DYNAMICAL REARRANGEMENT OF SYMMETRY

by

Giuseppe Vitiello

A thesis submitted to the University of
Wisconsin--Milwaukee in partial fulfillment of
the requirements for the degree of

DOCTOR OF PHILOSOPHY

1974

APPROVED BY:

Hiroomi Umezawa — Oct. 10, 1974
Major Professor — Date

Graduate School Dean — Date

DYNAMICAL REARRANGEMENT OF SYMMETRY

by

Giuseppe Vitiello

A thesis submitted to the University of
Wisconsin--Milwaukee in partial fulfillment of
the requirements for the degree of

DOCTOR OF PHILOSOPHY

1974

Dynamical rearrangement of symmetry, Ph.D. Thesis,
Abstract published in Diss. Ab. Intern. 36/02, 769-B (1975).

ACKNOWLEDGEMENTS

I wish to express my deep gratitude to Professor Hiroomi Umezawa for his inspiring discussions, help and guidance. I wish also to thank Prof. N. J. Papastamatiou and Dr. H. Matsumoto for their help and advice. Finally, my thanks to Mrs. M. N. Shah and Dr. J. K. Wyly for their collaboration.

ABSTRACT

DYNAMICAL REARRANGEMENT OF SYMMETRY

Giuseppe Vitiello, Ph.D.

University of Wisconsin--Milwaukee, 1974

Under the Supervision of Professor Hiroomi Umezawa

In Quantum Field Theory (Q.F.T.), the invariance of the theory is expressed as the invariance of field equations under certain transformations of Heisenberg fields. Since we are interested in physically relevant entities, we are faced with the problem of how the original invariance of the theory manifests itself at the level of observable (physical) particles. This is the problem studied in the present work.

In our analysis we start by the fundamental assumption that the set of physical field operators is an irreducible set of operators realized in the Fock space of physical particles. In Chapter I we give an outline of the self-consistent method in Q.F.T. In this method a mapping (the dynamical map) is introduced among Heisenberg fields and physical fields. The role of this mapping in the theory is fundamental since through it the dynamics described by the Heisenberg equations manifest its effects at the phenomenological level. Through the dynamical map we can thus express the Heisenberg field operators in terms of infields by collecting all the matrix elements of Heisenberg

operators. In Chapter II, we show that this expression takes a simple form when we use the path-integral to express matrix elements. We then ask what kind of transformations of infields reproduce the original invariant transformation of Heisenberg operators. The invariance of the theory requires that these transformations of infields are such that they leave the free field equations invariant. When the invariant transformations of the Heisenberg operators appear in a different form from that of the infield operators, we say that a dynamical rearrangement of symmetry has taken place. It is found that there is dynamical rearrangement of symmetry when spontaneous breakdown occurs. The generating functional of the Green's functions is modified by the addition of an infinitesimal ε-term which fixes the direction of the breaking. Since the dynamical rearrangement of symmetry concerns the general structure of the theory, we generalize our study to relativistic as well as to non-relativistic problems. In each of them the Goldstone theorem is proved and the role of the massless Goldstone particles in recovering the invariance of the theory is analyzed. It is found that these particles undergo a transformation, the boson transformation, which leaves the free field equations invariant. A Goldstone-type model is studied as an example of a model with Abelian symmetry. The spontaneous breakdown of SU(2) symmetry is also investigated. We analyze a relativistic model (iso-triplet scalar field) and a non-relativistic one (a ferromagnetic system). In terms of the path-integral method the infield transformations which induce the original SU(2) transformations of Heisenberg operators are identified. It is found that the algebra of infield transformations is the E(2) symmetry group algebra. It is shown that the discrepancy of two algebras is caused by the local nature of the observation in which one misses the infrared contributions. When the total (integrated) infrared effects are considered, the original symmetry group algebra is recovered. Exact expressions of symmetry generators in terms of physical operators are given.

An analysis of the spontaneous breakdown of gauge theories is also presented. Massless unphysical modes are found. Their role in recovering the invariance of the theory is shown.

The dynamical rearrangement of symmetry is analyzed in the framework of infinite unitarily inequivalent representations of the canonical variables. The occurrence of such representations and their physical usefulness is studied in Chapter III. A Q.F.T. for finite temperature is presented as an example and an application to superconductors is given. In particular the temperature dependence of the critical value of the Ginzburg-Landau parameter which separates type-II/1 from type-II/2 is computed. Agreement with experimental data is satisfactory.

Finally, a self-consistent formulation of the itinerant electron ferromagnet is presented in the Appendix. The pair approximation is used and the Goldstone boson (magnon) is studied as bound state of fermions by means of the Bethe-Salpeter equation.

APPROVED BY:

Major Professor   Date

TABLE OF CONTENTS

I. INTRODUCTION: THE SELF-CONSISTENT METHOD IN QUANTUM FIELD THEORY

1.1. The problem.

In the understanding of natural phenomena a central role has always been played by symmetry principles. This is due to the fact that very often the discovery of a symmetry principle means the possibility of a reduction to a simpler and clearer order among many phenomena or among many aspects of the same phenomenon, which is, after all, the task of any science. However, it is not always easy to recognize such symmetries. The reason for this difficulty can be understood in the following way: the fundamental symmetries can be distorted, "rearranged" when manifested at a phenomenological level. In particle physics, for example, the concept of internal symmetries, by which it is possible to group particles into families is well known. For example, when Heisenberg [1] classified protons and neutrons as "nucleons" it was clear that the underlying symmetry between the nucleons does not manifest as an exact symmetry, but as a"broken symmetry": the charge independence is indeed violated by the electromagnetic interaction. In general, all the various symmetry schemes and groupings, which are quite successful, also appear to be in some way "approximate" symmetry schemes, i.e. one has to disregard some phenomenological aspects, e.g. mass differences, which violate certain symmetry requirements. A way of looking at this situation is to interpret the observed deviations from the exact symmetry as a phenomenological distortion or rearrangement of the basic symmetry. Other examples of rearranged symmetries are easily found in solid state physics: the crystals manifest a periodic structure, but do not possess the translational invariance of the Hamiltonian of molecular gas. Similarly, in ferromagnetism the original rotational invariance manifests itself as full polarization and in superconductivity and superfluidity the phase invariance is the one that seems to disappear.

The crucial problem we are facing in the recognition of a symmetry and which is the object of the present study, is then an intrinsic duality of the description of the nature: one aspect of this duality concerns with original symmetries ascribed to "basic" entities, the other aspect concerns the corresponding rearranged symmetries of observable phenomena.

This duality in the description of nature was recognized soon in the Quantum Field Theory as the duality between fields and particles. Here, we are not going to give the historical development of this concept; we recall only, as an example, how fundamental this is in the renormalization theory, where the distinction among "bare" and "observed" particles is central, i.e. among basic fields and their "manifestation" in the presence of interaction. The approach we will use in the study of the phenomenon of the rearrangement of symmetry is the so called self-consistent method of Quantum Field Theory [2-4]. In this formalism the duality concept mentioned above plays a central role; in fact it is the basis and the starting point of the method.

In our analysis we will focus our attention mainly on problems concerning the general structure of the theory. Thus we will generalize our study to the relativistic as well as to the non-relativistic Quantum Field Theory. In the present chapter we give a brief account of the self-consistent method.

1.2. The Fock space of physical particles.

Let us start[1] by the fundamental assumption that the states of a physical system can be represented by vectors in a certain Hilbert space. Since we are interested in physically relevant entities, we must be able to choose the basis of the Hilbert space in such a way that any vector of the basis is a state of our system of definite number of physical particles, indeed such a number is one of the

[1] We will follow the approach for the construction of a Quantum Field Theory presented by H. Umezawa in his lectures at the University of Wisconsin-Milwaukee in 1972-73. In the following we will use the natural units $\hbar = c = 1$.

observables. At this point, however, we must specify what we mean by "physical particles": consider a scattering process between two or more particles. We can distinguish in such a process, a first stage in which we can identify by convenient measurements the kind, the number, the energy, etc., of the particles before they interact (incoming particles); a second stage, i.e. the one of interaction; a third stage in which again we can measure the kind, the number, the energy, etc. of the particles after the interaction (outgoing particles). What is observed is that in such a process the sum of the energies of the incoming particles is equal to the sum of the energies of the outgoing particles; we will refer to the incoming and to the outgoing particles as "physical particles" or else as "observed" or "free" particles, where the word "free" does not exclude the possibility of interaction among them. It only means that the total energy of the system is given by the sum of the energies of the observed particles. Furthermore, in analogy to the Quantum Mechanics, we require that the energy of the physical particles is determined as a certain function of their momenta. This requirement needs some care as we will see later. In solid state physics the physical particles are usually called quasiparticles.

Let us now consider the problem of the construction of the Hilbert space for the physical particles. First we classify the state of a single particle by the suffices (i,r), where i specifies the spatial distribution of the state while r specifies other freedoms (e.g. spin, charge, etc.). For brevity, we assume we are dealing with particles of one kind only (e.g. only electrons, or only protons, etc.). We must use wave packets to specify spatial distributions, because plane waves like $\exp(i\vec{k}\cdot\vec{x})$ are not normalizable. On the other hand, it is well known that an orthonormalized complete set of square-integrable functions $\{f_i(\vec{x}),\ i=1,2\ldots\}$ is a countable set. Thus, we introduce the annihilation operators for particles and their antiparticles in wave-packet states as

$$\begin{aligned} \alpha_i &= \int d^3k\ f_i(\vec{k})\alpha_{\vec{k}} \\ \beta_i &= \int d^3k\ f_i(\vec{k})\beta_{\vec{k}} \end{aligned} \tag{I.1}$$

where the orthonormalization condition for square-integrable functions of $f_i(\vec{k})$ is

$$\int d^3k\ f_i^*(\vec{k})f_j(\vec{k}) = \delta_{ij}\ . \tag{I.2}$$

Here $f_i(\vec{k})$ are the Fourier amplitudes of $f_i(\vec{x})$. In (I.1) we omitted for brevity the suffix r, and we introduced the annihilation operators $\alpha_{\vec{k}}$ and $\beta_{\vec{k}}$ for physical particles and their antiparticles of momentum $\vec{k}$, with (anti-)

commutation relations:

$$[\alpha_{\vec{k}}, \alpha^\dagger_{\vec{\ell}}]_\pm = \delta(\vec{k}-\vec{\ell})$$
$$[\beta_{\vec{k}}, \beta^\dagger_{\vec{\ell}}]_\pm = \delta(\vec{k}-\vec{\ell}) \tag{I.3}$$

and all other (anti-)commutators zero. As is well known we have anti-commutation relations for fermions, and commutation relations for bosons. $\alpha^\dagger_{\vec{\ell}}$ and $\beta^\dagger_{\vec{\ell}}$ are the hermitian conjugate of $\alpha_{\vec{\ell}}$ and $\beta_{\vec{\ell}}$.

Similarly, for α_i, β_i we have:

$$[\alpha_i, \alpha^\dagger_j]_\pm = \delta_{ij}$$
$$[\beta_i, \beta^\dagger_j]_\pm = \delta_{ij} \tag{I.4}$$

and all other (anti-)commutators are zero. Then we assume the existence of a <u>physical vacuum</u> state $|0\rangle$ defined as

$$\alpha_i|0\rangle = 0$$
$$\beta_i|0\rangle = 0 \ . \tag{I.5}$$

The Hilbert space of physical particles is now cyclically constructed by repeated applications of $\alpha^\dagger_i$, $\beta^\dagger_i$ on $|0\rangle$:

$$c\alpha^\dagger_i\alpha^\dagger_j \ldots|0\rangle = |\alpha_i\alpha_j \ldots\rangle$$

where c is a normalization constant.

In the following we consider for simplicity the operators α_i only. By using the previous definitions, and introducing the number operator N_i:

$$N_i = \alpha_i^\dagger \alpha_i \ , \tag{I.6}$$

we can write:

$$\begin{aligned}
\alpha_i |n_1,\ldots n_i,\ldots\rangle &= \sqrt{n_i}\ |n_1,\ldots n_i-1,\ldots\rangle \\
\alpha_i^\dagger |n_1,\ldots\ n_i,,\ldots\rangle &= \sqrt{n_i+1}\ |n_1,\ldots n_i+1,\ldots\rangle \\
N_i |n_1,\ldots n_i,\ldots\rangle &= n_i |n_1,\ldots n_i,\ldots\rangle
\end{aligned} \tag{I.7}$$

where $|n_1,\ldots n_i\ldots\rangle$ is a member of the orthonormalized basis of our Hilbert space $\{|n_1,n_2\ldots\rangle\}$:

$$\langle n_1',n_2',\ldots|n_1,n_2,\ldots\rangle = \delta_{n_1'n_1}\ \delta_{n_2'n_2}\cdots \ . \tag{I.8}$$

Any other state of the system is a superposition of these states

$$\underline{\xi} = \sum_{n_i} c(n_1,n_2,..)|n_1,n_2,\ldots\rangle \ . \tag{I.9}$$

We used a simplified notation, where $|n_1,n_2,..\rangle$ means a state of the system where n_ℓ particles are in the state

ℓ associated by some rule to a given couple (i,r). In this notation the vacuum $|0\rangle$ is $|0,0,\ldots\rangle$. n_i can be any non negative integer for bosons; 0 or 1 for fermions. In this latter case (I.7) give

$$\begin{aligned} \alpha_i|n_1,\ldots n_i,\ldots\rangle &= 0 \qquad \text{for } n_i = 0 \\ \alpha_i^\dagger|n_1,\ldots n_i\ldots\rangle &= 0 \qquad \text{for } n_i = 1 \,. \end{aligned} \tag{I.10}$$

Note that (I.7) and (I.10) are consistent with (I.3) and (I.4). One can also prove that the vectors of the Hilbert space so constructed are fully symmetrical states in the case of bosons and fully antisymmetrical states in the case of fermions. Furthermore, we note that by repeated application of α_i (and β_i) we can move from one member to another in the basis $\{|n_1,n_2,\ldots\rangle\}$; however the operators $\alpha_{\vec{k}}$ (and $\beta_{\vec{k}}$) do not map normalizable vectors on normalizable ones, due to (I.3); indeed

$$|\alpha_{\vec{k}}^\dagger|0\rangle|^2 = \langle 0|\alpha_{\vec{k}}\,\alpha_{\vec{k}}^\dagger|0\rangle = \delta(\vec{0})$$

which is not finite. Before considering the problem of how the annihilation operators α_i and β_i operate on the vectors of the Hilbert space, let us consider in more detail the mathematical nature of this space.

We introduced the set

$$\{|n_1,n_2,\ldots\rangle\} \tag{I.11}$$

and we used it as a basis of the Hilbert space (cf. eq. (I.9)). If we require that our space must be a separable Hilbert space, it is not correct to use the set $\{|n_1,n_2,\ldots\rangle\}$ as a basis, because it is not a countable set. To prove this, let us consider for simplicity a fermion system. Then, n_i can assume only the values 0 or 1. Then, we consider the set of numbers

$$\{0.n_1n_2\ldots\} \ , \tag{I.12}$$

where n_i can assume only the values 0 or 1. Using the binary system, we see that the set (I.12) covers all the values smaller than 1, i.e. it is a noncountable set. On the other hand, there is a one-to-one correspondence between the set (I.12) and the set (I.11) and thus we conclude that the latter is a noncountable one. Since the set (I.11) for bosons is larger than the one for fermions, in the case of a system of bosons the set (I.11) is also not countable. To remedy this situation, let us observe that it is very likely that we do not need states which contain an infinite number of particles. The number of particles can be as large as we want, but does not need to be infinite. We introduce then a <u>subset</u> of (I.11) as follows:

$$\{|n_1,n_2\ldots\rangle \ , \qquad \sum_i n_i = \text{finite}\} \ . \tag{I.13}$$

This set contains the vacuum $|0,0,\ldots\rangle$, but does not contain states like $|1,1,1,\ldots\rangle$ where $n_i = 1$ for all i.

We want to prove now that the set (I.13) is countable. Consider the state $|n_1,n_2,\ldots\rangle$ belonging to the set (I.13). Since this state contains only a finite number of particles, there is an integer number ℓ for which

$$\begin{cases} n_i = 0 & \text{for } i > \ell \\ n_\ell \neq 0 \end{cases} \tag{I.14}$$

Then, to each vector of (I.13) we can associate two numbers, i.e. ℓ and $N = \sum_i n_i$. For each product ℓN, there exists only a finite number of vectors in the set (I.13), because for each ℓN we can distribute a finite number of particles only in a finite number of states. This means that we can label the vectors in the set (I.13) as $\underline{\xi}_a$, with $a = 1,2,\ldots$ in such a way that ℓN is not decreasing for a increasing. The set (I.13) is thus countable. Now, we can consider a linear space H_F defined by

$$H_F = \left\{\underline{\xi} = \sum_{a=1}^{\infty} c_a\underline{\xi}_a, \ \sum_a |c_a|^2 = \text{finite}\right\} . \tag{I.15}$$

H_F is separable because the set $\{\underline{\xi}_a\}$ is countable. If $\underline{\zeta}$ and $\underline{\xi}$ are vectors of H_F

$$\underline{\xi} = \sum_{a=1}^{\infty} c_a \underline{\xi}_a \ , \qquad \sum_{a=1}^{\infty} |c_a|^2 = \text{finite} \tag{I.16}$$

$$\underline{\xi} = \sum_{a=1}^{\infty} b_a \underline{\xi}_a \ , \qquad \sum_{a=1}^{\infty} |b_a|^2 = \text{finite} \ , \tag{I.17}$$

the inner product is defined as

$$(\underline{\zeta},\underline{\xi}) = \sum_a c_a^* b_a \ , \tag{I.18}$$

where we used the fact that $\{\underline{\xi}_a\}$ is an orthonormalized set:

$$(\underline{\xi}_a,\underline{\xi}_b) = \delta_{ab} \ .$$

Since

$$|\underline{\xi}|^2 = (\underline{\xi},\underline{\xi}) = \sum_a |c_a|^2 = \text{finite} \ ,$$

the vectors of H_F have finite norm. The linear space H_F is called the Fock space of physical particles.

Now we come to the problem of how the annihilation and creation operators operate on H_F.

Let us assume that the particles under consideration are bosons. The argument can be easily extended to the case of fermions. Let us introduce the operators q_i and p_i defined as

$$q_i = \frac{1}{\sqrt{2}} (\alpha_i + \alpha_i^\dagger)$$
$$p_i = \frac{1}{i\sqrt{2}} (\alpha_i - \alpha_i^\dagger) \quad . \tag{I.19}$$

They satisfy the canonical commutation relation

$$[q_i, p_j] = i\delta_{ij} \quad . \tag{I.20}$$

Since we are considering a finite number of particles (cf. e.g. (I.13)), the present situation is very similar to that in Quantum Mechanics. In particular the Hilbert space in consideration is the oscillator realization of the canonical variables q_i and p_i. We know that it is a complete space and we can use the well known "unitarization" and "extension" procedure in which one needs to consider the operators

$$U_j(\sigma) = \exp[i\sigma p_j]$$
$$V_j(\sigma) = \exp[i\sigma q_j] \tag{I.21}$$

instead of p_i and q_i; σ is a real parameter. To summarize briefly this procedure, we consider the set D of all the finite summations of basic vectors of H_F:

$$D = \left\{ \underline{\xi}_N^{(D)} = \sum_{a=1}^{N} c_a \underline{\xi}_a \ , \quad N \text{ finite} \right\} . \tag{I.22}$$

The set D can be proved to be dense in H_F, i.e. it is characterized by the following properties:

- every vector of D belongs to H_F.
- every vector of H_F is either a member of D or limit of Cauchy sequence of vectors in D.

The last property can be expressed as

$$\underline{\xi} = \lim_{N\to\infty} \underline{\xi}_N \tag{I.23}$$

where $\underline{\xi}$ is a vector of H_F. We introduce then

$$U_j^M(\sigma) = \sum_{n=0}^{M} \frac{1}{n!} (i\sigma p_j)^n \tag{I.24}$$

where M is a positive and finite integer. Note that the action of any power of α_i and $\alpha_i^\dagger$ on a vector of the basis gives another vector of the basis. Thus, the action of any finite power of p_j on vectors in D creates a superposition of finite number of vectors in D, which is still a vector of D. Then it can be proved that the sequence of vectors $U_j^M(\sigma)\underline{\xi}_N^{(D)}$ has a limit for $M \to \infty$; thus, we define the operation of $U_j(\sigma)$ on D as

$$U_j(\sigma)\underline{\xi}_N^{(D)} = \lim_{M\to\infty} U_j^M(\sigma)\underline{\xi}_N^{(D)} .$$

Due to the unitarity of $U_j(\sigma)$,

$$|U_j(\sigma)\underline{\xi}_N^D| = |\underline{\xi}_N^D| \ , \tag{I.25}$$

from which we conclude that the operator $U_j(\sigma)$ is bounded, and therefore its definition can be "extended" on the whole H_F in the following way: let $\underline{\xi}$ be a vector of H_F; if it is a vector of D, the action of $U_j(\sigma)$ on $\underline{\xi}$ is well defined. If $\underline{\xi}$ is not a vector belonging to D, we can find in D a Cauchy sequence $\left\{\underline{\xi}_N^{(D)}\right\}$ whose limit is $\underline{\xi}$; then, we define the action of $U_j(\sigma)$ on $\underline{\xi}$ as

$$U_j(\sigma)\underline{\xi} = \lim_{N\to\infty} U_j(\sigma)\underline{\xi}_N^{(D)} \ . \tag{I.26}$$

In a similar way, we can define the action of $V_j(\sigma)$ on H_F. We therefore conclude that the Fock space of the physical particles is a representation of the unitary operators $U_j(\sigma)$ and $V_j(\sigma)$ with $j = 1,2,\ldots$ We also introduce $U(\bar{\sigma})$ and $V(\bar{\sigma})$ as

$$\begin{aligned} U(\bar{\sigma}) &= \exp\left[i \sum_{j=1}^{\infty} \sigma_j p_j\right] \\ V(\bar{\sigma}) &= \exp\left[i \sum_{j=1}^{\infty} \sigma_j q_j\right] \ , \end{aligned} \tag{I.27}$$

where we assume that only a finite number of σ_j are not zero. The operators $U(\bar{\sigma})$ and $V(\bar{\sigma})$ satisfy

$$U(\bar{\sigma})U(\bar{\tau}) = U(\bar{\sigma} + \bar{\tau}) \tag{I.28}$$

$$V(\bar{\sigma})V(\bar{\tau}) = V(\bar{\sigma} + \bar{\tau}) \tag{I.29}$$

$$U(\bar{\sigma})V(\bar{\tau}) = \exp[i\bar{\sigma}\cdot\bar{\tau}]V(\bar{\tau})U(\bar{\sigma}) \ , \tag{I.30}$$

where $(\bar{\sigma} + \bar{\tau}) \equiv \sum_j (\sigma_j + \tau_j)$ and $\bar{\sigma}\cdot\bar{\tau} \equiv \sum_j \sigma_j\tau_j$. The relation (I.30) reflects the canonical commutation relation (I.20). We observe that the knowledge of $U(\bar{\sigma})$ and $V(\bar{\sigma})$ can tell us about $p_j\underline{\xi}$ and $q_j\underline{\xi}$, respectively, whenever such vectors belong to H_F. Indeed

$$p_j\underline{\xi} = -i\left(\frac{d}{d\sigma_j} U(\bar{\sigma})\right)\underline{\xi}\Big|_{\bar{\sigma}=0} \tag{I.31}$$

$$q_j\underline{\xi} = -i\left(\frac{d}{d\sigma_j} V(\bar{\sigma})\right)\underline{\xi}\Big|_{\bar{\sigma}=0} \ . \tag{I.32}$$

Our next observation is that the Fock space introduced so far is an irreducible representation of the canonical variables q_j and p_j, i.e. of the annihilation and creation operators of physical particles [1,4,5]. In other words, any operator which commutes with $U(\bar{\sigma})$ and $V(\bar{\sigma})$ is a multiple of the identity operator. First, we can see that a vector $\underline{\zeta}$ of H_F is the null vector, i.e. $\underline{\zeta} = \underline{0}$, when and only when all the coefficients c_a (cf. e.g. (I.16)) are zero; one can see this from

$$(\underline{\xi}_a, \underline{\zeta}) = c_a \ ,$$

by using the orthonormality of the basis $\{\underline{\xi}_a\}$. Then, by using (I.5), we find that, if $\underline{\xi}$ is a vector of H_F,

$$\alpha_i \underline{\xi} = 0 \qquad \text{for all } i \ ,$$

when and only when

$$\underline{\xi} = c|0,0,\ldots\rangle \ ,$$

where c is an ordinary number. If A is an operator which commutes with q_i and p_i for all i, i.e. with α_i and $\alpha_i^\dagger$ for all i, then

$$\alpha_i A|0,0,\ldots\rangle = A\alpha_i|0,0,\ldots\rangle = 0 \ ,$$

i.e.

$$A|0,0,\ldots\rangle = c|0,0,\ldots\rangle \ ,$$

with c an ordinary number. Since any vector of the basis (I.13) is constructed by repeated operations of $\alpha_i^\dagger$, e.g.

$$|n_1,n_2,\ldots\rangle = f(n_1,n_2,\ldots)(\alpha_1^\dagger)^{n_1}(\alpha_2^\dagger)^{n_2}\ldots|0,0,\ldots\rangle, \tag{I.33}$$

with $f(n_1,n_2,\ldots)$ some function of n_i, we have

$$A|n_1,n_2,\ldots\rangle = c|n_1,n_2,\ldots\rangle \ ,$$

i.e., using (I.15),

$$A\underline{\xi} = c\underline{\xi}$$

for any $\underline{\xi}$ in H_F. This means that $A = cI$ with I the identity operator.

Let us note how the description of a system in terms of physical particles naturally leads to canonical variables whose irreducible representation is the above defined Fock space. As we will see in Chapter III, there exist infinitely many inequivalent representations of the canonical commutation relations, i.e. there are infinitely many Fock spaces which are unitarily inequivalent to each other [5]. This situation is peculiar to Quantum Field Theory: here we deal with an infinite number of canonical variables, contrary to what happens in Quantum Mechanics, when one has a finite number of variables only. For this reason, Von Neumann's theorem [6], which insures the unitary equivalence of all representations of canonical variables in Quantum Mechanics, cannot be applied to Quantum Field Theory. In Chapter III, we will analyze this problem in more detail. Here we want only to illustrate the present situation by recalling that in our construction of the Fock space we started by <u>choosing</u> in the <u>non separable</u> Hilbert space whose basis is the noncountable set (I.11), a separable <u>subset</u> whose basis is the countable set (I.13). The root of existence of infinitely many inequivalent representations is in the complete arbitrariness of our choice: there are indeed infinitely many ways of choosing a separable subset from the original non separable Hilbert space.

As we will see, the fact that the Fock space of physical particles is an irreducible representation of the canonical commutation relations, and the existence of infinitely many inequivalent representations of the canonical commutation relations, plays an important role in our analysis. Let us now rewrite our statements about the energy of a state of the system in the language of the Fock space. We consider a one-particle wave-packet state

$$\alpha_i^\dagger|0> = \int d^3k\ f_i(\vec{k})\alpha_{\vec{k}}^\dagger|0> \ .$$

Then, we introduce the energy operator H_o by requiring

$$H_o\alpha_i^\dagger|0> = \int d^3k\ E_k\ f_i(\vec{k})\alpha_{\vec{k}}^\dagger|0> \ . \tag{I.34}$$

Since this should be true for any square integrable function $f_i(\vec{k})$, we have

$$H_o\alpha_{\vec{k}}^\dagger|0> = E_k\ \alpha_{\vec{k}}^\dagger|0> \ . \tag{I.35}$$

Thus, for a many-particle system

$$H_o\alpha_{\vec{k}_1}^\dagger\cdots\alpha_{\vec{k}_n}^\dagger|0> = (E_{k_1}+\cdots+E_{k_n})\alpha_{\vec{k}_1}^\dagger\cdots\alpha_{\vec{k}_n}^\dagger|0> \ . \tag{I.36}$$

Since E_k is real, we require that H_o is self-adjoint

$$H_o^\dagger = H_o \ . \tag{I.37}$$

From (I.36) and (I.37) we can derive

$$\begin{aligned}
[H_o, \alpha_{\vec{k}}^\dagger] &= E_k \, \alpha_{\vec{k}}^\dagger \\
[H_o, \alpha_{\vec{k}}] &= -E_k \, \alpha_{\vec{k}} \ .
\end{aligned} \tag{I.38}$$

From (I.36), for $n = 0$, we find

$$H_o |0\rangle = 0 \ . \tag{I.39}$$

Putting

$$H_o = \sum_r \int d^3k \; E_k \, \alpha_{\vec{k}}^{r\dagger} \alpha_{\vec{k}}^r + H_1 \ , \tag{I.40}$$

where we restore the suffix r, (I.39), (I.38) and (I.3) give $H_1 = 0$. Thus,

$$H_o = \sum_r \int d^3k \; E_k (\alpha_{\vec{k}}^{r\dagger} \alpha_{\vec{k}}^r + \beta_{\vec{k}}^{r\dagger} \beta_{\vec{k}}^r) \ , \tag{I.41}$$

where we considered the operators $\beta_{\vec{k}}^r$, $\beta_{\vec{k}}^{r\dagger}$ also. (I.36) and (I.41) give exact meaning to the statement that the total energy of a system is given by the sum of the energy of the single "free" particles (cf. also (I.6) for the definition of the number operator).

In a similar way, we can introduce the momentum operator $\vec{P}_o$ as

$$\vec{P}_o = \sum_r \int d^3k\ \vec{k}(\alpha^{r\dagger}_{\vec{k}}\alpha^r_{\vec{k}} + \beta^{r\dagger}_{\vec{k}}\beta^r_{\vec{k}})\ . \tag{I.42}$$

Note that, although the operators H_o and $\vec{P}_o$ are well defined only on the dense set D, the operators

$$\exp[iH_o t] \tag{I.43}$$

and

$$\exp[i\vec{P}_o\vec{x}]\ , \tag{I.44}$$

with real t and $\vec{x}$, are well defined on the whole H_F. Let us introduce the "physical field" $\phi(x)$ defined as

$$\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[u(\vec{k})\alpha_{\vec{k}}\ e^{i\vec{k}\cdot\vec{x}-iE_k t} + \right.$$
$$\left. + v(\vec{k})\beta^\dagger_{\vec{k}}\ e^{-i\vec{k}\cdot\vec{x}+iE_k t}\right]\ . \tag{I.45}$$

In general, ϕ is a one-column matrix. The fact that the energy E_k of a physical particle is a certain function of its momentum means that the physical field $\phi(x)$ must solve a linear homogeneous equation:

$$\Lambda(\partial)\phi(x) = 0\ . \tag{I.46}$$

The differential operator $\Lambda(\partial)$ is in general a square matrix. The "wave functions" $u(\vec{k})$ and $v(\vec{k})$ are solutions of

$$\Lambda(ik)u(\vec{k}) = 0 \qquad \text{for } k_4 = -E_k \tag{I.47}$$

$$\Lambda(-ik)v(\vec{k}) = 0 \qquad \text{for } k_4 = E_k \tag{I.48}$$

with $k_\mu = (\vec{k}, iE_k)$. The field $\phi(x)$, as a superposition of $u(\vec{k})$ and $v(\vec{k})$ represents a general solution of (I.46). In relativistic theories the equation (I.46) always admits solution of negative and positive frequencies. In non-relativistic theories (e.g. solid state physics) the equation (I.46) can admit solutions of negative frequencies only (i.e. there can be no wave function $v(\vec{k})$). The equation (I.46) is called the free field equation. Since we are not going to study the equation (I.46) in detail, only the normalization of the wave functions are given here:

$$\begin{aligned} u^r_{\vec{k}}(x) &= u^r(\vec{k})\, e^{i\vec{k}\cdot\vec{x}-iE_kt} \\ v^r_{\vec{k}}(x) &= v^r(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}+iE_kt}, \end{aligned} \tag{I.49}$$

i.e.

$$\int d^3x\; u_{\vec{k}}^{r\dagger}(x)\Gamma_4(\partial,-\overleftarrow{\partial})v_{\vec{\ell}}^{s}(x) = 0$$

$$-\int d^3x\; u_{\vec{k}}^{r\dagger}(x)\Gamma_4(\partial,-\overleftarrow{\partial})u_{\vec{\ell}}^{s}(x) = \delta_{rs}\delta(\vec{k}-\vec{\ell}) \qquad \text{(I.50)}$$

$$-\int d^3x\; v_{\vec{k}}^{r\dagger}(x)\Gamma_4(\partial,-\overleftarrow{\partial})v_{\vec{\ell}}^{s}(x) = \pm\delta_{rs}\delta(\vec{k}-\vec{\ell})$$

where + is for fermions, - for bosons and the differential operator $\Gamma_4(\partial,-\overleftarrow{\partial})$ is uniquely determined by $\Lambda(\partial)$[2] [7].

1.3 The Heisenberg fields and the dynamical map.

In section 1.2 we introduced the concept of free field as one of the aspects of the duality in the description of nature. In the present section we introduce the other aspect of this duality, namely, the concept of Heisenberg field.

