couplings are independent from fluxes, this dependence only appearing in the normalisation factors $\gamma_{a,b,c}^i$. In particular the dependence on hypercharge fluxes is the only possible source of distinction between D-quark and charged lepton Yukawas which are equal before this flux is turned on. In what follows we will be using the uncorrected normalisation factors appendix A, since these corrections would only induce terms of order ϵ^2 in the physical Yukawa couplings. The same happens with the mixing discussed in section 2.4.3, which only affects the physical Yukawa couplings at higher order.

Our expressions for Yukawa couplings apply at the unification-string scale, presumably of order 10^{16} GeV, so that in order to compare with experimental fermion masses one needs to run the data up to the unification scale. An updated two-loop analysis for this running within the MSSM has been performed in ref. [70] from which we will take the data below. Here we will only discuss fermion masses for charged leptons and D-quarks, which are the ones relevant for the $SO(12)$ case studied here. Table 2.6 shows the relevant fermion mass ratios evaluated at the unification scale for various values of $\tan\beta$ (the ratio of the two Higgs vevs in the MSSM). We also show for reference the Yukawa couplings of the τ lepton and b and t quarks. Recall that $m_{\tau,b} = Y_{\tau,b}V\cos\beta$, $m_t = Y_tV\sin\beta$, with $V = \sqrt{V_u^2+V_d^2} \simeq 174$ GeV. As emphasised e.g. in ref. [70], the Yukawa couplings obtained at the GUT scale seem to depart from the predictions from the minimal $SU(5)$ GUT, which yield $Y_{\tau,\mu,e} = Y_{b,s,d}$. This happens not only, as is well known, for the first two generations but also for the third for which one has a sizeable departure from unification,

$\tan\beta$	10	38	50
m_d/m_s	$5.1 \pm 0.7 \times 10^{-2}$	$5.1 \pm 0.7 \times 10^{-2}$	$5.1 \pm 0.7 \times 10^{-2}$
m_s/m_b	$1.9 \pm 0.2 \times 10^{-2}$	$1.7 \pm 0.2 \times 10^{-2}$	$1.6 \pm 0.2 \times 10^{-2}$
m_e/m_μ	$4.8 \pm 0.2 \times 10^{-3}$	$4.8 \pm 0.2 \times 10^{-3}$	$4.8 \pm 0.2 \times 10^{-3}$
m_μ/m_τ	$5.9 \pm 0.2 \times 10^{-2}$	$5.4 \pm 0.2 \times 10^{-2}$	$5.0 \pm 0.2 \times 10^{-2}$
m_b/m_τ	0.73 ± 0.03	0.73 ± 0.03	0.73 ± 0.04
Y_τ	0.070 ± 0.003	0.32 ± 0.02	0.51 ± 0.04
Y_b	0.051 ± 0.002	0.23 ± 0.01	0.37 ± 0.02
Y_t	0.48 ± 0.02	0.49 ± 0.02	0.51 ± 0.04

Table 2.6: Running mass ratios of leptons and D-quarks at the unification scale from ref. [70]. The Yukawa couplings $Y_{\tau,b,t}$ at the unification scale are also shown.

see table 2.6. On the other hand, for large $\tan\beta$, which is going to be our case as we will see momentarily, there are additional large threshold corrections to Y_b from the low-energy SUSY thresholds. In particular the leading such corrections in the MSSM are given by

$$\delta Y_b = -\frac{g_3^2}{12\pi^2}\frac{\mu M_3}{m_{\tilde{b}}}\tan\beta - \frac{Y_t^2}{32\pi^2}\frac{\mu A_t}{m_{\tilde{t}}}\tan\beta \tag{2.201}$$

where M_3, $m_{\tilde{b}}$, $m_{\tilde{t}}$ are the gluino, sbottom and stop masses and μ, A_t are the higgsino mass and top trilinear parameter. These corrections may easily be of order 20% (see e.g. [70, 71]) and the sign depends on the relative signs of the soft terms. So from table 2.6 and taking into account these corrections we will take for the third generation ratio

$$\frac{Y_\tau}{Y_b} = 1.37 \pm 0.1 \pm 0.2 \tag{2.202}$$

at the unification scale. The low energy threshold corrections may render b/τ unification [70, 71] but only for particular choices of parameters, particularly of signs.

Since we do not know the corrections of order ϵ^2 in the matrix (2.198) we will not attempt to describe the first generation masses but will only require that one of the eigenvalues should be much smaller than the other two. We will thus only try to describe the hierarchies between the third and second generation. From the table one gets for the mass ratios

$$\frac{m_\mu}{m_\tau} = 4.8 - 6.1 \times 10^{-2} \quad , \quad \frac{m_s}{m_b} = (1.4 - 2.1) \times 10^{-2} \tag{2.203}$$

for $\tan\beta = 10 - 50$. One then has for the (lepton/quark) ratio

$$\frac{m_\mu/m_\tau}{m_s/m_b} \simeq 3.3 \pm 1 \tag{2.204}$$

We would like to see whether such hierarchies arise in our scheme.

The third generation Yukawa couplings

In order to compute the physical Yukawa couplings we need to compute the normalisation factors $\gamma^i_{a,b,c}$. In principle this requires full knowledge of the matter wavefunctions along the matter curves. If however we assume that the relevant wavefunctions are localised close to the (unique) intersection point of the three matter curves, one can use the locally convergent expressions for these normalisation factors given in appendix A. As we said, we will be using the uncorrected normalisation factors shown in that appendix, since the corrections would only induce terms of order ϵ^2 in the physical Yukawa couplings. Then the leading contribution to the third generation Yukawa couplings is given by the 33 entry of the above matrix. Using (2.159) we obtain

$$Y_{b,\tau} = (4\pi g_s)^{1/2}\,\sigma \left(-\frac{q_P^{a_{1,2}}(2\lambda_{a_{1,2}} + q_P^{a_{1,2}}(1+\zeta^2_{a_{1,2}}))}{m^4 + \lambda^2_{a_{1,2}}(1+\zeta^2_{a_{1,2}})}\right)^{1/2} \left(-\frac{q_P^{b_{2,3}}(-2\lambda_{b_{2,3}} + q_P^{b_{2,3}}(1+\zeta^2_{b_{2,3}}))}{m^4 + \lambda^2_{b_{2,3}}(1+\zeta^2_{b_{2,3}})}\right)^{1/2}$$
$$\left(-\frac{(2\zeta_c + q_P^c)(q_P^c + 2\zeta_c - 2\lambda_c) + (q_S^c + \lambda_c)^2}{m^4 + \zeta_c^2 + (\zeta_c - \lambda_c)^2}\right)^{1/2} \tag{2.205}$$

where the q_P, λ's and ζ's are defined in appendix A and all fluxes, as in previous sections, have been taken constant in the vicinity of the intersection point.

To estimate the value of the couplings we assume that the fluxes are such that $q_S^a \simeq 0$, $q_S^b \simeq 0$, and $q_S^c \ll q_P^c$, as we will indeed find in our numerical fits. Using the results in appendix A, we then find the approximate result

$$Y_{b/\tau} \simeq \left(8\,\pi g_s \sigma^2 \frac{q_P^{a_{1,2}}\, q_P^{b_{2,3}}\, q_S^c}{\lambda_a \lambda_b \lambda_c}\right)^{1/2} \tag{2.206}$$

The eigenvalues λ are expected to be generically of the order of the charges $q_P \simeq M$. Taking $q_S^c \ll q_P^c$ then shows that $Y_{b/\tau}$ is proportional to $\sigma g_s^{1/2}$ with a small coefficient $\mathcal{O}(1)$. The intersection slope σ is assumed to be small whereas g_s is constrained as we now explain. Flux quantisation requires $\int_{\Sigma_2}\langle F\rangle \simeq 2\pi$, so that taking $V_{\Sigma_2} \simeq V_S^{1/2}$ implies $M \simeq N_Y \simeq \tilde{N}_Y \simeq (2\pi)/V_S^{1/2}$. Next, the volume of the 7-brane surface S enters in the perturbative equality for the unification coupling (see e.g. [1])

$$\alpha_G \;=\; \frac{2\pi^2 g_s}{m_{st}^4 V_S} \tag{2.207}$$

Setting $\alpha_G \simeq 1/24$ leads to the estimate for the fluxes

$$\frac{M}{m_{st}^2} \;=\; \left(\frac{2\alpha_G}{g_s}\right)^{1/2} \;\simeq\; \frac{0.29}{g_s^{1/2}} \tag{2.208}$$

Having diluted fluxes imposes $M < m_{st}^2$. We then conclude that $g_s^{1/2}$ cannot be arbitrarily small. The conditions of small intersection slope and fluxes are needed to justify the effective description of the 7-brane theory as explained earlier.

Although (2.206) is just an approximation, it indicates that in the present scheme with the wavefunctions localised at the matter curve intersection point the third generation Yukawa couplings are large, of the same order of the Yukawa coupling of the top quark which we know on phenomenological grounds is of that order (see table 2.6). So e.g. in a MSSM scheme one expects a large $\tan\beta \simeq m_t/m_b \simeq 20-50$. In the computations below we show that the above qualitative statements remain true in our F-theory scheme.

Hierarchies of fermion masses

We have discussed above the Yukawa couplings for the third generation. The corrections discussed in previous sections are in principle able to generate Yukawa couplings and masses also for the first two generations. We would like to explore now to what extent the above results could be able to describe the observed structure of hierarchical lepton and D-quark masses. We will be interested now on the relative hierarchies among different D-quarks or different charged leptons, so that we will study the matrix

$$\frac{Y}{Y^{33}} = \begin{pmatrix} \mathcal{O}(\epsilon^2) & \mathcal{O}(\epsilon^2) & \epsilon\varrho^{-1}\frac{\gamma_a^1}{\gamma_a^3}\tilde{\theta}_{0x} \\ \mathcal{O}(\epsilon^2) & \epsilon\varrho^{-1}\frac{\gamma_a^2\gamma_b^2}{\gamma_a^3\gamma_b^3}(\tilde{\theta}_{0x}+\tilde{\theta}_{0y}) & \mathcal{O}(\epsilon^2) \\ \epsilon\varrho^{-1}\frac{\gamma_b^1}{\gamma_b^3}\tilde{\theta}_{0y} & \mathcal{O}(\epsilon^2) & 1 \end{pmatrix} \tag{2.209}$$

where we have divided the matrix (2.198) by the largest Y_{33} entry. It is easy to check that this matrix has eigenvalues

$$\begin{aligned} \lambda_1 &= 1+\mathcal{O}(\epsilon^2) \\ \lambda_2 &= \epsilon\varrho^{-1}\frac{\gamma_a^2\gamma_b^2}{\gamma_a^3\gamma_b^3}(\tilde{\theta}_{0x}+\tilde{\theta}_{0y})+\mathcal{O}(\epsilon^2) \\ \lambda_3 &= \mathcal{O}(\epsilon^2). \end{aligned}$$

This is interesting since we automatically get a hierarchy of masses of order $(1,\epsilon,\epsilon^2)$ from the start, without any further assumption. As shown in [57] if we had $\theta_0=0$ and we were left only with the corrections from θ_2, the matrix would be still rank one up to order ϵ^2 and the non-perturbative corrections would be unable to create the desired hierarchies. So in order to obtain hierarchies it turns out to be crucial the presence of a non-constant θ_0 as studied in the previous sections.

Identifying the first and second eigenvalues with the third and second generations one gets to leading order in ϵ at the unification scale

$$\frac{m_s}{m_b} = \left(-\frac{q_P^{a_1}q_P^{b_2}}{m_*^4}\right)^{1/2}\epsilon\,\varrho^{-1}(\tilde{\theta}_{0x}+\tilde{\theta}_{0y}) \tag{2.210}$$

$$\frac{m_\mu}{m_\tau} = \left(-\frac{q_P^{a_2}q_P^{b_3}}{m_*^4}\right)^{1/2}\epsilon\,\varrho^{-1}(\tilde{\theta}_{0x}+\tilde{\theta}_{0y}) \tag{2.211}$$

where we have used $\gamma_a^2/\gamma_a^3=\left(q_P^a/m_*^2\right)^{1/2}$ and $\gamma_b^2/\gamma_b^3=\left(-q_P^b/m_*^2\right)^{1/2}$. Using the expressions above together with the charges in table 2.3 we get for the ratio of ratios

$$\frac{m_\mu/m_\tau}{m_s/m_b} = \left(\frac{(M+\frac{1}{2}\tilde{N}_Y)(M+\tilde{N}_Y)}{(M-\frac{1}{3}\tilde{N}_Y)(M+\frac{1}{6}\tilde{N}_Y)}\right)^{1/2}. \tag{2.212}$$

This ratio is interesting because the dependence on the non-perturbative correction and extra flux-dependent factors cancel out yielding a result which only depends on the ratio

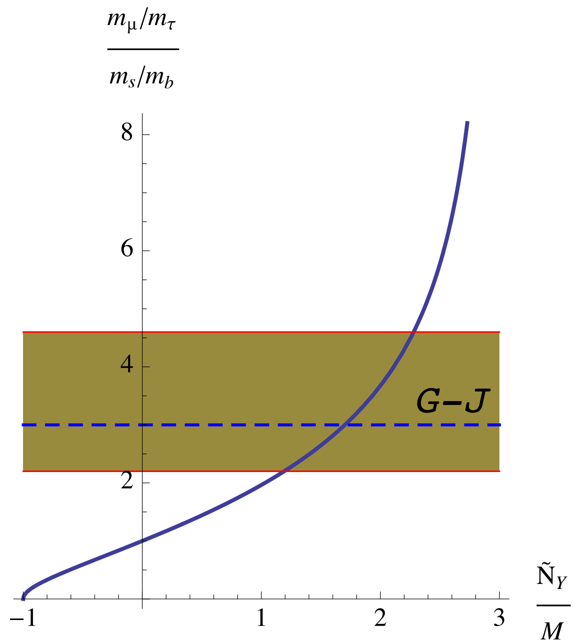

Figure 2.2: $\frac{m_\mu/m_\tau}{m_s/m_b}$ at the unification scale as a function of $\tilde{N}_Y/M$. The band shows the region consistent with the running of the low energy data up to the unification scale taken from ref. [70]. The dashed line is the Georgi-Jarlskog value [72].

$\tilde{N}_Y/M$. Experimentally one has from table 2.6 that $\frac{m_\mu/m_\tau}{m_s/m_b} = 3.3 \pm 1$ so that one has the constraint

$$\left(\frac{(1+\frac{\tilde{N}_Y}{2M})(1+\frac{\tilde{N}_Y}{M})}{(1-\frac{\tilde{N}_Y}{3M})(1+\frac{\tilde{N}_Y}{6M})}\right)^{1/2} = 3.3 \pm 1 \ . \tag{2.213}$$

This condition is displayed in figure 2.2. One observes that agreement with experiment may be obtained with a ratio of fluxes $\tilde{N}_Y/M = 1.8 \pm 0.6$, independently from the value of the rest of the parameters. Note that conditions for consistent local chirality are $M_y + q_Y^{a^+}\tilde{N}_Y > 0$ and $M_x + q_Y^{b^+}\tilde{N}_Y < 0$ which are satisfied as long as $-1 < \tilde{N}_Y/M < 3$. To reproduce the particular D-quark and charged lepton hierarchies we can substitute the last result in (2.210) and (2.211) which can be rewritten as

$$\frac{m_s}{m_b} = \left[\left(1-\frac{\tilde{N}_Y}{3M}\right)\left(1+\frac{\tilde{N}_Y}{6M}\right)\right]^{1/2} \frac{M}{m_*^2}\epsilon\,\varrho^{-1}(\tilde{\theta}_{0x}+\tilde{\theta}_{0y}) \tag{2.214}$$

$$\frac{m_\mu}{m_\tau} = \left[\left(1+\frac{\tilde{N}_Y}{2M}\right)\left(1+\frac{\tilde{N}_Y}{M}\right)\right]^{1/2} \frac{M}{m_*^2}\epsilon\,\varrho^{-1}(\tilde{\theta}_{0x}+\tilde{\theta}_{0y}) \tag{2.215}$$

From table 2.6 we see that $\frac{m_\mu}{m_\tau} = 5.4 \pm 0.6 \times 10^{-2}$ so that we get an estimate

$$\frac{M}{m^2}\,\epsilon\,(\tilde{\theta}_{0x}+\tilde{\theta}_{0y}) \simeq (2.3 \pm 0.2) \times 10^{-2} \tag{2.216}$$

where we have used the central value $\tilde{N}_Y/M = 1.8$ and the definition of ϱ. Thus taking $\tilde{N}_Y/M \simeq 1.8$ and the non-perturbative correction of the size given by eq.(2.216) one obtains values consistent with the experimental m_μ/m_τ and m_s/m_b ratios. Note that the number in eq.(2.216) is quite small, consistent with a non-perturbative origin.

$b-\tau$ (non)-unification

Hypercharge fluxes also violate the equality of the τ and b-quark Yukawas at unification. Indeed, using eq.(2.159) one gets

$$\frac{Y_\tau}{Y_b} = \frac{\gamma_{a_2}^3\gamma_{b_3}^3}{\gamma_{a_1}^3\gamma_{b_2}^3} \,. \tag{2.217}$$

with the expressions for the γ's given in appendix A. We would like to see now whether one can obtain the result $\frac{Y_\tau}{Y_b} = 1.37 \pm 0.1 \pm 0.2$ discussed above for some choice of fluxes consistent with the equations for the Yukawa hierarchies of second to third generation discussed above. Recall that the latter require $\tilde{N}_Y/M \simeq 1.2 - 2.4$. We have as free flux parameters M, N_Y, N_a, since N_b is determined by eq.(2.145) which sets $N_Y = 3(N_a - N_b)$. In figure 2.3 we show the value obtained for Y_τ/Y_b as a function of the fluxes M and N_Y and for values $N_a = \pm 1$, which is sufficient to show the general behavior. This is done for $\tilde{N}_Y/M = 1.8$ (upper plots) and 1.3 (lower plots), which correspond in turn to $\frac{m_\mu/m_\tau}{m_s/m_b} = 3, 2.2$ respectively, yielding consistent 2nd to 3rd generation mass hierarchies. A first conclusion is that correctly yielding the latter hierarchy requires in turn $Y_\tau/Y_b > 1$, as observed. One can easily have flux choices with $\frac{Y_\tau}{Y_b} = 1.37 \pm 0.1 \pm 0.2$, particularly for $N_a = -1$ and $\tilde{N}_Y/M = 1.3$ (lower left figure).

An example of consistent parameter choices is (the fluxes are given in m^2 units)

$$(M, N_a, N_Y, \tilde{N}_Y;\ \tilde{\epsilon}) \ = \ (2, 0, 0.1, 3.6;\ 7.5 \times 10^{-4}) \tag{2.218}$$

where $\tilde{\epsilon} = g_s^{1/2}\epsilon\,(\tilde{\theta}_{0x} + \tilde{\theta}_{0y})$. One obtains

$$\left(\frac{Y_\tau}{Y_b}, \frac{m_s}{m_b}, \frac{m_\mu}{m_\tau}, Y_b\right) = (1.38,\ 1.7\times 10^{-2},\ 5.4\times 10^{-2},\ 0.66 g_s^{1/2}\sigma) \tag{2.219}$$

in very good agreement with the experimental results in table 2.6. Note that the value for Y_b at the unification scale is consistent with those in table 2.6 for $g_s\sigma^2 \simeq 0.1$ which suggest indeed a large value of tanβ in the MSSM context. This is generally the case for all examples able to appropriately describe the mass ratios. Let us finally comment that the condition of diluted fluxes corresponds to e.g. $M/m_{st}^2 = (\sigma M)/m^2 < \mathcal{O}(1)$, and the same for the rest of the fluxes. This may be achieved in the above example and also for the examples in figure 2.3 by considering an appropriately small value for σ.

One can repeat the analysis in this chapter for the case of the non-SUSY Standard Model remaining below the string scale. Indeed, although we made our discussion in terms of superpotentials, the Yukawa coupling sector remains essentially unchanged in the presence of terms breaking SUSY below the string scale. In the case of the SM, extrapolating the low-energy masses up to the unification scale the ratios of the second to third generation masses remain similar to those shown in eq.(2.203), see e.g. ref. [73]. Concerning the b/τ ratio one gets around the unification scale $Y_\tau/Y_b = 1.73$, with no relevant low-energy thresholds giving additional contributions. The flux analysis above would equally apply to this non-SUSY case and there are wide ranges of fluxes consistent with the fermion hierarchies and τ/b ratio. The required fluxes tend to be however more diluted in this non-SUSY case.

We conclude that instanton effects are able to generate the observed second to third generation hierarchies of charged leptons and D-quarks, via the superpotential deformation (2.120). The hierarchies between second and third generations can then be easily

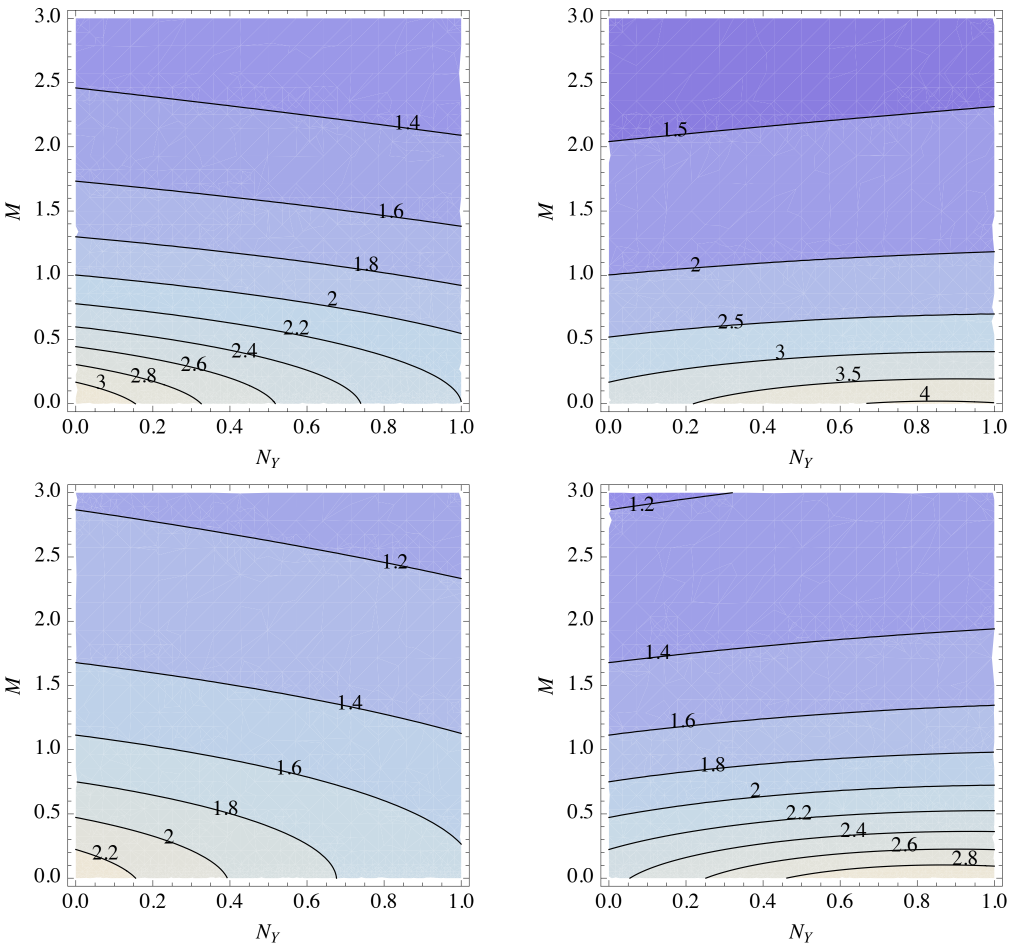

Figure 2.3: Y_τ/Y_b as a function of M, N_Y for $N_a = -1(1)$ left(right). Flux choices are consistent with $\frac{m_\mu/m_\tau}{m_s/m_b} = 3(2.2)$ for upper(lower) plots.

understood in this scheme. One can reproduce the values $(Y_\mu/Y_\tau)/(Y_s/Y_b) = 3.3 \pm 1.0$ at the unification scale, consistent with the low-energy data and, at the same time, the b/τ ratio with Yukawa couplings $Y_\tau/Y_b(m_{st}) \simeq 1.37 \pm 0.1 \pm 0.2$, as obtained from the RGE in the MSSM. These two attractive features are purely due to the hypercharge flux which explicitly breaks the underlying $SU(5)$ symmetry, which otherwise predicts equal masses for D-quarks and leptons of each generation at the unification scale. The Yukawa couplings of the third generation are large, corresponding to large values of $\tan\beta$ (for not too small $g_s\sigma^2$) within the context of the MSSM. In order to compute the masses for the first generation we would need to know the corrections to the Yukawa couplings at order ϵ^2 which, although feasible, is a more involved task.

2.5 Up-type Yukawas

2.5.1 The E_6 model

In the following we describe the E_6 local F-theory model which will serve to compute up-type quark Yukawa couplings. Similarly to the $SO(12)$ model, one may first consider the 7-brane Higgs background that defines the structure of matter curves and breaks the E_6 symmetry down to $SU(5)$, and then describe the background 7-brane flux that induces 4d chirality and breaks the GUT spectrum down to the MSSM.

Unlike in the $SO(12)$ case the Higgs background will be in part specified by a T-brane configuration and, as mentioned above, this implies that the Higgs and flux backgrounds are related by the equations of motion. As we will see in section 2.5.3 this feature of T-branes will have a direct impact on the zero mode wavefunctions localised at the matter curves, and this will in turn affect the physical Yukawa couplings computed in section 2.5.4.

Matter curves near the E_6 point

In the standard framework of $SU(5)$ local F-theory models, $\mathbf{10}_M \times \mathbf{10}_M \times \mathbf{5}_U$ Yukawa couplings are developed at points p where an enhanced E_6 symmetry occurs. This implies that in order to compute such Yukawas we must consider a 7-brane action where the fields Φ and A take values in the adjoint of E_6. Both Φ and A will have non-trivial background profiles along the 4-cycle S_{GUT}, and so the gauge symmetry group will not be E_6 but a subgroup that commutes with both $\langle\Phi\rangle$ and $\langle A\rangle$ at any point of S_{GUT}.

A local $SU(5)$ model with E_6 enhancement, dubbed E_6 model in the following, can be described by specifying the profiles $\langle\Phi\rangle$ and $\langle A\rangle$ in the vicinity of a $\mathbf{10}\times\mathbf{10}\times\mathbf{5}$ Yukawa point. By construction, $\langle\Phi\rangle$ and $\langle A\rangle$ are functions of S_{GUT} valued in the Lie algebra of E_6, and $\langle\Phi\rangle$ is such that at a generic point of this neighborhood it breaks the E_6 symmetry down to $SU(5)\times U(1)^n$, with $n = 0, 1, 2$. Then, by neglecting the effect of the worldvolume flux $\langle A\rangle$, we can identify $G_{S_{GUT}} = SU(5)$ as the GUT gauge group of this model. In addition, the profile $\langle\Phi\rangle$ will describe the different matter curves, that is the curves of S_{GUT} at which chiral modes in the representations $\mathbf{5}$ or $\mathbf{10}$ are localised.

This picture can be understood in more detail by expressing the local model data in terms of hermitian generators Q_α of E_6. These generators can be decomposed as $\{Q_\alpha\} = \{H_i, E_\rho\}$, where H_i generate the Cartan subalgebra of E_6 and E_ρ correspond to the roots of E_6. More precisely we have the usual relation

$$[H_i, E_\rho] \;=\; \rho_i E_\rho \tag{2.220}$$

where ρ_i is the i-th component of the root ρ. The 72 non-trivial roots are given by

$$(0, \underline{\pm1, \pm1, 0, 0, 0}) \tag{2.221}$$

where we should consider all possible permutation of the underlined vector entries, and

$$\frac{1}{2}(\pm\sqrt{3}, \pm1, \pm1, \pm1, \pm1, \pm1) \qquad \text{with even number of } +' \text{s}. \tag{2.222}$$

Near the up-type Yukawa point one can decompose the background profile of Φ as a linear combination of the above generators, with arbitrary functions of the 4-cycle S_{GUT}

as coefficients. If we parametrise the complex coordinates of S_{GUT} as (x, y) then we have that $\Phi = \Phi_{xy}\, dx \wedge dy$ and so in general

$$\langle \Phi_{xy} \rangle \;=\; \sum_i g_\alpha\, Q_\alpha \tag{2.223}$$

with $g_\alpha \equiv g_\alpha(x, \bar{x}, y, \bar{y})$ functions in the vicinity of the Yukawa point and $Q_i \in \{H_i, E_\rho\}$. For simplicity, the generators Q_α are often chosen to lie within the Cartan subalgebra of E_6, because then one can understand the background (2.223) as a configuration of intersecting 7-branes. For instance, one may consider the following background

$$\langle \Phi_{xy} \rangle \;=\; m^{3/2}\sqrt{x}\, P + \mu^2\,(bx - y)\, Q \tag{2.224}$$

where m and μ are real parameters with the dimension of mass, b is a complex adimensional parameter and P and Q are the following combinations of Cartan generators

$$P \;=\; \frac{1}{2}(\sqrt{3}H_1 + H_2 + H_3 + H_4 + H_5 + H_6) \tag{2.225}$$

$$Q \;=\; \frac{1}{2}\Big(\frac{5}{\sqrt{3}}H_1 - H_2 - H_3 - H_4 - H_5 - H_6\Big) \tag{2.226}$$

Given a background (2.223) one can analyse the symmetry breaking pattern of the local model and understand the structure of its matter curves [22,23]. The basic quantity to look at is $[\langle \Phi_{xy} \rangle, E_\rho]$, which will be a function valued on the Lie algebra of E_6 and tells us to which subgroup the initial E_6 group is broken. For instance, for the background (2.224) the set of generators that commute with $\langle \Phi_{xy} \rangle$ for all points of S_{GUT} is the set of roots

$$(0, \underline{1, -1, 0, 0, 0}) \tag{2.227}$$

as well as the Cartan generators. This implies that the subgroup of E_6 that remains as a gauge symmetry group is given by $SU(5) \times U(1)^2$, and the GUT gauge group can be identified with $G_{S_{GUT}} = SU(5)$.

At particular submanifolds of S_{GUT} there will be extra sets of roots that commute with $\langle \Phi_{xy} \rangle$, implying an enhancement of the bulk symmetry group. In particular we have that there is such enhancement for two different holomorphic curves, namely

$$\Sigma_{\mathbf{5}} = \{bx - y = 0\} \;\;\rightarrow\;\; \pm\frac{1}{2}(\sqrt{3}, \underline{1, -1, -1, -1, -1}) \tag{2.228}$$

$$\Sigma_{\mathbf{10}} = \{\mu^4(bx - y)^2 = m^3 x\} \;\;\rightarrow\;\; \pm(0, \underline{1, 1, 0, 0, 0}) \quad \text{or} \quad \pm\frac{1}{2}(-\sqrt{3}, \underline{1, 1, -1, -1, -1}) \tag{2.229}$$

where at the lhs we have displayed the matter curve or curve of enhancement and at the rhs the extra roots that commute with $\langle \Phi_{xy} \rangle$ at such curve. At the curve (2.228) there are ten additional roots that together with (2.227) and the Cartan subalgebra generate the group $SU(6) \times U(1)$. These extra roots transform as either a $\mathbf{5}$ or a $\bar{\mathbf{5}}$ representation of $SU(5)$, and so will the zero modes that are localised there [22,23]. Following the common practice one then dubs $bx - y = 0$ as the $\mathbf{5}$ matter curve $\Sigma_{\mathbf{5}}$ of the local model. At the curve (2.229) there are 20 extra unbroken roots transforming in the representations $\mathbf{10}$ and $\overline{\mathbf{10}}$ of $SU(5)$, enhancing the bulk symmetry group to $SO(10) \times U(1)$ and giving rise to a $\mathbf{10}$ matter curve $\Sigma_{\mathbf{10}}$. Finally, at the intersection point $p_{\rm up} = \{x = y = 0\}$ of both curves

$\langle \Phi_{xy} \rangle = 0$, and so the full E_6 symmetry remains unbroken. It is at this point where a Yukawa $\mathbf{10}_M \times \mathbf{10}_M \times \mathbf{5}_U$ must be generated via triple overlap of zero mode wavefunctions.

The **10** curve (2.229) requires some further explanation, as the roots that enhance the symmetry are not the same all over it. Indeed, at the branch $\sqrt{x} = bx - y$ we have that the roots in the first line of (2.229) are the ones that commute with the background, while for $-\sqrt{x} = bx - y$ the roots of the second line are the ones commuting with $\langle \Phi_{xy} \rangle$. While this make look puzzling, it was realised in [25] that the zero modes of the two branches of the **10** curve (2.229) are identified by the phenomenon of 7-brane monodromy as explained in section 2.3.3. In fact, it was also pointed out in [25] that such monodromy is necessary in order to achieve precisely one heavy generation of up-type quarks whenever $\langle \Phi_{xy} \rangle$ takes values in the Cartan of E_6, and a background similar to (2.224) was proposed as a candidate to obtain realistic up-like Yukawas. However, the analysis in [25,74] shows that it is not obvious to find non-singular solutions for the zero mode wavefunctions near the intersection point of matter curves in such monodromic 7-brane configurations. As this is the region of larger wavefunction overlap and the one that contributes most to the value of the Yukawa couplings, this complicates the computational and predictive power of such local model.

One can however consider an alternative background for the transverse position field Φ, based on the proposal made in [53] of describing up-like Yukawa couplings via T-branes. Indeed, let us consider the background

$$\langle \Phi_{xy} \rangle = m(E^+ + m\,xE^-) + \mu^2(bx - y)Q \tag{2.230}$$

where all quantities are as in (2.224) except for the generators $E^\pm$ whose corresponding roots, also denoted $E^\pm$, are defined as

$$E^\pm = \pm\frac{1}{2}(\sqrt{3}, 1, 1, 1, 1, 1) \tag{2.231}$$

and satisfy the relation $[E^+, E^-] = P$. More precisely, the triplet $\{E^+, E^-, P\}$ generates the $\mathfrak{su}(2)$ factor of a $\mathfrak{su}(5) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ maximal Lie subalgebra of $\mathfrak{e}_6$, under which the E_6 adjoint decomposes as

$$\mathbf{78} \to (\mathbf{24}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{3})_0 \oplus (\mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{10}, \mathbf{2})_{-1} \oplus (\overline{\mathbf{10}}, \mathbf{2})_1 \oplus (\mathbf{5}, \mathbf{1})_2 \oplus (\overline{\mathbf{5}}, \mathbf{1})_{-2}. \tag{2.232}$$

From this decomposition it is manifest that the pair of **10**'s described above transform as a doublet of the $SU(2)$ generated by $\{E^+, E^-, P\}$. In particular if we define

$$\begin{array}{ll} E_{\mathbf{10}^+} = (0, \underline{1, 1, 0, 0, 0}) & E_{\mathbf{10}^-} = \frac{1}{2}(-\sqrt{3}, \underline{1, 1, -1, -1, -1}) \\ E_{\overline{\mathbf{10}}^+} = -(0, \underline{1, 1, 0, 0, 0}) & E_{\overline{\mathbf{10}}^-} = -\frac{1}{2}(-\sqrt{3}, \underline{1, 1, -1, -1, -1}) \end{array} \tag{2.233}$$

we have the relations

$$[E^\pm, E_{\mathbf{10}^\mp}] = E_{\mathbf{10}^\pm}, \qquad [E^\pm, E_{\mathbf{10}^\pm}] = 0, \qquad [P, E_{\mathbf{10}^\pm}] = \pm E_{\mathbf{10}^\pm}. \tag{2.234}$$

Let us analyse the gauge symmetry group of this background and the structure of matter curves. Just as in the previous case we have to look at the commutant of $\langle \Phi \rangle$ as a function of the coordinates x, y. The gauge group is the commutant at generic points while the matter curves are identified by finding jumps in its rank [53]. For the background (2.230) one can easily check that the set of roots of the subalgebra $\mathfrak{su}(5) \oplus \mathfrak{u}(1) \subset \mathfrak{e}_6$ do commute at generic points in S and so we can identify the GUT gauge group with $SU(5)$.

Regarding the matter curves, we find that at $\Sigma_{\mathbf{5}} = \{bx - y = 0\}$ the roots $(\mathbf{5},\mathbf{1})_2 = \frac{1}{2}(\sqrt{3}, \underline{1,-1,-1,-1,-1}) = E_{\mathbf{5}}$ and $(\bar{\mathbf{5}},\mathbf{1})_{-2} = \frac{1}{2}(-\sqrt{3}, \underline{-1,1,1,1,1}) = E_{\bar{\mathbf{5}}}$ commute with $\langle\Phi\rangle$, since

$$[\langle\Phi\rangle, E_{\mathbf{5}}] \;=\; 2\mu^2(bx-y)E_{\mathbf{5}} \tag{2.235}$$

$$[\langle\Phi\rangle, E_{\bar{\mathbf{5}}}] \;=\; -2\mu^2(bx-y)E_{\bar{\mathbf{5}}} \tag{2.236}$$

and so at $\Sigma_{\mathbf{5}}$ the symmetry group enhances to $SU(6)\times U(1)$. Similarly, the action of $\langle\Phi\rangle$ on the sector $(\mathbf{10},\mathbf{2})_{-1}$ is given by

$$[\langle\Phi\rangle, R_+E_{\mathbf{10}^+} + R_-E_{\mathbf{10}^-}] = \begin{pmatrix} -\mu^2(bx-y) & m \\ m^2x & -\mu^2(bx-y)\end{pmatrix}\begin{pmatrix} R_+E_{\mathbf{10}^+} \\ R_-E_{\mathbf{10}^-}\end{pmatrix} \tag{2.237}$$

while for the conjugate sector $(\overline{\mathbf{10}},\mathbf{2})_1$ we have

$$[\langle\Phi\rangle, R'_+E_{\overline{\mathbf{10}}^+} + R'_-E_{\overline{\mathbf{10}}^-}] = \begin{pmatrix} \mu^2(bx-y) & -m^2x \\ -m & \mu^2(bx-y)\end{pmatrix}\begin{pmatrix} R'_+E_{\overline{\mathbf{10}}^+} \\ R'_-E_{\overline{\mathbf{10}}^-}\end{pmatrix} \tag{2.238}$$

where $R_\pm$, $R'_\pm$ are functions on S_{GUT}. At $\Sigma_{\mathbf{10}} = \{\mu^4(bx-y)^2 = m^3x\}$ the matrices in (2.237) and (2.238) have vanishing determinant so there are additional roots commuting with $\langle\Phi\rangle$, and therefore a jump in the rank of the symmetry group.[49] We therefore identify Σ_{10} with the **10** curve of this T-brane background. Notice that we arrive to the same matter curves $\Sigma_{\mathbf{5}}$, $\Sigma_{\mathbf{10}}$ if we consider the action $[\langle\bar\Phi\rangle,\cdot]$, and that as before they both meet at the Yukawa point $p_{\text{up}} = \{x = y = 0\}$.[50]

Finally, using the results of section 2.3.3 the above T-brane background is written, in real gauge as

$$\langle\Phi_{xy}\rangle = m(e^fE^+ + m\,xe^{-f}E^-) + \mu^2(bx-y)Q \tag{2.239}$$

with $f \equiv f(x,\bar{x},y,\bar{y})$ the Painlevé transcendent that appeared in 2.3.3. It is easy to check that everything works as before, and that we recover the same two matter curves $\Sigma_{\mathbf{5}}$ and $\Sigma_{\mathbf{10}}$.

Primitive worldvolume fluxes

On top of the non-primitive flux associated with the T-brane, the above model admits additional contributions to the background worldvolume flux $\langle F\rangle$ if they do not spoil the F-term and D-term conditions. The simplest way to introduce them is to consider primitive $(1,1)$ fluxes $\langle F\rangle$ in the Cartan of E_6. Considering such fluxes is important to complete the local F-theory model, not just because they will be generically there, but also because they play an important role for the phenomenology of the model. On the one hand they will generate 4d chirality for the $SU(5)$ spectrum, and on the other they will break the $SU(5)$ gauge group down to $SU(3)\times SU(2)\times U(1)_Y$.

[49] At Σ_{10} both roots $R \equiv E_{\mathbf{10}^+} + \frac{\mu^2}{m}(bx-y)E_{\mathbf{10}^-}$ and $R' \equiv \frac{\mu^2}{m}(bx-y)E_{\overline{\mathbf{10}}^+} + E_{\overline{\mathbf{10}}^-}$ commute with $\langle\Phi\rangle$ but since these are not conjugate to each other the enhanced algebra is a complex subalgebra of $\mathfrak{e}_6^{\mathbb{C}}$ that is not the complexification of a real algebra. Thus, we cannot associate a real gauge group to the matter curve $\Sigma_{\mathbf{10}}$ in agreement with the discussion in section 4.1 of [53].

[50] Note that for this local model $\langle\Phi_{xy}\rangle \neq 0$ at p_{up}, and so the symmetry group is no longer E_6 at the Yukawa point. As discussed in [53] this is a general feature of T-brane configurations, see also Appendix C of [58]. By abuse of terminology, we will still refer to this point as the E_6 point of the local model.

More precisely, let us consider the worldvolume flux

$$\langle F_Q \rangle = i\left[M(dx \wedge d\bar{x} - dy \wedge d\bar{y}) + N(dx \wedge d\bar{y} + dy \wedge d\bar{x})\right] Q \tag{2.240}$$

where the generator Q is given by (2.226), and M and N are flux densities near the Yukawa point that we will approximate by constants. It is easy to check that adding such flux will not spoil the equations of motion for any value of M, N, which will be considered as real parameters of the model in the following. The presence of such worldvolume flux will induce 4d chirality in the matter curves. Indeed, the modes of opposite chirality $\mathbf{5}$, $\bar{\mathbf{5}}$ and $\mathbf{10}$, $\overline{\mathbf{10}}$ feel the T-brane background in a similar way, and so whenever there is a zero mode solution for one chirality there will be a solution for the opposite chirality as well. This is no longer true for the background flux (2.240), that will select locally modes of one chirality or the other depending on the sign of M and N.

Besides inducing 4d chirality, worldvolume fluxes break the $SU(5)$ gauge group when switched on along the hypercharge generator. In general realistic GUT F-theory models will have such worldvolume flux, which we can represent locally as

$$\langle F_Y \rangle = i\left[\tilde{N}_Y(dy \wedge d\bar{y} - dx \wedge d\bar{x}) + N_Y(dx \wedge d\bar{y} + dy \wedge d\bar{x})\right] Q_Y \tag{2.241}$$

where N_Y, $\tilde{N}_Y$ are local flux densities and

$$Q_Y = \frac{1}{3}(H_2 + H_3 + H_4) - \frac{1}{2}(H_5 + H_6) \tag{2.242}$$

is the hypercharge generator. This flux will enter into the Dirac equation for the zero modes and, just as in the local $SO(12)$ model, it will be the only quantity that distinguishes between particles within the same $SU(5)$ multiplet but with different hypercharge, c.f. table 2.7 below.

Summary

Let us summarise the details of the E_6 model which we will use to compute up-like Yukawa couplings. If we parametrise the four-cycle S_{GUT} by the complex coordinates x, y, the Higgs background that breaks $E_6 \to SU(5) \times U(1)$ is given by

$$\langle \Phi_{xy} \rangle = m(e^f E^+ + m\, x e^{-f} E^-) + \mu^2(bx - y)Q \tag{2.243}$$

where m and μ are real parameters with the dimensions of mass, a, b are adimensional parameters and $E^{\pm}$ and Q are the E_6 roots given respectively by (2.231) and (2.226). The real function $f \equiv f(x, \bar{x})$ solves the equation (2.296) and can be approximated locally by (2.104). Finally, one may choose different values for the parameter b. For the sake of concreteness when computing physical Yukawas we will restrict to the case

$$b = 1 \tag{2.244}$$

although our discussion can be easily generalised to other values of b.

The worldvolume flux of this model will be given by

$$\langle F \rangle = \langle F_{\rm p} \rangle + \langle F_{\rm np} \rangle \tag{2.245}$$

where $\langle F_{\mathrm{np}} \rangle$ is the non-primitive flux that is necessary to compensate the contribution of $[\langle \Phi_{xy} \rangle, \langle \bar{\Phi}_{\bar{x}\bar{y}} \rangle]$ to the D-term equation, and reads

$$\langle F_{\mathrm{np}} \rangle \;=\; -i\partial\bar{\partial} f P \tag{2.246}$$

with the E_6 generator P given by (2.225). In addition we have that

$$\langle F_{\mathrm{p}} \rangle = iQ_R(dy \wedge d\bar{y} - dx \wedge d\bar{x}) + iQ_S(dx \wedge d\bar{y} + dy \wedge d\bar{x}) \tag{2.247}$$

is the primitive flux needed to generate chirality and further break the gauge group as $SU(5) \to SU(3) \times SU(2) \times U(1)_Y$. Here we have defined

$$Q_R = -MQ + \tilde{N}_Y Q_Y, \qquad Q_S = NQ + N_Y Q_Y \tag{2.248}$$

with Q_Y the hypercharge generator (2.242) and M, N, N_Y, $\tilde{N}_Y$ real flux densities. Because of the presence of the hypercharge generator, zero modes within the same $SU(5)$ multiplet but with different hypercharge will feel a different worldvolume flux, and this will translate into a different internal wavefunction profile for each of them. We have summarised in table 2.7 the different sectors that arise in the E_6 model together with their charges under the MSSM gauge group and the worldvolume flux operators (2.248). The latter charges are defined as

$$[Q_R, E_\rho] \;=\; q_R\, E_\rho, \qquad [Q_S, E_\rho] \;=\; q_S\, E_\rho \tag{2.249}$$

and so are given by a linear combination of flux densities.

Sector	Root	G_{MSSM}	q_R	q_S
$\mathbf{10}_1$	$(0,\underline{1,1,0},0,0) \oplus \frac{1}{2}(-\sqrt{3},\underline{1,1,-1},-1,-1)$	$(\bar{\mathbf{3}},\mathbf{1})_{\frac{2}{3}}$	$M + \frac{2}{3}\tilde{N}_Y$	$-N + \frac{2}{3}N_Y$
$\mathbf{10}_2$	$(0,\underline{1,0,0},\underline{1,0}) \oplus \frac{1}{2}(-\sqrt{3},\underline{1,-1,-1},\underline{1,-1})$	$(\mathbf{3},\mathbf{2})_{-\frac{1}{6}}$	$M - \frac{1}{6}\tilde{N}_Y$	$-N - \frac{1}{6}N_Y$
$\mathbf{10}_3$	$(0,0,0,0,1,1) \oplus \frac{1}{2}(-\sqrt{3},-1,-1,-1,1,1)$	$(\mathbf{1},\mathbf{1})_{-1}$	$M - \tilde{N}_Y$	$-N - N_Y$
$\mathbf{5}_1$	$\frac{1}{2}(\sqrt{3},-1,-1,-1,\underline{1,-1})$	$(\mathbf{1},\mathbf{2})_{-\frac{1}{2}}$	$-2M - \frac{1}{2}\tilde{N}_Y$	$2N - \frac{1}{2}N_Y$
$\mathbf{5}_2$	$\frac{1}{2}(\sqrt{3},\underline{1,-1,-1},-1,-1)$	$(\mathbf{3},\mathbf{1})_{\frac{1}{3}}$	$-2M + \frac{1}{3}\tilde{N}_Y$	$2N + \frac{1}{3}N_Y$
$\overline{\mathbf{10}}_1$	$(0,\underline{-1,-1,0},0,0) \oplus \frac{1}{2}(\sqrt{3},\underline{-1,-1,1},1,1)$	$(\mathbf{3},\mathbf{1})_{-\frac{2}{3}}$	$-M - \frac{2}{3}\tilde{N}_Y$	$N - \frac{2}{3}N_Y$
$\overline{\mathbf{10}}_2$	$(0,\underline{-1,0,0},\underline{-1,0}) \oplus \frac{1}{2}(\sqrt{3},\underline{-1,1,1},\underline{-1,1})$	$(\bar{\mathbf{3}},\mathbf{2})_{\frac{1}{6}}$	$-M + \frac{1}{6}\tilde{N}_Y$	$N + \frac{1}{6}N_Y$
$\overline{\mathbf{10}}_3$	$(0,0,0,0,-1,-1) \oplus \frac{1}{2}(\sqrt{3},1,1,1,-1,-1)$	$(\mathbf{1},\mathbf{1})_{1}$	$-M + \tilde{N}_Y$	$N + N_Y$
$\bar{\mathbf{5}}_1$	$\frac{1}{2}(-\sqrt{3},1,1,1,\underline{-1,1})$	$(\mathbf{1},\mathbf{2})_{\frac{1}{2}}$	$2M + \frac{1}{2}\tilde{N}_Y$	$-2N + \frac{1}{2}N_Y$
$\bar{\mathbf{5}}_2$	$\frac{1}{2}(-\sqrt{3},\underline{-1,1,1},1,1)$	$(\bar{\mathbf{3}},\mathbf{1})_{-\frac{1}{3}}$	$2M - \frac{1}{3}\tilde{N}_Y$	$-2N - \frac{1}{3}N_Y$
$\mathbf{X}^+, \mathbf{Y}^+$	$(0,\underline{1,0,0},\underline{-1,0})$	$(\mathbf{3},\mathbf{2})_{\frac{5}{6}}$	$\frac{5}{6}\tilde{N}_Y$	$\frac{5}{6}N_Y$
$\mathbf{X}^-, \mathbf{Y}^-$	$(0,\underline{-1,0,0},\underline{1,0})$	$(\bar{\mathbf{3}},\mathbf{2})_{-\frac{5}{6}}$	$-\frac{5}{6}\tilde{N}_Y$	$-\frac{5}{6}N_Y$

Table 2.7: Different sectors and charges for the E_6 model.

As we will see in section 2.5.3, the quantities q_R, q_S enter into the expressions for the internal wavefunctions of the MSSM chiral zero modes. In fact, these charges determine which sectors of those in table 2.7 have localised zero modes near the Yukawa point. In order to construct a local model with the MSSM chiral spectrum we need to impose that chiral modes only arise from the four first rows of table 2.7. This will impose some constraints on q_R and q_S, which will in turn impose constraints in the values of the flux densities M, N, N_Y, $\tilde{N}_Y$, as we briefly describe below. We refer the reader to appendix B of [58] where the local index formula (2.113) is applied to every sector.

One important constraint comes from avoiding the doublet-triplet splitting problem of 4d $SU(5)$ GUT models. Following [23], one can do so by adjusting the fluxes so that the sector of triplets $\mathbf{5}_2$, $\overline{\mathbf{5}}_2$ does not feel any net flux and it is then a non-chiral sector without any localised 4d modes. This condition amounts to imposing that $q_S(\mathbf{5}_2) = q_S(\overline{\mathbf{5}}_2) = 0$ or in other words that

$$N_Y + 6N \;=\; 0. \tag{2.250}$$

On the other hand, we would like to have a localised chiral mode in the sector $\mathbf{5}_1$ but not in $\overline{\mathbf{5}}_1$. This amounts to requiring that $q_S(\mathbf{5}_1) > 0$ which, using (2.250) translates into

$$N \;>\; 0. \tag{2.251}$$

In addition, we should require that there are localised chiral modes in the sector $\mathbf{10}_i$ but not in $\overline{\mathbf{10}}_i$ for $i = 1, 2, 3$. This can be understood in terms of the condition $q_R(\mathbf{10}_i) > 0$ with is achieved by imposing

$$M + q_Y \tilde{N}_Y \;>\; 0 \quad \text{for} \quad q_Y = \frac{2}{3}, -\frac{1}{6}, -1 \qquad \Rightarrow \qquad -\frac{3}{2} < \frac{\tilde{N}_Y}{M} < 6. \tag{2.252}$$

Non-perturbative effects

Finally, an essential piece of the model are the non-perturbative effects whose source is located at a 4-cycle $S_{\text{np}} \subset X$ defined by a holomorphic divisor function $h(x, y, z)$. As discussed in section 2.3.6 such effects will shift the tree-level superpotential to (2.120), where $\theta_0 = (4\pi^2 m_*)^{-1}[\log h]_{z=0}$. As the specific value for θ_0 depends on S_{np} and hence on the global completion of the local model, we will assume θ_0 to be a general holomorphic function on x, y that near the Yukawa point can be approximated by

$$\theta_0 \;=\; i(\theta_{00} + \theta_x x + \theta_y y). \tag{2.253}$$

In general, the presence of such non-perturbative effects will modify the local E_6 model described above, in the sense that the shift in the 7-brane superpotential modifies the F-term equations. These non-perturbative corrections to the 7-brane background for the E_6 model will be computed in subsection 2.5.3. Nevertheless, as shown in [57] such corrections to the background cancel each other out in the computation of holomorphic Yukawa couplings, and so one may still consider (2.243) and (2.245) for such purpose. Using this fact, in the next section we will show that the effect of (2.120) is to generate a hierarchical rank 3 matrix of up-type holomorphic Yukawa couplings.

2.5.2 Holomorphic Yukawas via residues

The purpose of this section is to compute the holomorphic piece of the $\mathbf{10} \times \mathbf{10} \times \mathbf{5}$ Yukawa couplings for the E_6 model above, and to show that the effect of the non-perturbative superpotential in (2.120) is to increase the rank of this Yukawa matrix from one to three. As pointed out in [53] and reviewed in section 2.3.2, holomorphic Yukawas in intersecting

7-brane models can be computed via an elegant residue formula that only depends on the 7-brane background data around the Yukawa point. Such residue formula was generalised to include the effect of the non-perturbative superpotential (2.120) in section 2.3.6.

