Jens Mund

The Spin-Statistics Theorem for Anyons and Plektons in d = 2+1

Dedicated to Klaus Fredenhagen on the occasion of his 60th birthday.

Received: 23 January 2008 / Accepted: 13 May 2008

Abstract We prove the spin-statistics theorem for massive particles obeying braid group statistics in three-dimensional Minkowski space. We start from first principles of local relativistic quantum theory. The only assumption is a gap in the mass spectrum of the corresponding charged sector, and a restriction on the degeneracy of the corresponding mass.

1 Introduction

The famous spin-statistics theorem relates the exchange statistics of a quantum field with the spin of its elementary excitations (22). Namely, it states that in the case of Bose/Fermi (para-) statistics there holds

$$e^{2\pi i s} = \mathrm{sign}\,\lambda,$$

where s is the spin of the particles and λ is the statistics parameter of the fields. In (4) a derivation from first principles without any non-observable quantities such as charge-carrying fields was found. However, basic input to this derivation was that the charge be localizable in bounded regions. In (2), Buchholz and Epstein extended the theorem to massive particles carrying a non-localizable charge. In the purely massive case, such charges are still localizable in space-like cones (3), i.e., cones in spacetime which extend to space-like infinity.[1] The analysis of Buchholz and Epstein was carried out in four-dimensional spacetime, in which case

Supported by FAPEMIG.

Departamento de Física, Universidade Federal de Juiz de Fora, 36036-900 Juiz de Fora, MG, Brazil. mund@fisica.ufjf.br

[1] More precisely, a space-like cone is a region in Minkowski space of the form $C = a + \cup_{\lambda>0}\lambda\mathscr{O}$, where a is the apex of C and $\mathscr{O}$ is a double cone whose closure does not contain the origin.

λ is a real number associated with a unitary representation of the permutation group ($\lambda > 0$ corresponding to Bosons and $\lambda < 0$ corresponding to Fermions). In three-dimensional spacetime, however, it may occur that the permutation group is replaced by the braid group, in which case the statistics parameter is a complex non-real number. The phase in its polar decomposition is called the statistics phase ω,

$$\omega := \frac{\lambda}{|\lambda|}. \tag{1}$$

In the case of non-real λ (i.e. $\omega \neq \pm 1$) one speaks of braid group statistics and calls the particles Plektons or, if the corresponding representation is Abelian, Anyons. Related to this phenomenon, in three-dimensional spacetime the spin of a particle needs not be integer or half-integer, but may assume any real value ("fractional" spin). In fact, the occurrence of braid group statistics is equivalent to the occurrence of "fractional" spin (7; 10).

In the present article, we prove that in this case the spin-statistics relation

$$e^{2\pi i s} = \omega \tag{2}$$

holds, starting from first principles and only assuming the following conditions on the mass spectrum. We consider a charged sector of a local relativistic quantum theory in three-dimensional Minkowski space, containing a massive particle with mass $m > 0$ and spin $s \in \mathbb{R}$. We assume that m is separated from the rest of the mass spectrum in its sector by a mass gap. We further assume that there are only finitely many "particle types" in its sector with this mass, and that they all have the same spin s. As a byproduct, we prove that the familiar symmetry between particles and antiparticles holds also in this case: Namely, that there is an equal number of antiparticle types (in the conjugate sector) with the same mass which all have the same spin $s \in \mathbb{R}$ (Proposition 1).

It should be noted that a "weak spin-statistics relation",

$$e^{4\pi i s} = \omega^2, \tag{3}$$

is known to hold (7; 10) under quite general conditions in the case of braid group statistics. It should also be noted that the strong spin-statistics relation (2) has been proved in (11) and in (14), but under a non-trivial hypothesis amounting to the Bisognano-Wichmann property, or modular covariance, of the charged fields (11) or the observables (14), respectively. In the present paper we do not need this hypothesis. In fact, we shall show in a subsequent paper (15) that the Bisognano-Wichmann property may be derived from first principles in a purely massive theory with braid group statistics, using the results of our present analyisis.

Our derivation will largely parallel that of Buchholz and Epstein (2). The crucial difference between the four-dimensional case considered in (2) and the present three-dimensional case lies in the structure of the Poincaré group and the irreducible massive representations of its universal covering group, which have been heavily used in (2). In particular, in four dimensions one has the so-called "covariant representation"[2], in which locally generated single particle wave functions have certain analyticity properties which are exploited in the proof. In three dimensions, however, there is no "covariant represention", and in the well-known

[2] This is a tensor product of the spin zero representation of the Poincaré group with a finite-dimensional representation of the (covering of the) Lorentz group.

Wigner representation the wave functions are *not* analytic. As a way out, we use here an equivalent representation found by the author in (17), which exhibits precisely the required analyticity properties. On the other hand, the representation of the translation subgroup in three dimensions does not differ essentially from that in four dimensions. Hence the results from (2) which use only the translations can directly be adapted to the three-dimensional case. This concerns in particular our Lemma 1 on the two-point functions.

The article is organized as follows. In Sect. 2 we specify in detail our framework, assumptions and results. In Sect. 3 we recall a result of Buchholz and Epstein (2) concerning analyticity of the two-point functions in momentum space, and extend their result on the particle-antiparticle symmetry to the present case. In Sect. 4, finally, we prove the spin-statistics theorem.

2 Framework, Assumptions and Results

We now specify our framework and make our assumptions and results precise.[3]

States and Fields Denoting the quantum numbers of our sector collectively by χ, the space of states of the sector corresponds to a Hilbert space $\mathscr{H}_\chi$. It is orthogonal to the vacuum Hilbert space $\mathscr{H}_0$ which contains a Poincaré invariant vector Ω, corresponding to the vacuum state. $\mathscr{H}_\chi$ carries a unitary representation of the universal covering group $\tilde{P}_+^\uparrow$ of the Poincaré group in $2+1$ dimensions, denoted by U_χ, satisfying the relativistic spectrum condition (positivity of the energy). Fields carrying charge χ are bounded operators from the vacuum Hilbert space $\mathscr{H}_0$ to $\mathscr{H}_\chi$. The linear space of these fields will be denoted by $\mathscr{F}_\chi$.

Localization Fields are localizable to the same extent to which the charges are localizable which they carry. In the case of braid group statistics, the charges cannot be localized in bounded regions of spacetime (4), but they can be localized, in the massive case, in regions which extend to infinity in some space-like direction, namely, in space-like cones (3). Now the manifold of space-like directions,

$$H := \{e \in \mathbb{R}^3,\ e \cdot e = -1\}, \tag{4}$$

is not simply connected in three dimensions (in contrast to the four-dimensional case): Given two space-like directions, there exists an infinity of non-homotopic paths in H from one to the other, distinguished by a winding number. It is precisely this fact which enables the occurrence of braid group statistics in three dimensions (see the remark after Eq. (15) below). To realize such statistics, the fields which create a charge localized in a given space-like cone C need additional information:

[3] Recall that in the case of braid group statistics there is no canonical way to construct a field algebra from the observables (18). But our framework, using a restricted notion of charged fields, can be set up starting from the standard assumptions (12) of local relativistic quantum theory on the observables plus weak Haag Duality, together with our assumptions on the mass spectrum. For the convenience of the reader, we sketch in Appendix A how this may be done and indicate the relation with the notions used in the literature (3; 5; 8).

Namely, a path in the set of space-like directions H starting from some fixed reference direction e_0 and "ending" in C.[4] We shall sketch this concept, which has been introduced in (9), in a slightly modified form introduced in (16). We say that a space-like cone C contains a space-like direction e if

[4] Two other possibilities are: To introduce a reference space-like cone from which all allowed localization cones have to keep space-like separated (this cone playing the role of a "cut" in the context of multivalued functions) (3); or a cohomology theory of nets of operator algebras as introduced by Roberts (19; 20; 21).

Fig. 1 $\hat{C}$ denotes the set of space-like directions contained in C, in the sense of Eq. (5). $(C,\tilde{e}_1)$ is equivalent with $(C,\tilde{e}_2)$, but inequivalent from $(C,\tilde{e}_3)$

$$C+e\subset C. \tag{5}$$

We say that a path $\tilde{e}$ in H *ends in* C if its endpoint is contained in C in the sense of Eq. (5). Two paths $\tilde{e}_1$ and $\tilde{e}_2$ starting at e_0 and ending in C will be called equivalent w.r.t. C iff the path $\tilde{e}_2*\tilde{e}_1^{-1}$ (the inverse of $\tilde{e}_1$ followed by $\tilde{e}_2$) is fixed-endpoint homotopic to a path which is contained in C. Figure 1 illustrates this concept. By a *path of space-like cones* we shall understand a pair

$$(C,\tilde{e})\,, \tag{6}$$

where C is a space-like cone and $\tilde{e}$ is the equivalence class w.r.t. C of a path in H starting at e_0 and ending in C. (We use the same symbol for a path and its equivalence class.) We shall use the notation $\tilde{C}$ for a path of space-like cones of the form $(C,\tilde{e})$. Such paths of space-like cones serve to label the localization regions of charged fields. Namely, for each $\tilde{C}$ there is a linear subspace $\mathscr{F}_\chi(\tilde{C})$ of $\mathscr{F}_\chi$, called the fields carrying charge χ localized in $\tilde{C}$. This family is isotonous in the sense that

$$\mathscr{F}_\chi(\tilde{C}_1)\subset\mathscr{F}_\chi(\tilde{C}_2)\qquad\text{if}\qquad\tilde{C}_1\subset\tilde{C}_2. \tag{7}$$

(We say that $\tilde{C}_1\doteq(C_1,\tilde{e}_1)$ is contained in $\tilde{C}_2\doteq(C_2,\tilde{e}_2)$, in symbols

$$\tilde{C}_1\subset\tilde{C}_2\,, \tag{8}$$

if $C_1\subset C_2$ and the corresponding paths $\tilde{e}_1$, $\tilde{e}_2$ are equivalent w.r.t. C_2.) The vacuum Ω has the Reeh-Schlieder property for the fields, i.e. for any path of space-like cones $\tilde{C}$ holds

$$\big(\mathscr{F}_\chi(\tilde{C})\,\Omega\big)^{-}=\mathscr{H}_\chi, \tag{9}$$

where the bar denotes the closure.