Although the physical particles undergo interaction processes, the language we set up in section 1.2 cannot describe such dynamical effects; the free field equations, in fact, do not contain any information about the interactions. Thus, we see that we need another source of information to describe the dynamics of a physical system. We assume the existence of basic entities, the Heisenberg

---

[2]For example, $\Gamma_4(\partial,-\overleftarrow{\partial}) = \gamma_4$ for the Dirac field and $\Gamma_4(\partial,-\overleftarrow{\partial}) = i\overleftrightarrow{\partial}_o$ for the scalar field. Note also that in (I.50) $u_{\vec{k}}^{r\dagger}(x)$ and $v_{\vec{k}}^{r\dagger}(x)$ are used for simplicity in place of $\bar{u}_{\vec{k}}^{r}(x) = u_{\vec{k}}^{r\dagger}(x)\eta$ and $\bar{v}_{\vec{k}}^{r}(x) = v_{\vec{k}}^{r\dagger}(x)\eta$ where η is the so called hermitization matrix [7].

fields, which satisfy basic relations characterizing the dynamics, namely, the Heisenberg equations. Denoting the Heisenberg fields by $\{\psi_i(x)\}$, we can <u>formally</u> write the "Heisenberg equations" as

$$\Lambda_i(\partial)\psi_i(x) = F(\psi_i(x)) \tag{I.51}$$

where $\Lambda_i(\partial)$ is a differential operator and F is some function of $\{\psi_i(x)\}$. (I.51) can be written in such a way that $\Lambda_i(\partial)$ can be made equal to the differential operator in the free field equation for ϕ_i (cf. eq. (I.46)). The Heisenberg equations describe the dynamical properties of our system. Let us stress, however, that the equation (I.51) is only a <u>formal relation</u> among the fields $\psi_i(x)$ unless we define their operational meaning, i.e. unless we specify the vector space on which they operate. In a traditional approach to Quantum Field Theory, the Heisenberg field operators are defined on a Hilbert space which is the Fock space of the so called "bare" particles. It is well known, however, that such a space is unitarily inequivalent to the Fock space of physical particles [4,5]. On the other hand, since we can observe only physical quantities, any kind of useful description of a system must be related to such physical quantities. In the present formulation, the only space we consider is, then, the Fock space of physical particles. To give a physical meaning to the description in terms of the Heisenberg fields, we introduce a mapping between such a description and the description in terms of physical fields. For this reason we require that the operators $\psi_i(x)$ must have well-defined matrix elements among states of the Fock space of physical particles H_F. Then we can read equation (I.51) as an equation among matrix elements.

To express mathematically the mapping between the two language levels we use in describing our system, we introduce the so called dynamical map [2-4], i.e. an expression of the Heisenberg field in terms of physical fields $\phi_j(x)$:

$$\psi_i(x) = \chi_i + \sum_j Z^{\frac{1}{2}}_{ij}\phi_j(x) + \sum_{j,k}\int d^4\xi_1\int d^4\xi_2 f_{ijk}(x,\xi_1,\xi_2)$$

$$:\phi_j(\xi_1)\phi_k(\xi_2): + \sum_{j,k,\ell}\int d^4\xi_1\int d^4\xi_2\int d^4\xi_3\ f_{ijk\ell}(x,\xi_1,\xi_2,\xi_3)$$

$$:\phi_j(\xi_1)\phi_k(\xi_2)\phi_\ell(\xi_3): +\ldots \qquad \text{(I.52)}$$

where $j,k,\ell\ldots$ are indices for different physical fields, χ_i is a c-number constant, which is zero unless $\psi_i(x)$ is a spinless field,[3] Z_{ij} is a c-number constant called the renormalization factor, the symbol :...: denotes normal product, ϕ stands for both ϕ and $\phi^\dagger$, $f_{ijk}(x,\xi_1,\xi_2)$, etc. are c-number functions, and the dots denote terms which

---

[3]Intuitively, χ_i is related to the square root of the density of the boson condensation.

contain higher order normal products. We will refer to χ_i, Z_{ij}, f_{ijk}, etc. as the coefficients of the dynamical map. We now clarify the meaning of (I.52). Let us start by saying that the equality in (I.52) must be read as an equality among matrix elements between states of the Fock space of physical particles, i.e. (I.52) is not an equality among field operators, but an equality among matrix elements.[4] We will call such kind of equalities weak equalities (or weak relations). Since $\{\psi_i(x)\}$ are defined by the dynamical map (I.52), which is a weak relation, then the Heisenberg equations (I.51) must also be read as weak relations. In (I.51) we can have products of fields $\psi(x)$. Thus, we define these products through the dynamical map by the following rule:

$$\left[\sum_{j,k}\int d^4\xi_1\int d^4\xi_2\; f_{ijk}(x,\xi_1,\xi_2):\phi_j(\xi_1)\phi_k(\xi_2):\right] \times$$

$$\times \left[\sum_{\ell,m}\int d^4\zeta_1\int d^4\zeta_2\; f'_{i\ell m}(y,\zeta_1,\zeta_2):\phi_\ell(\zeta_1)\phi_m(\zeta_2):\right]$$

$$\equiv \sum_{j,k,\ell,m}\int d^4\xi_1\int d^4\xi_2\int d^4\zeta_1\int d^4\zeta_2\; f_{ijk}(x,\xi_1,\xi_2)f_{i\ell m}(y,\zeta_1,\zeta_2)$$

$$:\phi_j(\xi_1)\phi_k(\xi_2)::\phi_\ell(\zeta_1)\phi_m(\zeta_2): \qquad \text{(I.53)}$$

etc.

---

[4] The presence of normal products in (I.52) is due to the fact that we are indeed interested in the computation of matrix elements.

However, since products of ψ at a same point $(x = y)$ may not be well defined, we need certain limiting procedures as

$$\lim_{\varepsilon \to 0} \psi(x)\ \psi(x+\varepsilon) \tag{I.54}$$

where the limit $\varepsilon \to 0$ must be taken according to well-specified rules. When (I.53) (together with well-specified rules for same-point products) is used, we can calculate matrix elements of products of $\psi_i(x)$ by calculating the well-defined matrix elements of the right-hand side of (I.53). Here we recall (cf. Sec. 1.2) the fact that matrix elements of physical fields are well-defined for states belonging to the set D dense in H_F, and by unitarization and extension procedures, for states belonging to the whole H_F.

Let us observe that there is no problem of convergence of the summation on the right hand side of (I.52) because all the normal product terms are linearly independent.

To better specify the nature of (I.52) we need to choose a particular class of physical fields $\{\phi_j\}$ among the infinite classes of such fields unitarily related to each other. Our choice here and in the following will be that of the incoming fields $\{\phi_j^{in}\}$. As a consequence of this choice, the coefficients in (I.52) must be of retarded nature, i.e. the domain of time integrations in (I.52) must be from $-\infty$ to t_x. This is equivalent to requiring that the physical incoming fields affect the Heisenberg fields $\psi_i(x)$ only from the past.[5] We can see, then, that time integrations in the dynamical map are well-defined only when contributions from $t \to -\infty$ vanish; and this shows that in taking the matrix elements of the terms of the map, it is essential that the states are not plane-wave states, but are wave-packet states (cf. eq. (I.1)).

To make this fundamental point more clear, let us assume translational invariance in (I.52), i.e. let us assume that the translation $\xi_\mu \to \xi_\mu + a_\mu$ in the physical fields (we consider, as we said, incoming fields) $\{\phi_i^{in}(\xi)\}$ induces the translation $x_\mu \to x_\mu + a_\mu$ in the Heisenberg fields $\psi_i(x)$. In general, the retarded functions $f_{ijk}(x,\xi_1,\xi_2)$, etc. will depend on the differences $x-\xi_1$, $x-\xi_2$, etc., and we write them as

$$f_{ijk}(x-\xi_1,x-\xi_2) = \theta(t_x-t_{\xi_1})\theta(t_x-t_{\xi_2})F_{ijk}(x-\xi_1,x-\xi_2) \qquad (I.55)$$

etc.

We consider now the matrix element $\langle 0|\psi_i(x)|\alpha_\ell\alpha_n\rangle$ as an example. Introducing the Fourier form of $F_{i\ell n}(x-\xi_1,x-\xi_2)$

[5] A unitary transformation will leave the dynamical mapping unaltered, while the coefficients of the map will be affected. If one would choose the outgoing fields $\{\phi_j^{out}\}$ as physical fields, then the coefficients must have advanced nature. We are assuming here the existence of a unitary operator, the S-matrix, which transforms $\phi^{in} \leftrightarrow \phi^{out}$ (cf. (I.80)).

as

$$F_{i\ell n}(x-\xi_1, x-\xi_2) =$$

$$= \int \frac{d^4p^{(1)}}{(2\pi)^4} \int \frac{d^4p^{(2)}}{(2\pi)^4} F_{i\ell n}(p^{(1)}, p^{(2)}) \exp\left[ip_\mu^{(1)}(x-\xi_1)_\mu + ip_\mu^{(2)}(x-\xi_2)_\mu\right], \quad (I.56)$$

with $p_\mu x_\mu \equiv \vec{p}\cdot\vec{x} - E_p t$, and using (I.1), (I.45) and (I.49), (I.52) gives:

$$<0|\psi_i(x)|\alpha_\ell \alpha_n> = \int \frac{d^4p^{(1)}}{(2\pi)^4} \int \frac{d^4p^{(2)}}{(2\pi)^4} \int \frac{d^3k^{(1)}}{(2\pi)^{3/2}} \int \frac{d^3k^{(2)}}{(2\pi)^{3/2}}$$

$$F_{i\ell n}(p^{(1)}, p^{(2)}) u(\vec{k}^{(1)}) u(\vec{k}^{(2)})$$

$$f_\ell(\vec{k}^{(1)}) f_n(k^{(2)}) \int_{-\infty}^{t_x} d^4\xi_1 \int_{-\infty}^{t_x} d^4\xi_2 \; e^{i\vec{k}^{(1)}\cdot\vec{\xi}_1 - i\omega_1 t_1}$$

$$e^{i\vec{k}^{(2)}\cdot\vec{\xi}_2 - i\omega_2 t_2} \; e^{ip_\mu^{(1)}(x-\xi_1)_\mu + ip_\mu^{(2)}(x-\xi_2)_\mu}$$

$$\pm (\ell \leftrightarrow n)\text{-term} \quad (I.57)$$

Where $(\ell \leftrightarrow n)$-term means the term obtained by exchanging ℓ with n; + is for fermions, - for bosons, $\omega_{1(2)} = \omega_{\vec{k}_{1(2)}}$ $t_{1(2)}$ means $t_{\xi_1(\xi_2)}$. Now, we note that the relation

$$\int dE \int_{-\infty}^{t} dt_1 \; f(E) \; e^{iEt_1} = \int dE \; f(E) \; \frac{e^{iEt}}{i(E-i\varepsilon)} \quad (I.58)$$

holds whenever f(E) is a square-integrable function; indeed, when f(E) is a square-integrable function, due to the Riemann-Lebesque theorem, we have

$$\lim_{t_1\to\pm\infty}\int dE\; f(E)\; e^{iEt_1} = 0\;. \tag{I.59}$$

(I.58) and (I.59) also give

$$\lim_{t\to-\infty}\int dE\; f(E)\;\frac{e^{iEt}}{i(E-i\varepsilon)} = 0 \tag{I.60}$$

and

$$\lim_{t\to+\infty}\int dE\; f(E)\;\frac{e^{iEt}}{i(E-i\varepsilon)} = 2\pi f(0) \tag{I.61}$$

In (I.58-61) the limit $\varepsilon\to+0$ is understood.

As an extension of (I.60) we can say that if F(t) is a retarded function, its Fourier transform F(E) can be written as

$$F(E) = \lim_{\varepsilon\to+0}\overline{F}(E-i\varepsilon)$$

with $\overline{F}(E)$ an analytic function in the lower half plane of complex E.

By using (I.58), (I.57) gives:

$$<0|\psi_i(x)|\alpha_\ell\alpha_n> = -\int\frac{dE_1}{(2\pi)}\int\frac{dE_2}{(2\pi)}\int\frac{d^3k^{(1)}}{(2\pi)^{3/2}}\int\frac{d^3k^{(2)}}{(2\pi)^{3/2}}$$

$$F_{i\ell n}(\vec{k}^{(1)},E_1;\vec{k}^{(2)},E_2)u(\vec{k}^{(1)})f_\ell(\vec{k}^{(1)})$$

$$u(\vec{k}^{(2)})f_n(\vec{k}^2)\ e^{i(\vec{k}^{(1)}+\vec{k}^{(2)})\cdot\vec{x}}\ \frac{e^{-i(\omega_1+\omega_2)t_x}}{(E_1-\omega_1-i\varepsilon)(E_2-\omega_2-i\varepsilon)}$$

$$\pm(\ell\leftrightarrow n)\text{-term}\ . \tag{I.62}$$

When $F_{i\ell n}(\vec{k}^{(1)},E_1;\vec{k}^{(2)},E_2) = F_{i\ell n}(p^{(1)},p^{(2)})$ with $\vec{p}^{(1)} = \vec{k}^{(1)}$, $p_4^{(1)} = iE_1$, $\vec{p}^{(2)} = \vec{k}^{(2)}$ and $p_4^{(2)} = iE_2$. As usual the limit $\varepsilon\to+0$ is understood. In a similar way we have

$$<\beta_\ell|\psi_i(x)|\alpha_n> = -\int\frac{dE_1}{(2\pi)}\int\frac{dE_2}{(2\pi)}\int\frac{d^3k^{(1)}}{(2\pi)^{3/2}}\int\frac{d^3k^{(2)}}{(2\pi)^{3/2}}$$

$$F_{i\ell n}(-\vec{k}^{(1)},E_1,\vec{k}^{(2)},E_2)\ v(\vec{k}^{(1)})f_\ell^*(\vec{k}^{(1)})u(\vec{k}^{(2)})f_n(\vec{k}^{(2)})$$

$$e^{i(-\vec{k}^{(1)}+\vec{k}^{(2)})\cdot\vec{x}}\quad\frac{e^{i(\omega_1-\omega_2)t_x}}{(E_1+\omega_1-i\varepsilon)(E_2-\omega_2-i\varepsilon)}$$

$$\pm\ (v\leftrightarrow u,\ f_\ell^*\leftrightarrow f_n,\ \omega_1\leftrightarrow-\omega_1,\ \omega_2\leftrightarrow-\omega_2)\text{-term} \tag{I.63}$$

and

$$<\beta_\ell|\psi_i(x)|\alpha_n\alpha_m> = \frac{i}{(2\pi)^3}\int\frac{d^4k^{(1)}}{(2\pi)^{3/2}}\int\frac{d^4k^{(2)}}{(2\pi)^{3/2}}\int\frac{d^4k^{(3)}}{(2\pi)^{3/2}}$$

$$F_{i\ell nm}(-k^{(1)},k^{(2)},k^{(3)})\ v(\vec{k}^{(1)})f_\ell^*(\vec{k}^{(1)})u(\vec{k}^{(2)})f_n(\vec{k}^{(2)})$$

$$u(\vec{k}^{(3)})f_m(\vec{k}^{(3)})\ e^{i(-\vec{k}^{(1)}+\vec{k}^{(2)}+\vec{k}^{(3)})\cdot\vec{x}}$$

$$\frac{e^{i(\omega_1-\omega_2-\omega_3)t_x}}{(E_1+\omega_1-i\varepsilon)(E_2-\omega_2-i\varepsilon)(E_3-\omega_3-i\varepsilon)} + \cdots \tag{I.64}$$

where $F_{i\ell nm}(-k^{(1)},k^{(2)},k^{(3)})$ is the Fourier amplitude of $F_{i\ell nm}(x-\xi_1,x-\xi_2,x-\xi_3)$ with $k_4^{(j)} = iE_j$, and the dots denote terms where suffices are exchanged. Thus, we see how all matrix elements of $\psi_i(x)$ can be obtained from (I.52). Further examples are:

$$<0|\psi_i(x)|0> = \chi_i \tag{I.65}$$

$$<0|\psi_i(x)|\alpha_j> = Z_{ij}^{\frac{1}{2}}\ u_j(x) \tag{I.66}$$

$$<\beta_j|\psi_i(x)|0> = Z_{ij}^{\frac{1}{2}}\ v_j(x) \tag{I.67}$$

etc.

where we used (I.49). There is no summation on repeated indices, and

$$u_i(x) \equiv \int \frac{d^3k}{(2\pi)^{3/2}} f_i(\vec{k}) u_{\vec{k}}(x) \tag{I.68}$$

In the previous examples, besides the fact that it is essential for states to be wave-packet states, we see that the time dependence of the Heisenberg fields (which means of the matrix elements of the Heisenberg fields) is controlled by the free energies; indeed the energy-like $\omega_1+\omega_2$ (cf. (I.62)) is nothing but the balance of energies of the initial and final free states. In this connection, let us study the limits $t\to\pm\infty$ of (I.52). As usual, the relations and the limits we consider are weak ones, i.e. relations and limits of matrix elements.

(I.65) and (I.66) show that

$$\lim_{t\to-\infty} (\psi_i(x) - \chi_i - \sum_j Z_{ij}^{1/2} \phi_j^{in}(x)) = 0 \tag{I.69}$$

Indeed, due to the retarded nature of the coefficients $f_{ij\ell}\cdots$, the limit $t\to-\infty$ excludes higher order terms (cf. eq. (I.57)).

As in the LSZ method, we introduce

$$a_{ij}(t) = -(2\pi)^3 \int d^3x \; u_j^+(x) \Gamma_4(\partial,-\overleftarrow{\partial})(\psi_i(x)-\chi_i) \; . \tag{I.70}$$

The field $(\psi_i(x)-\chi_i)$ is also called the interpolating field. Use of (I.68-70), (I.45), (I.50) and (I.1) gives

$$\lim_{t\to-\infty} a_{ij}(t) = Z_{ij}^{1/2} \; \alpha_j^{in} \quad \text{(no summation on j)} \tag{I.71}$$

This shows that the interpolating field approaches (weakly) the incoming field at $t \to -\infty$. Note that once more the use of smeared functions is essential: the limit (I.71) would be meaningless otherwise. To study the limit $t \to +\infty$ of $a_{ij}(t)$, we note that (I.61) gives

$$\lim_{t \to +\infty} \frac{e^{iEt}}{i(E - i\varepsilon)} = 2\pi\delta(E) \, . \tag{I.72}$$

(I.63) gives:

$$\begin{aligned} <\beta_\ell | a_{ij}(t) | \alpha_n> = (2\pi)^3 \int \frac{dE_1}{(2\pi)} \int \frac{dE_2}{(2\pi)} \int d^3k^{(1)} \int \frac{d^3k^{(2)}}{(2\pi)^{3/2}} \\ f_j^*(-\vec{k}^{(1)} + \vec{k}^{(2)}) \; u^\dagger(-\vec{k}^{(1)} + \vec{k}^{(2)}) \Gamma_4 (i(-k^{(1)} + k^{(2)}), -ik) \\ F_{i\ell n}(-\vec{k}^{(1)}, E_1; \vec{k}^{(2)}, E_2) \; v(\vec{k}^{(1)}) f_\ell^*(\vec{k}^{(1)}) \; u(\vec{k}^{(2)}) f_n(\vec{k}^{(2)}) \\ \frac{e^{i(\omega + \omega_1 - \omega_2)t_x}}{(E_1 + \omega_1 - i\varepsilon)(E_2 - \omega_2 - i\varepsilon)} \; \pm \; (v \leftrightarrow u, \; f_\ell^* \leftrightarrow f_n, \omega_1 \leftrightarrow -\omega_1, \omega_2 \leftrightarrow -\omega_2) \end{aligned} \tag{I.73}$$

where $k^{(1)} = (\vec{k}^{(1)}, i\omega_{\vec{k}}(1))$, $k^{(2)} = (\vec{k}^{(2)}, i\omega_{\vec{k}}(2))$, $k = (-\vec{k}^{(1)} + \vec{k}^{(2)}, i\omega)$, $\omega_1 = \omega_{\vec{k}}(2)$, $\omega_2 = \omega_{\vec{k}}(2)$, $\omega = \omega_{\vec{k}}(1)_{-\vec{k}}(2)$.

Due to the argument given after the equation (I.61), we can write

$$F_{i\ell n}(-\vec{k}^{(1)},E_1,\vec{k}^{(2)},E_2) = \frac{c_1^{i\ell n}(-\vec{k}^{(1)},E_1,\vec{k}^{(2)},E_2)}{2\pi i(\omega-E_1-E_2-i\varepsilon)} +$$

$$+ c_2^{i\ell n}(-\vec{k}^{(1)},E_1;\vec{k}^{(2)},E_2) \qquad \text{(I.74)}$$

where the c_2-term does not contain the factor $1/(\omega-E_1-E_2-i\varepsilon)$. Now, use of (I.74), (I.72) and (I.59) in (I.73) gives

$$\lim_{t\to+\infty} \langle\beta_\ell|a_{ij}(t)|\alpha_n\rangle =$$

$$(2\pi)^3\int d^3k^{(1)}\int\frac{d^3k^{(2)}}{(2\pi)^{3/2}}\, f_j^*(-\vec{k}^{(1)}+\vec{k}^{(2)})u^\dagger(-\vec{k}^{(1)}+\vec{k}^{(2)})$$

$$\Gamma_4(i(-k^{(1)}+k^{(2)}),-ik)c_1^{i\ell n}(-\vec{k}^{(1)},-\omega_1;\vec{k}^{(2)},\omega_2)\delta(\omega+\omega_1-\omega_2)$$

$$v(\vec{k}^{(1)})f_\ell^*(\vec{k}^{(1)})u(\vec{k}^{(2)})f_n(\vec{k}^{(2)})$$

$$\pm\ (v\leftrightarrow u,\ f_\ell^*\leftrightarrow f_n,\ \omega_1\leftrightarrow-\omega_1,\ \omega_2\leftrightarrow-\omega_2) \qquad \text{(I.75)}$$

Note that the right-hand side of (I.75) is zero when $\omega_2<\omega_1+\omega$.

In a similar way, we can show that other matrix elements of $a_{ij}(t)$ at $t=+\infty$ depend on factors similar to $c_1^{i\ell n}$. This means that, unless c_1-terms vanish,

$$\lim_{t\to+\infty} a_{ij}(t) \neq \lim_{t\to-\infty} a_{ij}(t)\ . \qquad \text{(I.76)}$$

Defining the annihilation operators of outgoing particles by

$$\lim_{t\to+\infty} a_{ij}(t) = Z^{\frac{1}{2}}_{ij}\ \alpha^{out}_j \quad \text{(no summation on j)} \tag{I.77}$$

(I.76) and (I.71) mean

$$\alpha^{in}_j \neq \alpha^{out}_j \tag{I.78}$$

The relation (I.78) shows how our definition of free particles does not mean that there is no interaction: (I.78) shows indeed that the in-fields are different from the out-fields and therefore that there exists interaction. Let us stress how the description in terms of physical particles given in Sec. 1.2 finds its expression in the dynamical map properties. We will refer to the in(out)-fields as the asymptotic limit of the interpolating fields.

We assume that in-fields and out-fields are related by a unitary operator S, called S-matrix:

$$\alpha^{out}_j = S^{-1}\ \alpha^{in}_j\ S \tag{I.79}$$

Then (I.78) shows that S is different from the identity operator when and only when the factors $c^{i\ell\cdots}_1$ vanish. Introducing the out-field operator $\phi^{out}(x)$ as the operator (I.45) with $\alpha_{\vec{k}}$ and $\beta_{\vec{k}}$ replaced by $\alpha^{out}_{\vec{k}}$ and $\beta^{out}_{\vec{k}}$, respectively, (I.79) is written as

$$\phi_j^{out}(x) = S^{-1} \phi_i^{in}(x) S . \tag{I.80}$$

It is not difficult to prove (and we omit the proof for the sake of brevity) that the vacuum state and the one-particle state are "stable" under S-matrix operation, i.e.

$$S|0> = |0> \tag{I.81}$$

and

$$S|\alpha_j> = |\alpha_j> \tag{I.82}$$

One can also express this by saying that one outgoing particle state is equal to one incoming particle state:

$$\alpha_j^{out}|0> = |\alpha_j> = \alpha_j^{in}|0> \tag{I.83}$$

The physical meaning of (I.81-83) is evident when one assumes that the probability for the reaction $|b> \rightarrow |a>$ is given by $|<a\text{-out}|b>|^2$, which is equal to $|<a|S|b>|^2$ (cf. (I.79)).[6] We see indeed that (I.81-83) mean that no reactions occur in vacuum or single particle states.

[6] The fact that the S-matrix elements involve only incoming and outgoing fields makes a specific choice of the interpolating fields immaterial in the theory [8].

Let us note that there is not necessarily a one-to-one correspondence between the sets $\{\psi_i\}$ and $\{\phi_j\}$. Assume the set of the Heisenberg fields has n members, while the set of in-fields has m members. It could happen that the asymptotic limit $t\to-\infty$ of $(\psi_i(x)-\chi_i)$ gives us a set of ℓ in-fields with $\ell<m$. This means that for some values of the index i, the asymptotic limit vanishes. In such a circumstance we say that we have ℓ elementary particles and $m-\ell$ composite particles [2,3,4,9,13]. We see, then, that composite particle fields will not appear in the linear part of the map (cf. eq. (I.69)). We will come back to this point in the next section.

We observe that the Heisenberg fields $\psi_i(x)$ given by (I.52) must satisfy the field equations (I.51), which describe the dynamics of the system under consideration. This condition determines the coefficients of the map that for this reason is called dynamical map; i.e., the mapping among $\psi_i(x)$ and $\phi_i(x)$ will be different in different dynamical situations, that is, for different Heisenberg field equations.

We already observed that the time dependence of the Heisenberg field is controlled by the physical fields in each term of the map. This implies that the coefficients of the dynamical map must be time independent. Let us consider again the free Hamiltonian H_o given in (I.41). By using the dynamical map (I.52) one can prove [2,4,9,10,11] that (cf.,e.g. (I.64) and (I.34))

$$\langle a|\,[H_o,\psi_i(x)]\,|b\rangle = \langle a|\,\frac{1}{i}\frac{\partial}{\partial t}\,\psi(x)\,|b\rangle \qquad \text{(I.84)}$$

where $|a\rangle$ and $|b\rangle$ are any two vectors belonging to a set D dense in H_F. On the other hand, if one introduces the Heisenberg Hamiltonian operator H such that

$$\langle a|[H,\psi_i(x)]|b\rangle = \langle a|\,\frac{1}{i}\frac{\partial}{\partial t}\,\psi(x)\,|b\rangle\ , \qquad \text{(I.85)}$$

by comparing (I.84) and (I.85) we can write

$$H = H_o + W_o \qquad \text{(I.86)}$$

where W_o is a c-number constant. Indeed, in deriving (I.86) we considered the fact that any quantity which commutes with the irreducible set $\{\phi_j^{in}\}$ must be a c-number constant. Due to (I.78), (I.86) does not imply that there is no interaction. We observe also that (I.86), where H is written in terms of ψ_i, while H_o is in terms of ϕ_j, is a direct consequence of the dynamical map (I.52), and thus is a weak equality.

Finally, we note that commutation relations among Heisenberg fields do not need to be postulated a priori, but can be calculated by using the dynamical map, and that there is no reason to expect that such commutators reduce to the canonical commutation relations at equal time. We have seen in Sec. 1.2 how the particle interpretation is naturally connected with a canonical formalism, in which one can represent the states of the system by means of vectors in a Hilbert space. This space is an irreducible representation of the canonical commutation relations and the vectors of the space can be represented by a series of integers corresponding to the number of particles present in each state. In a traditional formulation of Q.F.T., the particle interpretation essentially motivated the requirement for the Heisenberg fields commutators to be canonical. However, we observe that there is no reason to require a particle interpretation at the level of Heisenberg fields, since the only experimentally observable particles are not the so-called "bare" particles, but the physical particles. For these reasons, in the present formulation of Q.F.T., the only vector space we are interested in is the Fock space for physical particles on which <u>any</u> field operator should be defined. Thus one is naturally brought to express the Heisenberg fields through the dynamical map (I.52). Here there arises the question about the existence and the uniqueness of the map.

About the uniqueness, we can observe the following. Suppose the map exists. As we saw, the dynamics, i.e., the fundamental equations with specified boundary conditions, will determine the map uniquely. Thus, the problem of the uniqueness of the map is the problem of the uniqueness of the fundamental equations and this is, at the moment, an unsolved problem.

The existence of the map is related to the existence of an <u>irreducible</u> set of free fields, in terms of which any other operator, and thus also the Heisenberg operators, should be expressible. On the other hand, since in any physical problem the free fields are viewed as a phenomenological realization of a certain dynamics, it is reasonable to assume the existence of an irreducible set of such free fields. Then, if a self-consistent computation will not admit any solution, one could infer that the assumed fundamental equations do not describe the interaction accurately enough. In other words, the non-existence of the map will give us some information about the choice of the Heisenberg equations.

We can see, in any event, that the problems of the existence and uniqueness of the map need a deeper understanding and a more careful mathematical analysis. In particular, it will be very interesting to find necessary and sufficient conditions for the existence and uniqueness of the map.

1.4 The self-consistent method

In the previous sections we have seen that the experimental observations are described in terms of physical particles language. The dynamics of the physical systems is described in terms of Heisenberg fields, which are well defined in the Fock space of physical particles through the dynamical map. To build this Fock space we need to know what is our set of physical field operators. However, this set of physical field operators is determined by the dynamics, and this already presumes the knowledge of the Fock space of physical particles: we are clearly facing a problem of self-consistency.

A similar situation is present, for example, in the Lehmann-Symanzik-Zimmermann formalism [12], where the incoming fields are the asymptotic <u>weak</u> limit of the Heisenberg fields. However, to perform such limits one needs the knowledge of the Fock space associated with the incoming fields.

An outline of a self-consistent computation is the following [2,3,4,9]: the starting point is to assume certain free fields as candidates for our set of physical incoming fields by appealing to physical considerations and intuition. We then write the dynamical map (I.52). To solve the problem of finding convenient coefficients for the map, we consider matrix elements of both sides of (I.52) leaving the physical energy spectra unknown. The Heisenberg equations (I.51), which are equations relating to the matrix elements of $\psi_i(x)$, turn out to be a set of equations for the expansion coefficients of the map. The solutions of these equations will thus determine the map (I.52) together with the energy spectra of the physical particles. However, since the set of physical fields we started with, could not be a complete set, the set of coupled equations for the coefficients of the map could not admit any consistent solution; thus we need a self-consistent adjustment of the map. This can be done by introducing one or more new elements in the set of physical fields. Then the computation is repeated. The reason we need to introduce more free fields to get consistent solutions, has its root in our requirement that the Heisenberg fields must have well defined matrix elements in the Fock space of physical particles. This space is indeed an irreducible representation of the canonical commutation relations among physical field operators (cf. Sec. 1.2). This implies that any other operator defined in this space must be a function of the physical operators, or said in a different way, the physical operators form a complete set. When the set of physical infields we start with is not a complete set, then there will be at least one operator $\tilde{\psi}$ formed by the Heisenberg fields $\psi_i(x)$ whose asymptotic(weak) limit $t\to-\infty$ commutes with all the physical infields. The asymptotic weak limit of $\tilde{\psi}$ thus locates a new incoming field called field of composite particles (cf. Sec. 1.3) [2,4,9,13]. This concept of compositeness is based on the irreducibility of the set of physical fields. A successful example of self-consistent computation is given by the study of the deuteron [14,15]. Assume we are given a Heisenberg equation for the nucleon field. As an initial set of physical fields we use an isodoublet free Dirac field. We write the Heisenberg nucleon field in terms of normal products of the physical nucleon fields through the map. Then we consider the equations for the matrix elements of the Heisenberg nucleon field. We find that they do not admit a solution, unless another member is introduced in the initial set of physical fields. This new member turns out to be identifiable with the physical deuteron. From what we said above, we see that the deuteron can then be regarded as a composite particle. This description of the deuteron has led to reasonable agreement with the experimental results for low-energy neutron-proton scattering when the experimental value of the deuteron mass is used.

Until now, we have referred to the Heisenberg equations as the only conditions to be imposed in finding the map. It is clear, however, that the particular system under study will require that other conditions be fulfilled. Such conditions are to be analyzed case by case. A more general condition [9] is that of the microcausality,[7] which

---

[7] It is well known that local operators commute with each other on a space-like surface when their positions do not coincide.

requires that equal-time (anti-)commutators of Heisenberg fields are some polynomials of finite powers of space-derivatives of δ function. As we mentioned in Sec. 1.3, in the self-consistent formulation these commutators must be calculated (not postulated), thus the microcausality is a natural constraint on the map.

In a relativistic theory the coefficients of the map should be such that the Lorentz invariance of the theory is preserved. In a non relativistic theory we do not need such a condition. However, as we will see later, due to the possibility of using certain transformations (i.e. the boson transformations) we can still require that translational invariance must be satisfied.

Let us assume that the Heisenberg fields undergo a certain transformation T which leaves the Heisenberg equations invariant. Consequently, through the dynamical map, the free fields will undergo a transformation T'. Due to the invariance of the theory under T, the free field equations must be invariant under T'. When the two transformations T and T' have different forms, we say that the symmetry is <u>dynamically rearranged</u> [3,4,10,11,16-20]. This phenomenon will be analyzed in detail in Chapter II.

As we have seen the self-consistent method leads us to the choice of a set of physical operators. This means that in the self-consistent method we select a particular Fock space for physical particles among infinitely many unitarily inequivalent to each other. Thus our problem is one of self-consistency among the Heisenberg field equations and the Fock space of physical fields. The self-consistent conditions are related to the particular boundary conditions associated with the Heisenberg equations. In certain respects, the self-consistent method is analogous to the Hartree-Fock method where wave functions are self-consistently chosen. An example of a self-consistent computation has been presented for a relativistic model [9] to which we refer the reader for more details.

II. DYNAMICAL REARRANGEMENT OF SYMMETRY

2.1 Original symmetry and phenomenological symmetry.

As mentioned in Chapter I, many symmetry patterns, which we can recognize in the physical investigation, appear to be violated to a certain degree in the actual observations; usually one says that, in such circumstances, <u>the symmetry is broken</u>. This terminology suggests that the symmetry invariances eventually present in the theory are in some way lost at the phenomenological level. It is clear, however, that the invariant properties of the fundamental equations cannot simply disappear, since the theory must be internally consistent [2]. As a matter of fact in many cases we can recognize some symmetry patterns also at the level of the observation (e.g. crystals, ferromagnets, etc.; cf. Sec. 1.1).[8] Then, it is evident that the analysis of the problem of "broken symmetry" requires the study of the relation between original symmetry and phenomenological symmetry; we are thus still facing the problem of the mapping between the two languages used in Quantum Field Theory: the basic or Heisenberg field language and the physical field language. In the self-consistent method, the connection between original and

[8] In this connection we note that the use of a terminology as "hidden" or "secret" symmetry in place of "broken" symmetry [21] is only relatively recent.

phenomenological symmetry is quite obviously displayed through the dynamical map. Furthermore, due to the non-linear character of the dynamical map, which reflects the non-linear dynamical effects, one naturally expects that the original symmetries can <u>manifest</u> themselves through the mechanism of <u>dynamical rearrangement</u> at the level of physical fields. Indeed, to the comments on the dynamical map (cf. eq. (I.52)) given in Chapter I, we should add that the left and the right hand sides of (I.52) must have the same symmetry properties, although <u>not necessarily term-by-term</u>. To be more specific, let us consider a theory which is invariant under the transformation of the Heisenberg fields:

$$\psi(x) \rightarrow \psi'(x) = T[\psi(x)] \, , \tag{II.1}$$

where we have dropped the subscripts for convenience. The theory is said to be invariant under the transformation T when the basic dynamical equations are form-invariant under T. We can write the dynamical map schematically as

$$\psi(x) = \Phi[\phi(x)] \, . \tag{II.2}$$

Then we see that the transformation (II.1) can be performed only if the field $\phi(x)$ undergoes a transformation S

$$\phi(x) \rightarrow \phi'(x) = S[\phi(x)] \, , \tag{II.3}$$

such that

$$T[\psi(x)] = \Phi[S[\phi(x)]] \quad . \tag{II.4}$$

When the form of T is different from that of S we speak of dynamical rearrangement of symmetry [3,4,10,11,16-20]. Note that (II.2-4) are weak equalities. Let us note that from the above point of view the breakdown of symmetry is interpreted as a dynamical effect in the sense that the Lagrangian (or, if you want, the basic equations) is fully invariant, while because of the dynamics, the symmetry can appear in a different form at the physical level. In this case one speaks of spontaneous or dynamical breakdown of symmetry. Another approach is to assume a symmetry-violating term added to the original invariant Lagrangian: then one says that the symmetry is intrinsically broken. In this latter case, however, it has been shown [19] that a dynamical effect in the breaking of the symmetry can be still present. For example, in the Nambu model [22,23] if one introduces a bare mass term like $m_o\bar{\psi}\psi$ in the Lagrangian to break the γ_5-invariance, one finds that when the bare coupling constant is larger than a certain critical value, the asymmetric solution is still present in the limit $m_o \to 0$.