In order to apply the residue formula (2.127) to the E_6 model let us first gather the information which is relevant for computing the residue. Clearly, in order to compute the residue we only need to know the details of the model around the Yukawa point $p_{\rm up} = \{x = y = 0\}$, and so the local description of the E_6 model that was given in section 2.5.1 is justified. Moreover, from all the parameters that are involved in the local E_6 model only a few of them are relevant for computing (2.127). In fact, as can be deduced from our previous discussion in section 2.3.6 there are basically only two quantities which are relevant in the computation of the residue: the Higgs background $\langle\Phi^{\rm hol}\rangle$ that solves the equations of motion in the holomorphic gauge and in the absence of non-perturbative effects, and the holomorphic function θ_0 that encodes the information of such effects in the vicinity of the Yukawa point. For the reader's convenience we repeat both quantities here,

$$\langle\Phi^{\rm hol}_{xy}\rangle^{(0)} = m(E^+ + m\,xE^-) + \mu^2(bx-y)Q, \tag{2.254}$$

$$\theta_0 = i(\theta_{00} + \theta_x x + \theta_y y). \tag{2.255}$$

As discussed in section 2.5.1 the Higgs vev (2.254) specifies the two matter curves $\Sigma_{\mathbf{5}}$ and $\Sigma_{\mathbf{10}}$ where the chiral modes of the **5**-plets and **10**-plets are localised. For each of these two sectors we need to specify the pair (h, η) that will enter into the residue formula (2.127), and will couple to each other via the structure constants f_{abc} of E_6.

Sector 5

In this case the matter curve is given by $\Sigma_{\mathbf{5}} = \{bx - y = 0\}$ and there the localised zero modes may arise in two possible sectors: along $E_{\mathbf{5}} = \frac{1}{2}(\sqrt{3}, \underline{1,-1,-1,-1,-1})$ and along $E_{\bar{\mathbf{5}}} = \frac{1}{2}(-\sqrt{3}, \underline{-1,1,1,1,1})$. We will consider the case where, due to the presence of worldvolume fluxes, we have a chiral spectrum and a single zero mode in the sector **5** and none in $\bar{\mathbf{5}}$.[51] The action of (2.254) in this zero mode sector is such that

$$\Psi = 2\mu^2(bx-y). \tag{2.256}$$

It then only remains to specify the value of h for this sector. While in principle $h = h(x, y)$ may be any holomorphic function in the vicinity of $x = y = 0$, one may follow the philosophy in section 2.3.2 and apply a gauge transformation to remove any dependence of h on the complex coordinate $bx - y$. We then have that in this sector h can be taken to be an arbitrary holomorphic function of the orthogonal coordinate $x + by$. Because by assumption we only have one zero mode we will take it to be a constant, following the

[51] For vanishing hypercharge flux, and for the choice (2.244) the condition that chiral modes localised near the Yukawa point arise from the **5** sector is implemented by (2.251), while the fact that this is the only zero mode at this curve depends on the global aspects of the model, and we will take it as an assumption. When introducing the hypercharge flux N_Y and imposing the condition (2.250) the $SU(5)$ spectrum will be broken and there will only be a localised mode in the sector $\mathbf{5}_1$, namely the MSSM Higgs doublet H_u. The holomorphic Yukawas computed in this section will also be valid for the case, with the only replacement $\mathbf{5} \to \mathbf{5}_1$. See sections 2.5.3 and 2.5.4 for more details.

standard practice in the literature. We then have that

$$h_{\mathbf{5}}/\gamma_{\mathbf{5}} = 1 \tag{2.257}$$

$$i\eta_{\mathbf{5}}/\gamma_{\mathbf{5}} = \frac{1}{2\mu^2(bx-y)} - \epsilon\frac{\theta_x + b\theta_y}{4\mu^4(bx-y)^3} + \mathcal{O}(\epsilon^2) \tag{2.258}$$

with $\gamma_{\mathbf{5}}$ a real constant to be computed via wavefunction normalisation in the next section.

Sector 10

In this case the curve is given by $\Sigma_{\mathbf{10}} = \{\mu^4(bx-y)^2 = m^3x\}$ and the localised modes live in the root subspace spanned by (2.233). As before, we will assume that worldvolume fluxes are such there are exactly three chiral zero modes within the subspace spanned by $E_{\mathbf{10}^+} = (0, \underline{1,1,0,0,0})$ and $E_{\mathbf{10}^-} = \frac{1}{2}(-\sqrt{3}, \underline{1,1,-1,-1,-1})$ (see appendix B of [58] for more details) and that these will be our three families of **10**-plets in our $SU(5)$ GUT model.

The action of (2.254) in this sector is such that

$$\Psi\begin{pmatrix} E_{\mathbf{10}^+} \\ E_{\mathbf{10}^-}\end{pmatrix} = \begin{pmatrix} -\mu^2(bx-y) & m \\ m^2x & -\mu^2(bx-y)\end{pmatrix}\begin{pmatrix} E_{\mathbf{10}^+} \\ E_{\mathbf{10}^-}\end{pmatrix} \tag{2.259}$$

as can be read from (2.237). As before we need to specify $h_{\mathbf{10}}$, which now will be an $SU(2)$ doublet of arbitrary holomorphic functions. Again, by performing an appropriate holomorphic gauge transformation we can restrict ourselves to a very particular form for $h_{\mathbf{10}}$ since [53]

$$h_{\mathbf{10}} = \begin{pmatrix} h^+(x,y) \\ h^-(x,y)\end{pmatrix} - i\Psi\begin{pmatrix} \chi^+(x,y) \\ \chi^-(x,y)\end{pmatrix} = \begin{pmatrix} 0 \\ h(bx-y)\end{pmatrix} \tag{2.260}$$

for arbitrary $h^{\pm}$ and appropriate choices of $\chi^{\pm}$. While h can be any holomorphic function on the coordinate $bx-y$, under the assumption that we have three zero modes in this sector we can take them to be the monomials $\gamma^i_{\mathbf{10}}m_*^{3-i}(bx-y)^{3-i}$, with $\gamma^i_{\mathbf{10}}$ some normalisation factors to be fixed in the next section. We finally have that

$$h^i_{\mathbf{10}}/\gamma^i_{\mathbf{10}} = \begin{pmatrix} 0 \\ m_*^{3-i}(bx-y)^{3-i}\end{pmatrix} \tag{2.261}$$

$$\begin{aligned} i\eta^i_{\mathbf{10}}/\gamma^i_{\mathbf{10}} = & -\left[\frac{m_*^{3-i}(bx-y)^{3-i}}{\mu^4(bx-y)^2 - m^3x}\right]\begin{pmatrix} m \\ \mu^2(bx-y)\end{pmatrix} + \mathcal{O}(\epsilon^2) \\ & + \epsilon\frac{2\mu^4(\theta_x+b\theta_y)(bx-y) + m^3\theta_y}{(\mu^4(bx-y)^2 - m^3x)^3}m_*^{3-i}(bx-y)^{3-i}\begin{pmatrix} 2m\mu^2(bx-y) \\ (m^3x + \mu^4(bx-y)^2)\end{pmatrix} \\ & + \epsilon\frac{(\theta_x+b\theta_y)}{(\mu^4(bx-y)^2 - m^3x)^2}m_*^{3-i}(bx-y)^{2-i}\begin{pmatrix} 2m\mu^2(bx-y)(6-i) \\ m^3x(3-i) + (4-i)\mu^4(bx-y)^2\end{pmatrix} \end{aligned} \tag{2.262}$$

which has a rather complicated $\mathcal{O}(\epsilon)$ correction to $\eta^i_{\mathbf{10}}$. Nevertheless, the result that one obtains from applying the residue formula is still quite simple, as we will now see.

$\mathbf{10_M} \times \mathbf{10_M} \times \mathbf{5_U}$ Yukawas

Let us now apply the explicit expressions for $(h_{\mathbf{5}}, \eta_{\mathbf{5}})$ and $(h_{\mathbf{10}}, \eta_{\mathbf{10}})$ to the residue formula (2.127) for the Yukawa couplings. An important simplifications arises from the fact that the structure constants of E_6 satisfy

$$\mathrm{Tr}([E_{\mathbf{5}\,i}, E_{\mathbf{10}\,jk}^{\;M}]E_{\mathbf{10}\,lm}^{\;N}) = \epsilon_{ijklm}\epsilon^{MN} \tag{2.263}$$

where i,j,k,l,m are $\mathfrak{su}(5)$ indices and $M,N = \pm$ are $\mathfrak{su}(2)$ indices. As a result the non-trivial contributions to the $\mathbf{10}_M \times \mathbf{10}_M \times \mathbf{5}_U$ Yukawa will be of the form

$$Y \;=\; m_*^4\pi^2 \mathrm{Res}_{\,(0,0)}\left(\epsilon_{MN}\eta_{\mathbf{5}}\eta_{\mathbf{10}}^M h_{\mathbf{10}}^N\right) \;=\; m_*\pi^2 \mathrm{Res}_{\,(0,0)}\left(\eta_{\mathbf{5}}\eta_{\mathbf{10}}^+ h_{\mathbf{10}}^-\right) \tag{2.264}$$

where the contractions of the $SU(5)$ indices have been left implicit. In the first equality we have used that any other contribution will contain a term of the form $\epsilon_{MN}\eta_{\mathbf{10}}^M\eta_{\mathbf{10}}^N$ and so it will vanish identically, and in the second equality we have used that in our solution (2.261) $h_{\mathbf{10}}^+ = 0$. Hence, even if (2.262) has a complicated expression only the terms proportional to $E_{\mathbf{10}^+}$ will be relevant when computing up-like Yukawa couplings.

Let us proceed by computing (2.264) explicitly. At zeroth order in ϵ we have a contribution of the form

$$\begin{aligned} Y^{ij}_{\text{tree}} \;&=\; m_*^4\pi^2\gamma_{\mathbf{5}}\gamma_{\mathbf{10}}^i\gamma_{\mathbf{10}}^j\,\mathrm{Res}_{\,(0,0)}\left[\frac{m(m_*(bx-y))^{6-i-j}}{2\mu^2(bx-y)(\mu^4(bx-y)^2-m^3x)}\right] \\ &=\; -\frac{m_*^4\pi^2}{2m^2\mu^2}\gamma_{\mathbf{5}}\gamma_{\mathbf{10}}^i\gamma_{\mathbf{10}}^j\,\delta_{i3}\delta_{j3} \end{aligned} \tag{2.265}$$

and so at this level only Y^{33} is non-zero. At order $\mathcal{O}(\epsilon)$ we get a contribution of the form

$$Y^{ij}_{\text{np}} \;=\; \epsilon\,\frac{m_*^6\pi^2}{4m^2\mu^4}\left[b\theta_y+\theta_x\right]\gamma_{\mathbf{5}}\gamma_{\mathbf{10}}^i\gamma_{\mathbf{10}}^j\,\delta_{(i+j)4} \tag{2.266}$$

from the $\mathcal{O}(\epsilon)$ correction to $\eta_{\mathbf{5}}$. In fact, one can check that the $\mathcal{O}(\epsilon)$ correction to $\eta_{\mathbf{10}}$ do not contribute to (2.264) and that we are left with the following $\mathbf{10}_M \times \mathbf{10}_M \times \mathbf{5}_U$ Yukawa couplings,

$$Y^{ij} = \frac{\pi^2\gamma_{\mathbf{5}}}{4\rho_\mu\rho_m}\begin{pmatrix} 0 & 0 & \tilde{\epsilon}\rho_\mu^{-1}\gamma_{\mathbf{10}}^1\gamma_{\mathbf{10}}^3 \\ 0 & \tilde{\epsilon}\rho_\mu^{-1}\gamma_{\mathbf{10}}^2\gamma_{\mathbf{10}}^2 & 0 \\ \tilde{\epsilon}\rho_\mu^{-1}\gamma_{\mathbf{10}}^1\gamma_{\mathbf{10}}^3 & 0 & -2\gamma_{\mathbf{10}}^3\gamma_{\mathbf{10}}^3 \end{pmatrix} + \mathcal{O}(\epsilon^2) \tag{2.267}$$

where we have defined the slope densities

$$\rho_\mu \;=\; \frac{\mu^2}{m_*^2} \qquad\qquad \rho_m \;=\; \frac{m^2}{m_*^2} \tag{2.268}$$

as well as the non-perturbative parameter

$$\tilde{\epsilon} \;=\; \epsilon\,(\theta_x + b\theta_y). \tag{2.269}$$

As claimed, we obtain a Yukawa matrix such that in the absence of non-perturbative effects has rank one, but when taking them into account it increases its rank to three.[52]

[52]More precisely, the condition for rank enhancement is that $\tilde{\epsilon} \neq 0$, which seems to indicate that the pull-back of θ_0 along $\Sigma_{\mathbf{5}}$ must be non-trivial.

Note that the eigenvalues of this matrix display a hierarchical structure $(\mathcal{O}(1), \mathcal{O}(\tilde{\epsilon}), \mathcal{O}(\tilde{\epsilon}^2))$, as we will discuss in more detail in section 2.5.4.

An interesting feature of this Yukawa matrix it that its entries depend on very few parameters of the model, most notably $\tilde{\epsilon}$, ρ_μ and $\gamma^i_{\mathbf{10}}$. In fact the last set of parameters can be understood as wavefunction normalisation constants that cannot be determined from the analysis of this section. Instead, they can be calculated by computing zero mode fluctuations in a physical background and demanding that their 4d kinetic terms are canonically normalised, which is the task that we will endeavor in the next section. As we will see, $\gamma^i_{\mathbf{10}}$ will depend on the worldvolume flux densities of the model, and in particular in the hypercharge flux densities in (2.241). As U-quarks with different hypercharge feel F_Y differently, $\gamma^i_{\mathbf{10}}$ will take different values for each of them, and this will give rise to a rich structure of physical Yukawa couplings to be analysed in section 2.5.4.

2.5.3 Zero mode wavefunctions at the E_6 point

An remarkable aspect of the computations of the last section is that, in order to arrive at the Yukawa matrix (2.267), we did not have to fully solve for the chiral zero mode wavefunctions. Instead, we solved for the F-term equations and used the invariance of the superpotential under complexified gauge transformations. The price to pay for using that trick is that we do not have any physical criterium to fix the constants $\gamma_{\mathbf{5}}$, $\gamma^i_{\mathbf{10}}$ that appear in the Yukawa matrix because the wavefunctions that we are using are not in a physical gauge. As pointed out in [53] this is because via our previous computation we are only computing the holomorphic piece of the Yukawa couplings, and not their actual physical values. In order to compute physical Yukawa couplings we also need to solve the D-term equations for the zero mode wavefunctions and the demand that their corresponding 4d fluctuations have canonically normalised kinetic terms. This will fix the constants $\gamma_{\mathbf{5}}$, $\gamma^i_{\mathbf{10}}$ in terms of the data of the local model and provide us with the physical Yukawa matrix to be analysed in the next section.

As we will see, solving analytically for the zero mode D-term equations is a rather involved task, mainly because they involve the Painlevé transcendent f found in subsection 2.3.3. Nevertheless, we will be able to do so for a certain region of parameters of our local model, and we expect that our general conclusions are valid for other regions as well. We will first compute these physical wavefunctions in the absence of non-perturbative effects, which will already allow us to compute the normalisation factors $\gamma_{\mathbf{5}}$, $\gamma^i_{\mathbf{10}}$ to a good approximation. We will then include the corrections induced by non-perturbative effects, in the spirit of [49, 57], as explained earlier. As a cross-check of our results, we will use the corrected wavefunctions to rederive the Yukawa matrix (2.267), now with the factors $\gamma_{\mathbf{5}}$, $\gamma^i_{\mathbf{10}}$ fixed.

Perturbative zero-modes

In the absence of non-perturbative effects (i.e., $\epsilon = 0$) the zero mode equations reduce to (2.62-2.63) which can be solved in real gauge as explained in section 2.3.4. In fact, while the above equations are written for bosonic fluctuations, the same equations apply for the 7-brane fermionic zero modes, pairing up into 4d $\mathcal{N} = 1$ chiral multiplets $(a_{\bar{m}}, \psi_{\bar{m}})$ and $(\varphi_{xy}, \chi_{xy})$ with the same internal profile. In the following we will display the solutions to these equations for both the **5** and **10** sectors of the E_6 model. The details of the computation can be found in appendix A of [58].

- *Sector* 5_U

Following the results of section 2.3.4 one can easily solve the equations of motion, obtaining

$$\vec{\Psi}_{\mathbf{5}} = \gamma_{\mathbf{5}} \begin{pmatrix} i\frac{\zeta_{\mathbf{5}}}{2\mu^2} \\ i\frac{(\zeta_{\mathbf{5}}-\lambda_{\mathbf{5}})}{2\mu^2} \\ 1 \end{pmatrix} \chi_{\mathbf{5}}, \qquad \chi_{\mathbf{5}} = e^{\frac{q_R}{2}(|x|^2-|y|^2)-q_S(x\bar{y}+y\bar{x})+(x-y)(\zeta_{\mathbf{5}}\bar{x}-(\lambda_{\mathbf{5}}-\zeta_{\mathbf{5}})\bar{y}))} \tag{2.270}$$

with $\lambda_{\mathbf{5}}$ the lowest solution to

$$\lambda_{\mathbf{5}}^3 - (8\mu^4 + (q_R)^2 + (q_S)^2)\lambda_{\mathbf{5}} + 8\mu^4 q_S = 0 \tag{2.271}$$

and $\zeta_{\mathbf{5}} = \frac{\lambda_{\mathbf{5}}(\lambda_{\mathbf{5}}-q_R-q_S)}{2(\lambda_{\mathbf{5}}-q_S)}$, see appendix A for further details.

Notice that, because they depend on the hypercharge flux, q_R and q_S take different values for the two subsectors $\mathbf{5}_1$ and $\mathbf{5}_2$ of table 2.7, and so the same is true for $\lambda_{\mathbf{5}}$, $\zeta_{\mathbf{5}}$. In particular, imposing (2.250) we find that $q_S(\mathbf{5}_2) = 0$ and that the wavefunction for this sector is not localised along $\Sigma_{\mathbf{5}}$, as we briefly comment below.

- *Sector* 10_M

This sector is more involved because the zero modes lie along the root subspace spanned by $E_{\mathbf{10}^+} = (0, \underline{1,1,0,0,0})$ and $E_{\mathbf{10}^-} = \frac{1}{2}(-\sqrt{3}, \underline{1,1,-1,-1,-1})$ and so the appropriate ansatz is

$$\begin{pmatrix} a_{\bar{x}} \\ a_{\bar{y}} \\ \varphi_{xy} \end{pmatrix} = \vec{\Psi}_{\mathbf{10}^+} E_{\mathbf{10}^+} + \vec{\Psi}_{\mathbf{10}^-} E_{\mathbf{10}^-}. \tag{2.272}$$

Because $E_{\mathbf{10}^\pm}$ transform as a doublet of the $SU(2)$ generated by $\{E^+, E^-, P\}$, c.f.(2.234), it is useful to represent these wavefunction components with the following doublet notation

$$a = \begin{pmatrix} a^+ \\ a^- \end{pmatrix} \qquad \varphi = \begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix} \tag{2.273}$$

where $a_{\bar{x}}^\pm$, $a_{\bar{y}}^\pm$, $\varphi_{xy}^\pm$ belong respectively to $\vec{\Psi}_{\mathbf{10}^\pm}$. Then, following the strategy in section 2.3.4, we use the solution for the F-terms equations (2.62) to write a in terms of φ, and then substitute in the D-term equation (2.63) to find an equation for Ψ.

It is instructive to first consider the case where the primitive flux $\langle F_p \rangle$ in (2.247) is absent. The solution to the F-term equations is in fact quite similar to the one found in the previous section in the holomorphic gauge and reads

$$a = e^{fP/2}\bar{\partial}\xi \tag{2.274a}$$

$$\varphi = e^{fP/2}\left(h - i\Psi\xi\right) \tag{2.274b}$$

where ξ and h are also doublets with components $\xi^\pm$ and $h^\pm$ and

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \Psi = \begin{pmatrix} -\mu^2(x-y) & m \\ m^2 x & -\mu^2(x-y). \end{pmatrix} \tag{2.275}$$

In particular, notice that Ψ is the same matrix as in (2.259) after taking the choice (2.244). From (2.274b) we obtain

$$\xi = i\Psi^{-1}\left(e^{-fP/2}\varphi - h\right). \tag{2.276}$$

Finally, the D-term equation for the fluctuations (2.63) reads

$$\partial_x a_{\bar{x}} + \partial_y a_{\bar{y}} + \frac{1}{2}\partial_x f P a_{\bar{x}} - ie^{-fP/2}\Psi^\dagger e^{fP/2}\varphi = 0 \tag{2.277}$$

which by using (2.274a), and recalling that f only depends on $x, \bar{x}$, we find

$$\partial_x\partial_{\bar{x}}\xi + \partial_y\partial_{\bar{y}}\xi + \partial_x f P\partial_{\bar{x}}\xi - i\Lambda^\dagger\left(h - i\Psi\xi\right) = 0 \tag{2.278}$$

where we have defined

$$\Lambda = e^{fP}\Psi e^{-fP} = \begin{pmatrix} -\mu^2(x-y) & me^{2f} \\ m^2xe^{-2f} & -\mu^2(x-y) \end{pmatrix}. \tag{2.279}$$

To proceed it is convenient to make the following change of variables

$$U = e^{-fP/2}\varphi \qquad \Rightarrow \qquad \xi = i\Psi^{-1}\left(U - h\right) \tag{2.280}$$

and express (2.278) entirely in terms of the doublet U

$$\partial_x\partial_{\bar{x}}U + \partial_y\partial_{\bar{y}}U - (\partial_x\Psi)\Psi^{-1}\partial_{\bar{x}}U + (\partial_y\Psi)\Psi^{-1}\partial_{\bar{y}}U + \partial_x f\Psi P\Psi^{-1}\partial_{\bar{x}}U - \Psi\Lambda^\dagger U = 0 \tag{2.281}$$

so that the dependence on h drops completely. However, the D-term equation gives a coupled system of equations for U^+ and U^- that are quite involved to solve. Nevertheless, in the limit $m \gg \mu$ they decouple and one can prove that there is no localised mode for U^+, which we henceforth set to zero. Moreover, near the Yukawa point $p_{\rm up} = \{x = y = 0\}$ one can approximate $f = \log c + c^2m^2x\bar{x} + \ldots$ and solve analytically for U^-, finding $U^- = \exp(\lambda_{\mathbf{10}}x\bar{x})h$ with $\lambda_{\mathbf{10}}$ the negative solution to $c^2\lambda_{\mathbf{10}}^3 + 4c^4m^2\lambda_{\mathbf{10}}^2 - m^4\lambda_{\mathbf{10}} = 0$. At the end one finds the solution

$$\vec{\Psi}^j_{\mathbf{10}^+} = \gamma^j_{\mathbf{10}}\begin{pmatrix} \frac{i\lambda_{\mathbf{10}}}{m^2} \\ 0 \\ 0 \end{pmatrix} e^{f/2}\chi^j_{\mathbf{10}} \qquad \vec{\Psi}^j_{\mathbf{10}^-} = \gamma^j_{\mathbf{10}}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-f/2}\chi^j_{\mathbf{10}} \tag{2.282}$$

where $e^{f/2} = \sqrt{c}\, e^{m^2c^2x\bar{x}/2}$ and $\chi^j_{\mathbf{10}} = e^{\lambda_{\mathbf{10}}x\bar{x}}\, g_j(y)$, with g_j holomorphic functions of y.

Switching on the primitive worldvolume fluxes amounts to replacing $\partial_{x,y} \to D_{x,y}$ in the D-term equation, with $D_{x,y}$ the covariant derivatives corresponding to this flux, and similarly for $\bar{\partial}$ in the F-term equations. Still, in the limit $m \gg \mu$ and near the origin one finds a localised solution for U^- and the wavefunctions read

$$\vec{\Psi}^j_{\mathbf{10}^+} = \gamma^j_{\mathbf{10}}\begin{pmatrix} \frac{i\lambda_{\mathbf{10}}}{m^2} \\ -\frac{i\lambda_{\mathbf{10}}\zeta_{\mathbf{10}}}{m^2} \\ 0 \end{pmatrix} e^{f/2}\chi^j_{\mathbf{10}} \qquad \vec{\Psi}^j_{\mathbf{10}^-} = \gamma^j_{\mathbf{10}}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-f/2}\chi^j_{\mathbf{10}} \tag{2.283}$$

where $\lambda_{\mathbf{10}}$ is the negative solution to

$$m^4(\lambda_{\mathbf{10}} - q_R) + \lambda c^2\left(c^2m^2(q_R - \lambda_{\mathbf{10}}) - \lambda_{\mathbf{10}}^2 + q_R^2 + q_S^2\right) = 0 \tag{2.284}$$

and $\zeta_{\mathbf{10}} = -q_S/(\lambda_{\mathbf{10}} - q_R)$. The scalar wavefunctions $\chi_{\mathbf{10}}$ read

$$\chi^j_{\mathbf{10}} = e^{\frac{q_R}{2}(|x|^2-|y|^2)-q_S(x\bar{y}+y\bar{x})+\lambda_{\mathbf{10}}x(\bar{x}-\zeta_{\mathbf{10}}\bar{y})}\, g_j(y+\zeta_{\mathbf{10}}x) \tag{2.285}$$

where g_j holomorphic functions of $y+\zeta_{\mathbf{10}}x$, and $j=1,2,3$ label the different zero mode families. As usual, we choose such holomorphic representatives to be

$$g_j = m_*^{3-j}(y+\zeta_{\mathbf{10}}x)^{3-j}. \tag{2.286}$$

Finally, notice that within each family the wavefunctions differ for each of the sectors $\mathbf{10}_{1,2,3}$ of table 2.7 because they have different hypercharges and so q_R and q_S take different values for each. From the results of the previous sections we expect that this difference will only appear in the physical Yukawa couplings via different normalisation factors $\gamma^j_{\mathbf{10}}$, which we now proceed to discuss.

Normalisation factors

Having obtained explicit expressions for the zero mode wavefunctions one may now require that the 4d chiral modes have canonically normalised kinetic terms. The 4d kinetic terms for the wavefunctions of a sector ρ that one obtains via dimensional reduction are

$$K^{ij}_\rho = \langle \vec{\varphi}^{\,i}_\rho | \vec{\varphi}^{\,j}_\rho \rangle = m_*^4 \int_S \mathrm{Tr}\,(\vec{\varphi}^{\,i\,\dagger}_\rho \cdot \vec{\varphi}^{\,j}_\rho)\, \mathrm{dvol}_S \tag{2.287}$$

where i, j are family indices. To have canonically normalised kinetic terms we need to impose that $K^{ij}_\rho = \delta^{ij}$. In the case of the sector $\rho = \mathbf{5}$ there is only one family and we can easily achieve canonical kinetic terms by adjusting the value of the constant $\gamma_{\mathbf{5}}$. In this case the integral (2.287) reads

$$K_{\mathbf{5}} = m_*^4 |\gamma_{\mathbf{5}}|^2 ||\vec{v}_{\mathbf{5}}||^2 \int_S \chi^*_{\mathbf{5}} \chi_{\mathbf{5}}\, \mathrm{dvol}_S \tag{2.288}$$

with $\chi_{\mathbf{5}}$ given by (2.270), and $\vec{v}_{\mathbf{5}} = \frac{1}{2\mu^2}(i\zeta_{\mathbf{5}},\, i(\zeta_{\mathbf{5}}-\lambda_{\mathbf{5}}),\, 2\mu^2)^t$. Due to the convergence properties of $\chi_{\mathbf{5}}$ we can compute the above integral by extending the patch in which we define our local model to $\mathbb{C}^2$. We find that the required value for $\gamma_{\mathbf{5}}$ is

$$|\gamma_{\mathbf{5}}|^2 = -\frac{4}{\pi^2}\left(\frac{\mu}{m_*}\right)^4 \frac{(2\zeta_{\mathbf{5}}+q_R)(q_R+2\zeta_{\mathbf{5}}-2\lambda_{\mathbf{5}})+(q_S+\lambda_{\mathbf{5}})^2}{4\mu^4+\zeta_{\mathbf{5}}^2+(\zeta_{\mathbf{5}}-\lambda_{\mathbf{5}})^2}. \tag{2.289}$$

Here $\lambda_{\mathbf{5}}$ and $\zeta_{\mathbf{5}}$ are defined as in (2.270) and so depend on the worldvolume flux densities q_R and q_S, which are given in table 2.7 for both sectors $\mathbf{5}_1$ and $\mathbf{5}_2$. Hence in general both members of the $\mathbf{5}$-plet have different normalisation factors. In fact, for the sector $\mathbf{5}_2 = (\mathbf{3},\mathbf{1})_{1/3}$ that could contain a Higgs triplet we find that $\gamma_{\mathbf{5}_2}=0$ after we impose the condition (2.250).[53] That is because the integrand in (2.287) is not localised along the curve $\{x=y\}\subset\mathbb{C}^2$, which is turn related to the fact that this is a non-chiral sector of the model and one may assume that it only contains massive modes.

[53]For $q_S=0$ the parameter $\zeta_{\mathbf{5}}$ defined below eq.(2.270) reduces to $\zeta_{\mathbf{5}} = \frac{1}{2}(\lambda_{\mathbf{5}}-q_R)$ which upon substituting in (2.289) shows that $\gamma_{\mathbf{5}_2}=0$.

Notice that for the **10** sector (2.287) reads

$$\begin{aligned} K_{\mathbf{10}}^{ij} &= m_*^4 \int_S \mathrm{Tr}(\vec{\varphi}^{\,i}_{\mathbf{10}^+}{}^\dagger \vec{\varphi}^{\,j}_{\mathbf{10}^+} + \vec{\varphi}^{\,i}_{\mathbf{10}^-}{}^\dagger \vec{\varphi}^{\,j}_{\mathbf{10}^-}) \mathrm{dvol}_S \\ &= m_*^4 (\gamma_{\mathbf{10}}^i)^* \gamma_{\mathbf{10}}^j \sum_{\kappa=\pm} ||\vec{v}_{\mathbf{10}^\kappa}||^2 \int_S e^{\kappa f} (\chi_{\mathbf{10}}^i)^* \chi_{\mathbf{10}}^j \, \mathrm{dvol}_S \end{aligned} \tag{2.290}$$

with the vectors $\vec{v}_{\mathbf{10}^\pm}$ defined in (2.283). Because the integrand needs to be invariant under the rotation $(x,y) \to e^{i\alpha}(x,y)$ to have a non-vanishing result we deduce that $K_{\mathbf{10}}^{ij} = 0$ for $i \neq j$, and so we only need to adjust the constants $\gamma_{\mathbf{10}}^j$ in order to have canonical kinetic terms in this sector. In particular we obtain that the required result is

$$|\gamma_{\mathbf{10}}^j|^2 = -\frac{c}{m_*^2 \pi^2 (3-j)!} \frac{1}{\frac{1}{2\lambda_{\mathbf{10}} + q_R(1+\zeta_{\mathbf{10}}^2) - m^2 c^2} + \frac{c^2 \lambda_{\mathbf{10}}^2}{m^4} \frac{1}{2\lambda_{\mathbf{10}} + q_R(1+\zeta_{\mathbf{10}}^2) + m^2 c^2}} \left(\frac{q_R}{m_*^2}\right)^{4-j} \tag{2.291}$$

which not only depends on the family index j, but also on the sectors $\mathbf{10}_{1,2,3}$ of table 2.7, again via the flux densities q_R and q_S and the quantities $\lambda_{\mathbf{10}}$, $\zeta_{\mathbf{10}}$ that depend on them. Finally, notice that the effects of the non-primitive flux (2.247) in this sector appear through the dependence on the constant c.

Non-perturbative corrections

Let us now see how the presence of non-perturbative effects modifies the above wavefunction profile. As stated before, at the level of approximation that we are working these effects amount to adding the term proportional to ϵ in the F-term equation. This will modify the 7-brane background $\langle \Phi \rangle$ and $\langle A \rangle$ as well as the wavefunction profiles that were just computed for $\epsilon = 0$. These deformations are particularly involved for the T-brane sector of our background and as a consequence for the wavefunctions of the **10** matter curve. Nevertheless, as we will see the $\mathcal{O}(\epsilon)$ corrections to the **10**-plet wavefunctions only affect the Yukawa couplings at $\mathcal{O}(\epsilon^2)$, and so they can be neglected to the level of approximation of (2.267). In the next section we will see that we can reproduce (2.267) via the triple overlap of the $\mathcal{O}(\epsilon)$ corrected wavefunctions, now with explicit expressions for the normalisation factors $\gamma_{\mathbf{5}}$, $\gamma_{\mathbf{10}}^i$.

- *Corrections to the background*

Following section 2.5.2, we can solve the equations of motion for the background for $\epsilon \neq 0$ in the holomorphic gauge if we take $\langle A_{0,1} \rangle = 0$. Then, the Higgs background reads

$$\langle \Phi \rangle = \langle \Phi \rangle^{(0)} + \epsilon \partial \theta_0 \wedge \langle A^{1,0} \rangle^{(0)} \tag{2.292}$$

with $\langle \Phi \rangle^{(0)}$ and $\langle A^{1,0} \rangle^{(0)}$ the solutions to the equations of motion when $\epsilon = 0$ in holomorphic gauge, so in particular $\langle A^{1,0} \rangle^{(0)} = i\partial f\, P$ if we assume vanishing primitive fluxes. Thus, in our case,

$$\langle \Phi_{xy} \rangle = m(E^+ + m\, x E^-) + \epsilon\, \theta_y \partial_x f\, P + \mu^2 (x-y) Q \tag{2.293}$$

where we have used that $f = f(x, \bar{x})$ and taken the choice (2.244). One may now perform a complexified gauge transformation in order to go to a real gauge that satisfies the D-term up to $\mathcal{O}(\epsilon^2)$. For this we try the ansatz

$$g = e^{\frac{f}{2}P}e^{\frac{\epsilon}{2}(kE^+ + k^*E^-)} = e^{\frac{f}{2}P} + \frac{\epsilon}{2}(k\,e^{f/2}E^+ + k^*e^{-f/2}E^-) + \mathcal{O}(\epsilon^2) \tag{2.294}$$

with f as above and k a complex function of $x, \bar{x}$. From this transformation we obtain the physical background

$$\begin{aligned} \langle\Phi_{xy}\rangle &= m(e^f E^+ + m\,xe^{-f}E^-) + \epsilon\left[\theta_y\partial_x f + \frac{m}{2}(mxk - k^*)\right]P + \mu^2(x-y)Q + \mathcal{O}(\epsilon^2) \\ \langle A^{0,1}\rangle &= -\frac{i}{2}\bar{\partial}fP - i\frac{\epsilon}{2}\left(\bar{\partial}k\,e^f E^+ + \bar{\partial}k^*\,e^{-f}E^-\right) + \mathcal{O}(\epsilon^2) \end{aligned} \tag{2.295}$$

Inserting (2.295) into the D-term equation we recover that f has to satisfy the Painlevé equation while k satisfies a more complicated differential equation, namely

$$\begin{aligned} \cosh f\,\partial_x\partial_{\bar{x}}k + \partial_x f\partial_{\bar{x}}k\,e^f - \partial_x k\partial_{\bar{x}}f\,e^{-f} + me^f(m^2\bar{x}k^* - mk + 2\bar{\theta}_y\partial_{\bar{x}}f) \\ -m^2\bar{x}e^{-f}(m^2xk - mk^* + 2\theta_y\partial_x f) = 0. \end{aligned} \tag{2.296}$$

Using that near the origin $f = \log c + m^2c^2x\bar{x} + \ldots$ we find the solution

$$k = \bar{\theta}_y\,c^2mx + \theta_y\,\frac{1-c^2}{2c^2-1}m^2\bar{x}^2 + \ldots \tag{2.297}$$

where the dots stand for higher powers of $x, \bar{x}$.

Finally, let us restore the presence of primitive fluxes (2.247). As these fluxes commute with all the other elements of the background their presence does not modify the discussion above, and we can add their contribution to the corrected background independently. At the ends one finds

$$\begin{aligned} \langle\Phi_{xy}\rangle &= m(E^+ + m\,xE^-) + \epsilon\left[\theta_y\partial_x f + \frac{m}{2}(mxk - k^*)\right]P \qquad (2.298) \\ &\quad +\mu^2(x-y)Q + \ \epsilon\left[\theta_y(\bar{x}Q_R - \bar{y}Q_S) + \theta_x(\bar{x}Q_S + \bar{y}Q_R)\right] + \mathcal{O}(\epsilon^2) \\ \langle A^{0,1}\rangle &= \langle A_p^{0,1}\rangle - \frac{i}{2}\bar{\partial}fP - i\frac{\epsilon}{2}\left(\bar{\partial}k\,e^f E^+ + \bar{\partial}k^*\,e^{-f}E^-\right) + \mathcal{O}(\epsilon^2) \qquad (2.299) \end{aligned}$$

where $\langle A_p^{0,1}\rangle$ stands for the potential of the primitive flux (2.247) in a physical gauge. Notice that the $\mathcal{O}(\epsilon)$ corrections to the worldvolume flux lie along the non-commuting generators $E^\pm$, while for the Higgs background they lie along the Cartan of E_6.

In the following we solve for the wavefunctions for the pair (a, φ) and at first order in the non-perturbative parameter ϵ. That is, we will be looking for solutions to the system

$$\bar{\partial}_{\langle A\rangle}a = \mathcal{O}(\epsilon^2) \tag{2.300a}$$

$$\bar{\partial}_{\langle A\rangle}\varphi - i[a, \langle\Phi\rangle] + \epsilon\partial\theta_0 \wedge \partial_{\langle A\rangle}a = \mathcal{O}(\epsilon^2) \tag{2.300b}$$

$$\omega \wedge \partial_{\langle A\rangle}a - \frac{1}{2}[\langle\bar{\Phi}\rangle, \varphi] = \mathcal{O}(\epsilon^2) \tag{2.300c}$$

where $\langle A\rangle$ and $\langle\Phi\rangle$ are respectively specified by (2.298) and (2.299).

- *Sector 5*

The sector **5** is relatively simple due to the fact that its zero modes are not charged under the generators of the $\mathfrak{su}(2)$ algebra $\{E^\pm, P\}$. More precisely, for this sector $\langle A^{0,1}\rangle$

reduces to $\langle A_p^{0,1}\rangle$, and $\langle \Phi_{xy}\rangle$ to the second line of (2.298). As a result, solving the zero mode equations (2.300) for this sector is very similar to the analogous problem for the $SO(12)$ local model. Hence in the following we simply present the final result.

The solution to the non-perturbative zero mode equations is given by

$$\vec{\Psi}_{\mathbf{5}} = \gamma_{\mathbf{5}} \begin{pmatrix} i\frac{\zeta_{\mathbf{5}}}{2\mu^2} \\ i\frac{(\zeta_{\mathbf{5}}-\lambda_{\mathbf{5}})}{2\mu^2} \\ 1 \end{pmatrix} \chi_{\mathbf{5}}^{\text{np}}, \qquad \chi_{\mathbf{5}}^{\text{np}} = e^{\frac{q_R}{2}(|x|^2-|y|^2)-q_S(x\bar{y}+y\bar{x})+(x-y)(\zeta_{\mathbf{5}}\bar{x}-(\lambda_{\mathbf{5}}-\zeta_{\mathbf{5}})\bar{y}))}(1+\epsilon\Upsilon_{\mathbf{5}}) \tag{2.301}$$

with $\lambda_{\mathbf{5}}$, $\zeta_{\mathbf{5}}$ defined as in (2.270). The $\mathcal{O}(\epsilon)$ non-perturbative correction is

$$\Upsilon_{\mathbf{5}} = -\frac{1}{4\mu^2}(\zeta_{\mathbf{5}}\bar{x}-(\lambda_{\mathbf{5}}-\zeta_{\mathbf{5}})\bar{y})^2(\theta_x+\theta_y)+\frac{\delta_1}{2}(x-y)^2+\frac{\delta_2}{\zeta_{\mathbf{5}}}(x-y)(\zeta_{\mathbf{5}}y+(\lambda_{\mathbf{5}}-\zeta_{\mathbf{5}})x) \tag{2.302}$$

with the constants δ_1, δ_2 given by

$$\delta_1 = \frac{2\mu^2}{\lambda_{\mathbf{5}}^2}\{\bar{\theta}_x(q_R(\zeta_{\mathbf{5}}-\lambda_{\mathbf{5}})+q_S\zeta_{\mathbf{5}})+\bar{\theta}_y(q_R\zeta_{\mathbf{5}}-q_S(\zeta_{\mathbf{5}}-\lambda_{\mathbf{5}}))\} \tag{2.303}$$

$$\delta_2 = \frac{2\mu^2\zeta_{\mathbf{5}}}{\lambda_{\mathbf{5}}^2}\{\bar{\theta}_x(q_R+q_S)+\bar{\theta}_y(q_R-q_S)\}. \tag{2.304}$$

As in the $SO(12)$ model one can check that the corrections to the norm (2.289) only appear at $\mathcal{O}(\epsilon^2)$, because $\mathcal{O}(\epsilon)$ terms that appear in the integrand of (2.288) are not invariant under the rotation $(x,y)\to e^{i\alpha}(x,y)$.

- *Sector 10*

Similarly to the case of perturbative zero modes, finding the non-perturbative corrections to the wavefunctions of the sector **10** is in general rather involved. Nevertheless, taking the same approximations as in the perturbative case, one may understand how this corrections look like and argue that they will not be relevant for computing the matrix of physical Yukawa couplings.

The first step is to switch off the primitive fluxes and realise that, in the same way that $a=\bar{\partial}\xi$ and $\varphi=(h-i\Psi\xi-\epsilon\partial\theta_0\wedge\partial\xi)$ solve the F-term equations (2.300a) and (2.300b) in the holomorphic gauge, in the real gauge they are satisfied by

$$a = g\,\bar{\partial}\xi \tag{2.305a}$$

$$\varphi = g\,(h-i\Psi\xi-\epsilon\partial\theta_0\wedge\partial\xi) \;=\; g\,U\,dx\wedge dy \tag{2.305b}$$

with g given by (2.294) and Ψ given by (2.259). Here a, φ, χ are $SU(2)$ doublets as in eq.(2.274). The same applies to U, which can be expanded in powers of ϵ as

$$U \;=\; U^{(0)}+\epsilon\,U^{(1)}+\,\mathcal{O}(\epsilon^2) \tag{2.306}$$

where $U^{(0)}$ corresponds to solution found for $\epsilon=0$, namely

$$U_-^{(0)} \;=\; e^{\lambda_{\mathbf{10}}x\bar{x}}h(y) \qquad U_+^{(0)} \;=\; 0 \tag{2.307}$$

Then, one may solve for ξ as

$$\begin{aligned} \xi \;&=\; \xi^{(0)}+i\epsilon\Psi^{-1}\left[U^{(1)}+\partial_x\theta_0\partial_y\xi^{(0)}-\partial_y\theta_0\partial_x\xi^{(0)}\right]+\mathcal{O}(\epsilon^2) \\ \xi^{(0)} \;&=\; i\Psi^{-1}(U^{(0)}-h) \end{aligned} \tag{2.308}$$

and then solve for $U^{(1)}$ by inserting this expression into the D-term equation (2.300c). As in the perturbative case this problem can be easily solved in the limit $\mu \to 0$, obtaining that $U_-^{(1)} = 0$. As a result, in this limit we have the structure

$$\xi_+ \;=\; \xi_+^{(0)} + 0 + \mathcal{O}(\epsilon^2) \qquad\qquad \xi_- \;=\; 0 + \epsilon\,\xi_-^{(1)} + \mathcal{O}(\epsilon^2), \tag{2.309}$$

that is, the $\mathcal{O}(\epsilon)$ corrections to ξ are contained in the opposite doublet as the tree-level contribution. The same statement applies to a and φ. Indeed, we have that

$$\varphi_{xy} \;=\; g^{(0)}U^{(0)} + \epsilon\left(g^{(0)}U^{(1)} + g^{(1)}U^{(0)}\right) + \mathcal{O}(\epsilon^2) \tag{2.310}$$

where we have decomposed $g = g^{(0)} + \epsilon g^{(1)} + \mathcal{O}(\epsilon^2)$ as in (2.294). Then, because $g^{(0)}$ only involves P and $g^{(1)}$ involves $E^{\pm}$ we have

$$\varphi_+ \;=\; 0 + \epsilon\,\varphi_+^{(1)} + \mathcal{O}(\epsilon^2) \qquad\qquad \varphi_- \;=\; \varphi_-^{(0)} + 0 + \mathcal{O}(\epsilon^2) \tag{2.311}$$

Finally, a similar argument shows that $a_+ = a_+^{(0)} + \mathcal{O}(\epsilon^2)$ and $a_- = a_-^{(1)} + \mathcal{O}(\epsilon^2)$ and so the wavefunctions (2.283) have a correction of the form

$$\vec{\Psi}_{\mathbf{10}^+} \;=\; \begin{pmatrix} \bullet \\ \bullet \\ 0 \end{pmatrix} + \epsilon \begin{pmatrix} 0 \\ 0 \\ \bullet \end{pmatrix} + \mathcal{O}(\epsilon^2) \qquad \vec{\Psi}_{\mathbf{10}^-} \;=\; \begin{pmatrix} 0 \\ 0 \\ \bullet \end{pmatrix} + \epsilon \begin{pmatrix} \bullet \\ \bullet \\ 0 \end{pmatrix} + \mathcal{O}(\epsilon^2) \tag{2.312}$$

One can check that this structure remains even after we restore the presence of non-primitive fluxes. Then, since the $\mathcal{O}(\epsilon)$ correction vector is orthogonal to the 0^{th}-order solution, it is easy to see that no $\mathcal{O}(\epsilon)$ correction to the normalisation factors $\gamma_{\mathbf{10}}^j$ arises by plugging these corrected wavefunctions into (2.291).

2.5.4 Physical Yukawas and mass hierarchies

Given the above solutions for the non-perturbative wavefunctions one can insert them into (2.57) and compute their triple overlap to obtain the matrix of physical Yukawa couplings, i.e. the Yukawas in a basis where 4d kinetic terms are canonically normalised.

As we will see below the final result for the U-quark Yukawa matrix is

$$Y_U \;=\; \frac{\pi^2\gamma_{\mathbf{5}}}{4\rho_\mu\rho_m}\begin{pmatrix} 0 & 0 & \tilde{\epsilon}\rho_\mu^{-1}\gamma_L^1\gamma_R^3 \\ 0 & \tilde{\epsilon}\rho_\mu^{-1}\gamma_L^2\gamma_R^2 & 0 \\ \tilde{\epsilon}\rho_\mu^{-1}\gamma_L^3\gamma_R^1 & 0 & -2\gamma_L^3\gamma_R^3 \end{pmatrix} + \mathcal{O}(\tilde{\epsilon}^2) \tag{2.313}$$

where

$$\rho_\mu \;=\; \frac{\mu^2}{m_*^2} \qquad\qquad \rho_m \;=\; \frac{m^2}{m_*^2} \qquad\qquad \tilde{\epsilon} \;=\; \epsilon\,(\theta_x + \theta_y) \tag{2.314}$$

are all flux-independent parameters. The worldvolume flux dependence (and in particular the hypercharge dependence) is encoded in the normalisation factors $\gamma_{\mathbf{5}}$ and $\gamma_{R,L}^i$, where $\gamma_{\mathbf{5}}$ is given by (2.289) with the values of q_R, q_S for the sector $\mathbf{5}_1$ of table 2.7. Finally, γ_R^i is given by (2.291) using the values of q_R and q_S in the first row of table 2.7, and similarly for γ_L^i with the values in the second row.

We would like to see if this structure for Yukawa couplings allows to fit experimental fermion masses. Since our expressions apply at the GUT scale, presumably of order 10^{16} GeV, the data needs to be run up to this scale. Table 2.8 shows the result of doing so for the MSSM quark mass ratios, for different values of $\tan\beta$ as taken from ref. [70]. In the following we will analyse if this spectrum can be accommodated in our scheme.

$\tan\beta$	10	38	50
m_u/m_c	$2.7 \pm 0.6 \times 10^{-3}$	$2.7 \pm 0.6 \times 10^{-3}$	$2.7 \pm 0.6 \times 10^{-3}$
m_c/m_t	$2.5 \pm 0.2 \times 10^{-3}$	$2.4 \pm 0.2 \times 10^{-3}$	$2.3 \pm 0.2 \times 10^{-3}$
m_d/m_s	$5.1 \pm 0.7 \times 10^{-2}$	$5.1 \pm 0.7 \times 10^{-2}$	$5.1 \pm 0.7 \times 10^{-2}$
m_s/m_b	$1.9 \pm 0.2 \times 10^{-2}$	$1.7 \pm 0.2 \times 10^{-2}$	$1.6 \pm 0.2 \times 10^{-2}$
Y_t	0.48 ± 0.02	0.49 ± 0.02	0.51 ± 0.04
Y_b	0.051 ± 0.002	0.23 ± 0.01	0.37 ± 0.02

Table 2.8: Running mass ratios of quarks at the unification scale and for different values of *tan* β, as taken from ref. [70]. The Yukawa couplings $Y_{t,b}$ at the unification scale are also shown.

The physical Yukawa matrix

Let us first perform the computation of the physical Yukawa matrix. Inserting the zero-mode wavefunctions for the **5** and **10** sector into the cubic coupling (2.57) and applying the E_6 group theory relations we obtain

$$Y_U^{ij} = m_*^4 \int_S \det\left(\vec{\Psi}_{\mathbf{5}}, \vec{\Psi}^i_{\mathbf{10}^M}, \vec{\Psi}^j_{\mathbf{10}^N}\right) \epsilon_{MN}\, \mathrm{dvol}_S \tag{2.315}$$

with $M, N = \pm$ and ϵ_{MN} the $\mathfrak{su}(2)$ antisymmetric tensor. To obtain the Yukawas at zeroth order in ϵ we just need to plug into (2.315) the perturbative wavefunctions computed in subsection 2.5.3. One then finds the expression

$$Y_U^{(0)ij} = 2m_*^4 \gamma_{\mathbf{5}}\gamma^i_{\mathbf{10}}\gamma^j_{\mathbf{10}} \det\left(\vec{v}_{\mathbf{5}}, \vec{v}^i_{\mathbf{10}^+}, \vec{v}^j_{\mathbf{10}^-}\right) \int_S \chi_{\mathbf{5}}\chi^i_{\mathbf{10}}\chi^j_{\mathbf{10}}\, \mathrm{dvol}_S \tag{2.316}$$

where χ_ρ are the perturbative scalar wavefunctions of (2.270) and (2.285). As the product of these three wavefunctions is sharply localised around the origin one can replace the domain of integration by $\mathbb{C}^2$. Taking the holomorphic representatives for each family as in (2.286) one obtains

$$Y_U^{(0)33} = -\frac{\pi^2}{2\rho_\mu\rho_m}\gamma_{\mathbf{5}}\gamma^3_L\gamma^3_R \qquad\qquad Y_U^{(0)ij} = 0 \quad \text{for} \quad i \neq 3 \neq j \tag{2.317}$$

where the slope densities $\rho_{m,\mu}$ are defined as in (2.314), $\gamma_{\mathbf{5}}$ is the normalisation factor (2.289) evaluated for the sector $\mathbf{5}_1$ of table 2.7 and $\gamma^i_{R,L}$ are the normalisation factors (2.291) evaluated for the sectors $\mathbf{10}_{1,2}$ of the same table. Then, as expected from our construction, we obtain a rank 1 Yukawa matrix at the perturbative level, which moreover is in perfect agreement with the result of the residue computation of section 2.5.2.

The $\mathcal{O}(\epsilon)$ contribution to (2.315) can be written as

$$Y_U^{(1)ij} = 2m_*^4 \int_S \left[\det\left(\vec{\Psi}^{(1)}_{\mathbf{5}}, \vec{\Psi}^{(0)\,i}_{\mathbf{10}^+}, \vec{\Psi}^{(0)\,j}_{\mathbf{10}^-}\right) + \det\left(\vec{\Psi}^{(0)}_{\mathbf{5}}, \vec{\Psi}^{(1)\,i}_{\mathbf{10}^+}, \vec{\Psi}^{(0)\,j}_{\mathbf{10}^-}\right)\right.$$
$$\left. + \det\left(\vec{\Psi}^{(0)}_{\mathbf{5}}, \vec{\Psi}^{(0)\,i}_{\mathbf{10}^+}, \vec{\Psi}^{(1)\,j}_{\mathbf{10}^-}\right)\right] \mathrm{dvol}_S \tag{2.318}$$

where we have split the corrected wavefunction (2.301) as $\vec{\Psi}_{\mathbf{5}} = \vec{\Psi}_{\mathbf{5}}^{(0)} + \epsilon\, \vec{\Psi}_{\mathbf{5}}^{(1)}$ and similarly for the wavefunctions $\vec{\Psi}^{i}_{\mathbf{10}^\pm} = \vec{\Psi}^{(0)\,i}_{\mathbf{10}^\pm} + \epsilon \vec{\Psi}^{(1)\,i}_{\mathbf{10}^\pm}$ in the sector **10**.