Covariance There is a representation α_χ of the universal covering group $\tilde{P}_+^\uparrow$ of the Poincaré group $P_+^\uparrow$ by endomorphisms of $\mathscr{F}_\chi$, which implements the unitary representation U_χ in the sense that

$$\alpha_\chi(\tilde{g})(F)\,\Omega=U_\chi(\tilde{g})\,F\,\Omega \tag{10}$$

holds for all $\tilde{g}\in\tilde{P}_+^\uparrow$ and $F\in\mathscr{F}_\chi$. It acts covariantly on the fields in the following sense:

$$\alpha_\chi(\tilde{g}):\mathscr{F}_\chi(\tilde{C})\rightarrow\mathscr{F}_\chi(\tilde{g}\cdot\tilde{C}). \tag{11}$$

Fig. 2 $(C_1,\tilde{e}_1)$ and $(C_2,\tilde{e}_2)$ satisfy the hypothesis under which Eq. (15) holds

Here, $\tilde{g}\cdot\tilde{C}$ denotes the natural action of the universal covering of the Poincaré group on the paths of space-like cones, defined as follows. Let $\tilde{g}=(a,\tilde{\lambda})$, where a is a spacetime translation and $\tilde{\lambda}$ is an element of the universal covering group $\tilde{L}_+^\uparrow$ of the Lorentz group, projecting onto $\lambda\in L_+^\uparrow$. Then

$$\tilde{g}\cdot(C,\tilde{e}):=(g\cdot C,\tilde{\lambda}\cdot\tilde{e}), \tag{12}$$

where $\tilde{\lambda}\cdot\tilde{e}$ denotes the lift of the action of the Lorentz group on H to the respective universal covering spaces. Note that a 2π rotation acts non-trivially — it maps, for example, $(C,\tilde{e}_3)$ in Fig. 1 onto $(C,\tilde{e}_1)$.

Conjugate Charge There is a sector with the conjugate charge $\bar{\chi}$, for which all of the above-mentioned facts also hold. We shall denote the corresponding objects by $\mathscr{H}_{\bar{\chi}}$, $U_{\bar{\chi}}$, $\mathscr{F}_{\bar{\chi}}(\tilde{C})$, and $\alpha_{\bar{\chi}}$, respectively. In particular, $\mathscr{F}_{\bar{\chi}}(\tilde{C})$ is a linear space of operators mapping $\mathscr{H}_0$ onto $\mathscr{H}_{\bar{\chi}}$. There is a notion of operator adjoint, which associates with each field $F\in\mathscr{F}_\chi$ an adjoint field operator $F^\dagger\in\mathscr{F}_{\bar{\chi}}$, satisfying $(F^\dagger)^\dagger=F$ and preserving localization, i.e.

$$\big(\mathscr{F}_\chi(\tilde{C})\big)^\dagger=\mathscr{F}_{\bar{\chi}}(\tilde{C}). \tag{13}$$

The operation of adjoining intertwines the representations α_χ and $\alpha_{\bar{\chi}}$ in the sense that

$$\big(\alpha_\chi(\tilde{g})(F)\big)^\dagger=\alpha_{\bar{\chi}}(\tilde{g})(F^\dagger). \tag{14}$$

Statistics There is a complex number ω_χ of modulus one, the statistics phase of the sector χ, which (partly) characterizes the statistics of fields. Namely, suppose $\tilde{C}_1=(C_1,\tilde{e}_1)$ and $\tilde{C}_2=(C_2,\tilde{e}_2)$ are such that C_1 and C_2 are causally separated, and the path $\tilde{e}_1*\tilde{e}_2^{-1}$ goes "directly" from C_2 to C_1 in the mathematically positive sense.[5] (Note that this condition is independent of the choice of reference direction e_0. Figure 2 shows an example satisfying these conditions.) Then for $F_i\in\mathscr{F}_\chi(\tilde{C}_i)$, $i=1,2$, there holds

$$(F_2\Omega,F_1\Omega)=\omega_\chi\left(F_1^\dagger\Omega,F_2^\dagger\Omega\right). \tag{15}$$

Note that the hypothesis under which Eq. (15) holds is not symmetric in $\tilde{C}_1$ and $\tilde{C}_2$ just because of the condition on the paths $\tilde{e}_i$. Without this condition, Eq. (15) would imply $\omega_\chi\omega_{\bar{\chi}}=1$. But ω_χ and $\omega_{\bar{\chi}}$ are known to coincide (11), hence Eq. (15) would be be self-consistent only for $\omega_\chi=\pm1$, excluding braid group statistics.

[5] "Directly" means that it stays causally separated from the cone C_2 once it has left it; and "mathematically positive sense" means here the right-handed sense w.r.t. a future pointing time-like Minkowski vector.

Assumptions on the Particle Spectrum We consider a particle of strictly positive mass m and spin s in the sector χ, and assume that $\{m\}$ is separated from the rest of the mass spectrum in the sector χ by a mass gap. We further assume that there are only finitely many "particle types" in the sector χ with this mass, and that they all have the same spin s. More technically, let P_χ be the energy-momentum operator in the sector χ, i.e. the vector operator which generates the spacetime translations in the sense that $U_\chi(a) = \exp(ia \cdot P_\chi)$ for $a \in \mathbb{R}^3$, and let $M_\chi := P_\chi^2$ be the mass operator in the sector χ. This operator has as an eigenvalue the mass, m, of our particle. Our assumptions then are:

(A1) The mass m is strictly positive.
(A2) m is an isolated point in the spectrum of M_χ.
(A3) The restriction of the representation U_χ to the corresponding eigenspace is a finite multiple of the irreducible representation with mass m and spin s.

It is gratifying that the assumptions (A1) and (A2), together with the standard assumptions on the observables plus weak duality, imply the validity of our entire framework. In particular, they imply that the charge χ is localizable in space-like cones (3) and allow for the determination of the statistics phase ω_χ (namely, they exclude the so-called infinite statistics, $\lambda = 0$ (6)).

Results Under the above assumptions (A1) through (A3), we shall prove that the strong spin-statistics relation (2) holds in the case of braid group statistics (Theorem 1). As a byproduct, we prove that the familiar symmetry between particles and antiparticles holds also in this case. Namely, it is known that the mass spectrum of the conjugate sector $\bar{\chi}$ coincides with that of χ (6), and that the spins occurring in the eigenspace corresponding to mass m in the sector χ coincide with those in the conjugate sector $\bar{\chi}$ modulo one (11). What we show is that the spins actually coincide as real numbers, and that the degeneracies in the conjugate sectors $\chi, \bar{\chi}$ coincide — in other words, that the corresponding ray representations of the Poincaré group are unitarily equivalent (Proposition 1).

3 Momentum Space Two-Point Functions and Particle-Antiparticle Symmetry

Buchholz and Epstein's proof of the spin-statistics theorem in four dimensions relies on their result on the two-point functions in momentum space (2). The latter result extends straightforwardly to the present three-dimensional case, because it has been derived under precisely our conditions of covariance (11), the Reeh-Schlieder property (18), commutation relations as in Eq. (15) and a mass gap around $m > 0$, without referring to the representation of the Lorentz subgroup (which makes the crucial difference between three and four dimensions).

To state their result, some notation needs to be introduced. Fixing a Lorentz frame, spacetime points are written as $x = (x^0, x)$, and the Minkowski scalar product reads $(x^0, x) \cdot (y^0, y) = x^0 y^0 - x \cdot y$, where $x \cdot y$ denotes the standard scalar product in $\mathbb{R}^2$. The positive and negative mass shells $H_m^\pm$ are the set of momentum space points $p = (p_0, p)$ satisfying $p_0^2 - p \cdot p = m^2$ and $p_0 \gtrless 0$, respectively. The unique (up to a factor) Lorentz invariant measure on H_m^+ is denoted by $d\mu(p)$. The complexified mass shell H_m^c is defined as the set of $k = (k_0, k_1, k_2) \in \mathbb{C}^3$ satisfying

$k_0^2 - k_1^2 - k_2^2 = m^2$. Buchholz and Epstein consider a special class of space-like cones, namely, those of the form

$$C = C'', \tag{16}$$

where C is an open, salient cone with apex at the origin in the rest frame (which we shall occasionally identify with $\mathbb{R}^2$), and C'' denotes its causal completion. For a cone C of this form, let its dual C^* be defined by

$$C^* := \left\{ p \in \mathbb{R}^2 : p \cdot x > 0 \, \forall x \in C^{-} \setminus \{0\} \right\}. \tag{17}$$

Buchholz and Epstein use regularized fields, for which the functions $\tilde{g} \mapsto \alpha_\chi(\tilde{g})(F)$ are smooth. The set of smooth fields carrying charge χ and localized in $\tilde{C}$ shall be denoted by $\mathscr{F}_\chi^\infty(\tilde{C})$. The Reeh-Schlieder property (9) still holds for the smooth fields, also on the single particle space. More precisely, let $E_\chi^{(1)}$ be the spectral projector of the mass operator corresponding to the eigenvalue m, and let $\mathscr{H}_\chi^{(1)}$ be its range, i.e. the corresponding eigenspace. Then there holds

$$\left(E_\chi^{(1)} \mathscr{F}_\chi^\infty(\tilde{C}) \, \Omega \right)^{-} = \mathscr{H}_\chi^{(1)}. \tag{18}$$