In the following sections of this chapter, we analyze peculiar features of the dynamical rearrangement of symmetry. The symmetries we consider are internal continuous symmetries. If G is the symmetry group under which the basic equations are invariant and G^{in} is the symmetry group associated with the physical fields, an important question to ask is how the two groups are related to each other. We will find cases in which $G \neq G^{in}$ (cf. Sec. 2.5).

One reason why the observable symmetry group G^{in} can be different from the original symmetry group G is based on the fact that any macroscopic observation on an infinite system is a collection of local observations. Therefore, there always exists a possibility that in each local observation one misses an infinitesimal effect of the order of magnitude of $\frac{1}{v}$ (with the volume $v \to \infty$). This missing effect can be accumulated as a finite amount when it is integrated over the whole system. Such a locally infinitesimal effect is called the infrared effect [24,25], and when it is taken into account we can show that the original symmetry group is restored.

2.2 Asymmetric ground states.

In Sec. 2.1 we introduced the notion of the invariance of a theory under certain symmetry transformation as the invariance of the Heisenberg equations under such a transformation of the Heisenberg fields. The meaning of the dynamical rearrangement of symmetry is that a basic invariance of the theory cannot disappear but should be always present, although it could be manifest in a different form at the level of physical fields. Essentially, the invariance of a theory is understood as the invariance of operator relations and has thus definitely an algebraic character. Consequently, physical states may not possess the symmetry properties of the basic equations. This is what we actually observe in systems where spontaneous breakdown of symmetry occurs: the physical vacuum or, in the many body terminology, the physical ground state exhibits certain asymmetries in spite of the invariance of the theory [4,16-18,22,26,27]. (Note that the asymmetry of the ground state is not necessarily of "spatial" nature.) Furthermore, systematic structure in the ground state is recognizable as a manifestation of the dynamical rearrangement of symmetry [16-18]. It is then reasonable to expect that some kind of long-range correlation modes, which create such systematic macroscopic patterns, are present in the system. Let us recall that the symmetries under study are <u>continuous</u> symmetries and let us assume for the moment that there are no forces of Coulomb-type in our theory. In one of the following sections we will analyze the case where such forces exist.

Due to the long-range character of the correlation modes, we can describe these modes as <u>massless</u> particles existing among the physical particles.

The requirement that the theory must be invariant under the original transformation leads us to the invariance of the free field equations under the rearranged symmetry transformation of the physical fields. Thus we come naturally to the study of the symmetry generator written in terms of physical fields.

Since, as we said, in the case of spontaneous breakdown of symmetry the vacuum no longer possesses the symmetry properties of the basic equations, we have

$$D|0\rangle \neq 0 , \tag{II.5}$$

where the time-independent generator D is given by

$$D = \int d^3x \; j_o(x) \; . \tag{II.6}$$

Here the integration is extended over all the (infinite) volume and $j_\mu(x)$ is the Heisenberg current for which the conservation law

$$\partial_\mu j_\mu(x) = 0 \tag{II.7}$$

holds as a consequence of the invariance of the field equation.

It is clear from (II.5) and the translational invariance of the vacuum that the quantity

$$\langle 0|DD|0\rangle = \int d^3x \; \langle 0|j_o(x)D|0\rangle \tag{II.8}$$

is divergent. On the other hand, it is well known [5] that if $|a\rangle$ and $|b\rangle$ are any two state vectors of the physical Fock space H_F, then

$$\langle a|\exp\{i\theta D\}|b\rangle = 0 \ , \quad \text{for any } |a\rangle \text{ and } |b\rangle \ , \tag{II.9}$$

which means that $\exp\{i\theta D\}|b\rangle$ does not belong to H_F. In (II.9) θ represents symbolically the transformation parameters of the symmetry group in consideration.

Eq. (II.9) (and (II.5)) expresses the existence of infinitely many unitarily inequivalent representations of canonical variables in Q.F.T. Later we come back to this problem. At the moment let us note that there is spontaneous breakdown of symmetry when the transformations under consideration are not unitarily implementable [5,17,28]. Thus we need particular care in defining and using symmetry generators. Our prescription will be the following: any symmetry generators D (cf. (II.6)) will be understood as the following space-time limit

$$D \equiv \lim_{g(x)\to 1} D_g = \lim_{g(x)\to 1} \int d^3x \ g(x) j_o(x) \tag{II.10}$$

with $g(x)$ any square integrable function which is a solution of the free field equation. The operator D_g is well-defined as far as $g(x)$ is square integrable. The limit $g(x)\to 1$ must be taken only at the end of all the computation. We note that the difficulty in defining the generators with $g(x) = 1$ is not really important from the point of view of practical computation since we always need to consider not the limit of generator itself but only the limit of commutators of generators. Note also that the prescription (II.10) is equivalent to take a finite integration volume in (II.6) and take the limit $V \to \infty$ at the end of the computation [2,17,29].

By using the dynamical map, we write the Heisenberg current $j_\mu(x)$, i.e. $j_\mu[\psi(x)]$, in terms of the physical fields $\phi(x)$: $j_\mu[\phi(x)]$. This shows that the space-time dependence of j_μ is controlled by the physical fields. On the basis of the previous considerations in the present section we need to add to the set of physical fields also the physical massless fields, say $B_i(x)$, responsible for the systematic structure of the ground state. Thus, we can write $j_\mu(x)$ formally as $j_\mu[\phi(x), B(x)]$. Since the free field equations are linear and homogeneous, the simplest choice for the transformations of the physical fields which leave invariant the free field equations, is given by linear transformations of the massive fields and by transformations like

$$B_i(x) \to B_i(x) + c_i \ , \quad c_i = \text{c-number constant} \qquad \text{(II.11)}$$

for the massless fields $B_i(x)$. The symmetry generators written in terms of physical fields will then be bilinear in the massive fields and linear in the massless fields [17,18]. It is interesting to ask what is the most general set of transformations which leave invariant the free field equations. We can say that, if a transformation introduces terms non linear in some physical field $\phi_i(x)$, then in order that the transformation be invariant, each of these terms should include a projection operator which selects only energies on the single-particle energy shell [30].

Let us note that as a consequence of the time-independence of the symmetry generator, the <u>bilinear</u> part of the generator can only be a function of fields <u>with the same masses</u> [17,18]. This means that one cannot mix particles of different masses without supplying energy. Indeed the time independence of the generator means that it cannot supply any energy. We can express this fact by saying that there cannot be mass differences among physical fields which belong to the <u>same</u> irreducible representation of the symmetry group associated with phenomena [17,18]. This statement does not contradict the well known result that mass differences do arise in a theory with spontaneous breakdown of symmetry. The statement does not forbid the occurrence of mass differences, it only states that particles with different masses belong to different irreducible representations of the phenomenological symmetry group.

When there is spontaneous breakdown of symmetry, the vacuum is not an eigenvector of the symmetry generators.

Therefore the generators must contain a linear part in the physical massless fields $B_i(x)$. We conclude, therefore, that when spontaneous breakdown of symmetry occurs, the theory requires the existence of massless particles $B_i(x)$. Note that the transformations (II.11) are canonical if and only if $B_i(x)$ are bosons. The appearance of massless bosons when symmetry is spontaneously broken is the content of the so-called Goldstone theorem [26]. In the following sections we will prove this theorem by using the path-integral technique. It can be proved that the occurrence of a term linear in the massless fields $B_i(x)$ in at least one of the generators is also a sufficient condition for the spontaneous breakdown of symmetry [17].

Since the transformations (II.11) are canonical, the fields $\{B_i(x)+c_i\}$ have well defined transformation properties under the original symmetry group, or, in other words, they form an irreducible representation of such a group. Thus, in the presence of spontaneous breakdown it is necessary that there be just enough massless bosons to form an irreducible representation of the symmetry group [17].

The set of c-numbers $\{c_i\}$ can be considered as dynamical spurions which alone carry the symmetry information of the theory [4]. This can be seen in the following way [17,18]. If we consider the vacuum expectation of (B_i+c_i)

$$\langle 0|(B_i+c_i)|0\rangle = c_i \ , \tag{II.12}$$

it is evident that c_i can be considered as the square root of the Bose-Einstein condensation of the massless boson B_i. In other words, through the transformation (II.11) a boson condensation has been induced in the ground state of our system: we recover in this way the systematic pattern of the ground state. The spurion set $\{c_i\}$ is thus the <u>carrier of the original symmetry quantum numbers</u>. The boson condensation acts as a <u>printing</u> process of these symmetry properties on the ground state. Note that this printing does not require any supply of energy since the bosons are massless. It follows from these considerations that the original symmetry, i.e. the original conservation laws, can be recovered only if one takes into account the quantum numbers of these spurions [2].

Let us make a few more comments on the transformations (II.11). As already mentioned the symmetry generator D (cf. (II.6)) should be defined by means of the prescription (II.10). Similarly, the transformations (II.11) should be understood as the space-time limit of the transformations

$$B_i(x) \to B_i(x) + g(x)c_i \qquad \text{(II.13)}$$

where g(x) is the function introduced in (II.10). The non unitarity of the transformations (II.11) or (II.13) confirms the fact that the symmetry transformations are not unitarily implementable in the presence of spontaneous breakdown of symmetry. In terms of physical Fock spaces this means that the effect of the breakdown of the symmetry is, via dynamical rearrangement, a shift from one Fock space for physical particles to another unitarily inequivalent Fock space for physical particles.

The transformation (II.13) is called the boson transformation [31,32]. Due to the presence of the c-number function $g(x)$, this transformation is particularly interesting since the vacuum expectation value of $(B_i(x)+g(x)c_i)$ will be space-time dependent: (II.13) will correspond to the spontaneous breakdown of the translational invariance. The boson transformation has been very useful in the so-called "boson formulation of superconductivity" [32]. The transformation (II.11) can be viewed as a particular case of (II.13), namely its limit for $g(x) \to 1$.

In the next section we will prove that if the boson transformation is performed, the resultant transformation of the Heisenberg fields induced through the dynamical map leaves the Heisenberg equation invariant [33].

The boson transformation has a very interesting physical meaning: the dynamics of a physical system is described by the basic field equations with given boundary conditions. Different boundary conditions correspond to different solutions of our dynamics. Each solution is described in terms of states belonging to different (unitarily inequivalent) Fock spaces. Since different ground states are related to each other by the boson transformation with different choices of $g(x)$, the boson transformation relates the different dynamical solutions; in other words the Fock spaces corresponding to the different dynamical solutions are classified by $g(x)$.

Let us note that in a very natural way the existence of unitarily inequivalent representations of canonical variables and the occurence of the spontaneous breakdown of symmetry find a unified explanation in the framework of the self-consistent formulation of Quantum Field Theory. For this reason, we will treat in some detail the problem of the existence of unitarily inequivalent representations and its physical implications in Chapter III.

We close this section by noting that in the traditional formulation of Quantum Field Theory the spontaneous breakdown of symmetry is still related to the existence of unitarily inequivalent representations. The invariance of the theory under a certain symmetry group is the invariance of the canonical field equation

$$i[H,\psi(x)] = \frac{\partial}{\partial t}\,\psi(x) \qquad \text{(II.14)}$$

where H is the Hamiltonian. The invariance of the field equation leads to

$$[D,H] = 0 \qquad \text{(II.15)}$$

where D is the symmetry generator. On the other hand, in the presence of spontaneous breakdown of symmetry the vacuum is not an eigenstate of D, which contradicts (II.15). Here the "vacuum" is the vacuum in the Fock space of bare particles. To solve the contradiction one says that there are many degenerate vacuums each of which is a vacuum of a Fock space for bare particles unitarily inequivalent to other Fock spaces. It is however not well established what is the mathematical nature of a Hilbert space for bare particles which should have the many inequivalent Fock spaces as subspaces. In the self-consistent formulation these difficulties are bypassed by considering only Fock spaces for physical particles.

2.3 Spontaneous breakdown of symmetry in the path-integral formulation.

The path-integral or functional method [34] has been very useful in the investigation of spontaneously broken symmetries [35]. In this method symmetry transformations can be induced simply by changes of the functional integration variables. It has been realized, however, that special care is needed to be able to distinguish between symmetric and spontaneously broken solutions [30,36]. The reason for this lies in the fact that the field equations are not sufficient to determine the solutions uniquely. One needs to specify the boundary conditions of the problem.

If a field theory model admits a symmetric and a spontaneously broken solution, the corresponding boundary conditions are different. In the path-integral method one considers the so-called generating functional from which one can derive directly and unambiguously the Green's functions of the theory. In such a method therefore, one should be able to incorporate the boundary conditions in the same expression of the generating functional, since the knowledge of this functional is equivalent to the knowledge of the field equation solutions. To be more specific let us consider as an example a Goldstone-type model [26], namely a complex scalar field model [11]. We assume that the Lagrangian $L[\phi(x)]$ of the model is invariant under constant phase transformations of the Heisenberg field:

$$\phi(x) \to e^{i\alpha}\, \phi(x) \ . \tag{II.16}$$

In this model, for particular values of the parameters appearing in the Lagrangian, one can obtain two different types of solutions distinguished from each other by the vacuum expectation values of the field

$$\langle 0|\phi(x)|0\rangle = c \ . \tag{II.17}$$

When $c = 0$ the solutions are called symmetric, when $c \neq 0$ they are called asymmetric. For $c \neq 0$, unless one specifies the phase of c, one has an infinite number of asymmetric solutions.

The generating functional for this model is given by

$$W[J] = \frac{1}{N} \int [d\phi][d\phi^*]\exp\left\{i\int d^4x[L[\phi(x)] + J^*(x)\phi(x) + J(x)\phi^*(x)]\right\} , \qquad \text{(II.18)}$$

with

$$N = \int [d\phi][d\phi^*]\exp\left\{i\int d^4x\; L[\phi(x)]\right\} , \qquad \text{(II.19)}$$

and where $J(x)$ represents the field source.

Now we note that contributions to $W[J]$ come from "stationary points" in the $\phi(x)$-space, i.e. from points where the exponential has an extreme value. These points lie in a certain domain of the $\phi(x)$-space. Due to the invariance of the Lagrangian under the phase transformations (II.16) when $J(x) = 0$, the domain of the stationary points is a circle centered at $\phi(x) = 0$. The points of the circle are related by the transformation (II.16): the functional average $<\phi(x)>$ over such a circle is zero when $J(x) = 0$. Since in the path-integral method, the functional average of a certain quantity $F[\phi(x)]$, i.e. $<F[\phi(x)]>$, agrees with the vacuum expectation value of the chronological products of $\phi(x)$, $T(F[\phi(x)])$, we see that $W[J]$ as given in (II.18) can reproduce only the symmetric solution.

To obtain an asymmetric solution, we must thus modify the definition of W[J] to include the appropriate boundary condition. For example, W[J] will reproduce a broken solution when we restrict the integration domain to these points of the $\phi(x)$-space which have a non zero vacuum expectation value. In other words, we should pick up the points which give an asymmetric solution. To do this we introduce an invariance breaking ε-term in the exponent of (II.18) [30,33,36]. ε is an infinitesimal quantity to be taken as zero only at the end of the computations. In the case of the present model this procedure means to distort the circle of stationary points with respect to the origin and eventually to reduce it to a single point (for a fixed phase of c in (II.17)). The ε-term thus discriminates among the boundary conditions. In the complex scalar model we choose as ε-term the function

$$f[\phi(x)] = |\phi(x)-v|^2 \qquad \text{(II.20)}$$

with v a non vanishing real constant. This term breaks the phase invariance of the theory. In general the generating functional W[J] for spontaneously broken solutions should be

$$W[J] = \frac{1}{N}\int \Pi[d\phi_i]\exp\left\{i\int d^4x[L[\phi(x)] + \sum_i J_i(x)\phi_i(x) + \right.$$

$$\left. + i\varepsilon f[\phi_i(x)]]\right\}$$

where N is equal to the numerator of W[J] when $J_i(x)$ are all zero, and $f[\phi_i(x)]$ is a functional not invariant under the original invariant transformations of the theory. In the case of the complex scalar model, W[J] in (II.18) becomes:

$$W[J] = \frac{1}{N}\int[d\phi][d\phi^*]\exp\left\{i\int d^4x[L[\phi(x)] + J^*(x)\phi(x) + J(x)\phi^*(x) + i\varepsilon|\phi(x)-v|^2]\right\} \qquad (II.21)$$

with

$$N = \int[d\phi][d\phi^*]\exp\left\{i\int d^4x[L[\phi(x)] + i\varepsilon|\phi(x)-v|^2]\right\} \qquad (II.22)$$

Since each functional derivative $\delta/\delta J(x)$ ($\delta/\delta J^*(x)$) acting on W[J] generates the factor $i\phi^*(x)$ ($i\phi(x)$), the vacuum expectation value of any chronological product of $\phi^*(x)$ and $\phi(x)$, i.e. the Green's function, can be obtained by repeated operations of $\delta/\delta J(x)$ and $\delta/\delta J^*(x)$ followed by the limit $J \to 0$. To investigate the symmetry properties of the theory, we can thus derive the Ward-Takahashi identities [37] which express such properties. To this aim we make the change of variables (II.16) in the numerator of (II.21). Since the integral is unaltered by a change of variables, we must have

$$\frac{\partial W[J]}{\partial \alpha} = 0 \qquad (II.23)$$

which at $\alpha = 0$ leads to the basic identity

$$i\int d^4x \langle J_2(x)\psi(x) - J_1(x)\chi(x)\rangle_{\varepsilon,J} = \sqrt{2}\,\varepsilon v\int d^4x\langle\chi(x)\rangle_{\varepsilon,J} \quad \text{(II.24)}$$

The notation in (II.24) is as follows:

$$\langle F[\phi]\rangle_{J,\varepsilon} \equiv \frac{1}{N}\int [d\phi][d\phi^*]F[\phi]\exp\left\{i\int d^4x[L[\phi] + J^*\phi + \right.$$

$$\left. + J\phi^* + i\varepsilon|\phi - v|^2]\right\} \quad \text{(II.25)}$$

$$\phi(x) = \frac{1}{\sqrt{2}}[\psi(x) + i\chi(x)]$$

$$J(x) = \frac{1}{\sqrt{2}}[J_1(x) + iJ_2(x)]$$

It can be easily seen that J_1 is the source of $\psi(x)$ and J_2 the source of $\chi(x)$.

Further shorthand notations that will be extensively used are:

$$\langle F[\phi]\rangle_\varepsilon \equiv \langle F[\phi]\rangle_{\varepsilon,J=0} \quad \text{(II.26)}$$

$$\langle F[\phi]\rangle \equiv \lim_{\varepsilon\to 0}\langle F[\phi]\rangle_\varepsilon \quad \text{(II.27)}$$

Before we proceed with the investigation of (II.24), let us point out that

$$\langle\chi(x)\rangle = 0$$

due to the symmetry $\chi(x) \to -\chi(x)$ in the theory. Therefore the vacuum expectation value of $\phi(x)$ is due entirely to that of $\psi(x)$:

$$<\phi(x)> = \frac{1}{\sqrt{2}} <\psi(x)> \equiv \frac{1}{\sqrt{2}} \tilde{v} \qquad \text{(II.28)}$$

The second equality in (II.28) defines the quantity $\tilde{v}$.

By successive functional differentiations of (II.24) with respect to $J_1(x)$ and (or) $J_2(x)$, we can obtain all the identities relating Green's functions in this model. For example,

$$<\psi(x)>_\varepsilon = \sqrt{2}\ \varepsilon v \int d^4y\ <\chi(x)\chi(y)>_\varepsilon \qquad \text{(II.29)}$$

$$<\rho(x)\rho(y)>_\varepsilon - <\chi(x)\chi(y)>_\varepsilon = \sqrt{2}\ \varepsilon v \int d^4z <\chi(z)\chi(x)\rho(y)>_\varepsilon \quad \text{(II.30)}$$

where

$$\rho(x) = \psi(x) - <\psi(x)>_\varepsilon .$$

To rewrite these identities in momentum space, let us introduce the Fourier transforms:

$$<\chi(x)\chi(y)> = i(2\pi)^{-4} \int d^4p\ e^{-ip\cdot(x-y)}\ \Delta_\chi(p)$$

$$<\rho(x)\rho(y)> = i(2\pi)^{-4} \int d^4p\ e^{-ip\cdot(x-y)}\ \Delta_\rho(p)$$

$$<\chi(x)\chi(y)\rho(z)> = -(2\pi)^{-8}\int d^4p\, d^4q\, d^4r\, e^{-i(px+qy+rz)}$$

$$\times\, \delta(p+q+r)\Delta_\chi(p)\Delta_\chi(q)\Delta_\rho(r)\Gamma_{\chi\chi\rho}(p,q,r)$$

Here the metric used is $g_{oo} = -g_{11} = 1$. The propagation function Δ_χ, has the form:

$$\Delta_\chi(p) = \lim_{\varepsilon\to 0}\left[\frac{Z_\chi}{p^2-m_\chi^2+i\varepsilon a} + (\text{continuum contributions})\right]$$

where Z_χ is the wavefunction renormalization constant of the field χ. The continuum contribution comes from states with more than one particle. It can be shown [30,36] that the ε-term in W[J] generates the $-i\varepsilon$ prescription for the free propagator. As is well known, the pole singularities in the Green's functions are defined by putting an infinitesimal imaginary $i\eta$ $(\eta \geq 0)$. Here we introduced $a_\chi = \frac{\eta}{\varepsilon}$ with a_χ a real constant (renormalization of ε).

Eq. (II.29) implies that

$$\tilde{v} = \sqrt{2}\,\frac{Z_\chi}{a_\chi}\,v \qquad \text{with } m_\chi^2 = 0 \tag{II.31}$$

$$\tilde{v} = 0 \qquad \text{with } m_\chi^2 \neq 0$$

whereas (II.30) yields:

$$\Delta_\chi^{-1}(p) - \Delta_\rho^{-1}(p) = \tilde{v}\Gamma_{\chi\chi\rho}(0,p,-p)\ . \tag{II.32}$$

Eq. (II.31) is a statement of the Goldstone theorem [26]: if $\tilde{v} \neq 0$, $\chi(x)$ must be a zero mass field.

Eq. (II.32) gives the relation among the two propagation functions and the vertex function, and leads to restrictions on the renormalization constants. Two different cases can be distinguished:

(i) $m_\rho^2 = 0$. Then, since $m_\chi^2 = 0$ too (we assume $\tilde{v} \neq 0$), we obtain from (II.32):

$$\Gamma_{\chi\chi\rho}(o,p,-p)\Big|_{p^2=0} = 0 .$$

This possibility cannot be excluded without specifying the Lagrangian beyond the invariance requirement. A model for which this situation occurs is the massless free scalar model.

(ii) $m_\rho^2 \neq 0$. In this case (II.32) leads to the following relations:

$$\tilde{v}\Delta_\chi(p)\Gamma_{\chi\chi\rho}(o,p,-p)\Big|_{p^2=m_\rho^2} = 1$$

$$\tilde{v}\Delta_\rho(p)\Gamma_{\chi\chi\rho}(o,p,-p)\Big|_{p^2=0} = -1 .$$

Unless $\Gamma_{\chi\chi\rho}(r,s,t)$, with $r^2 = s^2 = 0$ and $t^2 = -m_\rho^2$, vanishes, ρ becomes unstable: $\rho \to \chi + \chi$.

From equation (II.31) and (II.29) we can now clearly see that it is the presence of the ε-term that generates the asymmetric solution. We note also that the ε-term prescription is equivalent in this model to the replacement $J \to J - i\varepsilon v$ in $W[J]$. One can then regard $J - i\varepsilon v$ as a new source J' and assume that J' does not vanish at $t \to \pm \infty$. However such a prescription is not a general one, since, for example, it cannot be applied to models involving fermion fields or composite Goldstone bosons.

Let us now comment on the role played by v in creating the spontaneously broken solution. At first sight, eq. (II.31) seems to indicate that $\tilde{v}$ depends on v; we wish to show, however, that $\tilde{v}$ is insensitive to the magnitude of v, as long as $\tilde{v} \neq 0$. Indeed, eq. (II.29) leads to:

$$\frac{\partial}{\partial v} \langle\psi(x)\rangle_\varepsilon = \sqrt{2}\, \varepsilon \int d^4y \langle\rho(x)\ \rho(y)\rangle_\varepsilon \qquad \text{(II.33)}$$

and therefore, unless $\rho(x)$ is massless too,

$$\lim_{\varepsilon\to 0} \frac{\partial}{\partial v} \langle\psi(x)\rangle_\varepsilon = \frac{\partial}{\partial v} \tilde{v} = 0$$

and $\tilde{v}$ is independent of v. Eq. (II.31) then shows that a_χ is proportional to v. On the other hand, the <u>phase</u> of v is crucial in picking out a particular spontaneously broken solution. To see this, let us replace v by $e^{i\alpha}v$ in (II.21); we can restore the ε-term to its original form by a change of variables $\phi \to e^{i\alpha}\phi$ and, since the Lagrangian is phase invariant, the new generating functional will differ from the old one by the replacement

$$J(x) \to J'(x) = e^{i\alpha}J(x) \; .$$

Therefore, if we denote with primes quantities referring to the new value of v, we have:

$$\tilde{v}' = \sqrt{2}\,\frac{1}{i}\,\frac{\delta}{\delta J(x)}\,W'[J] = e^{i\alpha}\tilde{v} \; .$$

In other words, the phase of $\tilde{v}$ is controlled by the phase of v. Since $\tilde{v}$ completely specifies the solution, we can rephrase our result as follows: <u>The phase of v determines the direction of the symmetry breaking, while its magnitude is irrelevant, as long as it is finite</u> [33]. This is a very satisfying result, since it implies that the ε-term that we added to the action does not introduce any arbitrary new constant into the theory.

As already mentioned, the invariance of a theory cannot disappear in the presence of spontaneous breakdown of symmetry. This feature of an invariant theory is expressed by the conservation of the local current corresponding to the symmetry transformation. In the path-integral formalism it is possible to derive Ward-Takahashi identities for the divergence of the current by inducing in the numerator of W[J] (eq. (II.21)) the local gauge transformation

$$\phi(x) \to e^{i\alpha(x)}\phi(x) \quad ,$$

which is not an invariant transformation for our model. Through a procedure based essentially on functional differentiations of $W[J]$, which we omit for brevity, it is possible [36] to verify the conservation of local current even in the presence of spontaneous breakdown. Always by using path-integral techniques, it has been possible [36] to show that expansions of asymmetric Green's functions in terms of symmetric Green's functions are expansions in powers of $\varepsilon|v|$. In the limit $\varepsilon \to 0$, each term of the expansions would approach zero. The expansion of asymmetric Green's functions in terms of symmetric Green's functions is then meaningless, as one might expect, since they correspond to orthogonal Hilbert spaces due to the spontaneous breakdown of symmetry. Thus, one cannot argue about the renormalizability of the spontaneously broken Green's functions on the basis of the properties of the symmetric solution. This means that one needs a new perturbative scheme for the renormalization procedure of theories with spontaneous breakdown of symmetry.

2.4 Dynamical rearrangement of symmetry and boson transformations.

We are now ready to study boson transformations and dynamical rearrangement of symmetry in the complex scalar model introduced in the previous section.

Our starting point is the dynamical map (cf. eq. (II.2)) and the possibility of expressing it in a compact form through the path-integral formulation. This is due to the fact that the matrix elements of local operators such as $\phi_H(x)$ can be obtained by suitable limiting operations (e.g. those of the LSZ formalism) from the Green's functions of the theory, while the latter are compactly summarized in the generating functional.

In the case of the considered Goldstone-type model, there is only one asymptotic field, which we shall denote by $\chi^{in}(x)$. It corresponds to $\chi(x)$, the imaginary part of the complex field $\phi(x)$:

$$\phi(x) = \frac{1}{\sqrt{2}} [\tilde{v}+\rho(x)+i\chi(x)] \qquad \text{(II.34)}$$

There is no asymptotic field corresponding to $\rho(x)$, since ρ becomes unstable in the presence of spontaneous breakdown. (The decay $\rho\to\chi\chi$ is dynamically and energetically allowed.)

In the Fock space of the in-fields, we define the S-operator by [30,33]:

$$S = \langle :\exp\left\{-iZ_{\chi}^{\frac{1}{2}}\int d^4x\ \chi^{in}(x)\vec{K}(x)\chi(x)\right\}: \rangle \ . \qquad \text{(II.35)}$$

The symbol K(x) denotes the d'Alambertian

$$K(x) = -\partial_\mu \partial^\mu$$

The arrow on K(x) signifies that it should be always acting on $\chi(x)$. Let us also remind the reader that the bracket on the right-hand side of (II.35) denotes functional average, not expectation value in the Fock space.

In a similar way, we can define the (unrenormalized) Heisenberg field operator $\phi_H(x)$ by means of the formula:

$$S\phi_H(x) = \langle \phi(x) : \exp\left\{ -iZ_\chi^{-\frac{1}{2}} \int d_4x \; \chi^{in}(x) \vec{K}(x) \chi(x) \right\} : \rangle \qquad \text{(II.36)}$$

This expression, which must be understood in the weak sense, is the functional equivalent of the familiar LSZ reduction formula. Apart from the factor S that appears on the left-hand side it corresponds to the dynamical mapping (II.2); S appears because the generating functional produces time-ordered Green's functions, instead of the retarded functions that should appear in an expansion in terms of in-field operators.

Let us now turn to the boson transformation. To this end, we introduce the in-field transformation:

$$\chi^{in}(x) \to \chi^{in}(x) + \alpha(x) \qquad \text{(II.37)}$$

where the c-number function $\alpha(x)$ satisfies the same field equation as $\chi^{in}(x)$:

$$K(x)\alpha(x) = 0 \tag{II.38}$$

The transformation (II.37) is called the boson transformation (cf. eq. (II.13)). We want to prove that the Heisenberg field $\phi'_H(x)$ which is obtained through the dynamical mapping (II.36) from the boson-transformed $\chi^{in}(x)$ is also a solution of the field equation (cf. Sec. 2.2). Let us start with the identity:

$$<\chi(x)Q> = \int d^4x' <\chi(x)\chi(x')> q(x') \tag{II.39}$$

which is valid for any operator Q and suitable choice of $q(x')$. Eq. (II.39), together with (II.29) and (II.31), implies the basic relation:

$$-i\,\frac{\tilde{v}}{Z_\chi}\int d^4x\ \alpha(x)K(x)<\chi(x)Q> = \lim_{\varepsilon\to 0}\sqrt{2}\ \varepsilon v\int d^4x\ \alpha(x)<\chi(x)Q>_\varepsilon \tag{II.40}$$

since the limit $\varepsilon \to 0$ serves to pick up the zero-mass pole and $\alpha(x)$ is a solution of the equation (II.38).[9] Next, define a new functional average with space-dependent ε-term:

[9]Nakanishi [38] used equation (II.40) with $\alpha(x)$ constant to derive the Ward-Takahashi identities in the presence of spontaneous breakdown by conventional field theory techniques.

$$<F[\phi(x)]>_{v(x)} \equiv$$

$$N'^{-1}\int [d\phi][d\phi^*]F[\phi(x)]\exp\left\{i\int d^4z[L[\phi]+i\varepsilon|\phi(z)-v(z)|^2\right\} \tag{II.41a}$$

with

$$N' = \int [d\phi][d\phi^*]\exp\left\{i\int d^4z[L[\phi]+i\varepsilon|\phi(z)-v(z)|^2\right\} \tag{II.41b}$$

and[10]

$$v(z) = v\left[1 + i\,\frac{Z_\chi^{\frac{1}{2}}}{\tilde{v}}\,\alpha(z)\right] \tag{II.42}$$

Then if we define the boson-transformed S and ϕ_H:

$$S[\chi^{in}(x) + \alpha(x)] \equiv$$

$$<:\exp\left\{-iZ_\chi^{-\frac{1}{2}}\int d^4x[\chi^{in}(x)+\alpha(x)]\vec{K}(x)\chi(x)\right\}:> \tag{II.43}$$

$$S\phi_H[x;\chi^{in}(x)+\alpha(x)] \equiv$$

$$<\phi(x):\exp\left\{-iZ_\chi^{-\frac{1}{2}}\int d^4x[\chi^{in}(x)+\alpha(x)]\vec{K}(x)\chi(x)\right\}:> \tag{II.44}$$

we find with the aid of (II.40) and the definitions (II.41), (II.42):

[10] Note that $Z_\chi^{\frac{1}{2}}/\tilde{v}$ is finite [33].

$$S[\chi^{in}(x)+\alpha(x)] =$$

$$C\langle :\exp\left\{-iZ^{-\frac{1}{2}}\int d^4x\ \chi^{in}(x)\vec{K}(x)\chi(x)\right\}:\rangle_{v(x)} \tag{II.45}$$

and

$$S\phi_H[x;\chi^{in}(x)+\alpha(x)] =$$

$$C\langle\phi(x):\exp\left\{-iZ_\chi^{-\frac{1}{2}}\int d^4x\ \chi^{in}(x)\vec{K}(x)\chi(x)\right\}:\rangle_{v(x)} \tag{II.46}$$

The factor

$$C \equiv \lim_{\varepsilon\to 0} \langle\exp\left\{\sqrt{2}\ \varepsilon v\ \frac{Z_\chi^{\frac{1}{2}}}{\tilde{v}} \int d^4x\ \alpha(x)\chi(x)\right\}\rangle_\varepsilon \tag{II.47}$$

comes from the denominator N' defined in (II.41b). Eqs. (II.45), (II.46) prove the assertion made earlier: the two field operators $\phi_H[x;\chi^{in}(x)]$ and $\phi_H[x;\chi^{in}(x)+\alpha(x)]$ differ by ε-terms, and therefore are solutions of the same field equations. We can rephrase our result as follows: when a theory allows spontaneous breakdown, there always exist solutions of the field equations with space and/or time-dependent vacuum; these solutions are obtained from the translationally invariant ones by the boson transformation (II.37). Such solutions play an important role in superconductivity [32], where the presence of persistent currents breaks the translational invariance of the vacuum. Notice also that the mass of the in-field does not change by the boson transformation because of (II.38). The situation can be visualized as a local Bose condensation at the χ^{in}-level, which induces the non-zero space-dependent vacuum expectation value of $\phi_H(x)$.