Performing the integral of the first term in (2.318) one obtains

$$Y_U^{(1)} \;=\; \frac{\pi^2\gamma_{\mathbf{5}}}{4\rho_\mu^2\rho_m}\begin{pmatrix} 0 & 0 & \gamma_L^1\gamma_R^3 \\ 0 & \gamma_L^2\gamma_R^2 & 0 \\ \gamma_L^3\gamma_R^1 & 0 & 0 \end{pmatrix}(\theta_x+\theta_y) \tag{2.319}$$

matching the result (2.267) for the case (2.244) that we are considering. Recall that in the computations of Section 2.5.2 the $\mathcal{O}(\epsilon)$ corrections to the Yukawa matrix came entirely from the corrections to the wavefunction in the **5** sector, while the corrections to the **10** sector did not contribute to the Yukawas. One can argue that the same will happen here as follows. First notice that due to the zero mode structure (2.312), $\vec{\varphi}^{(0)\,j}_{\mathbf{10}^-}$ and $\vec{\varphi}^{(1)\,i}_{\mathbf{10}^+}$ are proportional to each other up to multiplication by a complex function, and so the determinant in the second term of (2.318) vanishes identically. Second, for $\zeta_{\mathbf{10}} = 0$ the tree-level wavefunctions $\vec{\varphi}^{(0)\,i}_{\mathbf{10}^+}$ only have one non-vanishing component (c.f.(2.283)), and one can then see that the same applies to $\vec{\varphi}^{(1)\,i}_{\mathbf{10}^-}$, so that the third determinant in (2.318) vanishes as well. For $\zeta_{\mathbf{10}} \neq 0$ such determinant may not vanish identically, but its integral should vanish because $\zeta_{\mathbf{10}}$ is proportional to the flux q_S and the integral (2.315) should not depend explicitly on background worldvolume fluxes. Indeed, recall that (2.315) is equivalent to (2.57), which is invariant under complexified gauge transformations. Such transformations can be used to gauge away any dependence on the worldvolume flux, and so the result obtained for $q_S = 0$ should be true in general. In fact, one can use a complexified gauge transformation to take the wavefunctions computed in the previous section to the ones used in Section 2.5.2 in the residue formula, which is why both results match.[54]

Finally, adding up these two results as

$$Y_U \;=\; Y_U^{(0)} + \epsilon\, Y_U^{(1)} + \mathcal{O}(\epsilon^2) \tag{2.320}$$

we obtain (2.313), as claimed above. In the following we will analyse if given these up-like Yukawas we can reproduce the data in table 2.8.

The top quark Yukawa

The Yukawa for the top quark is given by the 33 entry of (2.313). To analyse its value it is useful to express the quantities ρ_μ and ρ_m as

$$\rho_\mu \;=\; \left(\frac{\mu}{m_*}\right)^2 \;=\; (2\pi)^{3/2}g_s^{1/2}\sigma_\mu \qquad\qquad \rho_m \;=\; \left(\frac{m}{m_*}\right)^2 \;=\; (2\pi)^{3/2}g_s^{1/2}\sigma_m \tag{2.321}$$

where $\sigma_\mu = (\mu/m_{st})^2$ and $\sigma_m = (m/m_{st})^2$ are the 7-brane intersection slopes measured in units of m_{st}, the scale that in the type IIB limit reduces to the string scale $m_{st} = 2\pi\alpha'$ and which is related to the F-theory scale as $m_{st}^4 = g_s(2\pi)^3 m_*^4$. We then have that

$$|Y_t| \;=\; (8\pi g_s)^{1/2}\sigma_m c\; \tilde{\gamma}_{\mathbf{5}_1}\tilde{\gamma}_{\mathbf{10}_1}\tilde{\gamma}_{\mathbf{10}_2} \tag{2.322}$$

[54] In relating wavefunctions by a complexified gauge transformation we assume that the definition of families in terms of monomials is preserved. That is, we assume that the choice of family representative (2.286) is mapped to (2.261) by a complexified gauge transformation that removes the flux dependence from the wavefunction.

where

$$\tilde{\gamma}_{\mathbf{5}_1} = \left(-\frac{(2\zeta_{\mathbf{5}_1} + q_R^{\mathbf{5}_1})(q_R^{\mathbf{5}_1} + 2\zeta_{\mathbf{5}_1} - 2\lambda_{\mathbf{5}_1}) + (q_S^{\mathbf{5}_1} + \lambda_{\mathbf{5}_1})^2}{4\mu^4 + \zeta_{\mathbf{5}_1}^2 + (\zeta_{\mathbf{5}_1} - \lambda_{\mathbf{5}_1})^2} \right)^{1/2} \tag{2.323}$$

$$\tilde{\gamma}_{\mathbf{10}_i} = \left(-\frac{q_R^{\mathbf{10}_i}}{\frac{m^4}{2\lambda_{\mathbf{10}_i} + q_R^{\mathbf{10}_i}(1+\zeta_{\mathbf{10}_i}^2) - m^2c^2} + \frac{c^2\lambda_{\mathbf{10}_i}^2}{2\lambda_{\mathbf{10}_i} + q_R^{\mathbf{10}_i}(1+\zeta_{\mathbf{10}_i}^2) + m^2c^2}} \right)^{1/2} \qquad i = 1, 2$$

with $q_{R,s}^{\mathbf{10}_i}$, $i = 1, 2$ the values of $q_{R,S}$ in the i^{th} row of table 2.7, $q_{R,S}^{\mathbf{5}_1}$ the ones in the fourth row, etc. Notice that this expression is quite similar to the one obtained for the third generation of down-like Yukawas in the $SO(12)$ model, except for an extra factor of $\sqrt{2}c$ which for the value ($c \sim 0.7$) is very close to 1. Hence in principle one expects that the Yukawa of the top and of the bottom are of the same order of magnitude, which in the scheme of the MSSM would favour a large tanβ.

From (2.322) one may proceed as in the $SO(12)$ model and estimate that primitive worldvolume flux densities are of the order

$$M, N \simeq 0.29\, g_s^{1/2} m_{st}^2 \tag{2.324}$$

with g_s not too small. In fact, the diluted flux approximation is one of the requirements that we need to impose in order to be able to trust the 7-brane effective action that led to the zero mode equations of Section 2.5.3. A further self-consistency restriction comes from the fact that the non-primitive flux (2.246) must be slowly varying in the region where wavefunctions are localised. As discussed in appendix A of [58], this leads to the condition

$$-\left(\frac{q_R}{2}\lambda_{\mathbf{10}} + \frac{m^2c^2}{2} \right) \frac{4c^4}{|2c^4 - 1|m^2} \gg 1. \tag{2.325}$$

Finally, recall that in order to simplify the **10**-plet zero mode equations restricted ourselves to the region of the parameter space such that $m \gg \mu$. All these approximations are only important for computing the normalisation factors for the wavefunctions, while the computation of holomorphic Yukawas in Section 2.5.2 is independent of them.

Given these restrictions one can see that one may accommodate a realistic value for the Yukawa of the top at the unifications scale. Indeed, if one for instance takes the values (in units of m_{st})

$$M = 0.3\,, \quad N = 0.03\,, \quad \tilde{N}_Y = 0.6\,, \quad N_Y = -0.18\,, \quad m = 0.5\,, \quad \mu = 0.1\,, \tag{2.326}$$

with $g_s = 1$ and $c = 0.7$ one obtains

$$Y_t = 0.5 \tag{2.327}$$

in quite good agreement with the values of table 2.8. One can also check that the wavefunctions are sufficiently localised in a region where the first two terms of (2.87) are a good approximation for the Painlevé transcendent.

Up-type quarks mass hierarchies

In order to analyse the flavour hierarchies among different U-quarks let us consider the matrix

$$\frac{Y_U}{Y^{33}} = \begin{pmatrix} 0 & 0 & -\frac{1}{2}\tilde{\epsilon}\rho_\mu^{-1}\frac{\gamma_L^1}{\gamma_L^3} \\ 0 & -\frac{1}{2}\tilde{\epsilon}\rho_\mu^{-1}\frac{\gamma_L^2\gamma_R^2}{\gamma_L^3\gamma_R^3} & 0 \\ -\frac{1}{2}\tilde{\epsilon}\rho_\mu^{-1}\frac{\gamma_R^1}{\gamma_L^3} & 0 & 1 \end{pmatrix} + \mathcal{O}(\tilde{\epsilon}^2) \tag{2.328}$$

whose eigenvalues are

$$\begin{aligned} \lambda_1 &= 1 + \mathcal{O}(\epsilon^2) \\ \lambda_2 &= -\epsilon\frac{1}{2\rho_\mu}\frac{\gamma_L^2\gamma_R^2}{\gamma_L^3\gamma_R^3}(\theta_x + \theta_y) + \mathcal{O}(\epsilon^2) \\ \lambda_3 &= \mathcal{O}(\epsilon^2), \end{aligned}$$

where we have used the expression for $\tilde{\epsilon}$ in (2.314). This yields automatically a hierarchy of U-quark masses of the form $(1, \epsilon, \epsilon^2)$ in fact quite similar to the one found in the $SO(12)$ model for the D-quarks and leptons. As in there, the quotient of quark masses of different families is rather simple. Namely identifying the first and second eigenvalues with the third and second generations of U-quarks we have

$$\begin{aligned} \frac{m_c}{m_t} &= \frac{1}{2}\left(\frac{q_R^{\mathbf{10}_1} q_R^{\mathbf{10}_2}}{\mu^4}\right)^{1/2} \epsilon\,(\theta_x + \theta_y) \qquad (2.329) \\ &= \frac{1}{2}\frac{M}{\mu^2}\left(1 + \frac{2\tilde{N}_Y}{3M}\right)^{1/2}\left(1 - \frac{\tilde{N}_Y}{6M}\right)^{1/2} \epsilon\,(\theta_x + \theta_y) \end{aligned}$$

where we have used that $q_R^{\mathbf{10}_1} = M + \frac{2}{3}\tilde{N}_Y$ and $q_R^{\mathbf{10}_2} = M - \frac{1}{6}\tilde{N}_Y$. Hence it is quite easy to accommodate the hierarchy between the charm and the top quark with a small non-perturbative parameter ϵ. In fact, one may consider the ratio of flux densities $\tilde{N}_Y/M \sim 1.8$ obtained in the $SO(12)$ model for the down-like Yukawa point p_{down} and apply it to this expression, since, if the two Yukawa points p_{up} and p_{down} are not far away the flux densities should be alike. One then obtains that a realistic mass ratio requires

$$\frac{M}{\mu^2}\epsilon\,(\theta_x + \theta_y) \simeq 4 \times 10^{-3} \tag{2.330}$$

which can be achieved by taking $\tilde{\epsilon} = \epsilon\,(\theta_x + \theta_y) \sim 10^{-4}$ as in the $SO(12)$ model and M and μ as in (2.326). Of course a more detailed analysis would require to embed both Yukawa points p_{up} and p_{down} in the same local model, possibly in a region of E_7 or E_8 enhancement.

3

Discrete flavour symmetries in D-brane models

In this chapter we explore an alternative route to flavour physics within string theory, that of discrete symmetries. We start by discussing some general aspects of discrete symmetries, distinguishing between gauge and global symmetries and the role of quantum gravitational effects. We review in some detail the field theory description of an abelian discrete gauge theory which gives some results that will be useful in subsequent sections where we explore non-abelian generalisations.

In section 3.2 we build an explicit example of non-abelian discrete gauge symmetry by compactifying on a manifold with discrete isometries and we show it is closely related to magnetised and intersecting D-brane models which are very interesting from the phenomenological point of view. In 3.3 we explore the possibility of having discrete symmetries acting on the flavour degrees of freedom for semi-realistic intersecting D6-brane models. In the next section we analyse the same problem from the T-dual perspective of magnetised D9-branes. Finally, we build some models which qualitatively resemble the MSSM and display non-abelian symmetries that pose strong constraints on the Yukawa couplings.

3.1 Generalities of discrete symmetries

Discrete symmetries are widely used in phenomenological models to forbid couplings or generate textures for them. Probably the simplest example is R parity in the MSSM, a discrete $\mathbb{Z}_2$ symmetry proposed *ad hoc* to forbid dangerous proton decay operators. In the context of flavour a standard mechanism to generate patterns in Yukawa couplings or PMNS matrix is to propose that these are somehow constrained by some approximate discrete symmetry. Depending on the particular group, the way it acts on the SM fields and how it is broken one can get different textures for the couplings. From this point of view the microscopic origin of the symmetry is obscure and one does not know if the symmetry is exact or approximate. In the case of approximate symmetries it can be also troublesome to compute the scale at which these are broken and the magnitude of the deviations from the exact case. In the context of string theory one should, in principle, be able to answer all of these questions for a given vacuum and compute the corrections, if any.

First of all, let us define what we mean by discrete gauge symmetry. When quantising a gauge theory, we identify states connected by a gauge transformation that is trivial at spatial infinity, dubbed *local* gauge transformations, so it acts trivially in the physical Hilbert space and we speak of an invariance rather than a symmetry. The so-called *global* gauge group consists of transformations that are constant at spatial infinity and it acts as a symmetry in the Hilbert space so states arrange in irreducible representations of such symmetry.[1] Applying Noether's theorem to the global group we get a conserved current and charge. A crucial feature of this charge is that it can be measured from arbitrarily far away by looking at the flux it generates. When the gauge symmetry is spontaneously broken it is the global part of the gauge group that is higgsed[2] and the associated charge is no longer conserved. However, a discrete subgroup may survive this breaking which we refer to as a *discrete gauge symmetry*. It acts as a symmetry in the space of states and the Hamiltonian is invariant under it which yields a conserved charge. Again, this charge has long-range effects as we will explicitly show in the next section.

Global symmetries, on the other hand, are simply symmetries of the dynamics that are not associated to any gauge invariance and can be both discrete and continuous. This means that the corresponding charge cannot be measured from arbitrarily far away due to the absence of a gauge field. This difference between global and gauge symmetries has important consequences when including quantum gravitational effects as we briefly mention in the following.

Indeed, there is a 'folk theorem' concerning any theory of quantum gravity that says that there are no global symmetries (see e.g. [88] for a recent review). The idea of the argument for continuous symmetries is as follows. Consider a black hole that was made out of matter charged under a global symmetry. Since there is no gauge field associated to the global charge it is not possible to measure it from infinity and every black hole of that kind looks the same, no matter the global charge it had originally. Upon evaporation no global charge is emitted and we end up with a small black hole. At some point Hawking's calculation can no longer be trusted and we really do not know what happens next. However, the important point is that we arrive at the same small black hole regardless of the global charge which means that the associated entropy is infinite. This is in conflict with the Covariant Entropy Bound that sets a limit on the amount of entropy there can be in a causal diamond [76]. Thus, global charges are not conserved, only gauge charges are.

This statement is expected to hold also for discrete symmetries although there is not such a clean argument. For perturbative string theory there is plenty of evidence that supports this claim, for instance, the $SL(2,\mathbb{Z})$ symmetry in Type IIB corresponds to large diffeomorphisms of M-theory on $\mathbf{T}^2$ as we saw in section 2.2.1 and the T-duality $\mathbb{Z}_2$ enhances to a $SU(2)$ gauge group at self-radius.

This means that if, for example, we embed the MSSM within string theory in such a way that R parity is a discrete gauge symmetry the proton will be stable. All the loop corrections, non-perturbative or any UV effects will respect the $\mathbb{Z}_2$ symmetry. This is clearly a good feature for certain phenomenological issues but it is unacceptable for others. For instance, if there is a discrete symmetry behind the structure of masses and mixings in the SM it should correspond to an approximate global symmetry since these do not seem to follow from an exact symmetry. The scale at which a given global symmetry is broken is model dependent and should be computed in a case by case basis.

[1] If the space has no boundary then the global gauge group is trivial which matches the fact that there can be no net charge is such spaces due to Gauss's law.

[2] Elitzur's theorem [75] states that only gauge invariant operators may acquire a vev so the local part of the gauge group is always preserved. The local gauge group may appear to be broken due to a choice of gauge.

3.1.1 Low energy description of a $\mathbb{Z}_p$ gauge theory

Let us analyse in some detail a simple theory that has an abelian discrete gauge symmetry which will hopefully clarify some of the features mentioned in the last section. Also, it will be useful when we go to the more involved case of non-abelian symmetries.

Consider a $U(1)$ gauge theory in 4d spontaneously broken by a scalar field Φ of charge p times the fundamental charge. The Lagrangian for such theory is

$$\mathcal{L} = -\frac{1}{2g^2} F \wedge *F + (d - ipA)\Phi \wedge *(d + ipA)\bar{\Phi} - V(|\Phi|) \tag{3.1}$$

with g the coupling constant, $V(|\Phi|)$ such that $\langle|\Phi|^2\rangle = v^2$ and the normalizations are such that the gauge group acts on the fields as

$$A' = A + d\chi, \qquad \Phi' = \Phi\, e^{ip\chi}. \tag{3.2}$$

The space of vacua is given by

$$\langle\Phi\rangle = v\, e^{i\phi_0} \tag{3.3}$$

with $\phi_0 \in [0, 2\pi)$ and $\phi_0 \sim \phi_0 + 2\pi$. Notice that in general two different vacua are connected by a gauge transformation that is non-trivial at spatial infinity so they correspond to different states. More explicitly, a constant gauge transformation acts on the vacuum as $\phi_0' = \phi_0 + p\chi$ and for generic values of χ we have that $\phi_0' \neq \phi_0$. However, since Φ has charge p there is a discrete set of values for χ that leaves the vacuum invariant. Namely, $\chi = 2\pi\frac{n}{p}$ with $n \in \mathbb{Z}$ which corresponds to a $\mathbb{Z}_p$ subgroup of the original $U(1)$. Thus, there is a $\mathbb{Z}_p$ gauge symmetry acting on the Hilbert space of the theory.

Let us compute the action for $V(|\Phi|) = \lambda(|\Phi|^2 - v^2)^2$. We expand the field Φ around the vacuum ϕ_0, $\Phi = (v+\rho)e^{i(\phi_0+\phi)}$, where ρ and ϕ are real scalars such that ϕ takes values on a circle of radius 2π. The Lagrangian is

$$\mathcal{L} = -\frac{1}{2g^2} F \wedge *F + d\rho \wedge *d\rho + (v+\rho)^2(d\phi - pA) \wedge *(d\phi - pA) + \lambda(4v^2\rho^2 + 4v\rho^3 + \rho^4). \tag{3.4}$$

We recognise $v^2(d\phi - pA)^2$ as a Stückelberg term which can be interpreted as a mass for the gauge boson (for a gauge where ϕ is constant) and signals a spontaneous breaking of the gauge symmetry. Also, the relative coefficient between the axion ϕ and the gauge field gives the charge of the axion under the $U(1)$ and therefore, the surviving discrete gauge group. Thus, we learn that a Stückelberg term of this kind is a distinctive feature for discrete gauge symmetries. Indeed, if we restrict to a theory that only contains the gauge field and an axion coupled via a Stückelberg term, the different vacua correspond to $A = 0$ and $\phi = \phi_0$. For the same reason as before a given vacuum is only invariant under $\mathbb{Z}_p$ which triggers the spontaneous breaking of the symmetry.

We now compute the deep IR limit of this theory in which we can analyse the objects that are charged under $\mathbb{Z}_p$. The IR corresponds to restricting to energies well below the symmetry breaking scale v so we should take the limit $v \to \infty$. The lightest excitations of this theory are those of the massive gauge field with $m_A \sim gv$ and the Higgs field ρ with $m_\rho \sim \lambda v$ so when $v \to \infty$ we do not expect to have any local degrees of freedom. Indeed, in this limit the equations of motion reduce to the constraint $d\phi - pA = 0$ and the theory becomes topological. Taking the limit $v \to \infty$ on (3.6) we arrive at the so-called BF Lagrangian

$$\mathcal{L} = -\frac{p}{2\pi} B \wedge dA \tag{3.5}$$

that has a discrete gauge symmetry $\mathbb{Z}_p$. Let us take the Lagrangian (3.4) and dualise the axion into a 2-form B using $4\pi v^2 d\phi = *dB = *H$, namely

$$\mathcal{L} = -\frac{1}{2g^2}F \wedge *F - \frac{1}{(4\pi)^2 v^2}H \wedge *H - \frac{p}{2\pi}B \wedge dA + \cdots \tag{3.6}$$

where the dots represent terms that are irrelevant in the IR. We can further dualise the gauge field A using $2\pi dA = g^2 * dV$

$$\mathcal{L} = -\frac{1}{(4\pi)^2 v^2}H \wedge *H + \frac{g^2}{8\pi^2}(dV - pB) \wedge *(dV - pB) + \cdots . \tag{3.7}$$

The Stückelberg term is now $(dV - pB)^2$ and has a dual interpretation to the one in (3.4). The field B is the potential of a $U(1)$ gauge theory that is higgsed down to $\mathbb{Z}_p$ due to the matter field V.

Thus, we have a theory with two different $U(1)$ gauge groups broken down to $\mathbb{Z}_p$, one corresponding to the original gauge field A and another one associated to the 2-form potential B. This allows us to find the charged objects in this theory. We know that a gauge field couples to particles via $\int_\Gamma A$ where Γ is the worldline of such particle and a 2-form potential naturally couples to strings by integration on the worldsheet Σ, $\int_\Sigma B$. The irreducible representations of $\mathbb{Z}_p$ are labeled simply by k mod(p) which gives the possible charges of these objects. This means that the consistent charged objects this theory can contain are particles and strings with charges k mod(p).

In the IR limit the only observables of the theory correspond to the holonomies that particles and strings induce among themselves. These are captured by the Wilson operators

$$W_{particle}(\Gamma, n_A) = \exp\left[in_A \oint_\Gamma A\right] \tag{3.8a}$$

$$W_{string}(\Sigma, n_B) = \exp\left[in_B \oint_\Sigma B\right] \tag{3.8b}$$

which can be interpreted as the phase that the wavefunction of particles and strings pick up when moving in a background with A and B fields, respectively. To compute the holonomy we can consider a probe particle of charge n_A moving in the background created by a string of charge n_B. The presence of a string of charge n_B with worldsheet Σ produces an extra term in the action $n_B \int_\Sigma B$ that modifies the equations of motion which now read

$$dA = 2\pi \frac{n_B}{p}\delta(\Sigma) \tag{3.9}$$

where $\delta(\Sigma)$ is a delta 2-form with support in Σ and all its legs conormal to Σ. The Aharonov-Bohm phase that a particle of charge n_A picks when moving along a closed loop $\Gamma = \partial D$ is[3]

$$\exp\left[in_A \int_\Gamma A\right] = \exp\left[in_A \int_D dA\right] = \exp\left[2\pi i \frac{n_A n_B}{p} \int_D \delta(\Sigma)\right] = \exp\left[2\pi i \frac{n_A n_B}{p} L(\Gamma, \Sigma)\right] \tag{3.10}$$

[3]For theories on topologically non-trivial spaces the path Γ need not be trivial in homology which gives an extra contribution to the phase. We neglect this observable since it is not distinctive of discrete gauge symmetries.

where $L(\Gamma, \Sigma)$ is the so-called linking number of Γ and Σ, a topological invariant that measures how many times the particle winds around the string. Of course, one can arrive at the same result by considering a string moving in the background created by a particle.

We end this section considering the operators that are dual to the Wilson loops (3.8), the 't Hooft operators. Naively, one would think these are given by $e^{i\phi(P)}$ and $e^{i\int_\Gamma V}$ but these operators are not invariant under the $U(1)$s associated to A and B respectively. In order to make them gauge invariant we need to attach particles ending on P and strings ending on Γ. More precisely, the 't Hooft operators are given by

$$T_{instanton}(P, m_B) = \exp\left[i m_B \left(\phi(P) + \sum_{i=1}^{p} \int_{\gamma_i} A \right) \right] \tag{3.11a}$$

$$T_{monopole}(\Gamma, m_A) = \exp\left[i m_A \left(\oint_\Gamma V + \sum_{i=1}^{p} \int_{\sigma_i} B \right) \right] \tag{3.11b}$$

where γ_i are paths that go from P to other instantons or extend to infinity and the surfaces σ_i go from Γ to other magnetic monopoles or infinity. The operator (3.11a) can be interpreted as a process in which a set of p particles come together in P and annihilate (or the time reversed process in which p particles are created at P). Similarly for (3.11a) we have that p strings can be created or destroyed.

In this section we have seen that there are several ways to characterise a field theory with an abelian discrete gauge symmetry. It can be regarded as a $U(1)$ theory spontaneously broken by a field of non-minimal charge p, a Stückelberg or BF theory proportional to p and a theory that contains particles and string with charges conserved modulo p that induce Aharonov-Bohm phases to each other.

3.1.2 Non-abelian generalisation

Since we are ultimately interested in having non-abelian discrete gauge symmetries let us generalise the previous section to the non-abelian case. A general analysis is beyond our scope so we just present some aspects that will be particularly useful for our purposes.

As a warm up example of a yet abelian case consider a $U(1)^N$ gauge theory that contains N axions $\phi^a \sim \phi^a + 2\pi$ coupled via Stückelberg terms, namely[4]

$$\mathcal{L} \supset (\partial_\mu \phi^a - k^a{}_c A^c_\mu)(\partial_\nu \phi^b - k^b{}_c A^c_\nu)\, \eta^{\mu\nu} \delta_{ab}. \tag{3.12}$$

with $k^b{}_a$ integer numbers. We ignored the kinetic terms for the gauge bosons since they do not play any role in our discussion. We know from our previous example that the vacua of this theory correspond to having constant axions. Thus, a given vacuum is a point in the space spanned by the axions (dubbed 'axionic manifold') which, in this case is $\mathbf{T}^N$. The action of the global gauge group on the axions is given by

$$\phi^a \to \phi^a + \sum_a k^a{}_b\, \chi^b \tag{3.13}$$

where χ^a is a constant in $[0, 2\pi)$ and corresponds to the element $e^{i\chi^a}$ in $U(1)_a$. Again, the symmetry that survives is given by the set of χ^a that leaves the vacuum invariant. This

[4]We consider the same number of gauge fields and axions for simplicity since this does not affect the idea we try to convey.

means that we should solve the equations

$$\phi^a \to \phi^a + \sum_b k^a{}_b \chi^b = \phi^a \mod 2\pi \tag{3.14}$$

for $a = 1, \ldots, N$. For $\det k \neq 0$ we find that $\chi^b = 2\pi (k^{-1})^b{}_a \, n^a$ with $n^a \in \mathbb{Z}$. In order to get only the inequivalent values of χ^a we must quotient these by $\chi^a \sim \chi^a + 2\pi$ which gives the surviving discrete symmetry $\mathbf{P}$, namely

$$\mathbf{P} = \frac{\Gamma}{\tilde{\Gamma}} \tag{3.15}$$

where we defined Γ as the lattice generated by the allowed values of χ^a and $\tilde{\Gamma} = 2\pi\mathbb{Z}^N$.

The key observation that allows to generalise this construction to non-abelian symmetries is to think of the Stückelberg terms in (3.12) as the gauging of the isometries of the axionic manifold $\mathbf{T}^N$. Since these isometries are abelian the potential discrete gauge symmetry that one may get is always abelian. On the other hand, one can consider axionic manifolds with non-abelian isometries which may yield non-abelian discrete gauge symmetries upon gauging, as follows.

Let $\mathcal{M}$ be an axionic manifold of dimension N and coordinates ϕ^a equipped with a metric G_{ab} with non-abelian isometries. We denote the Killing vectors by $X_A = X_A^a \partial_a$ with A running over $A = 1, \ldots, N$ where here N is the dimension of the Lie group of isometries $\mathrm{Iso}(\mathcal{M})$ (which coincides with the dimension of $\mathcal{M}$ itself since it is an axionic manifold). These satisfy the Lie algebra

$$[X_A, X_B] = f_{AB}{}^C X_C \tag{3.16}$$

where $[\ ,\ ]$ is the Lie bracket and $f_{AB}{}^C$ the structure constants. Notice that the group $\mathrm{Iso}(\mathcal{M})$ has a natural transitive action on $\mathcal{M}$ via the flow of the Killing vectors. More precisely, consider the integral curves of the Killing vectors

$$C^A(\lambda^A, p): \quad [0, 2\pi] \times \mathcal{M} \longrightarrow \mathcal{M} \tag{3.17}$$

that satisfy $\partial_{\lambda^A} C^A(\lambda^A, p) = X_A(p)$ and $C^A(0, p) = C^A(2\pi, p) = p$ where p is an arbitrary point in $\mathcal{M}$. Thus, for a given λ^A this map takes the point p to $C^A(\lambda^A, p) \in \mathcal{M}$. The group law translates into composition of maps and since it is non-abelian we have that, in general, $C^B(\lambda^B, C^A(\lambda^A, p)) \neq C^A(\lambda^A, C^B(\lambda^B, p))$.

The kinetic term for the axions ϕ^a reads

$$\mathcal{L} \supset G_{ab}(\phi) \partial_\mu \phi^a \partial^\mu \phi^b \tag{3.18}$$

and is invariant under the isometries generated by X_A if these are indeed Killing vectors, namely, $\mathcal{L}_{X_A} G = 0$.

Consider now adding a set of N non-commuting $U(1)$ gauge symmetries with generators T_A satisfying

$$[T_A, T_B] = i g_{AB}{}^C T_C \tag{3.19}$$

such that they gauge the isometries of $\mathcal{M}$ leading to a Stückelberg Lagrangian of the form

$$\mathcal{L} \supset G_{ab}(\phi)(\partial_\mu \phi^a - k^a{}_A A^A_\mu)(\partial_\nu \phi^b - k^b{}_B A^B_\nu)\, \eta^{\mu\nu} \tag{3.20}$$

where ${k^a}_A$ are functions of ϕ^a that relate the generators of the isometry and the generators in the gauge algebra. More precisely, demanding (3.20) to be invariant under the $U(1)$s yields $T_A \leftrightarrow {k^a}_A \partial_a$ so the functions ${k^a}_A$ tell you how the gauge group acts on $\mathcal{M}$ which, in general, is position dependent. Thus, consider another set of integral curves $\tilde{C}^A(\chi_A, p)$ associated to ${k^a}_A$ rather than X^a_A that satisfy $\partial_{\chi^A} \tilde{C}^A(\chi^A, p) = {k^a}_A(p)\partial_a$ with $\tilde{C}^A(0, p) = \tilde{C}^A(2\pi, p) = p$. Under a global gauge transformation $U = \exp(i\chi^A T_A)$ (no sum in A) a point $p \in \mathcal{M}$ transforms as

$$p \longrightarrow \tilde{C}^A(\chi^A, p) \tag{3.21}$$

which is different from p for generic values of χ^A. Since a given point in $\mathcal{M}$ corresponds to a particular vacuum in the field theory the gauging indeed breaks the gauge group. However, it may happen that $C^A(\chi^A, p) = p$ for a discrete set of values of χ^A in which case there is a residual discrete gauge symmetry. Since the composition of the integral curves does not commute we arrive at a non-abelian gauge symmetry.

In the following sections we present some examples of theories with non-abelian discrete gauge symmetries which are particular examples of this construction. In section 3.2 we consider a compactification on a manifold with discrete isometries whose lower-dimensional description is that of a compactification on a space with continuous isometries broken by metric fluxes. By performing a dimensional reduction one arrives precisely to a Lagrangian that contains a set of axions that span a manifold with non-abelian isometries together with the appropriate gaugings that yield a discrete gauge symmetry. In section 3.3.2 we study Type IIA toroidal orientifolds with intersecting D6-branes that also exhibit non-abelian discrete symmetries coming from these kind of gaugings.

Let us conclude this section with some remarks. For the abelian $\mathbb{Z}_p$ gauge theory we have seen that there are always charged particles and strings that induce Aharonov-Bohm phases among themselves. For a non-abelian discrete gauge symmetry the situation is somewhat similar, there are also particles and strings that affect each other for arbitrarily large distances. However, the situation is much more involved since their charges correspond to non-abelian representations of the discrete symmetry which makes their study quite complicated. An analysis of the charged states and their interactions is completely beyond our scope so we refer the reader to [77] for a detailed discussion on the subject.

3.2 Non-abelian discrete gauge symmetries from discrete isometries

In this section we consider a compactification in a manifold with discrete non-abelian isometries. We also discuss the lower-dimensional description in terms of gaugings as explained before as well as the relation to compactifications with magnetised and intersecting D-branes.

3.2.1 Compactification on twisted torus

In Kaluza-Klein compactification, isometries of the compactification manifold produce global symmetries in the worldsheet which correspond to gauge symmetries in the lower dimensional theory in the bulk. This is familiar for continuous isometries, but also holds for discrete isometries, suggesting a natural source for (possibly non-abelian) discrete gauge symmetries. We analyse a particular example that produces such non-abelian symmetries and that is closely related to the flavour symmetries studied in the subsequent sections.

Prototypical examples of compactification spaces with discrete isometries are twisted tori. For simplicity we focus on the case of a twisted torus $(\mathbf{T}^3)_M$ (where M denotes the first Chern class of the $\mathbf{S}^1$ fibration over the base $\mathbf{T}^2$). This space is a group manifold so it can be neatly displayed by the following coset construction (see e.g. [78, 79]). Consider the set $\mathcal{H}_3(\mathbb{R})$ of upper triangular matrices

$$g(x,y,z) = \begin{pmatrix} 1 & x & z+\frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad x,y,z \in \mathbb{R} \tag{3.22}$$

which forms a non-compact Heisenberg group under multiplication

$$g(x,y,z)g(x',y',z') = g(x+x', y+y', z+z' + \frac{1}{2}(xy'-x'y)). \tag{3.23}$$

A basis of e.g. right-invariant forms $\eta^x = dx$, $\eta^y = dy$, $\eta^z = dz - \frac{1}{2}(ydx - xdy)$ allows the introduction of a metric $ds^2 = (\eta^x)^2 + (\eta^y)^2 + (\eta^z)^2$ with an isometry group defined by right multiplication, and therefore given by $\mathcal{H}_3(\mathbb{R})$ itself. More precisely, we have that the Killing vectors of this metric are given by the left-invariant vectors of $\mathcal{H}_3(\mathbb{R})$, a simple basis for them being

$$X_L^x = \partial_x - \tfrac{1}{2}y\partial_z \qquad g(x,y,z) \to g(x+\lambda_x, y, z - \tfrac{1}{2}y\lambda_x) \tag{3.24a}$$
$$X_L^y = \partial_y + \tfrac{1}{2}x\partial_z \qquad g(x,y,z) \to g(x, y+\lambda_y, z + \tfrac{1}{2}x\lambda_y) \tag{3.24b}$$
$$X_L^z = \partial_z \qquad g(x,y,z) \to g(x,y,z+\lambda_z) \tag{3.24c}$$

where we have also specified the continuous isometries generated upon exponentiation of such Lie algebra elements.

The twisted torus is obtained as a left coset $(\mathbf{T}^3)_M = \mathcal{H}_3(\mathbb{R})/\mathcal{H}_3(M)$ of the non-compact space $\mathcal{H}_3(\mathbb{R})$ by the infinite discrete subgroup $\mathcal{H}_3(M)$ with elements of the form

$$\begin{pmatrix} 1 & Mn_x & Mn_z \\ 0 & 1 & Mn_y \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad n_x, n_y, n_z \in \mathbb{Z}. \tag{3.25}$$

In other words, by imposing the identifications

$$g(x,y,z) \sim g(x+M, y, z - \tfrac{M}{2}y) \sim g(x, y+M, z+\tfrac{M}{2}x) \sim g(x,y,z+M). \tag{3.26}$$

As the metric is made of right-invariant forms, $(\mathbf{T}^3)_M$ has a well-defined quotient metric. On the other hand, some of the isometries of the parent space $\mathcal{H}_3(M)$ are broken in $(\mathbf{T}^3)_M$. The quotient enjoys a continuous $U(1)$ isometry along the $\mathbf{S}^1$ fiber, generated by the invariant Killing vector $X_L^z = X_R^z = \partial_z$. However, the other two vectors X_L^x and X_L^y are not right-invariant, and so the corresponding continuous isometries disappear. Indeed, one can see that the action of X_L^x and X_L^y is in general different for different points of $\mathcal{H}_3(\mathbb{R})$ which are identified under (3.26). For instance,

$$\begin{aligned} e^{\lambda_x X_L^x} &: g(x,y,z) \to g(x+\lambda_x, y, z - \tfrac{1}{2}y\lambda_x) \\ e^{\lambda_x X_L^x} &: g(x, y+M, z+\tfrac{M}{2}x) \to g(x+\lambda_x, y+M, z+\tfrac{M}{2}x - \tfrac{1}{2}(y+M)\lambda_x) \\ &\qquad \sim g(x+\lambda_x, y, z - \tfrac{1}{2}y\lambda_x + M\lambda_x) \end{aligned} \tag{3.27}$$

and so these two actions are the same only if $\lambda_x \in \mathbb{Z}$. A similar statement holds for the parameter λ_y in (3.24b). Hence, one finds that the identifications (3.26) break two of the continuous isometries of the parent $\mathcal{H}_3(\mathbb{R})$, preserving only the discrete order-M actions generated by

$$e^{X_L^x} \,:\, g(x,y,z) \to g(x+1,y,z-\tfrac{1}{2}y) \quad , \quad e^{X_L^y} \,:\, g(x,y,z) \to g(x,y+1,z+\tfrac{1}{2}x). \tag{3.28}$$

Just like X_L^x and X_L^y, these generators do not commute, but rather produce an element of the $U(1)$ generated by X_L^z, and realise a discrete Heisenberg group $\mathbf{P} = H_M = \mathcal{H}_3(M = 1)/\mathcal{H}_3(M)$. This discrete non-abelian isometry group produces a discrete non-abelian gauge symmetry H_M in the lower-dimensional theory.

The above construction is a particular case of a more general setup (see e.g. [80,81]). Given a non-compact group G, the metric constructed with right-invariant forms has G itself as its isometry group (by right multiplication). In taking the coset G/H by a subgroup H, some of these isometries may survive (in continuous or discrete versions). In general, H is not a normal subgroup of G, so G/H is *not* a group, and cannot be the isometry group. To identify the correct isometry group, note that a point g_1 in G/H is, at the level of G, an equivalence class of points of the form $g_2 = g_1\gamma$, with $\gamma \in H$. An isometry R in G, mapping such g_1 and g_2 to g_1R and g_2R, is an isometry in G/H if the images are in the same equivalence class, namely if $g_2R = g_1R\gamma'$ for some $\gamma' \in H$. This requires R to satisfy $R^{-1}\gamma R = \gamma'$, namely conjugation by R should leave H invariant (although not necessarily pointwise). Those transformations form the so-called normaliser group N_H of H, and define the maximal subgroup of G such that H is normal in N_H. Since H acts trivially on G/H, the actual isometry group of G/H is N_H/H. It is easy to show that in the twisted torus the group $N_{\mathcal{H}_3(M)}/\mathcal{H}_3(M)$ corresponds to the one identified above, namely $H_M \times U(1)$.

It is natural to ask if, besides the above higher-dimensional description, there is a lower-dimensional description of the discrete gauge symmetry in terms of gauging of suitable scalars. Indeed, it is familiar that compactification on a twisted torus can alternatively be viewed as a compactification on $\mathbf{T}^3$ with metric fluxes, which can be described in terms of gauging a Heisenberg algebra [82]. The qualitative structure of the gauging is already manifest in the twisted torus metric, with $g_{xz} \sim y$ and $g_{yz} \sim x$, as follows. A gauge transformation of the KK gauge boson $V_\mu^x \sim g_\mu^x$ along the circle parametrised by y (i.e. a translation in y) shifts the vev of the scalar $\phi \sim g_{xz}$, and similarly for the KK gauge boson along x and the scalar g_{yz}. The integer M arises as the ratio of winding numbers of the map between full translations in the geometric circles, and the induced shifts in the scalar manifold. The non-abelian structure of the isometries of the scalar manifold makes the resulting discrete gauge symmetry non-abelian. This qualitative description can be fleshed out by performing the dimensional reduction explicitly; this was carried out in detail in the related but more interesting case of Type I magnetised toroidal compactification in [91]. See appendix C for a dimensional reduction of the T-dual setup of intersecting D6-branes.

3.2.2 Relation to magnetised D-branes and flavour symmetries

As described above, the twisted torus $(\mathbf{T}^3)_M$ can be regarded as an $\mathbf{S}^1$ fibration over a two-torus with first Chern class $\frac{M}{2\pi} dx \wedge dy$. Thus, instead of thinking about it as part of the physical space where we compactify we can look at it as the total space of a $U(1)$

bundle over $\mathbf{T}^2$ with M units of magnetic flux F. Namely, performing a dimensional reduction on the fibre we arrive at a flat two-torus together with a gauge field with some flux. This is realised in string theory by having a magnetised D-brane wrapped along $\mathbf{T}^2$ so we expect to have a non-abelian discrete gauge symmetry in this case too.

In the following we show that there is indeed a discrete symmetry by brute force, meaning that we compute the wavefunctions for massless fermions that are charged under such $U(1)$ and explicitly check that they arrange into a representation of the Heisenberg group identified above. Of course, this does not prove that such symmetry is gauge, this will be done in subsequent sections when we revisit this system. However, the advantage of this procedure is that in magnetised D-brane models these wavefunctions correspond to different flavours so we can already see the symmetry acting non-trivially on the flavour degrees of freedom.

Consider a toy model of a magnetised $\mathbf{T}^2$ with $F = 2\pi M dx \wedge dy$, $(M > 0)$. The Dirac equation for a massless Dirac fermion with charge $+1$ under $U(1)$ reads $\Gamma^\mu(\partial_\mu - iA_\mu)\Psi = 0$ so if we restrict to a left-handed fermion ψ and introduce the complex coordinate $z = x + \tau y$ with τ the complex structure of the torus it reduces to

$$\left(\bar{\partial} - \frac{i\pi M}{\mathrm{Im}\ \tau}\mathrm{Im}\ z\right)\psi = 0 \tag{3.29}$$

whose solutions are given in terms of Jacobi theta functions (see e.g. [50])

$$\psi^{j,M}(z,\bar{z}) = e^{i\pi M z \mathrm{Im}\ z/\mathrm{Im}\ \tau}\vartheta\begin{bmatrix} \frac{j}{M} \\ 0 \end{bmatrix}(Mz, M\tau), \qquad j = 0, 1, \ldots, M-1. \tag{3.30}$$

We defined the ϑ-functions as

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\nu,\tau) = \sum_{l\in\mathbb{Z}} e^{i\pi(l+a)^2\tau}e^{2\pi i(r+l)(\nu+b)}. \tag{3.31}$$

Let us check that this set of functions furnishes an irreducible representation of the Heisenberg group. Take the operators

$$A = \frac{1}{M}(\partial_x - i\pi M y), \quad B = \frac{1}{M}(\partial_y + i\pi M x), \quad C = \frac{2\pi i}{M} \tag{3.32}$$

which satisfy the Heisenberg algebra $[A, B] = C$ and therefore generate the discrete Heisenberg group upon integer exponentiation, i.e. $f(n_a, n_b, n_c) = \exp(n_a A + n_b B + n_c C)$ with $n_a, n_b, n_c \in \mathbb{Z}$. Acting with these operators on the wavefunctions one can check that

$$f(n_a, n_b, n_c)\psi^{j,M}(z,\bar{z}) = e^{\frac{2\pi i}{M}\left(n_c + \frac{n_a n_y}{2M}\right)}e^{\frac{2\pi i}{M}n_a j}\psi^{j+n_b,M}(z,\bar{z}) \tag{3.33}$$

so they indeed arrange in an irrep of H_M. Let us stress that the index j labels different copies of left-handed fermions with the same charge so it is a flavour index upon which the discrete symmetry acts.

Upon T-duality on one of the circles in $\mathbf{T}^2$ the magnetised D-branes turn into branes at angles where the units of flux are translated into the number of times the D-branes winds around one of the circles. This provides the starting point of the next section where we study intersecting brane models that exhibit such discrete non-abelian symmetry.

3.3 Intersecting branes

In this section we discuss the appearance of non-abelian flavour discrete symmetries in intersecting D6-brane models (see e.g. [1, 115]).

In order to make our discussion concrete we focus on intersecting D6-branes in the $\mathbb{Z}_2 \times \mathbb{Z}_2'$ orbifold which we review in the following and in appendix B, however, the main ideas can be generalised to other kinds of compactifications. Then, we analyse discrete gauge symmetries for intersecting D6-branes in the torus which will allow us to develop a geometrical intuition that is useful when dealing with orbifolds. We also discuss in some detail the distinction between exact and approximate symmetries. In the next section we consider the same problem from the T-dual perspective of magnetised D9-branes.

3.3.1 Intersecting branes and the $\mathbb{Z}_2 \times \mathbb{Z}_2'$ orbifold

Our examples of D-brane models with discrete flavour symmetries will be based on the toroidal $\mathbb{Z}_2 \times \mathbb{Z}_2'$ orientifold background analysed in [98] (see also [99,100,106]). As pointed out in there, in this background one can reproduce the main features of realistic D-brane models in Calabi-Yau compactifications, obtaining $\mathcal{N} = 1$ chiral vacua made up of rigid D-branes. In the following we will briefly review the construction of this class of models, emphasising those features which are more relevant for the analysis of discrete flavour symmetries. In doing so we will follow the notation and formalism of [98], which was mainly developed for models of intersecting D6-branes. Nevertheless, such models have a well-known T-dual description in terms of magnetised D-branes, a fact that we will exploit in section 3.4 to understand discrete flavour symmetries from an alternative viewpoint closer to the analysis of [91].

Branes at angles in the $\mathbb{Z}_2 \times \mathbb{Z}_2'$ orbifold

As emphasised in the literature (see [1, 97] for reviews on the subject) a quite successful approach to construct particle physics models from string theory is by considering models of intersecting D6-branes in type IIA Calabi-Yau compactifications. The simplest setup in which this approach can be implemented is by taking the compactification space to be a factorised six-torus $\mathbf{T}^6 = (\mathbf{T}^2)_1 \times (\mathbf{T}^2)_2 \times (\mathbf{T}^2)_3$ and adding sets of D6-branes that fill up 4d Minkowski space and wrap different three-dimensional slices of these six extra dimensions. Typically one considers that each D6-brane wraps a product of three one-cycles on the factorised $\mathbf{T}^6$, namely

$$[\Pi_\alpha] = \bigotimes_{i=1}^{3} \left(n_\alpha^i \ [a^i] + m_\alpha^i \ [b^i] \right) \qquad n_\alpha^i, m_\alpha^i \in \mathbb{Z} \text{ and coprime} \tag{3.34}$$

where $[a^i]$, $[b^i]$ correspond to the two fundamental one-cycles of $(\mathbf{T}^2)_i$. We show in figure 3.1.*i*) an example of two of these D6-branes a and b wrapping the three-cycles Π_a and Π_b respectively. If we now wrap N_a D6-branes on top of the three-cycle Π_a and N_b on top of Π_b we will have a 4d $U(N_a) \times U(N_b)$ gauge group upon dimensional reduction, with a 4d chiral fermion in the $(N_a, \bar{N}_b)$ representation at each intersection point. Hence, by considering several sets of D6-branes one can construct 4d effective theories similar to the Standard Model or extensions thereof [108–112].

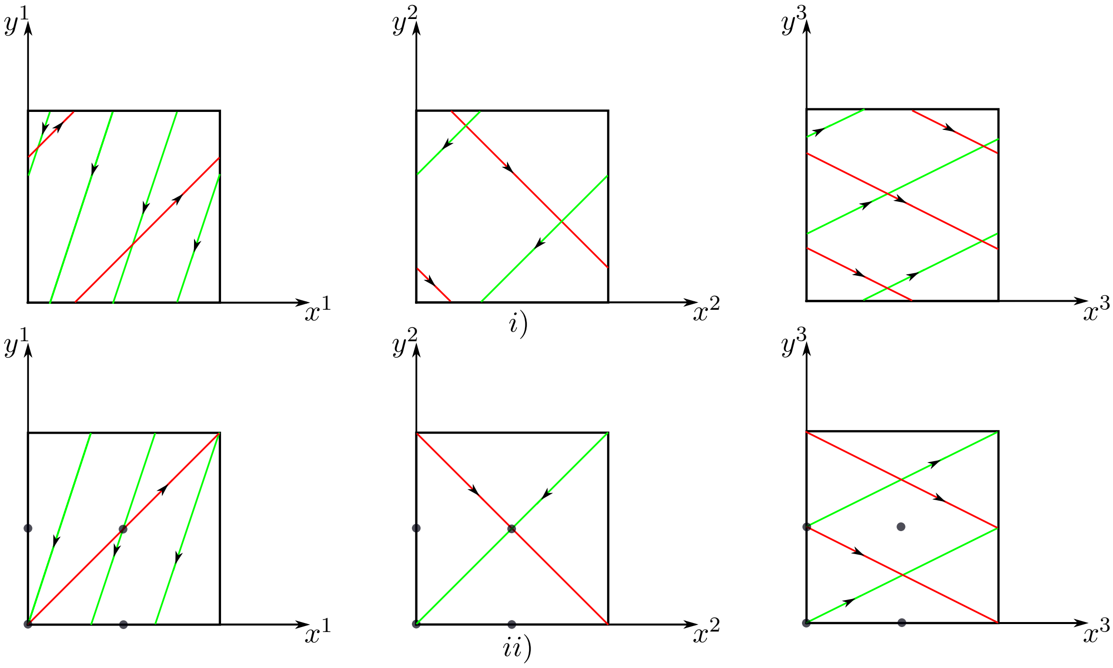

Figure 3.1: Branes with wrapping numbers $(n_a^i, m_a^i) = (1,1) \otimes (1,-1) \otimes (2,-1)$ (red) and $(n_b^i, m_b^i) = (-1,-3) \otimes (-1,-1) \otimes (2,1)$ (green) with $i)$ generic values of the position moduli and $ii)$ stuck at the fixed points in the $\mathbb{Z}_2 \times \mathbb{Z}_2'$ orbifold. The number of chiral families for $i)$ is $I_{ab}^{\mathbf{T}^6} = (-2) \times (-2) \times 4 = 16$ and for $ii)$ is $I_{ab} = 4$.

In this setup the relative orientation of a pair of D6-branes is specified in terms of three angles θ_{ab}^i, one per two-torus $(\mathbf{T}^2)_i$. One can render these two D6-branes mutually BPS by applying certain conditions to these angles [113]. However, in order to construct a consistent four-dimensional chiral and $\mathcal{N} = 1$ supersymmetric (and hence stable) model one needs the presence of negative tension objects like O6-planes and to replace the compactification manifold $\mathbf{T}^6$ by a toroidal orbifold of the form $\mathbf{T}^6/\Gamma$, with Γ a discrete symmetry group [114]. We will leave the effects of adding the O6-planes for section 3.5 and focus here on the implications of having D6-branes at angles in a toroidal orbifold rather than on $\mathbf{T}^6$.

In particular we will consider type IIA string theory compactified on the background $\mathbf{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2'$, the $\mathbb{Z}_2$ generators acting as

$$\Theta : \left\{ \begin{array}{l} z_1 \to -z_1 \\ z_2 \to -z_2 \\ z_3 \to z_3 \end{array} \right. \qquad \Theta' : \left\{ \begin{array}{l} z_1 \to z_1 \\ z_2 \to -z_2 \\ z_3 \to -z_3 \end{array} \right. \tag{3.35}$$

on the three complex coordinates of $\mathbf{T}^6 = (\mathbf{T}^2)_1 \times (\mathbf{T}^2)_2 \times (\mathbf{T}^2)_3$. This is the sort of background considered in [98] in order to construct semi-realistic models made of *rigid* D6-branes. These rigid or fractional D6-branes wrap three-cycles that are left invariant by (3.35) and so, unlike in the case of $\mathbf{T}^6$, it is not possible to displace them transversely.[5] As illustrated in figure 3.1.$ii)$ on each $(\mathbf{T}^2)_i$ a fractional D6-brane goes through two fixed

[5]The fact that fractional D6-branes are fully rigid is a consequence of the choice of discrete torsion in the $\mathbb{Z}_2 \times \mathbb{Z}_2'$ orbifold. Such choice implies that this orbifold contains collapsed three-cycles at the fixed loci of (3.35), see appendix B and ref. [98] for more details. In a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold with the opposite choice of discrete torsion fractional D6-branes are not rigid [114].

points of the action $z_i \to -z_i$. One can determine which are these two fixed points in terms of the wrapping numbers (n_a^i, m_a^i), as indicated in table 3.1. From the effective field theory viewpoint the fact that a D6-brane a is rigid implies that at low energies there will be no multiplets in the adjoint representation of $U(N_a)$. One can then build chiral $\mathcal{N}=1$ models where non-abelian gauge groups are asymptotically free [98], a required feature for realistic models in Calabi-Yau compactifications.

(n^i, m^i)	Fixed points on $(\mathbf{T}^2)_i$
(odd, odd)	$\{1,4\}$ or $\{2,3\}$
(odd, even)	$\{1,3\}$ or $\{2,4\}$
(even, odd)	$\{1,2\}$ or $\{3,4\}$

Table 3.1: Fixed points of a 1-cycle on a $\mathbf{T}^2/\mathbb{Z}_2$ in terms of its wrapping numbers.