The result of Buchholz and Epstein on the two-point functions, in the present context, is the following:

Lemma 1 (Buchholz, Epstein) *Let C_1 and C_2 be causally separated space-like cones of the form* (16) *such that $C_{12} := C_2 - C_1$ is a salient cone, and let $\tilde{C}_1, \tilde{C}_2$ be such that the hypothesis of Eq.* (15) *is satisfied. Then for any pair of fields $F_i \in \mathscr{F}_\chi^\infty(\tilde{C}_i)$, $i = 1,2$, there exists a function h which is analytic in the region*

$$\Gamma := \{ k = (k_0, k) \in H_m^c : Im \, k \in (C_{12})^* \} \tag{19}$$

and has smooth boundary values on the mass shells $H_m^\pm$ satisfying

$$\left(F_2 \Omega, U_\chi(x) E_\chi^{(1)} F_1 \Omega \right) = \int_{H_m^+} d\mu(p) \, h(p) \, e^{ip\cdot x}, \tag{20}$$

$$\omega_\chi \left(F_1^\dagger \Omega, U_{\bar{\chi}}(x) E_{\bar{\chi}}^{(1)} F_2^\dagger \Omega \right) = \int_{H_m^+} d\mu(p) \, h(-p) \, e^{ip\cdot x}. \tag{21}$$

Proof Replacing the factor "sign λ" in Eq. (2.2) of (2) by our ω_χ, Buchholz and Epstein's proof can be directly transferred to the present setting, since it uses only the conditions of covariance (11), space-like commutation relations (15), Reeh-Schlieder property (18) and a mass gap around $m > 0$. □

The lemma immediately implies the existence of antiparticles with the same mass m as the particles in the sector χ (which had been established in this generality already in (6)). Moreover, it implies a complete symmetry between particles and antiparticles, valid also in the present case of braid group statistics in three dimensions:

Proposition 1 (Particle-Antiparticle Symmetry) *The spins and multiplicities of the single particle spaces $\mathscr{H}_\chi^{(1)}$ and $\mathscr{H}_{\bar\chi}^{(1)}$ coincide. In particular, the restriction to $\mathscr{H}_{\bar\chi}^{(1)}$ of the representation $U_{\bar\chi}$ is equivalent with the restriction to $\mathscr{H}_\chi^{(1)}$ of the representation U_χ.*

Proof The proof requires only a slight modification from that of Buchholz and Epstein. Namely, the role of the square of the Pauli-Lubanski vector as a Casimir operator is, in $2+1$ dimensions, played by a scalar operator, the so-called Pauli-Lubanski scalar (1; 13) which is defined as follows. Let U be a representation of the universal covering of the Poincaré group in three spacetime dimensions, let L_0 denote the generator of the rotation subgroup in the representation U, and let L_i be the generator of the boosts in direction x^i, $i=1,2$. Let further J_μ be the vector operator $J_\mu = (-L_0, L_2, -L_1)$. The Pauli-Lubanski scalar of the representation U is defined as

$$W := J_\mu P^\mu, \tag{22}$$

where P^μ are the generators of the translation subgroup in the representation U. It has the following properties (1; 13): it commutes with the representation U, and has the value

$$W = -ms\mathbb{1} \tag{23}$$

if, and only if, U contains only irreducible representations whose masses and spins have the product value ms. Considering now the representations U_χ and $U_{\bar\chi}$, we denote their Pauli-Lubanski scalars as W_χ and $W_{\bar\chi}$, respectively. The key point is that for each field $F \in \mathscr{F}_\chi^\infty(\tilde C)$, there is a field $\delta_\chi(F) \in \mathscr{F}_\chi^\infty(\tilde C)$ such that, due to covariance (10), there holds[6]

$$W_\chi F\Omega = \delta_\chi(F)\Omega. \tag{24}$$

The same holds for the conjugate sector $\bar\chi$. Let now, for $i=1,2$, $\tilde C_i$ and $F_i \in \mathscr{F}_\chi^\infty(\tilde C_i)$ satisfy the hypothesis of Lemma 1. Then

$$E_\chi^{(1)}\left(\delta_\chi(F_1) + msF_1\right)\Omega = E_\chi^{(1)}(W_\chi + ms\mathbb{1})F_1\Omega = 0$$

by Eq. (23). Lemma 1 and the Reeh-Schlieder property (18) then imply that also

$$E_{\bar\chi}^{(1)}\Big((\delta_\chi(F_2))^\dagger + msF_2^\dagger\Big)\Omega = 0. \tag{25}$$

[6] Namely, δ_χ is the "derivation" on $\mathscr{F}_\chi^\infty$ defined by

$$\delta_\chi(F) := -\frac{d}{ds}\frac{d}{dt}\sum_{\mu=0}^{2}\alpha_\chi\left(\tilde\lambda^{(\mu)}(t)\,T(se_{(\mu)})\right)(F)\,\big|_{s=t=0},$$

where $T(\cdot)$ is the translation subgroup, $e_{(\mu)}$ are the unit vectors in the given Lorentz frame, $\tilde\lambda^{(0)}(-t)$ is the rotation subgroup, $\tilde\lambda^{(1)}(t)$ is the boost subgroup in direction $e_{(2)}$ and $\tilde\lambda^{(2)}(-t)$ is the boost subgroup in direction $e_{(1)}$.

But by Eq. (14), the adjoint of $\delta_\chi(F_2)$ is $\delta_{\bar\chi}(F_2^\dagger)$, and therefore $(\delta_\chi(F_2))^\dagger\Omega = W_{\bar\chi}F_2^\dagger\Omega$. Then Eq. (25) reads

$$E_{\bar\chi}^{(1)}(W_{\bar\chi} + ms\mathbb{1})F_2^\dagger\Omega = 0.$$

This shows that only spin s occurs in the single particle space $\mathscr{H}_{\bar\chi}^{(1)}$, as claimed. The proof of the claim that not only the spin, but also the multiplicity n coincides then proceeds precisely as in (2). □

4 The Spin-Statistics Theorem

We now prove the spin-statistics theorem. Our line of reasoning parallels that of Buchholz and Epstein (2), which uses heavily the representation of the covering group of the Poincaré group. Since this representation has completely different (analyticity) properties in three dimensions, the corresponding details have to be worked out differently in the present case.

By our assumption (A3), the representation $U_\chi|\mathscr{H}_\chi^{(1)}$ is equivalent to n copies of the irreducible representation of the universal covering group of the Poincaré group with mass $m > 0$ and spin $s \in \mathbb{R}$. Let us denote this representation by U. It acts on the Hilbert space $L^2(H_m^+, d\mu) \otimes \mathbb{C}^n$, elements of which are functions ("wave functions")

$$\psi : H_m^+ \times \{1,\dots,n\} \to \mathbb{C}, \quad (p,\alpha) \mapsto \psi(p,\alpha)$$

with finite norm w.r.t. the scalar product

$$(\psi,\phi) = \int_{H_m^+} d\mu(p) \sum_{\alpha=1}^{n} \overline{\psi(p,\alpha)}\, \phi(p,\alpha).$$

The representation U acts in this space as

$$\Big(U(a,\tilde\lambda)\psi\Big)(p,\alpha) = e^{is\Omega(\tilde\lambda,p)}\, e^{ia\cdot p}\, \psi(\lambda^{-1}p,\alpha), \tag{26}$$

where λ is the Lorentz transformation onto which $\tilde\lambda$ projects, and $\Omega(\tilde\lambda,p) \in \mathbb{R}$ is the Wigner rotation. The latter satisfies the so-called cocycle identities

$$\Omega(1,p) = 1, \qquad \Omega(\tilde\lambda\tilde\lambda',p) = \Omega(\tilde\lambda,p) + \Omega(\tilde\lambda',\lambda^{-1}p), \tag{27}$$

and for the subgroup $\tilde r(\cdot)$ of rotations (which is not isomorphic to $SO(2)$ but to $\mathbb{R}$) holds

$$\Omega\,(\tilde r(\omega),p) = \omega \quad \text{ for all } \omega \in \mathbb{R},\, p \in H_m^+. \tag{28}$$

By Proposition 1, $U_{\bar\chi}$ is also equivalent to this representation. Thus, there are isometric isomorphisms V_χ and $V_{\bar\chi}$ from $\mathscr{H}_\chi^{(1)}$ and $\mathscr{H}_{\bar\chi}^{(1)}$ onto $L^2(H_m^+, d\mu) \otimes \mathbb{C}^n$, which intertwine the representations $U_\chi|\mathscr{H}_\chi^{(1)}$ and $U_{\bar\chi}|\mathscr{H}_{\bar\chi}^{(1)}$, respectively, with U.