We are now in a position to prove the basic theorem about the dynamical rearrangement of phase symmetry in our model: <u>The only in-field transformation</u>

$$\chi^{in}(x) \rightarrow \tilde{\chi}^{in}(x,\alpha) \qquad \text{(II.48)}$$

<u>which, through the dynamical mapping, induces a constant phase transformation of the Heisenberg field $\phi_H(x)$, is the boson transformation (II.37) with $\alpha(x) \rightarrow$ const</u>. [33]. Precisely speaking, the problem of dynamical rearrangement in this model involves finding an operator $\tilde{\chi}^{in}(x,\alpha)$, which is a function of $\chi^{in}(x)$ and a phase parameter α, such that the following two conditions are satisfied:

$$S[\tilde{\chi}^{in}(x,\alpha)] = S[\chi^{in}(x)] \qquad \text{(II.49)}$$

$$S\phi_H[\tilde{\chi}^{in}(x,\alpha)] = e^{i\alpha} S\phi_H[\chi^{in}(x)] \qquad \text{(II.50)}$$

The operators on the left-hand sides of the above equations are defined through the dynamical mapping (II.35), (II.36):

$$S[\tilde{\chi}^{in}(x,\alpha)] = \langle :\exp\left\{-i\, A\, [\tilde{\chi}^{in}(x,\alpha)]\right\}: \rangle \qquad \text{(II.51)}$$

$$S\phi_H[\tilde{\chi}^{in}(x,\alpha)] \;=\; <\phi(x):\exp\left\{-i\,A\,[\tilde{\chi}^{in}(x,\alpha)]\right\}:> \qquad \text{(II.52)}$$

with

$$A\,[\tilde{\chi}^{in}(x,\alpha)] \;\equiv\; Z_\chi^{-\frac{1}{2}}\int d^4x\;\tilde{\chi}^{in}(x,\alpha)\overleftarrow{K}(x)\chi(x) \qquad \text{(II.53)}$$

In addition, since the transformation (II.48) must be a symmetry transformation at the in-field level, we must impose the condition:

$$K(x)\tilde{\chi}^{in}(x,\alpha) \;=\; 0 \qquad \text{(II.54)}$$

The constraints (II.49), (II.50) lead to the following conditions by differentiation with respect to α:

$$\begin{aligned} <:(-i)\int d^4z\;Z_\chi^{-\frac{1}{2}}\,\frac{\partial\tilde{\chi}^{in}(z,\alpha)}{\partial\alpha}\;\times \\ \overleftarrow{K}(z)\chi(z)\exp\left\{-i\,A\,[\tilde{\chi}^{in}(x,\alpha)]\right\}:> \;=\; 0 \end{aligned} \qquad \text{(II.55)}$$

and

$$\begin{aligned} <\phi(x):(-i)\int d^4z\;Z_\chi^{-\frac{1}{2}}\,\frac{\partial\tilde{\chi}^{in}(z,\alpha)}{\partial\alpha}\,\overleftarrow{K}(z)\chi(z)\exp\left\{-i\,A\,[\tilde{\chi}^{in}(x,\alpha)]\right\}:> \\ =\; i<\phi(x):\exp\left\{-i\,A\,[\tilde{\chi}^{in}(x,\alpha)]\right\}:> \end{aligned} \qquad \text{(II.56)}$$

On the other hand, the basic identity (II.24) of the model, when evaluated with $J_1(x) = 0$ and $J_2(x) =$

$-Z_\chi^{-\frac{1}{2}}\tilde{\chi}^{in}(x,\alpha)K(x)$,[11] yields

$$<:(-i)Z_\chi^{-\frac{1}{2}}\int d^4z\ \tilde{\chi}^{in}(z,\alpha)\overleftrightarrow{K}(z)(\tilde{v}+\rho(z))\exp\left\{-i\ A\ [\tilde{\chi}^{in}(x,\alpha)]\right\}:> =$$

$$<:(-i)\int d^4z\ \frac{\tilde{v}}{Z_\chi}\ K(x)\chi(x)\exp\left\{-i\ A\ [\tilde{\chi}^{in}(x,\alpha)]\right\}:> \qquad \text{(II.57)}$$

Eq. (II.40) was used on the right-hand side of this relation. Since $\rho(z)$ has no zero-mass pole and $\tilde{\chi}^{in}(z,a)$ has momentum-space support confined on the hypersurface $k^2 = 0$ by virtue of (II.54), the left-hand side of (II.57) is zero. Therefore

$$<:(-i)\int d^4z\ \frac{\tilde{v}}{Z_\chi}\ K(x)\chi(x)\exp\left\{-i\ A\ [\tilde{\chi}^{in}(x,\alpha)]\right\}:> = 0 \qquad \text{(II.58)}$$

In a similar fashion, we can use the identity obtained by applying $-i(\frac{\delta}{\delta J_1(x)} + i\ \frac{\delta}{\delta J_2(x)})$ on both sides of (II.36) to prove that:

$$<\phi(x):(-i)\int d^4z\ \frac{\tilde{v}}{Z_\chi}\ K(z)\chi(z)\exp\left\{-i\ A\ [\tilde{\chi}^{in}(x,\alpha)]\right\}$$

$$= i<\phi(x):\exp\left\{-i\ A\ [\tilde{\chi}^{in}(x,\alpha)]\right\}:> \qquad \text{(II.59)}$$

Comparison of (II.55), (II.56) with (II.58), (II.59) leads to the following necessary and sufficient condition

---

[11] We are allowed to use a q-number source function because of the presence of normal ordering in (II.57), (II.59).

on $\tilde{\chi}^{in}$:

$$\frac{\partial}{\partial\alpha}\,\tilde{\chi}^{in}(x,\alpha) = \frac{\tilde{v}}{Z_{\alpha}^{\frac{1}{2}}}$$

This differential equation, with the initial condition

$$\tilde{\chi}^{in}(x,\alpha{=}0) = \chi^{in}(x) \qquad \text{(II.60)}$$

implies that:

$$\tilde{\chi}^{in}(x,\alpha) = \chi^{in}(x) + \frac{\tilde{v}}{Z_{\chi}^{\frac{1}{2}}}\,\alpha \qquad \text{(II.61)}$$

Clearly, this solution satisfies the requirement (II.54). The in-field transformation (II.61) must be understood as the limit of the boson transformation

$$\chi^{in}(x) \to \chi^{in}(x) + \frac{\tilde{v}}{Z_{\chi}^{\frac{1}{2}}}\,\alpha g(x) \qquad \text{(II.62)}$$

with g(x) a <u>normalizable</u> c-number function that tends to 1, and satisfies

$$K(x)g(x) = 0\ .$$

For example, the transition from (II.57) to (II.58) is possible only if $\tilde{\chi}^{in}(x,\alpha)$ on the left-hand side is understood in the sense of (II.62). The matrix elements of $S[\chi^{in}{+}\alpha]$, $S\phi_H[\chi^{in}{+}\alpha]$ are <u>not</u> the same as those obtained from $S[\chi^{in}+\alpha g(x)]$, $S\phi_H[\chi^{in}+\alpha g(x)]$ in the limit $g(x)\to 1$, with $g(x)$ a normalizable function. Of course, local operators like $\chi^{in}(x)$ are well-defined in the Fock space only when they are smeared out by normalizable wavepackets.

This situation is closely related to the fact that the generator of (II.61):

$$I = \int d^3x\, \dot{\chi}^{in}(x)\, \frac{\tilde{v}}{Z_\chi^{\frac{1}{2}}}$$

is ill-defined, whereas

$$I(g) = \int d^3x\, \frac{\tilde{v}}{Z_\chi^{\frac{1}{2}}}\, [g(x)\, \overleftrightarrow{\frac{\partial}{\partial t}}\, \chi^{in}(x)]$$

is well-defined (cf. Sec. 2.2). In fact, the Heisenberg field transformation is implemented through:

$$\lim_{g\to 1} \left\{ e^{i\alpha I(g)}\, \phi_H(x) e^{-i\alpha I(g)} \right\} = e^{i\alpha}\, \phi_H(x) \qquad \text{(II.63)}$$

In summary, the phase transformation of the Heisenberg field:

$$\phi_H(x) \to e^{\alpha}\, \phi_H(x)$$

is induced by the in-field transformation

$$\chi^{in}(x) \to \chi^{in}(x) + \frac{\tilde{v}}{Z_\chi^{\frac{1}{2}}}\, \alpha g(x)$$

with g(x) a normalizable solution of

$$K(x)g(x) = 0$$

when the limit $g(x) \to 1$ is performed after all matrix elements of $\phi_H(x)$ have been calculated.

The fact that the invariant transformation has different forms at the level of the Heisenberg fields and the level of the in-field expresses the dynamical rearrangement of the phase symmetry. Our results show the crucial role played by the boson transformation of the Goldstone field in recovering the original symmetry of the Heisenberg field operator (cf. Sec. 2.2).

Let us note that although the form of the original transformation is different from that of the in-field transformation, still the (abelian) phase group is unchanged through the dynamical rearrangement of symmetry. This result has been confirmed also in the study of the Nambu model [22] and of superconductivity [30,32]. There are however examples [24,25,39] in which the non-abelian symmetry group of the original transformations is different from that of the physical field transformations. Examples like these are studied in the next section.

2.5 Change of symmetry group through dynamical rearrangement of symmetry.

In this section we investigate the relation between transformations of in-fields and those of the Heisenberg fields in the spontaneous breakdown of a non-abelian symmetry group, studying an iso-triplet Lorentz-scalar field as an example in the Relativistic Quantum Field Theory [24] and a ferromagnetic system as an example in many body [25].

There appear Goldstone bosons perpendicular to the symmetry breaking direction. The symmetric associated with the asymptotic fields proves to be not SU(2), but E(2), which contains an abelian subgroup.

This subgroup consists of translations of the Goldstone fields which leave the free field equations invariant. We will also analyze the problem of how such abelian transformations can induce non-abelian transformations of the Heisenberg operators.

By use of the path-integral method we derive the Ward-Takahashi identities and the asymptotic field transformations which induce the SU(2) transformations of Heisenberg fields. We find that in order to achieve well defined SU(2) transformations of Heisenberg fields, the space-time independent translations of the Goldstone bosons which induce them must be considered as certain limits of space-time dependent transformations. The non-commutativity among these limiting processes is the origin of the non-abelian nature of the SU(2) transformations.

The fact that the symmetry group associated with the in-fields is not SU(2) but E(2) is due to certain infrared effects. Since the vectors in the Hilbert space are constructed from smeared asymptotic fields, some infrared contributions become unobservable, leading to the change of algebra mentioned above (cf. Secs. 2.1 and 2.4).

a) Relativistic Quantum Field Theory model.

We will consider a model with SU(2) symmetry. The Lagrangian is assumed to be a function of an iso-triplet Lorentz-scalar field $\vec{\phi}(x)$ and to be invariant under the transformation

$$\vec{\phi}(x) \to e^{i\vec{\alpha}\cdot\vec{t}}\,\vec{\phi}(x) \qquad \text{(II.64)}$$

i.e.

$$L(\vec{\phi}(x)) = L(e^{i\vec{\alpha}\cdot\vec{t}}\,\vec{\phi}(x)) \qquad \text{(II.65)}$$

where t_i is the 3×3 matrix:

$$(t_i)_{jk} = i\varepsilon_{jik} \qquad (i,j,k = 1,2,3) \qquad \text{(II.66)}$$

and α_i are real constants, and the $\{\varepsilon_{ijk}\}$ are the SU(2) structure constants.

The generating functional is

$$W[\vec{J}] = \frac{1}{N}\int [d\vec{\phi}]\exp\left\{i\int d^4x\ [L(\vec{\phi}(x)) + \vec{J}(x)\cdot\vec{\phi}(x) + \frac{i\varepsilon}{2}(\vec{\phi}(x)-\vec{v})^2]\right\} \qquad \text{(II.67)}$$

with

$$N = \int [d\vec{\phi}]\exp\left\{i\int d^4x\ [L(\vec{\phi}(x)) + \frac{i\varepsilon}{2}(\vec{\phi}(x)-\vec{v})^2]\right\}$$

The ε-term is introduced in order to generate the spontaneous breakdown of symmetry in the direction of $\vec{v}$ (cf. Sec. 2.3). We can choose without loss of generality $\vec{v} = (0,0,v)$ with v a real constant. Let us now perform the change of variables (2.1) on the numerator of $W[\vec{J}]$. Since the integral must be invariant under such a change, one must have:

$$-i\,\frac{\partial W[\vec{J}]}{\partial \alpha_i} = 0$$

which leads to

$$\int d^4x <i\ \vec{J}(x)\ t_i\ \vec{\phi}(x) + \varepsilon\vec{v}\ t_i\ \vec{\phi}(x)>_{\varepsilon,J} = 0\ , \qquad \text{(II.68)}$$

where the notation (II.25) has been used.

Repeated operations of $\delta/\delta\vec{J}$ on (II.68) lead to the Ward-Takahashi identities. We will here list some of these relations which are used later on:

$$\langle\phi_3(x)\rangle_\varepsilon = \varepsilon v \int d^4y\ \langle\phi_1(x)\ \phi_1(y)\rangle_\varepsilon \tag{II.69a}$$

$$\langle\phi_3(x)\rangle_\varepsilon = \varepsilon v \int d^4y\ \langle\phi_2(x)\ \phi_2(y)\rangle_\varepsilon \tag{II.69b}$$

$$\langle\phi_1(x)\ \phi_1(y)\rangle_\varepsilon = \langle\phi_2(x)\ \phi_2(y)\rangle_\varepsilon \tag{II.69c}$$

$$\langle\phi_1(x)\phi_1(y)\rangle_\varepsilon - \langle\phi_3(x)\phi_3(y)\rangle_\varepsilon = -\varepsilon v \int d^4z \langle\phi_1(z)\phi_1(x)\phi_3(y)\rangle_\varepsilon \tag{II.69d}$$

and

$$\langle\phi_2(x)\phi_2(y)\rangle_\varepsilon - \langle\phi_3(x)\phi_3(y)\rangle_\varepsilon = -\varepsilon v \int d^4z \langle\phi_2(z)\phi_2(x)\phi_3(y)\rangle_\varepsilon \tag{II.69e}$$

where the notation (II.26), (II.27) has been used.

The identities (II.69a,b) lead to the Goldstone theorem: when $\langle\phi_3\rangle \neq 0$, $\phi_1(x)$ and $\phi_2(x)$ must be massless fields. We shall introduce $\tilde{v}$ by $\lim_{\varepsilon\to 0} \langle\phi_3(x)\rangle_\varepsilon = \tilde{v}$. Since the propagators of $\phi_1(x)$ and $\phi_2(x)$ are written as

$$\langle\phi_i(x)\phi_i(y)\rangle_\varepsilon = i(2\pi)^{-4}\int d^4p\ \frac{Z_i}{p^2+i\varepsilon a_i}\ e^{-ip(x-y)} + \text{c.c.} \qquad (i = 1,2) \tag{II.70}$$

we find

$$\tilde{v}(\varepsilon) = i\varepsilon v \Delta_{\phi_i}(\varepsilon,0) \text{ which is, for } \varepsilon\to 0,\ \tilde{v} = v\ \frac{Z_i}{a_i} \qquad (i = 1,2) \tag{II.71}$$

Here c.c. stands for continuum contribution, and $\Delta_{\phi_i}(\varepsilon,p) \equiv Z_i/(p^2+i\varepsilon a_i)$. Now equation (II.69c) says that $Z_1 = Z_2$ ($\equiv Z$), which is a result of a $1\leftrightarrow 2$ symmetry in equation (II.66).

Recalling that ϕ_1 and ϕ_2 are massless, we find that an arbitrary local operator Q must satisfy the following relation (cf. eq. (II.40)):

$$\lim_{\varepsilon\to 0} \varepsilon v\int d^4y\langle\phi_i(y)Q\rangle = -i\,\frac{\tilde{v}}{Z}\int d^4y(-\partial_y^2)\langle\phi_i(y)Q\rangle \qquad (i = 1,2) \tag{II.72}$$

Applying this relation to (II.68), we have

$$\int d^4x\ \langle i\vec{J}(x)t_i\vec{\phi}(x)\rangle_J = i\frac{\tilde{v}}{Z}\int d^4x(-\partial_x^2)i\varepsilon_{3ik}\langle\phi_k(x)\rangle_J \tag{II.73}$$

where $\langle F\rangle_J$ means $\langle F\rangle_{\varepsilon,J}$ with $\varepsilon\to 0$.

Taking the derivative $-i\,\dfrac{\delta}{\delta\vec{J}(y)}$ of both sides of (II.73), we obtain

$$\begin{aligned}\langle t_i\vec{\phi}(x)\rangle_J &+ \int d^4y\ \langle i\vec{J}(y)t_i\vec{\phi}(y)\vec{\phi}(x)\rangle_J \\ &= i\,\frac{\tilde{v}}{Z}\int d^4y\ (-\partial_y^2)i\varepsilon_{3ik}\langle\phi_k(y)\vec{\phi}(x)\rangle_J\end{aligned} \tag{II.74}$$

The expressions (II.73) and (II.74) are the Ward-Takahashi identities which are not explicitly ε-dependent.

The identities (II.69d) and (II.69e) give us information on the $\phi_3(x)$-propagator. The symmetry requirements alone cannot determine the mass of $\phi_3(x)$: both values $m_3 = 0$ and $m_3 \neq 0$ are allowed even when $\langle\phi_3\rangle \neq 0$. To study this we introduce the following decomposition

$$\phi_3(x) = \rho(x) + \tilde{v}(\varepsilon)$$

with

$$\langle\rho(x)\rangle_\varepsilon = 0.$$

Making use of the momentum representation of propagators and vertex functions defined as

$$\langle\rho(x)\rho(y)\rangle = i(2\pi)^{-4}\int d^4p\ \Delta_\rho(p)e^{-ip\cdot x}$$

and

$$\langle\phi_1(x)\phi_1(y)\rho(z)\rangle = -(2\pi)^{-8}\int d^4p\ d^4q\ d^4r\ \Delta_{\phi_1}(p)\Delta_{\phi_1}(q)\Delta_\rho(r)$$

$$\times\ \Gamma_{\phi_1\phi_1\rho}(p,q,r)\exp\left\{-i(px+qy+rz)\right\}\delta(p+q+r)$$

the identity (II.69d) is rewritten, using equation (II.71) for $\varepsilon \neq 0$

$$\Delta_{\phi_1}^{-1}(p) - \Delta_\rho^{-1}(p) = \tilde{v}\ \Gamma_{\phi_1\phi_1\rho}\ (o,\ p,\ -p) \qquad \text{(II.75)}$$

Since $\tilde{v} \neq 0$, ρ is massless when and only when

$$\Gamma_{\phi_1\phi_1\rho}(o,\ p,\ -p)\Big|_{p^2=0} = 0 \qquad \text{(II.76)}$$

The massless scalar model $L(\vec{\phi}(x)) = \frac{1}{2}\,\partial_\mu\phi_i(x)\partial_\mu\phi_i(x)$ is the simplest example of this kind. We are not going to study the possibility of a massless ρ-particle.

If ρ is massive, the vertex $\Gamma_{\phi_1\phi_1\rho}(o,p,-p)$ does not vanish when $p^2 = 0$. Unless $\Gamma_{\phi_1\phi_1\rho}(p,q,r)$, with $p^2 = q^2 = 0$ and $r^2 = -m_\rho^2$, vanishes, ρ becomes unstable due to the decay $\rho \to \phi_1 + \phi_2$. In the following analysis we examine only this case. We then have only two massless in-fields corresponding to $\phi_1(x)$ and $\phi_2(x)$.

Let us note that $\tilde{v}$ is independent of the value of v. This can be shown simply as follows:

$$\frac{\partial\tilde{v}}{\partial v} = \lim_{\varepsilon\to 0}\ \varepsilon\int d^4y\ \langle\rho(x)\rho(y)\rangle_\varepsilon = 0$$

where $\rho(x)$ is considered massive. The relations of current conservation can be derived in the manner given in reference [36].

Now we discuss which transformations of in-fields induce the SU(2) transformations of the Heisenberg operators. Since there exist only two in-fields, their transformations are expected to be quite different from the original transformation of Heisenberg operators.

We introduce the in-field operators $\phi_1^{in}(x)$ and $\phi_2^{in}(x)$ which satisfy the following free field equations:

$$(-\partial_x^2)\phi_1^{in}(x) = (-\partial_x^2)\phi_2^{in}(x) = 0 \tag{II.77}$$

As we did in the previous section, we write the s-matrix S, and the Heisenberg operator $\vec{\phi}_H(x)$ as

$$S(\phi_1^{in},\phi_2^{in}) = \langle :\exp\left\{-iA(\phi_1^{in},\phi_2^{in})\right\}:\rangle \tag{II.78}$$

and

$$S\vec{\phi}_H(x;\phi_1^{in},\phi_2^{in}) = \langle\vec{\phi}(x) : \exp\left\{-iA(\phi_1^{in},\phi_2^{in})\right\}:\rangle \tag{II.79}$$

where

$$\begin{aligned} A(\phi_1^{in},\phi_2^{in}) = \int d^4x\ [&Z^{-\frac{1}{2}}\phi_1^{in}(x)(-\partial_x^2)\phi_1(x) \\ &+ Z^{-\frac{1}{2}}\phi_2^{in}(x)(-\partial_x^2)\phi_2(x)] \end{aligned} \tag{II.80}$$

We shall look for the transformation

$$\left.\begin{aligned} \phi_1^{in'}(x,\lambda) &= \Phi_1(x;\lambda,\alpha_i,\phi_1^{in}(x),\phi_2^{in}(x)) \\ \phi_2^{in'}(x,\lambda) &= \Phi_2(x;\lambda,\alpha_i,\phi_1^{in}(x),\phi_2^{in}(x)) \end{aligned}\right\} \tag{II.81}$$

where λ and the α_1 are our choice of transformation parameters, and which satisfies the following requirements:

$$(-\partial_x^2)\phi_1^{in'}(x,\lambda) = (-\partial_x^2)\phi_2^{in'}(x,\lambda) = 0 \tag{II.82}$$

$$\left.\begin{aligned} S(\phi_1^{in'},\phi_2^{in'}) &= S(\phi_1^{in},\phi_2^{in}) \\ S\phi_H(x;\phi_1^{in'},\phi_2^{in'}) &= e^{i\lambda\vec{\alpha}\cdot\vec{t}}\, S\phi_H(x;\phi_1^{in},\phi_2^{in}) \end{aligned}\right\} \tag{II.83}$$

Taking $\partial/\partial\lambda$ of the relations in (II.83), we obtain

$$\begin{aligned} <:-i\int d^4x\, Z^{-\frac{1}{2}} &\left[\frac{\partial\phi_1^{in'}(x,\lambda)}{\partial\lambda}(-\partial_x^2)\phi_1(x) + \right. \\ &\left. + \frac{\partial\phi_2^{in'}(x,\lambda)}{\partial\lambda}(-\partial_x^2)\phi_2(x)\right]\exp\left\{-iA(\phi_1^{in'},\phi_2^{in'})\right\}:> = 0 \end{aligned} \tag{II.84a}$$

and

$$\begin{aligned} <\vec{\phi}(x):-i\int d^4x\, Z^{-\frac{1}{2}} &\left[\frac{\partial\phi_1^{in'}(x,\lambda)}{\partial\lambda}(-\partial_x^2)\phi_1(x) + \right. \\ &\left. + \frac{\partial\phi_2^{in'}(x,\lambda)}{\partial\lambda}(-\partial_x^2)\phi_2(x)\right]\exp\left\{-iA(\phi_1^{in'},\phi_2^{in'})\right\}:> \\ &= <i\alpha\cdot t\vec{\phi}(x) : \exp\left\{-iA(\phi_1^{in'},\phi_2^{in'})\right\}:> \end{aligned} \tag{II.84b}$$

On the other hand, with the following choice of source currents

$$J_1(x) = -Z^{-\frac{1}{2}}\phi_1^{in'}(x,\lambda)(-\partial_x^2)$$

$$J_2(x) = -Z^{-\frac{1}{2}}\phi_2^{in'}(x,\lambda)(-\partial_x^2)$$

and

$$J_3(x) = 0$$

in (II.73) and (II.74), we obtain the following identities:

$$<:\int d^4x \, \frac{\tilde{v}}{Z} \, (-\partial_x^2)\phi_2(x) \, \exp\{-iA\}:> = 0 \tag{II.85a}$$

$$<:\int d^4x \, \frac{\tilde{v}}{Z} \, (-\partial_x^2)\phi_1(x) \, \exp\{-iA\}:> = 0 \tag{II.85b}$$

$$<:\int d^4x \, Z^{-\frac{1}{2}}\left\{\phi_2^{in'}(x,\lambda)(-\partial_x^2)\phi_1(x) - \phi_1^{in'}(x,\lambda)(-\partial_x^2)\phi_2(x)\right\} \times \exp\{-iA\}:> = 0 \tag{II.85c}$$

$$-<\vec{\phi}(x) : \int d^4y \, \frac{\tilde{v}}{Z} \, (-\partial_y^2)\phi_2(y)\exp\{-iA\}:> = t_1<\vec{\phi}(x):\exp\{-iA\}:>, \tag{II.86a}$$

$$<\vec{\phi}(x) : \int d^4y \, \frac{\tilde{v}}{Z} \, (-\partial_y^2)\phi_1(y)\exp\{-iA\}:> = t_2<\vec{\phi}(x):\exp\{-iA\}:> , \tag{II.86b}$$

$$-<\vec{\phi}(x) : \int d^4y \, Z^{-\frac{1}{2}}\left\{\phi_2^{in'}(y,\lambda)(-\partial_y^2)\phi_1(y) - \phi_1^{in'}(y,\lambda)(-\partial_y^2)\phi_2(y)\right\} \times \exp\{-iA\}:> = t_3<\vec{\phi}(x):\exp\{-iA\}:> \tag{II.86c}$$

Here use was made of the relation

$$<\int d^4x\ \phi_i^{in'}(x)(-\partial_x^2)\phi_3(x)Q> = 0 \qquad (i = 1,2) \tag{II.87}$$

which comes from the fact that ρ is unstable. Comparing (II.84b) with (II.86a,b,c) we find that

$$\frac{\partial\phi_1^{in'}(x,\lambda)}{\partial\lambda} = \alpha_3\phi_2^{in'}(x,\lambda) - \alpha_2 \frac{\tilde{v}}{Z^{\frac{1}{2}}} \tag{II.88a}$$

and

$$\frac{\partial\phi_2^{in'}(x,\lambda)}{\partial\lambda} = -\alpha_3\phi_1^{in'}(x,\lambda) + \alpha_1 \frac{\tilde{v}}{Z^{\frac{1}{2}}} \tag{II.88b}$$

These conditions together with (II.85a,b,c) are enough to obtain (II.84a). Solving (II.88a,b) with the initial conditions

$$\phi_1^{in'}(x,\lambda=0) = \phi_1^{in}(x)$$

and

$$\phi_2^{in'}(x,\lambda=0) = \phi_2^{in}(x)$$

we have

$$\begin{aligned}\phi_1^{in'}(x,\lambda) &= \phi_1^{in}(x)\cos\lambda\alpha_3 + \phi_2^{in}(x)\sin\lambda\alpha_3 \\ &+ \frac{\tilde{v}}{Z^{\frac{1}{2}}}\,[\alpha_3^{-1}(1-\cos\lambda\alpha_3)\alpha_1-\alpha_3^{-1}(\sin\lambda\alpha_3)\alpha_2]\end{aligned} \tag{II.89a}$$

and

$$\phi_2^{in'}(x,\lambda) = -\phi_1^{in}(x)\sin\lambda\alpha_3 + \phi_2^{in}(x)\cos\lambda\alpha_3$$
$$+ \frac{\tilde{v}}{Z^{1/2}} [\alpha_3^{-1}(\sin\lambda\alpha_3)\alpha_1 + \alpha_3^{-1}(1-\cos\lambda\alpha_3)\alpha_2] . \quad \text{(II.89b)}$$

Note that these transformations do not change the free field equations. This is the dynamical rearrangement corresponding to a rotation by the amount $\lambda\sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2}$ around the axis with directors $(\alpha_1,\alpha_2,\alpha_3)$. Putting $\lambda = 1$ in (II.83) and (II.89a,b) we have

$$\phi_H(\phi_1^{in'}(x,\lambda=1),\ \phi_2^{in'}(z,\lambda=1))$$
$$= e^{i\vec{\alpha}\cdot\vec{t}}\ \phi_H(\phi_1^{in}(x),\phi_2^{in}(x)) . \quad \text{(II.90)}$$

The transformation (II.89a,b) is a combination of a c-number translation and a rotation mixing $\phi_1^{in}(x)$ and $\phi_2^{in}(x)$. The c-number translation must be understood as the following limit of a space-time dependent transformation:

$$\phi_1^{in'}(x,\lambda) = \lim_{g(x)\to 1} [\phi_1^{in}(x)\cos\lambda\alpha_3 + \phi^{in}(x)\sin\lambda\alpha_3$$
$$+ \frac{\tilde{v}}{Z^{1/2}} \left\{\alpha_3^{-1}(1-\cos\lambda\alpha_3)\alpha_1 - \alpha_3^{-1}(\sin\lambda\alpha_3)\alpha_2\right\}g(x)] \quad \text{(II.91)}$$

with

$$(-\partial_x^2)g(x) = 0 \quad \text{(II.92)}$$

This is because, in a rigorous sense, $\phi_1^{in'}(x)$ and $\phi_2^{in'}(x)$ in the identities (II.85)-(II.87) must be considered as the limits of smeared operators. In particular, (II.87) is not valid unless Q is considered as an operator smeared by a normalizable function.

It is worth noting that, even without taking the limit $g(x)\to 1$, $\phi_i^{in'}(x,\lambda)$ satisfies the equation for $\phi_i^{in}(x)$. Since an operator with exactly zero momentum is not defined, one must consider the constant translational transformation as a limit like (II.91). In this connection we study the generator of the transformations (II.89a,b):

$$I(\alpha_1,\alpha_2,\alpha_3) = \int d^3x[\alpha_1(-\frac{\tilde{v}}{Z^{\frac{1}{2}}})\dot{\phi}_2^{in}(x) + \alpha_2\frac{\tilde{v}}{Z^{\frac{1}{2}}}\dot{\phi}_1^{in}(x) + \alpha_3(\dot{\phi}_2^{in}(x)\phi_1^{in}(x) - \dot{\phi}_1^{in}(x)\phi_2^{in}(x))] \tag{II.93}$$

Though linear parts of this are not well defined (cf. Sec. 2.2), a smeared out generator

$$I(\alpha_1,\alpha_2,\alpha_3;g) = \int d^3x[(-\alpha_1\frac{\tilde{v}}{Z^{\frac{1}{2}}}\phi_2(x) + \alpha_2\frac{\tilde{v}}{Z^{\frac{1}{2}}}\phi_1(x))\overleftrightarrow{\partial}_t g(x) + \alpha_3(\dot{\phi}_2^{in}(x)\phi_1^{in}(x) - \dot{\phi}_1^{in}(x)\phi_3^{in}(x))] \tag{II.94}$$

is well defined and also is time independent because of (II.92). Then, (II.90) is understood as

$$\lim_{g(x)\to 1} e^{-iI(\alpha_1,\alpha_2,\alpha_3;g)} \vec{\phi}_H(\phi_1^{in}(x),\phi_2^{in}(x)) e^{iI(\alpha_1,\alpha_2,\alpha_3;g)}$$

$$= e^{i\vec{\alpha}\cdot\vec{t}}\, \vec{\phi}_H(\phi_1^{in}(x),\phi_2^{in}(x)) \tag{II.95}$$

We note now that any SU(2)-transformation can be expressed as a successive rotation around the first, second and third axis:

$$e^{i\vec{\alpha}\cdot\vec{t}} = e^{i\beta_1 t_1}\, e^{i\beta_2 t_2}\, e^{i\beta_3 t_3} \;.$$

The previous discussion indicates that the limit $g(x)\to 1$ must be taken in each rotation. According to (II.89) the above rotations are induced by the following in-field transformations respectively:

$$\begin{cases} \phi_1^{in'}(x) = \phi_1^{in}(x) \\ \phi_2^{in'}(x) = \phi_2^{in}(x) + \dfrac{\tilde{v}}{Z^{\frac{1}{2}}}\beta_1 \end{cases} \tag{II.96a}$$

$$\begin{cases} \phi_1^{in'}(x) = \phi_1^{in}(x) - \dfrac{\tilde{v}}{Z^{\frac{1}{2}}}\beta_2 \\ \phi_2^{in'}(x) = \phi_2^{in}(x) \end{cases} \tag{II.96b}$$

and

$$\begin{cases} \phi_1^{in'}(x) = \phi_1^{in}(x)\cos\beta_3 + \phi_2^{in}(x)\sin\beta_3 \\ \phi_2^{in'}(x) = -\phi_1^{in}(x)\sin\beta_3 + \phi_2^{in}\cos\beta_3 \end{cases} \tag{II.96c}$$

(II.96a) and (II.96b) show that each of these translations of the Goldstone bosons corresponds to an independent rotation. Generators of these transformations are given by

$$D_1^{in} = -\lim_{g(x)\to 1} \int \frac{\tilde{v}}{Z^{\frac{1}{2}}} (\dot{\phi}_2^{in}(x)g(x) - \dot{g}(x)\phi_2^{in}(x))d^3x \qquad \text{(II.97a)}$$

$$D_2^{in} = -\lim_{g(x)\to 1} \int \frac{\tilde{v}}{Z^{\frac{1}{2}}} (\dot{g}(x)\phi_1^{in}(x) - \dot{\phi}_1^{in}(x)g(x))d^3x \qquad \text{(II.97b)}$$

and

$$D_3^{in} = -\int (\dot{\phi}_1^{in}(x)\phi_2^{in}(x) - \dot{\phi}_2^{in}(x)\phi_1^{in}(x))d^3x \qquad \text{(II.97c)}$$

which form the algebra

$$\begin{cases} [D_1^{in}, D_2^{in}] = 0 \\ [D_2^{in}, D_3^{in}] = i\, D_1^{in} \\ [D_3^{in}, D_1^{in}] = i\, D_2^{in} \end{cases} \qquad \text{(II.98)}$$

However, we know that the SU(2) algebra has the form

$$[D_i, D_j] = i\varepsilon_{ijk} D_k \qquad (i,j,k = 1,2,3) \qquad \text{(II.99)}$$

where D_i are the generators for the SU(2) transformations of the Heisenberg operators.