Chirality and the orbifold projection

Let us now consider N_a D6-branes wrapping the three-cycle Π_a of $\mathbf{T}^6$ and N_b of them wrapping Π_b, with wrapping numbers

$$\begin{array}{lcccc} \Pi_a & : & (n_a^1, m_a^1) & (n_a^2, m_a^2) & (n_a^3, m_a^3) \\ \Pi_b & : & (n_b^1, m_b^1) & (n_b^2, m_b^2) & (n_b^3, m_b^3) \end{array} \tag{3.36}$$

as stated above, this D6-brane sector yields a 4d $U(N_a) \times U(N_b)$ gauge group at low energies, together with $\mathcal{N}=1$ chiral multiplets in the $(N_a, \bar{N}_b)$ representation, one per each point of intersection of these two three-cycles. The chirality and multiplicity of these multiplets is given respectively by the sign and by the absolute value of the topological intersection number I_{ab}, which in the case of $\mathbf{T}^6 = (\mathbf{T}^2)_1 \times (\mathbf{T}^2)_2 \times (\mathbf{T}^2)_3$ is given by

$$I_{ab}^{\mathbf{T}^6} = I_{ab}^1 I_{ab}^2 I_{ab}^3 = \prod_{i=1}^{3} (n_a^i m_b^i - n_b^i m_a^i). \tag{3.37}$$

If we now consider rigid D6-branes in $\mathbf{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2'$ the chiral spectrum will be different, because the $\mathbb{Z}_2 \times \mathbb{Z}_2'$ action relates several of the intersection points of the two D6-branes and these will no longer be independent degrees of freedom. In particular, one should project out all those zero modes that are not invariant under the $\mathbb{Z}_2 \times \mathbb{Z}_2'$ orbifold action, as we now describe.

Let us consider the case where $I_{ab}^{\mathbf{T}^6} \neq 0$, so that in the toroidal case we have a net number of chiral fermions in the ab sector. Since the D6-branes wrap BPS and factorisable three-cycles, the massless spectrum in the ab sector is given by $|I_{ab}^{\mathbf{T}^6}|$ 4d chiral fermions in the representation $(N_a, \bar{N}_b)$, whose 4d chirality is given by sign$(I_{ab}^{\mathbf{T}^6})$. In fact, by applying the usual CFT rules for computing the open string spectrum between two intersecting D-branes [110, 113], one can associate to each intersection the piece of 10d massless fermion whose SO(8) weight representation is given by [115]

$$r_{ab} = (r_0; r_1, r_2, r_3) = \frac{1}{2}(s_1 s_2 s_3; -s_1, -s_2, -s_3) \tag{3.38}$$

where the first entry indicates the 4d chirality, and the other three correspond to the compact extra dimensions. Here $s_i = \text{sign}(\vartheta^i_{ab})$, with ϑ^i_{ab} the angle of intersection in $(\mathbf{T}^2)_i$ measured anti-clockwise. Notice that $s_i = \text{sign}(I^i_{ab})$ and so 4d chirality is indeed given by $\text{sign}(I^{\mathbf{T}^6}_{ab})$. In our conventions $I^{\mathbf{T}^6}_{ab} > 0$ corresponds to 4d left-handed fermions.

When introducing the orbifold projection, some of these massless fields will be projected out. In particular, we must require that the internal fermionic wavefunctions are invariant under the action of the $\mathbb{Z}_2 \times \mathbb{Z}'_2$ generators. These act on a fermion with Lorentz indices (3.38) as

$$\begin{array}{rcl} \Theta: & \Psi(z_1,z_2,z_3) \mapsto & e^{i\pi(r_1-r_2)}\Psi(-z_1,-z_2,z_3) = s_1 s_2 \Psi(-z_1,-z_2,z_3) \\ \Theta': & \Psi(z_1,z_2,z_3) \mapsto & e^{i\pi(r_2-r_3)}\Psi(z_1,-z_2,-z_3) = s_2 s_3 \Psi(z_1,-z_2,-z_3). \end{array} \tag{3.39}$$

A generic open string wavefunction, which is basically a delta function localised at the intersection point, will not be invariant under such transformations, and so one must form linear combinations that transform appropriately under internal coordinate reversal.

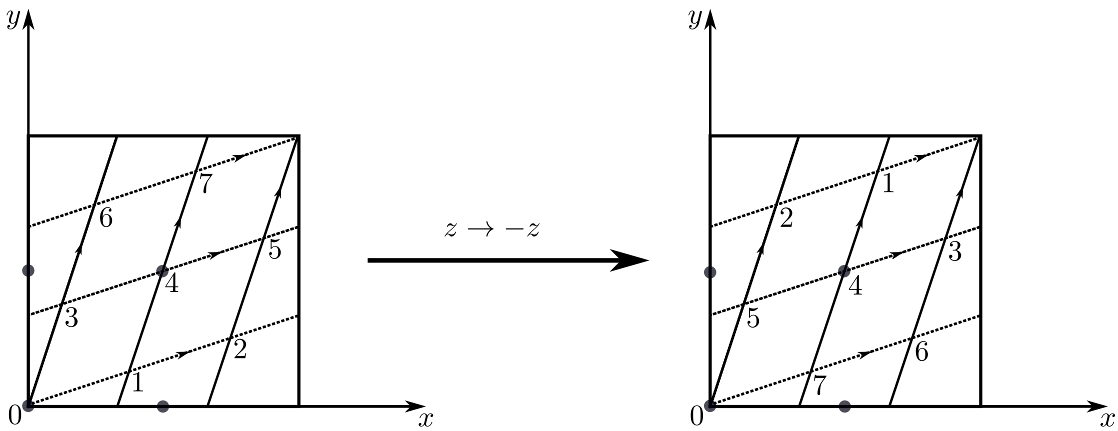

Figure 3.2: Space inversion in one of the tori and D-branes with wrapping numbers $(1,3)$ and $(3,1)$. Some of the intersection points are invariant while others get exchanged. The even combination of points are $\{0,\ 1+7,\ 2+6,\ 5+3,\ 4\}$ and the odd ones $\{1-7,\ 2-6,\ 5-3\}$. Notice that the number of even and odd points is in agreement with (3.40).

In order to describe these combination of wavefunctions let us first consider two D-branes wrapping 1-cycles of $(\mathbf{T}^2)_i$ going through the origin, and see how their intersection points transform under the $\mathbb{Z}_2$ action generated by $z \mapsto -z$. As shown in figure 3.2, one can take linear combinations of delta functions at the intersection points, in order to form wavefunctions which are even or odd under the orbifold action. Such linear combinations have coefficients ± 1 and the number of even and odd points for a given intersection number I^i_{ab} is given by

$$I^i_e = \frac{1}{2}\left(I^i_{ab} + s_i \rho_i\right) \qquad I^i_o = \frac{1}{2}\left(I^i_{ab} - s_i \rho_i\right) \tag{3.40}$$

where $s_i = \text{sign}(I^i_{ab})$ and

$$\rho_i \equiv \left\{ \begin{array}{ll} 1 & \text{for } I^i_{ab} \text{ odd} \\ 2 & \text{for } I^i_{ab} \text{ even.} \end{array} \right. \tag{3.41}$$

Going back to intersecting D6-brane on $(\mathbf{T}^2)_1 \times (\mathbf{T}^2)_2 \times (\mathbf{T}^2)_3 / \mathbb{Z}_2 \times \mathbb{Z}'_2$, from (3.39) it is clear that we need to impose that our wavefunctions satisfy

$$\Psi(z_1,z_2,z_3) = s_1 s_2 \Psi(-z_1,-z_2,z_3) = s_2 s_3 \Psi(z_1,-z_2,-z_3) \tag{3.42}$$

and so depending on the signs of the two-tori intersection numbers I_{ab}^i we will have to impose different projections. In particular we have that for $I_{ab}^B \neq 0$ and

- $s_1 s_2 > 0,\ s_2 s_3 > 0 \quad \Longrightarrow \quad \Psi = \psi_e^{j_1} \psi_e^{j_2} \psi_e^{j_3} \quad \text{or} \quad \psi_o^{j_1} \psi_o^{j_2} \psi_o^{j_3}$
- $s_1 s_2 > 0,\ s_2 s_3 < 0 \quad \Longrightarrow \quad \Psi = \psi_e^{j_1} \psi_e^{j_2} \psi_o^{j_3} \quad \text{or} \quad \psi_o^{j_1} \psi_o^{j_2} \psi_e^{j_3}$
- $s_1 s_2 < 0,\ s_2 s_3 > 0 \quad \Longrightarrow \quad \Psi = \psi_o^{j_1} \psi_e^{j_2} \psi_e^{j_3} \quad \text{or} \quad \psi_e^{j_1} \psi_o^{j_2} \psi_o^{j_3}$
- $s_1 s_2 < 0,\ s_2 s_3 < 0 \quad \Longrightarrow \quad \Psi = \psi_e^{j_1} \psi_o^{j_2} \psi_e^{j_3} \quad \text{or} \quad \psi_o^{j_1} \psi_e^{j_2} \psi_o^{j_3}$

where $\psi_e^{j_i}$ runs over even combinations of intersection points on $(\mathbf{T}^2)_i$, and $\psi_o^{j_i}$ is an odd combination of delta-wavefunctions in $(\mathbf{T}^2)_i$. For the first case above we have that the number of generations after the orbifold projection is given by $I_{ab} = I_e^1\, I_e^2\, I_e^3 + I_o^1\, I_o^2\, I_o^3$ or

$$I_{ab} \;=\; \frac{1}{4}\left[I_{ab}^1 I_{ab}^2 I_{ab}^3 + s_2 s_3 \rho_2 \rho_3 I_{ab}^1 + s_1 s_3 \rho_1 \rho_3 I_{ab}^2 + s_1 s_2 \rho_1 \rho_2 I_{ab}^3\right].$$

Hence, after imposing that $s_1 s_2 > 0$ and $s_2 s_3 > 0$ we recover the result

$$I_{ab} = \frac{1}{4}\left[I_{ab}^1 I_{ab}^2 I_{ab}^3 + I_{ab}^1\, \rho_2 \rho_3 + I_{ab}^2\, \rho_1 \rho_3 + I_{ab}^3\, \rho_1 \rho_2\right]. \tag{3.43}$$

A different choice of signs s_1, s_2, s_3 will select different parities for the wavefunctions of each two-torus, and so a different total number of chiral fermions. Nevertheless, the final expression for I_{ab} will again be given by (3.43). Notice that this result matches eq.(B.10), which has been obtained in appendix B by means of the topological techniques of [98].

The same statement holds if we consider the case where the toroidal intersection number $I_{ab}^{\mathbf{T}^6}$ vanishes, as we now briefly discuss. Let us for instance consider the case where only $I_{ab}^1 = 0$.[6] Then instead of (3.38) we have $I_{ab}^2 I_{ab}^3$ fermions of the form

$$r_{ab} \;=\; \frac{1}{2}\,(s_2 s_3; -, -s_2, -s_3) \quad \text{and} \quad \frac{1}{2}\,(-s_2 s_3; +, -s_2, -s_3) \tag{3.44}$$

that is, a non-chiral spectrum. The orbifold action reads

$$\begin{array}{rcl} \Theta: & \Psi(z_1, z_2, z_3) \;\mapsto\; \pm s_2 \Psi(-z_1, -z_2, z_3) \\ \Theta': & \Psi(z_1, z_2, z_3) \;\mapsto\; s_2 s_3 \Psi(z_1, -z_2, -z_3) \end{array} \tag{3.45}$$

and so we arrive at the following wavefunctions

- $s_2 > 0,\ s_3 > 0 \quad \Longrightarrow \quad \Psi = \psi_e^{j_2} \psi_e^{j_3} \quad \text{or} \quad \psi_o^{j_2} \psi_o^{j_3} \quad \Longrightarrow \quad I_{ab} = I_e^2 I_e^3 - I_o^2 I_o^3$
- $s_2 > 0,\ s_3 < 0 \quad \Longrightarrow \quad \Psi = \psi_e^{j_2} \psi_o^{j_3} \quad \text{or} \quad \psi_o^{j_2} \psi_e^{j_3} \quad \Longrightarrow \quad I_{ab} = I_e^2 I_o^3 - I_o^2 I_e^3$
- $s_2 < 0,\ s_3 > 0 \quad \Longrightarrow \quad \Psi = \psi_o^{j_2} \psi_e^{j_3} \quad \text{or} \quad \psi_e^{j_2} \psi_o^{j_3} \quad \Longrightarrow \quad I_{ab} = I_o^2 I_e^3 - I_e^2 I_o^3$
- $s_2 < 0,\ s_3 < 0 \quad \Longrightarrow \quad \Psi = \psi_o^{j_2} \psi_o^{j_3} \quad \text{or} \quad \psi_e^{j_2} \psi_e^{j_3} \quad \Longrightarrow \quad I_{ab} = I_o^2 I_o^3 - I_e^2 I_e^3$

[6] The case with two vanishing intersection numbers I_{ab}^i does not correspond to D6-branes preserving $\mathcal{N} = 1$ supersymmetry, and will not be considered here, while the case which all three $I_{ab}^i = 0$ is trivial.

where the relative minus sign in the expression for I_{ab} comes from the fact that the two fermions in (3.44) have opposite 4d chirality. Again, in each of the four cases above we find that the index of net chirality I_{ab} matches the expression (3.43) with $I^1_{ab} = 0$.

To summarise, by looking at the action of the orbifold on the open string degrees of freedom we can recover the chiral index obtained in [98] via topological methods. While we have focused on the fermionic modes, the same result is obtained by looking at the light scalars at the D6-brane intersections. The method used here to compute the chiral index I_{ab} is perhaps more involved that the one in [98], but it also carries more information. First, for the case where $I^{\mathbf{T}^6}_{ab} = 0$ not only does it compute the net chiral index (3.43) but also detects massless particles of opposite chirality that contribute with opposite signs to the index.[7] Second, this method not only gives the 4d massless spectrum, but also the explicit expression for the open string wavefunctions in the internal dimensions of the compactification. As we will now see, this will be crucial for studying in detail the discrete flavour symmetries that appear in this class of models.

3.3.2 Discrete flavour symmetries for intersecting branes

Having reviewed D-brane models on $\mathbf{T}^6/\mathbb{Z}_2 \times \mathbb{Z}'_2$ and how family replication arises for them, we now turn to show the emergence of discrete flavour symmetries in such models. More precisely, we will describe how the Dihedral group D_4 and products of it arise in this context, and how the different families transform non-trivially under them.

As discussed in [91], D_4 and other non-abelian discrete flavour symmetries naturally arise in the context of D-brane models, and in particular for models of magnetised D-branes on $\mathbf{T}^{2n}$. There one can detect discrete *gauge* flavour symmetries in terms of a 4d effective Lagrangian obtained via dimensional reduction. As shown in appendix C such Lagrangian can also be obtained from models of intersecting D-branes on $\mathbf{T}^{2n}$, and so in principle one can apply the 4d methods of [91] to detect discrete gauge symmetries. However, it turns out that in models of intersecting D-branes the presence of discrete flavour symmetries can be detected geometrically as well. This is particularly useful to describe them in $\mathbf{T}^6/\mathbb{Z}_2 \times \mathbb{Z}'_2$, where applying dimensional reduction is not obvious for certain sectors. In the following we will apply such geometric approach first for a toroidal background and then for $\mathbf{T}^6/\mathbb{Z}_2 \times \mathbb{Z}'_2$. Finally, this geometric picture allows to quickly detect when there is an exact discrete flavour symmetry and when such symmetry is just approximate, as we briefly discuss.

Flavour symmetries on the torus

Let us consider type IIA string theory compactified on $\mathbf{T}^6 = (\mathbf{T}^2)_1 \times (\mathbf{T}^2)_2 \times (\mathbf{T}^2)_3$. Such manifold contains 6 continuous isometries $(x_i, y_i) \to (x_i + \lambda_{x_i}, y_i + \lambda_{y_i})$, $i = 1, 2, 3$ which, upon dimensional reduction of the metric, manifest as a $U(1)^6$ gauge group in the 4d effective theory. We will represent such 4d gauge bosons respectively as $V^{x_i}_\mu$ and $V^{y_i}_\mu$.

Let us now introduce a D6-brane wrapping a factorisable three-cycle Π_a of the form (3.34). Geometrically, it is clear that the presence of such three-cycle breaks the invariance under translations along the three directions of $\mathbf{T}^6$ transverse to the D6-brane world-

[7] This spectrum is only computed at tree-level in the string coupling g_s, so one expects that vector-like pairs of massless particles will gain a mass by means of quantum corrections, unless some discrete symmetry forbids such mass term. See section 3.5 for an example.

volume, while in the three directions parallel to Π_a the translational isometries remain unbroken. From the effective field theory viewpoint three generators of the initial $U(1)^6$ gauge group become massive via a Stückelberg mechanism, in which the D6-brane scalars ϕ_a^i that parametrise the transverse displacement of Π_a in $(\mathbf{T}^2)_i$ are eaten by the generators of the corresponding isometry. Following appendix C, the Stückelberg Lagrangian reads

$$\mathcal{L}_{\rm St} \;=\; -\frac{1}{2}\sum_{i=1}^{3}\left(\partial_\mu \phi_a^i - m_a^i V_\mu^{x_i} + n_a^i V_\mu^{y_i}\right)^2 \tag{3.46}$$

and so the bulk gauge symmetry $U(1)^6$ is broken down to $U(1)^3 \times \mathbb{Z}_{q_1} \times \mathbb{Z}_{q_2} \times \mathbb{Z}_{q_3}$. The $U(1)$ factors are generated by the massless combinations $n_a^i V_\mu^{x_i} + m_a^i V_\mu^{y_i}$, while the factors $\mathbb{Z}_{q_i}$ are the discrete remnants of the broken $U(1)$ symmetries generated by $n_a^i V_\mu^{y_i} - m_a^i V_\mu^{x_i}$. This symmetry breaking pattern is similar to the one studied in [92], section 2.5, from where one deduces that $q_i = (n_a^i)^2 + (m_a^i)^2$.

Needless to say, adding more D6-branes will further break the translational symmetry. In particular, one would expect that by adding a D6-brane on a three-cycle Π_b that intersects Π_a transversally all continuous symmetries are broken. Indeed, one then finds that the Lagrangian reads

$$\mathcal{L}_{\rm St} \;=\; -\frac{1}{2}\sum_{i=1}^{3}\left(\partial_\mu \phi_a^i - m_a^i V_\mu^{x_i} + n_a^i V_\mu^{y_i}\right)^2 + \left(\partial_\mu \phi_b^i - m_b^i V_\mu^{x_i} + n_b^i V_\mu^{y_i}\right)^2 \tag{3.47}$$

and so if $I_{ab}^i = n_a^i m_b^i - n_b^i m_a^i \neq 0 \;\forall i$ then all gauge bosons $V_\mu^{x_i}, V_\mu^{y_i}$, become massive. In fact, as discussed in appendix C the remaining discrete gauge symmetry is given by

$$\mathcal{T}^{ab}_{\mathbf{T}^6} \;=\; \mathbb{Z}_{I^1_{ab}} \times \mathbb{Z}_{I^2_{ab}} \times \mathbb{Z}_{I^3_{ab}}. \tag{3.48}$$

Finally, additional D6-branes on three-cycles Π_c, Π_d, etc may further break this symmetry.

While understanding discrete symmetries from the viewpoint of the effective theory is quite powerful, it is instructive to develop a more geometrical picture of their meaning. For this, let us focus on one of the $\mathbf{T}^2$ factors of $\mathbf{T}^6$. As shown in figure 3.3a, the absence of D-branes implies a $U(1)^2$ gauge symmetry that corresponds to invariance of the background upon translation in the x and y coordinates. Adding a D-brane a on a 1-cycle $n[a] + m[b]$ partially breaks this translational symmetry (fig. 3.3b): infinitesimal translations in the direction $\vec{v}_\parallel = (n, m)$ leave the geometry invariant while those along $\vec{v}_\perp = (-m, n)$ do not. Nevertheless, finite translations along $\vec{v}_\perp$ do leave the geometry invariant and these, upon quotienting by the coordinate identifications of $\mathbf{T}^2$, generate a discrete group $\mathbb{Z}_q$ with $q = n^2 + m^2$. One then obtains a gauge group $U(1) \times \mathbb{Z}_q$. Adding a second D-brane b that intersects the first one (fig. 3.3c) will totally break the invariance under infinitesimal translations. Still, a discrete translational symmetry remains, given by the cyclic permutation of the intersection points of the two D-branes, and this generates a $\mathbb{Z}_{I_{ab}}$ gauge symmetry. Applying this result to each $(\mathbf{T}^2)_i$ factor of $\mathbf{T}^6$ we obtain (3.48).[8]

[8]In general, given a $\mathbf{T}^{2n}$ geometry we have a $U(1)^{2n}$ translational symmetry. If we introduce two n-cycles Π_a^n, Π_b^n that are each a $\mathbf{T}^n \subset \mathbf{T}^{2n}$ and that intersect transversally, then the group of translational symmetry is broken to $\mathcal{T} = \Gamma/\hat{\Gamma}$, where Γ is the lattice generated by the intersection points and $\hat{\Gamma}$ is the lattice of coordinate identifications that defines $\mathbf{T}^{2n}$. When $\mathbf{T}^{2n}$ is factorisable $\mathcal{T}$ is a direct product of discrete subgroups, as in (3.48).

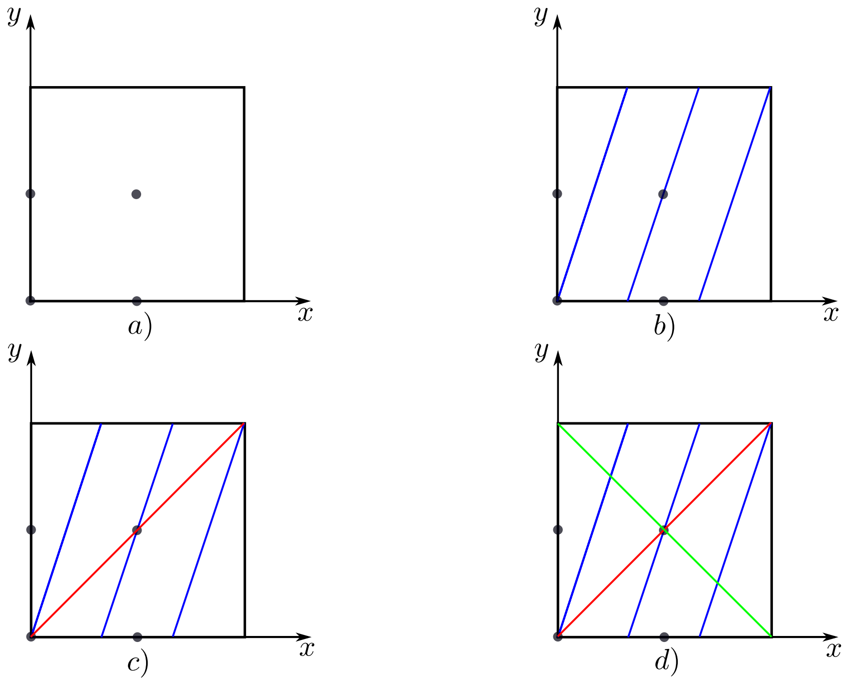

Figure 3.3: $\mathbf{T}^2/\mathbb{Z}_2$ with a) no branes b) one brane on the cycle (1,3) c) two branes on (1,3) and (1,1) d) three branes on (1,3), (1,1) and (1,-1).

From this geometrical perspective one can also see that we are indeed dealing with a flavour symmetry, that acts on the intersection points of each $\mathbf{T}^2$ as the shift generator

$$g_{\mathcal{T}} = \begin{pmatrix} & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ 1 & & & & \end{pmatrix}. \tag{3.49}$$

Finally, this symmetry is further broken if we include additional D-branes (fig. 3.3d). One can check that if we add a D-brane c then the fundamental region of $\mathbf{T}^2$ will be divided into d identical regions, with $d = \text{g.c.d.}(I_{ab}, I_{bc}, I_{ca})$ [116]. The remaining discrete gauge symmetry is then $\mathbb{Z}_d$, which corresponds to the common factor $\mathbb{Z}_{I_{ab}} \cap \mathbb{Z}_{I_{bc}} \cap \mathbb{Z}_{I_{ca}}$ of the symmetries for each pair of D-branes. Going back to the case of $\mathbf{T}^6 = (\mathbf{T}^2)_1 \times (\mathbf{T}^2)_2 \times (\mathbf{T}^2)_3$, we conclude that for a system of three D6-branes the translational symmetry is given by

$$\mathcal{T}^{abc}_{\mathbf{T}^6} = \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \mathbb{Z}_{d_3} \tag{3.50}$$

with $d_i = \text{g.c.d.}(I^i_{ab}, I^i_{bc}, I^i_{ca})$. This kind of symmetries will constrain the values of the Yukawa couplings of this sector, as pointed out in [91, 103, 116].

In fact, the above is not the complete flavour symmetry of the model, as there are further bulk symmetries that are broken by the presence of the D6-branes. Besides the 4d $U(1)^6$ gauge symmetry arising from the metric there will be a 4d $U(1)^6$ gauge symmetry that comes from the B-field, and is generated by the 4d gauge bosons $B^{x_i}_\mu$, $B^{y_i}_\mu$ that arise upon dimensional reduction. From appendix C, the Stückelberg Lagrangian for a single

D6-brane reads

$$\mathcal{L}_{\mathrm{St}} = -\frac{1}{2}\sum_{i=1}^{3}\left(\partial_\mu \xi_a^i - n_a^i B_\mu^{x_i} - m_a^i B_\mu^{y_i}\right)^2 \tag{3.51}$$

with ξ_a^i the Wilson line modulus of the D6-brane on $(\mathbf{T}^2)_i$. This action also has a simple geometrical interpretation, namely that acting with a B-field gauge generator induces a Wilson line on the D6-brane via pull-back on its worldvolume Π_a. A gauge transformation along $-m_a^i B_\mu^{x_i} + n_a^i B_\mu^{y_i}$ will have vanishing pull-back and will remain a symmetry of the background, while one along $n_a^i B_\mu^{x_i} + m_a^i B_\mu^{y_i}$ will be detected by the D6-brane and the corresponding $U(1)$ symmetry will be broken to a discrete subgroup. One can again see that the remaining symmetry is given by $U(1)^3 \times \mathbb{Z}_{q_1} \times \mathbb{Z}_{q_2} \times \mathbb{Z}_{q_3}$.

Adding further D6-branes will generalise this Lagrangian to

$$\mathcal{L}_{\mathrm{St}} = -\frac{1}{2}\sum_{\alpha}\sum_{i=1}^{3}\left(\partial_\mu \xi_\alpha^i - n_\alpha^i B_\mu^{x_i} - m_\alpha^i B_\mu^{y_i}\right)^2 \tag{3.52}$$

with $\alpha = a, b, c, \ldots$. For a system of two D6-branes a and b the symmetry is broken to

$$\mathcal{W}_{\mathbf{T}^6}^{ab} = \mathbb{Z}_{I_{ab}^1} \times \mathbb{Z}_{I_{ab}^2} \times \mathbb{Z}_{I_{ab}^3} \tag{3.53}$$

which in principle looks similar to (3.48) but the action of the generators on the flavour degrees of freedom is quite different. In this case the generator of the flavour symmetry acts on the intersection points of each $\mathbf{T}^2$ as the clock generator[9]

$$g_{\mathcal{W}} = \begin{pmatrix} 1 & & & & \\ & e^{2\pi i \frac{1}{N}} & & & \\ & & e^{2\pi i \frac{2}{N}} & & \\ & & & \ddots & \\ & & & & e^{2\pi i \frac{N-1}{N}} \end{pmatrix} \tag{3.54}$$

with $N = I_{ab}^i$. As it is easy to check the generators (3.49) and (3.54) do not commute, and so with their combined action they end up generating the discrete non-abelian group of the form $H_N \simeq (\mathbb{Z}_N \times \mathbb{Z}_N) \rtimes \mathbb{Z}_N$ for each $\mathbf{T}^2$, or more precisely

$$\mathbf{P}_{\mathbf{T}^6}^{ab} = H_{I_{ab}^1} \times H_{I_{ab}^2} \times H_{I_{ab}^3} \tag{3.55}$$

which is the result obtained in the T-dual picture of magnetised D9-branes [91, 103]. Finally, for a triplet of D6-branes this symmetry is reduced to

$$\mathbf{P}_{\mathbf{T}^6}^{abc} = H_{d_1} \times H_{d_2} \times H_{d_3} \tag{3.56}$$

with again $d_i = \text{g.c.d.}(I_{ab}^i, I_{bc}^i, I_{ca}^i)$

[9] The action of B_μ is equivalent to switching on a Wilson line, $A^\alpha = d\chi^\alpha = 2\pi d\zeta^\alpha$, where $\zeta^\alpha \sim \zeta^\alpha + 1$ is the coordinate of the D6-brane α along the corresponding $\mathbf{T}^2$. An open string located at $\zeta^\alpha = j/N$ and with charge q_α will have its phase shifted as $e^{iq_\alpha \chi^\alpha} = e^{2\pi i q_\alpha j/N}$, from where the action (3.54) follows.

Flavour symmetries on $\mathbb{Z}_2 \times \mathbb{Z}_2'$

Let us now consider the case of type IIA string theory compactified on $\mathbf{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2'$. Unlike the case of $\mathbf{T}^6$ the orbifold background does not have any continuous isometry even in the absence of D-branes. Hence the dimensional reduction that led to effective actions of the form (3.47) or (3.52) does not apply, and we need to use a different method to determine which are the discrete flavour symmetries that can arise in this case. Notice that the same will be true in Calabi-Yau compactifications, as these manifolds do not contain any continuous isometry either.

Fortunately in our discussion of $\mathbf{T}^6$ we have developed an alternative method for detecting discrete flavour symmetries. For instance, in the case of translational isometries the flavour symmetry was understood as the group of isometries of the manifold that is also preserved by the D-brane configuration. This observation can also be applied to the $\mathbf{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2'$ orbifold, whose group of translational isometries is discrete and given by $\mathbb{Z}_2^6$. Upon dimensional reduction this will give rise to a 4d discrete $\mathbb{Z}_2^6$ gauge group that will be broken to a subgroup by the inclusion of D6-branes, and this subgroup will be part of the discrete flavour symmetry of the model.

To get an idea of this symmetry breaking let us again consider the toy example $\mathbf{T}^2/\mathbb{Z}_2$. The $\mathbb{Z}_2$ quotient is generated by $z \mapsto -z$ and so there are four fixed points that break the continuous isometry group $U(1)^2$ of $\mathbf{T}^2$ down to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The generators of this discrete group are the actions $z \mapsto z + 1/2$ and $z \mapsto z + \tau/2$, with τ the complex structure of the torus, that interchange the fixed points at $\{0, 1/2, \tau/2, (1+\tau)/2\}$ among them.

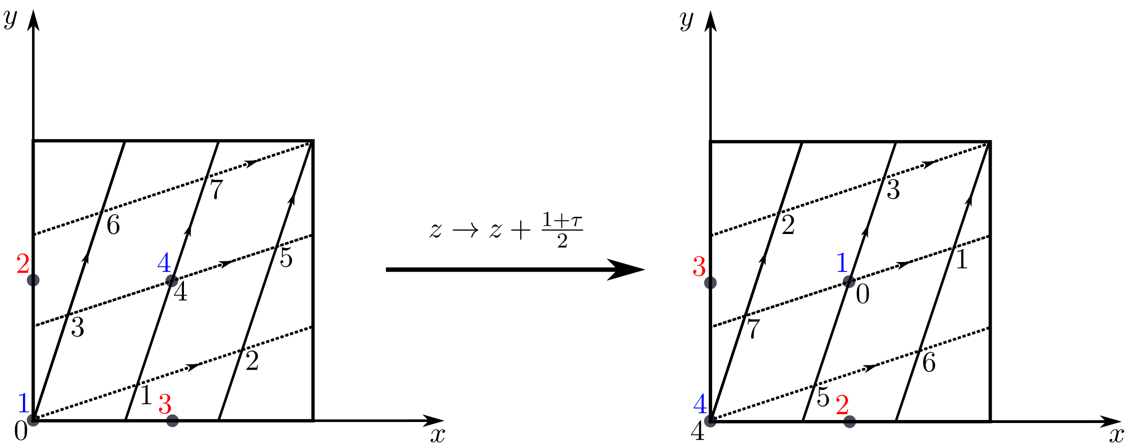

Figure 3.4: Translation $z \to z + \frac{1+\tau}{2}$ in a square torus. This is the generator of the shift symmetry in the intersecting brane picture.

Let us now introduce D-branes in this background. In our toy example fractional D-branes are represented by 1-cycles that pass through two of the four fixed points of $\mathbf{T}^2/\mathbb{Z}_2$. It is then clear that the presence of a single D-brane breaks the group of translations $\mathbb{Z}_2 \times \mathbb{Z}_2$ down to $\mathbb{Z}_2$, where this latter $\mathbb{Z}_2$ interchanges the two fixed points that the D-brane goes through. For instance, as shown in figure 3.4 a D-brane whose wrapping numbers (n, m) are both odd will go through the fixed points {1,4} or {2,3} (see table 3.1). The symmetry of this system is then the $\mathbb{Z}_2$ generated by $z \to z + \frac{1+\tau}{2}$ that interchanges the fixed points as $1 \leftrightarrow 4$ and $2 \leftrightarrow 3$. As figure 3.4 also shows, this $\mathbb{Z}_2$ symmetry will still be preserved after we introduce a second D-brane, provided that it also goes through the fixed points

{1,4} or {2,3} or, in other words, if its wrapping numbers (n, m) are both odd as well. In general, a pair of fractional 1-cycles on $\mathbf{T}^2/\mathbb{Z}_2$ will preserve a $\mathbb{Z}_2$ translational symmetry if they belong to the same row of table 3.1, which is equivalent to asking that the intersection number $I_{ab} = n_a m_b - n_b m_a$ is even. Finally, three or more 1-cycles will preserve the same $\mathbb{Z}_2$ symmetry if they all belong to the same row of table 3.1, or in other words if all the pairwise intersection numbers $I_{ab}, I_{bc}, I_{ca}, \ldots$ are even.

One may now generalise these observations to the case of $(\mathbf{T}^2)_1 \times (\mathbf{T}^2)_2 \times (\mathbf{T}^2)_3/\mathbb{Z}_2 \times \mathbb{Z}_2'$, as each $(\mathbf{T}^2)_i$ factor will behave like our toy example. Instead of our previous result (3.48) for a pair of D6-branes on $\mathbf{T}^6$ we now have that each $(\mathbf{T}^2)_i$ factor contributes at most with a $\mathbb{Z}_2$ symmetry, and only if the intersection number I^i_{ab} is even. Hence

$$\mathcal{T}^{ab}_{\mathbf{T}^6/\mathbb{Z}_2\times\mathbb{Z}_2'} = \mathbb{Z}_{\rho_1} \times \mathbb{Z}_{\rho_2} \times \mathbb{Z}_{\rho_3} \tag{3.57}$$

with ρ_i defined as in (3.41). Similarly, for a system of three D6-branes we have

$$\mathcal{T}^{abc}_{\mathbf{T}^6/\mathbb{Z}_2\times\mathbb{Z}_2'} = \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \mathbb{Z}_{d_3} \tag{3.58}$$

where now $d_i = \text{g.c.d.}(2, I^i_{ab}, I^i_{bc}, I^i_{ca})$.

Just like in the case of $\mathbf{T}^6$, this will not be the whole flavour symmetry group. There will also be a symmetry group generated by discrete gauge transformations of the B-field, whose gauge bosons B^{x_i,y_i}_μ are projected out infinitesimally by the orbifold. Just like for finite translations, these finite B-field transformations generate a $\mathbb{Z}_2^6$ group on $\mathbf{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2'$ that is broken to a subgroup when D6-branes are introduced. One can check that this subgroup $\mathcal{W}^{ab}_{\mathbf{T}^6/\mathbb{Z}_2\times\mathbb{Z}_2'}$ is isomorphic to (3.57) when two D6-branes are introduced, and similarly for $\mathcal{W}^{abc}_{\mathbf{T}^6/\mathbb{Z}_2\times\mathbb{Z}_2'}$ and (3.58) for a triplet of D6-branes. As in the $\mathbf{T}^6$ case, the two $\mathbb{Z}_2$ subgroups that arise from $(\mathbf{T}^2)_i$ do not commute, but rather generate the non-abelian group $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \simeq H_2$, which is nothing but the Dihedral group D_4. The final symmetry group for a pair of D6-branes would then be given by

$$\mathbf{P}^{ab}_{\mathbf{T}^6/\mathbb{Z}_2\times\mathbb{Z}_2'} = H_{\rho_1} \times H_{\rho_2} \times H_{\rho_3} = D_4^{[\rho_1-1]} \times D_4^{[\rho_2-1]} \times D_4^{[\rho_3-1]} \tag{3.59}$$

where $D_4^{[0]}$ is the trivial group and $D_4^{[1]} = D_4$, while for a D6-brane triplet we should have

$$\mathbf{P}^{abc}_{\mathbf{T}^6/\mathbb{Z}_2\times\mathbb{Z}_2'} = D_4^{[d_1-1]} \times D_4^{[d_2-1]} \times D_4^{[d_3-1]} \tag{3.60}$$

with $d_i = \text{g.c.d.}(2, I^i_{ab}, I^i_{bc}, I^i_{ca})$.

We will rederive this result in the next section, where we will use the T-dual framework of magnetised D-branes to obtain (3.59) and (3.60), as well as to make contact with the results of [101–104]. In addition to deriving the symmetry group we will use the magnetised picture to classify under which representations do the chiral families transform on each model.

Exact versus approximate symmetries

The previous discussion is quite useful in order to draw a notion of exact and approximate discrete symmetry for this class of models. By exact symmetry it is meant a discrete *gauge* symmetry of the 4d effective field theory, in the sense of [117–124]. The non-abelian discrete symmetries discussed in [91] are of this sort, the procedure to detect the gauge nature of a discrete symmetry being the construction of the effective 4d Lagrangian. For the case of the $\mathbb{Z}_2 \times \mathbb{Z}_2'$ orbifold, the construction of such 4d Lagrangian is beyond the scope of this paper, and so we will instead adopt a different approach and discuss the exactness of a discrete flavour symmetry by means of the geometric intuition developed above.

As pointed out in the last subsection, the $\mathbb{Z}_2 \times \mathbb{Z}_2'$ orbifold background has a group of translational isometries given by $\mathcal{T}_{\mathbb{Z}_2 \times \mathbb{Z}_2'} = \mathbb{Z}_2^6$. Coming from isometries of the internal manifold, this group naturally translates as a $\mathbb{Z}_2^6$ gauge group in the 4d effective theory. Adding a fractional D6-brane will break this group down to $\mathbb{Z}_2^3$, where each $\mathbb{Z}_2$ factor comes from a different $(\mathbf{T}^2)_i$. This $\mathbb{Z}_2^3$ symmetry group can be understood as the group of translations that leaves both the orbifold background and the D-brane invariant, and so it is a natural candidate for a 4d discrete gauge symmetry of the orbifold plus D-brane background. The question is now if the whole set of D-branes in a given model will respect such symmetry as well, or in other words if the whole orbifold plus D-brane backgrounds will be invariant under this $\mathbb{Z}_2^3$ translational symmetry or a subgroup thereof.[10]

By looking at figure 3.4 it is clear that a group of two or more D-branes will be invariant under a $\mathbb{Z}_2$ shift symmetry of $(\mathbf{T}^2)_i$ if all of them go through the same pair of fixed points. In fact, the condition for the symmetry to be exact is weaker, and we only need to require that all the intersection numbers $I^i_{\alpha\beta}$ between D-branes in this two-torus are even. The same is true for the $\mathbb{Z}_2$ discrete symmetry that arises from the B-field, and so for the whole D_4 that both $\mathbb{Z}_2$ actions generate. Let us denote as $D_4^{(i)}$ the dihedral flavour symmetry that may arise from $(\mathbf{T}^2)_i$, we then have that

$$D_4^{(i)} \text{ is exact} \quad \Longleftrightarrow \quad I^{(i)}_{\alpha\beta} \text{ is even } \forall \alpha, \beta \tag{3.61}$$

and so the group (3.60) may become a gauge or exact discrete gauge symmetry of the 4d effective theory depending on the whole set of D-brane intersection numbers.[11]

It may seem that the exactness condition (3.61) is kind of restrictive when constructing explicit D-brane models. However, as follows from the results of [98], one typically needs that D6-branes go through the same fixed points in order to satisfy the RR twisted tadpole conditions necessary to construct an anomaly-free consistent model. From this viewpoint, the stringy consistency conditions of the model render natural the appearance of exact discrete flavour symmetries in the low energy theory, as the examples of section 3.5 illustrate.

To be more precise let us consider a $\mathbb{Z}_2 \times \mathbb{Z}_2'$ model with K stacks of fractional D6-branes, and let us separate them in two subgroups $A = \{a_1, a_2, a_3, \dots\}$ and $B = \{b_1, b_2, \dots\}$, wrapping three-cycles Π_{a_i}, Π_{b_j} of the orbifold. The group A will yield a 4d gauge group $\prod_i U(N_{a_i})$, whose chiral spectrum will be specified by the intersection numbers $I_{a_k a_l}$. Typically, demanding that this sector of the theory is free of chiral anomalies by itself will impose cancellation of RR twisted tadpoles within the group A of D6-branes,

[10] Another important element of a D-brane model is the orientifold planes or O-planes, which we have so far ignored. One can check that their presence does not further break these discrete symmetries, at least for the class of models with rectangular $(\mathbf{T}^2)_i$ that we will consider in section 3.5.

[11] Strictly speaking if (3.61) is true then $D_4^{(i)}$ is a global symmetry of the 2d action of the BCFT theory, which becomes a local symmetry in target space to all orders in string perturbation theory. One should still check that this symmetry is preserved at the non-perturbative level in the string coupling, something that here is assumed.

and this will most likely happen when all the D6-branes in A respect the same $D_4^{(i)}$ symmetry in $(\mathbf{T}^2)_i$. There will then be a discrete symmetry group of the form (3.60) acting on this sector and constraining its couplings in the 4d effective field theory.

We may in particular consider the case where the group A of D6-branes contains the spectrum of the Standard Model or an extension thereof, while the group B of additional D6-branes contains an extra (hopefully hidden) sector of the theory. If all the $(\mathbf{T}^2)_i$ intersection numbers $I^i_{a_k a_l}$ are even there will be a flavour symmetry group $D_4^{(i)}$ acting on the visible sector of the theory. However, the extra sector B may not respect such symmetry, and if there is a single intersection number $I^i_{a_k b_l}$ which is odd then $D_4^{(i)}$ will not be an exact symmetry of the model. Nevertheless, it can still be considered an *approximate* symmetry of the sector A, because all the couplings within this sector must still respect this symmetry at least at tree-level. In particular, in a supersymmetric model the holomorphic Yukawa couplings of this sector will be constrained by the flavour symmetry at all orders in perturbation theory. The Kähler potential, on the other hand, may already get symmetry-breaking corrections at the perturbative level by effects involving the D-branes b_l that do not respect the symmetry (e.g., massive open strings attached to b_l running in loops). It would be interesting to see if the scenarios and techniques that apply to approximate continuous symmetries, see e.g. [125,126], could also be at work for this case.

3.4 Magnetised branes

An interesting feature of the D-brane models analysed in the previous sections is that they have a well-known T-dual description in terms of magnetised D-branes, which is a fruitful arena for understanding flavour symmetries. Indeed, as emphasised in [50], the framework of magnetised D-branes allows to compute 4d effective couplings by first solving for the internal wavefunction of the chiral modes, and then calculating their overlap over the extra dimensions of the compactification. As pointed out in [103], by inspection of these zero mode wavefunctions one can understand the flavour symmetries present in the model. Finally, it was shown in [91] how to obtain from this framework a 4d effective Lagrangian describing such discrete gauge flavour symmetries.

In the following we will rederive our previous results in the dual context of magnetised D-brane models, both in $\mathbf{T}^6$ and in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. This will allow to perform a more systematic analysis of the discrete flavour symmetries, and in particular to see under which representation the different families of chiral multiplets transform.[12]

3.4.1 Non-abelian flavour symmetries from magnetisation

Let us first consider type IIB string theory compactified on the factorised six-torus $\mathbf{T}^6 = (\mathbf{T}^2)_1 \times (\mathbf{T}^2)_2 \times (\mathbf{T}^2)_3$ and N D9-branes filling the whole of 10d space-time. We may add an non-trivial magnetisation $\langle F_2 \rangle$ along the coordinates of $\mathbf{T}^6$ without breaking 4d Poincaré invariance. In particular, we may choose a $U(N)$ Yang-Mills field strength of the

[12] Although more systematic, the field theory framework of magnetised D9-branes is less general than the framework of intersecting D6-branes, because it fails to capture the actual 4d effective theory when the magnetic fluxes are not diluted and/or when anti-D9-branes or D-branes of lower dimension are present. In this sense, the results of this section can be seen as complementary to the ones obtained previously in the context of intersecting D6-branes.

form

$$F_2 = \sum_{i=1}^{3} \frac{\pi i}{\text{Im } \tau^i} \begin{pmatrix} \frac{m_a^i}{n_a^i}\mathbb{I}_{N_a} & & & \\ & \frac{m_b^i}{n_b^i}\mathbb{I}_{N_b} & & \\ & & \frac{m_c^i}{n_c^i}\mathbb{I}_{N_c} & \\ & & & \ddots \end{pmatrix} dz^i \wedge d\bar{z}^i \tag{3.62}$$

where $z^i = dx^i + \tau^i dy^i$ is the complexified coordinate of $(\mathbf{T}^2)_i$, $N_\alpha = n_\alpha^1 n_\alpha^2 n_\alpha^3$, $N = \sum_\alpha N_\alpha$. Each block within (3.62) can be seen as a different D9-brane with 'magnetic numbers' $n_\alpha^i, m_\alpha^i \in \mathbb{Z}$ and with gauge group $U(d_\alpha^1 d_\alpha^2 d_\alpha^3)$, $d_\alpha^i = \text{g.c.d.}(n_\alpha^i, m_\alpha^i)$ [127]. One can then describe a pair of D9-branes in terms of these magnetic numbers

$$\begin{array}{lcccc} D9_a & : & (n_a^1, m_a^1) & (n_a^2, m_a^2) & (n_a^3, m_a^3) \\ D9_b & : & (n_b^1, m_b^1) & (n_b^2, m_b^2) & (n_b^3, m_b^3) \end{array} \tag{3.63}$$

in a rather analogous fashion to (3.36). In fact, both configurations are mapped to each other by performing three T-dualities, as have been used extensively in the literature. In this correspondence, the matter localised at the D6-brane intersections $\Pi_a \cap \Pi_b$ is mapped to the set of zero modes that arise from a $N_a \times N_b$ submatrix of the 10d U(N) adjoint fields (Ψ, A_M) [50]. In the following we will assume that $d_\alpha^i = \text{g.c.d.}(n_\alpha^i, m_\alpha^i) = 1$ and, in particular, that $n_\alpha^i = 1\ \forall \alpha, i$.[13] This greatly simplifies the analysis, since then $N_\alpha = 1$ and the internal profile of the 4d chiral zero modes is an scalar wavefunctions ψ^j instead of a matrix of wavefunctions.

As pointed out in [103], by inspection of such zero mode wavefunctions one can guess the flavour symmetry of a model of magnetised D-branes. However, as we will now show, one can directly characterise this flavour symmetry group by looking at the symmetries of the D-brane configuration, without solving for any zero mode. For simplicity, let us first consider a $\mathbf{T}^2$ and a $U(2)$ gauge sector with a magnetisation

$$F_2 = \frac{\pi i}{\text{Im } \tau} \begin{pmatrix} m_a & \\ & m_b \end{pmatrix} dz \wedge d\bar{z} \tag{3.64}$$

that breaks the gauge symmetry down to $U(1)_a \times U(1)_b$. In general this system is interpreted as two magnetised D-branes a and b, whose zero modes in the ab sector feel the relative flux $M = m_a - m_b = -I_{ab}^{\mathbf{T}^2}$. Even if F_2 is constant we have a vector potential of the form

$$A(x,y) = \pi \begin{pmatrix} m_a & \\ & m_b \end{pmatrix} (xdy - ydx) \tag{3.65}$$

which breaks the invariance under translations. More precisely we have that

$$\begin{array}{rcl} A(x+\lambda_x, y) & = & A(x,y) + \pi\lambda_x\,(m_a X_a + m_b X_b)dy, \\ A(x, y+\lambda_y) & = & A(x,y) - \pi\lambda_y\,(m_a X_a + m_b X_b)dx \end{array} \tag{3.66}$$

where

$$X_a = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \quad \text{and} \quad X_b = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}. \tag{3.67}$$

[13] While more involved, one can generalise the analysis for the case $n_\alpha^i > 1$, along the lines of [50, 105]. The case where some of the $n_\alpha^i = 0$ also makes sense, and describes a model with D7, D5 or D3-branes. This case, however, it is difficult to analyse from the field theory viewpoint and it is then more convenient to analyse the discrete flavour symmetries from the T-dual framework of intersecting D6-branes.

Eq.(3.66) can be interpreted as the fact that, in the presence of the background flux (3.64), an arbitrary translation is no longer a symmetry of the theory because $\langle A \rangle$ is not invariant under it. Nevertheless, from (3.66) we see that this variation is equivalent to a linear gauge transformation, which can in turn be interpreted as a Wilson line. For certain discrete choices of λ_x, λ_y such Wilson line will be trivial, and this will correspond to a discrete symmetry of the configuration.

To properly see this point let us replace the gauge potential A by a gauge covariant object such as the covariant derivative iD. In addition, we must take into account that in a gauge theory translations of the form $x \to x + \lambda_x$ are generated by $\exp(\lambda_x D_x)$. The gauge covariant version of (3.66) is then

$$e^{\lambda_j D_j} iD_k e^{-\lambda_j D_j} = iD_k + \lambda_j F_{jk} \tag{3.68}$$

where $j, k = x, y$, and we have used that $[D_j, D_k] = -iF_{jk}$. In fact, translations are not the only possible gauge transformations that we can perform but, just like in the case of D-branes at angles, there are also the ones generated by the 4d gauge bosons $B_\mu^{x,y}$ that arise from the B-field. These act on the covariant derivative as a diagonal linear gauge transformation, namely

$$e^{\mu_j B_j} iD_k e^{-\mu_j B_j} = iD_k + \mu_j \delta_{jk} \mathbb{I}_2, \qquad B_x = 2\pi i\, x\, \mathbb{I}_2,\ B_y = 2\pi i\, y\, \mathbb{I}_2 \tag{3.69}$$

Finally, we can write both (3.68) and (3.69) in the form

$$\begin{array}{rclcl} e^{\lambda_x D_x + \mu_y B_y}\, iD\, e^{-\lambda_x D_x - \mu_y B_y} & = & \Xi_y\, iD\, \Xi_y^{-1}, & \quad & \Xi_y = e^{2\pi i (\xi_{y,a} X_a + \xi_{y,b} X_b) y} \\ e^{\lambda_y D_y + \mu_x B_x}\, iD\, e^{-\lambda_y D_y - \mu_x B_x} & = & \Xi_x\, iD\, \Xi_x^{-1}, & \quad & \Xi_x = e^{2\pi i (\xi_{x,a} X_a + \xi_{x,b} X_b) x} \end{array} \tag{3.70}$$

where

$$\xi_{y,a} = \lambda_x m_a + \mu_y \qquad \xi_{y,b} = \lambda_x m_b + \mu_y \tag{3.71}$$

$$\xi_{x,a} = -\lambda_y m_a + \mu_x \qquad \xi_{x,b} = -\lambda_y m_b + \mu_x \tag{3.72}$$

with $\xi_{x,\alpha}$ representing a Wilson line for the gauge group $U(1)_\alpha$ along the coordinate x, and similarly for $\xi_{y,\alpha}$. Notice that no Wilson lines are induced for $U(1)_a$ if $\mu_y = -\lambda_x m_a$ and $\mu_x = \lambda_y m_a$, and so this gauge sector remains invariant under this particular combined action of the bulk gauge transformations. In other words, the magnetised D-brane $\alpha = a$ breaks the original $U(1)^4$ symmetry of the bulk down to $(U(1) \times \mathbb{Z}_q)^2$, similarly to the previous case of a D-brane wrapping a 1-cycle. A similar statement can be made for the D-brane b, and it can all be encoded in the 4d effective field theory via the following Stückelberg Lagrangian

$$\mathcal{L}_{\rm St} = -\frac{1}{2} \sum_{\alpha=a,b} \left\{ \left(\partial_\mu \xi_{x,\alpha} + m_\alpha V_\mu^y - B_\mu^x\right)^2 + \left(\partial_\mu \xi_{y,\alpha} - m_\alpha V_\mu^x - B_\mu^y\right)^2 \right\} \tag{3.73}$$

which is a particular case of (C.16), derived in appendix C from dimensional reduction. Here $\xi_{x,\alpha}$, $\xi_{y,\alpha}$ represent the 4d scalar fluctuations corresponding to the Wilson lines of $U(1)_\alpha$, $V_\mu^{x,y}$ are the gauge bosons that arise from the metric and $B_\mu^{x,y}$ from the B-field.

One can now interpret (3.70) as advanced before, whenever $(\xi_{y,a}, \xi_{y,b}) \in \mathbb{Z}^2$ we have a trivial Wilson line shift in the rhs of (3.70), and so the corresponding gauge transformation generated by a V_μ^x and B_μ^y is a symmetry of the system. One can check that there are

$M = m_a - m_b$ inequivalent values of (λ_x, μ_y) that correspond to $(\xi_{y,a}, \xi_{y,b}) \in \mathbb{Z}^2$, and that such values generate a residual $\mathbb{Z}_M$ symmetry. Similarly, there are M values of (λ_y, μ_x) such that $(\xi_{x,a}, \xi_{x,b}) \in \mathbb{Z}^2$, and these generate an additional $\mathbb{Z}_M$ symmetry. Finally, because of (3.68) these two $\mathbb{Z}_M$ symmetries do not commute, and we end up with a non-abelian symmetry group given by $H_M \simeq (\mathbb{Z}_M \times \mathbb{Z}_M) \rtimes \mathbb{Z}_M$.