Following Buchholz and Epstein, we now fix two causally separated (paths of) space-like cones $\tilde{C}_1$, $\tilde{C}_2$ as in the hypothesis of Lemma 1, and pick n smooth field operators localized in either one of these cones, $F_{i,\beta} \in \mathscr{F}^\infty_\chi(\tilde{C}_i)$, $\beta = 1,\dots,n$. We then consider, for $i = 1, 2$, the wave functions

$$\psi_{i,\beta} := V_\chi E^{(1)}_\chi F_{i,\beta}\Omega \qquad \text{and} \qquad \psi^c_{i,\beta} := V_{\bar\chi} E^{(1)}_{\bar\chi} F^\dagger_{i,\beta}\Omega \tag{29}$$

in $L^2(H^+_m, d\mu)\otimes\mathbb{C}^n$, and complex $n\times n$ matrices $\Psi_i(p)$ and $\Psi^c_i(p)$ defined by

$$\Psi_i(p)_{\alpha\beta} := \psi_{i,\beta}(p,\alpha) \qquad \text{and} \qquad \Psi^c_i(p)_{\alpha\beta} := \psi^c_{i,\beta}(p,\alpha) \tag{30}$$

for $p\in H^+_m$. We assume that the matrices $\Psi_i(p)$ are invertible for p in some open set on the mass shell. (This is possible due to the Reeh-Schlieder property.) Lemma 1 asserts that for each pair α,β there is a smooth function $h_{\alpha\beta}$, analytic in Γ, such that

$$h_{\alpha\beta}(p) = \sum_{\gamma=1}^n \overline{\psi_{2,\alpha}(p,\gamma)}\,\psi_{1,\beta}(p,\gamma) \equiv (\Psi_2(p)^*\,\Psi_1(p))_{\alpha\beta}\,,$$

$$h_{\alpha\beta}(-p) = \omega_\chi \sum_{\gamma=1}^n \overline{\psi^c_{1,\beta}(p,\gamma)}\,\psi^c_{2,\alpha}(p,\gamma) \equiv \omega_\chi\ (\Psi^c_1(p)^*\,\Psi^c_2(p))_{\beta\alpha}\,,$$

where the star * denotes the matrix adjoint. (Note that this implies that the matrices $\Psi_i(p)$ and $\Psi^c_i(p)$ are invertible for almost all p.) In other words, by Lemma 1 the smooth matrix valued function on the mass shell

$$p \mapsto \Psi_2(p)^*\,\Psi_1(p) =: M(p) \tag{31}$$

has an analytic extension into the subset Γ of the complexified mass shell described in (19), with smooth boundary value on the negative mass shell given by[7]

$$M(-p) = \omega_\chi\ (\Psi^c_1(p)^*\,\Psi^c_2(p))^T\,, \tag{32}$$

where the superscript T denotes matrix transposition. Buchholz and Epstein now proceed to show that, in the case of Bosons and Fermions, the wave function matrices $\Psi_1(p)$ and $\Psi_2(p)^*$ separately have analytic extensions. This is not so in the present case. However, we show that their transforms under certain boosts behave analytically in the boost variable, which exhibits the underlying modular covariance and is sufficient for our purpose.

Let us recall the relevant geometric notions. We denote the one-parameter group of boosts in 1-direction by $\lambda_1(\cdot)$, acting in p-space as

$$\lambda_1(t) = \begin{pmatrix} \cosh(t) & \sinh(t) & 0 \\ \sinh(t) & \cosh(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{33}$$

This matrix-valued function has an analytic extension into $\mathbb{C}$ satisfying (12)

$$\lambda_1(t+i\theta) = (\,j(\theta) + i\sin(\theta)\,\sigma)\,\lambda_1(t), \tag{34}$$

[7] The letter p shall be reserved for points on the positive mass shell, so $-p$ is on the negative mass shell.

where $j(\theta) = \mathrm{diag}(\cos\theta, \cos\theta, 1)$ and σ maps (p_0, p_1, p_2) to $(p_1, p_0, 0)$. In particular,

$$\lambda_1(\pm i\pi) = j, \tag{35}$$

where $j \equiv \mathrm{diag}(-1,-1,1)$ acts as the reflection of p_0 and p_1, leaving p_2 unchanged. Note that j maps H_m^+ onto H_m^- and satisfies $j^2 = \mathbb{1}$.

From now on we shall suppose that the dual of the "difference cone" C_{12} contains the negative 1-axis, that is:

$$\mathbb{R}^- \times \{0\} \subset (C_{12})^* . \tag{36}$$

In this case, for any $p \in H_m^+$ and any z in the strip

$$G := \mathbb{R} + i(0,\pi), \tag{37}$$

the point $\lambda_1(-z)p$ is in the subset Γ of the complexified mass shell described in Lemma 1. (This is so because its imaginary part is the image under σ of a point in the past cone, hence of the form (q_0, q) with $q \in \mathbb{R}^- \times \{0\}$.) Hence by Lemma 1 and Eq. (31), for fixed $p \in H_m^+$ the smooth matrix-valued function

$$t \mapsto \Psi_2(\lambda_1(-t)p)^* \Psi_1(\lambda_1(-t)p) \equiv M(\lambda_1(-t)p) \tag{38}$$

has an analytic extension into the strip G, and, by Eqs. (32) and (35), its boundary value at $t = i\pi$ is

$$M(\lambda_1(-t)p)|_{t=i\pi} \equiv M(jp) = \omega_\chi \left(\Psi_1^c(-jp)^* \Psi_2^c(-jp)\right)^T . \tag{39}$$

We need analyticity of Ψ_1 and Ψ_2 separately. However, it turns out that it is not $\Psi_i(\lambda_1(-t)p)$ which is analytic, but rather the matrices $\Psi_i(t;p)$, $i = 1,2$, defined by

$$\Psi_i(t;p)_{\alpha\beta} := \left(U(\tilde\lambda_1(t))\psi_{i,\beta}\right)(p,\alpha) \tag{40}$$

$$\equiv e^{is\Omega(\tilde\lambda_1(t),p)}\Psi_i(\lambda_1(-t)p)_{\alpha\beta}. \tag{41}$$

Here, $\tilde\lambda_1(\cdot)$ denotes the unique lift to $\tilde L_+^\uparrow$ of the one-parameter group $\lambda_1(\cdot)$. The Wigner rotation factor in the last equation is independent of α, β and i, and therefore cancels in Eq. (38). Hence Eq. (38) implies that

$$t \mapsto \Psi_2(t;p)^* \Psi_1(t;p) \equiv M(\lambda_1(-t)p) \tag{42}$$

has an analytic extension into the strip G with boundary value given by Eq. (39).

Lemma 2 *For any $p \in H_m^+$, the smooth matrix-valued functions $t \mapsto \Psi_1(t;p)$ and $t \mapsto \Psi_2(t;p)^*$ extend to analytic functions on the strip G with smooth boundary values at the upper boundary $\mathbb{R} + i\pi$.*

(Note that $\Psi_2(t;p)$ is analytically continued *after* conjugation.)

Proof The proof uses the same reasoning as (2, Sect. 3). Let us denote, for brevity, $f_1(t) := \Psi_1(t;p)$, $f_2(t) := \Psi_2(t;p)^*$ and $h(t) := M(\lambda_1(-t)p)$. We know, by Eq. (42), that $t \mapsto f_2(t)f_1(t) \equiv h(t)$ has an analytic extension into the strip G. The equation

$$f_1(t+t_0)_{\alpha\beta} = \Big(U(\tilde{\lambda}_1(t)) V_\chi E_\chi^{(1)} \alpha_\chi(\tilde{\lambda}_1(t_0))(F_{1,\beta})\Omega \Big)(p,\alpha)$$

shows that $f_1(t+t_0)$ is of the same form as $f_1(t)$, with $F_{1,\beta}$ substituted by $\alpha_\chi(\tilde{\lambda}_1(t_0))(F_{1,\beta})$. Now for t_0 sufficiently small, $\tilde{\lambda}_1(t_0)\cdot\tilde{C}_1$ still satisfies (together with $\tilde{C}_2$) the hypothesis of Lemma 1 and condition (36). Hence, the same reasoning as above shows that there is a matrix-valued function $h_{t_0}(t)$ analytically extendible in t into the strip G, such that

$$f_2(t)f_1(t+t_0) = h_{t_0}(t) \tag{43}$$

for t_0 sufficiently small. Smoothness of $F_{1,\beta}$ implies that f_1 is smooth and that $h_{t_0}(t)$ is smooth in t_0. The above equation implies that

$$f_2(t)\,\frac{\mathrm{d}}{\mathrm{d}t}f_1(t) = \hat{h}(t) := \frac{\mathrm{d}}{\mathrm{d}t_0}h_{t_0}(t)\big|_{t_0=0}. \tag{44}$$

The last two equations imply the following differential equation for f_1:

$$f_1(t)^{-1}\,\frac{\mathrm{d}}{\mathrm{d}t}f_1(t) = h(t)^{-1}\,\hat{h}(t). \tag{45}$$

The right-hand side is meromorphic in the strip G and continuous on its closure G^- (up to isolated points). Hence f_1 can be integrated along any path γ in G^- starting from the real (=lower) boundary, as long as the path does not cross zeroes of the determinant of $h(z)$, yielding an analytic extension $f_{1,\gamma}$ along γ. If γ crosses a zero z_0 of det $h(z)$, we make use of the following observation: Eq. (43) implies the relation

$$f_1(t) = f_1(t+t_0)\,h(t+t_0)^{-1}\,h_{-t_0}(t+t_0), \tag{46}$$

which extends from real t to values in the strip G, along the path γ. Since the zeroes of det $h(z)$ are isolated, the determinant of $h(z_0+t_0)$ is non-zero for t_0 sufficiently small. Thus, the function $f_{1,\gamma}$ can be continuously (and hence analytically) continued into z_0 by the (analytic extension of the) above equation. Hence, f_1 extends analytically along any path into the strip. But the latter is simply connected, hence the analyic extensions are independent of the paths, proving the claimed analyticity of $t \mapsto \Psi_1(t;p)$. Smoothness of the boundary value at $\mathbb{R}+i\pi$ follows from Eq. (46). Analyticity of $\Psi_2(t;p)^* \equiv f_2(t)$ is shown along the same lines. $\square$