To investigate the change of the SU(2) algebra into the E(2) algebra (II.98) we recall the fact than one needs the limiting process $g(x)\to 1$ in order to make the transformations well defined. Since this limiting procedure acts as a suitable infrared cut-off for the Goldstone bosons, we expect that an SU(2) algebra may be recovered when we take into account these infrared effects which are missing in local observations. The following shows that this is indeed the case. We start by decomposing the in-fields into the sum of a non-zero momentum (or smeared) part plus a small momentum (or soft) part

$$\phi_i^{in}(x) = \phi_{i,s}^{in}(x) + \phi_{i,\eta}^{in}(x) \qquad (i = 1,2) \qquad \text{(II.100)}$$

Here η is an infinitesimal parameter which indicates the order of the infrared cut-off; i.e.

$$\phi_{i,\eta}^{in}(\vec{x}) = \frac{\eta}{2}\int_{-\infty}^{+\infty} dt\ e^{-\eta|t|}\ \phi_i^{in}(x) \qquad \text{(II.101)}$$

In terms of creation and annihilation operators of $\phi_i^{in}(x)$, $\phi_{i,\eta}^{in}(x)$ is written as

$$\phi_{i,\eta}^{in}(\vec{x}) = \frac{\eta}{(2\pi)^{3/2}}\int\frac{d\ \vec{k}}{\sqrt{2k}}\ \frac{\eta}{\vec{k}^2+\eta^2}\ (a_{\vec{k}}\ e^{i\vec{k}\cdot\vec{x}} + a_{\vec{k}}^{\dagger}\ e^{-i\vec{k}\cdot\vec{x}})$$

which, in the limit $\eta\to 0$, becomes

$$\phi_{i,\eta}^{in}(\vec{x}) = \frac{\eta}{2(2\pi)^{\frac{1}{2}}}\int\frac{d\ \vec{k}}{\sqrt{2k}}\ \delta(\vec{k})(a_{\vec{k}}\ e^{i\vec{k}\cdot\vec{x}} + a_{\vec{k}}^{\dagger}\ e^{-i\vec{k}\cdot\vec{x}}) \qquad \text{(II.102)}$$

We observe that $\phi^{in}_{i,\eta}(\vec{x})$ is independent of x in the limit $\eta \to 0$ and is of order η.

Now we will evaluate the Heisenberg fields up to first order in η. By the use of the fact that $\phi^{in}_{i,\eta}(\vec{x})$ becomes independent of $\vec{x}$ for small η, we have

$$S\vec{\phi}_H(x) = \langle\vec{\phi}(x) : \exp\left\{-iA(\phi^{in}_{i,s} + \phi^{in}_{i,\eta})\right\}:\rangle$$

$$= S\vec{\phi}_{H,s}(x) - i\,\frac{Z^{\frac{1}{2}}}{\tilde{v}}\,[St_2\vec{\phi}_{H,s}(x)\phi^{in}_{1,\eta} - St_1\vec{\phi}_{H,s}(x)\phi^{in}_{2,\eta}] \tag{II.103}$$

where

$$S\vec{\phi}_{H,s}(x) = \langle\vec{\phi}(x) : \exp\left\{-iA(\phi^{in}_{i,s})\right\}:\rangle \tag{II.104}$$

Here we have used (II.77) and (II.86a,b). Note that

$$\lim_{\eta\to 0}\,[\text{matrix elements of } S\vec{\phi}_H(x)]$$

$$= \lim_{\eta\to 0}\,[\text{matrix elements of } S\vec{\phi}_{H,s}(x)]$$

because $\phi^{in}_{1,\eta}$ is of order η. Multiplying both sides of (II.103) by S^{-1} we have

$$\vec{\phi}_H(x) = \vec{\phi}_{H,s}(x) - i\,\frac{Z^{\frac{1}{2}}}{\tilde{v}}\,[t_2\vec{\phi}_{H,s}(x)\phi^{in}_{1,\eta} - t_1\vec{\phi}_{H,s}(x)\phi^{in}_{2,\eta}] \tag{II.105}$$

Since the generators D_i are given in terms of $\vec{\phi}_H(x)$ as

$$D_i = -\frac{1}{2} i \int d^3x\, \vec{\phi}_H(x)\, t_i \overleftrightarrow{\partial}_t\, \vec{\phi}_H(x)$$

they are expressed, to first order in η, as

$$D_1 = D_{1,s} + \frac{Z^{\frac{1}{2}}}{\tilde{v}} D_{3,s}\, \phi^{in}_{1,\eta} \qquad \text{(II.106a)}$$

$$D_2 = D_{2,s} + \frac{Z^{\frac{1}{2}}}{\tilde{v}} D_{3,s}\, \phi^{in}_{2,\eta} \qquad \text{(II.106b)}$$

and

$$D_3 = D_{3,s} - \frac{Z^{\frac{1}{2}}}{\tilde{v}} [D_{1,s}\, \phi^{in}_{1,\eta} + D_{2,s}\, \phi^{in}_{2,\eta}] \qquad \text{(II.106c)}$$

where the $D_{i,s}$ are given by

$$D_{i,s} = -\frac{1}{2} i \int d^3x\, \vec{\phi}_{H,s}(x)\, t_i \overleftrightarrow{\partial}_t\, \vec{\phi}_H(x) \qquad \text{(II.107)}$$

Since the operators $D_{i,s}$ are supposed to generate the in-field transformations (II.96a,b,c) in the limit $\eta \to 0$, we may write them as

$$D_{1,s} = -\int d^3x\, \frac{\tilde{v}}{Z^{\frac{1}{2}}} (\dot{\phi}^{in}_2(x) g_\eta(x) - \dot{g}_\eta(x) \phi^{in}_2(x)) \qquad \text{(II.108a)}$$

$$D_{2,s} = -\int d^3x\, \frac{\tilde{v}}{Z^{\frac{1}{2}}} (\dot{g}_\eta(x) \phi^{in}_1(x) - \dot{\phi}^{in}_1(x) g_\eta(x)) \qquad \text{(II.108b)}$$

and

$$D_{3,s} = -\int d^3x (\dot{\phi}^{in}_{1,s}(x) \phi^{in}_{2,s}(x) - \dot{\phi}^{in}_{2,s}(x) \phi^{in}_{1,s}(x)) \qquad \text{(II.108c)}$$

where $g_\eta(x)$ is a function satisfying $\partial_x^2 g_\eta(x) = 0$ and $\lim_{\eta\to 0} g_\eta(x) = 1$. Note that $\lim_{\eta\to 0} D_1 = \lim_{\eta\to 0} D_{1,s} = D_1^{in}$, $\lim_{\eta\to 0} D_2 = \lim_{\eta\to 0} D_{2,s} = D_2^{in}$, $\lim_{\eta\to 0} D_{3,s} = D_3^{in}$ and

$$D_3^{in} \equiv -\int d^3x\, (\dot\phi_1^{in}(x)\phi_2^{in}(x) - \dot\phi_2^{in}(x)\phi_1^{in}(x))$$

$$= D_{3,s} - \frac{Z^{\frac{1}{2}}}{\tilde v}[D_{1,s}\phi_{1,\eta}^{in} + D_{2,s}\phi_{2,\eta}^{in}] = D_3$$

to first order in η.

In order to discuss the commutation relations of the $\{D_i\}$, η must be chosen independently for each rotation, since the limit $g(x)\to 1$ (i.e. $\eta\to 0$) is taken in every rotation. The following commutation relations are derived from (II.108a,b,c) and the definition of $\phi_{i,\eta}^{in}$:

$$[D_{1,s}, D_{2,s'}] = 0 \tag{II.109a}$$

$$[D_{2,s}, D_{3,s'}] = i\, D_{1,\, s\cap s} \tag{II.109b}$$

$$[D_{3,s}, D_{1,s'}] = i\, D_{2,\, s\cap s'} \tag{II.109c}$$

$$[D_3, D_{1,s}] = i\, D_{2,s} \qquad [D_3, D_{2,s}] = -i\, D_{1,s} \tag{II.110}$$

$$[D_{1,s}, \phi_{2,\eta'}] = i\,\frac{\tilde v}{Z^{\frac{1}{2}}}\,\theta(\eta'-\eta)\ , \tag{II.111a}$$

$$[D_{2,s}, \phi_{1,\eta'}] = -i\,\frac{\tilde v}{Z^{\frac{1}{2}}}\,\theta(\eta'-\eta)\ , \tag{II.111b}$$

$$[D_{3,s},\ \phi_{1,\eta'}] = i\ \phi_{2,\eta'}\ \theta(\eta'-\eta)\ , \qquad \text{(II.111c)}$$

$$[D_{3,s},\ \phi_{2,\eta'}] = -i\ \phi_{1,\eta'}\ \theta(\eta'-\eta)\ . \qquad \text{(II.111d)}$$

Here $s \cap s'$ is the common part of s and s' which is the complement of $\eta'' = \max(\eta,\eta')$.

Then we have

$$[D_1,D_2] = [D_{1,s} + \frac{Z^{\frac{1}{2}}}{\tilde{v}}\ D_{3,s}\ \phi^{in}_{1,\eta},\ D_{2,s'} + \frac{Z^{\frac{1}{2}}}{\tilde{v}}\ D_{3,s'}\ \phi^{in}_{2,\eta'}]$$

$$= i\theta(\eta'-\eta)\left\{D_{3,s'} - \frac{Z^{\frac{1}{2}}}{\tilde{v}}\ D_{1,s'}\ \phi^{in}_{1,\eta'} - \frac{Z^{\frac{1}{2}}}{\tilde{v}}\ D_{2,s'}\ \phi^{in}_{2,\eta'}\right\}$$

$$+\ i\theta(\eta-\eta')\left\{D_{3,s} - \frac{Z^{\frac{1}{2}}}{\tilde{v}}\ D_{1,s}\ \phi^{in}_{1,\eta} - \frac{Z^{\frac{1}{2}}}{\tilde{v}}\ D_{2,s}\ \phi^{in}_{2,\eta}\right\}$$

$$= i\ D_3\ .$$

$$[D_2,D_3] = [D_{2,s} + \frac{Z^{\frac{1}{2}}}{\tilde{v}}\ D_{3,s}\ \phi^{in}_{2,\eta},\ D_3]$$

$$= i\left\{D_{1,s} + \frac{Z^{\frac{1}{2}}}{\tilde{v}}\ D_{3,s}\ \phi^{in}_{1,\eta}\right\}$$

$$= i\ D_1$$

and

$$[D_3,D_1] = [D_3,\ D_{1,s} + \frac{Z^{\frac{1}{2}}}{\tilde{v}}\ D_{3,s}\ \phi^{in}_{1,\eta}]$$

$$= i\ D_2\ .$$

Summarizing, we found that, although

$$[\lim_{\eta\to 0} D_1, \lim_{\eta'\to 0} D_2] = 0 \, , \tag{II.112a}$$

$$[D_3, \lim_{\eta\to 0} D_1] = i \lim_{\eta\to 0} D_2 \, , \tag{II.112b}$$

$$[D_3, \lim_{\eta\to 0} D_2] = -i \lim_{\eta\to 0} D_1 \, , \tag{II.112c}$$

we have

$$\lim_{\eta\to 0} \lim_{\eta'\to 0} [D_i, D_j] = i \, \varepsilon_{ijk} D_k \, . \tag{II.113}$$

This shows that the infrared effect is responsible for the difference of the two algebras, (II.98) and (II.99): the non zero value of $[D_1,D_2]$ is due to the fact that the η-limit and the commutation operation cannot be interchanged.

Since all observable results are manifested through the in-fields, the algebra (II.98) is the one which is directly related to observations.

We have seen that two different sets of generators, $\{D_i\}$ and $\{D_i^{in}\}$, generate the same transformations of the Heisenberg fields. To understand this we note that the infrared term by which the two sets of generators differ is global in nature although locally infinitesimal. Such an object commutes with any local operator but it may contribute to the commutators among the generators.

The same situation is also found in the scale invariant theory [39]; if the algebra does not change, massiveness of any particle is forbidden. A change of the algebra is the reason why some particles become massive when the scale symmetry is spontaneously broken. In the present case, the SU(2) algebra changes into an E(2) algebra: $(\phi_1^{in}, \phi_2^{in})$ forms a doublet of this algebra and the ϕ_3 field becomes a massive singlet of this algebra, although this particle is unstable. In other words, the Goldstone particles form a non unitary irreducible representation of the symmetry group E(2) associated with the in-fields. Note also that the symmetry properties of the theory under isospin rotation are solely carried by the Goldstone particles at the level of physical fields (cf. Sec. 2.2). The original SU(2) symmetry transformations are recovered when we consider an infrared effect which is missing in local observations.

Among the Ward-Takahashi identities, those which are influenced by the ϵ-term are the relations with the soft boson limit. It is worth noting that these relations with the soft boson limit manifest not the SU(2) symmetry, but an E(2) symmetry for the in-fields. This may be significant when one studies relations in the soft pion limit in particle physics.

b) The ferromagnetic systems.

Now we turn our attention to the case of a ferromagnet [25]. Our considerations are restricted to the

$T = 0^oK$ case.

The ferromagnetic systems are characterized by the Lagrangian which is made of the electron field $\psi(x)$.[12]

$$\psi(x) = \begin{pmatrix} \psi_\uparrow(x) \\ \psi_\downarrow(x) \end{pmatrix}$$

The Lagrangian is invariant under the spin rotation

$$\psi(x) \to e^{i\theta_i\lambda_i}\,\psi(x) \qquad i = 1,2,3 \qquad \text{(II.114)}$$

Here θ_i are real parameters and λ_i are the spin matrices $\lambda_i = \frac{1}{2}\sigma_i$ where σ_i are the Pauli matrices.

The generating functional is

$$W[J,j,n] = \frac{1}{N}\int[d\psi][d\psi^\dagger]\exp\Big\{i\int d^4x[\,L\,[\psi(x)] + J^\dagger(x)\psi(x)$$
$$+ \psi^\dagger(x)J(x) + j^\dagger(x)\ S_\psi^{(-)}(x) + S_\psi^{(+)}(x)j(x) \qquad \text{(II.115)}$$
$$+ S_\psi^{(3)}(x)\ n(x) - i\varepsilon S_\psi^{(3)}(x)]\Big\}$$

where

$$N = \int[d\psi][d\psi^\dagger]\exp\Big\{i\int d^4x[\,L\,[\psi(x)] - i\varepsilon S_\psi^{(3)}(x)]\Big\}\ . \qquad \text{(II.116)}$$

---

[12] $\psi(x)$ is the Heisenberg electron field. In the following, physical particle fields will be called quasifields or simply quasiparticles.

Here $S_\psi^{(\alpha)}(x)$ $(\alpha = +,-,3)$ is the spin density which is made of $\psi(x)$. In the following we do not need the explicit form of $S_\psi^{(\alpha)}(x)$ in terms of $\psi(x)$. The electron fields ψ, $\psi^\dagger$ and their sources $J^\dagger$ and J anticommute, while the sources of spin fields j and $j^\dagger$ are c-numbers. Also $S_\psi^{\pm}(x) \equiv S_\psi^{(1)}(x) \pm iS_\psi^{(2)}$.

The presence of sources j, $j^\dagger$ and n for the spin fields $S_\psi^{(\alpha)}(x)$ are not required for calculation of the Green's functions. However, use of j, $j^\dagger$ and n makes it possible to study the behavior of spin in ferromagnets without specifying the explicit dependence of $S_\psi^{(\alpha)}(x)$ on $\psi(x)$.

In writing down (II.115), we have in our minds both the cases of localized spins and itinerant electrons. In case of localized spin, the integration in the exponent of (II.115) should be read as

$$\sum_\ell \int dt\,[L[\psi(x_\ell)] + J^\dagger(x_\ell)\psi(x_\ell) + \psi^\dagger(x_\ell)J(x_\ell) + j^\dagger(x_\ell)S_\psi^{(-)}(x_\ell) + S_\psi^{(+)}(x_\ell)j(x_\ell) + S_\psi^{(3)}(x_\ell)n(x_\ell) - i\epsilon S_\psi^{(3)}(x_\ell)] \ . \tag{II.117}$$

In general, for the case of localized spin the following replacement is understood

$$\int d^4x \; F(x) \rightarrow \sum_\ell \int dt \; F(x_\ell)$$

for any quantity $F(x)$.

Let us now put $J = 0$ and $n = 0$ and perform the change of variables (II.114) in the numerator of eq. (II.115). When θ_i are infinitesimal the change of variables (II.114) should induce rotation of the spin fields:

$$S_\psi^{(i)}(x) \to S_\psi^{(i)}(x) - \theta_j \, \varepsilon_{ijk} \, S_\psi^{(k)}(x) \ . \qquad \text{(II.118)}$$

Here ε_{ijk} is the completely antisymmetric tensor: $\varepsilon_{ijk} = (-1)^p$ where p is the number of permutations of 1, 2, and 3. Since a change of variables does not influence the integration, we have

$$\frac{\partial W}{\partial \theta_\ell} = 0 \ . \qquad \text{(II.119)}$$

This gives

$$\begin{aligned}\int d^4x < [\varepsilon_{1\ell k}(j(x) + j^\dagger(x)) + i\varepsilon_{2\ell k}(j(x) - j^\dagger(x)) \\ - i\varepsilon \ \varepsilon_{3\ell k}] S_\psi^k(x) >_{\varepsilon, j} = 0 \ .\end{aligned} \qquad \text{(II.120)}$$

Operating $\delta/\delta j(y)$ and $\delta/\delta j^\dagger(y)$ on this and then putting $j = 0$, we obtain

$$<S_\psi^{(1)}(y)>_\varepsilon = <S_\psi^{(2)}(y)>_\varepsilon = 0 \qquad \text{(II.121)}$$

$$\varepsilon \int d^4x < S_\psi^{(2)}(x) \ S_\psi^{(1)}(y)>_\varepsilon = 0 \qquad \text{(II.122)}$$

$$<S_\psi^{(3)}(y)>_\varepsilon = \varepsilon \int d^4x <S_\psi^{(1)}(x)\ S_\psi^{(1)}(y)>_\varepsilon \qquad \text{(II.123)}$$

$$<S_\psi^{(3)}(y)>_\varepsilon = \varepsilon \int d^4x <S_\psi^{(2)}(x)\ S_\psi^{(2)}(y)>_\varepsilon \ . \qquad \text{(II.124)}$$

We used the usual notation (II.25-27).

Let us now write

$$<S_\psi^{(i)}(x)\ S_\psi^{(i)}(y)>_\varepsilon = i \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\ \rho_i(p)$$
$$\times \left(\frac{1}{p_o-\omega_p+i\varepsilon a_i} - \frac{1}{p_o+\omega_p-i\varepsilon a_i}\right) + \text{continuum contribution.} \qquad \text{(II.125)}$$

In eq. (II.125) ω_p is the energy of a quasiparticle which is a bound state of electrons. We will prove that $\rho_i(p) \neq 0$ which proves the existence of such a bound state. The continuum contribution comes from those states which contain more than one quasiparticle. The spectral functions $\rho_i(p)$ cannot be negative because $S_\psi^{(i)}$ are hermitian.

When we consider the case of localized spins, eq. (II.125) should be replaced by

$$<S_\psi^{(i)}(x_\ell) S_\psi^{(i)}(x_k)>_\varepsilon = iv \int \frac{dp_o}{(2\pi)} \int \frac{d^3p}{(2\pi)^3}\, e^{-ip(x_\ell - x_k)}\ \rho_i(p)$$
$$\times \left(\frac{1}{p_o-\omega_p+i\varepsilon a_i} - \frac{1}{p_o+\omega_p-i\varepsilon a_i}\right) + \text{continuum contribution} \qquad \text{(II.126)}$$

in which v is the volume of unit lattice and the integration of $\vec{p}$ is confined to the domain

$$-\frac{\pi}{d} < p_i < \frac{\pi}{d}$$

where d is the lattice length. The equations (II.123) and (II.124) should be read as

$$<S_\psi^{(3)}(x_k)>_\varepsilon = \varepsilon \sum_\ell \int dt <S_\psi^{(i)}(x_\ell) \; S_\psi^{(i)}(x_k)>_\varepsilon \; .$$

Also, operating $(\delta/\delta j^\dagger(z))(\delta/\delta j(y))$ and $(\delta/\delta j(z))(\delta/\delta j^\dagger(y))$ on (II.120), putting then $j = 0$ and subtracting, we obtain

$$<S_\psi^{(1)}(x) \; S_\psi^{(1)}(y)>_\varepsilon = <S_\psi^{(2)}(x) \; S_\psi^{(2)}(y)>_\varepsilon$$

which gives

$$\rho_1(p) = \rho_2(p) \; ; \qquad a_1 = a_2 \tag{II.127}$$

The magnetization is given by $g\mu_B<S_\psi^{(3)}(x)>$ where μ_B is the Bohr magneton. We shall use the notation

$$M(\varepsilon) \equiv <S_\psi^{(3)}(x)>_\varepsilon \tag{II.128}$$

together with

$$M = \lim_{\varepsilon \to 0} M(\varepsilon) \; . \tag{II.129}$$

Equations (II.123) and (II.124) then say that there should be a bound state of gapless energy:

$$\omega_p = 0 \quad \text{at} \quad p = 0 . \tag{II.130}$$

Indeed equations (II.123) and (II.124) give

$$M(\varepsilon) = i\varepsilon\Delta_i(\varepsilon,0) \qquad i = 1,2 \tag{II.131}$$

which can lead to non vanishing M with $\varepsilon \to 0$, only when $\omega_p = 0$ at $p = 0$. We further have

$$M = \frac{2\rho}{a} . \tag{II.132}$$

In eq. (II.131) we used

$$\Delta_i(\varepsilon,p) = \rho_i(p) \left(\frac{1}{p_o-\omega_p+i\varepsilon a_i} - \frac{1}{p_o+\omega_p-i\varepsilon a_i}\right) \tag{II.133}$$

and in eq. (II.132) we put $\rho_1(0) = \rho_2(0) = \rho$ and $a_1 = a_2 = a$.

In the case of localized spins, derivation of eq. (II.131) requires the formula:

$$\frac{V}{(2\pi)^3} \sum_{\ell} e^{-ipx_\ell} = \delta^{(3)}(p) \tag{II.134}$$

Summarizing, we have shown that the relation (II.128) along with non zero M requires the existence of gapless bosons, i.e. the magnons. Note that without the ε-term we cannot realize the case $M \neq 0$.

Let us now calculate ρ. Since M is the local spin density in the third direction, the total spin in this direction is NM where N is the number of lattice points. Then the ground state expectation value of $\vec{S}^2$ is given by

$$<0|\vec{S}^2|0> = NM(NM+1) \tag{II.135}$$

Assuming $t_k < t_\ell$ in eq. (II.126b) with $i = 1,2$ and then performing the limit $t_k \to t_\ell$ (same results can be obtained by assuming $t_\ell < t_k$), we find, using eq. (II.134) that

$$<0|S^{(i)}\ S^{(i)}|0> = \rho N\ , \quad \text{for } i = 1,2\ . \tag{II.136}$$

Thus

$$<0|\vec{S}^2|0> = 2\rho N + (NM)^2 \tag{II.137}$$

Comparing this with (II.135) we get

$$\rho = \frac{1}{2} M \tag{II.138}$$

This gives

$$\frac{M}{2\rho^{\frac{1}{2}}} = (\frac{M}{2})^{\frac{1}{2}} \tag{II.139}$$

(II.138) also shows that $a = 1$ (cf. (II.132)).

To study the dynamical rearrangement of symmetry let us introduce the field for the magnons

$$B(x) = \int \frac{d^3k}{(2\pi)^{3/2}} B_{\vec{k}}\ e^{i\vec{k}\cdot\vec{x} - i\omega_k t} \tag{II.140}$$

and

$$B^{\dagger}(x) = \int \frac{d^3k}{(2\pi)^{3/2}} B^{\dagger}_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}+i\omega_k t} \tag{II.141}$$

The commutation relations are

$$\left.\begin{aligned} &[B(x),B^{\dagger}(y)]_{t_x=t_y} = \delta(\vec{x}-\vec{y}) \\ &[B(x),B(y)] = [B^{\dagger}(x),B^{\dagger}(y)] = 0 \end{aligned}\right\} \tag{II.142}$$

(II.140) and (II.142) satisfy the equations:

$$\left.\begin{aligned} K(\vec{\partial})B^{\dagger}(x) &= 0 \\ B(x)K(-\overleftarrow{\partial}) &= 0 \end{aligned}\right\} \tag{II.143}$$

with

$$K(\vec{\partial}) = -(i \frac{\vec{\partial}}{\partial t} + \omega) \tag{II.144}$$

Here arrows on the derivatives indicate the directions in which the derivatives operate.

We write also the free field equations for the quasi-electron $\phi(x)$ as

$$\left.\begin{aligned} \Lambda(\vec{\partial})\phi(x) &= 0 \\ \phi^{\dagger}(x)\Lambda(-\overleftarrow{\partial}) &= 0 \end{aligned}\right\} \tag{II.145}$$

Let us denote the S-matrix and the spin-density operators for the Heisenberg electron field by $\mathcal{S}$ and $s^{(i)}(x)$ respectively. We write

$$\mathcal{S}(\phi,\phi^{\dagger},B,B^{\dagger}) = \langle :\exp\{-iA(\phi,\phi^{\dagger},B,B^{\dagger})\}:\rangle \tag{II.146}$$

and

$$\mathcal{S}s^{(i)}(x,\phi,\phi^{\dagger},B,B^{\dagger}) = \langle S_{\psi}^{(i)}(x):\exp\{-iA(\phi,\phi^{\dagger},B,B^{\dagger})\}:\rangle \tag{II.147}$$

where

$$\begin{aligned} A(\phi,\phi^{\dagger},B,B^{\dagger}) = \int d^4x\{&\rho^{-\frac{1}{2}}B(x)K(\vec{\partial})S_{\psi}^{(-)}(x) \\ &+ \rho^{-\frac{1}{2}}S_{\psi}^{(+)}(x)K(-\overleftarrow{\partial})B^{\dagger}(x) + Z^{-\frac{1}{2}}\phi^{\dagger}(x)\Lambda(\vec{\partial})\psi(x) \\ &+ Z^{-\frac{1}{2}}\psi^{\dagger}(x)\Lambda(-\overleftarrow{\partial})\phi(x)\} \end{aligned} \tag{II.148}$$

and $i = 1,2,3$.

Here Z is the wave-function renormalization of the electron. (II.147) together with (II.146) gives us the expressions of the spin-density operator in terms of quasiparticles.[13]

Our task is to study the following question: how do ϕ, $\phi^{\dagger}$, B, $B^{\dagger}$ in (II.146) and (II.147) transform so that

[13] In the case of localized spins (II.146) and (II.147) are still valid due to the time integration on the entire domain $(-\infty,+\infty)$ and due to the relation (II.134).

spin rotation of $S^{(i)}(\phi,\phi^\dagger,B,B^\dagger)$ is induced? Let us write the transformed fields $\phi_\theta(x)$, $\phi_\theta^\dagger(x)$, $B_\theta(x)$, $B_\theta^\dagger(x)$ as

$$\left.\begin{aligned} \phi_\theta(x) &= \Phi(x,\theta_i,\phi,\phi^\dagger,B,B^\dagger) \\ B_\theta(x) &= \tilde{B}(x,\theta_i,\phi,\phi^\dagger,B,B^\dagger) \\ \text{etc.} & \end{aligned}\right\} \quad i = 1,2,3$$

and require that they satisfy the equations for quasi-particles (II.145) and (II.143):

$$\left.\begin{aligned} \Lambda(\vec{\partial})\phi_\theta(x) &= 0 \\ K(\vec{\partial})B_\theta^\dagger(x) &= 0 \\ \phi_\theta^\dagger(x)\Lambda(-\overleftarrow{\partial}) &= 0 \\ B_\theta(x)K(-\overleftarrow{\partial}) &= 0 \end{aligned}\right\} \tag{II.149}$$

and that

$$\frac{\partial}{\partial\theta_\ell} S(\phi_\theta,\phi_\theta^\dagger,B_\theta,B_\theta^\dagger) = 0 \tag{II.150}$$

$$\frac{\partial}{\partial\theta_\ell} S^{(i)}(x,\phi_\theta,\phi_\theta^\dagger,B_\theta,B_\theta^\dagger) = -\epsilon_{i\ell k}\, S^{(k)}(x,\phi_\theta,\phi_\theta^\dagger,B_\theta,B_\theta^\dagger) \ . \tag{II.151}$$

We note that when the choices

$$\left.\begin{aligned} J(x) &= -\Lambda(-\overleftrightarrow{\partial})\phi_\theta(x)\ Z^{-\frac{1}{2}} \\ j(x) &= -K(-\overleftrightarrow{\partial})B_\theta^\dagger(x)\ \rho^{-\frac{1}{2}} \\ n(x) &= 0 \end{aligned}\right\} \tag{II.152}$$

are made, then W[J,j,n] becomes the transformed S-matrix $S(\phi_\theta,\phi_\theta^\dagger,B_\theta,B_\theta^\dagger)$. By a computation similar to that for the isotriplet model, we can then solve the equations (II.150) and (II.151) [25]. Thus, we find the following transformations for the quasiparticles:

$$\left.\begin{aligned} B_\theta(x) &= B(x) + i\theta_1\left(\frac{M}{2}\right)^{\frac{1}{2}} \\ B_\theta^\dagger(x) &= B^\dagger(x) - i\theta_1\left(\frac{M}{2}\right)^{\frac{1}{2}} \\ \phi_\theta(x) &= \phi(x) \\ \phi_\theta^\dagger(x) &= \phi^\dagger(x) \end{aligned}\right\} \text{ for } \theta_2 = \theta_3 = 0 \tag{II.153}$$

$$\left.\begin{aligned} B_\theta(x) &= B(x) - \theta_2\left(\frac{M}{2}\right)^{\frac{1}{2}} \\ B_\theta^\dagger(x) &= B_\theta^\dagger(x) - \theta_2\left(\frac{M}{2}\right)^{\frac{1}{2}} \\ \phi_\theta(x) &= \phi(x) \\ \phi_\theta^\dagger(x) &= \phi^\dagger(x) \end{aligned}\right\} \text{ for } \theta_1 = \theta_3 = 0 \tag{II.154}$$

$$\left.\begin{aligned}
B_\theta(x) &= e^{-i\theta_3}\, B(x) \\
B^\dagger_\theta(x) &= e^{i\theta_3}\, B^\dagger(x) \\
\phi_\theta(x) &= e^{i\theta_3\lambda_3}\, \phi(x) \\
\phi^\dagger_\theta(x) &= \phi^\dagger(x)\, e^{-i\theta_3\lambda_3}
\end{aligned}\right\} \text{ for } \theta_1 = \theta_2 = 0 \,, \qquad \text{(II.155)}$$

Thus we conclude that the spin rotation for the electron Heisenberg field is induced by the E(2) transformations (II.153-155) of the quasiparticle fields.

We now note that c-number translations in the transformations for B in (II.153) and (II.154) must be understood as the limit for $f(x) \to 1$ of the transformations

$$B(x) \to B_\theta(x) = \lim_{f\to 1} [B(x) + if(x)\theta_1(\tfrac{M}{2})^{\frac{1}{2}}] \qquad \text{(II.156)}$$

$$B(x) \to B_\theta(x) = \lim_{f\to 1} [B(x) - f(x)\theta_2(\tfrac{M}{2})^{\frac{1}{2}}] \qquad \text{(II.157)}$$

respectively. Here $f(x)$ stands for any square-integrable function. Since $B_\theta(x)$ must satisfy the magnon equation, it is necessary that $f(x)$ satisfies the magnon equation. Note that the magnon equations are invariant under the transformations in (II.156-157) even before the limit $f(x) \to 1$ is taken, exhibiting the E(2) invariance of the theory.

The generators of the transformations (II.153-155) (with θ_1 and θ_2 replaced by $\theta_1 f(x)$ and $\theta_2 f(x)$ respectively) are

$$\left.\begin{aligned} s_f^{(1)} &= \left(\frac{M}{2}\right)^{\frac{1}{2}} \int d^3x [B(x)f(x) + B^\dagger(x)f^*(x)] \\ s_f^{(2)} &= -i \left(\frac{M}{2}\right)^{\frac{1}{2}} \int d^3x [B(x)f(x) - B^\dagger(x)f^*(x)] \\ s^{(3)} &= d^3x[\phi^\dagger(x)\lambda_3\phi(x) - B^\dagger(x)B(x)] \ . \end{aligned}\right\} \qquad \text{(II.158)}$$

As was pointed out previously, presence of the function $f(x)$ is essential for $s_f^{(1)}$ and $s_f^{(2)}$ to be well defined. We also note that the generators (II.158) are time independent since $f(x)$ satisfies the magnon equation.