It is instructive to apply this discrete symmetry to the chiral zero mode wavefunctions of this magnetised system, and in particular to those in the bifundamental representation $(+1, -1)$ of $U(1)_a \times U(1)_b$, which is where families of chiral matter arise from. On these modes $B_{x,y}$ act trivially, so the above discrete symmetry is implemented by[14]

$$e^{\frac{n_x}{M} D_x} \quad \text{and} \quad e^{\frac{n_y}{M} D_y} \qquad\qquad n_x, n_y = 0, \ldots, M-1 \tag{3.74}$$

with these operators acting on the zero modes obtained by solving the internal Dirac or Laplace equations on $\mathbf{T}^2$ [50]

$$\psi^{j,M}(z, \bar{z}) = \begin{cases} e^{i\pi M z \mathrm{Im} z/\mathrm{Im}\tau}\, \vartheta \begin{bmatrix} \frac{j}{M} \\ 0 \end{bmatrix} (Mz, M\tau) & \text{if} \quad M > 0 \\ e^{i\pi M \bar{z} \mathrm{Im} \bar{z}/\mathrm{Im}\tau}\, \vartheta \begin{bmatrix} \frac{j}{M} \\ 0 \end{bmatrix} (M\bar{z}, M\bar{\tau}) & \text{if} \quad M < 0 \end{cases} \tag{3.75}$$

$j = 0, 1, \ldots, |M| - 1$ running over independent zero mode solutions. One can check that

$$g_{\mathcal{W}}^{n_x} = e^{\frac{n_x}{M} D_x} \psi^{j,M} = e^{2\pi i \frac{n_x j}{M}} \psi^{j,M} \qquad g_{\mathcal{T}}^{n_y} = e^{\frac{n_y}{M} D_y} \psi^{j,M} = \psi^{j+n_y,M} \tag{3.76}$$

so that if we consider the vector of wavefunctions

$$\Psi = \begin{pmatrix} \psi^{0,M} \\ \vdots \\ \psi^{M-1,M} \end{pmatrix} \tag{3.77}$$

we have that the group elements $g_{\mathcal{T}}$ and $g_{\mathcal{W}}$ act as (3.49) and (3.54) respectively, generating the discrete Heisenberg group $H_M \simeq (\mathbb{Z}_M \times \mathbb{Z}_M) \rtimes \mathbb{Z}_M$ as mentioned above.

If we now consider the full $\mathbf{T}^6 = (\mathbf{T}^2)_1 \times (\mathbf{T}^2)_2 \times (\mathbf{T}^2)_3$ magnetised D9-brane system, we obtain that the zero mode wavefunctions for the $D9_a D9_b$ sector are [50]

$$\psi_{ab}^{j_1,j_2,j_3} = \psi_{ab}^{j_1,-I_{ab}^1}(z_1, \bar{z}_1) \cdot \psi_{ab}^{j_2,-I_{ab}^2}(z_2, \bar{z}_2) \cdot \psi_{ab}^{j_3,-I_{ab}^3}(z_3, \bar{z}_3) \tag{3.78}$$

with $I_{ab}^i = m_b^i - m_a^i$ and $\psi^{j,M}$ as in (3.75). The number of zero modes in the ab sector is given by $|I_{ab}^{\mathbf{T}^6}| = |I_{ab}^1||I_{ab}^2||I_{ab}^3|$, as expected from the T-dual intersecting D6-brane system, and their 4d chirality is again given by $\mathrm{sign}(I_{ab}^{\mathbf{T}^6})$. From our discussion on $\mathbf{T}^2$ it follows that each index j_i transforms in the fundamental representation of $H_{I_{ab}^i}$. We then obtain

[14] In [91] the alternative set of operators was considered

$$\begin{aligned} & e^{\frac{n_x}{M} X_x} \qquad\qquad X_x = \partial_x - \pi i \left(m_a X_a + m_b X_b \right) y \\ & e^{\frac{n_y}{M} X_y} \qquad\qquad X_y = \partial_y + \pi i \left(m_a X_a + m_b X_b \right) x \end{aligned}$$

in order to implement the action of the flavour symmetry group on wavefunctions. Both choices are in fact equivalent as they differ by a trivial Wilson line shift.

again that the flavour symmetry group of this sector is given by $\mathbf{P}^{ab}_{\mathbf{T}^6} = H_{I^1_{ab}} \times H_{I^2_{ab}} \times H_{I^3_{ab}}$, as in (3.55).

Let us now consider magnetised D9-branes in a $\mathbf{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold background,[15] again with the $\mathbb{Z}_2$ generators acting as (3.35). Because of the presence of the orbifold fixed loci, the $U(1)^6$ translational symmetry of $\mathbf{T}^6$ is broken down to the discrete subgroup $\mathbb{Z}_2^6$, and this reduces the set of operators of the form (3.74) that are compatible with the symmetries of the background.

For our purposes it is instructive to again consider the toy example $\mathbf{T}^2/\mathbb{Z}_2$ with $\mathbb{Z}_2$ action generated by $z \mapsto -z$. As before, this background has four fixed points at $\{0, 1/2, \tau/2, (1+\tau)/2\}$ that are interchanged by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry group generated by the discrete translations $z \mapsto z + 1/2$ and $z \mapsto z + \tau/2$. Let us now consider the magnetised $U(2)$ sector (3.64) in this background. The $\mathbf{T}^2/\mathbb{Z}_2$ discrete isometry $z \mapsto z + 1/2$ is implemented by $\exp(\frac{1}{2}D_x)$, while $z \mapsto z + \tau/2$ is implemented by $\exp(\frac{1}{2}D_y)$. From the discussion above, we know that these operators correspond to symmetries of the magnetised system only if they belong to (3.74), or in other words if M is even. We then find that a pair of magnetised D-branes respects the orbifold translational $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry if and only if $I_{ab} =$ even, exactly as we found in the T-dual picture of intersecting D-branes.

As it is clear from (3.76), for M even the group elements $g_{\mathcal{T}}^{M/2}$ and $g_{\mathcal{W}}^{M/2}$ will generate a discrete flavour symmetry group acting on the zero mode wavefunctions. Because the group action is non-abelian and in general it describes a discrete Heisenberg group, we can identify the flavour group with $H_2 \simeq (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \simeq D_4$.

It was shown in [107] that the families of wavefunctions indeed arrange themselves in representations of the dihedral group D_4. We summarise the results in the following.

Summary

In general we find that the wavefunctions that correspond to bifundamental fields $(N_a, \bar{N}_b)$ are of the form

$$\psi_{ab}^{j_1,j_2,j_3} = \psi_{ab}^{j_1} \cdot \psi_{ab}^{j_2} \cdot \psi_{ab}^{j_3} \tag{3.79}$$

where $\psi_{ab}^{j_i}$ lives in $(\mathbf{T}^2)_i$. Depending on the signs of the intersection numbers I^i_{ab} these wavefunctions will be even or odd under the action $z_i \mapsto -z_i$, as discussed below (3.42). If $\psi_{ab}^{j_i}$ is even the index j_i will run over I^i_e values and if it is odd over I^i_o values, with I^i_e and I^i_o defined in (3.40). Moreover, if I^i_{ab} is even, this index will transform under a specific representation of the flavour symmetry group D_4 of $(\mathbf{T}^2)_i$. This representation will depend on the value of I^i_{ab} and the wavefunction parity, as shown in table 3.2.

[15] More precisely, we consider the type IIB orbifold background mirror symmetric to our previous type IIA $\mathbf{T}^6/\mathbb{Z}_2 \times \mathbb{Z}'_2$ background. These two backgrounds are quite similar but not exactly the same, because upon three T-dualities the choice of discrete torsion of a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold is reversed. As a result, the fixed points of the type IIB $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold considered in this section contain collapsed two and four-cycles instead of collapse three-cycles.

$\lvert I^i_{ab}\rvert$	$\psi^{j_i}_{\text{even}}$ dim $= \lvert I^i_{ab}\rvert/2+1$	$\psi^{j_i}_{\text{odd}}$ dim $= \lvert I^i_{ab}\rvert/2-1$
$4s+2$	$\overset{s+1}{\oplus}\ \mathbf{R}_2$	$\overset{s}{\oplus}\ \mathbf{R}_2$
$8s+4$	$\overset{s+1}{\oplus}(+,+)\overset{s+1}{\oplus}(+,-)\overset{s+1}{\oplus}(-,+)\overset{s}{\oplus}(-,-)$	$\overset{s}{\oplus}(+,+)\overset{s}{\oplus}(+,-)\overset{s}{\oplus}(-,+)\overset{s+1}{\oplus}(-,-)$
$8s+8$	$\overset{s+2}{\oplus}(+,+)\overset{s+1}{\oplus}(+,-)\overset{s+1}{\oplus}(-,+)\overset{s+1}{\oplus}(-,-)$	$\overset{s}{\oplus}(+,+)\overset{s+1}{\oplus}(+,-)\overset{s+1}{\oplus}(-,+)\overset{s+1}{\oplus}(-,-)$

Table 3.2: Different family representations under the flavour symmetry group D_4 on each $(\mathbf{T}^2)_i$. Here $\mathbf{R}_2$ stands for the 2-dimensional irreducible representation of the dihedral group D_4 while $(\pm,\pm')$ stands for the one-dimensional representation in which the two generators of D_4 act respectively as $\pm\mathbb{I}$ and $\pm'\mathbb{I}$.

3.5 Examples

In this section we illustrate our analysis via a couple of semi-realistic examples. More precisely, we will consider two intersecting D6-brane models on the $\mathbb{Z}_2\times\mathbb{Z}_2'$ orbifold with a Pati-Salam gauge group. The first example is a four generation model already constructed in [98], with a $D_4\times D_4\times D_4$ symmetry group constraining its Yukawa couplings. The second example is a new, three generation model with a D_4 symmetry group.

One important ingredient of these models is the presence of orientifold planes, that allow to construct consistent and stable D-brane configurations. While the presence of O-planes does not change the discrete symmetries of a $\mathbb{Z}_2\times\mathbb{Z}_2'$ orbifold background, it does affect the D-brane content of a model and the associated 4d chiral spectrum. Hence, before presenting our examples we briefly review the effect of adding an orientifold projection to the $\mathbb{Z}_2\times\mathbb{Z}_2'$ orbifold.

3.5.1 Orientifolding

In general, in order to build consistent, stable and 4d Poincaré invariant models based on intersecting or magnetised D-branes in Calabi-Yau compactifications we need to include the presence of negative tension objects that cancel the positive tension of the D-branes. The simplest way to do so is to include the presence of the non-dynamical, negative tension objects known as orientifold planes. In the case of type IIA string theory compactified on $(\mathbf{T}^2)_1\times(\mathbf{T}^2)_2\times(\mathbf{T}^2)_3$ this is achieved by modding out the theory by $\Omega\mathcal{R}$, where Ω stands for the worldsheet parity operator and $\mathcal{R}$ for the anti-holomorphic involution $\mathcal{R}\,:\, z_i\mapsto\bar{z}_i$. For this the D-brane configuration has to be invariant under the action of $\Omega\mathcal{R}$, and so for each D6-brane wrapping the three-cycle (3.34) there must be another D6-brane wrapping $\Pi_{\alpha'}=\mathcal{R}\Pi_\alpha$. If as in [108] we consider that each $(\mathbf{T}^2)_i$ has a rectangular geometry, a $U(N_a)$ gauge group will arise from wrapping N_a D6-branes on Π_a and also on $\Pi_{a'}$, where

$$\begin{array}{lcccc} \Pi_a & : & (n_a^1,m_a^1) & (n_a^2,m_a^2) & (n_a^3,m_a^3) \\ \Pi_{a'} & : & (n_a^1,-m_a^1) & (n_a^2,-m_a^2) & (n_a^3,-m_a^3). \end{array} \tag{3.80}$$

In order to obtain a gauge group $U(N_a)\times U(N_b)$ we also need to place N_b D6-branes on Π_b and $\Pi_{b'}$. The spectrum of 4d left-handed chiral fermions in bifundamental representations is then given by [108]

$$I_{ab}^{\mathbf{T}^6}(N_a,\bar{N}_b)\quad+\quad I_{ab'}^{\mathbf{T}^6}(N_a,N_b) \tag{3.81}$$

where $I_{ab'}^{\mathbf{T}^6} = I_{ab'}^1 I_{ab'}^2 I_{ab'}^3 = \prod_{i=1}^3 (n_a^i m_b^i + n_b^i m_a^i)$. In addition there are 4d chiral fermions arising from the intersection of Π_a with its orientifold image $\Pi_{a'}$, that transform in the symmetric and antisymmetric representations of $U(N_a)$, namely we have

$$\begin{array}{c}\square\\[-4pt]\square\end{array}_a \, \frac{1}{2}(I_{aa'}^{\mathbf{T}^6} - 8 m_a^1 m_a^2 m_a^3) \quad + \quad \square\!\square_a \, \frac{1}{2}(I_{aa'}^{\mathbf{T}^6} + 8 m_a^1 m_a^2 m_a^3). \tag{3.82}$$

The same orientifold projection can be performed for the toroidal orbifold $\mathbf{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2'$, also by modding out the theory by $\Omega\mathcal{R}$. Again, a rigid D6-brane wrapping Π_a will have an orientifold image wrapping $\Pi_{a'}$. It is easy to see that Π_a and $\Pi_{a'}$ will go through the same fixed points on each $(\mathbf{T}^2)_i$, and so adding D6-brane orientifold images will not break the discrete flavour symmetry of the model any further. In order to obtain the chiral spectrum in this background one must consider (3.81) and (3.82) and project out all the chiral modes that are not invariant under the orbifold action. Following our discussion of section 3.3.1, it is easy to see that (3.81) is replaced by

$$I_{ab}(N_a, \bar{N}_b) \quad + \quad I_{ab'}(N_a, N_b) \tag{3.83}$$

where I_{ab} is given by (3.43) and similarly for $I_{ab'}$ with the replacement $b \to b'$. The orbifold projection of (3.82) is less straightforward but one can check that it amounts to

$$\begin{array}{c}\square\\[-4pt]\square\end{array}_a \, \frac{1}{2}(I_{aa'} + 4\, I_{aO6}^{\mathbf{T}^6}) \quad + \quad \square\!\square_a \, \frac{1}{2}(I_{aa'} - 4\, I_{aO6}^{\mathbf{T}^6}) \tag{3.84}$$

where for computing $I_{aa'}$ we use again the expression (3.43), but with the wrapping numbers of $\Pi_{a'}$ instead of Π_b. On the other hand, $I_{aO6}^{\mathbf{T}^6}$ is the $\mathbf{T}^6$ intersection number (3.37) between Π_a and the three-cycle Π_{O6}, with

$$[\Pi_{O6}] \;=\; -2\left([a^1]\cdot[a^2]\cdot[a^3] + [a^1]\cdot[b^2]\cdot[b^3] + [b^1]\cdot[a^2]\cdot[b^3] + [b^1]\cdot[b^2]\cdot[a^3]\right). \tag{3.85}$$

The three-cycle (3.85) has a geometrical interpretation, namely that the orientifold projection $\Omega\mathcal{R}$ introduces a set of O6-planes that are located at the fixed point loci of $\Omega\mathcal{R}$, $\Omega\mathcal{R}\Theta$, $\Omega\mathcal{R}\Theta'$ and $\Omega\mathcal{R}\Theta\Theta'$, and adding up the homology classes of all these three-cycles we can associate a total homology class $[\Pi_{O6}]$ for the O6-plane. If each $(\mathbf{T}^2)_i$ has a rectangular geometry such homology class is given by (3.85). We refer the reader to [98] for other cases in which some $(\mathbf{T}^2)_i$ is not rectangular, and for a generalisation of eqs.(3.83), (3.84) to these cases.[16]

The importance of introducing O6-planes is that they allow to construct consistent and supersymmetric D6-brane models [114]. In general, a D6-brane model will be consistent only if the RR-tadpole condition

$$\sum_\alpha N_\alpha([\Pi_\alpha^F] + [\Pi_{\alpha'}^F]) \;=\; 4[\Pi_{O6}] \tag{3.86}$$

is satisfied. Here the index α runs over each of the D6-branes of the model, Π_α^F stands for the fractional three-cycles described in appendix B and $\Pi_{\alpha'}^F$ is the image of Π_α^F under $\mathcal{R}$. As discussed in appendix B we can describe a D6-brane on Π_α^F in terms of $\mathbf{T}^6$ wrapping

[16] Our conventions are such that a positive intersection number I_{ab} signals a net amount of $|I_{ab}|$ 4d left-handed chiral fermions in the representation $(N_a, \bar{N}_b)$, while a negative intersection signals $|I_{ab}|$ fermions in the same representation but with opposite chirality. In [98] this chirality convention is reversed.

numbers (n_α^i, m_α^i). One can then see that the condition for a D6-brane model to preserve the $\mathcal{N}=1$ supersymmetry of the $\mathbf{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2' \times \Omega\mathcal{R}$ background is [113]

$$\theta_\alpha^1 + \theta_\alpha^2 + \theta_\alpha^3 \,=\, 0 \text{ mod } 2\pi, \quad \forall\, \alpha \tag{3.87}$$

with $\theta_\alpha^i \,=\, \tan^{-1}\frac{m_\alpha^i R_{y_i}}{n_\alpha^i R_{x_i}}$. As shown in [98], both conditions (3.85) and (3.87) are equivalent to simple expressions in terms of the $\mathbf{T}^6$ wrapping numbers (n_α^i, m_α^i) and can be satisfied simultaneously. In the next subsection we will consider a set of D6-branes which are a Pati-Salam subsector of a D6-brane model built in [98] satisfying both conditions.

3.5.2 A global Pati-Salam four-generation model

As an example of D-brane model with non-trivial flavour symmetry group let us consider the intersecting D6-brane model in table 8 of [98], which is based on the orientifold background $\mathbf{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2' \times \Omega\mathcal{R}$. In particular, we will consider the subsector given by the D6-branes a_1, a_2 and a_3 in that model, whose wrapping numbers we display in table 3.3.

N_α	(n_α^1, m_α^1)	(n_α^2, m_α^2)	(n_α^3, m_α^3)
$N_{a_1}=4$	$(1,0)$	$(0,1)$	$(0,-1)$
$N_{a_2}=2$	$(1,0)$	$(2,1)$	$(4,-1)$
$N_{a_3}=2$	$(-3,2)$	$(-2,1)$	$(-4,1)$

Table 3.3: Wrapping numbers for the four-generation Pati-Salam model of [98].

The gauge group that arises from this set of D-branes is given by $U(4)\times U(2)\times U(2)$, and as shown in [98] the chiral spectrum contains four families of left-handed chiral fermions in the representations $(\mathbf{4},\mathbf{2},\mathbf{1})+(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})$. We then have a four-generation Pati-Salam model, and because the supersymmetry conditions (3.87) amount to imposing

$$\begin{aligned} 2U^2 \,&=\, U^3 \\ \tan^{-1}\left(\tfrac{2U^1}{3}\right) + \tan^{-1}\left(\tfrac{U^2}{2}\right) + \tan^{-1}\left(\tfrac{U^3}{4}\right) \,&=\, \pi \end{aligned} \tag{3.88}$$

with $U^i \,=\, R_{y_i}/R_{x_i}$, we can find a continuum of supersymmetric solutions. The matter spectrum then contains $4(\mathbf{4},\mathbf{2},\mathbf{1})+4(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})$ $\mathcal{N}=1$ left-handed chiral multiplets.

Flavour group and representations

Let us analyse this Pati-Salam model in light of the results of section 3.3.1. Table 3.4 shows the toroidal intersection numbers $I_{\alpha\beta}^i$ for each of the relevant sectors of this model. Notice that all these numbers are even, and so in (3.60) one has that $d_1 = d_2 = d_3 = 2$. That is, the flavour symmetry group of this sector is given by

$$\mathbf{P}^{a_1 a_2 a_3}_{\mathbf{T}^6/\mathbb{Z}_2\times\mathbb{Z}_2'} \,=\, D_4^{(1)} \times D_4^{(2)} \times D_4^{(3)} \tag{3.89}$$

where the factor $D_4^{(i)}$ arises from the symmetries on the two-torus $(\mathbf{T}^2)_i$. While (3.89) corresponds to a symmetry of the Pati-Salam sector, it does not need to be respected by the whole D-brane model. Indeed, in order to satisfy the consistency conditions (3.86)

Sector	$U(4)\times U(2)_L\times U(2)_R$	$I^1_{\alpha\beta}$	$I^2_{\alpha\beta}$	$I^3_{\alpha\beta}$	Projection	$I_{\alpha\beta}$
a_1a_2	$(\mathbf{4},\bar{\mathbf{2}},\mathbf{1})$	0	-2	4	$-I_e^2\,I_o^3$	2
a_1a_2'	$(\mathbf{4},\mathbf{2},\mathbf{1})$	0	-2	4	$-I_e^2\,I_o^3$	2
a_1a_3	$(\mathbf{4},\mathbf{1},\bar{\mathbf{2}})$	2	2	-4	$I_e^1\,I_e^2\,I_o^3$	-4
a_1a_3'	$(\mathbf{4},\mathbf{1},\mathbf{2})$	-2	2	-4	$-$	0
a_2a_3	$(\mathbf{1},\mathbf{2},\bar{\mathbf{2}})$	2	4	0	$I_e^1\,I_e^2$	6
a_2a_3'	$(\mathbf{1},\mathbf{2},\mathbf{2})$	-2	0	-8	$-I_e^1\,I_e^3$	-10
a_2a_2'	$(\mathbf{1},\mathbf{1}_{+2},\mathbf{1})$	0	-4	8	$I_o^2\,I_e^3-I_e^2\,I_o^3$	-5+9
a_3a_3'	$(\mathbf{1},\mathbf{1},\mathbf{1}_{+2})\,\&(\mathbf{1},\mathbf{1},\mathbf{3})$	12	4	8	$I_e^1\,I_e^2\,I_e^3+I_o^1\,I_o^2\,I_o^3$	105+15

Table 3.4: Bulk intersection numbers together and the wavefunctions surviving the orbifold action in the four-generation Pati-Salam model. The last column shows the total intersection number. A positive intersection number indicates a left-handed $\mathcal{N}=1$ chiral multiplet, and a negative one a right-handed chiral multiplet. The a_1a_1' sector is non-chiral and contains a 4d $\mathcal{N}=2$ hypermultiplet in the antisymmetric representation of $U(4)$ [98].

we will need to add extra sets of D6-branes to those of table 3.4, and in order for (3.89) to be an exact symmetry of the model all the intersection numbers $I^i_{\alpha\beta}$ involving these extra D6-branes also need to be even. In general one would not expect this to be the case and, indeed, by looking at the completion of this model given by table 8 of [98] one realises that there is always some D6-brane β of this extra set such that $I^i_{a_j\beta}=$ odd for any given i. The flavour symmetry group (3.89) is then broken by the presence of the other D-branes of this model, and can only be thought as an approximate symmetry of the Pati-Salam sector of table 3.4. Nevertheless, even if not exact this symmetry will constrain the Yukawa couplings of this model at tree-level and, because of $\mathcal{N}=1$ supersymmetry, at all orders in perturbation theory. It is then useful to analyse under which representation of the discrete flavour group (3.89) transform each of the chiral modes of table 3.4.

Let us for instance consider the sector a_1a_2 of this model, which contains 2 copies of left-handed multiplets in the representation $(\mathbf{4},\bar{\mathbf{2}},\mathbf{1})$. In order to see how these two copies arise we must compute the combination of even or odd points on each two-torus that survive the orbifold projection. Following our discussion of section 3.3.1 and due to the particular signs of $I^i_{a_1a_2}$ for $i=1,2,3$ one is instructed to keep the wavefunctions of the form $\psi^{j_2}_{\text{even}}\psi^{j_3}_{\text{odd}}$ or $\psi^{j_2}_{\text{odd}}\psi^{j_3}_{\text{even}}$, counted with the appropriate chirality. More precisely, the chiral index in this sector is given by $I_o^2I_e^3-I_e^2I_o^3=0-(-2)=2$, in agreement with [98]. In fact, because $I^2_{a_1a_2}=-2$ there are no odd wavefunctions in $(\mathbf{T}^2)_2$ and so the wavefunctions that correspond to this sector are of the form

$$\psi^{j_2}_{a_1a_2}=\psi^{j_2}_{\text{even}}\cdot\psi^{j_3}_{\text{odd}}\qquad j_2=0,1,\quad j_3=0. \tag{3.90}$$

By looking at table 3.2 one can see how these chiral modes transform under the flavour group $\mathbf{P}^{a_1a_2a_3}$. On the one hand the index j_2 transforms in the 2-dimensional representation $\mathbf{R}_2$ of $D_4^{(2)}$, and on the other hand the index j_3 only takes one value and transforms as $(-,-)$ under the flavour subgroup $D_4^{(3)}$.

Geometrically, one can understand this result by drawing both D6-branes and labelling their intersection points as p_2^j with $j=0,1$ in $(\mathbf{T}^2)_2$ and p_3^k with $k=0,1,2,3$ in $(\mathbf{T}^2)_3$, see Figure 3.5. Clearly, the two points in the second torus are even under the

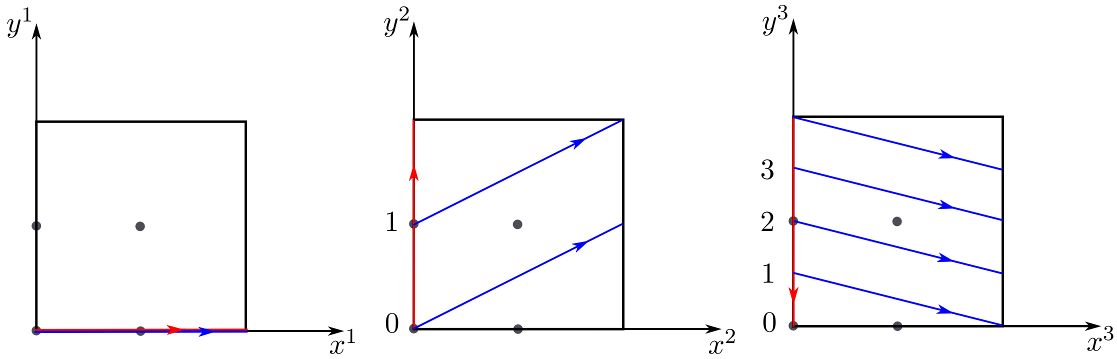

Figure 3.5: Branes a_1 (red) and a_2 (blue) with labels for the different intersection points.

orbifold action and there are no odd points. In the third torus we find three even points, namely, $p_3 = \{0, 1+3, 2\}$ and an odd one given by $p_3 = \{1-3\}$. Since the orbifold action selects the points whose parity are (odd,even) or (even,odd), we have that the surviving points are $(p_2, p_3) = (\{0\}, \{1-3\})$ and $(p_2, p_3) = (\{1\}, \{1-3\})$ which correspond to the two different chiral modes in (3.90). One can now see how the translation $z_2 \mapsto z_2 + \tau/2$ interchanges these two points, while they pick up a minus sign under the translation $z_3 \mapsto z_3 + \tau/2$. Adding up the action of the discrete B-field transformation (see footnote 9) we indeed recover that these two zero modes transform as $\mathbf{R}_2 \otimes (-,-)$ under $D_4^{(2)} \times D_4^{(3)}$. Finally, it is easy to see that the D6-brane intersections, which are the line $\{y_1 = 0\}$ in $(\mathbf{T}^2)_1$, are invariant under the translation $z_1 \mapsto z_1 + 1/2$ and in general by the full action of $D_4^{(1)}$. The final result has been summarised in table 3.5, together with the representations for the other sectors of the form $a_i a_j$ and $a_i a_{j'}$ with $i \neq j$, that can be treated similarly.

Sector	Field	$D_4^{(1)}$	$D_4^{(2)}$	$D_4^{(3)}$
a_1a_2	$F_L = (\mathbf{4}, \bar{\mathbf{2}}, \mathbf{1})$	$\mathbf{1}$	$\mathbf{R}_2$	$(-,-)$
a_1a_2'	$F_L' = (\mathbf{4}, \mathbf{2}, \mathbf{1})$	$\mathbf{1}$	$(-,-)$	$\mathbf{R}_2$
a_1a_3	$F_R = (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})$	$\mathbf{R}_2$	$\mathbf{R}_2$	$(-,-)$
a_2a_3	$H = (\mathbf{1}, \mathbf{2}, \bar{\mathbf{2}})$	$\mathbf{R}_2$	$\mathbf{1} \oplus (+,-) \oplus (-,+)$	$\mathbf{1}$
a_2a_3'	$H' = (\mathbf{1}, \bar{\mathbf{2}}, \bar{\mathbf{2}})$	$\mathbf{R}_2$	$\mathbf{1}$	$\mathbf{1}^2 \oplus (+,-) \oplus (-,+) \oplus (-,-)$

Table 3.5: Representations of the Pati-Salam fields under the flavour symmetry group.

Yukawa couplings

Given the above representations under the flavour symmetry group one can now consider the Yukawa couplings

$$\begin{aligned} Y : (a_1a_2) \otimes (a_1a_3) \otimes (a_2a_3) &\longrightarrow (\mathbf{4}, \bar{\mathbf{2}}, \mathbf{1}) \otimes (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}) \otimes (\mathbf{1}, \mathbf{2}, \bar{\mathbf{2}}) \\ Y' : (a_1'a_2) \otimes (a_1a_3) \otimes (a_2'a_3) &\longrightarrow (\mathbf{4}, \mathbf{2}, \mathbf{1}) \otimes (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}) \otimes (\mathbf{1}, \bar{\mathbf{2}}, \bar{\mathbf{2}}) \end{aligned} \tag{3.91}$$

which are allowed by gauge invariance.[17] It however happens that several of these couplings are not allowed by the discrete flavour symmetry (3.89), as we will now see.

Let us first consider the coupling Y in (3.91). In principle, Y has $2\times4\times6$ independent components since there are 6 different Higgses that appear in this set of Yukawa couplings. Nevertheless, in general the discrete symmetries in each torus will reduce the number of independent Yukawas. On the one hand, invariance under $D_4^{(1)}$ forces us to choose the singlet in $\mathbf{1}\otimes\mathbf{R}_2\otimes\mathbf{R}_2 = \mathbf{1}\oplus(+,-)\oplus(-,+)\oplus(-,-)$ which reduces by a factor 4 the number of independent Yukawas. On the other hand, under $D_4^{(2)}$ the coupling Y behaves as follows

$$\begin{aligned}\mathbf{R}_2\otimes\mathbf{R}_2\otimes(\mathbf{1}\oplus(+,-)\oplus(-,+)) &= (\mathbf{1}\oplus(+,-)\oplus(-,+)\oplus(-.-))\otimes(\mathbf{1}\oplus(+,-)\oplus(-,+))\\ &= \mathbf{1}^3\oplus(+,-)^3\oplus(-,+)^3\oplus(-,-)^3\end{aligned}$$

which reduces by another factor of 4 the number of independent Yukawas. Finally, since $D_4^{(3)}$ does not impose further constraints we conclude that there are only $\frac{2\times4\times6}{4\times4} = 3$ independent components in Y. In other words, at tree-level there will only be three independent Yukawas within this sector. More precisely one obtains the following Yukawa couplings $Y_{ijk}F_{L,i}F_{R,j}H_k$ where

$$Y_{ijk}H_k = \begin{pmatrix} aH_0+cH_2 & bH_1 & aH_3+cH_5 & bH_4 \\ bH_1 & aH_0-cH_2 & bH_4 & aH_3-cH_5 \end{pmatrix}. \tag{3.92}$$

The row index i runs over the two families of left-handed multiplets F_L in the a_1a_2 sector, while the index j runs over the four families of right-handed multiplets F_R. For concreteness we have displayed the definition of these multiplets in terms of D-brane intersections in table 3.6.

$F_{L,i}$	$F_{R,j}$	H_k
$(\psi^0)_2\cdot(\psi^1-\psi^3)_3$	$(\psi^0)_1\cdot(\psi^0)_2\cdot(\psi^1-\psi^3)_3$	$(\psi^0)_1\cdot(\psi^0+\psi^2)_2$
$(\psi^1)_2\cdot(\psi^1-\psi^3)_3$	$(\psi^0)_1\cdot(\psi^1)_2\cdot(\psi^1-\psi^3)_3$	$(\psi^0)_1\cdot(\psi^0-\psi^2)_2$
	$(\psi^1)_1\cdot(\psi^0)_2\cdot(\psi^1-\psi^3)_3$	$(\psi^0)_1\cdot(\psi^1+\psi^3)_2$
	$(\psi^1)_1\cdot(\psi^1)_2\cdot(\psi^1-\psi^3)_3$	$(\psi^1)_1\cdot(\psi^0+\psi^2)_2$
		$(\psi^1)_1\cdot(\psi^0-\psi^2)_2$
		$(\psi^1)_1\cdot(\psi^1+\psi^3)_2$

Table 3.6: Wavefunctions of the fields in the Yukawa couplings (3.92). Here $(\psi^j)_i$ stands for a delta-function localised at the j^{th} intersection of the D6-branes a_1 and a_2 in $(\mathbf{T}^2)_i$.

Considering now the Yukawa couplings Y' in (3.91), one finds that the effect of the discrete flavour symmetry is even more dramatic since there is no combination which is invariant under the factor $D_4^{(2)}$. As a result these Yukawa couplings will vanish and (3.92) will be the only set of Yukawas at the perturbative level. Hence, this four-generation Pati-Salam model will in fact have two families whose mass is generated perturbatively. It would be interesting to see how non-perturbative effects can generate the Yukawa couplings for the remaining two generations.

[17] This includes those abelian discrete gauge symmetries that remain after the $U(1)$ factors of the gauge group are broken by a Stückelberg mechanism [90].

Mass terms and net chirality

Besides Yukawa couplings, discrete flavour symmetries may forbid other kinds of couplings like mass terms between vector-like pairs of of zero modes. In the model at hand such kind of pairs arise in the sector a_2a_2', whose total intersection number is given by $I_{a_2a_2'} = 4$. This signals that we have a net chirality of four left-handed chiral multiplets in the representation $(\mathbf{1}, \mathbf{1}_{+2}, \mathbf{1})$, where $\mathbf{1}_{+2}$ stands for an antisymmetric representation of $U(2)$.[18] However, this net chirality does not signal the actual content of open string zero modes of this sector. A careful analysis using the rules of subsection 3.3.1 shows that in fact there are nine left-handed chiral multiplets (the ones arising from the wavefunctions of the form (even,odd)) and five right-handed chiral multiplets (the ones from the sector (odd,even)) in the representation $(\mathbf{1}, \mathbf{1}_{+2}, \mathbf{1})$.

Typically, one would not worry about this mismatch between the zero mode content and the net chiral index, because the ten extra zero modes not accounted by $I_{a_2a_2'}$ naturally arrange into five vector-like pairs that form singlets under the gauge group $U(4)\times U(2)_L \times U(2)_R$. Hence, one expects that the presence of loop corrections or extra compactification ingredients like background fluxes will generate a mass term for these pairs not protected by gauge invariance.

Nevertheless given a flavour symmetry one needs to check that these pairs of opposite chirality zero modes also form singlets under the discrete flavour group. For the model at hand, table 3.7 shows the charges of the different points (or wavefunctions) under the flavour group (3.89) for the sector a_2a_2'. From there one can see that one cannot form a

Sector	$D_4^{(1)}$	$D_4^{(2)}$	$D_4^{(3)}$
(even,odd)	$\mathbf{1}$	$\mathbf{1}\oplus(+,-)\oplus(-,+)$	$(+,-)\oplus(-,+)\oplus(-,-)$
(odd,even)	$\mathbf{1}$	$(-,-)$	$\mathbf{1}^2\oplus(+,-)\oplus(-,+)\oplus(-,-)$

Table 3.7: Representations under the dihedral groups of the zero modes in a_2a_2'.

vector-like pair that is a singlet under the factor $D_4^{(2)}$. As a result, a mass term for any vector-like pair is forbidden by the discrete flavour symmetry. Even if in this particular case the flavour symmetry is approximate, the effect generating such mass term must also break the flavour symmetry (like e.g. non-perturbative effects), and so we expect that such masses for vector-like pairs are smaller that the ones allowed by all sort of symmetries.

The centre of the flavour group

While the flavour symmetry group (3.89) is non-abelian, its centre can be compared with other discrete abelian groups present in this model. In particular it can be compared with the $\mathbb{Z}_N$ discrete gauge symmetries contained in the U(1) factors of $U(4)\times U(2)_L\times U(2)_R$. These discrete gauge symmetries are discussed in appendix C of [107] following the general prescription of [90]. The result is that they are trivial in the sense that they reduce to the centre of the gauge group $SU(4)\times SU(2)_L\times SU(2)_R$, which is generated by the elements

$$g_4 = \mathrm{diag}(i,i,i,i), \qquad g_{2,L} = \mathrm{diag}(-1,-1)_L, \qquad g_{2,R} = \mathrm{diag}(-1,-1)_R. \tag{3.93}$$

[18] More precisely, one computes the spectrum of this sector by applying eqs.(3.84), with $I_{a_2O6} = 4$. Hence one obtains a net number of four chiral multiplets in the antisymmetric of $U(2)_L$ and no matter in the symmetric representation of $U(2)_L$.

Let us denote the centre of the gauge group by $Z(G)$ and the centre of $D_4^{(1)} \times D_4^{(2)} \times D_4^{(3)}$ by $Z(P)$. We would like to know if any subgroup of $Z(P)$ is contained in $Z(G)$ when acting on the Pati-Salam model. Both groups are finite so they have a finite collection of subgroups and this can be answered by direct computation. Table 3.8 shows the charges of the visible sector under every $\mathbb{Z}_2$ subgroup of $Z(P)$ and $Z(G)$.

Sector	$\mathbb{Z}_2^{(1)}$	$\mathbb{Z}_2^{(2)}$	$\mathbb{Z}_2^{(3)}$	$\mathbb{Z}_{2,C}$	$\mathbb{Z}_{2,L}$	$\mathbb{Z}_{2,R}$
a_1a_2	$+$	$-$	$+$	$-$	$-$	$+$
a_1a_3	$-$	$-$	$+$	$-$	$+$	$-$
a_2a_3	$-$	$+$	$+$	$+$	$-$	$-$
a_1a_2'	$+$	$+$	$-$	$-$	$-$	$+$
a_2a_3'	$-$	$+$	$+$	$+$	$-$	$-$
a_2a_2'	$+$	$+$	$+$	$+$	$+$	$+$
a_3a_3'	$+$	$+$	$+$	$+$	$+$	$+$

Table 3.8: Charges of the visible sector under $\mathbb{Z}_2$ subgroups of $Z(P)$ and $Z(G)$.

Looking at Table 3.8 we see that $\mathbb{Z}_2^{(1)}$ and $\mathbb{Z}_{2,R}$ are the same. Also, the $\mathbb{Z}_2$ generated by the product of the generators of $\mathbb{Z}_2^{(3)}$ and $\mathbb{Z}_{2,C}$ is equivalent to $\mathbb{Z}_2^{(2)}$ which shows that $\mathbb{Z}_2^{(2)}$ and $\mathbb{Z}_2^{(3)}$ are not independent but are related by a gauge transformation. We thus find the discrete flavour group is actually $D_4^{(1)} \times D_4^{(2)} \times D_4^{(3)}/(\mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{(3)})$, and that all the couplings forbidden by this symmetry should be understood in terms of this quotient.

3.5.3 A local Pati-Salam three-generation model

Besides the four-generation model of [98], one may construct other models in $\mathbb{Z}_2 \times \mathbb{Z}_2'$ with semi-realistic spectrum that also display a non-trivial discrete flavour symmetry. In the following we analyse a simple three-generation Pati-Salam model where families transform with non-abelian representations under a Dihedral flavour group.

The D6-brane content of the model is shown in table 3.9, where the wrapping numbers n, l are arbitrary positive integers. Again, this D6-brane content is not sufficient to

N_α	(n_α^1, m_α^1)	(n_α^2, m_α^2)	(n_α^3, m_α^3)
$N_a = 4$	$(1, 0)$	$(1, 1)$	$(1, \text{-}1)$
$N_b = 2$	$(n, \text{-}3)$	$(0, 1)$	$(3, \text{-}1)$
$N_c = 2$	$(l, \text{-}1)$	$(\text{-}2, 1)$	$(\text{-}1, \text{-}1)$

Table 3.9: Wrapping numbers for the three-generation Pati-Salam model.

satisfy the RR-tadpole conditions (3.86), and extra D-branes should be added in order to construct a complete model. We will then consider it as a local $\mathbb{Z}_2 \times \mathbb{Z}_2'$ model, whose discrete flavour symmetry may or may not be broken by the extra D-branes that complete it.

It is easy to see that the gauge group that arises from this D6-brane content is again given by $U(4) \times U(2)_L \times U(2)_R$, and that now the supersymmetry conditions amount to

$$\begin{aligned} U^2 &= U^3 \\ \tan^{-1}\left(\frac{3U^1}{n}\right) + \tan^{-1}\left(\frac{U^3}{3}\right) &= \frac{\pi}{2} \\ \tan^{-1}\left(\frac{U^1}{l}\right) + \tan^{-1}\left(\frac{U^2}{2}\right) - \tan^{-1}U^3 &= 0 \end{aligned} \tag{3.94}$$

where $U^i = R_{y_i}/R_{x_i}$. One can solve these equations by setting $n > l > 0$, $U^1 = \sqrt{\frac{n(n-l)}{2}}$ and $U^2 = U^3 = \sqrt{\frac{2n}{n-l}}$, hence finding again a $\mathcal{N} = 1$ Pati-Salam model.

The chiral spectrum of this model can be found by computing the intersection numbers on each two-torus and applying the results of subsection 3.3.1. The result is displayed in table 3.10, from where it is manifest that all the intersection points in the third torus are even. We then conclude there is a flavour symmetry group of the form

$$\mathbf{P}^{abc}_{\mathbf{T}^6/\mathbb{Z}_2\times\mathbb{Z}_2'} = D_4 \tag{3.95}$$

where D_4 is generated by translations and B-field transformations on $(\mathbf{T}^2)_3$. The zero mode spectrum in the bc sector depends on the integer wrapping numbers $n > l > 0$. In the table we have considered the choice $n = 2$, $l = 1$, which gives a minimal Higgs sector.

Sector	$U(4)\times U(2)_L\times U(2)_R$	$I^1_{\alpha\beta}$	$I^2_{\alpha\beta}$	$I^3_{\alpha\beta}$	Projection	$I_{\alpha\beta}$
ab	$(\bar{\mathbf{4}},\mathbf{2},\mathbf{1})$	-3	1	2	$I^1_o\, I^2_e\, I^3_e$	-2
ab'	$(\bar{\mathbf{4}},\bar{\mathbf{2}},\mathbf{1})$	3	-1	4	$I^1_o\, I^2_e\, I^3_o$	-1
ac	$(\mathbf{4},\mathbf{1},\bar{\mathbf{2}})$	-1	3	-2	$I^1_e\, I^2_o\, I^3_e$	2
ac'	$(\mathbf{4},\mathbf{1},\mathbf{2})$	-1	1	0	$I^1_e\, I^2_e$	1
bc	$(\mathbf{1},\bar{\mathbf{2}},\mathbf{2})$	1	2	-4	$I^1_e\, I^2_e\, I^3_o$	-2
bc'	$(\mathbf{1},\mathbf{2},\mathbf{2})$	5	2	2	$I^1_e\, I^2_e\, I^3_e$	12
aa'	$(\mathbf{6}_{+2},\mathbf{1},\mathbf{1})$	0	-2	2	$-$	0
bb'	$(\mathbf{1},\mathbf{1}_{+2},\mathbf{1})$	12	0	6	$I^1_e\, I^3_e - I^1_o\, I^3_o$	18
cc'	$(\mathbf{1},\mathbf{1},\mathbf{1}_{+2})$	2	4	-2	$-$	0

Table 3.10: Bulk intersection numbers of the model of table 3.9 with $n = 2$ and $l = 1$, together with the points surviving the orbifold action and the total intersection number.

Similarly to the previous example we can easily extract the representation of these chiral Pati-Salam families under the flavour symmetry group D_4. We present the result of this analysis in table 3.11, which shows that in this model one generation is different in the sense that it transforms under an abelian representation of D_4, while the other two form a doublet of the fundamental representation $\mathbf{R}_2$ of the Dihedral group.

The only Yukawas allowed by gauge invariance (including anomalous $U(1)$'s) are

$$Y : ab \otimes ac \otimes bc \quad \longrightarrow \quad (\bar{\mathbf{4}},\mathbf{2},\mathbf{1})\otimes(\mathbf{4},\mathbf{1},\bar{\mathbf{2}})\otimes(\mathbf{1},\bar{\mathbf{2}},\mathbf{2}) \tag{3.96}$$

$$Y' : ab' \otimes ac \otimes bc' \quad \longrightarrow \quad (\bar{\mathbf{4}},\bar{\mathbf{2}},\mathbf{1})\otimes(\mathbf{4},\mathbf{1},\bar{\mathbf{2}})\otimes(\mathbf{1},\mathbf{2},\mathbf{2}) \tag{3.97}$$

and under the discrete D_4 these coupling behave as

$$Y \quad : \quad \mathbf{R}_2\otimes\mathbf{R}_2\otimes[(-,-)\oplus(-,-)] = \mathbf{1}\oplus\mathbf{1}\oplus\ldots \tag{3.98}$$

$$Y' \quad : \quad (-,-)\otimes\mathbf{R}_2\otimes\left(\oplus^6\mathbf{R}_2\right) = \overset{6}{\oplus}\,\mathbf{1}\oplus\ldots \tag{3.99}$$

Sector	Fields	D_4
ab	$F_R = (\bar{\mathbf{4}}, \mathbf{2}, \mathbf{1})$	$\mathbf{R}_2$
ab'	$F'_R = (\bar{\mathbf{4}}, \bar{\mathbf{2}}, \mathbf{1})$	$(-,-)$
ac	$F_L = (\mathbf{4}, \mathbf{1}, \bar{\mathbf{2}})$	$\mathbf{R}_2$
ac'	$F'_L = (\mathbf{4}, \mathbf{1}, \mathbf{2})$	$(+,+)$
bc	$H = (\mathbf{1}, \bar{\mathbf{2}}, \mathbf{2})$	$(-,-) \oplus (-,-)$
bc'	$H' = (\mathbf{1}, \mathbf{2}, \mathbf{2})$	$\overset{6}{\oplus}\, \mathbf{R}_2$

Table 3.11: D_4 representations.

where the dots stand for nontrivial representations of D_4, and we used $\mathbf{R}_2 \otimes \mathbf{R}_2 = \mathbf{1} \oplus (+,-) \oplus (-,+) \oplus (-,-)$. We then conclude that there are a total of eight independent parameters in the Yukawa couplings given by Y and Y'.

4

$U(1)$ Mixing and D-brane Linear Equivalence

In this chapter we study the kinetic mixing between different $U(1)$s in Type II D-brane models. We start looking at Calabi-Yau orientifold compactifications of Type IIA and compute the open-closed mixing by dimensional reduction of the D6-brane effective action. Then we explore the structure of the space of magnetic monopoles of the compactification and show that the mixing can be understood in terms of the Witten effect. Following this idea we propose a supergravity formula to compute the open-open kinetic mixing that may be regarded as the computation in the closed string channel that accounts for the one-loop corrections in the dual open string channel. We then consider the same problem in the context of Type IIB with magnetised D7-branes which is phenomenologically interesting since it is closely related to the F-theory GUTs. To do so we use the formalism of generalised complex geometry which is a useful tool to describe D-branes with worldvolume fluxes and we see that the analysis works in a similar way to the IIA case with the appropriate generalisations. This approach allows to compute the mixing of the hypercharge $U(1)$ with the $U(1)$s in the closed string sector in F-theory $SU(5)$ GUTs (in the perturbative limit). Again, we propose an expression for the open-open mixing in 'closed string variables' that takes into account the one-loop correction.

4.1 U(1) kinetic mixing for intersecting D6-branes

In this section we discuss the kinetic mixing between U(1)s in type IIA orientifold compactifications. In particular, we consider models made up of D6-branes wrapping special Lagrangian cycles (sLags) of Calabi-Yau threefolds, and describe the kinetic mixing of open string U(1)'s with RR U(1)'s. We derive the expression for such mixing by means of the Witten effect, recovering the results of [143] from a purely type IIA perspective. Finally, we point out the relation between open string U(1)'s and the relative cohomology of the compactification manifold, and propose a supergravity description for the kinetic mixing between open string U(1)'s at the one-loop level.

4.1.1 Type IIA Calabi-Yau orientifolds with D6-branes

Let us consider an orientifold of type IIA string theory on $\mathbb{R}^{1,3} \times \mathcal{M}_6$ with $\mathcal{M}_6$ a Calabi-Yau 3-fold. The orientifold projection is obtained by modding out by the action $\Omega_p(-1)^{F_L}\sigma$ where Ω_p is the worldsheet parity, F_L is the space-time fermion number for the left-movers and σ is an antiholomorphic involution of $\mathcal{M}_6$ acting as $z_i \to \bar{z}_i$ on local coordinates, which introduces O6-planes at the fixed loci. Therefore, the action of the involution on the Kähler form J and holomorphic 3-form Ω of $\mathcal{M}_6$ is given by

$$\sigma J = -J, \qquad \sigma\Omega = \bar{\Omega}. \tag{4.1}$$

The supersymmetry conditions for a D6-brane wrapping a 3-cycle π in $\mathcal{M}_6$ are

$$J|_\pi = 0, \qquad \mathrm{Im}\,\Omega|_\pi = 0. \tag{4.2}$$

Since the D6-brane charge lies in the homology group $H_3(\mathcal{M}_6, \mathbb{Z})$, cancellation of the total charge in the compact internal space can be written as

$$\sum_\alpha N_\alpha([\pi_\alpha] + [\pi^*_\alpha]) - 4[\pi_O] = 0 \tag{4.3}$$

where α is an index that runs over the set of branes, N_α is the total number of branes on π_α and $\pi^*_\alpha = \sigma\pi_\alpha$ is the cycle wrapped by the orientifold image of α. Finally, π_O is the fixed locus of the involution σ where the O6-planes lie and the factor -4 is due to the O-plane RR charge, assumed to be negative.

The 4d massless spectrum that arises from the closed string sector of the compactification can be computed upon dimensional reduction of the 10d type IIA supergravity action, and is given in terms of harmonic forms on $\mathcal{M}_6$. It is then useful to introduce a basis of harmonic forms of definite parity under the involution σ,

$$\begin{array}{ccccc} & & \sigma-\text{even} & & \sigma-\text{odd} \\ 2-\text{forms} & \omega_i & i = 1, \ldots, h^{1,1}_+ & \omega_{\hat{i}} & \hat{i} = 1, \ldots, h^{1,1}_- \\ 3-\text{forms} & \alpha_I & I = 0, \ldots, h^{1,2} & \beta^I & I = 0, \ldots, h^{1,2} \\ 4-\text{forms} & \tilde{\omega}^{\hat{i}} & \hat{i} = 1, \ldots, h^{1,1}_- & \tilde{\omega}^i & i = 1, \ldots, h^{1,1}_+ \end{array}$$

normalised such that

$$\int_{\mathcal{M}_6} \omega_i \wedge \tilde{\omega}^j = l_s^6\, \delta_i^j, \qquad \int_{\mathcal{M}_6} \omega_{\hat{i}} \wedge \tilde{\omega}^{\hat{j}} = l_s^6\, \delta_{\hat{i}}^{\hat{j}}, \qquad \int_{\mathcal{M}_6} \alpha_I \wedge \beta^J = l_s^6\, \delta_I^J. \tag{4.4}$$

For instance, in order to reduce to 4d the RR forms C_3 and C_5 one can expand them as

$$C_3 = A_1^i \wedge \omega_i + \mathrm{Re}\,(N^I)\alpha_I \tag{4.5}$$
$$C_5 = C_{2,I} \wedge \beta^I + V_{1,i} \wedge \tilde{\omega}^i \tag{4.6}$$

where we have taken into account the intrinsic parity of C_3 and C_5 (respectively even and odd) under the orientifold action. One then obtains that C_3 gives rise to $h^-_{1,1}$ axions Re (N^I) and to $h^+_{1,1}$ gauge bosons A_1^i, while C_5 contains their 4d dual degrees of freedom.