Lemma 2 allows for the definition of "geometric Tomita operators" acting on the matrix-valued functions Ψ_1 and Ψ_2. Namely, we define for $p \in H_m^+$,

$$\hat{\Psi}_1(p) := \overline{\Psi_1(t;-jp)|_{t=i\pi}}, \quad \check{\Psi}_2(p) := \left(\overline{\Psi_2(t;-jp)}\right)\big|_{t=i\pi}, \tag{47}$$

where complex conjugation is understood componentwise. (Note that Ψ_1 is first analytically continued to $t = i\pi$ and then conjugated, while Ψ_2 is first conjugated and then continued.) We now have

$$\hat{\Psi}_1(p)^*\,\check{\Psi}_2(p) = \left\{\Psi_2(t;-jp)^*\Psi_1(t;-jp)\right\}^T\big|_{t=i\pi}$$

by definition. But the function in curly brackets coincides, by Eq. (42), with $M(-\lambda_1(-t)$
$jp)$ whose analytic continuation into $t = i\pi$ is $M(-p)$ by Eq. (35). Using Eq. (39), we therefore have

$$\hat{\Psi}_1(p)^*\,\check{\Psi}_2(p) = \omega_\chi\,\Psi_1^c(p)^*\,\Psi_2^c(p). \tag{48}$$

We want to find a relation between $\hat{\Psi}_1$ and Ψ_1^c, constituting a Bisognano-Wichmann property on the single particle level (Proposition 2). The proof of this relation relies on the fact that the matrix-valued function $\hat{\Psi}_1$ transforms under Lorentz transformations (close to unity) just like Ψ_1 (Lemma 3). The proof of this transformation behaviour is the crucial and difficult point in our analysis, since the Wigner rotation factor spoils the analyticity needed for the definition of $\hat{\Psi}_1$. Observe that for $\lambda \in L_+^\uparrow$ sufficiently small, λC_1 is contained in a space-like cone of the form $(C_1^\lambda)''$, which satisfies, together with C_2, the hypothesis of Lemma 1 and the condition (36), $\mathbb{R}^- \times \{0\} \subset (C_2 - C_1^\lambda)^*$. Let $\mathscr{U}_{12}$ be a neighbourhood of the identity in $L_+^\uparrow$ consisting of such λ. The set of $\tilde{\lambda} \in \tilde{L}_+^\uparrow$ which project onto $\mathscr{U}_{12}$ has an infinity of connected components, differing by 2π-rotations. Let now $\tilde{\mathscr{U}}_{12}$ be the one containing the identity. This ensures that for $\tilde{\lambda} \in \tilde{\mathscr{U}}_{12}$, the paths $\tilde{\lambda} \cdot \tilde{C}_1$ and $\tilde{C}_2$ have the correct relative winding number so as to satisfy the hypothesis of Eq. (15). Then, for $\tilde{\lambda} \in \tilde{\mathscr{U}}_{12}$, the wave function[8]

$$\psi_{1,\beta}^\lambda := U(\tilde{\lambda})\psi_{1,\beta} \equiv V_\chi E_\chi^{(1)} \alpha_\chi(\tilde{\lambda})(F_{1,\beta})\,\Omega \tag{49}$$

is of the same form as $\psi_{1,\beta}$, with $F_{1,\beta}$ substituted by $\alpha_\chi(\tilde{\lambda})(F_{1,\beta})$, and Lemma 2 applies, asserting that the matrix-valued function

$$t \mapsto \Psi_1^\lambda(t;p)_{\alpha\beta} := \Big(U(\tilde{\lambda}_1(t))\psi_{1,\beta}^\lambda\Big)(p,\alpha)$$

has an analytic extension into G, with continuous boundary value at $\mathbb{R} + i\pi$. This allows for the definition of

$$\widehat{\Psi_1^\lambda}(p) := \overline{\Psi_1^\lambda(t;-jp)|_{t=i\pi}}, \tag{50}$$

in analogy with Eq. (47).

Lemma 3 *There is a neighbourhood $\tilde{\mathscr{U}}$ of the unit in $\tilde{L}_+^\uparrow$ such that for all $\tilde{\lambda} \in \tilde{\mathscr{U}}$ and $p \in H_m^+$ there holds*

$$\widehat{\Psi_1^\lambda}(p) = e^{is\Omega(\tilde{\lambda},p)}\,\hat{\Psi}_1(\lambda^{-1}p)\,. \tag{51}$$

Proof The claimed equation is equivalent with

$$e^{is\Omega(\tilde{\lambda}_1(t)\tilde{\lambda},-jp)}\,\psi_{1,\beta}(-\lambda^{-1}\lambda_1(-t)jp,\alpha)\big|_{t=i\pi}$$

$$= e^{-is\Omega(\tilde{\lambda},p)}\,e^{is\Omega(\tilde{\lambda}_1(t),-j\lambda^{-1}p)}\,\psi_{1,\beta}(-\lambda_1(-t)j\lambda^{-1}p,\alpha)\big|_{t=i\pi}. \tag{52}$$

Now the function $t \mapsto e^{is\Omega(\tilde{\lambda}_1(t)\tilde{\lambda},q)}$ has branch points in the strip G, see Lemma C.1 of (17). Hence none of the (t-dependent) factors in the above equation possesses an analytic extension into the strip by its own. However, we have constructed in (17) a function living on the mass shell which compensates the singularities of the Wigner rotation factor. In Appendix B, we adopt the results of (17) to the present situation, leading to the following assertion (cf. Lemma B.2). Let

$$u_{\frac{\pi}{2}}(p) := e^{is\frac{\pi}{2}}\left(\frac{p_0-p_2}{m}\cdot\frac{p_0-p_2+m+ip_1}{p_0-p_2+m-ip_1}\right)^s \quad \text{and} \tag{53}$$

$$\omega(\tilde{\lambda},p) := e^{is\Omega(\tilde{\lambda},p)}\,u_{\frac{\pi}{2}}(\lambda^{-1}p). \tag{54}$$

[8] We use a superscript λ instead of $\tilde{\lambda}$, which causes no confusion since we have a one-to-one correspondence between $\mathscr{U}_{12}$ and $\tilde{\mathscr{U}}_{12}$.

Then the function $t \mapsto \omega(\tilde\lambda_1(t)\tilde\lambda, p)$ has an analytic extension into the strip G for all $\tilde\lambda$ in a neighbourhood $\tilde{\mathscr{U}}_0$ of the unit. Further, at $t = i\pi$ it has the boundary value

$$\omega(\tilde\lambda_1(i\pi)\tilde\lambda, p) = e^{i\pi s}\, e^{is\Omega(j\tilde\lambda\tilde\lambda_0 j,p)}\, u\left(j(\lambda\lambda_0)^{-1}jp\right), \text{ where} \tag{55}$$

$$u(p) := \left(\frac{p_0-p_1}{m}\cdot\frac{p_0-p_1+m-ip_2}{p_0-p_1+m+ip_2}\right)^s. \tag{56}$$

Here, $\lambda_0 := r(\pi/2)$ is the rotation about $\pi/2$, and $\tilde\lambda_0 := \tilde r(\pi/2)$, where $\tilde r(\cdot)$ is the unique lift to $\tilde L_+^\uparrow$ of the one-parameter group of rotations. Further, $\tilde\lambda \mapsto j\tilde\lambda j$ is the unique lift (23) of the adjoint action of j on $L_+^\uparrow$ to an automorphism of the universal covering group. To apply this result, we rewrite the claimed Eq. (52) as follows:

$$\begin{aligned}&\omega(\tilde\lambda_1(t)\tilde\lambda, -jp)\cdot\phi(-\lambda^{-1}\lambda_1(-t)jp)\,\big|_{t=i\pi}\\ &\quad= \left(e^{-is\Omega(\tilde\lambda,p)}\,\omega(\tilde\lambda_1(t), -j\lambda^{-1}p)\right)\cdot\phi(-\lambda_1(-t)j\lambda^{-1}p)\,\big|_{t=i\pi},\end{aligned} \tag{57}$$

where

$$\phi(p) := u_{\frac{\pi}{2}}(p)^{-1}\,\psi_{1,\beta}(p,\alpha). \tag{58}$$

Lemma B.2 then asserts that for $\tilde\lambda \in \tilde{\mathscr{U}}_0$ the first factor $\omega(\tilde\lambda_1(t)\tilde\lambda, -jp)$ on the left hand side of Eq. (57) is analytic in G and has the boundary value

$$e^{i\pi s}\, e^{-is\Omega(\tilde\lambda\tilde\lambda_0,p)}\, u\left(-j(\lambda\lambda_0)^{-1}p\right) \tag{59}$$

at $t = i\pi$. (Here we have used that the Wigner rotation satisfies the identity

$$\Omega(j\tilde\lambda j, p) = -\Omega(\tilde\lambda, -jp), \tag{60}$$

see (17, Lemma B.2).) Similarly, the first factor $e^{-is\Omega(\tilde\lambda,p)}\,\omega(\tilde\lambda_1(t), -j\lambda^{-1}p)$ on the right hand side of Eq. (57) is analytic, with boundary value

$$e^{i\pi s}\, e^{-is\Omega(\tilde\lambda,p)}\, e^{-is\Omega(\tilde\lambda_0,\lambda^{-1}p)}\, u\left(-j(\lambda\lambda_0)^{-1}p\right) \tag{61}$$

at $t = i\pi$. Due to the cocycle identity (27), this coincides with the boundary value (59) of the first factor on the left hand side of Eq. (57).