The generators (II.158) satisfy the following commutation relations:

$$\left.\begin{aligned} [s_f^{(1)}, s_f^{(2)}] &= iM\int d^3x |f(x)|^2 = \text{const.} I \\ [s^{(3)}, s_f^{(1)}] &= is_f^{(2)} \\ [s^{(3)}, s_f^{(2)}] &= -is_f^{(1)} \end{aligned}\right\} \ . \qquad \text{(II.159)}$$

or, in terms of $s_f^{(\pm)} \equiv s_f^{(1)} \pm i\, s_f^{(2)}$,

$$\left.\begin{aligned} [s_f^{(+)}, s_f^{(-)}] &= 2M\int d^3x |f(x)|^2 = \text{const.} I \\ [s^{(3)}, s_f^{(\pm)}] &= \pm s_f^{(\pm)} \end{aligned}\right\} \qquad \text{(II.160)}$$

On the other hand, the spin operators $S_\ell^{(+)}$, $S_\ell^{(-)}$ and $S^{(3)}$ satisfy the algebraic relations of SU(2), generating the spin group (rotation group) transformations (II.114):

$$\left.\begin{aligned} [S^{(+)}, S^{(-)}] &= 2S^{(3)} \\ [S^{(3)}, S^{(\pm)}] &= \pm\, S^{\pm} \end{aligned}\right\} \qquad \text{(II.161)}$$

where $S^{(\pm)} = S^{(1)} \pm iS^{(2)}$.

Now we will study what causes the change of the original spin-rotation algebra (II.161) into the algebra (II.160).[14]

Let us decompose the magnon field B(x) into the sum of two parts as

$$B(x) = B_t(x) + B_\eta(x) \qquad \text{(II.162)}$$

where B_η contains only momenta smaller than η while momenta in B_t are larger than η. Here η is infinitesimal. There are many ways of constructing such B_η, for example

$$B_\eta(x) = \frac{\eta}{2}\int_{-\infty}^{+\infty} dt\; e^{-\eta|t|}\; B(x) \; . \qquad \text{(II.163)}$$

Using (II.140), we can write

[14]The algebra (II.160) is properly denoted as h_4 algebra [40].

$$B_\eta(x) = \frac{\eta}{2(2\pi)^{\frac{1}{2}}} \int d^4k\ \delta_\eta(k) B_{\vec{k}}\ e^{i\vec{k}\cdot\vec{x}} \qquad \text{(II.164)}$$

Here $\delta_\eta(k)$ is a function which approaches $\delta(k)$ in the limit $\eta \to 0$. Therefore $B_\eta(x)$ is of order η and independent of x in the limit $\eta \to 0$.

As already done in the case of the isotriplet model, using (II.147) we can write $SS^{(i)}(x,\phi,\phi^\dagger,B,B^\dagger)$, up to the first order in η to obtain:

$$\left.\begin{aligned} S^{(1)}(y) &= s_t^{(1)}(y) + \left(\frac{1}{2M}\right)^{\frac{1}{2}}(B_\eta + B_\eta^\dagger) s_t^{(3)}(y) \\ S^{(2)}(y) &= s_t^{(2)}(y) - i\left(\frac{1}{2M}\right)^{\frac{1}{2}}(B_\eta - B_\eta^\dagger) s_t^{(3)}(y) \\ S^{(3)}(y) &= s_t^{(3)}(y) + \left(\frac{1}{2M}\right)^{\frac{1}{2}}[i(B_\eta - B_\eta^\dagger) s_t^{(2)}(y) \\ &\qquad - (B_\eta + B_\eta^\dagger) s_t^{(1)}(y)] \ . \end{aligned}\right\} \qquad \text{(II.165)}$$

Note that, in the limit $\eta \to 0$, the matrix elements of $S^{(i)}(y)$ are equal to those of $s_t^{(i)}(y)$:

$$\langle i|S^{(i)}(y)|j\rangle = \langle i|s_t^{(i)}(y)|j\rangle \ . \qquad \text{(II.166)}$$

In particular, this for $i = 3$ gives

$$\langle 0|s_t^{(3)}(y)|0\rangle = \langle 0|S^{(3)}(y)|0\rangle = M \ . \qquad \text{(II.167)}$$

We can therefore write:

$$S_t^{(3)}(y) = M + :s_t^{(3)}(y): \qquad \text{(II.168)}$$

(II.165) show that, when we express the spin-density operators $S^{(i)}(y)$ in terms of quasiparticles and then ignore the infrared operators B_η and $B_\eta^\dagger$, we obtain $s_t^{(i)}(y)$. Therefore, the space-integration of $s_t^{(i)}(y)$ must be the generators in (II.158) in which $B(x)$ is replaced by $B_t(x)$. Therefore, we can write (II.165) as

$$S_f^{(1)} = s_t^{(1)} + \left(\frac{1}{2M}\right)^{\frac{1}{2}}(B_\eta+B_\eta^\dagger)s_t^{(3)} \qquad \text{(II.169)}$$

$$S_f^{(2)} = s_t^{(2)} - i\left(\frac{1}{2M}\right)^{\frac{1}{2}}(B_\eta-B_\eta^\dagger)s_t^{(3)} \qquad \text{(II.170)}$$

$$S_f^{(3)} = s_t^{(3)} + \left(\frac{1}{2M}\right)^{\frac{1}{2}}[i(B_\eta-B_\eta^\dagger)s_t^{(2)} - (B_\eta+B_\eta^\dagger)s_t^{(1)}] \qquad \text{(II.171)}$$

Here

$$s_t^{(1)} = \left(\frac{M}{2}\right)^{\frac{1}{2}} \int d^3x(B_t(x)f(x) + B_t^\dagger(x)f^*(x))$$

$$s_t^{(2)} = -i\left(\frac{M}{2}\right)^{\frac{1}{2}} \int d^3x(B_t(x)f(x) - B_t^\dagger(x)f^*(x))$$

$$s_t^{(3)} = \int d^3x:\phi^\dagger(x)\lambda_3\phi(x)f(x): + \int d^3x(M-B_t^\dagger(x)B_t(x))f(x)$$

according to (II.158). Here $f(x)$ is the square-integrable function which appeared in (II.158) and which is extremely close to 1. The spin-rotation generators $S^{(i)}$ should be given by $S_f^{(i)}$ in the limit $f \to 1$.

Use of (II.168) gives

$$s_t^{(1)} + \left(\frac{1}{2M}\right)^{\frac{1}{2}} (B_\eta + B_\eta^\dagger) s_t^{(3)}$$

$$= s_f^{(1)} + \left(\frac{1}{2M}\right)^{\frac{1}{2}} (B_\eta + B_\eta^\dagger) : s_t^{(3)} :, \text{ etc.}$$

because $B = B_t + B_\eta$. $s_f^{(1)}$ is given in (II.158). In this way we can rewrite (II.169) and (II.170) as

$$\left.\begin{aligned} S_f^{(1)} &= s_f^{(1)} + \left(\frac{1}{2M}\right)^{\frac{1}{2}} (B_\eta + B_\eta^\dagger) : s_t^{(3)} : \\ S_f^{(2)} &= s_f^{(2)} - i\left(\frac{1}{2M}\right)^{\frac{1}{2}} (B_\eta - B_\eta^\dagger) : s_t^{(3)} : \end{aligned}\right\} \qquad \text{(II.172)}$$

where $s_f^{(1)}$ and $s_f^{(2)}$ are given in (II.158).

Our task now is to show that the spin-operators $S_f^{(i)}$ in (II.169-171) indeed satisfy the commutators for the rotation group (II.161) when the limit $f \to 1$ is taken. We first note the following commutation relations:

$$
\left.\begin{aligned}
[s_f^{(1)}, s_f^{(2)}] &= iM\int d^3x\,|f(x)|^2 \\
[:s_t^{(3)}:, s_t^{(1)}] = [s_t^{(3)}, s_f^{(1)}] &= i\ s_t^{(2)} \\
[:s_t^{(3)}:, s_t^{(2)}] = [s_t^{(3)}, s_f^{(2)}] &= -i\ s_t^{(1)} \\
[s_f^{(1)}, B_\eta(x)] &= -(\tfrac{M}{2})^{\frac{1}{2}} f_\eta^*(x) \\
[s_f^{(1)}, B_\eta^\dagger(x)] &= (\tfrac{M}{2})^{\frac{1}{2}} f_\eta(x) \\
[s_f^{(2)}, B_\eta(x)] &= -i(\tfrac{M}{2})^{\frac{1}{2}} f_\eta^*(x) \\
[s_f^{(2)}, B_\eta^\dagger(x)] &= -i(\tfrac{M}{2})^{\frac{1}{2}} f_\eta(x) \\
[s^{(3)}, B_\eta(x)] &= B_\eta(x) \\
[s^{(3)}, B_\eta^\dagger(x)] &= -B_\eta^\dagger(x)\ .
\end{aligned}\right\} \qquad \text{(II.173)}
$$

Here $f_\eta(x)$ is the infrared part of the square-integrable function $f(x)$:

$$f(x) = f_t(x) + f_\eta(x)$$

In other words, $f_\eta(x)$ contains only momenta smaller than η and therefore has domain of range $1/\eta$. Thus $f_\eta(x)$ vanishes in the limit $\eta \to 0$ because $f(x)$ is square-integrable. Let us note that, since we consider commutators of two generators (i.e. two successive rotations) we need two infrared cut-off η and $\bar{\eta}$: two limits $\eta \to 0$ and $\bar{\eta} \to 0$ are to be performed successively. Let us assume that the limit $\bar{\eta}\to 0$ is to be performed before the limit $\eta\to 0$ and therefore assume that $\eta >> \bar{\eta}$ (we find a same result when we exchange the order of two limits). To take care of the locally infinitesimal effect, the space integration must extend to infinity. We must, therefore, take the limit $f\to 1$ before η and $\bar{\eta}$ tend to zero in order to recognize the differences between $S_f^{(i)}$ and $s_f^{(i)}$. Using (II.172-173) we find

$$[S_f^{(1)},S_f^{(2)}] = iM\int d^3x\,|f(x)|^2$$

$$+ i\,\frac{1}{2}\,(f_\eta^*(x)+f_\eta(x)+f_{\bar{\eta}}^*(x)+f_{\bar{\eta}}(x)):s_t^{(3)}: \tag{II.174}$$

$$- \left(\frac{1}{2M}\right)^{\frac{1}{2}}(B_{\bar{\eta}} - B_{\bar{\eta}}^\dagger)s_t^{(2)} - i\left(\frac{1}{2M}\right)^{\frac{1}{2}}(B_\eta+B_\eta^\dagger)s_t^{(1)}$$

Since $\bar{\eta} << \eta$ implies that $|f_{\bar{\eta}}|<<|f_\eta|$ for the square-integrable function f, we ignore $f_{\bar{\eta}}$ and $f_{\bar{\eta}}^*$ in the second term in the right hand side of (II.174). Taking the limit $f\to 1$, (II.174) leads to

$$[S^{(1)},S^{(2)}] = i\,S^{(3)} \tag{II.175}$$

where (II.168) and (II.165) are taken into consideration.

We also obtain

$$[S^{(3)},S^{(1)}] = i\,S^{(2)},\quad [S^{(3)},S^{(2)}] = -i\,S^{(1)} \tag{II.176}$$

Thus we found that

$$\lim_{\eta\to 0}\lim_{\tilde{\eta}\to 0}\lim_{f\to 1}[S_f^{(i)},S_f^{(j)}] = i\,\varepsilon_{ijk}\,S^{(k)} \tag{II.177}$$

while

$$\begin{aligned}
\lim_{f\to 1}\lim_{\tilde{\eta}\to 0}[S_f^{(1)},S_f^{(2)}] &= i\,M\lim_{f\to 1}\int d^3x|f(x)|^2 \\
&= \text{const.}I \\
\lim_{f\to 1}\lim_{\tilde{\eta}\to 0}[S_f^{(3)},S_f^{(1)}] &= \lim_{f\to 1}\lim_{\tilde{\eta}\to 0} i\,S^{(2)} \\
\lim_{f\to 1}\lim_{\tilde{\eta}\to 0}[S_f^{(3)},S_f^{(2)}] &= -\lim_{f\to 1}\lim_{\tilde{\eta}=0} i\,S^{(1)}\ .
\end{aligned} \tag{II.178}$$

where $\tilde{\eta} \equiv$ minimum of η and $\bar{\eta}$.

Eq. (II.177), in which the limit $f\to 1$ is performed before the limit $\tilde{\eta}\to 0$ and $\eta\to 0$, corresponds to the rotational group symmetry, while (II.178) correspond to the E(2) group symmetry. We have thus proved that the differences between $S^{(i)}$ and $s^{(i)}$ are due to the infrared effects: $\lim f\to 1$ and $\lim \tilde{\eta}\to 0$ are not commutable. The infrared term, although locally infinitesimal, gives, however, a finite global contribution to the commutators of the generators $S^{(i)}$ of the electron transformation. Its locally infinitesimal nature makes it, instead, commutable with any local operator and thus it does not contribute to the generators commutator for the quasiparticles, which are directly related to the (local) observations.

Let us note that the E(2) symmetry (II.160) is related with observable results, since quasiparticles are related to observable energy levels. The fact that the magnons are associated with the E(2) symmetry can be expressed by saying that the magnons form an irreducible non unitary representation of the E(2) symmetry group.

Although we assumed the electron model, all the arguments in connection with the magnons are true in any model for the spin, because we did not assume any specific form of spin density operator $S^{(i)}(x)$. In case of the electron model, quasielectron ϕ appears in addition to the magnons. It is remarkable that under the E(2) transformations in (II.153) and (II.154), ϕ does not change at all; magnon is the only agent for the transformation generated by $S^{(1)}$ and $S^{(2)}$.

It is also easy to show that the E(2) transformations of quasiparticles in (II.153-155) induce the spin transformation on the Heisenberg operator of the electron ψ_H, i.e. $\psi_H(x) \to \exp(i\theta_i\lambda_i)\psi_H(x)$. To do this we use the path-integral formalism to express $\psi_H(x)$ in terms of ϕ and B.

$$S\psi_H(x) = \langle\psi(x):\exp[-iA(\phi,\phi^\dagger,B,B^\dagger)]:\rangle$$

We then perform E(2) transformations (II.153-155); this results in the spin rotation of ψ_H. The proof for this follows the same steps of argument for $S^{(i)}(x)$.

Dyson concluded [42] that interaction between low frequency magnons is very small. We note that this is a manifestation of the E(2) symmetry.

Indeed we proved (cf. (II.150)) that the S-matrix is invariant under the transformation $B \to B + \text{const.}$ (cf. (II.153-155)); this implies that the magnon operator B always appears with its derivatives in the S-matrix, and thus magnon interaction disappears in the zero momentum limit. In this connection we note that the study of interactions among quasiparticles requires the study of relations among vertices with many external lines. This can be done by the path-integral technique, too [24,30,33, 36].

We analyze now the previous results in connection with the Holstein and Primakoff [43] commutation relations for spin operators.

Holstein and Primakoff in 1940 introduced a method to diagonalize the Hamiltonian in the exchange interaction model of a ferromagnet, by introducing second quantized creation and annihilation operators a_ℓ^* and a_ℓ for magnons. In terms of such operators, the spin angular momentum operators are given by [43]

$$\left.\begin{aligned} S_\ell^{(+)} &= S_\ell^{(1)} + iS_\ell^{(2)} = (2S)^{\frac{1}{2}}(1 - a_\ell^* a_\ell/2S)^{\frac{1}{2}}\, a_\ell \\ S_\ell^{(-)} &= S_\ell^{(1)} - iS_\ell^{(2)} = (2S)^{\frac{1}{2}}\, a_\ell^*(1 - a_\ell^* a_\ell/2S)^{\frac{1}{2}} \\ S_\ell^{(3)} &= S - a_\ell^* a_\ell \quad . \end{aligned}\right\} \qquad \text{(II.179)}$$

Here S is the eigenvalue of the third component of total spin and $S^{(\pm)}$, $S^{(3)}$ satisfy the spin group algebra. The operators a_ℓ^* and $a_{\ell'}$ satisfy the usual commutation relations for bosons:

$$\left.\begin{aligned} [a_\ell, a_{\ell'}^*] &\equiv a_\ell a_{\ell'}^* - a_{\ell'}^* a_\ell = \delta_{\ell\ell'} \\ [a_\ell, a_{\ell'}] &= [a_\ell^*, a_{\ell'}^*] = 0 \ . \end{aligned}\right\} \qquad \text{(II.180)}$$

A crucial role in the Holstein-Primakoff method is played by the approximation used in writing the Hamiltonian in terms of a_ℓ^* and a_ℓ: essentially they neglected all terms which were not bilinear in a_ℓ^* and a_ℓ. In this way they were able to construct a linear formalism suitable for practical calculations. In this approximation, the factor $(1 - a_\ell^* a_\ell/2S)^{\frac{1}{2}}$ in eq. (II.179) is put equal to 1; therefore the operators $S^{(\pm)}$ are replaced by $s^{(\pm)}$ which are defined by

$$\left.\begin{aligned} s_\ell^{(+)} &= (2S)^{\frac{1}{2}}\, a_\ell \\ s_\ell^{(-)} &= (2S)^{\frac{1}{2}}\, a_\ell^* \\ s_\ell^{(3)} &= S_\ell^{(3)} = S - a_\ell^* a_\ell \end{aligned}\right\} \qquad \text{(II.181)}$$

These operators satisfy the following commutation relations

$$\left.\begin{aligned} [s_\ell^{(3)}, s_{\ell'}^{(+)}] &= s_\ell^{(+)}\delta_{\ell\ell'} \\ [s_\ell^{(3)}, s_{\ell'}^{(-)}] &= -s_\ell^{(-)}\delta_{\ell\ell'} \\ [s_\ell^{(+)}, s_{\ell'}^{(-)}] &= 2S\delta_{\ell\ell'} = \text{const.}I.\delta_{\ell\ell'} \end{aligned}\right\} \qquad \text{(II.182)}$$

where I is the unit matrix.

Let us now introduce

$$\left.\begin{aligned} S^{(\alpha)} &= \sum_\ell S_\ell^{(\alpha)} \\ s^{(\alpha)} &= \sum_\ell s_\ell^{(\alpha)} \end{aligned}\right\} \qquad \text{(II.183)}$$

with $\alpha = +, -, 3$ and $s^\pm = s^{(1)} \pm is^{(2)}$.

Since $S_\ell^{(\alpha)}$ and $s_\ell^{(\alpha)}$ are the operators associated with the position x_ℓ, equations in (II.183) mean

$$\left.\begin{aligned} S^{(\alpha)} &= \sum_\ell S^{(\alpha)}(x_\ell) \\ s^{(\alpha)} &= \sum_\ell s^{(\alpha)}(x_\ell) \ . \end{aligned}\right\} \qquad \text{(II.184)}$$

Note that $S^{(\alpha)}$ are the spin-rotation generators with the algebra (II.161).

The operators $s_\ell^{(1)}$ and $s_\ell^{(2)}$ respectively generate the "field transformations"

$$\left.\begin{aligned} a_\ell &\to a_\ell + i\left(\frac{S}{2}\right)^{\frac{1}{2}} \theta_1 \\ a_\ell &\to a_\ell - i\left(\frac{S}{2}\right)^{\frac{1}{2}} \theta_2 \end{aligned}\right\} \quad \text{(for all } \ell\text{)} \qquad \text{(II.185)}$$

while $s^{(3)}$ induces rotation around the third axis:

$$a_\ell \to e^{i\theta_3} a_\ell \quad . \qquad \text{(II.186)}$$

Here θ_1, θ_2 and θ_3 are continuous parameters of the transformations. Our previous results mean that without any approximation, when we express the spin density operators $S^{(\alpha)}(x_\ell)$ in terms of quasiparticles and sum them over all the space points, then the result is not $S^{(\alpha)}$ but $s^{(\alpha)}$ with the boson Heisenberg operator a_ℓ replaced by the free field operator of the quasiboson (i.e. the magnon).

Although our conclusion is that $S^{(\alpha)}$ becomes $s^{(\alpha)}$ when expressed in terms of quasiparticles, i.e. in the quasiparticle picture, this does not justify the Holstein-Primakoff approximation. In the Holstein-Primakoff article, the operator a_ℓ acts as the Heisenberg operator and not as an operator for quasiboson. When a_ℓ is expressed in terms of the quasiboson, i.e. magnon B, the expression is a very complicated one. In fact, a_ℓ is an infinite power series in normal products of B and $B^\dagger$. Thus we are not considering the linear approximation, which has been clarified by

Dyson[15] [42]. To establish the quasiparticle picture, we need to consider infinite power expansions of Heisenberg operators in terms of creation-annihilation operators of quasiparticles (i.e. the dynamical map). Through this process the quasiboson (magnon) emerges as a bound state of electrons (cf. Appendix).

Finally, we recall that our arguments are completely general; no assumption is made on the Lagrangian except its invariance under the spin rotation transformation (II.114); furthermore, our study covers the cases of localized spin (such as the Heisenberg model) and of continuous spin distribution (such as the itinerant electron).

Although the path-integral method presents various results which are independent of specific model and of any approximation, it rarely helps us in computing model-dependent quantities. In the framework of the self-consistent method, an example of model-dependent calculation for ferromagnetism is given in the Appendix.

In closing this section we observe that in both the examples studied, i.e. the iso-triplet model and the ferromagnetic system, the three-parameter SU(2) symmetry is not

---

[15]The representation considered by Holstein and Primakoff is a nonlinear representation of the rotation group. We have seen that if we express the spin-density operators in terms of quasiparticles, this naturally leads us to the linear representation of E(2) group.

For an analysis from the viewpoint of algebraic realization of spin-wave theory see ref. [41].

simply reduced to the cylindrical rotation symmetry of one parameter, but is replaced by the E(2) symmetry of three parameters. As already mentioned the E(2) symmetry is the one which concerns observations and it can be detected through low-energy theorems.

2.6 Spontaneous breakdown of gauge symmetry.

We have seen that spontaneous breakdown of symmetry implies the existence of massless bosons, the Goldstone particles, in the hypothesis that there are no long range forces in the theory. Intuitively speaking, the phenomenological systematic structure of the vacuum is due to the presence of these massless bosons which act as long range correlation modes.

In particle physics, it seems hard, however, to explain the observed symmetry violations in terms of Goldstone bosons since there is no experimental observation of such massless particles. In non-relativistic phenomena it is known [44] that the presence of the Coulomb interaction affects the Goldstone theorem in the sense that the excitation modes have finite mass. Intuitively, this fact suggests that the role of the Goldstone mode in the creation of the systematic structure of the ground state is played by the long range Coulomb force. Following these ideas, it has been shown [45] that in particle physics the presence of a gauge field affects a spontaneously broken symmetry theory by eliminating the massless Goldstone particles (Higgs phenomenon). By taking advantage of the Higgs phenomenon, a unified theory of weak and electromagnetic interactions has been proposed [46] and many investigations of spontaneously broken gauge theories have been done, especially in connection with the renormalizability of such theories [35].

It has been observed [47] that if long range forces are present in the theory, the conservation of the total charge associated with the symmetry transformation is not valid, since surface contributions of the current do not vanish. This fact invalidates the Goldstone theorem and Goldstone particles completely disappear from the theory. This situation is, however, unsatisfactory since the Goldstone bosons play the crucial role of preserving the local conservation of currents associated with the invariance of the theory [2,4] (cf. Sec. 2.2), and the absence of such Goldstone particles would make the theory internally inconsistent from the point of view of the invariance properties. For example, in the case of a chiral-gauge invariant Nambu-type model, in which the helicity current is coupled with an axial vector gauge field, Freundlich and Lurié [48] found that the gauge invariance vanishes completely from the theory when expressed in terms of physical fields. Obviously this conclusion is not acceptable since the invariance of the theory cannot simply disappear and furthermore, it would be impossible the dynamical mapping of the Heisenberg fields, which do transform under chiral-gauge transformation, in terms of physical fields all invariant under such a transformation. Starting from these criticisms Aurilia, Takahashi and Umezawa [49] found that, although a manifestly covariant solution of the field equations is consistent only with the symmetric solution of the mass equation, however, there exists a non-covariant solution associated with a long range mode which plays the role of a Goldstone particle. In non-relativistic theories a similar situation is found. In particular, in superconductivity [32,50] it is found that a boson condensation is allowed also in the presence of Coulomb force; in this sense the Goldstone theorem is still valid and the gapless energy modes recover the symmetry properties of the theory.

On the other hand, in the relativistic case Nakanishi found that in the Landau-gauge formalism the theory can be put in a manifestly covariant from and that the Goldstone bosons are present as unphysical particles [51].

In the present section we analyze the Higgs phenomenon by means of the path-integral technique, in which as usual we do not need to specify the particular model. Our only assumption on the Lagrangian $L(\phi(x),\phi^*(x),A_\mu(x))$ is that it is invariant under global gauge transformation (first kind gauge transf.)

$$\phi(x) \to e^{i\alpha}\,\phi(x) \tag{II.187a}$$

$$A_\mu(x) \to A_\mu(x)\ , \tag{II.187b}$$

and under local gauge transformation (second kind gauge transf.)

$$\phi(x) \to e^{ie_o\Lambda(x)}\,\phi(x) \tag{II.188a}$$

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\Lambda(x)\ . \tag{II.188b}$$

$\phi(x)$ is a complex scalar field and $A_\mu(x)$ is a gauge vector boson field; the constant α and the c-number real function $\Lambda(x)$ are the parameters of the gauge transformations (II.187) and (II.188) respectively; e_o is a constant (charge). It is also assumed that $\Lambda(x) \underset{|x|\to\infty}{\to} 0$. The generating functional is [52]

$$\begin{aligned} W[K,J_\mu] = \frac{1}{N}\int [d\phi][d\phi^*][dA_\mu][dB]\exp\Big\{i\int d^4x[L[\phi(x),\phi^*(x),A_\mu(x)] \\ + K^*(x)\phi(x) + \phi^*(x)K(x) + J_\mu(x)A_\mu(x) + B(x)\partial^\mu A_\mu(x) \\ + i\varepsilon|\phi(x) - v|^2\Big\} \end{aligned} \tag{II.189}$$

where N is equal to the numerator of (II.189) with $K = J_\mu = 0$. The ε-term is introduced to violate the first kind gauge transformation (II.187). The term $B(x)\partial^\mu A_\mu(x)$ is equivalent to the gauge condition $\partial^\mu A_\mu(x) = 0$; indeed

$$\int [dB] \exp\left\{ i \int d^4x \; B(x)\partial^\mu A_\mu(x) \right\} = \Pi_x \; \delta(\partial^\mu A_\mu(x)) \; . \qquad \text{(II.190)}$$

Let us introduce the notation

$$\phi(x) = \frac{1}{\sqrt{2}} [\psi(x)+i\chi(x)] \; ; \; K(x) = \frac{1}{\sqrt{2}} [K(x)+iK_2(x)] \qquad \text{(II.191)}$$

and assume v real and different from zero.

By performing the transformations (II.187) and (II.188) in the numerator of (II.189) we obtain

$$\begin{aligned} & i\int d^4x \langle K_2(x)\psi(x) - K_1(x)\chi(x)\rangle_{\varepsilon,K,J_\mu} \\ & \quad = \sqrt{2}\; \varepsilon v \int d^4x \; \langle\chi(x)\rangle_{\varepsilon,K,J_\mu} \end{aligned} \qquad \text{(II.192)}$$

$$\begin{aligned} & i\langle\partial^2 B(x) - \partial^\mu J_\mu(x) + e_o(K_2(x)\psi(x) - K_1(x)\chi(x)\rangle_{\varepsilon,K,J_\mu} \\ & \quad = \sqrt{2}\; \varepsilon v e_o \; \langle\chi(x)\rangle_{\varepsilon,K,J_\mu} \end{aligned} \qquad \text{(II.193)}$$

By operating $\delta/\delta K_2(y)$ on (II.192), we have

$$\langle\psi(y)\rangle_\varepsilon = \sqrt{2}\; \varepsilon v \int d^4x \; \langle\chi(x)\chi(y)\rangle_\varepsilon \qquad \text{(II.194)}$$

By putting $\psi(x) = \tilde{v} + \rho(x)$, with $\langle\rho(x)\rangle = 0$, we can see that, if

$$\lim_{\varepsilon\to 0} \langle\psi(x)\rangle_\varepsilon \equiv \lim_{\varepsilon\to 0} \tilde{v}(\varepsilon) \equiv \tilde{v} \neq 0 \; , \qquad \text{(II.195)}$$

the χ-propagator must have a massless pole (Goldstone theorem):

$$<\chi(x)\chi(y)>_\varepsilon = \frac{i}{(2\pi)^4}\int d^4p\ e^{-ip(x-y)}\ \Delta_\chi(\varepsilon,p) \qquad \text{(II.196)}$$

with

$$\Delta_\chi(\varepsilon,p) = Z_\chi\left|\frac{1}{p^2+i\varepsilon a_\chi} + \tilde{\Delta}_\chi(\varepsilon,p)\right| \qquad \text{(II.197)}$$

where Z_χ is the wave-function renormalization of $\chi(x)$ and $\tilde{\Delta}_\chi(\varepsilon,p)$ is the continuum contribution part of the propagator. (II.194) gives

$$\tilde{v} = \sqrt{2}\ v\ \frac{Z_\chi}{a_\chi}\ . \qquad \text{(II.198)}$$

From (II.193), by convenient differentiations, we obtain

$$i<\partial_x^2 B(x)\partial_y^2 B(y)>_\varepsilon = \sqrt{2}\ \varepsilon v e_o <\chi(x)\partial_y^2 B(y)>_\varepsilon \qquad \text{(II.199)}$$

$$i<\partial_x^2 B(x)\chi(y)>_\varepsilon + e_o<\psi(x)>_\varepsilon\delta(x-y) = \sqrt{2}\ \varepsilon v e_o<\chi(x)\chi(y)>_\varepsilon \qquad \text{(II.200)}$$

$$i<\partial_x^2 B(x)A_\mu(y)>_\varepsilon - \partial^\mu\delta(x-y) = \sqrt{2}\ \varepsilon v e_o<\chi(x)A_\mu(y)>_\varepsilon \qquad \text{(II.201)}$$

Let us note that, due to the symmetry $\chi \to -\chi$, $A_\mu \to -A_\mu$ and $B \to -B$, it must be $<\rho(x)B(y)> = <\rho(x)\chi(y)> = <\rho(x)A_\mu(x)> = <\chi(x)> = <A_\mu(x)> = <B(x)> = 0$. By using

(II.198) and writing the propagators for fields A(x) and B(x) in the momentum representation as

$$\langle A(x)B(y)\rangle = \frac{i}{(2\pi)^4}\int d^4p\; e^{-ip(x-y)}\,\Delta_{AB}(p) \qquad \text{(II.202)}$$

with $\Delta_{AB}(p) = \Delta_{BA}(-p)$, (II.199-201) together with (II.197) give [52]

$$\left.\begin{aligned}
\Delta_{\chi\chi}(p) &= Z_\chi\left[\frac{1}{p^2+i\varepsilon a_\chi} + \tilde{\Delta}_{\chi\chi}(p)\right] \\
\Delta_{B\chi}(p) &= -\frac{e_o\tilde{v}}{p^2+i\varepsilon a_\chi} \\
\Delta_{BB}(p) &= \frac{(e_o\tilde{v})^2}{Z_\chi}\left[\frac{1}{p^2+i\varepsilon a_\chi} - \frac{1}{p^2}\right] \\
\Delta_{BA_\mu}(p) &= -ip_\mu\,\frac{1}{p^2}
\end{aligned}\right\} \qquad \text{(II.203)}$$

where we considered only lower orders in ε and the term $1/p^2$ is defined as $\lim_{\eta\to 0} 1/(p^2+i\eta)$. It was also assumed that $\Delta_{\chi A_\mu}(\varepsilon,p)$ does not have a pole term like $p_\mu/(p^2+i\varepsilon a_\chi)$. Due to (II.203) we can write the Heisenberg fields in terms of in-fields as follows:

$$\left.\begin{aligned}
\chi(x) &= Z_\chi^{\frac{1}{2}}\,\chi^{in}(x) + \dots \\
B(x) &= -\frac{e_o\tilde{v}}{Z_\chi^{\frac{1}{2}}}\,\chi^{in}(x) + \frac{e_o\tilde{v}}{Z_\chi^{\frac{1}{2}}}\,b^{in}(x) + \dots \\
A_\mu(x) &= Z_3^{\frac{1}{2}}\,U_\mu^{in}(x) - \frac{Z_\chi^{\frac{1}{2}}}{e_o\tilde{v}}\,\partial^\mu b^{in}(x) + \dots
\end{aligned}\right\} \qquad \text{(II.204)}$$

We put

$$<\chi^{in}(x)\chi^{in}(y)> = \frac{i}{(2\pi)^4}\int d^4p \ e^{-ip(x-y)} \frac{1}{p^2+i\varepsilon a_\chi} \qquad (II.205)$$

$$<b^{in}(x)b^{in}(y)> = \frac{i}{(2\pi)^4}\int d^4p \ e^{-ip(x-y)} \frac{-1}{p^2+i\eta} \qquad (II.206)$$

and

$$[\chi^{in}(x),b^{in}(y)] = 0 \ . \qquad (II.207)$$

Z_3 is the wave function renormalization constant for the field $U_\mu(x)$ and $b^{in}(x)$ is a negative norm state. The presence of this state in the theory can be ascribed to the fact that we are using the gauge $\partial^\mu A_\mu(x)$ [51,52]. Eqs. (II.204) can be written also in the form

$$\left.\begin{aligned}
&\chi(x) = Z_\chi^{\frac{1}{2}}\,\chi^{in}(x) + \ldots \\
&b(x) = B(x) + \frac{e_o\tilde{v}}{Z_\chi}\,\chi(x) = \frac{e_o\tilde{v}}{Z_\chi^{\frac{1}{2}}}\,b^{in}(x) + \ldots \\
&U_\mu(x) = A_\mu(x) + \frac{Z_\chi}{(e_o\tilde{v})^2}\,\partial^\mu b(x) \\
&\quad = A_\mu(x) + \frac{Z_\chi}{(e_o\tilde{v})^2}\,\partial^\mu B(x) + \frac{1}{e_o\tilde{v}}\,\partial^\mu\chi(x) = Z_3^{\frac{1}{2}}\,U_\mu^{in}(x) \\
&\quad + \ldots
\end{aligned}\right\} \qquad (II.208)$$

where dots mean higher order terms.