A convenient basis of harmonic p-forms is given by those that have integer cohomology class, that is whose integrals over any p-cycle are integer numbers. In particular we will choose the two-forms ω_i such that $[\omega_i] \in H^2_+(\mathcal{M}_6, \mathbb{Z})$, etc.[1] This automatically implies that a D2-brane wrapping a 2-cycle Λ_2 will have integer electric charges under the RR U(1)'s. More precisely, a D2-brane wrapping a two-cycle Λ_2^j whose class is Poincaré dual to $[\tilde{\omega}^j]$ will have electric charge δ^{ji} under the RR U(1) generated by A_1^i. Another

[1] In general $H^2(\mathcal{M}_6, \mathbb{Z})$ may not decompose as $H^2_+(\mathcal{M}_6, \mathbb{Z}) \oplus H^2_-(\mathcal{M}_6, \mathbb{Z})$, but the latter may only be a sublattice of the former. In this case one should introduce appropriate factors of 2 in (4.4). For simplicity, in the following we will assume that (4.4) even when $[\omega_i]$, $[\tilde{\omega}^i]$, $[\omega_{\hat{i}}]$, $[\tilde{\omega}^{\hat{i}}]$ all belong to integer cohomology.

consequence of this choice is that the gauge kinetic mixing for RR U(1)'s takes the simple form [146]

$$f_{ij} \;=\; -i\mathcal{K}_{ij\hat{k}}T^{\hat{k}} \tag{4.7}$$

where we have defined the complexified Kähler moduli $T^{\hat{k}}$ by

$$J_c \;\equiv\; B_2 + iJ \;=\; T^{\hat{k}}\omega_{\hat{k}} \tag{4.8}$$

while

$$\mathcal{K}_{ij\hat{k}} \;\equiv\; \frac{1}{l_s^6}\int_{\mathcal{M}_6}\omega_i\wedge\omega_j\wedge\omega_{\hat{k}} \tag{4.9}$$

are the triple intersection numbers, which in this basis are simply integers.

Regarding the open string sector of the compactification, the 4d massless spectrum that arises from a single D6-brane α wrapping a 3-cycle π_α is given by

$$A_1^\alpha \;=\; \frac{\pi}{l_s}(A_1^{4d,\alpha} + \theta_\alpha^j\,\zeta_j) \tag{4.10}$$

$$\phi_\alpha \;=\; \phi_\alpha^j X_j \tag{4.11}$$

where $A_1^{4d,\alpha}$ is a 4d gauge vector field (we will henceforth suppress the superscript 4d)[2]. Moreover, θ_α^j are the components of the corresponding Wilson line moduli with

$$\frac{\zeta_j}{2\pi} \in \mathrm{Harm}^1(\pi_\alpha,\mathbb{Z}) \tag{4.12}$$

and ϕ_α^j are the D6-brane position moduli, namely the components of a normal deformation of the brane preserving the sLag conditions (4.2) with

$$X_i \in N(\pi_\alpha) \text{ such that } \mathcal{L}_{X_i}J = \mathcal{L}_{X_i}\mathrm{Im}\Omega = 0 \tag{4.13}$$

where $\mathcal{L}_{X_i}$ is the Lie derivative along X_i. These two scalar fields together form a 4d complex modulus, namely

$$\Phi_\alpha^j = \theta_a^j + \lambda_i^j\phi_\alpha^i \tag{4.14}$$

with λ_i^j a complex matrix relating $\{\zeta_j\}$ and $\{X_i\}$ and defined by

$$\iota_{X_i}J_c|_{\pi_\alpha} = \lambda_i^j\zeta_j \tag{4.15}$$

where J_c is the complexified Kähler form (4.8). It is straightforward to generalise this spectrum to the case of a stack of N_α D6-branes wrapping π_α, so that the 4d gauge group is given by $U(N_\alpha)$ and Φ_α^j transform in its adjoint representation. Finally, 4d chiral multiplets may arise from the transverse intersections of π_α with its orientifold image π_α^* as well as with other 3-cycles wrapped by the remaining D6-branes of the compactification [1, 97].

[2] The $\frac{1}{l_s}$ is introduced to keep $A_1^{4d,\alpha}$ and θ_α^j dimensionless and the factor of π for later convenience. The field ϕ_a^j is related to the normal coordinate by $y_a^j = \frac{l_s}{2}\phi_a^j$ so ϕ_a^j is also dimensionless.

4.1.2 Separating two D6-branes

In order to discuss kinetic mixing between open an closed string U(1)'s let us follow [143] and first consider type IIA strings on $\mathbb{R}^{1,3} \times \mathcal{M}_6$, without any orientifold projection, and suppose that we have two D6-branes a and b wrapping the same sLag 3-cycle $\pi_a = \pi_b$. This leads to a gauge group $U(2)$ in 4d, which breaks down to $U(1)_a \times U(1)_b$ when these two 3-cycles are separated. Nevertheless, only a linear combination of these two $U(1)$'s remains massless at low energies, while the other one becomes massive due to the Stückelberg mechanism.

Indeed, let us consider the CS action for a single D6-brane wrapping a 3-cycle π_α, which is obtained from a 3-cycle π after a small normal deformation of the form $\phi_\alpha = \phi_\alpha^j X_j$.[3] We have that (see [143–145] for further details)

$$
\begin{aligned}
S_{CS}^\alpha &\supset \mu_6 \int_{\mathbb{R}^{1,3}\times\pi_\alpha} \left[\mathcal{F}_2^\alpha \wedge C_5 + \frac{1}{2}\mathcal{F}_2^\alpha \wedge \mathcal{F}_2^\alpha \wedge C_3\right] \\
&= \mu_6 \int_{\mathbb{R}^{1,3}\times\pi} e^{\mathcal{L}_{\phi_\alpha}} \left[\mathcal{F}_2^\alpha \wedge C_5 + \frac{1}{2}\mathcal{F}_2^\alpha \wedge \mathcal{F}_2^\alpha \wedge C_3\right]
\end{aligned}
\tag{4.16}
$$

with $\mu_6 = \frac{2\pi}{l_s^7}$ the D6-brane charge, $\mathcal{L}_{\phi_\alpha} = \phi_\alpha^j \mathcal{L}_{X_j}$ the Lie derivative along such deformation and $\mathcal{F}_2^\alpha = \frac{l_s^2}{2\pi} F_2^\alpha + B_2$. In the absence of orientifold projection the RR 5-form potential C_5 has the expansion

$$
C_5 = C_{2,I} \wedge \beta^I + \tilde{C}_2^I \wedge \alpha_I + V_{1,i} \wedge \tilde{\omega}^i \tag{4.17}
$$

where now $i = 1, \dots, h^{1,1}$ runs over all harmonic 2-forms in $\mathcal{M}_6$. We now consider two 3-cycles π_a and π_b that are deformations of π and wrap a D6-brane on each of them. The full CS action then contains the following piece

$$
S_{CS}^a + S_{CS}^b \supset \frac{\pi}{l_s^6}\left[\int_{\mathbb{R}^{1,3}} (F_2^a + F_2^b) \wedge C_{2,I} \int_\pi \beta^I + \int_{\mathbb{R}^{1,3}} (F_2^a + F_2^b) \wedge \tilde{C}_2^I \int_\pi \alpha_I\right] \tag{4.18}
$$

where we have used that the integrals of β^I, α_I only depend on the homology class of the 3-cycle, and in particular that $\int_\pi e^{\mathcal{L}_{\phi_\alpha}} \beta^I = \int_\pi \beta^I$, same for α_I. For a non-trivial $[\pi]$ some of these integrals will be non-vanishing, and so the combination $U(1)_a + U(1)_b$ will develop a BF coupling and therefore a Stückelberg mass, while the orthogonal combination

$$
U(1)_{(a-b)} = \frac{1}{2}\left[U(1)_a - U(1)_b\right] \tag{4.19}
$$

will remain massless.

We can now read off the kinetic mixing of (4.19) with the RR U(1)'s from the remaining terms of $S_{CS}^a + S_{CS}^b$. For this it is useful to consider the following expansion for the RR potentials

$$
C_3 = A_1^i \wedge \omega_i + \dots \tag{4.20}
$$

$$
C_5 = \tilde{A}_1^i \wedge *_6\, \omega_i + \dots \tag{4.21}
$$

[3] Such normal deformation should be small enough so that π and π_α have the same topology, and in particular the same number of non-trivial 1-cycles.

where the dots represent terms that do not contain 4d gauge bosons. This new expansion for C_5 is chosen so that $F_2^{\mathrm{RR},i} \equiv dA_1^i = *_4 d\tilde{A}_1^i$. Plugging the expansion (4.21) into (4.16) and projecting into the combination $F_2^{(a-b)} \equiv \frac{1}{2}[F_2^a - F_2^b]$ we obtain

$$S_{CS}^a + S_{CS}^b \supset -\frac{\pi}{2l_s^5}\int_{\mathbb{R}^{1,3}} F_2^{(a-b)} \wedge *F_2^{\mathrm{RR},i}\int_\pi (\iota_{\phi_a}J - \iota_{\phi_b}J)\wedge \omega_i + \ldots \tag{4.22}$$

where we have only kept terms linear in the deformations ϕ_α. Here we have used that $\mathcal{L}_\phi = d\iota_\phi + \iota_\phi d$ and that $*_6\omega_i = c_i J^2 - J\wedge\omega_i$ with $c_i = (3\int_{\mathcal{M}_6}\omega_i\wedge J^2)/(2\int_{\mathcal{M}_6}J^3)$. Using the definition (4.15) we can recast this result as

$$\mathrm{Re}\ f_{i(a-b)} \ = \ \frac{1}{4l_s^3}(\phi_a^k - \phi_b^k)\,\mathrm{Im}\ (\lambda_k^j)\int_\pi \zeta_j\wedge\omega_i \tag{4.23}$$

where the basis of 1-forms $\{\zeta_j\}$ is defined as in (4.12).

The imaginary part of $f_{i(a-b)}$ is obtained from plugging (4.20) into (4.16), which gives

$$S_{CS}^a + S_{CS}^b \supset \frac{\pi}{2l_s^5}\int_{\mathbb{R}^{1,3}} F_2^{(a-b)}\wedge F_2^{\mathrm{RR},i}\int_\pi \Big[(\theta_a^j - \theta_b^j)\zeta_j + (\iota_{\phi_a}B - \iota_{\phi_b}B)\Big]\wedge\omega_i + \ldots \tag{4.24}$$

where we again integrated by parts and kept terms linear in the deformations. Comparing to the general expression (**??**) we conclude that

$$\mathrm{Im}\ f_{i(a-b)} \ = \ -\frac{1}{4l_s^3}\Big[(\theta_a^j - \theta_b^j) + (\phi_a^k - \phi_b^k)\,\mathrm{Re}\ (\lambda_k^j)\Big]\int_\pi \zeta_j\wedge\omega_i \tag{4.25}$$

Adding this result to (4.23) we obtain that

$$f_{i(a-b)} = -\frac{i}{4l_s^3}(\Phi_a^j - \Phi_b^j)\int_\pi \zeta_j\wedge\omega_i \tag{4.26}$$

which as expected is a holomorphic function of the D6-brane moduli. Notice that the mixing vanishes for $\Phi_a^j = \Phi_b^j$, which corresponds to the case where the two branes are on top of each other and the gauge group enhances to $SU(2)$.

In order to arrive at (4.26) we assumed that π_a and π_b are obtained from deforming the same 3-cycle π. As a result, they are not only in the same homology class but are also homotopic. However, since the vanishing of the Stückelberg mass depends only on the homology of the cycles and not on their homotopy class, one would like to have an expression for the kinetic mixing that applies in the general case. In [143] such a formula was found to be

$$f_{i(a-b)} = -\frac{i}{2l_s^4}\int_\Sigma\left(J_c + \frac{l_s^2}{2\pi}\tilde{F}_2^{(a-b)}\right)\wedge\omega_i \tag{4.27}$$

where Σ is a 4-chain such that $\partial\Sigma = \pi_a - \pi_b$, and $\tilde{F}_2^{(a-b)}$ is such that

$$\int_\Sigma \tilde{F}_2^{(a-b)}\wedge\omega_i \equiv \frac{\pi}{l_s}\left[\int_{\pi_a}A_1^a\wedge\omega_i - \int_{\pi_b}A_1^b\wedge\omega_i\right]. \tag{4.28}$$

It can be easily shown that this 4-chain expression reproduces (4.26) for homotopic branes. Moreover, following [143] one can see that the kinetic mixing needs to be of the form (4.27) by performing the M-theory lift of these compactifications. In the next section we will arrive at (4.27) from yet a different viewpoint, without using any M-theory lift.

Notice that the open-closed kinetic mixing vanishes if

$$\int_{\pi} \omega_i \wedge \zeta_j = \int_{\rho_j} \omega_i = 0 \tag{4.29}$$

where the 2-cycle $\rho_j \subset \pi$ is Poincaré dual to ζ_j. This is true only if none of the 2-cycles of π are non-trivial in $\mathcal{M}_6$. By the results of [147], this is equivalent to saying that the 3-cycles π_a and π_b are linearly equivalent. Hence, in the present case linear equivalence of D-branes translates into a vanishing kinetic mixing with RR photons. In the following sections we will see how this statement can be generalised to more involved D-brane configurations.

Orientifolding

Let us now include the effect of the orientifold projection. Because in this case C_5 has the expansion (4.6), instead of (4.18) we obtain

$$\begin{aligned} S^a_{CS} + S^{a^*}_{CS} + S^b_{CS} + S^{b^*}_{CS} \supset\ & \frac{1}{2}\frac{\pi}{l_s^6} \int_{\mathbb{R}^{1,3}} (F_2^a + F_2^b) \wedge C_{2,I} \left[\int_{\pi} \beta^I - \int_{\pi^*} \beta^I \right] \\ =\ & \frac{\pi}{l_s^6} \int_{\mathbb{R}^{1,3}} (F_2^a + F_2^b) \wedge C_{2,I} \int_{\pi} \beta^I \end{aligned} \tag{4.30}$$

where the extra factor of $1/2$ arises due to the orientifold projection, and we have used the fact that $F_{\alpha^*} = -F_\alpha$ for the 7d gauge field. As a result, now $U(1)_a + U(1)_b$ will develop a Stückelberg mass if and only if $[\pi] \neq [\pi^*]$.

Let us assume that this is the case and compute the kinetic mixing for the massless $U(1)$ (4.19), for which one can obtain expressions similar to the unorientifolded case. Indeed, we have that

$$S^a_{CS} + S^{a^*}_{CS} + S^b_{CS} + S^{b^*}_{CS} \supset \frac{\pi}{2l_s^5} \int_{\mathbb{R}^{1,3}} F_2^{(a-b)} \wedge F_2^{\mathrm{RR},i} \cdot (\theta_a^j - \theta_b^j) \int_{\pi} \zeta_j \wedge \omega_i \tag{4.31}$$

from where we can deduce that the kinetic mixing again takes the form (4.26). In terms of a 4-chain formula we would again arrive to (4.27) if the 4-chain Σ is defined in the quotient space $\mathcal{M}_6/\{1,\sigma\}$. We should multiply this expression by an extra factor of $1/2$ if instead Σ is defined in the covering space $\mathcal{M}_6$ and such that $\partial\Sigma = \pi_a - \pi_b - \pi_a^* + \pi_b^*$.[4]

On the other hand, for $[\pi] = [\pi^*]$ both $U(1)_a$ and $U(1)_b$ remain massless. One should then be able to write the kinetic mixing of each $U(1)_\alpha$ with the RR $U(1)$'s individually. Indeed, one finds that the expression analogous to (4.27) is

$$f_{i\alpha} = -\frac{i}{2l_s^4} \int_{\Sigma_\alpha} \left(J_c + \frac{l_s^2}{2\pi} \tilde{F}_2^\alpha \right) \wedge \omega_i \tag{4.32}$$

where Σ_α is defined in the quotient space as the projection of a 4-chain Σ'_α which in the covering space $\mathcal{M}_6$ satisfies $\partial\Sigma'_\alpha = \pi_\alpha - \pi_\alpha^*$.

[4] In this case the 4-chain sigma can be divided into two pieces as $\Sigma = \Sigma_{ab} - \Sigma_{ab}^*$, where $\partial\Sigma_{ab} = \pi_a - \pi_b$ and Σ_{ab}^* is its orientifold image.

4.1.3 Kinetic mixing via the Witten effect

Let us now describe an alternative derivation for the kinetic mixing formula (4.27), based on the Witten effect [148]. This effect is the fact that for $U(1)$ gauge theories that break CP and contain magnetic monopoles, the latter acquire an electric charge proportional to the CP breaking term. For theories whose CP violating effect is a θ-term this electric charge can be computed exactly, namely

$$Q_E = -\frac{\theta}{2\pi}e. \tag{4.33}$$

This can be generalised to theories that contain multiple $U(1)$'s and whose action is described by (**??**). The lattice of charges is then [149]

$$\begin{aligned} Q_I^E &= n_I^e - 16\pi^2 \text{Im } f_{IJ}\, n_J^m \\ Q_I^M &= 4\pi n_I^m \end{aligned} \tag{4.34}$$

where $I = 1, \ldots, K$ runs over the set of massless $U(1)$'s and $n_e^I \in \mathbb{Z}/2$, $n_m^I \in \mathbb{Z}$ are the charges that appear in the action when we include this particle. In other words, including a particle with charges (n_I^e, n_I^m) amounts to consider the action

$$S = S_{4d,\,U(1)} + Q_I^E \frac{1}{l_s}\int_W A^I + Q_I^M \frac{1}{l_s}\int_W \tilde{A}^I \tag{4.35}$$

where W is the worldline of the particle in 4d and $\tilde{A}^I$ is the gauge field dual to A^I. Notice however that the physical charges are the ones that we get when the kinetic mixing Re f_{pq} is diagonalised.

Our strategy to determine the open-closed $U(1)$ mixing will be to compute the electric charges of 4d magnetic monopoles that arise in D6-brane models. Given the above facts, such electric charge should be proportional to the imaginary part of the gauge kinetic function. Finally, by performing an electric-magnetic duality in one of the $U(1)$'s we will also be able to obtain the real part of f_{IJ}. As before we will first consider the case of parallel D6-branes without O6-planes and subsequently include the orientifold projection.

In the system of two homotopic D6-branes wrapping π_a and π_b, the 4d monopole with unit charge under $U(1)_{a-b} = \frac{1}{2}\left[U(1)_a - U(1)_b\right]$ is given by a D4-brane wrapping $W \times \Sigma$, where W is the worldline of the monopole in 4d and $\partial\Sigma = \pi_a - \pi_b$. In order to compute the electric charge under the closed string $U(1)$'s we can dimensionally reduce the D4-brane CS action to obtain

$$S_{CS}^{D4} \supset \mu_4 \int_{W\times\Sigma} C_3 \wedge \mathcal{F}^{D4} = \mu_4 \int_{W\times\Sigma} A_1^i \wedge \omega_i \wedge \mathcal{F}^{D4} = Q_i^E \frac{1}{l_s}\int_W A_1^i \tag{4.36}$$

where the electric charges are given by

$$Q_i^E = \frac{2\pi}{l_s^4}\int_\Sigma \mathcal{F}^{D4} \wedge \omega_i. \tag{4.37}$$

This term is precisely (minus) the imaginary part of the mixing in (4.27).[5] In particular, the field strength in $\mathcal{F}^{D4} = B_2 + \frac{l_s^2}{2\pi}F_2^{D4}$ is such that $F_2^{D4}|_{\pi_\alpha} = F_2^\alpha$, as the monopole must

[5]One has to be careful to make the overall coefficient match. The D4-brane on $W \times \Sigma$ has magnetic charge 1 under the open string $U(1)$ which does not agree with the convention in eq.(4.34). Thus, one should introduce an additional factor of 4π to account for such mismatch in both conventions which indeed reproduces the imaginary part of the mixing (4.27).

interpolate between the two D6-brane configurations. More precisely, F_2^{D4} is the curvature of a line bundle on Σ such that on its boundary $\partial\Sigma$ it reduces to the line bundles on the corresponding D6-brane. Such line bundle is nothing but the Wilson lines A_1^α, and so one recovers (4.28) by simply identifying $\tilde{F}_2^{(a-b)}$ with F_2^{D4}. Notice that (similarly to the 4-chain Σ) there are many F^{D4} that have the appropriate boundary conditions. As we will see in the next section the line bundle extension F_2^{D4} appears naturally in the context of generalised complex geometry.

Let us now consider the 4d dual vector boson $\tilde{A}_1^i$ that appears in the expansion (4.21) of the RR potential C_5. Looking at the appropriate term on the D4-brane CS action we obtain

$$S_{CS}^{D4} \supset \mu_4 \int_{W\times\Sigma} C_5 = -\mu_4 \int_{W\times\Sigma} \tilde{A}_1^i \wedge J \wedge \omega_i = \tilde{Q}_i^E \frac{1}{l_s} \int_W \tilde{A}_1^i \tag{4.38}$$

where we have again used that $*_6\omega_i = c_i J^2 - J\wedge\omega_i$ and assumed that $c_i = 0$, which will be automatically satisfied in the orientifold case. The electric charges in the dual basis are then given by

$$\tilde{Q}_i^E = -\frac{2\pi}{l_s^4} \int_\Sigma J \wedge \omega_i \tag{4.39}$$

which as expected reproduces the real part of the mixing (4.27).

Notice that in this derivation we do not need to assume that the two 3-cycles π_a and π_b are homotopic, or that they relatively close to a reference 3-cycle π. The only requirement is that they are homologous so the Stückelberg mass vanishes and a 4-chain Σ exists. This then provides an alternative way to derive the expression (4.27) from first principles without having to perform the lift to M-theory. Finally, it is straightforward to extend this derivation to the orientifold case and again obtain (4.27), except that now the 4-chain Σ and the field strength F_2^{D4} should connect the 3-cycles π_α and π_α^* and be evaluated in the orientifold quotient space.

4.1.4 General case

From the above discussion it is clear how to generalise these results to arbitrary D6-brane configurations. Indeed, let us consider K stacks of D6-branes, each stack containing N_α D6-branes wrapped on π_α and their corresponding orientifold images on π_α^*, and such that the RR tadpole condition (4.3) is satisfied. We will find a massless $U(1)_X$ for each linear combination

$$\pi_X = \sum_{\alpha=1}^{K} n_{X\alpha} N_\alpha \pi_\alpha, \qquad n_{X\alpha} \in \mathbb{Z} \tag{4.40}$$

such that $[\pi_X] - [\pi_X^*]$ is trivial in $H_3(\mathcal{M}_6, \mathbf{R})$.[6] Thus, the number of massless $U(1)$'s is given by $K - r$ where r is dimension of the vector subspace generated by $[\pi_\alpha] - [\pi_\alpha^*]$ within $H_3^-(\mathcal{M}_6, \mathbb{Z})$. In order to fix the normalisation we pick a basis of $U(1)$'s given by

$$\hat{U}(1)_\alpha = \frac{1}{L_\alpha} \sum_{\beta=1}^{K} n_{\alpha\beta} U(1)_\beta \tag{4.41}$$

[6] If $[\pi_X] - [\pi_X^*]$ is trivial in $H_3(\mathcal{M}_6, \mathbf{R})$ but not in $H_3(\mathcal{M}_6, \mathbb{Z})$ then for this $U(1)$ to be massless it must have a component of RR $U(1)$ [143]. Throughout our discussion we assume that Tor $H_3(\mathcal{M}_6, \mathbb{Z}) = 0$ so that this possibility is not realised.

with $n_{\alpha\beta} \in \mathbb{Z}$ such that g.c.d$(n_{\alpha 1}, n_{\alpha 2}, \ldots, n_{lK}) = 1$ for all $\alpha = 1, \ldots, K$ and orthogonal, namely $n_{\alpha\gamma} n_{\gamma\beta} = L_\alpha \delta_{\alpha\beta}$.

For a massless $U(1)_X$ we can associate the formal linear combination of 3-cycles (4.40), and we know that there exists a 4-chain Σ_X such that $\partial \Sigma_X = \pi_X - \pi_X^*$. Wrapping a D4-brane on Σ_X corresponds to considering a 4d magnetic monopole of $\hat{U}(1)_X$ which, due to the normalisation (4.41), has magnetic charge $n_X^m = 1$. Dimensionally reducing the CS action for such monopole we will find its charges with respect to the closed string $U(1)$'s from where we can read off the kinetic mixing, namely

$$f_{iX} \;=\; \frac{1}{2\, l_s^4} \int_{\Sigma_X} (J - i\mathcal{F}^{D4}) \wedge \omega_i \tag{4.42}$$

where the integral is evaluated in the orientifold quotient space. This expression is slightly subtle in the sense that it may depend on some discrete choices related to the pair $(\Sigma_X, \mathcal{F}^{D4})$. Such subtleties can be easily removed after a proper understanding of the space of monopoles of the compactification, as we discuss in the following.

4.1.5 Monopoles and relative homology

Besides the general formula (4.42) for the gauge kinetic mixing between open and closed string U(1)'s, the previous discussion gives us an overall picture of the set of monopoles that appear in type IIA compactifications with D6-branes. On the one hand, monopoles charged under closed string U(1)'s are classified by D4-branes wrapping orientifold-odd 4-cycles, or in other words by the homology group $H_4^-(\mathcal{M}_6, \mathbb{Z})$. On the other hand, open string U(1) monopoles are classified by D4-branes wrapping odd 4-chains Σ_X ending on the D6-branes 3-cycles π_α and their orientifold images π_α^*, whose formal union we will denote as π_{D6}. The appropriate homology group that classifies such 4-chains is the relative homology group $H_4^-(\mathcal{M}_6, \pi_{D6}, \mathbb{Z})$, which includes $H_4^-(\mathcal{M}_6, \mathbb{Z})$ as a subgroup. In fact, we can identify $H_4^-(\mathcal{M}_6, \pi_{D6}, \mathbb{Z})$ as the lattice of integral U(1) magnetic charges, that contains not only monopoles charged under open string U(1)'s and closed string U(1)'s, but also bound states of those.

Indeed, notice that the formula (4.42) is slightly ambiguous, in the sense that we can have two different 4-chains Σ_X and Σ'_X with the same boundary, and so the expression for the rhs integral could be different for Σ_X and Σ'_X. Let us temporarily simplify this formula by setting $\mathcal{F}^{D4} = B$. Then, if these two chains differ by a trivial 4-cycle (that is if they are they belong to the same class of $H_4^-(\mathcal{M}_6, \pi_{D6}, \mathbb{Z})$) then we have that $\int_{\Sigma_X} J_c \wedge \omega_i = \int_{\Sigma'_X} J_c \wedge \omega_i$ and so we get the same result for the rhs of (4.42) independently of which chain we choose. If on the other hand Σ_X and Σ'_X differ by a 4-cycle Λ_4^j such that $[\Lambda_4^j] \in H_4^-(\mathcal{M}_6, \mathbb{Z})$, then the two integrals will differ by

$$-\frac{i}{l_s^4} \int_{\Lambda_4^j} J_c \wedge \omega_i \;=\; -\frac{i}{l_s^4} \int_{\mathcal{M}_6} J_c \wedge \omega_i \wedge \omega_j \;=\; f_{ij} \tag{4.43}$$

where ω_j is Poincaré dual to Λ_4^j and represents a closed string $U(1)_j$, and f_{ij} is the kinetic mixing (4.7) between $U(1)_i$ and $U(1)_j$. The correct way to interprets this fact is that, if a D4-brane wrapping Σ_X corresponds to a 4d monopole with unit charge under $U(1)_X$, then a D4-brane wrapping Σ'_X has unit charge under $U(1)_X$ but also under $U(1)_j$, and so it is equivalent to a bound state of open and closed string U(1) monopoles. Therefore, via the Witten effect it will obtain a electric and magnetic charge under $U(1)_i$ which is not given by the kinetic mixing f_{iX}, but rather by the sum of mixings $f_{iX} + f_{ij}$, see eq.(4.34). In general, it is easy to see that the integral $\int_{\Sigma_X} J_c \wedge \omega_i$ will only depend on the homology class $[\Sigma_X] \in H_4^-(\mathcal{M}_6, \pi_{D6}, \mathbb{Z})$ which as stated before is nothing but the lattice of integral U(1) magnetic charges of the 4d effective theory. Hence, in order to properly use eq.(4.42) we first need to take a basis for $H_4^-(\mathcal{M}_6, \pi_{D6}, \mathbb{Z})$ and identify those 4d monopoles that have unit charge under $U(1)_X$ but no integer charge under the closed string U(1)'s, and then apply eq.(4.42) with a 4-chain Σ_X in the corresponding relative homology class.

Let us now restore the full dependence of $\mathcal{F}^{D4}$ in (4.42) and let us see which further source of ambiguity that gives. Even if we keep Σ_X within the same relative homology class there are infinite discrete choices of F_2^{D4} such that the appropriate boundary conditions

$$\int_{\Sigma_X} F_2^{D4} \wedge \omega_i = \int_{\Sigma_X} \tilde{F}_2^X \wedge \omega_i \equiv \frac{1}{L_X} \sum_{\beta=1}^{K} n_{X\beta} \int_{\pi_\beta} A_1^\beta \wedge \omega_i \tag{4.44}$$

are satisfied. Indeed, let us consider the case where the 4-chain Σ_X contains a non-trivial 2-cycle Λ_2^j such that $[\Lambda_2^j]$ is non-trivial in $H_2^+(\mathcal{M}_6, \mathbb{Z})$. By Poincaré duality on Σ_X, one may then consider a 2-form F_2^j on Σ_X which is the curvature of a vanishing line bundle on $\partial\Sigma_X$ and satisfying

$$\int_{\Sigma_X} F_2^j \wedge \gamma = \int_{\Lambda_2^j} \gamma \tag{4.45}$$

for any closed 2-form γ on $\mathcal{M}_6$. Then it is easy to see that if one takes $F_2^{D4} = [F_2^{D4}]_0 + nF_2^j$ with $[F_2^{D4}]_0$ satisfying (4.44), eq.(4.37) reads

$$Q_i^E = \frac{1}{l_s^2} \int_{\Sigma_X} \tilde{F}_2^X \wedge \omega_i + n\,\delta^{ij} \tag{4.46}$$

where we have assumed that $[\Lambda_2^j]$ is Poincaré dual to $[\tilde{\omega}^j]$ and so $\int_{\Lambda_2^j} \omega_i = \delta^{ij}$. This result is easily interpreted as the fact that the piece of flux $n\,F_2^j$ induces a the charge of n D2-branes wrapping Λ_2^j on the D4-brane on Σ_X, so this D4-brane is actually a 4d particle with unit magnetic charge under $\hat{U}(1)_X$ and electric charge n under $U(1)_j$. Therefore comparing (4.34) and (4.46) one concludes that $\mathrm{Im}\, f_{iX} = -\frac{2\pi}{l_s^4} \int_{\Sigma_X} \left(B + \frac{l_s^2}{2\pi} \tilde{F}_2^X \right) \wedge \omega_i$ and that in eq.(4.42) one must assume that F_2^{D4} does not induce any non-trivial D2-brane charge.

To summarise, we find that the set of monopoles in a type IIA orientifold compactification is classified by the relative homology group $H_4^-(\mathcal{M}_6, \pi_{D6}, \mathbb{Z})$, where π_6 is the formal sum of the D6-brane locations. The dimension of this lattice is the total number of massless U(1)'s, open and closed, of the compactification, and so in some sense the space of 4d U(1)'s should also be classified by this same relative homology group. This is rather natural if we interpret the whole discussion above from the viewpoint of M-theory. Indeed, lifting the type IIA compactifiaciton to M-theory in a 7-dimensional manifold $\mathcal{M}_7$ we have that $H_4^-(\mathcal{M}_6, \pi_{D6}, \mathbb{Z})$ lifts to the homology group $H_5(\mathcal{M}_7, \mathbb{Z})$, and that the U(1) magnetic monopoles become M5-branes wrapping non-trivial 5-cycles in $\mathcal{M}_7$ (see appendix E for a simple example). The U(1)'s themselves are classified by harmonic two-forms in $\mathcal{M}_7$, hence (assuming no torsion in homology) by the Poincaré dual group $H^2(\mathcal{M}_7, \mathbb{Z})$. Finally, in M-theory the kinetic mixing between U(1)'s is given by the simple formula

$$f_{\alpha\beta} = -\frac{2\pi i}{l_M^9} M^I \int_{\mathcal{M}_7} \phi_I \wedge \omega_\alpha \wedge \omega_b \tag{4.47}$$

where l_M is the M-theory characteristic length and ϕ_I, $I = 1, \ldots, b_3(\mathcal{M}_7)$ runs over the harmonic 3-forms of $\mathcal{M}_7$, and M^I are the complex moduli associated to them. Following [143], from this formula one can reproduce the gauge kinetic mixing between type IIA closed string U(1)'s (4.7) and open and closed string U(1)'s (4.42). It is therefore natural to wonder if one could also reproduce the gauge kinetic mixing between two open string U(1)'s by means of a similar expression. As we will see in the following, this can be done if one describes open string U(1)'s via 2-forms on $\mathcal{M}_6$, that instead to be harmonic representatives of $H^2(\mathcal{M}_6, \mathbb{Z})$ belong to the cohomology $H^2(\mathcal{M}_6 - \pi_{D6}, \mathbb{Z})$, related by Lefschetz duality to the group $H_4(\mathcal{M}_6, \pi_{D6}, \mathbb{Z})$ classifying the monopoles.

4.1.6 Open-closed U(1) mixing and linear equivalence

The concept of linear equivalence is usually formulated to relate different p-cycles π_p of a d dimensional manifold $\mathcal{M}_d$, being stronger than equivalence in homology. While typically one applies this concept to divisor submanifolds of a complex manifold, one may extend such definition to more general cases following [147], see also Appendix D.

Indeed, let us consider two p-cycles π_p^a and π_p^b that live in the same homology class of $\mathcal{M}_d$. One can then write down the differential equation

$$d\varpi^{(a-b)} = \delta_{d-p}(\pi_p^a) - \delta_{d-p}(\pi_p^b) \tag{4.48}$$

where $\delta_{p-d}(\pi_p^\alpha)$ is a bump $(d-p)$-form localised on top of the p-cycle π_p^α and transverse to it. Because $[\pi_p^a] = [\pi_p^b]$ we know that ϖ is globally well-defined $(d-p-1)$-form. While there are in principle many solutions to this equation, one may in addition require that

$$d^*\varpi^{(a-b)} = 0 \qquad \text{and} \qquad \int_{\Lambda_{d-p-1}} \varpi^{(a-b)} \in \mathbb{Z} \tag{4.49}$$

which fixes ϖ up to an harmonic representative of the cohomology group $H^{d-p-1}(\mathcal{M}_d, \mathbb{Z})$. From a mathematical viewpoint, this allows to identify ϖ as the connection of a gerbe. From a physical viewpoint we will see that they are natural conditions when we want to relate ϖ with an open string U(1).

Given (4.48) and (4.49), it is easy to see that ϖ admits the following global Hodge decomposition

$$\varpi^{(a-b)} = \omega + d^*H \tag{4.50}$$

where ω is a harmonic $(d-p-1)$-form and H_{d-p} is a globally well-defined $(d-p)$ form. We then say that the two p-cycles π_p^a and π_p^b are *linearly equivalent* if $[\omega] \in H^{d-p-1}(\mathcal{M}_6, \mathbb{Z})$, or in other words if the two components of ϖ are separately quantised.

While the above definition is rather abstract, one may detect linear equivalence in a rather simple way as follows. Given the two p-cycles π_p^a and π_p^b let us construct a $(p+1)$-chain $\Sigma^{(a-b)}$ such that $\partial\Sigma^{(a-b)} = \pi_p^a - \pi_p^b$. Then, from the discussion in Appendix D one can see that

$$\int_{\Sigma^{(a-b)}} \tilde{\omega}_{p+1} = \int_{\mathcal{M}_6} \tilde{\omega}_{p+1} \wedge \varpi^{(a-b)} \mod \mathbb{Z} \tag{4.51}$$

for any closed $(p+1)$-form $\tilde{\omega}_{p+1}$ with integer cohomology class $[\tilde{\omega}_{p+1}] \in H^{p+1}(\mathcal{M}_6, \mathbb{Z})$. Moreover, if $\tilde{\omega}_{p+1}$ is harmonic we have that we can replace $\varpi^{(a-b)} \to \omega$ in the rhs of (4.51). Hence if we take $\tilde{\omega}_{p+1}$ to be harmonic and with integer homology class we have that π_p^a and π_p^b are linearly equivalent if and only if

$$\int_{\Sigma^{(a-b)}} \tilde{\omega}_{p+1} \in \mathbb{Z} \qquad \forall\, \Sigma^{(a-b)} \text{ such that } \partial\Sigma^{(a-b)} = \pi_p^a - \pi_p^b \tag{4.52}$$

Actually, this criterion for linear equivalence can be refined if we restrict the class of chains that enter into eq.(4.52), and such refinement will allow to relate the above definitions with the computation of open-closed $U(1)$ kinetic mixing. For concreteness, let us consider a simple case of interest discussed in the previous sections. Namely, we consider two D6-branes wrapping two homologous 3-cycles π_3^a and π_3^b of $\mathcal{M}_6$. One can then write the differential equation

$$d\varpi_2^{(a-b)} = \delta_3(\pi_3^a) - \delta_3(\pi_3^b) \tag{4.53}$$

which is nothing but (4.48) for the particular case $d = 6$, $p = 3$. Requiring that ϖ_2 is co-closed and quantised as in (4.49) one obtains

$$\varpi_2^{(a-b)} = c^j \omega_j + d^* H \tag{4.54}$$

where $\{\omega_j\}$ is a basis of harmonic 2-forms with integer cohomology class and $c^j \in \mathbf{R}$. This definition of ϖ_2 only fixes the value of c^j mod $\mathbb{Z}$, and so one can always define ϖ_2 such that $c_j \in [0, 1)$, $\forall\, j$. With this choice there is a 4-chain $\Sigma^{(a-b)}$ such that

$$\int_{\Sigma^{(a-b)}} \tilde{\omega} = \int_{\mathcal{M}_6} \tilde{\omega} \wedge \varpi_2^{(a-b)} \tag{4.55}$$

for any closed 4-form $\tilde{\omega}$, and without the need of the mod $\mathbb{Z}$ that appears in eq.(4.51).

We can also see (4.55) as a consequence of Lefschetz duality between the groups $H_4(\mathcal{M}_6, \pi_3^a \cup \pi_3^b, \mathbb{Z})$ and $H^2(\mathcal{M}_6 - \{\pi_3^a \cup \pi_3^b\}, \mathbb{Z})$. Indeed, the 2-forms (4.54) are harmonic representatives of the cohomology group $H^2(\mathcal{M}_6 - \{\pi_3^a \cup \pi_3^b\}, \mathbb{Z})$, which contains $H^2(\mathcal{M}_6, \mathbb{Z})$. Changing the value of the coefficients c^j by an integer number amounts to change the cohomology class by an element of $H^2(\mathcal{M}_6, \mathbb{Z})$, so choosing $c_j \in [0, 1)$, $\forall\, j$ means choosing a particular class in $H^2(\mathcal{M}_6 - \{\pi_3^a \cup \pi_3^b\}, \mathbb{Z})$. Then, restricting the 4-chain Σ^{a-b} to the dual class in $H_4(\mathcal{M}_6, \pi_3^a \cup \pi_3^b, \mathbb{Z})$ allows to write down (4.55) without any mod $\mathbb{Z}$ ambiguity.

As we saw when discussing monopoles, restricting the 4-chain Σ to a particular relative homology class is also needed when computing the open-closed kinetic mixing from the chain integral (4.42). In fact one can see that, if we ignore the contribution of the Wilson lines, in the present case such chain integral reads

$$-\frac{i}{l_s^4} \int_{\Sigma^{(a-b)}} J_c \wedge \omega_i = -\frac{i}{l_s^4} \int_{\mathcal{M}_6} J_c \wedge \omega_i \wedge \varpi_2^{(a-b)} = -i\mathcal{K}_{ij\hat{k}} T^{\hat{k}} c^j = f_{ij} c^j \tag{4.56}$$

Finally, for $c_j \in [0, 1)$ linear equivalence between π_3^a and π_3^b amounts to require that $c_j = 0$, $\forall\, j$. That is, the equivalence $\pi_3^a \sim \pi_3^b$ corresponds to the vanishing of the kinetic mixing $f_{i(a-b)}$ of $\frac{1}{2}[U(1)_a - U(1)_b]$ with any $\mathrm{U}(1)_i$ from the closed string sector.

This statement is of course modified if we include the Wilson line dependence in the kinetic mixing $f_{i(a-b)}$. In general we have that

$$f_{i(a-b)} = -\frac{i}{2l_s^4}\int_\Sigma \left(J_c + \frac{l_s^2}{2\pi}\tilde{F}_2^{(a-b)}\right) \wedge \omega_i \; = \; f_{ij}c^j - \frac{i}{4\pi l_s^4}\sum_j (\theta_a^j - \theta_b^j)\int_{\mathcal{M}_6} \tilde{\omega}^j \wedge \omega_i \quad (4.57)$$

where $\tilde{\omega}^j$ is obtained from ζ_j in (4.10) as follows. First consider the Poincaré dual class $[\rho_j] = \mathrm{PD}_\pi[\zeta_j]$ of two-cycles within the three-cycle π. Then consider $[\rho_j]$ as a class of $H_2(\mathcal{M}_6, \mathbb{Z})$ and take its Poincaré dual four-form class $[\tilde{\omega}^j] \in H^4(\mathcal{M}_6, \mathbb{Z})$ and in particular the harmonic representative $\tilde{\omega}^j$. Hence, having vanishing kinetic mixing imposes a further condition on the Wilson line moduli, a condition that we will dub *generalised linear equivalence* for reasons that will become clear in the next section.

Now, because the Wilson line dependence can also be written as $\int_\Sigma F_2^{\mathrm{D4}} \wedge \omega_i$ with F_2^{D4} a bundle extension on the D4-brane monopole connecting the two D6-branes it follows that $\tilde{\omega}^j = \varpi_2^{(a-b)} \wedge \eta^j$ for some two-form $[\eta^j] \in H^2(\mathcal{M}_6, \mathbb{Z})$. We can then express the kinetic mixing as

$$f_{i(a-b)} \; = \; -\frac{i}{2l_s^4}\int_{\mathcal{M}_6} \left(J_c + \frac{l_s^2}{2\pi}F_2\right) \wedge \omega_i \wedge \varpi_2^{(a-b)} \tag{4.58}$$

where we have defined $l_s^2 F_2 \equiv \sum_j (\theta_a^j - \theta_b^j)\eta^j$. This expression is quite similar to the initial one (4.27), with the replacement of the integral over the 4-chain $\Sigma^{(a-b)}$ by the Poincaré dual harmonic form $\varpi_2^{(a-b)}$. Recall that in (4.27) the 4-chain sigma must lie in a particular relative homology class of $H_4(\mathcal{M}_6, \pi_3^a \cup \pi_3^b, \mathbb{Z})$, so that a monopole on that chain has no initial magnetic charge under the closed string U(1)'s, and only acquires it via the Witten effect. In (4.58) this is implemented by imposing that $\varpi_2^{(a-b)}$ is of the form (4.54) with $c_j \in [0, 1)$. Moreover, in this expression we have replaced the Wilson line integral (4.28) that defines $\tilde{F}_2^{(a-b)}$, by an integral over the whole manifold $\mathcal{M}_6$ involving the two-form F_2. One possible way to interpret this is via the trading of the Wilson line background by a shift in the B-field background, as done in [150].

The above discussion can be easily generalised for the case where we have more than two D6-branes wrapping 3-cycles the same homology class. For each massless $\mathrm{U}(1)_X$ of the open string sector given by a linear combination of 3-cycles (4.40) such that $[\pi_X] = 0$ we can define the 2-form ϖ_X by

$$d\varpi_X \; = \; \sum_\alpha n_{X\alpha} N_\alpha \delta_3(\pi_\alpha) \tag{4.59}$$

and such that ϖ_2 co-closed and has integer relative homology class. This fixes ϖ_X to be of the form (4.54), where again we impose that $c_j \in [0, 1)$. Then we find that the kinetic mixing between $U(1)_X$ and a $U(1)_i$ of the closed string sector is given by

$$f_{iX} \; = \; \frac{1}{2l_s^4}\int_{\mathcal{M}_6} (J - i\mathcal{F}) \wedge \omega_i \wedge \varpi_X \tag{4.60}$$

where the contribution proportional to J vanishes if the combination π_X is linearly trivial, since then $c_j = 0, \forall\, j$. Here $\mathcal{F} = B + \frac{l_s^2}{2\pi}F_2^X$, where $l_s^2 F_2^X = L_X^{-1}\sum_\beta n_{X\beta}\theta_\beta^j \eta_\beta^j$ (c.f. (4.44)) and $\tilde{\omega}_\beta^j = \varpi_X \wedge \eta_\beta^j$, with $\tilde{\omega}_\beta^j$ obtained from each D6-brane β harmonic one-form ζ_j as described before. Notice that (4.60) reduces to (4.42) if we apply Poincaré duality in relative homology and replace $\int_{\mathcal{M}_6} \varpi_X \wedge \ \to\ \int_{\Sigma_X}$, and then take $\mathcal{F}^{D4} = \mathcal{F}|_{\Sigma_X}$. Finally, one can generalise this expression to the case of orientifold compactifications by modifying (4.59) in the obvious way and by taking ϖ_X, F_2^X with the appropriate orientifold parity, or more precisely by taking into account that $[\varpi_X] \in H^2_+(\mathcal{M}_6, \pi, \mathbb{Z})$ and $[\frac{1}{2\pi} F_2^X] \in H^2_-(\mathcal{M}_6, \mathbb{Z})$.

4.1.7 Open-open U(1) mixing from supergravity

The expression (4.60) is very suggestive from the viewpoint of M-theory, because by doing the simple replacements

$$\begin{array}{rcl} \mathcal{M}_6 & \to & \hat{\mathcal{M}}_7 \\ J_c + F_2 & \to & M^I \phi_I \\ \varpi_X & \to & \omega_X \end{array} \tag{4.61}$$

with ω_X an harmonic 2-form on $\hat{\mathcal{M}}_7$ one recovers the M-theory expression (4.47) for the kinetic mixing of two U(1)'s. All these replacements are standard when lifting a type IIA compactification to M-theory except perhaps the last one, which suggests that a harmonic representative of $H^2_+(\mathcal{M}_6, \pi, \mathbb{Z})$ becomes an harmonic representative of $H^2(\hat{\mathcal{M}}_7, \mathbb{Z})$ upon the M-theory lift. This is however not too surprising in view of our previous discussion of section 4.1.5, where in order to match the monopoles in a type IIA compactification and its M-theory lift we concluded that $H_4^-(\mathcal{M}_6, \pi, \mathbb{Z})$ should be identified with $H_5(\hat{\mathcal{M}}_7, \mathbb{Z})$.

In fact, a similar sort of lift has been recently discussed in [150] in the context of F-theory compactified in K3, in which one-forms on the $\mathbf{P^1}$ base that were co-closed and closed up to the 7-brane locations were obtained from integrating harmonic 2-forms along the elliptic fiber of a K3 manifold. In a similar spirit, one would expect that the harmonic 2-forms of a G_2-manifold $\hat{\mathcal{M}}_7$ become the 2-forms ω_i and ϖ_2^X when reducing them on the M-theory circle, and that the latter are only closed up to the location of the D6-branes.[7] In this sense, and in the same way that one expands the M-theory three-form potential A_3 in a basis of harmonic two-forms in order to obtain the 4d U(1) gauge bosons, one could consider expanding the type IIA RR three-form potential as

$$C_3 \ = \ A_1^i \wedge \omega_i + A_1^X \wedge \varpi_X + \ldots \tag{4.62}$$

where ω_i run over a basis of harmonic representatives of $H^2_+(\mathcal{M}_6, \mathbb{Z})$ and ϖ_X complete the basis of harmonic representatives of the relative cohomology group $H^2_+(\mathcal{M}, \pi, \mathbb{Z})$. Just like ω_i can be thought as the internal wavefunction for the closed string U(1) gauge bosons A_1^i, then ϖ_X can be understood as the internal wavefunction for the massless open string U(1) gauge bosons A_1^X.

In particular, from (4.47) one would expect to be able to write an expression for the open string U(1) kinetic mixing in terms of the 2-forms ϖ and F_2. It is easy to convince oneself that the appropriate expression should then be of the form

$$f_{XY} \ = \ \int_{\mathcal{M}_6} (J - i\mathcal{F}) \wedge \varpi_X \wedge \varpi_Y \tag{4.63}$$

where $\mathcal{F} = B + \frac{l_s^2}{2\pi}(F_2^X + F_2^Y)$, with $F_2^{X,Y}$ defined as in below (4.60). Compared to the M-theory formula (4.47), we again see that ϖ lift to an integer harmonic two-form in

[7] Appendix E shows this correspondence in detail for the particular case of several parallel D6-branes in flat space and their M-theory lift to a Taub-NUT space.

$\hat{\mathcal{M}}_7$, while F_2 lifts to a harmonic three-form contained in Re $(M^I)\phi_I$. This matches our intuition: massless D6-brane U(1) lift in M-theory to certain 5-cycles, and their Wilson line lift to integrals of the three-form potential A_3 over such 5-cycles.

Now, if this is the correct expression for the gauge kinetic mixing one may wonder at which level it is being computed. In the following we will argue that (4.60) corresponds to the one-loop corrected gauge kinetic mixing between two massless open string U(1)'s. Indeed, at tree level we have gauge kinetic functions for models of intersecting D6-branes *i)* depend only on the complex structure moduli and the periods of the RR potential C_3 *ii)* do not depend on the D6-brane moduli and *iii)* are diagonal for appropriate linear combinations of U(1) generators like the one in (4.41). On the contrary, the gauge kinetic functions computed from (4.63) are such that *a)* depend on the complexified Kähler moduli via the presence of J_c *b)* depend on the D6-brane moduli via the presence of ϖ_X, which depends on the D6-brane positions, and of F_2^X which depends on the D6-brane Wilson lines *c)* are typically non-vanishing for $X \neq Y$. As pointed out in the literature (see e.g. [151]) these last three features do appear for open string kinetic functions when one computes their one-loop threshold corrections. Finally, notice that the 2-forms ϖ_X are defined via (4.59) as a sort of backreaction of a particular linear combination of D6-brane sources, just like one would do for the RR field strength $F_2 = dC_1$ except that now the backreacting sources appear with certain signs and multiplicities. Hence, following the general philosophy of [152] it is reasonable to think that the backreaction of localised sources in type II supergravity captures the physics of the open string channel at the one-loop level, giving further evidence that (4.60) should describe threshold corrected gauge kinetic functions. It would be nice to compare this expression with explicit computations for D6-brane gauge kinetic mixings at one-loop level, as performed in [151] (see also [153]).

4.2 Linear equivalence of magnetized branes

In our analysis of type IIA compactifications we have found that the kinetic mixing between open and closed string U(1)'s can be computed by means of a simple chain formula, whose physical meaning can be understood as the Witten effect applied to D-brane U(1) monopoles. Moreover, we found that just like the set of monopoles is classified by a relative homology group, the set of U(1)'s is classified by a dual cohomology group. In particular, one is able to characterize the massless open string U(1)'s in terms of a pair of bulk two-forms (ϖ, F_2) from which one can reproduce the previous chain formula for the kinetic mixing. This kinetic mixing will vanish when the set of D6-branes defining the U(1) are linearly equivalent, or in other words when ϖ is orthogonal to any harmonic 2-from in the compactification manifold. Finally, the pair (ϖ, F_2) not only allows to compute open-closed U(1) mixing by means of integrals over the compactification manifold, but also to propose a similar bulk formula to compute open-open U(1) mixing.

In this section we would like to extend this picture to type IIB compactifications, as well as to F-theory ones. The novelty of these compactifications is that they contain magnetized branes, and so the typical object to consider is a bound state of 3, 5 and 7-branes. However, just like in the type IIA case, we will be able to describe open-closed kinetic mixing by using the Witten effect, and to characterize open string U(1)'s in terms of bulk forms ϖ. Also, the fact that the open-closed mixing vanishes can be translated into a version of linear equivalence adapted to brane bound states. Finally, we will derive a supergravity-like formula to compute kinetic mixing between open string U(1)'s, and apply it to F-theory GUT models with U(1) gauge factors beyond the hypercharge.

4.2.1 Type IIB orientifolds with O3/O7-planes

Let us then consider type IIB string theory compactified on $\mathbf{R}^{1,3} \times \mathcal{M}_6$, where again $\mathcal{M}_6$ is a Calabi-Yau manifold, and mod it out by the orientifold action $\Omega_p(-1)^{F_L}\sigma$, where now σ is a holomorphic involution $\mathcal{M}_6$ such that

$$\sigma J = J \ , \qquad \sigma\Omega = -\Omega \tag{4.64}$$

The fixed loci of σ are points and/or complex four-cycles of $\mathcal{M}_6$, where O3 and O7-planes are respectively located. The cancellation of RR tadpoles imposes that

$$\sum_\alpha N_\alpha([\pi_\alpha] + [\pi_\alpha^*]) - 8[\pi_O] = 0 \tag{4.65}$$

where π_α are the complex four-cycles wrapped by the D7-branes in the model, π_α^* are their orientifold images, and π_O the divisors wrapped by the orientifold planes. Besides (4.65) one needs to impose that the total D5 and D3-brane charges of the compactification vanish.