We now know that for any $\tilde\lambda \in \tilde{\mathscr{U}} := \tilde{\mathscr{U}}_0 \cap \tilde{\mathscr{U}}_{12}$ both sides of Eq. (57) are analytic in the strip G, and the same holds for the first factor on each side. Further, we know that the boundary values at $t = i\pi$ of the first factors coincide. It follows that the second factors, namely the functions

$$f_1(t) = \phi(-\lambda^{-1}\lambda_1(-t)jp) \qquad \text{and} \qquad f_2(t) = \phi(-\lambda_1(-t)j\lambda^{-1}p), \tag{62}$$

also have an analytic extension into the strip. It only remains to show that their boundary values at $t = i\pi$ coincide. To this end, note that the analyticity of the two functions (62) holds for all $p \in H_m^+$ and λ in the projection of $\tilde{\mathscr{U}}$ onto $L_+^\uparrow$, which we shall denote by $\mathscr{U}$. Hence we can analytically continue the function ϕ into the subset

$$\Gamma_0 := \{\lambda \lambda_1(z) p : \; p \in H_m^+, z \in G, \lambda \in \mathscr{U}\}$$

of the complexified mass shell H_m^c along paths of the form $\lambda\lambda_1(z(t))p$. Now a straightforward calculation shows that every $k = \lambda\lambda_1(z)p \in \Gamma_0$ can be uniquely written in the form $k = r\lambda_1(i\theta)r^{-1}q$, where r is a rotation, $\theta \in (0,\pi)$ and $q \in H_m^+$. By restricting λ to a smaller neighbourhood if necessary, one can achieve $r \in \mathscr{U}$. Letting θ go to zero then defines a deformation retraction of Γ_0 onto the mass hyperboloid. Hence Γ_0 is simply connected, which implies that our analytic continuation of ϕ is path-independent, yielding an analytic function $\hat{\phi}$ on Γ_0, continuous at the real boundary H_m^-, such that $f_1(z) = \hat{\phi}(-\lambda^{-1}\lambda_1(-z)jp)$ and $f_2(z) = \hat{\phi}(-\lambda_1(-z)j\lambda^{-1}p)$. But the points $-\lambda^{-1}\lambda_1(-i\pi)jp$ and $-\lambda_1(-i\pi)j\lambda^{-1}p$ coincide, namely with $-\lambda^{-1}p$, hence $f_1(i\pi) = f_2(i\pi)$. This completes the proof. □

Proposition 2 *The following "Bisognano-Wichmann property" holds: There is a regular $n \times n$ matrix D such that for all $p \in H_m^+$ there holds*

$$\hat{\Psi}_1(p) = D\,\Psi_1^c(p). \tag{63}$$

It will become clear in the proof of Theorem 1 that D is isometric.

Proof The proof goes again along the lines of (2), but uses our Lemma 3. Let p be in the dense set of points satisfying $\det \Psi_1^c(p) \neq 0$, and let $D(p)$ be the matrix

$$D(p) := \hat{\Psi}_1(p)\,\Psi_1^c(p)^{-1}.$$

Due to Eq. (48), $D(p)$ is independent of the specific choice of operators $F_{1,\beta}$ from which $\hat{\Psi}_1(p)$ and $\Psi_1^c(p)$ are constructed. In particular, for $\tilde{\lambda} \in \tilde{\mathscr{U}}_{12}$, we may substitute $F_{1,\beta}$ by $\alpha_\chi(\tilde{\lambda})(F_{1,\beta})$ as in Eq. (49), yielding substitution of $\hat{\Psi}_1(p)$ by $\widehat{\Psi_1^\lambda}(p)$ and of $\Psi_1^c(p)_{\alpha\beta}$ by

$$\Psi_1^{\lambda,c}(p)_{\alpha\beta} := \Big(U(\tilde{\lambda})\, V_{\bar{\chi}} E_{\bar{\chi}}^{(1)} F_{1,\beta}^\dagger\, \Omega \Big)(p,\alpha).$$

Hence we have

$$D(p) = \widehat{\Psi_1^\lambda}(p)\,\Psi_1^{\lambda,c}(p)^{-1} = \hat{\Psi}_1(\lambda^{-1}p)\,\Psi_1^c(\lambda^{-1}p)^{-1} = D(\lambda^{-1}p).$$

(In the second equation we have used that, by Lemma 3, $\widehat{\Psi_1^\lambda}(p)$ and $\Psi_1^{\lambda,c}(p)$ have the same transformation dependence on $\tilde{\lambda}$, namely $\Psi_1^{\lambda,c}(p) = e^{is\Omega(\tilde{\lambda},p)}\Psi_1^c(\lambda^{-1}p)$ and Eq. (51).) This shows that $D(p)$ is locally constant, and, since p was arbitrary, constant.

□

As a corollary, we get a relation between $\check{\Psi}_2(p)$ and $\Psi_2^c(p)$.

Corollary 1 *For all $p \in H_m^+$ there holds*

$$\check{\Psi}_2(p) = e^{2\pi is} D\,\Psi_2^c(p). \tag{64}$$

Proof Let us choose our paths $\tilde{C}_1$ and $\tilde{C}_2$ so as to satisfy $\tilde{C}_1 = \tilde{r}(\pi)\cdot\tilde{C}_2$, where $\tilde{r}(\cdot)$ denotes the one-parameter group of rotations in $\tilde{L}_+^\uparrow$. (This is compatible with the hypothesis of Lemma 1.) Then the wave function

$$\psi_{2,\beta}^{\pi} := U(\tilde{r}(\pi))\,\psi_{2,\beta} \equiv V_\chi E_\chi^{(1)}\,\alpha_\chi(\tilde{r}(\pi))(F_{2,\beta})\,\Omega \tag{65}$$

is of the same form as $\psi_{1,\beta}$, with $F_{1,\beta}$ substituted by $\alpha_\chi(\tilde{r}(\pi))(F_{2,\beta})$. Hence, Lemma 2 allows for the analytic extension into $t = i\pi$,

$$\widehat{\Psi_2^\pi}(p)_{\alpha\beta} := \overline{\Big(U(\tilde{\lambda}_1(t))\,\psi_{2,\beta}^{\pi}\Big)\,(-jp,\alpha)|_{t=i\pi}}\;.$$

Now the group relation $\tilde{\lambda}_1(t)\tilde{r}(\pi) = \tilde{r}(\pi)\tilde{\lambda}_1(-t)$ implies that

$$\begin{aligned}\widehat{\Psi_2^\pi}(p)_{\alpha\beta} &= \overline{\Big(U(\tilde{r}(\pi))U(\tilde{\lambda}_1(-t))\,\psi_{2,\beta}\Big)\,(-jp,\alpha)|_{t=i\pi}} \\ &\equiv e^{-i\pi s}\,\overline{\Big(U(\tilde{\lambda}_1(-t))\,\psi_{2,\beta}\Big)\,(-r(-\pi)jp,\alpha)|_{t=i\pi}}\;.\end{aligned} \tag{66}$$

(In the last equation we have used relation (28).) The group relation $r(-\pi)j = jr(\pi)$ and the identity $\overline{f(-t)|_{t=i\pi}} = \bar{f}(t)|_{t=i\pi}$, holding for the analytic extension of a function $\bar{f}$, yield

$$\widehat{\Psi_2^\pi}(p) = e^{-is\pi}\,\check{\Psi}_2(r(\pi)p)\,. \tag{67}$$

On the other hand, Proposition 2 asserts that

$$\widehat{\Psi_2^\pi}(p) = D\,\Psi_2^{\pi,c}(p), \tag{68}$$

where $\Psi_2^{\pi,c}(p)$ is defined just as $\Psi_1^c(p)$ with $F_{1,\beta}^\dagger$ substituted by $\alpha_{\bar{\chi}}(\tilde{r}(\pi))(F_{2,\beta}^\dagger)$. But using Eq. (28) yields $\Psi_2^{\pi,c}(p) = \exp(i\pi s)\,\Psi_2^c(r(-\pi)p)$. Hence, taking into account that $r(\pi) = r(-\pi)$, Eqs. (67) and (68) imply the claimed Eq. (64). □

This implies our main result, the relation between spin and statistics for anyons and plektons:

Theorem 1 (Spin-Statistics Theorem) *The spin s and statistics phase* ω_χ *are related by*

$$e^{2\pi is} = \omega_\chi.$$

Proof Substituting Eqs. (63) and (64) into Eq. (48), yields

$$D^* D\, e^{2\pi is} = \omega_\chi \mathbb{1}\;,$$

since the matrices $\Psi_i^c(p)$ are invertible for almost all p. Uniqueness of the polar decomposition then implies the claim, and also implies that D is isometric. □