Note that (II.193) for $K = J_\mu = \varepsilon = 0$ gives

$$-\partial^2 B(x) = 0 \tag{II.209}$$

which gives

$$-\partial^2 \chi^{in}(x) = -\partial^2 b^{in}(x) = 0 \tag{II.210}$$

due to (II.204). Since $\partial_\mu A_\mu(x) = 0$, (II.204) gives

$$\partial_\mu U_\mu^{in}(x) = 0 \tag{II.211}$$

To better understand the role played by χ^{in} and b^{in} let us study the dynamical rearrangement of symmetry. The free field equations for our in-fields χ^{in}, b^{in}, U_μ^{in} are the eqs. (II.210-211). Assuming a mass m_ρ for $\rho^{in}(x)$, the free field equation for $\rho^{in}(x)$ is

$$(-\partial^2 - m_\rho^2)\rho^{in}(x) = 0 \tag{II.212}$$

As usual we write the s-matrix S and the Heisenberg fields as

$$S = \langle :\exp\left\{-iA(\chi^{in}(x), \rho^{in}(x), b^{in}(x), U_\mu^{in}(x))\right\}:\rangle \tag{II.213}$$

$$SA_\mu^H(x) = \langle A_\mu(x) :\exp\left\{-iA(\chi^{in}(x), \rho^{in}(x), b^{in}(x), U_\mu^{in}(x))\right\}:\rangle \tag{II.214}$$

$$S\phi_H(x) = \langle \phi(x) :\exp\left\{-iA(\chi^{in}(x), \rho^{in}(x), b^{in}(x), U_\mu^{in}(x))\right\}:\rangle \tag{II.215}$$

with

$$A(\chi^{in}(x),\rho^{in}(x),b^{in}(x),U_\mu^{in}(x)) =$$

$$\int d^4x [Z_\chi^{-\frac{1}{2}}\chi^{in}(x)(-\partial_x^2)\chi(x) + Z_\rho^{-\frac{1}{2}}\rho^{in}(x)(-\partial_x^2-m_\rho^2) \qquad \text{(II.216)}$$

$$\rho(x) + \frac{Z_\chi^{\frac{1}{2}}}{e_o\tilde{v}} b^{in}(x)(-\partial_x^2)b(x) + Z_3^{-\frac{1}{2}}U_\mu^{in}(x)(-\partial_x^2)U^\mu(x)].$$

We look now for transformations of the in-fields

$$\left.\begin{array}{l} \chi^{in}(x) \to \chi_\lambda^{in}(x) = \tilde{\chi}(x,\lambda;\chi^{in},\rho^{in},b^{in},U_\mu^{in}) \\ \rho^{in}(x) \to \rho_\lambda^{in}(x) = \tilde{\rho}(x,\lambda;\chi^{in},\rho^{in},b^{in}U_\mu^{in}) \\ \text{etc.} \end{array}\right\} \qquad \text{(II.217)}$$

with certain parameter λ, such that χ_λ^{in}, ρ_λ^{in}, b_λ^{in} and U_λ^{in} satisfy the free field equations (II.210-212) and such that

$$\frac{\partial}{\partial\lambda} S(\lambda) = 0 \qquad \text{(II.218)}$$

$$\frac{\partial}{\partial\lambda} SA_\mu^H(\lambda,x) = \frac{\partial}{\partial\lambda} [S(A_\mu^H(x)+\lambda\partial^\mu\alpha(x))] = \partial^\mu\alpha(x)S \qquad \text{(II.219)}$$

$$\frac{\partial}{\partial\lambda} S\phi_H(\lambda,x) = ie_o\, \alpha(x)\, S\phi_H(x) \qquad \text{(II.220)}$$

where $\alpha(x)$ is a real c-number function. We also used the notation $F(\lambda,x) \equiv F(\chi_\lambda^{in}(x),\rho_\lambda^{in}(x),b_\lambda^{in}(x),U_{\mu,\lambda}^{in}(x))$. The equations (II.218-220) can be solved by the same method used to solve the equations (II.84) and (II.150-151) in Sec. 2.5. In the present case, however, we first modify $W[K,J_\mu]$ given in the equation (II.189) by introducing the term $J_B(x)\partial^2 B(x)$ in the exponential. The introduction of such a term does not modify the previous equations since in the Ward-Takahashi identities the field $B(x)$ appears always with the derivative operator ∂_x^2. Then our choice for the sources is

$$\left.\begin{aligned}
J_\mu(x) &= -Z_3^{-\frac{1}{2}}\, U_\mu^{in}(-\partial_x^2) \\
K_1(x) &= -Z_\rho^{-\frac{1}{2}}\, \rho^{in}(x)(-\partial_x^2-m_\rho^2) \\
K_2(x) &= -Z_\chi^{-\frac{1}{2}}[\chi^{in}(x)(-\partial_x^2) + b^{in}(x)(-\partial_x^2)] \\
\partial^2 J_B(x) &= -\frac{Z_\chi^{\frac{1}{2}}}{e_o\tilde{v}}\, b^{in}(x)(-\partial_x^2)
\end{aligned}\right\} \qquad \text{(II.221)}$$

When we assume that $(-\partial_x^2)\alpha(x) = 0$, we find [52] the following transformations for the in-fields:

$$\left.\begin{aligned}
U_\mu^{in}(x) &\to U_\mu^{in}(x) \\
\rho^{in}(x) &\to \rho^{in}(x) \\
b^{in}(x) &\to b^{in}(x) \\
\chi^{in}(x) &\to \chi^{in} + e_o \frac{\tilde{v}}{Z^{\frac{1}{2}}}\,\alpha(x)
\end{aligned}\right\} \qquad \text{(II.222)}$$

for $\lambda \to 1$. This shows that the (local) gauge transformation of the Heisenberg fields are induced through the boson transformation of the massless in-field χ^{in}.

Let us observe now that due to (II.209) we can define the positive-frequency part $B^{(+)}(x)$ of $B(x)$ and define the physical states $|phys>$ by [51]

$$B^{(+)}(x)|phys> = 0 \ . \qquad \text{(II.223)}$$

On the other hand, (II.203) shows that χ^{in} does not commute with B, unless $\tilde{v} = 0$. Thus

$$B^{(+)}(y)\chi^{in}(x)|phys> = \chi^{in}(x)B^{(+)}(y)|phys> + c|phys> \qquad \text{(II.224)}$$

where c is a c-number created by the commutation of χ^{in} and B. Then, (II.224) shows that $\chi^{in}(x)$ is a unphysical field since

$$B^{(+)}(y)(\chi^{in}(x)|phys>) \neq 0 \ . \qquad \text{(II.225)}$$

In conclusion we have shown that in a spontaneously broken theory a Goldstone mode is present even in the presence of a guage field, although this mode is unphysical. It is due to these Goldstone (unphysical) particles that the invariance of the theory can be consistently recovered at the level of physical fields.

We considered above the case of an abelian gauge theory.

It will be interesting to consider also non-abelian gauge theories and to analyze the connection, if any, between the unphysical Goldstone mode and the fictitious scalar field introduced by Faddeev and Popov [54] in the perturbative treatment of a Yang-Mills [55] field theory.

III. INEQUIVALENT REPRESENTATIONS OF CANONICAL COMMUTATION RELATIONS

3.1 Existence of unitarily inequivalent representations of canonical commutation relations.

For systems with a finite number of degrees of freedom, different irreducible representations of the canonical (anti-) commutation relations are unitarily equivalent to each other. This is the content of the well-known Von Newmann's theorem [6].

In Quantum Mechanics, the number of degrees of freedom is always finite for any system and the Von Neumann's theorem holds. A different situation occurs in Quantum Field Theory where systems have always infinitely many degrees of freedom; it has been shown, indeed, that there are infinitely many unitarily inequivalent representations of the canonical (anti-) commutation relations [5]. This feature of Q.F.T. has been widely studied [2,29,56,57], and here we want to illustrate it by considering a boson system as an example [2,29,52]. The extension to the case of fermions is straightforward.

We consider as hamiltonian of our system the following one:

$$H = \int d^3k \, [\omega_k (a^\dagger_{\vec{k}} a_{\vec{k}} + b^\dagger_{\vec{k}} b_{\vec{k}}) + \nu_k (a_{\vec{k}} b_{-\vec{k}} + b^\dagger_{-\vec{k}} a^\dagger_{\vec{k}})] \quad . \tag{III.1}$$

The commutation relations for $a_{\vec{k}}$ and $b_{\vec{k}}$ are:

$$[a_{\vec{k}}, a^{\dagger}_{\vec{\ell}}] = [b_{\vec{k}}, b^{\dagger}_{\vec{\ell}}] = \delta(\vec{k}-\vec{\ell}) \qquad \text{(III.2)}$$

and all other commutators zero.

We denote by $H_F(a,b)$ the Fock space cyclically built by operations of $a^{\dagger}_{\vec{k}}$ and $b^{\dagger}_{\vec{k}}$ on the vacuum $|0\rangle\rangle$. This is defined by

$$a_{\vec{k}}|0\rangle\rangle = b_{\vec{k}}|0\rangle\rangle = 0 \qquad \text{(III.3)}$$

$H_F(a,b)$ is an irreducible representation of (III.2).

Let us recall that $a_{\vec{k}}$, $a^{\dagger}_{\vec{k}}$, $b_{\vec{k}}$ and $b^{\dagger}_{\vec{k}}$ do not map normalizable vectors into normalizable ones (cf. Sec. 1.2); thus, we need to introduce

$$\begin{aligned} a_i &= \int \frac{d^3k}{(2\pi)^{3/2}} f_i(\vec{k}) a_{\vec{k}} \\ b_i &= \int \frac{d^3k}{(2\pi)^{3/2}} f_i(\vec{k}) b_{\vec{k}} \end{aligned} \qquad \text{(III.4)}$$

with square-integrable functions $f_i(\vec{k})$.

We want to show that there exist infinitely many unitarily inequivalent representations of (III.2).

Let us consider the transformation (called the Bogoliubov transformation)

$$a_{\vec{k}} = \alpha_{\vec{k}} \cosh \theta_k + \beta^{\dagger}_{-\vec{k}} \sinh \theta_k$$

$$b_{\vec{k}} = \beta_k \cosh \theta_k + \alpha^{\dagger}_{-\vec{k}} \sinh \theta_k \tag{III.5}$$

with

$$\cosh 2\theta_k = \frac{\omega_k}{\sqrt{\omega_k^2 - \nu_k^2}}$$

$$\sinh 2\theta_k = -\frac{\nu_k}{\sqrt{\omega_k^2 - \nu_k^2}} \tag{III.6}$$

It is easy to see that by using (III.5), (III.1) becomes

$$H = \int d^3k \; E_k(\alpha^{\dagger}_{\vec{k}}\alpha_{\vec{k}} + \beta^{\dagger}_{\vec{k}}\beta_{\vec{k}}) + W_o \tag{III.7}$$

with

$$E_k = \sqrt{\omega_k^2 - \nu_k^2} \tag{III.8}$$

and

$$W_o = \int d^3k \left(\sqrt{\omega_k^2 - \nu_k^2} - \omega_k\right) .$$

By requiring that

$$[\alpha_{\vec{k}}, \alpha^{\dagger}_{\vec{\ell}}] = [\beta_{\vec{k}}, \beta^{\dagger}_{\vec{\ell}}] = \delta(\vec{k}-\vec{\ell}) \tag{III.9}$$

and all other commutators of α and β are zero, we see that

(III.5) preserve the commutations (III.2).

We introduce the operators

$$\alpha_i = \int \frac{d^3k}{(2\pi)^{3/2}} f_i(\vec{k})\alpha_{\vec{k}}$$
$$\beta_i = \int \frac{d^3k}{(2\pi)^{3/2}} f_i(\vec{k})\beta_{\vec{k}} \tag{III.10}$$

By defining the (α,β)-vacuum $|0>$ as

$$\alpha_{\vec{k}}|0> = \beta_{\vec{k}}|0> = 0 \ , \tag{III.11}$$

we can construct the Fock space $H_F(\alpha,\beta)$ by cyclical operations of $\alpha_i^\dagger$ and $\beta_i^\dagger$ on $|0>$. Since (III. 5) preserve the canonical commutators (III.2), $H_F(\alpha,\beta)$ is also an irreducible representation of (III.2).

Let us now assume that there exists a unitary operator $G(\theta)$ which generates the transformations (III.5):

$$a_{\vec{k}} = G(\theta)\alpha_{\vec{k}}G^{-1}(\theta)$$
$$b_{\vec{k}} = G(\theta)\beta_{\vec{k}}G^{-1}(\theta) \ . \tag{III.12}$$

In this hypothesis, one can prove that the $|0>$ is related to $|0>>$ by

$$|0> = \exp\left[-\delta(\vec{0})\int d^3k \log\cosh\theta_k\right]$$
$$\times \quad \exp\left[\int d^3k \tanh\theta_k a^\dagger_{\vec{k}}\, b^\dagger_{-\vec{k}}\right]|0>> \ . \tag{III.13}$$

Since $\delta(\vec{0}) \equiv \delta(\vec{k}-\vec{\ell})|_{\vec{k}=\vec{\ell}} = \infty$, (III.13) means that $|0\rangle$ cannot be expressed in terms of states of $H_F(a,b)$, unless $\theta_k = 0$ for all values of k. This means that any state of $H_F(\alpha,\beta)$ cannot be expressed in terms of states of $H_F(a,b)$: the two spaces $H_F(a,b)$ and $H_F(\alpha,\beta)$ are orthogonal to each other. <u>We conclude thus that the transformation (III.5) cannot be implemented by a unitary operator $G(\theta)$</u>, i.e. the two irreducible representations of the canonical commutation relations (III.2) (or (III.9)), $H_F(\alpha,\beta)$ and $H_F(a,b)$, are unitarily inequivalent to each other. We also note that $\alpha_{\vec{k}}$ and $\beta_{\vec{k}}$ are characterized by the parameter θ_k defined in (III.6). By different choice of this parameter, provided that the transformations (III.5) preserve the canonical commutators (III.2), we can construct a different inequivalent representation of (III.2). Thus, there are infinitely many such representations. Note that the result (III.13) is independent of the choice of the parameter θ_k except for the case $\theta_k = 0$. It is also interesting to note that the inequivalence of the representations is expressed in (III.13) through the exponential in $\delta(\vec{0})$; since

$$\delta(\vec{0}) = \frac{1}{(2\pi)^3}\int d^3x = \frac{\text{Volume of the system}}{(2\pi)^3} ,$$

we see that the orthogonality of $H_F(a,b)$ and $H_F(\alpha,\beta)$ comes from the fact that the volume is infinite[16] (i.e. there is

[16]Note that in many body problems it is not unrealistic to

an infinite number of degrees of freedom). Let us suppose that our system has a finite volume: then in (III.13) the factor $\exp[-\delta(\vec{0})]$ will not appear and furthermore the integrations on the momentum $\vec{k}$ should be replaced by summations on a countable set of discrete momenta, say $\vec{k}_i$. We then see that when the volume becomes infinite, the generator $G(\theta)$ transforms vectors of the separable space $H_F(a,b)$ into vectors with infinite number of quanta a and b which belong to the non-separable Hilbert space H (cf. Sec. 1.2). The space $H_F(\alpha,\beta)$ is also a separable subspace of H. The orthogonality between $H_F(a,b)$ and $H_F(\alpha,\beta)$ is the orthogonality between two different separable subspaces of the non-separable Hilbert space H. We can think of $|0\rangle$ as a state where a and b bosons are condensed. Since $|0\rangle$ is translational invariant we can obtain a non-vanishing density of bosons only if infinitely many of them are condensed. Thus we conclude that a locally observable boson condensation in a translational invariant system can be achieved only by going to a space ($H_F(\alpha,\beta)$) unitary inequivalent to the original space ($H_F(a,b)$) where no bosons are condensed. Note that, due to the orthogonality between $H_F(\alpha,\beta)$ and $H_F(a,b)$, α and β are not annihilation operators in $H_F(a,b)$ and a and b are not annihilation operators in $H_F(\alpha,\beta)$.

16(cont'd) talk of "infinite" volume: at boundary surfaces the potential is not infinite. Thus wave-packets can spread outside and a continuous distribution of momentum is allowed.

Let us observe that the hamiltonian (III.7) has the form of the free particle hamiltonian (I.41) plus the c-number W_o. $H_F(\alpha,\beta)$ is then the Fock space for physical particles introduced in Sec. 1.2. Let us assume that in choosing the parameter θ_k in (III.5), we use (III.7) as a condition to be satisfied. Then we can look at the set of operators $\{\alpha_{\vec{k}},\beta_{\vec{k}}\}$ as the set of physical operators, and we can read the transformation (III.5) as (simple) dynamical maps. All the procedure to construct $H_F(\alpha,\beta)$ is then equivalent to the one followed in the self-consistent method introduced in Sec. 1.4. We see in this way how the self-consistent method is actually a procedure to select a Fock space for physical particles ($H_F(\alpha,\beta)$) among infinitely many unitarily inequivalent spaces (corresponding to different θ_k). The selection is made by the requirement that the free hamiltonian must take the form (III.7); this requirement determines an appropriate θ_k, i.e. it determines the coefficients of the maps (cf. (III.5)). Since (III.5) are not unitary transformations we see that in general the dynamical map is not unitary.[17]

Recall that in the self-consistent method the canonical commutators (III.2) are not required a priori (cf. Chap. I). In general, the condition (III.7) can be satisfied by many

---

[17] The unitarily inequivalence of $H_F(a,b)$ and $H_F(\alpha,\beta)$ is known as Haag's theorem [5,58] in a relativistic Q.F.T. However the same situation is present in non relativistic theories, too [3,10,25,30,32,59].

θ_k, determined by different boundary conditions. The Fock space $H_F(\theta_k)$ ($\equiv H_F(\alpha,\beta)$) corresponding to each θ_k is then a realization of a different physical situation: thus, we conclude that unitary inequivalence means physical inequivalence. While in Quantum Mechanics different choices of Hilbert spaces give the same physical results (due to the Von Neumann's theorem) in Quantum Field Theory different (inequivalent) representations of the canonical commutation relations give different physical situations. One example of this is given by our study of spontaneous breakdown of symmetry in Chap. II: different boundary conditions ($c = 0$ in the symmetric case, $c \neq 0$ in the asymmetric one, (cf. eq. (II.17)) lead to different (unitarily inequivalent) Fock spaces for physical particles. That is, we have a space with the vacuum invariant ($c = 0$ case) and a space with the vacuum not invariant ($c \neq 0$ case) under the original symmetry transformation.

H. Umezawa, Y. Takahashi and S. Kamefuchi [29] have also investigated the possibility of describing mass spectra by relating different mass values to different inequivalent representations.

Another example in which one could "use" the inequivalent representations is the problem of unstable states in Q.F.T. Spontaneously decaying particles are uncomfortable objects in Q.F.T. since one does not know how to treat in a rigorous way unstable states[18] [60]. Let us observe, however, that any unstable state of mean life λ, is stable if "observed" in a time interval $\Delta t < \lambda$. Suppose that one could parametrize different inequivalent Fock spaces by intervals Δt. Then, given $\Delta t < \lambda$, it should be possible to select a Fock space $H_F(\Delta t)$ in such a way that states of mean life λ are described as stable states in $H_F(\Delta t)$. Physical results will be, in general, functions of the parameter Δt.

The occurrence of infinitely many inequivalent representations has been also useful in the formulation of a Q.F.T. at finite temperature [61]. This is the subject of the following section.

3.2 Quantum Field Theory for finite temperature.

Quantum Field Theory methods and techniques have been widely used in many body problems. One of the principle reasons for this is the fact that states of a system with a large number of interacting particles can be described with relative simplicity as vectors of the Fock space. On the other hand, as Matsubara observed [62], the statistical average $<A>$ of a certain quantity A has properties similar to the vacuum expectation value of A. A method of computing the partition function Z by using the Feynmann diagram

[18] Note that a one-particle state is "stable" (cf. eqs. (I.82-83)).

technique, and a Green's function method for Statistical Mechanics are based on this observation [63]. However, it should be noted that an essential feature of Q.F.T., namely the canonical transformation technique, cannot be incorporated in the Green's function formalism. Indeed by a canonical transformation one is able to change the representation of the canonical variables, i.e. to go from one Hilbert space to another one, and this is not possible in a theory formulated in terms of Green's functions which presume a specific choice of the Hilbert space. Furthermore, a canonical transformation U, which commutes with the hamiltonian H, has no effect when trace operations are involved, e.g.

$$\mathrm{Tr}[A\, e^{-\beta H}] = \mathrm{Tr}[U^{-1} A U\, e^{-\beta H}] \ .$$

On the basis of the considerations in the previous section, we will show that the above difficulties can be bypassed [61] in a Quantum Field Theory for finite temperature.

We start by observing that the field equations for Heisenberg fields are temperature independent. Thus any temperature dependence can be introduced only through temperature dependent boundary conditions. On the other hand, we want to construct a field theory which should give us Statistical Mechanics results. Then, as a boundary condition we require that vacuum expectation values of observables lead to the grand ensemble averages. Denoting by A any observable, our requirement is

$$<0|A|0> = <A> \tag{III.14}$$

with

$$<A> = Z^{-1}(\beta)\mathrm{Tr}[A\ e^{-\beta H}] \ , \qquad \beta = \frac{1}{k_B T} \tag{III.15}$$

where Z is the partition function

$$Z(\beta) = \mathrm{Tr}[e^{-\beta H}] \ , \tag{III.16}$$

k_B is the Boltzmann constant and the hamiltonian H includes the chemical potential. The condition (III.14) suggests to us that the vacuum $|0>$ must be temperature dependent. Thus, our problem is to parametrize the unitarily inequivalent Fock spaces by using the temperature as parameter. Then the boundary condition (III.14) will select the appropriate space. Let us denote the temperature dependent vacuum as $|0(\beta)>$ and write (III.14) as

$$<0(\beta)|A|0(\beta)> = Z^{-1}(\beta) \sum_n <n|A|n>\ e^{-\beta E_n} \tag{III.17}$$

with

$$H|n> = E_n|n> \tag{III.18}$$

$$\langle n|m\rangle = \delta_{nm} \tag{III.19}$$

The state $|n\rangle$ is a temperature independent state belonging to a Fock space H_F^o. It can be shown [61] that a convenient way to construct the state $|0(\beta)\rangle$, such that (III.17) is satisfied, is to introduce a "fictitious" dynamical system identical to the one we want to study. Quantities associated with such a fictitious system are denoted by a tilde. The fictitious system is characterized by

$$\tilde{H}|\tilde{n}\rangle = E_n|\tilde{n}\rangle \ , \tag{III.20}$$

$$\langle\tilde{n}|\tilde{m}\rangle = \delta_{nm} \tag{III.21}$$

where E_n in (III.20) and (III.18) are the same by definition.

We denote by $|n,\tilde{m}\rangle$ a state of the space spanned by the direct product of $|n\rangle$ and $|\tilde{m}\rangle$. We have

$$\begin{aligned} \langle\tilde{m},n|A|n',\tilde{m}'\rangle &= \langle n|A|n'\rangle\delta_{\tilde{m}\tilde{m}'} \\ \langle\tilde{m},n|A|n',\tilde{m}'\rangle &= \langle\tilde{m}|\tilde{A}|\tilde{m}'\rangle\delta_{nn'} \end{aligned} \tag{III.22}$$

The state $|0(\beta)\rangle$ is then constructed as

$$|0(\beta)\rangle = Z^{-\frac{1}{2}}(\beta) \sum_n e^{-\beta E_n/2} |n,\tilde{n}\rangle \ . \tag{III.23}$$

By using (III.22) and (III.23) we easily can verify that the condition (III.17) is fulfilled. We note that in

(III.23) the states $|n\rangle$ and $|\tilde{n}\rangle$ appear always in pairs and that the function of the states $|\tilde{n}\rangle$ is to pick up the diagonal part of A (cf. eq. (III.29)). To better clarify the previous construction of $|0(\beta)\rangle$ let us consider as an example an ensemble of free bosons with frequency ω. The extension to the fermion case is easy [61]. The hamiltonian is

$$H = \omega a^{\dagger} a$$

with

$$[a,a^{\dagger}] = 1 \ , \quad [a,a] = 0 \ .$$

The Fock space H_F^0 has vacuum $|0\rangle$ and state of n $(n = 0,1,2...)$ bosons $1/\sqrt{n!}\ (a^{\dagger})^n|0\rangle$. The eigenvalues of H are $n\omega$. The fictitious system is also considered:

$$\tilde{H} = \omega\tilde{a}^{\dagger}\tilde{a} \ ; \quad [\tilde{a},\tilde{a}^{\dagger}] = 1 \ , \quad [\tilde{a},\tilde{a}] = 0$$

and we assume that

$$[a,\tilde{a}] = [a,\tilde{a}^{\dagger}] = 0 \ .$$

Eq. (III.23) is now

$$|0(\beta)\rangle = Z^{-\frac{1}{2}}(\beta)\sum_n e^{-\beta n\omega/2}\,\frac{1}{n!}\,(a^{\dagger})^n(\tilde{a}^{\dagger})^n|0,\tilde{0}\rangle \ . \quad \text{(III.24)}$$

By introducing

$$u(\beta) = (1-e^{-\beta\omega})^{-\frac{1}{2}} \equiv \sqrt{1+f_B(\omega)}$$

$$v(\beta) = (e^{\beta\omega}-1)^{-\frac{1}{2}} \equiv \sqrt{f_B(\omega)} \tag{III.25}$$

$$G_B = -i\theta(\beta)(\tilde{a}a - a^{\dagger}\tilde{a}^{\dagger}) \tag{III.26}$$

with $\theta(\beta)$ given by

$$\cosh\,\theta(\beta) = u(\beta)\ , \tag{III.27}$$

(III.24) is rewritten as

$$|0(\beta)\rangle = \sqrt{1-e^{-\beta\omega}}\ \exp(e^{-\beta\omega/2}\ a^{\dagger}\tilde{a}^{\dagger})|0,\tilde{0}\rangle$$

$$= u^{-1}(\beta)\ \exp\left(\frac{v(\beta)}{u(\beta)}\ a^{\dagger}\tilde{a}^{\dagger}\right)|0,\tilde{0}\rangle \tag{III.28}$$

Temperature dependent operators can be defined as

$$a(\beta) = e^{-iG_B}\ a\ e^{iG_B} = u(\beta)a - v(\beta)\tilde{a}^{\dagger}$$

$$\tilde{a}(\beta) = e^{-iG_B}\ \tilde{a}\ e^{iG_B} = u(\beta)\tilde{a} - v(\beta)\tilde{a}^{\dagger} \tag{III.29}$$

Eq. (III.29) and (III.28) then show that

$$a(\beta)|0(\beta)\rangle = \tilde{a}(\beta)|0(\beta)\rangle = 0\ , \tag{III.30}$$

and we see that $|0(\beta)\rangle$ acts as a vacuum for $a(\beta)$ and $\tilde{a}(\beta)$.

Consequently, the temperature dependent Fock space $H_F(\beta)$ can be cyclically built from $|0(\beta)\rangle$; a generic state is $(1/\sqrt{n!m!})\ (a^\dagger(\beta))^n(\tilde{a}^\dagger(\beta))^m|0(\beta)\rangle$. We note that $|0(\beta)\rangle$ is related to $|0,\tilde{0}\rangle$ by a Bogoliubov transformation, as shown in (III.28). Then, by modifying the parameter temperature we can go from one representation of the canonical commutation relations to another inequivalent representation: different (temperature dependent) boundary conditions are thus incorporated in our formalism. Note also that states of $H_F(\beta)$ are eigenstates neither of H nor of $\tilde{H}$; however it can be shown that $\hat{H} \equiv H - \tilde{H}$ is diagonal, which tells us that the average energies $\langle H\rangle$ and $\langle\tilde{H}\rangle$ are equal. The boson average number is given by

$$\langle 0(\beta)|a^\dagger a|0(\beta)\rangle = v^2(\beta) = f_B(\omega) \qquad \text{(III.31)}$$

which reproduces the well-known Bose-Einstein distribution. On the other hand, it can be seen that for the one-particle state we have

$$a^\dagger(\beta)|0(\beta)\rangle = \frac{1}{\sqrt{f_B(\omega)}}\,\tilde{a}|0(\beta)\rangle = \frac{1}{\sqrt{1+f_B(\omega)}}\,a^\dagger|0(\beta)\rangle. \qquad \text{(III.32)}$$

Furthermore (III.28) shows that in the vacuum state $|0(\beta)\rangle$ the number of particles with and without the tilde is the same. From (III.32) we see that a one-particle state is built up from the thermal equilibrium state $|0(\beta)\rangle$ by adding one particle or by eliminating one particle with tilde. Thus we can conclude that the particle with tilde is a hole of the physical particle; in the equilibrium state $|0(\beta)\rangle$ there are equal numbers of particles and holes; one quantum excited state in $|0(\beta)\rangle$ can be created in two equivalent ways: by adding a particle or by eliminating a hole. It is very interesting that this excitation process is temperature dependent (cf. eq. (III.32)).

The previous considerations, which are analyzed in detail in ref. [61], give us a physical interpretation of the formalism presented. At T = 0, our physics can be described by means of the space H_F^o. Since there cannot be thermal excitations, we do not need to consider states for the "holes" (the tilde-states). As the temperature increases excitation or condensation processes can be induced. This introduces new degrees of freedom in our system. These new freedoms are expressed by the introduction of the tilde-Fock spaces, which are unitarily inequivalent to each other and to the original space H_F^o. Indeed, when it is considered that particles and holes carry momentum, (III.28) should read as

$$|0(\beta)\rangle = \pi_{\vec{k}} \frac{1}{\cosh\theta_k} \exp[\tanh \theta_k \, a^\dagger_{\vec{k}} \, \tilde{a}^\dagger_{-\vec{k}}]\,|0,\tilde{0}\rangle \ . \qquad \text{(III.32)}$$

The dependence of θ_k on k is due to the dependence of ω on k (cf. eq. (III.25)). Then one has

$$<0,\tilde{0}|0(\beta)> = \prod_{\vec{k}} \frac{1}{\cosh\theta_k} = \exp[-\sum_{\vec{k}} \log \cosh\theta_k] . \quad \text{(III.33)}$$

As the volume $V \to \infty$, we can use

$$\sum_{\vec{k}} \to \frac{V}{(2\pi)^3} \int d\vec{k} .$$

Then (III.33) gives for $V \to \infty$

$$<0,\tilde{0}|0(\beta)> = 0 . \quad \text{(III.34)}$$

The value of the parameter θ_k can be chosen in such a way that the vacuum expectation values of the hamiltonian H and of the number operator $N \equiv \sum_{\vec{k}} a^{\dagger}_{\vec{k}} a_{\vec{k}}$ are constant.

One can also introduce the "thermodynamical potential"

$$\Omega = <0(\beta)| \, [-\frac{1}{\beta} K + H] |0(\beta)> \quad \text{(III.35)}$$

where K is the "entropy" (divided by the Boltzmann constant k_B):

$$K = -\sum_{\vec{k}} [a^{\dagger}_{\vec{k}} a_{\vec{k}} \log \sinh^2 \theta_k - a_{\vec{k}} a^{\dagger}_{\vec{k}} \log \cosh^2 \theta_k] .$$

By imposing that

$$\frac{\partial\Omega}{\partial\theta_k} = 0 , \quad \text{(III.36)}$$

one finds

$$\sinh^2\theta_k = f_B(\omega_k) \ , \tag{III.37}$$

which is a condition on θ_k.

By means of the present formalism one can also give a description of quasiparticles at finite temperature with spectra depending on temperature. The bound-state problem can be treated by using the Bethe-Salpeter equation and in superconductivity at finite temperature it has been shown [61,32] that the Goldstone boson is a bound state of two electrons with spin up and down. A detailed account of the formulation of superconductivity at non zero temperature is given in ref. [32]. A microscopic computation of the boson characteristic function at non-zero temperature [64] gives reasonable agreement with experimental data for T not too close to T_c.

Let us consider the problem of first-order phase transition at H_{c1}.

Recently, Auer and Ullmaier [65] have experimentally investigated the problem of phase transition at the lower critical field H_{c1} in type-II superconductors. By measuring the magnetization curves for many samples with different values of κ, the Ginzburg-Landau parameter, they have been able to determine the temperature dependence of the critical value of κ. This quantity, which is denoted by κ_c, separates type-II/1 superconductors, which exhibit a first-order phase transition at H_{c1}, from type-II/2 superconductors which exhibit a second-order phase transition.

In some recent articles [66-69] it has been shown that the problem of a first-order phase transition at H_{c1} can be theoretically investigated by means of the boson formulation of superconductivity [32]. The magnetization curves have been theoretically computed for many values of κ; these computations have shown that for values of κ sufficiently small a minimum of H in the B-H curves appears, implying that the transition is first order. The limit value of κ for which the minimum disappears gives the critical value κ_c [66,68]. The interaction energy between two flux lines has been computed [67,69]; it has been shown that there is a critical value of κ such that for $\kappa > \kappa_c$ the interaction is always repulsive, while for $\kappa < \kappa_c$ the interaction is attractive for a certain range of values of the distance between the flux lines. In all these articles the analysis was restricted to the case of zero temperature. For temperature different from zero it is found that the magnetic properties of type-II superconductors around H_{c1} are very well described by the following form of the boson characteristic function [32,66,67]:

$$c(k) = [1+\alpha(T)\xi_o^2(T)k^2 + \delta(T)\xi_o^4(T)k^4]^{-1} \qquad \text{(III.38)}$$

where $\alpha(T)$ is a function of temperature which is fixed by the microscopic formulation, and $\delta(T)$ is a phenomenological parameter. ξ_o in Eq. (III.38) is the BCS temperature-dependent coherence length: $\xi_o(T) = v_F/\pi\Delta(T)$. Since $c(k)$ is positive definite, so are $\alpha(T)$ and $\delta(T)$ [32].