Dimensionally reducing the 10d type IIB supergravity action one encounters a series of massless fields that arise from the closed string sector of the theory. As before, these are classified by the harmonic forms of $\mathcal{M}_6$ with a definite parity under the action of σ. We now take the following basis of harmonic forms with integer cohomology class

$$\begin{array}{lllll} & & \sigma-\text{even} & & \sigma-\text{odd} \\ 2-\text{forms} & \omega_i & i=1,\ldots,h_+^{1,1} & \omega_{\hat{i}} & \hat{i}=1,\ldots,h_-^{1,1} \\ 3-\text{forms} & \alpha_I & I=0,\ldots,h_+^{1,2} & \tilde{\alpha}_{\hat{I}} & \hat{I}=0,\ldots,h_-^{1,2} \\ & \beta^I & I=0,\ldots,h_+^{1,2} & \tilde{\beta}^{\hat{I}} & \hat{I}=0,\ldots,h_-^{1,2} \\ 4-\text{forms} & \tilde{\omega}^i & i=1,\ldots,h_+^{1,1} & \tilde{\omega}^{\hat{i}} & \hat{i}=1,\ldots,h_-^{1,1} \end{array}$$

and with normalization

$$\int_{\mathcal{M}_6} \omega_i \wedge \tilde{\omega}^j = l_s^6 \delta_i^j, \quad \int_{\mathcal{M}_6} \omega_{\hat{i}} \wedge \tilde{\omega}^{\hat{j}} = l_s^6 \delta_{\hat{i}}^{\hat{j}}, \quad \int_{\mathcal{M}_6} \alpha_I \wedge \beta^J = l_s^6 \delta_I^J, \quad \int_{\mathcal{M}_6} \tilde{\alpha}_{\hat{I}} \wedge \tilde{\beta}^{\hat{J}} = l_s^6 \delta_{\hat{I}}^{\hat{J}} \tag{4.66}$$

The next step is to expand the RR four-form potential C_4 in the σ-even harmonic forms ω_i, α_I, β^I, $\tilde{\omega}^i$. Since the field strength of C_4 must satisfy the 10d self-duality condition $\hat{F}_5 = *_{10}\hat{F}_5$, it is convenient to define the following basis of complex harmonic three-forms

$$\gamma_I \ = \ \alpha_I + i f_{IJ} \beta^J \tag{4.67}$$

where f_{IJ} are function of the complex structure moduli chosen so that γ_I is a $(2,1)$-form. Then it is easy to see that if we expand the four-form potential as

$$C_4 = \sum_I (A_1^I \wedge \text{Re } \gamma_I - V_1^I \wedge \text{Im } \gamma_I) + \sum_i \left(C_2^i \wedge \omega_i - \text{Re}(T^i)\tilde{\omega}^i\right) \tag{4.68}$$

then 10d self-duality of $\hat{F}_5$ implies that $*_4 dA_1^I = dV_1^I$. We then obtain $h_+^{1,2}$ vector multiplets A_1^I and $h_+^{1,1}$ axions Re (T^i), the other 4d modes being dual degrees of freedom. Finally, using (4.66) it is easy to check that f_{IJ} is precisely the gauge kinetic function for the closed string U(1)'s [?, ?, ?]. Notice for instance that a D3-brane wrapping a three-cycle in the Poincaré dual class of α_J will not only have unit magnetic charge under the closed string U(1) generated by A_1^J, but also an electric charge under A_1^I proportional to Im f_{IJ}, as expected from the Witten effect.

Besides above spectrum there will be 4d massless fields arising from the open string sector of the compactification. In particular, the sector of a single D7-brane wrapping a holomorphic 4-cycle S_α of $\mathcal{M}_6$ is given by

$$\begin{aligned} A_1^\alpha &= \frac{1}{l_s}(A_1^{4d,\alpha} + a_\alpha^j A_j + c.c.) && (4.69)\\ \phi_\alpha &= \Phi_\alpha + c.c. = \Phi_\alpha^m X_m + c.c. && (4.70) \end{aligned}$$

where A_j are a basis of $H^{(0,1)}(S_\alpha)$ and X_m are holomorphic sections of the normal bundle of S_α, which are in one to one correspondence with the homology group $H^{(2,0)}(S_\alpha)$ by contraction with Ω. As a result, the Wilson line moduli a_α^j and the position moduli Φ_α^j are each complex 4d scalar fields by themselves.[8] One can easily generalize this spectrum to the case of a stack of N_α D7-branes, as well as to include the effect of the orientifold projection. We refer the reader to [154] for further details in this direction.

4.2.2 Separating two D7-branes

For simplicity let us first consider two D7-branes wrapping four-cycles S_a and S_b of $\mathcal{M}_6$, and threaded respectively by worldwolume fluxes $\bar{F}_a$ and $\bar{F}_b$, leaving the action of the orientifold for later. Like with the D6-branes we assume that S_α, $\alpha = a, b$ lie in the same homology class and that they can both be obtained after normal deformations ϕ_α of a reference four-cycle S. As before, we can reduce the CS action for these branes and read off the terms that yield Stückelberg masses for the open string $U(1)$'s as well as their kinetic mixing with the closed string sector. The relevant terms of the CS action now are

$$S_{CS}^\alpha \supset \mu_7 \int_{\mathbb{R}^{1,3}\times S_\alpha} P[C_6 \wedge \mathcal{F}_2^\alpha + \frac{1}{2} C_4 \wedge \mathcal{F}_2^\alpha \wedge \mathcal{F}_2^\alpha] \qquad (4.71)$$

which upon dimensional reduction yield [154]

$$\begin{aligned} S_{CS}^\alpha \supset\ & \frac{1}{l_s^7}\int_{\mathbb{R}^{1,3}} \tilde{C}_2^i \wedge F_2^\alpha \int_{S_\alpha} \tilde{\omega}^i + \frac{1}{2\pi l_s^7}\int_{\mathbb{R}^{1,3}} C_2^i \wedge F_2^\alpha \int_{S_\alpha} \bar{F}_2^\alpha \wedge \omega^i && (4.72)\\ +\ & \frac{1}{2\pi l_s^6}\int_{\mathbb{R}^{1,3}} F_2^\alpha \wedge F_2^{RR,I} \int_{S_\alpha} \text{Re } (a_\alpha \wedge \gamma_I + \iota_{\Phi_\alpha}\gamma_I \wedge \bar{F}_2^\alpha) \\ -\ & \frac{1}{2\pi l_s^6}\int_{\mathbb{R}^{1,3}} F_2^\alpha \wedge * F_2^{RR,I} \int_{S_\alpha} \text{Im } (a_\alpha \wedge \gamma_I + \iota_{\Phi_\alpha}\gamma_I \wedge \bar{F}_2^\alpha) \end{aligned}$$

where we have used

$$\begin{aligned} C_4 &= \tilde{C}_2^i \wedge \omega^i + A_1^I \wedge \text{Re } \gamma_I - V_1^I \wedge \text{Im } \gamma_I + \ldots && (4.73)\\ C_6 &= \tilde{C}_2^i \wedge \tilde{\omega}^i + \ldots && (4.74) \end{aligned}$$

with ω^i, $\tilde{\omega}^i$ running over all (1,1) and (2,2)-forms of $\mathcal{M}_6$. We have also used that $a_\alpha = a_\alpha^j A_j$ is a (0,1)-form and $\bar{F}_2^\alpha$, $\iota_{\Phi_\alpha}\gamma_I$ are (1,1)-forms of S_α. Finally, we ommited

[8] We normalize the fields Φ_α as $\Phi_\alpha = \frac{y_\alpha}{l_s}$ where y_α are the transverse coordinates to the brane α.

any contribution proportional to the B-field since it can be simply recovered via gauge invariance by replacing $\frac{l_s^2}{2\pi}F \to \mathcal{F}$.

The terms in the first line of (4.72) correspond to the Stückelberg mass for the open string $U(1)$ and the second line gives the kinetic mixing with the closed string sector. Taking into account both D7-branes a and b, the combination $U(1)_a + U(1)_b$ will be massive due to the first term in (4.72) and in order to keep $U(1)_{(a-b)} \equiv \frac{1}{2}[U(1)_a - U(1)_b]$ massless we will impose that $[\bar{F}_2^a] = [\bar{F}_2^b] \equiv [\bar{F}_2]$. Indeed, notice that the Stückelberg mass for $U(1)_{(a-b)}$ is proportional to

$$\int_S (\bar{F}_2^a - \bar{F}_2^b) \wedge \omega^i \tag{4.75}$$

and so setting both worldvolume flux equal on S prevents this U(1) from getting a mass. There is a subtlety in this statement which is particularly important for the case of the hypercharge $U(1)_Y$ in $SU(5)$ F-theory GUTs. Namely, in order to keep $U(1)_{(a-b)}$ massless the class $[\bar{F}_2^a - \bar{F}_2^b]$ should only be zero as an element of $H^2(\mathcal{M}_6)$ and it could be non-trivial in $H^2(S)$ [155]. We will assume that the fluxes are the same also in S and leave the more involved case for the next section.

An analogous computation to the one in the previous section shows that the kinetic mixing between $U(1)_{(a-b)}$ and the closed string $U(1)_I$ is given by

$$f_{I(a-b)} = -\frac{i}{4\pi^2 l_s^4}(a_a^j - a_b^j)\int_S A_j \wedge \gamma_I - \frac{i}{4\pi^2 l_s^4}(\Phi_a^m - \Phi_b^m)\int_S \iota_{X_m}\gamma_I \wedge \bar{F}_2 \tag{4.76}$$

Comparing this expression to (4.27) and (4.42) one can guess what the mixing is in the case where the branes are not homotopic, namely

$$f_{I(a-b)} = -\frac{i}{2\pi l_s^5}\int_\Gamma \gamma_I \wedge \tilde{\mathcal{F}} \tag{4.77}$$

where Γ is a 5-chain with $\partial\Gamma = S_a - S_b$ and $\tilde{\mathcal{F}}$ is a 2-form defined on Γ such that $d\tilde{\mathcal{F}} = 0$ and $\tilde{\mathcal{F}}|_{S_\alpha} = \bar{\mathcal{F}}_\alpha$. We implicitly include the contribution coming from the Wilson lines in the boundary $\partial\Gamma$ as in (4.28) (see also the comment below eq.(4.37)). Finally, notice that $\mathcal{F}$ also includes the contribution from the B-field, but that it vanishes in the case that $H = dB = 0$, since then B is an harmonic (1,1)-form and so $\gamma_I \wedge B \equiv 0$.

As before, we can also arrive at (4.77) by computing the electric charge of a 4d $U(1)_{(a-b)}$-monopole with respect to the closed string $U(1)$'s. Indeed, consider a D5-brane wrapped on $W \times \Gamma$ where W is a worldline in $\mathbb{R}^{1,3}$, and with worldvolume flux $\tilde{\mathcal{F}}$ along Γ. This corresponds to a monopole in 4d with unit magnetic charge under $U(1)_{(a-b)}$. The CS action for this objects has a piece of the form

$$S_{CS}^{D5} \supset \mu_5 \int_{W\times\Gamma} C_4 \wedge \tilde{\mathcal{F}} = \mu_5 \int_\Gamma \tilde{\mathcal{F}} \wedge \mathrm{Re}\ \gamma_I \int_W A_1^I - \mu_5 \int_\Gamma \tilde{\mathcal{F}} \wedge \mathrm{Im}\ \gamma_I \int_W V_1^I \tag{4.78}$$

so the induced electric and magnetic charges under $U(1)_I$ are

$$Q_I^E = \frac{2\pi}{l_s^5}\int_\Gamma \mathrm{Re}\ \gamma_I \wedge \tilde{\mathcal{F}} \qquad Q_I^M = -\frac{2\pi}{l_s^5}\int_\Gamma \mathrm{Im}\ \gamma_I \wedge \tilde{\mathcal{F}} \tag{4.79}$$

which reproduces (4.77) by virtue of the Witten effect.

The effect of the orientifold projection is also quite similar to the case of D6-branes. We have take into account that ω^i and $\tilde{\omega}^i$ in (4.72) only run over σ-even forms of $\mathcal{M}_6$, and we need to include the orientifold image for every D7-brane. This leads to

$$f_{I(a-b)} = -\frac{i}{2\pi l_s^5}\int_\Gamma \gamma_I \wedge \tilde{\mathcal{F}} \tag{4.80}$$

with Γ an orientifold-odd chain such that $\partial\Gamma = S_a - S_b - S_a^* + S_b^*$. Again, the contribution from the B-field vanishes in case that $H = 0$, which we will assume in the following.

4.2.3 Monopoles and generalized cycles

Let us rephrase the previous example in the language of generalized complex geometry. This will allow us to easily extend the above expression for the U(1) mixing to more general situations, and in particular to the case of interest in F-theory GUTs to be analyzed in section **??**. In addition, this formalism properly treats D-branes with worldvolume fluxes, and so it allows to generalize the concept of linear equivalence of submanifolds [147] to bound states of D-branes, as well as to derive a supergravilty-like formula for its kinetic mixing. Finally, generalized geometry has been shown to be the right framework to include the effect of background fluxes in both type IIA and type IIB $\mathcal{N} = 1$ compactifications, and hence it should be the appropriate tool to understand U(1) kinetic mixing in the context of moduli stabilization.

In generalized complex geometry a Dp-brane is a pair $(\Sigma, \mathcal{F})$ where Σ is a $(p+1)$-cycle and $\mathcal{F}$ is the worldvolume field strength together with the B-field, $\mathcal{F} = F + B$.[9] Thus, the two D7-branes of the last section correspond to $(S_a, \mathcal{F}_a)$ and $(S_b, \mathcal{F}_b)$ and the condition to have a massless $U(1)$ is that there should be a linear combination of the two that is trivial in generalized homology. Indeed, in the previous example we had

$$(S_a, \mathcal{F}_a) - (S_b, \mathcal{F}_b) = \hat{\partial}(\Gamma, \tilde{\mathcal{F}}) \tag{4.81}$$

where $\hat{\partial}$ is the generalized boundary operator defined in appendix F, while Γ and $\tilde{\mathcal{F}}$ are the five-chain and worldvolume flux defined below (4.77). In particular, $\tilde{\mathcal{F}}$ satisfies

$$d\tilde{\mathcal{F}} = 0, \qquad \tilde{\mathcal{F}}|_{S_a} = \mathcal{F}_a, \qquad \tilde{\mathcal{F}}|_{S_b} = \mathcal{F}_b. \tag{4.82}$$

where we have used that $H = 0$. In order to compute the kinetic mixing we look at the CS action of a D5-brane wrapped on $(W \times \Gamma, \mathcal{F})$ which can be written as

$$S_{CS} = \mu_5 \int_{W\times\Gamma} C \wedge e^{\tilde{\mathcal{F}}} \tag{4.83}$$

where C is the RR polyform. Since the closed string $U(1)$'s arise from C_4 it suffices to look at that term. Upon dimensional reduction it yields

$$\begin{aligned} S_{CS} \;&\supset\; \mu_5 \int_\Gamma \text{Re}\ \gamma_I \wedge e^{\tilde{\mathcal{F}}} \int_W A_1^I \; - \mu_5 \int_\Gamma \text{Im}\ \gamma_I \wedge e^{\tilde{\mathcal{F}}} \int_W V_1^I \\ &=\; \mu_5\, j_{(\Gamma,\tilde{\mathcal{F}})}(\text{Re}\ \gamma_I) \int_W A_1^I \; - \mu_5\, j_{(\Gamma,\tilde{\mathcal{F}})}(\text{Im}\ \gamma_I) \int_W V_1^I \end{aligned} \tag{4.84}$$

[9] See appendix F for some basic definitions of generalized submanifolds and their homology [157].

where we have introduced the current associated to the generalized chain $(\Gamma, \tilde{\mathcal{F}})$, dubbed $j_{(\Gamma,\tilde{\mathcal{F}})}$, and used the fact that the D5 flux $\tilde{\mathcal{F}}$ is magnetic so it has no component along the time direction W. Thus, the $U(1)_I$ electric and magnetic charges of the D5-monopole are

$$Q_E^I = \frac{2\pi}{l_s^5}\, j_{(\Gamma,\tilde{\mathcal{F}})}(\mathrm{Re}\ \gamma_I) \qquad Q_M^I = -\frac{2\pi}{l_s^5}\, j_{(\Gamma,\tilde{\mathcal{F}})}(\mathrm{Im}\ \gamma_I) \tag{4.85}$$

from where we can read off the kinetic mixing. Applying this formalism to this example is somewhat cumbersome but it shows that the only relevant information for the mixing is in the so-called generalized chain $(\Gamma, \tilde{\mathcal{F}})$, and that this procedure for computing the mixing can be generalized to more involved setups.

We now consider a case which is closely related to the hypercharge $U(1)_Y$ in F-theory GUTs and for which the formalism of generalized geometry turns out to be quite useful. Suppose again that we have two D7-branes wrapped on four-cycles S_a and S_b, both homotopic to S. This implies that our D7-branes are described by the generalized cycles $(S_a, \mathcal{F}_a)$ and $(S_b, \mathcal{F}_b)$ such that $[S_a] = [S_b] \equiv [S]$. In addition we assume that $[\mathcal{F}_a] = [\mathcal{F}_b]$ in the cohomology of the total space but with $[\mathcal{F}_a]$ and $[\mathcal{F}_b]$ different in $H^2(S)$.[10]In that case we get a massless $U(1)$ so there must exist a generalized chain that connects the D7-branes but it cannot be the same one as before. Indeed, suppose it were $(\Gamma, \tilde{\mathcal{F}})$. The chain Γ connecting the submanifolds is topologically $S \times I$ where I is an interval with coordinate $t \in [0, 1]$ so Γ can be sliced in 4-cycles $S_t \simeq S$. Thus, the 2-form $\tilde{\mathcal{F}}$ on $S \times I$ defines a family of 2-forms $\tilde{\mathcal{F}}_t$ on S_t such that $\tilde{\mathcal{F}}_0 = \mathcal{F}_a$ and $\tilde{\mathcal{F}}_1 = \mathcal{F}_b$. Since $\tilde{\mathcal{F}}$ is continuous and $d\tilde{\mathcal{F}} = 0$ we have that $[\tilde{\mathcal{F}}_0] = [\tilde{\mathcal{F}}_1]$ in $H^2(S)$, contradicting our assumption.

This means that we have to look for another candidate to be the generalized chain associated to the massless $U(1)$. The strategy to find it is to realize that the quantised part of the worldvolume flux $\bar{F}_\alpha$ induces a D5-brane charge that can be related by Poincaré duality to a 2-cycle class $[\Pi_\alpha]$ of $H_2(S_\alpha, \mathbb{Z})$. If instead of two magnetized D7-branes we had two D7-branes on S_a, S_b with $\bar{F}_a = \bar{F}_b = 0$ and two D5-branes on the representative 2-cycles Π_a, Π_b, then we would have the same D-brane charges as in the magnetized system,[11] and it would be simple to connect them by means of a five-chain Γ and a three-chain Σ such that $\partial\Sigma = \Pi_a - \Pi_b$. So the only ingredient that we need is a generalized chain that interpolates between a magnetized D7-brane and a D7+D5-brane pair.

Such gereneralized chain is given by the following equation

$$(S_a, \mathcal{F}_a) = (S_a, 0) + (\Pi_a, 0) + \hat{\partial}\left[-(\Gamma_a, \tilde{\mathcal{F}}_a) + (\Gamma_a, \tilde{\mathcal{F}}_{\Pi_a})\right] \tag{4.86}$$

where $\Pi_a \subset S_a$ is any 2-cycle such that $[\Pi_a] = \mathrm{P.D.}\,[F_a]$, Γ_a is a 5-chain with $\partial\Gamma_a = S'_a - S_a$ and

$$d\tilde{\mathcal{F}}_a = 0 \qquad \tilde{\mathcal{F}}_a|_{S_a} = \bar{F}_a \qquad \tilde{\mathcal{F}}_a|_{S'_a} = \bar{F}'_a \tag{4.87}$$

$$d\tilde{\mathcal{F}}_{\Pi_a} = \delta^{(3)}_{\Gamma_a}(\Pi_a) \qquad \tilde{\mathcal{F}}_{\Pi_a}|_{S_a} = 0 \qquad \tilde{\mathcal{F}}_{\Pi_a}|_{S'_a} = \bar{F}'_a \tag{4.88}$$

where for simplicity we have removed the presence of the B-field, which anyway will not appear in our final result. This looks rather messy but its interpretation as a physical process is simple. The term $-\hat{\partial}(\Gamma_a, \tilde{\mathcal{F}}_a)$ corresponds to moving the brane from S_a to a reference four-cycle S'_a keeping the worldvolume flux fixed. The second term $\hat{\partial}(\Gamma_a, \tilde{\mathcal{F}}_{\Pi_a})$

[10] We are also assuming that $\int_{S_a} \mathcal{F}_a \wedge \mathcal{F}_a = \int_{S_b} \mathcal{F}_b \wedge \mathcal{F}_b$.

[11] We are ignoring induced D3-brane charges since they become irrelevant in the orientifold case.

is responsible for moving the brane back to S_a and removing the flux leaving the remnant $(S_a, 0) + (\Pi_a, 0)$ which represents a D7 on S_a and a D5 on Π_a each of them without fluxes. The fact that we have to move the brane back and forth is a technicality that regularizes the differential equations, and eventually we will take the limit in which we do not move the brane at all (see the discussion in appendix G).

Doing the same with the brane b we arrive at

$$(S_a, \mathcal{F}_a) - (S_b, \mathcal{F}_b) = \hat{\partial}\left[(\Gamma, 0) + (\Sigma, 0) - (\Gamma_a, \tilde{\mathcal{F}}_a) + (\Gamma_b, \tilde{\mathcal{F}}_b) + (\Gamma_a, \tilde{\mathcal{F}}_{\Pi_a}) - (\Gamma_b, \tilde{\mathcal{F}}_{\Pi_b})\right] \tag{4.89}$$

with Σ a 3-chain such that $\partial\Sigma = \Pi_a - \Pi_b$. This equation describes a process in which the worldvolume fluxes are turned into D5-branes at both S_a and S_b, and these are connected to each other by means of the three-chain Σ, while the D7s are connected via Γ. Thus, we have found the generalized chain $(\mathfrak{S}, \mathfrak{F})$ associated with the open string massless $U(1)$.

According to our previous discussion the kinetic mixing with the closed string $U(1)_I$ will be proportional to $j_{(\mathfrak{S},\mathfrak{F})}(\gamma_I)$. One still needs to show that the result is independent of the arbitrary choices made to define $(\mathfrak{S}, \mathfrak{F})$. We relegate the proof to Appendix G, in which we also derive the following more convenient expression

$$j_{(\mathfrak{S},\mathfrak{F})}(\gamma_I) = \int_\Sigma \gamma_I + \int_{S_a} \gamma_I \wedge A_{\Pi_a} - \int_{S_b} \gamma_I \wedge A_{\Pi_b}. \tag{4.90}$$

with $dA_{\Pi_a} = \mathcal{F}_a - \delta^2_{S_a}(\Pi_a)$ and $dA_{\Pi_b} = \mathcal{F}_b - \delta^2_{S_b}(\Pi_b)$. Again, one can show that this expression is independent of the choice of Π_a and Π_b. Notice that these equations do not fix the harmonic piece of A_{Π_a} and A_{Π_b}. In order to match the result obtained from dimensional reduction we take them to be the Wilson lines in each D7-brane. Introducing the normalization factor we finally find

$$f_{I(a-b)} = -\frac{i}{4\pi^2 l_s^3}\left[\int_\Sigma \gamma_I - \int_{S_a} \gamma_I \wedge A_{\Pi_a} + \int_{S_b} \gamma_I \wedge A_{\Pi_b}\right]. \tag{4.91}$$

In Appendix G we show that this expression reduces to (4.77) when $[\bar{F}_a] = [\bar{F}_b] \in H^2(S)$. Finally we can easily extend this analysis to the orientifold case, where we have

$$f_{I(a-b)} = -\frac{i}{8\pi^2 l_s^3}\left[\int_\Sigma \gamma_I - \int_{S_a} \gamma_I \wedge A_{\Pi_a} + \int_{S_a^*} \gamma_I \wedge A_{\Pi_a^*} + \int_{S_b} \gamma_I \wedge A_{\Pi_b} - \int_{S_b^*} \gamma_I \wedge A_{\Pi_b^*}\right] \tag{4.92}$$

with Σ a 3-chain such that $\partial\Sigma = \Pi_a - \Pi_b - \Pi_a^* + \Pi_b^*$.

5

Conclusions - Conclusiones

Conclusions

In this thesis we have analysed the generation and structure of Yukawa couplings in F-theory models of $SU(5)$ unification taking into account the presence of non-perturbative effects. In particular, we have shown how non-perturbative effects can induce non-trivial Yukawas for the two lighter families of quarks and leptons, and naturally produce a hierarchical mass structure among the three families.

The non-perturbative effects that had been considered so far, i.e. $\epsilon \int \theta_1 \Phi F^2$, vanish identically for the phenomenologically interesting cases of $SO(2N)$ and $E_{6,7,8}$ enhancement so we have included the more general superpotential (2.118) and discussed which are the leading corrections to the Yukawa couplings. This approach shows that there are two different scenarios, the one that includes $\epsilon \int \theta_0 F^2$ and the one in which the first non-trivial correction is given by $\int \theta_2 \Phi^2 F^2$. We have seen, by means of a residue calculation, that the former is more satisfactory since it naturally leads to a pattern of masses $(\mathcal{O}(\epsilon^2), \mathcal{O}(\epsilon), \mathcal{O}(1))$ unlike the latter in which the hierarchy looks like $(\mathcal{O}(\epsilon^2), \mathcal{O}(\epsilon^2), \mathcal{O}(1))$ [57]. Furthermore, by computing physical wavefunctions we have seen the importance of the hypercharge flux, that breaks the grand unified gauge group, in the masses of the elementary particles. Indeed, since the way the SM particles couple to this flux depends on their hypercharge, the Yukawa couplings do not follow the naive GUT relations and the unification scale. In particular, this allows to have different couplings for the down-type quarks and charged leptons already at M_{GUT}.

Given the local nature of our approach, the masses depend on several free parameters that can only be fixed in a global computation (intersection angles, flux densities and strength of the non-perturbative effects). By scanning in such parameter space for the $SO(12)$ model we are able to find a region that yields the correct value for Y_τ and Y_b as well as the ratios between the third and second generations. In general, this region is such that the strength of the intersection and fluxes are comparable. This computation is not able to address the first generation since this would require going to next order in perturbation theory in ϵ, which looks quite challenging.

Regarding the E_6 model, the combination of T-branes and non-perturbative effects allows to naturally generate a large top Yukawa coupling as well as the correct hierarchy between the different families. In particular, we are able to find parameters such that the top mass is in agreement with experiment and the correct ratio between the two heavier generations is achieved. It is interesting to notice that the fact that we use T-branes introduces new features in the model. In particular, for intersecting branes the Higgs and gauge bundles are completely unrelated, however, the T-branes couple these two and generate a non-primitive flux that affects the physical Yukawas.

Strictly speaking, in order to compare these results for the down and up-type particles one has to build a single model that includes both at the same time. However, our calculations suggest that the bottom and top couplings at the unification scale are similar which points at large $\tan\beta$ in an MSSM scheme. It would be interesting to build a model that allows to analyse these issues in more detail which would enable the study not only masses but also mixings. In principle, one can do this by considering an E_8 singularity at a point which is generically broken to $SU(5)$ by the geometry of the branes and that generates the correct structure of matter curves. Then, by including appropriate gauge fluxes one can generate chirality and break the gauge group. Finally, similar non-perturbative effects should generate non-vanishing Yukawas for the lighter generations. At the holomorphic level this seems to be more or less straightforward, however, the computation of physical wavefunctions looks challenging from the technical point of view since the presence of T-branes complicates the D-terms very much.

We have also studied the appearance of discrete flavour symmetries in Type II D-brane models. In particular, we have first analysed those that arise from the breaking of a continuous symmetry since one can work in a purely field theoretical context. Indeed, by looking at the BF couplings of the effective action it is possible to compute the surviving gauge symmetry as we review. We have seen this in detail for intersecting D6-branes and magnetised D9-branes in toroidal compactifications. This has allowed us to develop a geometrical understanding of these symmetries which does not require to know the effective action in detail. This is particularly useful in the case in which the discrete symmetries do not come from the breaking of a continuous symmetry in an obvious way. A phenomenologically interesting case is that of D-brane models in CY compactifications that enjoy discrete isometries.

For intersecting branes one finds that there is a symmetry whenever the embedding of the D6-branes preserve an underlying symmetry of the closed string sector which can be discrete or continuous. Since the D6-branes host the SM fields one can check that it may happen that such symmetries act non-trivially on them and, in particular, this action can be different for each generation. We study this explicitly in a class of models based on the $\mathbb{Z}_2 \times \mathbb{Z}_2'$ toroidal orientifold which has discrete (abelian) isometries that do not arise from the breaking of a continuous one in an obvious way. Then, when including the D6-branes, these may the preserve the said symmetry which acts on the flavour degrees of freedom. It is interesting to notice that the presence of the branes can make the original abelian symmetry into a non-abelian one as our example shows. Furthermore, this point of view allows to determine wether the symmetries are exact or approximate in a rather simple way.

We also study the same problem in the mirror dual setup of magnetised D9-branes. In this case the embedding of the branes respects the symmetries of the bulk trivially. However, the magnetisation may or may not respect them and we are able to understand when this can happen in a geometric way too. Since the geometry contains a non-trivial gauge flux one should demand that the gauge bundle respects this symmetry which can be checked by looking simply at the background covariant derivative. By analysing the open string wavefunctions in these toroidal orbifolds we classify the representations that may appear which becomes very useful when studying the Yukawa couplings in a given model. We provide two Pati-Salam models and work out in detail their symmetries and how these constrain the Yukawa couplings.

It would be interesting to explore other examples and perform a more systematic study to see what symmetries may appear. Also, since the measured Yukawa operators do not follow an exact symmetry, it seems reasonable to study in more detail the approximate symmetries and, in particular, the leading corrections in perturbation theory to the Kähler potential and the non-perturbative ones to the superpotential in supersymmetric models.

Finally, we have explored the kinetic mixing between the different $U(1)$s in Type II D-brane models. We have started by computing the open-closed mixing in intersecting D6-brane models by simple dimensional reduction. Then, we have seen how this mixing can be rephrased in terms of the Witten effect for the open string magnetic monopoles which yields a more general expression and shows the importance of the relative (co)homology groups in this kind of constructions. Also, using this approach one can write such formula in a way that resembles the mixing for the closed string sector which allows to propose a supergravity computation for the mixing in the open string sector that accounts for the one-loop correction.

Following the same argument we tackle the kinetic mixing in the context of magnetised D7-brane compactifications that is particularly interesting since it is closely related to the F-theory models. We have seen that the appropriate language for such situations is generalised complex geometry and we have extended the concept of linear equivalence of submanifolds to the case of magnetised D-branes that nicely captures the open-closed mixing. Again, by virtue of the Witten effect we have computed the mixing of the hypercharge $U(1)$ in F-theory $SU(5)$ GUTs with the closed string $U(1)$s. We have proposed a formula for the open string mixing in 'closed string variables' in this setup too.

This seems to suggest that, roughly speaking, one can include the open string degrees of freedom by performing a dimensional reduction of supergravity, which in principle only accounts for the closed string ones, if we consider the appropriate relative cohomology groups instead of the usual cohomology. This is perhaps not so surprising since in the M/F-theory uplift of Type IIA/B compactifications with D6/D7-branes respectively these are purely geometrical configurations. It would be interesting to examine in more detail the role of relative (co)homology in Type II compatifications with D-branes and see if this is just a nice mathematical way of arranging the relevant data or if it is really useful in practice. In particular, it would be very interesting to check wether the proposed formulas for the open-open mixing work in simple examples.

Conclusiones

En esta tesis hemos analizado la generación y estructura de los operadores de Yukawa en modelos de unificación $SU(5)$ en Teoría F teniendo en cuenta la presencia de efectos no perturbativos. En particular, hemos mostrado como efectos no perturbativos pueden inducir Yukawas para las dos generaciones más ligeras de quarks y leptones y producen una estructura de masas jerárquica entre las tres familias.

Los efectos no perturbativos que se habían considerado hasta ahora, i.e. $\epsilon \int \theta_1 \Phi F^2$, se anulan idénticamente para los cases fenomenológicamente interesantes de $SO(2N)$ y $E_{6,7,8}$ por lo que hemos incluido el superpotencial más general (2.118) y discutido cuáles son las correcciones dominantes a los acoplos de Yukawa. Este método muestra que hay dos escenarios diferentes: el que incluye $\epsilon \int \theta_0 F^2$ y en el que la primera corrección no nula viene dada por $\int \theta_2 \Phi^2 F^2$. Hemos visto que el primero es más satisfactorio ya que de manera natural conduce a un patrón de masas $(\mathcal{O}(\epsilon^2), \mathcal{O}(\epsilon), \mathcal{O}(1))$ a diferencia del segundo en que la jerarquía es de tipo $(\mathcal{O}(\epsilon^2), \mathcal{O}(\epsilon^2), \mathcal{O}(1))$ [57]. Además, al calcular las funciones de onda físicas hemos apreciado la importancia del flujo e hipercarga, que rompe el grupo de gran unificación, en las masas de las partículas elementales. En efecto, como la manera en que las partículas del Modelo Estándar se acoplan a dicho flujo depende de su hipercarga, los operadores de Yukawa no siguen relaciones simples a la escala de unificación. En particular, esto permite tener acoplos diferentes para los quarks tipo down y los leptones cargados incluso a la escala M_{GUT}.

Dada la naturaleza local de nuestro modelo, las masas dependen de varios parámetros libres que solo se pueden fijar en un cálculo global (ángulos de intersección, densidades de flujos y magnitud de los efectos no perturbativos). Explorando dicho espacio de parámetros para el modelo $SO(12)$ encontramos una región que conduce a un valor correcto para Y_τ y Y_b así como para los cocientes entre la tercera y segunda generaciones. En general, dicha región es tal que la magnitud de las intersecciones y los flujos son comparables. Este cálculo no es capaz de describir la primera generación ya que implicaría ir a segundo orden en ϵ lo que parece bastante difícil.

En cuanto al modelo E_6, la combinación de T-branas y efectos no perturbativos permite generar un Yukawa para el top grande así como una jerarquía correcta entre las diferentes familias. En particular, somos capaces de encontrar parámetros tales que tanto la masa del top como el cociente con la segunda generación están de acuerdo con el experimento. Es intresante notar que el hecho de usar T-branas introduce nuevas características en el modelo. En particular, para branas intersecantes los fibrados de Higgs y gauge son completamente independientes, sin embargo, las T-branas los acoplan lo que genera un flujo no primitivo que afecta a los Yukawa físicos.

Estrictamente hablando, para comparar estos resultados para las partículas tipo down y up, uno tendría que construir un modelo único que incluya ambas a la vez. Sin embargo, nuestros cálculos muestran que los acoplos para el bottom y top son similares a la escala de unificación de que apunta a un $\tan\beta$ grande en un esquema tipo MSSM. Sería interesante construir un modelo que permita analizar estas cuestiones en detalle lo que abriría la puerta a estudiar no solo masas sino también mezclas. En principio, esto se puede hacer considerando un modelo E_8 que se rompe a $SU(5)$ genéricamente debido a la geometría de las branas. Además, añadiendo flujos se puede generar quiralidad así como romper el grupo gauge. Finalmente, al incluir efectos no-perturbativos se deberían generar Yukawas no nulos para el resto de familias. A nivel holomorfo esto parece más o menos sencillo, sin embargo, el cálculo de las funciones de onda físicas no parece fácil a nivel técnico ya que la presencia de T-branas complica los términos D en exceso.

También se ha estudiado la aparición de simetrías discretas de sabor en modelos de D-branas en la Tipo II. En particular, hemos analizado aquellas que provienen de la ruptura de simetrías continuas ya que se puede trabajar en teoría de campos. En efecto, mirando los acoplos BF en la acción efectiva se pueden calcular las simetrías que sobreviven como se revisa. Hemos visto esto en detalle para D6-branas intersecantes y D9-branas magnetizadas en compactificaciones toroidales. Esto nos ha permitido desarrollar una intuición geométrica para estas simetrías que no requiere el conocimiento de la acción efectiva en detalle. Esto es particularmente útil en los casos en los que las simetrías discretas no vienen de la ruptura de simetrías continuas de manera obvia. Un caso interesante fenomenológicamente consiste en modelos de D-branas en compactificaciones en CY con isometrías discretas.

Para branas intersecantes se encuentra que existe una simetría siempre que la inclusión de las D6-branas preserve la simetría subyacente del sector de cuerda cerrada, que puede ser tanto continua como discreta. Como las D6-branas contienen los campos del Modelo Estándar se puede ver que es posible que dichas simetrías actúen de manera no trivial sobre ellos y, en particular, sicha acción puede ser diferente para cada generación. Estudiamos esto explícitamente en una clase de modelos basados en el orientifold toroidal $\mathbb{Z}_2 \times \mathbb{Z}_2'$ que contiene isometrías discretas (abelianas) que no provienen de simetrías continuas de manera obvia. Al incluir las branas, éstas pueden preservar dicha simetría que actúa sobre los grados de libertad de sabor. Es interesante notar que la presencia de las branas puede convertir la simetría abeliana original en no-abeliana, como muestra nuestro ejemplo. Además, desde este punto de vista se puede determinar de manera sencilla si las simetrías son exactas o aproximadas.

También estudiamos el mismo problema en el contexto dual de D9-branas magnetizadas. En este caso la inclusión de las branas respeta las simetrías de cuerda cerrada trivialmente. Sin embargo, es posible que la magnetización no lo haga lo cual también se puede entender de manera geométrica. Debido a que la geometría contiene un flujo, se tiene que exigir que el fibrado gauge también respete la simetría lo cual se puede comprobar simplemente a través de la derivada covariante. Analizando las funciones de onda de cuerda abierta para los orbifolds toroidales clasificamos las representaciones que pueden aparecer lo cual se muy útil a la hora de estudiar los acoplos de Yukawa en modelos concretos. Construimos dos modelos Pati-Salam y analizamos en detalle sus simetrías y cómo éstas afectan a los acoplos de Yukawa.

Sería interesante explorar diferentes ejemplos y realizar un estudio sistemático de qué simetrías pueden aparecer. Además, debido a que los operadores de Yukawa medidos no siguen una simetría exacta, parece razonable estudiar en más detalle las simetrías aproximadas y, en particular, las correciones dominantes al potencial Kähler en teoría de perturbaciones así como las correcciones no perturbativas al superpotencial en modelos supersimétricos.

Finalmente, hemos explorado la mezcla cinética entre los diferentes $U(1)$s en modelos de D-branas de la tipo II. Hemos comenzado calcuando la mezcla abierta-cerrada en modelos de branas intersecantes usando simplemente reducción dimensional. Además, hemos visto como dicha mezcla puede reformularse en términos del efecto Witten para los monopolos magnéticos de cuerda abierta lo que conduce a una expresión más general y muestra la importancia de los grupos de cohomología relativa en este tipo de construcciones. Utilizando este método hemos escrito la fórmula para la mezcla de manera que se parece al caso de cuerda cerrada lo que nos ha permitido proponer una expresión en supergravedad para la mezcla en el sector de cuerda abierta que tiene en cuenta la correción a un loop.

Siguiendo el mismo razonamiento hemos estudiado la mezcla cinética en el contexto de D7-branas magnetizadas que es particularmente interesante ya que está estrechamente relacionado con los modelos de Teoría F. Hemos visto que el lenguaje apropiado para estas situaciones es la geometría compleja generalizada y hemos extendido el concepto de equivalencia lineal de subvariedades a D-branas con flujos que captura de manera elegante la mezcla abierta-cerrada. De nuevo, a través del efecto Witten hemos calculado la mezcla del $U(1)$ de hipercarga en modelos de unificación $SU(5)$ en Teoría F con los $U(1)$s de cuerda cerrada. También se ha propuesto en este caso una formula para la mezcla abierta en variables de cuerda cerrada.

Esto parece sugerir que, a grandes rasgos, uno puede incluir los grados de libertad de cuerda abierta haciendo una reducción dimensional de supergravedad, que en principio solo tiene en cuenta los de cuerda cerrada, si se consideran los grupos de cohomología relativa adecuados en vez de la cohomología usual. Es posible que esto no resulte sorprendente ya que la descripcion en Teoría M/F de compactificaciones de la Tipo IIA/B con D6/7-branas respectivamente viene dada por una configuración puramente geométrica. Sería interesante examinar en más detalle el papel de la (co)homología relativa en compactificaciones de la TIpo II con D-branas y ver si es solo una manera elegante de organizar la información relevante o si es útil en la práctica. En particular, sería muy interesante comprobar si la fórmula propuesta para la mezcla abierta-abierta funciona en ejemplos sencillos.

A

Physical wavefunctions and normalisation factors

In this appendix we write down explicitly the perturbative wavefunctions and normalisation factors for the $SO(12)$ and E_6 models of chapter 2.

$SO(12)$ model

The physical wavefunctions for the $SO(12)$ model were explicitly computed in [57] and we simply quote the results here since they can be computed using the general methods explained in section 2.3.4 and are very similar to the $U(3)$ toy model.

Sector a

For the a sector we have

$$\vec{\Psi}_{a_p^+} = \begin{pmatrix} -i\frac{\lambda_{a_p}}{m^2} \\ \zeta_{a_p}\frac{i\lambda_{a_p}}{m^2} \\ 1 \end{pmatrix} e^{\lambda_{a_p}x(\bar{x}-\zeta_{a_p}\bar{y})} e^{\frac{M+q_Y\tilde{N}_Y}{2}\left(|x|^2-|y|^2\right)-q_S\mathrm{Re}(x\bar{y})} f_{a_p}(y+\zeta_{a_p}x)E_{a_p^+} \tag{A.1}$$

with $\zeta_{a_p} = -\frac{q_S}{\lambda_{a_p}-q_P}$ and λ_{a_p} is the lowest (negative) solution to the cubic

$$\lambda_{a_p}^3 - \left(m^4+q_P^2+q_S^2\right)\lambda_{a_p} + m^4 q_P = 0. \tag{A.2}$$

The normalisation factor reads

$$|\gamma_{a_p}^i|^2 = -\frac{1}{2(3-i)!\pi^2}\left(\frac{m}{m_*}\right)^4 \frac{q_P(2\lambda_{a_p}+q_P(1+\zeta_{a_p}^2))}{m^4+\lambda_{a_p}^2(1+\zeta_{a_p}^2)}\left(\frac{q_P}{m_*^2}\right)^{3-i} \tag{A.3}$$

where the family functions are monomials, i.e. $f_{a_p,j}(y+\zeta_{a_p}x) = m_*^{3-j}(y+\zeta_{a_p}x)^{3-j}$ for $j=0,1,2$.

Sector b

The physical wavefunctions for the b sector are

$$\vec{\Psi}_{b_p^+} = \begin{pmatrix} -\zeta_{b_p}\frac{i\lambda_{b_p}}{m^2} \\ \frac{i\lambda_{b_p}}{m^2} \\ 1 \end{pmatrix} e^{\lambda_{b_p}y(\bar{y}-\zeta_{b_p}\bar{x})} e^{\frac{M-q_Y\tilde{N}_Y}{2}\left(|y|^2-|x|^2\right)-q_S\mathrm{Re}(x\bar{y})} f_{b_p}(x+\zeta_{b_p}y)E_{b_p^+} \tag{A.4}$$

with $\zeta_{b_p} = -\frac{q_S}{\lambda_{b_p}+q_P}$ and λ_{b_p} is the lowest (negative) solution to the cubic

$$\lambda_{b_p}^3 - \left(m^4 + q_P^2 + q_S^2\right)\lambda_{b_p} - m^4 q_P = 0. \tag{A.5}$$

The normalisation factor is

$$|\gamma_{b_p}^i|^2 = -\frac{1}{2(3-i)!\pi^2}\left(\frac{m}{m_*}\right)^4 \frac{q_P(-2\lambda_{b_p} + q_P(1+\zeta_{b_p}^2))}{m^4 + \lambda_{b_p}^2(1+\zeta_{b_p}^2)}\left(-\frac{q_P}{m_*^2}\right)^{3-i}. \tag{A.6}$$

Sector c

For the c sector the wavefunctions are

$$\vec{\Psi}_{c_p^+} = \begin{pmatrix} i\frac{\lambda_{c_p}}{m^2} \\ \frac{i(\zeta_{c_p}-\lambda_{c_p})}{m^2} \\ 1 \end{pmatrix} e^{\left(\frac{1}{2}q_Y\tilde{N}_Y + \zeta_{c_p}\right)|x|^2 + \left(\lambda_{c_p} - \zeta_{c_p} - \frac{1}{2}q_Y\tilde{N}_Y\right) - \left(\frac{1}{2}q_S - \zeta_{c_p} + \lambda_{c_p}\right)x\bar{y} - \left(\frac{1}{2}q_S + \zeta_{c_p}\right)\bar{x}y} E_{c_p^+} \tag{A.7}$$

where we took the holomorphic family function to be 1 since we only want a single generation in the Higgs sector. Here $\zeta_{c_p} = \frac{\lambda_{c_p}(\lambda_{c_p}-q_P-q_S)}{2(\lambda_{c_p}-q_S)}$ and λ_{c_p} is the lowest (negative) solution to the cubic

$$\lambda_{c_p}^3 - \left(2m^4 + q_P^2 + q_S^2\right)\lambda_{c_p} + 2m^4 q_S = 0. \tag{A.8}$$

The normalisation factor reads

$$|\gamma_{c_p}|^2 = -\frac{1}{2\pi^2}\left(\frac{m}{m_*}\right)^4 \frac{(2\zeta_{c_p} + q_P)(q_P + 2\zeta_{c_p} - 2\lambda_{c_p}) + (q_S + \lambda_{c_p})^2}{m^4 + \zeta_{c_p}^2 + (\zeta_{c_p} - \lambda_{c_p})^2} \tag{A.9}$$

X,Y bosons

Unlike the previous cases, this sector does not correspond to any matter curve since they are not charged under the Higgs field $\langle\Phi\rangle$. This means that they are not localised at any curve within S_{GUT} and are usually referred to as bulk modes. However, the computation of the wavefunctions is analogous to the one for localised matter. The wavefunctions are

$$\Psi_{X,Y^\pm} = e^{-\frac{5}{12}\lambda\left(|x|^2+|y|^2\right)} f_{X,Y^\pm} \tag{A.10}$$

with $f_{X,Y^+} = f_{X,Y^+}(\bar{u}, v)$ and $f_{X,Y^-} = f_{X,Y^-}(u, \bar{v})$ where we defined the rotated coordinates

$$u = cx + sy, \quad v = -sx + cy; \qquad c = \frac{N_Y}{\sqrt{N_Y^2 + (\tilde{N}_Y - \lambda)^2}}, \quad s = \frac{\tilde{N}_Y - \lambda}{\sqrt{N_Y^2 + (\tilde{N}_Y - \lambda)^2}} \tag{A.11}$$

and $\lambda = \sqrt{N_Y^2 + \tilde{N}_Y^2}$. As expected, the holomorphic functions $f_{X,Y^\pm}$ depend on two coordinates instead of one since these are bulk modes.

These wavefunctions for massive gauge bosons have been used in [36] to compute dimension 6 proton decay operators in models with intermediate scale supersymmetry.

E_6 model

The physical wavefunctions for the E_6 model can also be computed using the methods of section 2.3.4. We gather them here for convenience.

Sector $\mathbf{10}_M$

This sector is charged under the T-brane so it transforms in a non-abelian representation under the broken part of the gauge group. A similar case was solved in detail in the examples of section 2.3.4. In order to have an analytic solution we take the limit $\mu \ll m$ which decouples the wavefunctions in the doublet of $SU(2)$ where the T-brane lives. These are

$$\begin{aligned}
\vec{\Psi}^{j}_{\mathbf{10}^+} &= \gamma^j_{\mathbf{10}} \begin{pmatrix} \frac{i\lambda_{\mathbf{10}}}{m^2} \\ -\frac{i\lambda_{10}\zeta_{\mathbf{10}}}{m^2} \\ 0 \end{pmatrix} e^{\frac{f}{2}} e^{\frac{q_R}{2}(|x|^2-|y|^2)-q_S(x\bar{y}+y\bar{x})+\lambda_{\mathbf{10}}x(\bar{x}-\zeta_{\mathbf{10}}\bar{y})} g_j(y+\zeta_{\mathbf{10}}x) \\
\vec{\Psi}^{j}_{\mathbf{10}^-} &= \gamma^j_{\mathbf{10}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-\frac{f}{2}} e^{\frac{q_R}{2}(|x|^2-|y|^2)-q_S(x\bar{y}+y\bar{x})+\lambda_{\mathbf{10}}x(\bar{x}-\zeta_{\mathbf{10}}\bar{y})} g_j(y+\zeta_{\mathbf{10}}x)
\end{aligned} \tag{A.12}$$

where we defined $\zeta_{\mathbf{10}} = -\frac{q_S}{\lambda_{10}-q_R}$ and $\lambda_{\mathbf{10}}$ is the lowest solution to

$$m^4(\lambda_{\mathbf{10}} - q_R) + \lambda_{\mathbf{10}}c^2\left(c^2m^2(q_R-\lambda_{\mathbf{10}}) - \lambda_{\mathbf{10}}^2 + q_R^2 + q_S^2\right) = 0. \tag{A.13}$$

Also, to arrive at this solution we take the Painlevé trascendent to be $f = \log c + c^2c^2|x|^2$ which is a good approximation near the origin. See [58] for more details on when this is indeed a valid approximation.

The normalisation factor is given by

$$|\gamma^j_{\mathbf{10}}|^2 = -\frac{c}{m_*^2\pi^2(3-j)!}\frac{1}{\frac{1}{2\lambda_{\mathbf{10}}+q_R(1+\zeta_{\mathbf{10}}^2)-m^2c^2} + \frac{c^2\lambda_{\mathbf{10}}^2}{m^4}\frac{1}{2\lambda_{\mathbf{10}}+q_R(1+\zeta_{\mathbf{10}}^2)+m^2c^2}}\left(\frac{q_R}{m_*^2}\right)^{4-j}. \tag{A.14}$$

Sector $\mathbf{5}_U$

This sector is not charged under the T-brane so the wavefunctions are of the same kind as those in the previous model. One finds

$$\vec{\Psi}_{\mathbf{5}} = \gamma_{\mathbf{5}} \begin{pmatrix} i\frac{\zeta_{\mathbf{5}}}{2\mu^2} \\ i\frac{(\zeta_{\mathbf{5}}-\lambda_{\mathbf{5}})}{2\mu^2} \\ 1 \end{pmatrix} e^{\frac{q_R}{2}(|x|^2-|y|^2)-q_S(x\bar{y}+y\bar{x})+(x-y)(\zeta_{\mathbf{5}}\bar{x}-(\lambda_{\mathbf{5}}-\zeta_{\mathbf{5}})\bar{y})} \tag{A.15}$$

with $\zeta_{\mathbf{5}} = \frac{\lambda_{\mathbf{5}}(\lambda_{\mathbf{5}}-q_R-q_S)}{2(\lambda_{\mathbf{5}}-q_S)}$ and $\lambda_{\mathbf{5}}$ the lowest solution to

$$\lambda_{\mathbf{5}}^3 - (8\mu^4 + q_R^2 + q_S^2)\lambda_{\mathbf{5}} + 8\mu^4 q_S = 0. \tag{A.16}$$

Finally, the normalisation factor is

$$|\gamma_{\mathbf{5}}|^2 = -\frac{4}{\pi^2}\left(\frac{\mu}{m_*}\right)^4 \frac{(2\zeta_{\mathbf{5}}+q_R)(q_R+2\zeta_{\mathbf{5}}-2\lambda_{\mathbf{5}}) + (q_S+\lambda_{\mathbf{5}})^2}{4\mu^4+\zeta_{\mathbf{5}}^2+(\zeta_{\mathbf{5}}-\lambda_{\mathbf{5}})^2}. \tag{A.17}$$

B
The $\mathbb{Z}_2 \times \mathbb{Z}_2'$ orbifold

Let us consider type IIA string theory compactified on the toroidal orbifold background $\mathbf{T}^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, the $\mathbb{Z}_2$ generators acting as

$$\Theta : \begin{cases} z_1 \to -z_1 \\ z_2 \to -z_2 \\ z_3 \to z_3 \end{cases} \qquad \Theta' : \begin{cases} z_1 \to z_1 \\ z_2 \to -z_2 \\ z_3 \to -z_3 \end{cases} \tag{B.1}$$

on the three complex coordinates of $\mathbf{T}^6 = (\mathbf{T}^2)_1 \times (\mathbf{T}^2)_2 \times (\mathbf{T}^2)_3$. Besides such action one must specify the choice of discrete torsion that relates these two $\mathbb{Z}_2$ group generators. As explained in [128] there are two inequivalent choices, whose twisted homologies are either $(h_{11}^{\text{tw.}}, h_{21}^{\text{tw.}}) = (48, 0)$ or $(h_{11}^{\text{tw.}}, h_{21}^{\text{tw.}}) = (0, 48)$. Similarly to [98] we will consider the second case, and dub it $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold with discrete torsion or $\mathbb{Z}_2 \times \mathbb{Z}_2'$. Such background then contains 96 collapsed three-cycles at the fixed loci of (B.1).

Let us now add space-time filling D6-branes wrapping supersymmetric three-cycles of this toroidal orbifold. In terms of a factorized $\mathbf{T}^6$ geometry these can be described as the product of three one-cycles

$$[\Pi_a] = \bigotimes_{i=1}^{3} \left(n_a^i\ [a^i] + m_a^i\ [b^i]\right) \qquad n_a^i, m_a^i \in \mathbb{Z} \text{ and coprime} \tag{B.2}$$

where $[a^i]$, $[b^i]$ are the homology classes of the fundamental one-cycles of $(\mathbf{T}^2)_i$. Notice that the $\mathbf{T}^6$ homology class $[\Pi_a]$ is invariant under the orbifold action (B.1), and so one can consider three-cycle representatives Π_a also invariant under (B.1). A D6-brane wrapping an invariant three-cycle will suffer the orbifold projection on its Chan-Paton degrees of freedom, resulting into fractional D6-branes with non-vanishing charge under the RR twisted sector of the theory. Geometrically, on each $(\mathbf{T}^2)_i$ a fractional D6-branes goes through two fixed points of the action $z_i \to -z_i$, and it wraps collapsed three-cycles that correspond to such fixed points (see fig. B.1). Precisely because of this, fractional D6-branes are 'rigid', they cannot be taken away from a fixed locus of the action (B.1), and so they do not contain the deformation moduli typical of D-branes in toroidal compactifications.