A Justification of the Assumptions

We assume the standard assumptions on the algebra $\mathscr{A}$ of local observables (12) plus weak Haag Duality of the vacuum representation (3, Eq. (1.11)), and consider a covariant representation π_χ of $\mathscr{A}$ which is strictly massive in the sense of our assumptions (A1) and (A2). As shown in (3), π_χ is then localizable in space-like cones, i.e., equivalent to the vacuum representation when restricted to the causal complement of a space-like cone. One can then enlarge the algebra of observables to the so-called universal algebra $\mathscr{A}_{\text{uni}}$ (9; 8) and find an endomorphism ρ of $\mathscr{A}_{\text{uni}}$ such that the (unique lift of the) representation π_χ is equivalent to the representation $\pi_0 \circ \rho$, where π_0 is the vacuum representation of $\mathscr{A}_{\text{uni}}$ acting in a vacuum Hilbert space $\mathscr{H}_{00}$. The endomorphism ρ is localized in some specific space-like cone C_0 in the sense that

$$\rho(A) = A \qquad \text{if } A \in \mathscr{A}_{\text{uni}}(C_0'), \tag{A.1}$$

where C_0' denotes the causal complement of C_0. The endomorphism ρ has a conjugate $\bar{\rho}$ such that $\bar{\rho}\rho$ contains the identity representation ι of $\mathscr{A}_{\text{uni}}$ (3). We shall choose a corresponding intertwiner $R \in \mathscr{A}_{\text{uni}}$, with the normalization convention of (5), i.e. R is not isometric but satisfies $R^*R = |\lambda_\chi|^{-1}\mathbb{1}$ (5, Eq. (3.14)). Associated with ρ is the statistics operator ε_ρ, which describes the interchange of two charges localized in causally separated space-like cones. Using the notions of our Sect. 2, it is constructed as follows. We fix the reference direction e_0 so as to be contained, in the sense of Eq. (5), in C_0. Let $\tilde{C}_1 = (C_1, \tilde{e}_1)$ and $\tilde{C}_2 = (C_2, \tilde{e}_2)$ be paths of space-like cones satisfying the hypothesis of Eq. (15). Let further U_i, $i = 1,2$, be (heuristically speaking) charge transporters which transport the charge ρ from C_0 to C_i along the path $\tilde{e}_i$. This means the following. U_i is an intertwiner such that $\operatorname{Ad} U_i \circ \rho$ is localized in C_i (instead of C_0) in the sense of Eq. (A.1), and at the same time is an observable localized in I_i, where I_i is a space-like cone (or the complement of one) containing the complete path $\tilde{e}_i(t)$, $t \in [0,1]$, in the sense of Eq. (5). Then

$$\varepsilon_\rho := \rho(U_1^*)U_2^*U_1\rho(U_2). \tag{A.2}$$

The corresponding statistics parameter λ_χ and statistics phase ω_χ are then defined by the relations

$$\phi(\varepsilon_\rho) = \lambda_\chi \mathbb{1}\,, \quad \omega_\chi = \frac{\lambda_\chi}{|\lambda_\chi|}, \tag{A.3}$$

respectively. (They depend only on the equivalence class of ρ, i.e., on its sector χ.) Here, ϕ is the left inverse of ρ, that is a positive linear endomorphism of $\mathscr{A}_{\text{uni}}$ satisfying

$$\phi\left(\rho(A)B\rho(C)\right) = A\phi(B)C, \quad \phi(\mathbb{1}) = \mathbb{1}\,. \tag{A.4}$$

It can be expressed as (3; 5)

$$\phi(A) = |\lambda_\chi| R^*\bar{\rho}(A)R. \tag{A.5}$$

(The factor $|\lambda_\chi|$ appears here in contrast to (3) because we have chosen the normalization convention of R as in (5).)

We now identify the objects and notions of our Sect. 2 within the frame indicated above and with objects derived within this framework in (5; 3; 9; 8). Our sectors χ and $\bar{\chi}$ are just the equivalence classes of the representations $\pi_0 \circ \rho$ and $\pi_0 \circ \bar{\rho}$, respectively. Our Hilbert spaces $\mathscr{H}_0$, $\mathscr{H}_\chi$ and $\mathscr{H}_{\bar{\chi}}$ are the fibres $\{\iota\} \times \mathscr{H}_{00}$, $\{\rho\} \times \mathscr{H}_{00}$ and $\{\bar{\rho}\} \times \mathscr{H}_{00}$ of the vector bundle $\mathscr{H}$ of generalized state vecors introduced in (5), see also (3), respectively. The respective scalar products are inherited by that of $\mathscr{H}_{00}$. Our vacuum vector Ω is identified with the Poincaré invariant vector Ω_0 inducing the vacuum state:

$$\Omega = (\iota, \Omega_0) \in \mathscr{H}_0.$$

The spaces of our fields $\mathscr{F}_\chi$ and $\mathscr{F}_{\bar{\chi}}$ are defined as the subspaces $\{\rho\} \times \mathscr{A}_{\text{uni}}$ and $\{\bar{\rho}\} \times \mathscr{A}_{\text{uni}}$, respectively, of the field bundle $\mathscr{F}$ introduced in (5). A generalized field operator $F = (\rho, B) \in \mathscr{F}_\chi$ then acts on a generalized state vector $(\iota, \psi) \in \mathscr{H}_0$ as

$$(\rho, B)\,(\iota, \psi) := (\rho, \pi_0(B)\psi) \in \mathscr{H}_\chi.$$

The adjoint $F^\dagger$ of a generalized field operator $F=(\rho,B)\in\mathscr{F}_\chi$ is defined by

$$(\rho,B)^\dagger := (\bar\rho,\bar\rho(B^*)R), \tag{A.6}$$

where B^* is the C^*-adjoint of B in $\mathscr{A}_{\text{uni}}$.

The notion of localized generalized field operators has been introduced in (5) in the case of permutation group statistics. The extension to the case of braid group statistics needs a refinement, which has been introduced in (9), see also (8). There, $\mathscr{K}$ denotes the class of space-like cones or causal complements thereof, and a path in $\mathscr{K}$ is a finite sequence $(I_0,\dots,I_n)$, $I_k\in\mathscr{K}$, such that either $I_k\subset I_{k-1}$ or $I_k\supset I_{k-1}$, $k=1,\dots,n$. We say that such path starts at C_0 if $I_0=C_0$. The relation to our notion of paths of space-like cones, Eq. (6) is as follows. Our $(C,\tilde e)$ corresponds to a path $(I_0,\dots,I_n)$ in $\mathscr{K}$ starting at C_0 if $\tilde e$, considered as a path in H, has the decomposition $\tilde e=\gamma_n*\cdots*\gamma_0$ such that $\gamma_k(t)$ is contained in I_k in the sense of Eq. (5) for all $t\in[0,1]$ and $k=0,\dots,n$. With this identification, our space of localized fields $\mathscr{F}_\chi(\tilde C)$ is defined as

$$\mathscr{F}_\chi(\tilde C) := \mathscr{F}_\chi\cap\mathscr{F}(\tilde C),$$

where $\mathscr{F}(\tilde C)$ is the space of generalized field operators localized along $\tilde C$ as defined in (9; 8). $\mathscr{F}_{\bar\chi}(\tilde C)$ is defined analogously. The fact that the adjoint preserves localization, Eq. (13), is just Eq. (6.37) in (3) (which strengthens Lemma 4.3 in (5)).

Our representations U_χ and α_χ of the universal covering group of the Poincaré group in $\mathscr{H}_\chi$ and $\mathscr{F}_\chi$, respectively, are defined as follows. Let $U(\tilde g)$ and $\alpha(\tilde g)$ be the representations in $\mathscr{H}$ and $\mathscr{F}$ as defined in (5, Eqs. (4.3) and (4.4)) in the case of permutation group statistics, and (8, Eqs. (2.18) and (2.19)) in the case of braid group statistics, respectively. Then we define

$$U_\chi(\tilde g) := U(\tilde g)\big|\mathscr{H}_\chi \quad\text{ and }\quad \alpha_\chi(\tilde g) := \alpha(\tilde g)\big|\mathscr{F}_\chi.$$

The covariance condition (11) is just Eq. (4.7) in (5). Our Eq. (14), relating the adjoint, α_χ and $\alpha_{\bar\chi}$ (defined analogously), is just Eq. (4.20) in (5). The fact that Eqs. (11), (13) and (14) also hold in the case of braid group statistics has been shown in (16).

Our Eq. (15), fixing the significance of the statistics phase ω_χ, corresponds to Eq. (6.5) in (5) in the case of permutation group statistics. But since we are not aware of literally the same equation in the literature in the case of braid group statistics, we give a direct proof, transferring their arguments to this case.

Lemma A.1 *Let $\tilde C_1=(C_1,\tilde e_1)$ and $\tilde C_2=(C_2,\tilde e_2)$ be paths of space-like cones satisfying the hypothesis of Eq.* (15)*. Let further $F_i=(\rho,B_i)\in\mathscr{F}_\chi(\tilde C_i)$, $i=1,2$. Then there holds Eq.* (15)*, namely,*

$$(F_2\Omega,F_1\Omega) = \omega_\chi\left(F_1^\dagger\Omega,F_2^\dagger\Omega\right).$$

Proof $(\rho,B_i)\in\mathscr{F}_\chi(\tilde C_i)$ means that there are unitary charge transporters U_i satisfying precisely the hypothesis of Eq. (A.2), and that $A_i:=U_iB_i$ is an observable localized in C_i, $i=1,2$. Denoting $\rho_i:=\text{Ad }U_i\circ\rho$, we then have

$$\begin{aligned}\rho(B_2^*)\,\varepsilon_\rho^*\,\rho(B_1) &= \rho(B_2^*U_2^*)\,U_1^*\,U_2\rho(U_1B_1) = \rho(A_2^*)\,U_1^*\,U_2\rho(A_1)\\ &= U_1^*\rho_1(A_2^*)\rho_2(A_1)U_2 = U_1^*A_2^*A_1U_2 = U_1^*A_1A_2^*U_2 = B_1\,B_2^*.\end{aligned} \tag{A.7}$$

(We have used that ρ_i are localized in C_i in the sense of Eq. (A.1) and that A_1 and A_2^* commute due to locality of the observables.) Applying the left inverse ϕ to Eq. (A.7), using the explicit formula (A.5) for the left inverse and taking into account that ϕ preserves the C^*-adjoint, yields

$$\bar\lambda_\chi\,B_2^*B_1 = |\lambda_\chi|\,R^*\bar\rho(B_1B_2^*)R.$$

Using this equation, we get

$$\begin{aligned}(F_2\Omega,F_1\Omega) &= (\Omega_0,\pi_0(B_2^*B_1)\,\Omega_0) = (\bar\lambda_\chi)^{-1}\,|\lambda_\chi|\,(\Omega_0,\pi_0(R^*\bar\rho(B_1B_2^*)R)\Omega_0)\\ &= \omega_\chi\,(\pi_0(\bar\rho(B_1^*)R)\Omega_0,\pi_0(\bar\rho(B_2^*)R)\Omega_0) = \omega_\chi\left(F_1^\dagger\Omega,F_2^\dagger\Omega\right),\end{aligned}$$

since $(\bar\lambda_\chi)^{-1}|\lambda_\chi|=\omega_\chi$. This completes the proof. □

B An Analytic Cocycle for the Massive Irreducible Representations of $\tilde{P}_+^\uparrow$ in 2+1 Dimensions

In (17), we have shown that the Wigner rotation factor $\exp(is\Omega(\tilde{\lambda},p))$ is non-analytic in the sense that the function $t\mapsto \exp(is\Omega(\tilde{\lambda}_1(t)\tilde{\lambda},p))$ has singularities in the strip G for any fixed $p\in H_m^+$ and $\tilde{\lambda}\in\tilde{L}_+^\uparrow$ in a neighbourhood of the unit. These singularities are in fact branch points if s is not an integer (see Lemma C.1 in (17)). However, we have constructed a function $u(p)$ living on the mass shell which compensates the singularities of the Wigner rotation factor. In more detail, our function is given by

$$u(p) := \left(\frac{p_0-p_1}{m}\cdot\frac{p_0-p_1+m-ip_2}{p_0-p_1+m+ip_2}\right)^s, \quad p_0 := (p_1^2+p_2^2+m^2)^{\frac{1}{2}}. \tag{B.1}$$

(Note that p_0-p_1 is strictly positive for all $p\in H_m^+$, hence the argument in brackets lies in the cut complex plane $\mathbb{C}\backslash\mathbb{R}_0^-$. The power of $s\in\mathbb{R}$ is then defined via the branch of the logarithm on $\mathbb{C}\backslash\mathbb{R}_0^-$ with $\ln 1=0$.) We then define a map $c:\tilde{L}_+^\uparrow\times H_m^+\to\mathbb{C}\backslash\{0\}$ by

$$c(\tilde{\lambda},p) := u(p)^{-1}\, e^{is\Omega(\tilde{\lambda},p)}\, u(\lambda^{-1}p)\,. \tag{B.2}$$

In group theoretical terms, the map $c(\cdot,\cdot):\tilde{L}_+^\uparrow\times H_m^+\to\mathbb{C}\backslash\{0\}$ is a cocycle which is equivalent to the Wigner rotation factor. To state its analyticity properties, we need some more notation. Let W_1 be the wedge region

$$W_1 := \left\{x\in\mathbb{R}^3; x^1 > \left|x^0\right|\right\}, \tag{B.3}$$

and let the reference direction e_0 be specified as $e_0=(0,0,-1)$. Denote by $\tilde{W}_1$ the pair $(W_1,\tilde{e}_1)$, where $\tilde{e}_1$ is the equivalence class of a path in H starting from the reference direction e_0 and staying within W_1 in the sense of Eq. (5). If $\tilde{e}$ is a path in H ending at a direction e contained in W_1 in the sense of Eq. (5), and $\tilde{e}$ is equivalent to $\tilde{e}_1$ w.r.t W_1, we write

$$\tilde{e}\in\tilde{W}_1. \tag{B.4}$$

Let further $\tilde{e}_0$ be the constant path at e_0. We found the following result.

Lemma B.1 **(17)** *Let $\tilde{\lambda}$ be an element of $\tilde{L}_+^\uparrow$ such that $\tilde{\lambda}\cdot\tilde{e}_0\in\tilde{W}_1$ in the sense of Eq.* (B.4). *Then for all $p\in H_m^+$ the function*

$$t\mapsto c(\tilde{\lambda}_1(t)\tilde{\lambda},p)$$

has an analytic extension into the strip $\mathbb{R}+i(0,\pi)$. This extension satisfies the boundary condition

$$c(\tilde{\lambda}_1(i\pi)\tilde{\lambda},p) = e^{i\pi s}\,\overline{c(\tilde{\lambda},-jp)} \tag{B.5}$$

$$\equiv e^{i\pi s}\,c(j\tilde{\lambda}j,p). \tag{B.6}$$

(The very last equation is not contained in (17), but follows directly from the identity (60) and the fact that the function u satisfies $u(-jp)=\overline{u(p)}$.)

Let us rewrite this result for the present purpose, namely, the proof of Lemma 3. Lemma 3 needs an analyticity statement for $\tilde{\lambda}$ in a neighbourhood of the unit (namely the set $\mathscr{U}_{12}$), whereas the set of $\tilde{\lambda}$ satisfying the hypothesis of Lemma B.2 is *not* a neighbourhood of the unit (since e_0 is at the boundary of W_1). To this end, we fix a Lorentz transformation λ_0 which maps e_0 into W_1, and let $\tilde{\lambda}_0$ be the (unique) element of $\tilde{L}_+^\uparrow$ over λ_0 such that $\tilde{\lambda}_0\cdot\tilde{e}_0\in\tilde{W}_1$ in the sense of Eq. (B.4). (For example, a rotation about $\pi/2$ would do.) We then define

$$u_{\lambda_0}(p) := e^{is\Omega(\tilde{\lambda}_0,p)}\,u(\lambda_0^{-1}p)\equiv u(p)\,c(\tilde{\lambda}_0,p), \tag{B.7}$$

and a corresponding cocycle

$$c_{\lambda_0}(\tilde{\lambda},p) := u_{\lambda_0}(p)^{-1}\, e^{is\Omega(\tilde{\lambda},p)}\, u_{\lambda_0}(\lambda^{-1}p)\,. \tag{B.8}$$

Lemma B.2 i) *Let $\tilde{\lambda}$ be an element of $\tilde{L}_+^{\uparrow}$ such that $\tilde{\lambda}\tilde{\lambda}_0 \cdot \tilde{e}_0 \in \tilde{W}_1$ in the sense of Eq.* (B.4). *Then for all $p \in H_m^+$ the function*

$$f(t) := e^{is\Omega(\tilde{\lambda}_1(t)\tilde{\lambda},p)}\, u_{\lambda_0}(\lambda^{-1}\lambda_1(-t)p) \tag{B.9}$$

has an analytic extension into the strip $\mathbb{R} + i(0,\pi)$, continuous at the boundary. At $t = i\pi$, this extension has the boundary value

$$f(i\pi) = e^{i\pi s}\,\overline{u_{\lambda_0}(-jp)\, c_{\lambda_0}(\tilde{\lambda}, -jp)} \tag{B.10}$$

$$\equiv e^{i\pi s}\, e^{is\Omega(j\tilde{\lambda}\tilde{\lambda}_0 j,p)}\, u\big(j(\lambda\lambda_0)^{-1} jp\big)\,. \tag{B.11}$$

ii) *If λ_0 is the rotation about $\pi/2$, then the set of $\tilde{\lambda}$ satisfying the hypothesis of* (i) *is a neighbourhood of the unit. Further, in this case u_{λ_0} is given by*

$$u_{\lambda_0}(p) = e^{is\frac{\pi}{2}} \left(\frac{p_0 - p_2}{m} \cdot \frac{p_0 - p_2 + m + ip_1}{p_0 - p_2 + m - ip_1}\right)^s =: u_{\frac{\pi}{2}}(p)\,. \tag{B.12}$$

Proof Ad *i*) By definition of the cocycle c_{λ_0}, $f(t)$ coincides with $u_{\lambda_0}(p)\, c_{\lambda_0}(\tilde{\lambda}_1(t)\tilde{\lambda}, p)$. Since our definitions imply the identity

$$u_{\lambda_0}(p)\, c_{\lambda_0}(\tilde{\lambda}, p) = u(p)\, c(\tilde{\lambda}\tilde{\lambda}_0, p) \tag{B.13}$$

for all $\tilde{\lambda} \in \tilde{L}_+^{\uparrow}$, we have

$$f(t) = u(p)\, c(\tilde{\lambda}_1(t)\tilde{\lambda}\tilde{\lambda}_0, p). \tag{B.14}$$

Lemma B.1 then asserts that for $\tilde{\lambda}\tilde{\lambda}_0 \cdot \tilde{e}_0 \in \tilde{W}_1$, this function is analytic in the strip G, and has the boundary value

$$f(i\pi) = e^{i\pi s}\, u(p)\,\overline{c(\tilde{\lambda}\tilde{\lambda}_0, -jp)}. \tag{B.15}$$

Using $u(p) = \overline{u(-jp)}$ and once again Eq. (B.13), yields Eq. (B.10) of the lemma. On the other hand, substituting Eq. (B.6) into Eq. (B.15) and using the defining relation (B.2), yields Eq. (B.11) of the lemma.

Ad *ii*) A rotation $r(\frac{\pi}{2})$ about $\pi/2$ maps e_0 into the interior of the wedge W_1. Hence the set of $\tilde{\lambda}$ satisfying the hypothesis of (*i*) is a neighbourhood of the unit. Further, the corresponding $\tilde{\lambda}_0$ is just $\tilde{r}(\frac{\pi}{2})$, where $\tilde{r}(\cdot)$ is the lift of the one-parameter group of rotations to $\tilde{L}_+^{\uparrow}$. Hence $\Omega(\tilde{\lambda}_0, p) = \pi/2$ by Eq. (28). Together with $r(\frac{\pi}{2})^{-1}(p_0, p_1, p_2) = (p_0, p_2, -p_1)$, this implies Eq. (B.12). □

Acknowledgements It is a pleasure for me to thank Klaus Fredenhagen for drawing my attention to the article of Buchholz and Epstein on my search for a PCT theorem for anyons. Further, I gratefully acknowledge financial support by FAPEMIG and by the Graduiertenkolleg "Theoretische Elementarteilchenphysik" (Hamburg).

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