Following [98] the homology class of a fractional D6-brane is of the form

$$[\Pi_a^F] = \frac{1}{4}[\Pi_a^B] + \frac{1}{4}\left(\sum_{I,J \in S_\Theta^a} \epsilon_{a,IJ}^{\Theta}\,[\Pi_{IJ,a}^{\Theta}]\right) + \frac{1}{4}\left(\sum_{J,K \in S_{\Theta'}^a} \epsilon_{a,JK}^{\Theta'}\,[\Pi_{JK,a}^{\Theta'}]\right) + \frac{1}{4}\left(\sum_{I,K \in S_{\Theta\Theta'}^a} \epsilon_{a,IK}^{\Theta\Theta'}\,[\Pi_{IK,a}^{\Theta\Theta'}]\right) \tag{B.3}$$

where $[\Pi_a^B]$ stands for a *bulk* three-cycle, that is a $\mathbf{T}^6$ three-cycle of the form (B.2) that is inherited in the orbifold quotient. Bulk three-cycles correspond to the untwisted RR charges of the orbifold, and the intersection number between them is given by

$$I_{ab}^B = [\Pi_a^B] \cdot [\Pi_b^B] = 4 \prod_{i=1}^{3} (n_a^i\, m_b^i - m_a^i\, n_b^i), \tag{B.4}$$

where the factor of 4 arises from taking into account the $\mathbb{Z}_2 \times \mathbb{Z}_2'$ orbifold action. Beside the bulk cycles there are 32 collapsed three-cycles for each of the three twisted sectors. Their homology class is of the form

$$[\Pi_{IJ,\,a}^g] = 2[e_{IJ}^g] \otimes \left(n_a^{i_g} [a^{i_g}] + m_a^{i_g} [b^{i_g}] \right) \tag{B.5}$$

where $g = \Theta, \Theta', \Theta\Theta'$ runs over all twisted sectors, and $i_g = 3, 1, 2$, respectively. Here e_{IJ}^g $I, J \in \{1, 2, 3, 4\}$ stand for the 16 fixed points on $(\mathbf{T}^2)_i \times (\mathbf{T}^2)_j / \mathbb{Z}_2$, where $\mathbb{Z}_2 = \{1, g\}$ and $(\mathbf{T}^2)_{i,j}$ are the two-tori such that $g : (z_i, z_j) \mapsto (-z_i, -z_j)$ (see fig. B.1). These fixed points correspond to the $\mathbb{Z}_2$ singularities of a **K3** surface in its orbifold limit $\mathbf{T}^4/\mathbb{Z}_2^2$, and each can be blown up to a $\mathbf{P}^1$ whose homology class is given by $[e_{IJ}^g]$. Finally, a^{i_g}, b^{i_g} stand for the fundamental one-cycles of $(\mathbf{T}^2)_{i_g}$, the two-torus which is left invariant under the action of g. Gathering all these facts together, one can compute the intersection number of two collapsed three-cycles as

$$[\Pi_{IJ,\,a}^g] \cdot [\Pi_{KL,\,b}^h] \;=\; 4\, \delta_{IK} \delta_{JL} \delta^{gh} \,(n_a^{i_g}\, m_b^{i_g} - m_a^{i_g}\, n_b^{i_g}) \tag{B.6}$$

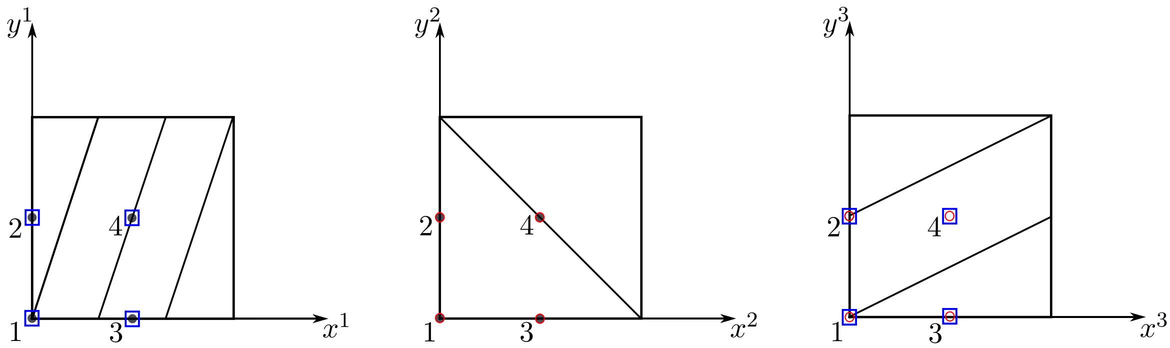

Figure B.1: Fractional brane passing through 4 fixed points for each twisted sector. Fixed points are denoted by dots in the Θ sector, by circles in the Θ' sector and squares in the $\Theta\Theta'$ sector.

The homology class (B.3) is given by a particular linear combination of bulk and collapsed three-cycles, which is determined as follows. In the covering space $(\mathbf{T}^2)^3$ a BPS D6-brane looks like as a product of three 1-cycles with wrapping numbers (n_a^i, m_a^i) and constant slope, see figure B.1. Fractional D6-branes must be invariant under (B.1), and so on each two-torus they must pass through two fixed points of $(\mathbf{T}^2)_i/\mathbb{Z}_2$ with $\mathbb{Z}_2 = \{1, z_i \mapsto -z_i\}$. Which are these fixed points depends on the wrapping numbers (n_a^i, m_a^i), see table 3.1 in the main text.

Let us now consider a particular twisted sector, say $g = \Theta$. The collapsed three-cycles of this sector are related to the fixed points e_{IJ}^Θ of $(\mathbf{T}^2)_1 \times (\mathbf{T}^2)_2 / \{1, \Theta\}$. A fractional D6-brane will pass through 4 fixed points e_{IJ}^Θ. More precisely, the index I will take two different values specified by (n_a^1, m_a^1) and one of the choices in table 3.1, while J will be constrained by (n_a^2, m_a^2). This subset of 2×2 elements $\{(I,J)\} \subset \{1,2,3,4\} \times \{1,2,3,4\}$ is denoted as S_Θ^a in (B.3), and similar definitions apply to $S_{\Theta'}^a$ and $S_{\Theta\Theta'}^a$. It is easy to see that given the bulk wrapping numbers (n_a^i, m_a^i), $i = 1,2,3$ there are eight different choices for specifying S_Θ^a, $S_{\Theta'}^a$ and $S_{\Theta\Theta'}^a$. From the viewpoint of the covering space $(\mathbf{T}^2)^3$, these choices correspond to the 2^3 different locations that an invariant three-cycle can have.

Besides S_g^a one needs to specify the signs $\epsilon_{a,IJ}^\Theta$, $\epsilon_{a,JK}^{\Theta'}$, $\epsilon_{a,IK}^{\Theta\Theta'} = \pm 1$ that appear in (B.3). These signs are not arbitrary but must fulfill several consistency conditions discussed in [98]. One finds that there are essentially 8 inequivalent choices for these signs. In general the set of fixed points is given by

$$\begin{aligned} S_\Theta &= \{\{I_1, I_2\} \times \{J_1, J_2\}\} \\ S_{\Theta'} &= \{\{J_1 J_2\} \times \{K_1 K_2\}\} \\ S_{\Theta\Theta'} &= \{\{K_1 K_2\} \times \{I_1 I_2\}\} \end{aligned} \tag{B.7}$$

where $i_\alpha, j_\alpha, k_\alpha$, $\alpha = 1,2$ represent fixed point coordinates in the first, second and third $\mathbf{T}^2$ factors, respectively. If we fix $\epsilon_{I_1 J_1}^\Theta = \epsilon_{J_1 K_1}^{\Theta'} = \epsilon_{K_1 I_1}^{\Theta\Theta'} = +1$ then all the other ϵ's depend on only three independent signs. More precisely we have that $\epsilon_{I_2 J_1}^\Theta = \epsilon_{K_1 I_2}^{\Theta\Theta'} = \epsilon_I$, $\epsilon_{I_1 J_2}^\Theta = \epsilon_{J_2 K_1}^{\Theta'} = \epsilon_J$, $\epsilon_{J_1 K_2}^{\Theta'} = \epsilon_{K_2 I_1}^{\Theta\Theta'} = \epsilon_K$ and $\epsilon_{I_2 J_2}^\Theta = \epsilon_I \epsilon_J$, $\epsilon_{J_2 K_2}^{\Theta'} = \epsilon_J \epsilon_K$, $\epsilon_{K_2 I_2}^{\Theta\Theta'} = \epsilon_K \epsilon_I$, with $\epsilon_I, \epsilon_J, \epsilon_K = \pm 1$. The choice of these three signs can be interpreted as the choice of discrete Wilson lines for a fractional D6-brane along each one-cycle.

Having fixed S_g^a and $\epsilon_{a,IJ}^g$ as above, there are four inequivalent choices of wrapping numbers (n_a^I, m_a^I) which correspond to the same bulk three-cycle Π_a^B but to different fractional three-cycle Π_a^F. These are given by

$$\begin{array}{ccc} (n_a^1, m_a^1) & (n_a^2, m_a^2) & (n_a^3, m_a^3) \\ (-n_a^1, -m_a^1) & (-n_a^2, -m_a^2) & (n_a^3, m_a^3) \\ (n_a^1, m_a^1) & (-n_a^2, -m_a^2) & (-n_a^3, -m_a^3) \\ (-n_a^1, -m_a^1) & (n_a^2, m_a^2) & (-n_a^3, -m_a^3) \end{array} \tag{B.8}$$

and can be interpreted as the four different $\mathbb{Z}_2 \times \mathbb{Z}_2'$ twisted charges that a fractional D6-brane can have. Indeed, it is easy to see that one obtains a pure bulk D6-brane by adding these four fractional D6-branes. One can further support this claim by computing the chiral spectrum between two fractional D6-branes, as we now proceed to show.

Chiral index

Given two stacks of fractional D6-branes wrapped on Π_a^F and Π_b^F one can easily compute the chiral spectrum of open strings with one endpoint on each of them. Indeed, let us consider N_a D6-branes wrapped on Π_a^F and N_b D6-branes on Π_b^F. Then the chiral spectrum will be given by I_{ab} left-handed chiral multiplets in the bifundamental $(N_a, \bar{N}_b)$ representation of $SU(N_a) \times SU(N_b)$. Here $I_{ab} = [\Pi_a^F] \cdot [\Pi_b^F]$ is the topological intersection number of the two three-cycles, and can be computed from (B.4), (B.6) and the fact that an intersection number between a bulk and a collapsed three-cycle vanishes.

For instance, let us consider the case where Π_a^F, Π_b^F are such that they have trivial discrete Wilson lines ($\epsilon_{a,b}^g = 1$ in (B.3)) and they both intersect the origin of $(\mathbf{T}^2)^3$ ($I_1 = J_1 = K_1 = 1$ in (B.7) for $S_g^{a,b}$). Then the intersection number I_{ab} is specified by the bulk wrapping numbers

$$\begin{array}{cccc} \Pi_a^F & : & (n_a^1, m_a^1) \quad (n_a^2, m_a^2) \quad (n_a^3, m_a^3) \\ \Pi_b^F & : & (n_b^1, m_b^1) \quad (n_b^2, m_b^2) \quad (n_b^3, m_b^3) \end{array} \tag{B.9}$$

More precisely we find that

$$I_{ab} = [\Pi_a^F] \cdot [\Pi_b^F] = \frac{1}{4} \left[I_{ab}^1 I_{ab}^2 I_{ab}^3 + I_{ab}^1 \,\rho_2 \rho_3 + I_{ab}^2 \,\rho_1 \rho_3 + I_{ab}^3 \,\rho_1 \rho_2 \right] \qquad \text{(B.10)}$$

where $I_{ab}^i = n_a^i m_b^i - n_b^i m_a^i$ and ρ_i is defined as in (3.41). Despite the factor of $1/4$ one can check that such intersection number is always an integer, as required by consistency. Notice that the bulk intersection number I_{ab}^B remains unchanged if we replace Π_a^F by any of the other bulk wrapping numbers in (B.8). The intersection numbers I_{ab}^i for each individual two-torus do however depend on this choice, and so does the total intersection number I_{ab}. As show in the main text this formula is reproduced by considering those linear combinations of intersection points invariant under the orbifold action. As one can also see from that discussion, the four different type of projection that depend on the signs s_1, s_2, s_3 can be obtained by considering the pair of D6-branes (B.9) and then replacing Π_a^F by any of the other bulk wrapping numbers in (B.8). This shows in more detail that each of these D6-branes has a different Chan-Paton factor, because for the same bulk embedding the open strings ending in Π_a feel a different orbifold action, adding up to the regular representation of $\mathbb{Z}_2 \times \mathbb{Z}_2'$.

C
Flavour symmetries from dimensional reduction

Let us consider the dimensional reduction that yields the Stückelberg lagrangians (3.46-3.51) in 4d for D6-branes at angles in type IIA on $\mathbf{T}^2\times\mathbf{T}^2\times\mathbf{T}^2$ which shows the appearance of discrete symmetries. In [91] this was done for a toroidal compactification of Type I with magnetic fluxes. In our case we should consider the Type IIA supergravity together with the DBI action for the branes at angles to get the full non-Abelian structure. We will, however, take a simpler approach and consider only the DBI part to derive the abelian part of the symmetry.

Consider a D6-brane wrapping a factorisable three-cycle $\Pi_a = (n_a^1, m_a^1)\otimes(n_a^2, m_a^2)\otimes(n_a^3, m_a^3)$. The DBI action for such a D6-brane is

$$S_6 = -\mu_6 \int_{M_4} d^4x \int_{\Pi_a} d^3q\, e^{-\Phi}\sqrt{-\det(P[G]+P[B]-kF)} \tag{C.1}$$

with $k = 2\pi\alpha'$ and $P[\cdot]$ is the pullback on the worldvolume of the brane which looks like

$$P[A]_{\alpha\beta} = A_{\alpha\beta} + A_{ij}\partial_\alpha\phi^i\partial_\beta\phi^j + \partial_\alpha\phi^i A_{i\beta} + \partial_\beta\phi^i A_{\alpha i} \tag{C.2}$$

where α, β are indices on the brane and i, j are transverse. ϕ^i are the embedding functions of the brane in the bulk. Using the Taylor expansion of the determinant

$$\det(1+M) = 1 + \mathrm{Tr}\, M + \frac{1}{2}[\mathrm{Tr}\, M]^2 - \frac{1}{2}\mathrm{Tr}\, M^2 + \dots \tag{C.3}$$

we can expand the action (C.1) in derivatives. Namely,

$$\begin{aligned} S_6 \;=\; -\mu_6 \int_{M_4} d^4x \int_{\Pi_a} d^3q\, e^{-\Phi_0}\sqrt{-\det G_{\alpha\beta}}\,\Big(1 + \frac{1}{2}G^{\alpha\beta}G_{ij}\partial_\alpha\phi^i\partial_\beta\phi^j + G^{\alpha\beta}\partial_\alpha\phi^i G_{i\beta} \\ - \frac{k}{2}B_{\alpha\beta}F^{\alpha\beta} + \frac{k^2}{4}F_{\alpha\beta}F^{\alpha\beta} + \dots\Big) \end{aligned} \tag{C.4}$$

where we only kept the terms quadratic in fluctuations. Since the brane is wrapping the cycle Π_a we take the following rotated coordinates in $\mathbf{T}^2\times\mathbf{T}^2\times\mathbf{T}^2$

$$q^l = x^l\cos\theta_l + y^l\sin\theta_l, \qquad p^l = -x^l\sin\theta_l + y^l\cos\theta_l, \qquad \tan\theta_l = \frac{m_a^l}{n_a^l} \tag{C.5}$$

with x^l, y^l real coordinates on $(\mathbf{T}^2)_l$ for $l = 1,2,3$, the q^l's are along the brane and the p^l's transverse to it. Going back to the action (C.4) we get the following terms

$$\begin{aligned} S_6 \;\supset\; -\mu_6 \int_{M_4} d^4x \int_{\Pi_a} d^3q\, e^{-\Phi_0}\sqrt{-\det G_{\alpha\beta}}\,\Big(\frac{1}{2}G^{\mu\nu}G_{pp}\partial_\mu\phi^p\,\partial_\nu\phi^p + G^{\mu\nu}\partial_\mu\phi^p\, G_{p\nu} \\ - kB_{\mu q}F^{\mu q} + \frac{k^2}{2}F_{\mu q}F^{\mu q} + \dots\Big)\,. \end{aligned} \tag{C.6}$$

In this expression the indices μ, ν are in 4d, while p and q run through p^l and q^l respectively. The first line yields[1]

$$\mathcal{L}_{\rm St} \;=\; -\frac{1}{2}\sum_{i=1}^{3}\left(\partial_\mu \phi_a^i - m_a^i V_\mu^{x_i} + n_a^i V_\mu^{y_i}\right)^2 \tag{C.7}$$

where we defined $\phi_a^i = \sqrt{n_i^2 + m_i^2}\,\phi^i$ so that $\phi_a^i \sim \phi_a^i + 1$ following the conventions in [91]. This is the Lagrangian (3.46) that describes the spontaneous breaking of the continuous isometry group $U(1)^6$ of the torus to $U(1)^3 \times \mathbb{Z}_{q_1} \times \mathbb{Z}_{q_2} \times \mathbb{Z}_{q_3}$ with $q_i = (n_a^i)^2 + (m_a^i)^2$ due to the presence of the brane.[2] Also, ϕ_a^i provide the longitudinal degree of freedom to the massive gauge bosons $-m_a^i V_\mu^{x_i} + n_a^i V_\mu^{y_i}$.

Furthermore, from the second line in (C.6) one finds the following contribution to the low energy action

$$\mathcal{L}_{\rm St} \;=\; -\frac{1}{2}\sum_{i=1}^{3}\left(\partial_\mu \xi_a^i - n_a^i B_\mu^{x_i} - m_a^i B_\mu^{y_i}\right)^2 \tag{C.8}$$

where $\xi_a^i = \sqrt{n_i^2 + m_i^2}\,A^i$ and we have rescaled the B-field as $B \to k^{-1}B$, c.f.(3.51).

This analysis shows that the presence of a single brane breaks the continuous $U(1)^{12}$ gauge symmetry that arises from the reduction of the metric and B-field down to $U(1)^6$ plus some discrete part.[3] It is clear that adding more branes will generically break the gauge symmetry completely. Indeed, the Lagrangian for a set of intersecting branes will include the terms

$$\mathcal{L}_{\rm St} \;=\; -\frac{1}{2}\sum_{\alpha}\sum_{i=1}^{3}\left\{\left(\partial_\mu \phi_\alpha^i - m_\alpha^i V_\mu^{x_i} + n_\alpha^i V_\mu^{y_i}\right)^2 + \left(\partial_\mu \xi_\alpha^i - n_\alpha^i B_\mu^{x_i} - m_\alpha^i B_\mu^{y_i}\right)^2\right\} \tag{C.9}$$

where α runs over the branes. Unless all the branes are parallel in a given torus this will Higgs the continuous part of the gauge group completely. Nevertheless, there can be a discrete remnant which we discuss in the following.

Let us restrict to the case where there are only two branes a and b and focus on the part of the action (C.9) that involves $V_\mu^{x_i}$, $V_\mu^{y_i}$. Following [91], one can see that the discrete gauge group coming from the i-th torus is

$$\mathcal{T}_i^{ab} = \frac{\Gamma_i}{\hat{\Gamma}_i} \tag{C.10}$$

where Γ_i is the lattice of the i-th torus and $\hat{\Gamma}_i$ is the lattice generated by the intersection points. Namely,

$$\Gamma_i = \langle (1,0), (0,1) \rangle, \qquad \hat{\Gamma}_i = \frac{1}{I_{ab}^i}\langle (n_a^i, m_a^i), (n_b^i, m_b^i) \rangle. \tag{C.11}$$

[1] The kinetic term of the gauge bosons V_μ^x and V_μ^y that complete the Stückelberg Lagrangian can be obtained from dimensional reduction of the closed string sector of the theory.

[2] See section 2.5 in [92] for a discussion on the discrete part of this group.

[3] In case that we have an orientifold background the bulk symmetry is not $U(1)^{12}$ but $U(1)^6 \times \mathbb{Z}_2^6$. Indeed, because the O6-planes are located along $y^i = 0, \frac{1}{2}$ for $i = 1, 2, 3$, the U(1) symmetries generated by $V_\mu^{y_i}$ and $B_\mu^{x_i}$ are broken down to $\mathbb{Z}_2$, obtaining a residual $\mathbb{Z}_2 \times \mathbb{Z}_2$ gauge group for each $(\mathbf{T}^2)_i$.

One can check that indeed $\mathcal{T}_i^{ab} = \mathbb{Z}_{I_{ab}^i}$ and since the three-cycles the D6-branes wrap are factorisable we have

$$\mathcal{T}_{\mathbf{T}^6}^{ab} = \mathbb{Z}_{I_{ab}^1} \times \mathbb{Z}_{I_{ab}^2} \times \mathbb{Z}_{I_{ab}^3} \tag{C.12}$$

which reproduces eq.(3.48) in the main text. A completely analogous argument shows that the second term in (C.9) yields

$$\mathcal{W}_{\mathbf{T}^6}^{ab} = \mathbb{Z}_{I_{ab}^1} \times \mathbb{Z}_{I_{ab}^2} \times \mathbb{Z}_{I_{ab}^3} \tag{C.13}$$

in agreement with eq.(3.53).

These two groups, $\mathcal{T}_{\mathbf{T}^6}^{ab}$, $\mathcal{W}_{\mathbf{T}^6}^{ab}$, do not commute as can be seen from their action on the wavefunctions of chiral matter in the ab sector. Instead they generate the non-Abelian discrete group

$$\mathbf{P}_{\mathbf{T}^6}^{ab} = H_{I_{ab}^1} \times H_{I_{ab}^2} \times H_{I_{ab}^3} \tag{C.14}$$

with $H_N \simeq (\mathbb{Z}_N \times \mathbb{Z}_N) \rtimes \mathbb{Z}_N$.

Magnetised D-branes

In order to connect with the dimensional reduction of Type I with magnetised D9-branes performed in section 6.2 of [91] we T-dualise the above setup along the directions y^i for $i = 1, 2, 3$. Thus, as usual D6 branes at angles turn into D9 branes with magnetic fluxes given by

$$F_{x^i y^i}^a = \frac{m_a^i}{k n_a^i} \mathbb{I}_{n_a^i}. \tag{C.15}$$

Notice that since the three-cycles the D6-branes wrap are factorisable the 'nondiagonal' fluxes $F_{z^i \bar{z}^j}$ are zero for $i \neq j$. This means that just like before the dimensional reduction on $\mathbf{T}^6$ factorises in $(\mathbf{T}^2)_1 \times (\mathbf{T}^2)_2 \times (\mathbf{T}^2)_3$. More precisely, the above computation gives in this case

$$\mathcal{L}_{\rm St} = -\frac{1}{2} \sum_\alpha \sum_{i=1}^3 \left\{ \left(\partial_\mu \xi_{x,\alpha}^i - n_\alpha^i B_\mu^{x_i} - m_\alpha^i V_\mu^{y_i}\right)^2 + \left(\partial_\mu \xi_{y,\alpha}^i + m_\alpha^i V_\mu^{x_i} - n_\alpha^i B_\mu^{y_i}\right)^2 \right\} \tag{C.16}$$

where the axions $\xi_{x,\alpha}^i$, $\xi_{y,\alpha}^i$ correspond to the Wilson lines on the worldvolume of the brane along x^i and y^i respectively and have periodic identifications $\xi_{q,\alpha}^i \sim \xi_{q,\alpha}^i + 1$. In order to cancel the total D9-charge we must include an orientifold projection acting trivially on the tori which introduces O9-planes. Also, the B-field does not survive the projection so we can safely set it to zero[4] in (C.16) which reproduces the result in [91], namely,

$$\mathcal{L}_{\rm St} = -\frac{1}{2} \sum_\alpha \sum_{i=1}^3 \left\{ \left(\partial_\mu \xi_{x,\alpha}^i - \frac{m_\alpha^i}{n_\alpha^i} V_\mu^{y_i}\right)^2 + \left(\partial_\mu \xi_{y,\alpha}^i + \frac{m_\alpha^i}{n_\alpha^i} V_\mu^{x_i}\right)^2 \right\} \tag{C.17}$$

with $\xi_{q,\alpha}^i \sim \xi_{q,\alpha}^i + 2/n_\alpha^i$.[5]

[4] Actually, a subgroup $\mathbb{Z}_2^6$ survives the orientifold projection, as in the type IIA case (see footnote 3).
[5] The factor 2 appears due to the orientifold projection.

D

Linear equivalence of p-cycles

In this appendix we review the definition of linear equivalence as given in [147] for general cycles. We start by introducing the concept of p-gerbe with connection and then give the definition of linear equivalence of cycles.

From bundles to p-gerbes

Roughly speaking a p-gerbe is a generalisation in higher dimension of a line bundle. In order to motivate its definition we start by giving three equivalent characterisations of the topology of a line bundle $\mathcal{L}$ on a manifold X,

- A cohomology class in $H^2(X, \mathbb{Z})$,
- A real codimension 2 submanifold M of X,
- An element in the Čech cohomology group $\check{H}^1(X, U(1))$.

The cohomology class characterising $\mathcal{L}$ is its first Chern class $c_1(\mathcal{L})$ while the real codimension 2 submanifold is the Poincaré dual of $c_1(\mathcal{L})$. Finally the element in $\check{H}^1(X, U(1))$ specifies the transition functions of the bundle, namely taking an open cover $\{\mathcal{U}_\alpha\}$ of X such that $\mathcal{L}$ is trivial over each set $\mathcal{U}_\alpha$ given $g \in \check{H}^1(X, U(1))$ we get the functions

$$g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to U(1) , \tag{D.1}$$

such that $g_{\alpha\beta}(x) g_{\beta\alpha}(x) = 1$ in $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ and furthermore

$$g_{\alpha\beta}(x) g_{\beta\gamma}(x) g_{\gamma\alpha}(x) = 1 , \quad \forall x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma . \tag{D.2}$$

The natural generalisation of the previous characterisations of a line bundle are

- A cohomology class in $H^{p+2}(X, \mathbb{Z})$,
- A real codimension $p+2$ submanifold M of X,
- An element in the Čech cohomology group $\check{H}^{p+1}(X, U(1))$.

We will take one of the three equivalent characterisations as a definition of a p-gerbe. We now will endow p-gerbes with a connection in a way similar to how we endow line bundles with a connection and relate the curvature of this connection to the cohomology class in $H^{p+2}(X, \mathbb{Z})$.

Connections on p-gerbes

It is again useful to start recalling how a connection is built on a line bundle. Given a open cover $\{\mathcal{U}_\alpha\}$ of X a connection on a line bundle $\mathcal{L}$ with transitions functions $g \in \check{H}^1(X, U(1))$ is a set of 1-forms A_α defined on $\mathcal{U}_\alpha$ that on double intersections $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ satisfy

$$i(A_\beta - A_\alpha) = g_{\alpha\beta}^{-1} dg_{\alpha\beta} \,. \tag{D.3}$$

In particular since $d(g_{\alpha\beta}^{-1} dg_{\alpha\beta}) = 0$ we have that there is a global closed two form (the curvature of the bundle) satisfying

$$F|_{\mathcal{U}_\alpha} = dA_\alpha \,. \tag{D.4}$$

We can now adapt this procedure to the case of a p-gerbe. Let us call $C^q(\mathcal{U}, \mathcal{F})$ the set of q Čech cochains with values in a sheaf $\mathcal{F}$ for the open covering $\mathcal{U}$. Then given the transition functions of the p-gerbe $\mathcal{G}$ we can build the element $\varpi^{(1)} \in C^p(\mathcal{U}, \Omega^1(X))$ satisfying

$$(\delta\varpi^{(1)})_{\alpha_1 \ldots \alpha_p} = (g^{-1} dg)_{\alpha_1 \ldots \alpha_p} \,, \tag{D.5}$$

where $g \in \check{H}^{p+1}(X, U(1))$ are the transition functions of the gerbe $\mathcal{G}$. We can iteratively arrive at the definition of the curvature of the connection

$$(d\varpi^{(q)})_{\alpha_1 \ldots \alpha_{p-q}} = (\delta\varpi^{(q+1)})_{\alpha_1 \ldots \alpha_{p-q}} \,, \tag{D.6}$$

where at each stage we have $\varpi^{(q)} \in C^{p-q+1}(\mathcal{U}, \Omega^q(X))$ and δ is the Čech coboundary operator. This can be repeated until we arrive at $\varpi^{(p+1)}$ which is an element of $C^0(\mathcal{U}, \Omega^{p+1}(X))$, we define $G = d\varpi^{(p+1)}$ to be the curvature of the p-gerbe. It is a globally defined and closed $p+2$ form whose class $[G] \in H^{p+2}(X, \mathbb{R})$ is the image of the characteristic class of the gerbe $\mathcal{G}$ under the natural inclusion $i : H^{p+2}(X, \mathbb{Z}) \to H^{p+2}(X, \mathbb{R})$.

We can give a direct construction of the connection on a p-gerbe associated to a codimension $p+2$ submanifold which enters directly in the definition of linear equivalence of submanifolds. Given a codimension $p+2$ submanifold M we denote its Poincaré dual $p+2$-form as $\delta(M)$. We define a connection as a 0 Čech cochain, ϖ, satisfying the following differential equations

$$d\varpi_\alpha^{(p+1)} = \delta(M)|_{\mathcal{U}_\alpha} \,. \tag{D.7}$$

This does not specify completely the connection since the addition of a closed form produces a different solution to the previous differential equation. We can partially fix this ambiguity by imposing the following conditions

$$d^*\varpi^{(p+1)} = 0 \,, \qquad \int_{\Lambda_{p+1}} \varpi^{(p+1)} \in \mathbb{Z} \,, \tag{D.8}$$

where $\Lambda_{p+1} \in H_{p+1}(X \setminus M, \mathbb{Z})$. These conditions will still be satisfied if we add to ϖ an integral harmonic form but this ambiguity is not important in characterising the connection on the p-gerbe. Once these conditions are imposed we see that the connection has the following Hodge decomposition

$$\varpi^{(p+1)} = \omega + d^* H \,, \tag{D.9}$$

where ω is the harmonic part of the connection and $dd^*H = \delta(M)$. We define the holonomy of a p-gerbe as the group given by

$$hol(\varpi^{(p+1)}, \Sigma_{p+1}) = \exp\left(2\pi i \int_{\Sigma_{p+1}} \omega\right) \tag{D.10}$$

for every $(p+1)$-cycle Σ_{p+1} in X.

Choosing a basis of non trivial $(p+1)$-cycles we get a complete characterization of the holonomy of the connection. Since the holonomy and curvature data completely specify the connection we have that a connection on a gerbe is trivial if both its holonomy and its curvature are zero.

Linear equivalence of submanifolds

We now give the definition of linear equivalence of submanifolds and discuss how this reproduces the usual definition of linear equivalence between divisors. We say that two submanifolds M and N are linearly equivalent if the gerbe $\mathcal{G}_M\mathcal{G}_N^{-1}$ is trivial.[1] We can extend this definition to linear combinations of submanifolds as it is usually done in the case of divisors, namely, two linear combinations of submanifolds $M = \sum_i a_i M_i$ and $N = \sum_i b_i N_i$ are linearly equivalent if the connection on the gerbe $\prod_i \mathcal{G}_{M_i}^{a_i} \prod_j \mathcal{G}_{N_j}^{-b_j}$ has is trivial.

This definition of linear equivalence is rather abstract so in the following we give an alternative characterisation which is more useful in practice. The p-gerbe $\mathcal{G}_M\mathcal{G}_N^{-1}$ has a connection satisfying the following differential equation

$$d\varpi_\alpha = [\delta(M) - \delta(N)]\,|_{\mathcal{U}_\alpha}\,. \tag{D.11}$$

A first condition that we need to impose on the connection of the gerbe in order to be trivial is that its curvature is zero, this happens when the two submanifolds are in the same homology class so that the right hand side of (D.11) is an exact form and the connection ϖ is globally well defined. All we need to check is that the gerbe has trivial holonomy which, according to our definition, happens when the harmonic part of the connection ϖ is integral. More precisely, from (D.8) we see that linear equivalence amounts to

$$\omega \in H^{p+1}(U,\mathbb{Z}) \quad \Leftrightarrow \quad d^*H \in H^{p+1}(U,\mathbb{Z}) \tag{D.12}$$

where $U \equiv X \setminus (M \cup N)$.[2] Following [147] one can show that this condition is equivalent to

$$\int_\Gamma \theta \in \mathbb{Z}\,, \tag{D.13}$$

where $\partial\Gamma = M - N$ and θ is a harmonic form in X with integral cohomology class. The proof goes as follows. The condition (D.12) is equivalent to

$$\int_U \varphi \wedge d^*H \in \mathbb{Z} \tag{D.14}$$

[1]Given a gerbe $\mathcal{G}$ we define its dual $\mathcal{G}^{-1}$ to be the gerbe whose transition functions are the inverse of the ones of the gerbe $\mathcal{G}$. Moreover we can introduce a product in the space of p-gerbes defining the product of two p-gerbes $\mathcal{M}$ and $\mathcal{N}$ with transitions functions $g_\mathcal{M}$ and $g_\mathcal{N}$ to be the p-gerbe with transition functions $g_\mathcal{M} g_\mathcal{N}$. Note that the notion of dual and product agree with the known ones in the case of line bundles.

[2]We need to remove M and N from X because the form d^*H has poles on these submanifolds.

with φ a $(d-p-1)$-form with integral class in the compactly supported cohomology of U, where d is the dimension of X. Moreover, given a chain Γ connecting both cycles, i.e. $\partial\Gamma = M - N$, we can decompose its Poincaré dual as $\delta(\Gamma) = A + dB + d^*H$ where we fixed the coexact term using that $d\delta(\Gamma) = \delta(\partial\Gamma) = \delta(M) - \delta(N) = dd^*H$. Now, since Γ belongs to $H_{d-p-1}(U, \mathbb{Z})$ we have that

$$\int_\Gamma \varphi \in \mathbb{Z}. \tag{D.15}$$

Also, since φ is closed we can write

$$\int_\Gamma \varphi = \int_X \varphi \wedge A + \int_X \varphi \wedge d^*H, \tag{D.16}$$

so (D.14) is met if and only if $\int_X \varphi \wedge A \in \mathbb{Z}$ for all such φ. Finally, for harmonic forms θ we have that $\int_\Gamma \theta = \int_X \theta \wedge A = \int_X \varphi \wedge A$ since one can show that θ is cohomologous to a closed form with compact support in U. Thus, M and N are linearly equivalent if and only if $\int_\Gamma \theta \in \mathbb{Z}$ for every harmonic form θ.

E
M-theory lift of D6-branes

In this appendix we show how the harmonic two-forms $\varpi \in H^2(\mathcal{M}_6 - \{\pi_{D6}\}, \mathbb{Z})$ appear from the M-theory lift of a configuration of D6-branes. Here π_{D6} is the set of all branes.

The M-theory lift of N parallel D6-branes in flat spacetime is given by a purely geometric configuration. More explicitly, we have that the lift corresponds to the space $\mathbb{R}^{1,3} \times \mathbb{R}^3 \times \mathbf{TN}_N$ where $\mathbf{TN}_N$ is the N-center Taub-NUT space with metric

$$ds^2 = V d\vec{r} \cdot d\vec{r} + \frac{1}{V}(d\psi + \vec{\omega} \cdot d\vec{r})^2 \tag{E.1}$$

with $\vec{r} \in \mathbb{R}^3$ and $\psi \sim \psi + 4\pi$. Also,

$$V = \frac{1}{R} + \sum_{\alpha=1}^{N} V_\alpha, \qquad V_\alpha = \frac{1}{|\vec{r} - \vec{a}_\alpha|} \tag{E.2}$$

and

$$\vec{\omega} = \sum_{\alpha=1}^{N} \vec{\omega}_\alpha, \qquad \vec{\nabla} V_\alpha = \vec{\nabla} \times \vec{\omega}_\alpha. \tag{E.3}$$

This geometry is a $\mathbf{S}^1$ fibration over $\mathbb{R}^3$ and the radius of the fiber is given by the $g_{\psi\psi}$ component of the metric, namely, $\frac{1}{V}$. Near the location of the D6-branes $\vec{a}_\alpha$ the radius goes like $\frac{1}{V_\alpha} = |\vec{r} - \vec{a}_\alpha|$ so the fiber shrinks. On the other hand, asymptotically far away from them it has constant radius R which determines the string coupling constant in Type IIA, namely $g_s^{IIA} = \frac{R}{\sqrt{\alpha'}}$.

An alternative way to understand this space is to regard it as the total space of a $U(1)$ gauge bundle over $\mathbb{R}^3$ with N magnetic monopoles at $\vec{a}_\alpha$. Then, the one-forms $\omega_\alpha \equiv \vec{\omega}_\alpha \cdot d\vec{r}$ are the gauge fields associated to the monolopole α. Thus, their field strengths are given by $F_\alpha = d\omega_\alpha$ and satisfy the equations,

$$dF_\alpha = -4\pi\delta_3(\vec{r} - \vec{a}_\alpha), \qquad d * F = 0. \tag{E.4}$$

There are N cohomologically independent harmonic normalizable 2-forms that, when reducing the C_3 potential, yield the gauge fields associated with the D6-branes. These are [156]

$$4\pi\Omega_\alpha = d\chi_\alpha, \qquad \chi_\alpha = \frac{V_\alpha}{V}(d\psi + \omega) - \omega_\alpha. \tag{E.5}$$

which are orthogonal, i.e. $\int \Omega_\alpha \wedge \Omega_\beta = R^2 \delta_{\alpha\beta}$. It is useful to take the linear combinations

$$\Omega_{\alpha\beta} = \Omega_\alpha - \Omega_\beta, \qquad \Omega_{CM} = \sum_{\alpha=1}^{N} \Omega_\alpha \tag{E.6}$$

since $\Omega_{\alpha\beta}$ has compact support while the center of mass 2-form Ω_{CM} does not.

For every harmonic 2-form $\Omega_{\alpha\beta}$ with compact support there is a non-trivial 2-cycle in homology $\pi_{\alpha\beta}$ (which is compact by definition). Topologically, these cycles are 2-spheres given by $\pi_{\alpha\beta} \simeq \pi^{-1}(I_{\alpha\beta})$ where $\pi : \mathbf{TN}_N \to \mathbb{R}^3$ is the projection of the fibration and $I_{\alpha\beta}$ is a path in the base that connects $\vec{a}_\alpha$ and $\vec{a}_\beta$. When we take the perturbative limit $R \to 0$ these 2-cycles become 1-chains in the base which are just the paths $I_{\alpha\beta}$. In Type IIA the M-theory circle disappears from the description so these can be regarded as 1-cycles in $H_1(\mathbb{R}^3, P)$ where P is the set of points $\vec{a}_\alpha$. Thus, by Poincaré-Lefschetz duality there should be $N-1$ independent 2-forms in $H^2(\mathbb{R}^3 - P)$. Let us see this explicitly.

Consider the 2-forms $\Omega_{\alpha\beta}$ in $\mathbf{TN}_N$, namely

$$\begin{aligned} 4\pi\Omega_{\alpha\beta} &= d\left(\frac{V_\alpha - V_\beta}{V}(d\psi + \omega)\right) - d(\omega_\alpha - \omega_\beta) \qquad \text{(E.7)} \\ &= -\frac{R^2(V_\alpha - V_\beta)}{R^2V^2} dV \wedge (d\psi + \omega) + \frac{R}{RV} d((V_\alpha - V_\beta)(d\psi + \omega)) - d(\omega_\alpha - \omega_\beta). \end{aligned}$$

In the perturbative limit, $R \to 0$ and $RV \to 1$ so only the last term survives which yields

$$\Omega_{\alpha\beta} \xrightarrow{R\to 0} \tilde{\Omega}_{\alpha\beta} = \frac{1}{4\pi} d(\omega_\beta - \omega_\alpha). \qquad \text{(E.8)}$$

Using eq.(E.4) we see that $d\tilde{\Omega}_{\alpha\beta} = \delta_3(\vec{r} - \vec{a}_\alpha) - \delta_3(\vec{r} - \vec{a}_\beta)$ so it is a closed 2-form in $\mathbb{R}^3 - P$ and $d * \tilde{\Omega}_{\alpha\beta} = 0$. One could think that since $\tilde{\Omega}_{\alpha\beta}$ is given by the exterior derivative of $\omega_\beta - \omega_\alpha$ it is trivial in cohomology, however, the one-forms ω_α are not globally well-defined, instead they have to be defined in different patches and glued together using a gauge transformation. Thus, an alternative way of writing this 2-form is $\tilde{\Omega}_{\alpha\beta} = d^* H_{\alpha\beta}$ where $H_{\alpha\beta}$ is a globally well-defined 3-form with a singularity of the form $\frac{1}{r}$ at the points $\vec{a}_\alpha$ and $\vec{a}_\beta$. Thus, putting everything together we have that

$$\tilde{\Omega}_{\alpha\beta} = d^* H_{\alpha\beta}, \qquad d\tilde{\Omega}_{\alpha\beta} = \delta_3(\vec{r} - \vec{a}_\alpha) - \delta_3(\vec{r} - \vec{a}_\beta), \qquad d^*\tilde{\Omega}_{\alpha\beta} = 0. \qquad \text{(E.9)}$$

This shows that $\tilde{\Omega}_{\alpha\beta}$ are precisely the 2-forms $\varpi^{(\alpha-\beta)}$ in the main text. Since this is a non-compact toy model there is no harmonic 2-form in $\tilde{\Omega}_{\alpha\beta}$ which means that in flat space any two points are linearly equivalent. When embedded in a compact model such harmonic 2-form must be added to $\tilde{\Omega}_{\alpha\beta}$ to ensure that it is integral.

F
Generalized homology

In this appendix we give the basic definitions of the generalized homology introduced in [157]. We refer the reader to [157] for a detailed discussion on the subject.

The RR charges of D-branes in the presence of a NSNS H-flux are classes in twisted K-theory but it is generally very difficult to compute them in concrete examples. On the other hand, generalized homology captures some aspects of the RR charges beyond usual homology and is much easier to work with.

In the context of generalized complex geometry a Dp-brane can be described as a submanifold Σ carrying a gauge bundle $\mathcal{F} = F + B$ which is a generalized submanifold $(\Sigma, \mathcal{F})$.[1] In other to define a homology theory for these objects we need to define chains and a boundary operator that squares to zero.

Consider a Dp-brane wrapping a $(p+1)$-submanifold with a $U(1)$ gauge field that may have a Dirac monopole on a $(p-2)$-submanifold $\Pi \subset \Sigma$ which must satisfy $\partial\Pi \subset \partial\Sigma$. The field strength $\mathcal{F}_\Pi$ thus satisfies

$$d\mathcal{F}_\Pi = H|_\Sigma + \delta_\Sigma(\Pi) \tag{F.1}$$

in the presence of a H-flux. Since $\mathcal{F}_\Pi$ should be globally well-defined we have that

$$\text{P.D.}[H|_\Sigma] + [\Pi] = 0. \tag{F.2}$$

Thus, the pair $(\Sigma, \mathcal{F}_\Pi)$ is a generalized submanifold and a generalized chain is defined to be a formal sum of these pairs. We restrict the sum to contain only even or odd dimensional submanifolds as suggested by Type IIB and IIA string theories respectively.

We define the generalized boundary operator in such a way that that its action is dual to the action of the H-twisted exterior derivative on forms. Therefore, inspired in the CS action for a D-brane, we associate a current $j_{(\Sigma,\mathcal{F}_\Pi)}$ to each chain by

$$j_{(\Sigma,\mathcal{F}_\Pi)}(C) \equiv \int_\Sigma C|_\Sigma \wedge e^{\mathcal{F}_\Pi} \tag{F.3}$$

where C is an arbitrary polyform of definite parity. The derivative $d_H = d + H\wedge$ acts on the current in the following way

$$(d_H j_{(\Sigma,\mathcal{F}_\Pi)})(C) = \int_\Sigma d_H C|_\Sigma \wedge e^{\mathcal{F}_\Pi} = \int_{\partial\Sigma} C|_\Sigma \wedge e^{\mathcal{F}_\Pi|_{\partial\Sigma}} - \int_\Pi C|_\Pi \wedge e^{\mathcal{F}_\Pi|_\Pi} \tag{F.4}$$

where we used Stoke's theorem. Thus,

$$d_H j_{(\Sigma,\mathcal{F}_\Pi)} = j_{(\partial\Sigma,\mathcal{F}_\Pi|_{\partial\Sigma})} - j_{(\Pi,\mathcal{F}_\Pi|_\Pi)} \tag{F.5}$$

[1] We restrict ourselves to abelian D-branes.

so we define the generalized boundary operator $\hat{\partial}$ by imposing

$$d_H j_{(\Sigma,\mathcal{F}_\Pi)} = j_{\hat{\partial}(\Sigma,\mathcal{F}_\Pi)} \tag{F.6}$$

which leads to

$$\hat{\partial}(\Sigma,\mathcal{F}_\Pi) \equiv (\partial\Sigma,\mathcal{F}_\Pi|_{\partial\Sigma}) - (\Pi,\mathcal{F}_\Pi|_\Pi). \tag{F.7}$$

One can check that $\hat{\partial}^2 = 0$ which allows to define the generalized homology as $\mathrm{Ker}(\hat{\partial})/\mathrm{Im}(\hat{\partial})$.

In order to preserve RR gauge invariance D-branes can only wrap generalized chains that are closed [157]. Then for any two generalized cycles that are in the same homology class there is a physical process that connects them although it may not be energetically favorable. In order to define the energy of a generalized cycle one can introduce a generalized calibration [157]. Of particular relevance for our discussion is the effect of dissolving D(p-2)-branes in Dp-branes which is nicely captured in this formalism since both situations correspond to different representatives of the same homology class as is shown in chapter 4.

G

Details on the computation of $j_{(\mathfrak{S},\mathfrak{F})}$

Here we derive the equation (4.90) starting with the generalized chain $(\mathfrak{S},\mathfrak{F})$ and show it has all the desired properties.

First, one can check that a change in S'_a, $\mathcal{F}'_a$ or B_a corresponds to choosing a different generalized chain $(\mathfrak{S}',\mathfrak{F}') = (\mathfrak{S},\mathfrak{F}) + \hat{\partial}(\mathfrak{s},\mathfrak{f})$ so the associated current is $j_{(\mathfrak{S}',\mathfrak{F}')} = j_{(\mathfrak{S},\mathfrak{F})} + d\, j_{(\mathfrak{s},\mathfrak{f})}$. Since γ_I is harmonic we have that $d\, j_{(\mathfrak{s},\mathfrak{f})}(\gamma_I) = 0$ which shows that our result is independent of all these choices.

Now we take the limit where B_a and B_b go to zero that yields a simpler and more trasparent expression which is manifestly independent of S'_a, $\mathcal{F}'_a$ or B_a. Let us focus on the contribution due to $\hat{\partial}[-(B_a,\tilde{\mathcal{F}}_a) + (B_a,\tilde{\mathcal{F}}_{\Pi_a})]$, namely

$$j_{(\mathfrak{S},\mathfrak{F})}(\gamma_I) \supset \int_B \gamma_I \wedge (\tilde{\mathcal{F}}_\Pi - \tilde{\mathcal{F}}). \tag{G.1}$$

where we dropped the subscript a to simplify the notation. Let us define $\mathcal{H} = \tilde{\mathcal{F}}_\Pi - \tilde{\mathcal{F}}$ which satisfies

$$d\mathcal{H} = \delta^3_B(\Pi), \qquad \mathcal{H}|_S = -\mathcal{F}, \qquad \mathcal{H}|_{S'} = 0. \tag{G.2}$$

We have that $B = S \times I_L$ with I_L an interval of lenght L with coordinate $t \in [0, L]$ and a slicing of B in S_t with $S_0 = S$ and $S_L = S'$. Since the integral above does not depend on B it can not depend on L so we may take the limit $L \to 0$ to make it manifestly independent of B. We define

$$\begin{aligned} \gamma_I &= \gamma_I|_{S_t} + \tilde{\gamma}_I & \text{(G.3)} \\ \mathcal{H} &= \mathcal{H}|_{S_t} + \tilde{\mathcal{H}}. & \text{(G.4)} \end{aligned}$$

The equation for $\mathcal{H}$ translates into

$$d_S(\mathcal{H}|_{S_t}) + d_{I_L}(\mathcal{H}|_{S_t}) + d_S\tilde{\mathcal{H}} = \delta^2_{S_0}(\Pi) \wedge \delta(t)dt, \qquad \mathcal{H}|_{S_0} = -\mathcal{F}, \qquad \mathcal{H}|_{S_L} = 0 \tag{G.5}$$

where we used the fact that $d = d_{I_L} + d_S$ with d_{I_L} and d_S the exterior derivatives on I_L and S_t respectively. From the boundary conditions for $\mathcal{H}$ we find that

$$\lim_{L\to 0} d_S(\mathcal{H}|_{S_t}) = 0, \qquad \lim_{L\to 0} d_{I_L}(\mathcal{H}|_{S_t}) = \mathcal{F} \wedge \delta(t)dt. \tag{G.6}$$

Thus, the differential equation for $L \to 0$ is

$$d_S\tilde{\mathcal{H}} = (\delta^2_{S_0}(\Pi) - \mathcal{F}) \wedge \delta(t)dt \tag{G.7}$$

so we necessarily have that

$$\lim_{L\to 0} \tilde{\mathcal{H}} = -A_\Pi \wedge \delta(t)dt \tag{G.8}$$

with A_Π a 1-form in S that satisfies

$$d_S A_\Pi = \mathcal{F} - \delta^2_S(\Pi). \tag{G.9}$$

Going back to the integral (G.1) we find

$$\int_B \gamma_I \wedge (\tilde{\mathcal{F}}_\Pi - \tilde{\mathcal{F}}) = \int_{S\times I_L} (\gamma_I|_{S_t} \wedge \tilde{\mathcal{H}} + \tilde{\gamma}_I \wedge \mathcal{H}|_{S_t}) \tag{G.10}$$

where only the first term contributes in the limit $L \to 0$ since it contains a delta function unlike the second one. Therefore,

$$\int_B \gamma_I \wedge (\tilde{\mathcal{F}}_\Pi - \tilde{\mathcal{F}}) = -\int_{S\times I_L} \gamma_I|_S \wedge \tilde{A}_\Pi \wedge \delta(t)dt = \int_S \gamma_I \wedge \tilde{A}_\Pi. \tag{G.11}$$

Notice there is minus sign due to the orientation of S in $\partial B = S' - S$.

Using this result we may write

$$j_{(\mathfrak{S},\mathfrak{F})}(\gamma_I) = \int_\Sigma \gamma_I + \int_{S_a} \gamma_I \wedge A_{\Pi_a} - \int_{S_b} \gamma_I \wedge A_{\Pi_b}. \tag{G.12}$$

The only thing left is to show that (G.12) is independent of the choice of Π_a and Π_b. Consider $\Pi'_a = \Pi_a + \partial\sigma_a$ and $\Pi'_b = \Pi_b + \partial\sigma_b$ which also changes Σ into $\Sigma' = \Sigma + \sigma_a - \sigma_b$. One can readily show that the formula above is independent of σ_a and σ_b. Finally, a change $\Sigma' = \Sigma + \pi$ with π a closed 3-cycle in $\mathcal{M}_6$ does change the expression above but just by integer number which can be interpreted as a redefinition of the $U(1)$ sector.

As a further check of this result let us derive eq.(4.79) starting from $j_{(\mathfrak{S},\mathfrak{F})}(\gamma_I)$ when $[F_a] = [F_b] \in H^2(S)$. More explicitly, we show that

$$\int_\Gamma \gamma \wedge \tilde{\mathcal{F}} = \int_\Sigma \gamma + \int_{B_a} \gamma \wedge H_a - \int_{B_b} \gamma \wedge H_b. \tag{G.13}$$

Since nothing depends on B_a and B_b we choose $B_a = -B_b = -\Gamma$. Also, since $[F_a] = [F_b] \in H^2(S)$ we have that we may take $\Sigma \subset \Gamma$ so

$$\int_\Sigma \gamma + \int_{B_a} \gamma \wedge H_a - \int_{B_b} \gamma \wedge H_b = \int_\Gamma \gamma \wedge (\delta^2_\Gamma(\Sigma) - H_a + H_b). \tag{G.14}$$

The quantity $\mathcal{Q} \equiv \delta^2_\Gamma(\Sigma) + H_a - H_b$ satisfies the equation

$$d\mathcal{Q} = \delta^3_{S_b}(\Pi_b) - \delta^3_{S_a}(\Pi_a) + d\delta^2_\Gamma(\Sigma) = 0 \tag{G.15}$$

where we used $d\delta^2_\Gamma(\Sigma) = \delta^3_\Gamma(\partial\Sigma) = \delta^3_{S_a}(\Pi_a) - \delta^3_{S_b}(\Pi_b)$. The boundary conditions are $\mathcal{Q}|_{S_a} = \mathcal{F}_a$ and $\mathcal{Q}|_{S_b} = \mathcal{F}_b$ so $\mathcal{Q} = \tilde{\mathcal{F}}$.

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