In order to compute $\alpha(T)$ we must calculate the Bethe-Salpeter amplitude for the boson state; according to the results of the general formulation, $\alpha(T)$ is given by the following expression:

$$\alpha(T) = -\frac{\pi^2\Delta^2}{8\, v_F^4}\left[\frac{\partial^4}{\partial \ell^4}\frac{\overline{R}(\ell)}{R(\ell)}\right]_{\ell=0}, \qquad \text{(III.39)}$$

where

$$R(\ell) = \frac{\lambda}{4}\int\frac{d^3k}{(2\pi)^3}\frac{1}{E_+E_-}\left[\frac{1-f_+-f_-}{E_++E_-}+\frac{f_+-f_-}{E_+-E_-}\right], \qquad \text{(III.40)}$$

$$\overline{R}(\ell) = \frac{\lambda}{4}\int\frac{d^3k}{(2\pi)^3}\frac{(\varepsilon_+-\varepsilon_-)^2}{E_+E_-}\left[\frac{1-f_+-f_-}{E_++E_-}+\frac{f_+-f_-}{E_+-E_-}\right]. \qquad \text{(III.41)}$$

In eqs. (III.40) and (III.41) λ is the coupling constant of the BCS hamiltonian, and the symbols $f_\pm$, $E_\pm$, $\varepsilon_\pm$ stand for

$$f_\pm = f_{k\pm\frac{1}{2}\ell}, \quad E_\pm = E_{k\pm\frac{1}{2}\ell}, \quad \varepsilon_\pm = \varepsilon_{k\pm\frac{1}{2}\ell}.$$

f_k is the Fermi distribution function

$$f_k = \left[1 + e^{\beta E_k}\right]^{-1},$$

with $\beta = 1/k_BT$, k_B being the Boltzmann constant; E_k is the energy spectrum of quasifermions which has the well-known form

$$E_k = \sqrt{\omega_k^2 + \Delta^2(T)} \ ,$$

with

$$\varepsilon_k = \frac{1}{2m} (k^2 - k_F^2) \ .$$

It should be noted that in the expression (III.39) it is neglected the fact that at nonzero temperature the collective modes have a finite life time; this effect becomes more important when T approaches the critical temperature T_c.

Computations of the expression (III.39) show that $\alpha(T)$ has the following form:

$$\alpha(T) = \alpha(0)[1-\gamma(T)] \ , \tag{III.42}$$

where $\alpha(0)$ is the value of α at $T = 0^oK$ [70]:

$$\alpha(0) = \frac{2\pi^2}{45} \tag{III.43}$$

and

$$\gamma(T) = \left[\frac{\Delta(0)}{2k_B T_c}\right]^2 \left[\frac{\kappa_B(T)}{\kappa_B(0)}\right]^2 \frac{1}{t^2} \int_0^\infty \frac{\mathrm{tgh}(\frac{\beta E}{2})}{\cosh^2(\frac{\beta E}{2})} \frac{d\varepsilon}{E} \ . \tag{III.44}$$

In the expression (III.44) $t = T/T_c$ is the reduced temperature; $\kappa_B(T)$ is defined as

$$\kappa_B(T) = \lambda_L(T)/\xi_o(T) \tag{III.45}$$

where $\lambda_L(T)$ is the temperature-dependent London penetration depth [71]:

$$\left[\frac{\lambda_L(0)}{\lambda_L(T)}\right]^2 = 1 + 2 \int_0^\infty d\varepsilon \, \frac{\partial f}{\partial E} \ . \qquad \text{(III.46)}$$

The integrations in (III.44) and (III.46) are calculated numerically.

According to the results of the boson formulation the interaction energy between two flux lines is given by the following expression:

$$E(d) = \frac{\Phi^2}{8\pi^2} \int_0^\infty kF(k)J_o(kd)dk \ , \qquad \text{(III.47)}$$

where Φ is the unit quantum flux, J_o is the Bessel function, d is the distance between the centers of the flux lines and F(k) is expressed in terms of the boson characteristic function as

$$F(k) = \frac{c(k)}{\lambda_L^2 k^2 + c(k)} \ . \qquad \text{(III.48)}$$

Equations (III.47), (III.48) and (III.38) lead to the conclusion that a first order transition at H_{c1} will occur provided that the two following conditions are satisfied [69]:

$$0 \leq \delta(T) \leq \frac{1}{4} \alpha^2(T) \ , \qquad \text{(III.49)}$$

$$\kappa_B(T) < \kappa_{Bc}(T) \ , \tag{III.50}$$

where $\kappa_{Bc}(T)$ is given by:

$$\kappa_{Bc}^2(T) = \frac{\alpha(T)[2\alpha^2(T)-9\delta(T)]+2[\alpha^2(T)-3\delta(T)]^{3/2}}{\alpha^2(T)-4\delta(T)} \ . \tag{III.51}$$

Condition (III.49) comes from the requirement that δ must be positive and that for $\kappa \to \infty$ the transition is second-order. $\kappa_{Bc}(T)$ gives the critical value for κ_B which separates type-II/1 from type-II/2. The agreement with the experimental data is rather good up to $t = 0.6$ but becomes worse for higher temperatures [64]. One of the principal reasons may lie in the fact that we have neglected the finite life time of collective mode; another reason may be that the form (III.38) for the boson characteristic function is not quite accurate for temperatures close to T_c.

It will be interesting to use the formalism of the present section to extend the results for ferromagnetic systems (Sec. 2.5 and Appendix) to finite temperature. In particular, we hope to compute a value for the magnetization which fits experimental data in a better way than the presently available theoretical results. On the other hand, since in our formulation the boson excitations are automatically taken into account, it should be possible to recover the $T^{3/2}$ dependence of the magnetization also for the itinerant electron model of ferromagnetism. While there are reasonable theoretical arguments for the $T^{3/2}$ law in the case of the Heisenberg ferromagnet, there is no proof for the itinerant electron case. In this respect, it is interesting to note the presence of the $B^\dagger B$ term in the spin density $S^{(3)}$ (cf. eq. (II.158)). Indeed, the vacuum expectation value of such a term at $T \neq 0$, for low temperature, immediately gives

$$\langle B^\dagger(x)B(x)\rangle_{T\neq 0} \propto T^{3/2} .$$

(Recall that the $T^{3/2}$ term in the case of the Heisenberg ferromagnet comes from the a^*a term in $S^{(3)}$ (cf. eq. (II.181)) and see e.g. ref.[43]). Another theoretical improvement using the above formalism would be to extend the Landau phase approximation and to construct a theory which excludes the existence of one and two dimensional ferromagnets.

APPENDIX:
SELF-CONSISTENT FORMULATION OF ITINERANT ELECTRON FERROMAGNETISM

In Sec. 2.5 we have studied the dynamical rearrangement of symmetry in ferromagnetism by means of the path-integral method. Our arguments were completely general and both the cases of localized spin (Heisenberg model) and of continuous spin distribution (the itinerant electron model) were considered.

In this Appendix we study the itinerant electron case in the framework of the self-consistent method by considering a practical model. The magnon, as a bound state of electrons, will be treated by the Bethe-Salpeter equations. In our computation we will consider the pair approximation, i.e. we will consider only those processes which conserve the number of fermion pairs.

We consider the hamiltonian [72]

$$H = \hbar\int d^3x\,[\psi_\uparrow^\dagger\varepsilon(\partial)\psi_\uparrow + \psi_\downarrow^\dagger\varepsilon(\partial)\psi_\downarrow + \lambda\psi_\uparrow^\dagger\psi_\downarrow^\dagger\psi_\downarrow\psi_\uparrow - \mu(\psi_\uparrow^\dagger\psi_\uparrow + \psi_\downarrow^\dagger\psi_\downarrow)] \tag{A.1}$$

where $\psi_{\uparrow,\downarrow}$ are the Heisenberg fields for the electrons:

$$\psi(x) = \begin{pmatrix}\psi_\uparrow(x)\\ \psi_\downarrow(x)\end{pmatrix} \tag{A.2}$$

and

$$\varepsilon(\partial) = -\frac{1}{2m}(\vec{\nabla}^2 + k_F^2) \tag{A.3}$$

with $\vec{k}_F$ the Fermi momentum and m the electron mass. The last term in (A.1) is introduced to eliminate the self-energy of the electron.

Here we assume the equal time anticommutation relations for the Heisenberg electron field $\psi(x)$:

$$\begin{aligned}[\psi_s(x), \psi^\dagger_{s'}(y)]_+ \delta(t_x - t_y) &\equiv (\psi_s(x)\psi_{s'}(y) + \\ &+ \psi_{s'}(y)\psi_s(x))\delta(t_x - t_y) = \delta(x-y)\delta_{ss'}\end{aligned} \tag{A.4}$$

s and s' stay for ↑ or ↓. We recall that in the self-consistent method one should not assume the canonical anticommutators (A.4), but find by computation what are the anticommutators for the Heisenberg electron fields. In this sense our computation is not completely self-consistent. Use of (A.4) leads us to the field equation

$$(\varepsilon(\partial) + \frac{1}{i}\frac{\partial}{\partial t} - \tilde{M}\sigma_3)\psi(x) = -\ s - \tilde{M}\sigma_3\psi(x) + \mu\psi(x) \ . \tag{A.5}$$

Here σ_3 is the 3rd Pauli matrix, s is given by

$$s \equiv \begin{pmatrix} s_\uparrow \\ s_\downarrow \end{pmatrix} \equiv \lambda \begin{pmatrix} \psi^\dagger_\downarrow \psi_\downarrow \psi_\uparrow \\ \psi^\dagger_\uparrow \psi_\uparrow \psi_\downarrow \end{pmatrix} , \tag{A.6}$$

and the term $-\tilde{M}\sigma_3$ added to both sides of (A.5) acts as energy counterterm. Its presence is justified by the fundamental requirement that the total hamiltonian H must be equal, up to a c-number constant, to the free hamiltonian (cf. Chap. I). In other words, the interaction should not create any self-energy for the quasielectron. This is equivalent to require that the quasielectron field

$$\phi(x) = \begin{pmatrix} \phi_{\uparrow}(x) \\ \\ \phi_{\downarrow}(x) \end{pmatrix} \tag{A.7}$$

satisfies the free field equation

$$\left(\frac{1}{i}\frac{\partial}{\partial t} + E(\vec{\nabla}^2)\right)\phi(x) = 0 \tag{A.8}$$

with

$$E(\vec{\nabla}^2) \equiv (\varepsilon(\partial) - \tilde{M}\sigma_3) \tag{A.9}$$

The anticommutation relations for $\phi(x)$ are

$$[\phi(x),\phi^{\dagger}(y)]_{+}\delta(t_x - t_y) = \delta(x-y)I \tag{A.10}$$

where I is the unit matrix.

Let us note that the quasielectron energy (cf. (A.9) and (A.3))

$$E_{k\uparrow,\downarrow} = (\varepsilon_k \mp \tilde{M}) \ , \tag{A.11}$$

with

$$\varepsilon_k = \frac{1}{2m}\,(k^2 - k_F^2) \ , \tag{A.12}$$

can be positive or negative. We then associate with the positive energy the annihilation operator $\alpha_{\vec{k}\uparrow,\downarrow}$ and with the negative energy the creation operator $\beta^\dagger_{-\vec{k}\downarrow,\uparrow}$ with anti-commutators

$$[\alpha_{\vec{k}s}, \alpha^\dagger_{\vec{\ell}s'}]_+ = [\beta_{\vec{k}s}, \beta^\dagger_{\vec{\ell}s'}]_+ = \delta(\vec{k}-\vec{\ell})\delta_{ss'} \ ; \tag{A.13}$$

all other anticommutators are zero.

The quasielectron field $\phi(x)$ is then written as

$$\phi_{\uparrow,\downarrow}(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \,[\alpha_{\vec{k}\uparrow,\downarrow}\ \theta(E_{k\uparrow,\downarrow}) + \beta^\dagger_{-\vec{k}\downarrow,\uparrow}\ \theta(-E_{k\uparrow,\downarrow})]\ \ e^{i\vec{k}\cdot\vec{x}-iE_{k\uparrow,\downarrow}t} \tag{A.14}$$

We now introduce the 2×2 matrix

$$S(x-y) \equiv \langle 0|T[\phi(x),\phi^\dagger(y)]|0\rangle \ , \tag{A.15}$$

where T[...] means chronological product of the fields ϕ. $S(x-y)$ is the Green's function of (A.8):

$$i(\frac{1}{i}\,\frac{\partial}{\partial t_x} + E(\vec{\nabla}^2_x))S(x-y) = \delta(x-y)\ I \ . \tag{A.16}$$

We have

$$S(x-y) = \begin{bmatrix} S_{\uparrow}(x-y) & 0 \\ 0 & S_{\downarrow}(x-y) \end{bmatrix} \tag{A.17}$$

with

$$S_{11}(x-y) \equiv S_{\uparrow}(x-y) \equiv \langle 0|T[\phi_{\uparrow}(x),\phi_{\uparrow}^{\dagger}(x)]|0\rangle =$$

$$= i\int \frac{d^4k}{(2\pi)^4} \left[\frac{1}{E-E_{k\uparrow}+i\varepsilon}\,\theta(k^2-Q_{\uparrow}^2) + \right. \tag{A.18}$$

$$\left. + \frac{1}{E-E_{k\uparrow}-i\varepsilon}\,\theta(Q_{\uparrow}^2-k^2)\right] e^{i\vec{k}(\vec{x}-\vec{y})-iE(t_x-t_y)}$$

where the limit $\varepsilon \to 0$ is understood and

$$Q_{\uparrow,\downarrow}^2 \equiv k_F^2 \pm 2m\tilde{M} \tag{A.19}$$

(A.18) with $\uparrow$ replaced with $\downarrow$ gives $S_{22}(x-y) \equiv S_{\downarrow}(x-y)$.

In the self-consistent method one has to find the coefficients of the dynamical map in order to find the dynamical solutions. In the present case our problem is to express the Heisenberg electron field in terms of the quasiparticles. We choose as candidates for quasiparticles, the quasielectrons. Since we are mainly interested in quantities bilinear in the Heisenberg electron field we consider the following dynamical map

$$T[\psi(x),\psi^{\dagger}(y)] = \chi(x-y) + \int d^3q\int d^3p\; T^{(1)}(p,q;x,y)\alpha_{\vec{q}\uparrow}^{\dagger}\alpha_{\vec{p}\uparrow}$$

$$+ \int d^3q\int d^3p\; T^{(2)}(p,q;x,y)\alpha_{\vec{q}\uparrow}^{\dagger}\beta_{\vec{p}\downarrow} + \ldots \tag{A.20}$$

where dots denote other normal product terms bilinear in α and β plus higher order normal product terms. In (A.20) we exclude terms which do not conserve the spin of the quasi-electrons. The coefficients of the map, which are c-number 2×2 matrices, are given by

$$\langle 0|T[\psi(x),\psi^{\dagger}(y)]|0\rangle = \chi(x-y) \qquad \text{(A.21)}$$

$$\langle \alpha_{\vec{\ell}\uparrow}|T[\psi(x),\psi^{\dagger}(y)]|\alpha_{\vec{\ell}\uparrow}\rangle = T^{(1)}(\ell,\ell;x,y) \qquad \text{(A.22)}$$

etc.

Our problem is now to calculate the coefficients of the map. We will derive the Bethe-Salpeter (B-S) equations and try to solve them in the pair approximation. To find the B-S equation for $\chi(x-y)$ we consider

$$\Lambda(\vec{\partial}_x)\chi(x-y)\Lambda(-\overleftarrow{\partial}_y) = \Lambda(\vec{\partial}_x)\langle 0|T[\psi(x),\psi^{\dagger}(y)]|0\rangle\Lambda(-\overleftarrow{\partial}_y) \qquad \text{(A.23)}$$

where

$$\Lambda(\vec{\partial}) \equiv -\left[\frac{1}{i}\frac{\partial}{\partial t} + E(\vec{\nabla})\right] \qquad \text{(A.24)}$$

Use of the field equation (A.5) gives [72], in the pair approximation,

$$\chi(x-y) = S(x-y) - i\int d^4\xi\ S(x-\xi)[\tilde{M}\sigma_3 - \mu I + \chi']S(\xi-y) \qquad \text{(A.25)}$$

Here χ' denotes a 2×2 matrix of elements

$$\chi'_{11} = \lambda\langle 0|\psi_{\downarrow}^{\dagger}(x)\psi_{\downarrow}(x)|0\rangle; \qquad \chi'_{12} = -\lambda\langle 0|\psi_{\downarrow}^{\dagger}(x)\psi_{\uparrow}(x)|0\rangle,$$

$$\chi'_{21} = -\lambda\langle 0|\psi_{\uparrow}^{\dagger}(x)\psi_{\downarrow}(x)|0\rangle = \chi'^{\dagger}_{12}; \quad \chi'_{22} = \lambda\langle 0|\psi_{\uparrow}^{\dagger}(x)\psi_{\uparrow}(x)|0\rangle. \tag{A.26}$$

Since the interaction does not create any self-energy for the quasielectron, it must be

$$\langle 0|T[\psi(x),\psi^{\dagger}(y)]|0\rangle = \langle 0|T[\phi(x),\phi^{\dagger}(y)]|0\rangle \tag{A.27}$$

i.e., from (A.21) and (A.15),

$$\chi(x-y) = S(x-y) \ . \tag{A.28}$$

(A.25) then shows that

$$\tilde{M}\sigma_3 - \mu I + \chi' = 0 \tag{A.29}$$

i.e.

$$\tilde{M} = \frac{\lambda}{2}\langle 0|\psi^{\dagger}(x)\sigma_3\psi(x)\ |0\rangle \tag{A.30}$$

$$\mu = \frac{\lambda}{2}\langle 0|\psi^{\dagger}(x)\psi(x)|0\rangle \tag{A.31}$$

(A.30) gives us the magnetization $\tilde{M}$ which must be different from zero for a ferromagnetic system. (A.30) is the boundary condition under which the field equation (A.5) must be solved. (A.30) and (A.27) give

$$\tilde{M} = \frac{\lambda}{2} <0|\phi^{\dagger}(x)\sigma_3\phi(x)|0> \tag{A.32}$$

i.e., by using (A.14) and the definition (A.19),

$$1 = \frac{\lambda}{2\tilde{M}} \int_{Q_{\downarrow}}^{Q_{\uparrow}} \frac{d^3k}{(2\pi)^3} \quad . \tag{A.33}$$

(A.33) is called the gap equation. When (A.30) is satisfied, an energy difference $2\tilde{M}$ appears between spin up and spin down quasielectrons. Let us study now the B-S amplitude

$$G(x,y) = <j|T[\psi(x),\psi^{\dagger}(y)]|i> - <0|T[\psi(x),\psi^{\dagger}(y)]|0>\delta_{ij} \tag{A.34}$$

with the states $|i>$ and $|j>$ satisfying $<i|j> = \delta_{ij}$. We consider then the relation

$$\begin{aligned} \Lambda(\vec{\partial}_x)G(x,y)\Lambda(-\overleftarrow{\partial}_y) = \Lambda(\vec{\partial}_x)\Big\{&<j|T[\psi(x),\psi^{\dagger}(y)]|i> - \\ &- <0|T[\psi(x),\psi^{\dagger}(y)]|0>\delta_{ij}\Big\}\Lambda(-\overleftarrow{\partial}_y) \end{aligned} \tag{A.35}$$

By using the field equation (A.5), (A.35) gives [72]:

$$G(x,y) = G^{o}(x,y) - i\int d^4\xi \; S(x-\xi)F(\xi)S(\xi-y) \tag{A.36}$$

where the 2×2 matrix $F(\xi)$ is

$$F(\xi) = \begin{pmatrix} -\lambda G_{22}(\xi,\xi) & \lambda G_{12}(\xi,\xi) \\ \lambda G_{21}(\xi,\xi) & -\lambda G_{11}(\xi,\xi) \end{pmatrix} \quad , \tag{A.37}$$

and $G^o(x,y)$ is defined as

$$G^o(x,y) = \langle j|T[\phi(x),\phi^\dagger(y)]|i\rangle - \langle 0|T[\phi(x),\phi^\dagger(y)]|0\rangle\delta_{ij} \quad (A.38)$$

This shows that $G^{(o)}(x,y)$ is different from zero only in the following cases (s and s' are $\uparrow$ or $\downarrow$):

$$\begin{array}{lll} & |j\rangle & |i\rangle \\ a) & |0\rangle & |\alpha_{\vec{\ell}-\vec{p},s},\ \beta_{\vec{p},s'}\rangle \\ b) & |\alpha_{-\vec{p},s},\ \beta_{-\vec{\ell}+\vec{p},s'}\rangle & |0\rangle \\ c) & |\alpha_{-\vec{p},s}\rangle & |\alpha_{\vec{\ell}-\vec{p},s'}\rangle \\ d) & |\beta_{-\vec{\ell}+\vec{p},s}\rangle & |\beta_{\vec{p},s'}\rangle \end{array} \quad (A.39)$$

In these cases, (A.38) gives:

$$G^o_{ss'}(x,y) = \frac{1}{(2\pi)^3} G^o_{ss'}(\ell,E)\, e^{i\vec{q}\cdot\vec{x}}\, e^{i\vec{p}\cdot\vec{y}}\, e^{-iE^s_x t_x}\, e^{-i(E-E^s_x)t_y} \quad (A.40)$$

where the notation is $E = E_i - E_j$,

$$\begin{pmatrix} G^o_{\uparrow\uparrow} & G^o_{\uparrow\downarrow} \\ G^o_{\downarrow\uparrow} & G^o_{\downarrow\downarrow} \end{pmatrix} \equiv \begin{pmatrix} G^o_{11} & G^o_{12} \\ G^o_{21} & G^o_{22} \end{pmatrix}, \quad (A.41)$$

and

$$E_x^s = E_{\ell-p,s} > 0;\ \vec{q} = \vec{\ell}-\vec{p};\ E_{p,s'} < 0;\ E = E_{\ell-p,s} + |E_{p,s'}|;$$

in the case (a)

$$E_x^s = E_{-\ell+p,s} < 0;\ \vec{q} = \vec{\ell}-\vec{p};\ E_{-p,s'} > 0;\ E = -|E_{-\ell+p,s}|-E_{p,s'};$$

in the case (b)

$$E_x^s = E_{\ell-p,s} > 0;\ \vec{q} = \vec{\ell}-\vec{p};\ E_{-p,s'} > 0;\ E = E_{\ell-p,s} - E_{-p,s'};$$

in the case (c)

$$E_x^s = E_{-\ell+p,s} < 0;\ \vec{q} = \vec{\ell}-\vec{p};\ E_{p,s'} < 0;\ E = -|E_{-\ell+p,s}|+|E_{p,s'}|;$$

in the case (d)

Writing G(x,x) as

$$G(x,x) = \frac{1}{(2\pi)^3} G(\ell,E)\, e^{i\vec{\ell}\cdot\vec{x}-iEt}\ , \tag{A.42}$$

(A.36) and (A.40) give

$$\begin{aligned}
G_{11}(\ell,E) &= G^o_{11}(\ell,E) + Q^{11}(\ell,E)G_{22}(\ell,E)\\
G_{12}(\ell,E) &= G^o_{12}(\ell,E) - Q^{12}(\ell,E)G_{12}(\ell,E)\\
G_{21}(\ell,E) &= G^o_{21}(\ell,E) - Q^{21}(\ell,E)G_{21}(\ell,E)\\
G_{22}(\ell,E) &= G^o_{22}(\ell,E) + Q^{22}(\ell,E)G_{11}(\ell,E)
\end{aligned} \tag{A.43}$$

where $Q^{ij}(\ell,E)$ is defined by

$$Q^{ij}(x) = i\lambda S^{(i)}(x) S^{(j)}(-x) =$$

$$\int \frac{d^3\ell}{(2\pi)^3} \frac{dE}{(2\pi)} Q^{ij}(\ell,E)\, e^{i\vec{\ell}\cdot\vec{x}-iEt} \qquad (i,j = 1,2) \tag{A.44}$$

with $S^{(1)}(x) = S_{11}(x)$; $S^{(2)}(x) = S_{22}(x)$; (cf. (A.18)). We also introduce the notation $G^{(1)} = G_{12}$ and $G^{(2)} = -G_{21}$. (A.44) and (A.18) gives:

$$Q^{11}(\ell,E) = \lambda \int_{\tilde{Q}^-_\uparrow}^{\tilde{Q}^+_\uparrow} \frac{d^3k}{(2\pi)^3} \frac{1}{E+E_{-\uparrow}-E_{+\uparrow}}$$

$$Q^{12}(\ell,E) = -\lambda \int_{\tilde{Q}^+_\downarrow}^{\tilde{Q}^-_\uparrow} \frac{d^3k}{(2\pi)^3} \frac{1}{E+E_{-\downarrow}-E_{+\uparrow}}$$

$$Q^{21}(\ell,E) = \lambda \int_{\tilde{Q}^-_\downarrow}^{\tilde{Q}^+_\uparrow} \frac{d^3k}{(2\pi)^3} \frac{1}{E+E_{-\uparrow}-E_{+\downarrow}} = Q^{12}(-\ell,-E) \tag{A.45}$$

$$Q^{22}(\ell,E) = \lambda \int_{\tilde{Q}^-_\downarrow}^{\tilde{Q}^+_\downarrow} \frac{d^3k}{(2\pi)^3} \frac{1}{E+E_{-\downarrow}-E_{+\downarrow}} \,.$$

where $E_{\pm\uparrow,\downarrow} \equiv E_{|k\pm\frac{\ell}{2}|\uparrow,\downarrow}$, and

$$\tilde{Q}^{\pm}_{\uparrow,\downarrow} \equiv \tilde{Q}_{\uparrow,\downarrow} \pm \frac{\ell}{2}\cos\theta$$

$$\tilde{Q}^{2}_{\uparrow,\downarrow} \equiv \frac{1}{4}\left(\ell^2\cos^2\theta - \ell^2 + 4Q^2_{\uparrow,\downarrow}\right) \,. \tag{A.46}$$

with $Q_{\uparrow,\downarrow}$ given in (A.19).

Let us now turn our attention to the "spin current"

$$\vec{j}^{(i)}(x) = \frac{1}{2mi}[\psi^\dagger(x)\vec{\nabla}\sigma_i\psi(x) - (\vec{\nabla}\psi^\dagger(x))\sigma_i\psi(x)] =$$

$$= -\frac{1}{2mi}[(\vec{\nabla}_x-\vec{\nabla}_y)[\sigma_i\psi(x)\psi^\dagger(y)]]_{x=y} + c_j^{(i)} \quad i=1,2,3 \tag{A.47}$$

and to the "spin density"

$$\rho^{(i)}(x) = \psi^\dagger(x)\sigma_i\psi(x) = -\mathrm{Tr}[\sigma_i\psi(x)\psi^\dagger(y)]_{x=y} + c_\rho^{(i)}, \quad i=1,2,3 \tag{A.48}$$

where $c_j^{(i)}$ and $c_\rho^{(i)}$ are c-numbers created by commutations among ψ and $\psi^\dagger$. Use of the field equation (A.5) leads to

$$\vec{\nabla}\cdot\vec{j}^{(i)}(x) + \frac{\partial}{\partial t}\rho^{(i)}(x) = 0 , \quad i=1,2,3. \tag{A.49}$$

We have, in any of the cases (A.39),

$$\langle j|\vec{j}^{(i)}(x)|i\rangle = -\frac{1}{2mi}\mathrm{Tr}[(\vec{\nabla}_x-\vec{\nabla}_y)\sigma_i G^0(x,y)]_{x=y} +$$

$$+ \delta\vec{j}^{(i)}(x) \qquad i=1,2,3 \tag{A.50}$$

with

$$\delta\vec{j}^{(i)}(x) = \frac{1}{2m}[(\vec{\nabla}_x-\vec{\nabla}_y)\int d^4\xi\ \mathrm{Tr}[\sigma_i S(x-\xi)F(\xi)S(\xi-y)]]_{x=y} \tag{A.51}$$

where $F(\xi)$ is defined in (A.37) and can be written as

$$F(x) = \frac{1}{(2\pi)^3} F(\ell,E)\, e^{i\vec{\ell}\cdot\vec{x}-iEt} \tag{A.52}$$

Similarly, we have

$$<j|\rho^{(i)}(x)|i> = -Tr[\sigma_i G^o(x,y)]_{x=y} + \delta\rho^{(i)}(x),\ i=1,2,3 \tag{A.53}$$

with

$$\delta\rho^{(i)}(x) = [i\int d^4\xi\ Tr[\sigma_i S(x-\xi)F(\xi)S(\xi-y)]]_{x=y}\ . \tag{A.54}$$

Use of (A.18) and similar equations for $S_{22}(x-y)$ gives

$$\delta\vec{j}^{(1)}(x) = -\frac{1}{(2\pi)^3}\frac{\lambda}{m} e^{i\vec{\ell}\cdot\vec{x}-iEt}\left\{\int_{\tilde{Q}_\downarrow^-}^{\tilde{Q}_\uparrow^+}\frac{d^3k}{(2\pi)^3}\vec{k}\,\frac{G^{(2)}(\ell,E)}{E+E_{-\uparrow}-E_{+\downarrow}} + \right.$$
$$\left. + \int_{\tilde{Q}_\downarrow^+}^{\tilde{Q}_\uparrow^-}\frac{d^3k}{(2\pi)^3}\vec{k}\,\frac{G^{(1)}(\ell,E)}{E+E_{-\downarrow}-E_{+\uparrow}}\right\}$$

$$\delta\vec{j}^{(2)}(x) = -\frac{i}{(2\pi)^2}\frac{\lambda}{m} e^{i\vec{\ell}\cdot\vec{x}-iEt}\left\{\int_{\tilde{Q}_\downarrow^-}^{\tilde{Q}_\uparrow^+}\frac{d^3k}{(2\pi)^3}\vec{k}\,\frac{-G^{(2)}(\ell,E)}{E+E_{-\uparrow}-E_{+\downarrow}} + \right.$$
$$\left. + \int_{\tilde{Q}_\downarrow^+}^{\tilde{Q}_\uparrow^-}\frac{d^3k}{(2\pi)^3}\vec{k}\,\frac{G^{(1)}(\ell,E)}{E+E_{-\downarrow}-E_{+\uparrow}}\right\} \tag{A.55}$$

$$\delta\vec{j}^{(3)}(x) = -\frac{1}{(2\pi)^3}\frac{\lambda}{m} e^{i\vec{\ell}\cdot\vec{x}-iEt}\left\{\int_{\tilde{Q}_\uparrow^-}^{\tilde{Q}_\uparrow^+}\frac{d^3k}{(2\pi)^3}\vec{k}\,\frac{G_{22}(\ell,E)}{E+E_{-\uparrow}-E_{+\uparrow}} - \right.$$
$$\left. - \int_{\tilde{Q}_\downarrow^-}^{\tilde{Q}_\downarrow^+}\frac{d^3k}{(2\pi)^3}\vec{k}\,\frac{G_{11}(\ell,E)}{E+E_{-\downarrow}-E_{+\downarrow}}\right\}$$

In the same way,

$$\delta\rho^{(1)}(x) = -\frac{1}{(2\pi)^3}\,\lambda\, e^{i\vec{\ell}\cdot\vec{x}-iEt}\left\{\int_{\tilde{Q}_{\downarrow}^{-}}^{\tilde{Q}_{\uparrow}^{+}} \frac{d^3k}{(2\pi)^3}\,\frac{G^{(2)}(\ell,E)}{E+E_{-\uparrow}-E_{+\downarrow}} + \right.$$

$$\left. + \int_{\tilde{Q}_{\downarrow}^{+}}^{\tilde{Q}_{\uparrow}^{-}} \frac{d^3k}{(2\pi)^3}\,\frac{G^{(1)}(\ell,E)}{E+E_{-\downarrow}-E_{+\uparrow}}\right\} \tag{A.56}$$

etc.

We introduce next the "free spin current"

$$\vec{j}_o^{(i)}(x) = \frac{1}{2mi}[\phi^\dagger(x)\vec{\nabla}\sigma_i\phi(x) - (\vec{\nabla}\phi^\dagger(x))\sigma_i\phi(x)] =$$

$$= -\frac{1}{2mi}\,[(\vec{\nabla}_x-\vec{\nabla}_y)\mathrm{Tr}[\sigma_i\phi(x)\phi^\dagger(y)]]_{x=y} + c_{oj}^{(i)} \qquad i = 1,2,3 \tag{A.57}$$

and the "free spin density"

$$\rho_o^{(i)}(x) = \phi^\dagger(x)\sigma_i\phi(x) = -\mathrm{Tr}[\sigma_i\phi(x)\phi^\dagger(y)]_{x=y} + c_{o\rho}^{(i)}\;, \qquad i = 1,2,3\;. \tag{A.58}$$

Then

$$<j|\vec{j}_o^{(i)}(x)|i> = -\frac{1}{2mi}\,\mathrm{Tr}[(\vec{\nabla}_x-\vec{\nabla}_y)\sigma_i G^o(x,y)]_{x=y} \tag{A.59}$$

$$<j|\rho_o^{(i)}(x)|i> = -\mathrm{Tr}[\sigma_i G^o(x,y)]_{x=y}\;. \tag{A.60}$$

Thus

$$\vec{j}^{(i)}(x) = \vec{j}_o^{(i)}(x) + \delta\vec{j}^{(i)}(x)$$
$$\rho^{(i)}(x) = \rho_o^{(i)}(x) + \delta\rho^{(i)}(x) \tag{A.61}$$

where $\vec{j}^{(i)}$, $\vec{j}_o^{(i)}$, $\rho^{(i)}$, $\rho_o^{(i)}$ must be understood as the corresponding matrix elements $\langle j|\vec{j}^{(i)}(x)|i\rangle$, etc.

By using (A.43), (A.5), (A.8), (A.55), (A.56) and (A.61) we can prove that the conservation law (A.49) is still valid although we consider the current and the density as expressed in the pair approximation in (A.61).

Let us note that, while

$$\vec{\nabla}\vec{j}_o^{(1)}(x) + \frac{\partial}{\partial t}\rho_o^{(1)}(x) = -\ 2i\tilde{M}(\phi_\uparrow^\dagger\phi_\downarrow - \phi_\downarrow^\dagger\phi_\uparrow) \tag{A.62}$$

and similar equation for $i = 2$, for $i = 3$ we have

$$\vec{\nabla}\vec{j}_o^{(3)}(x) + \frac{\partial}{\partial t}\rho_o^{(3)}(x) = 0 \ , \tag{A.63}$$

which is the conservation law for the "free spin current" in the 3rd direction. We can show also that

$$\rho^{(1)}(x)\Big|_{\ell=0} = \rho^{(2)}(x)\Big|_{\ell=0} = \delta\rho^{(3)}(x)\Big|_{\ell=0} = 0. \tag{A.64}$$

We observe that, for the momentum $\vec{\ell}$ different from zero, the gap equation (A.33) can be generalized as

$$1 = -\lambda\int_{\tilde{Q}_{\downarrow}^{+}}^{\tilde{Q}_{\uparrow}^{-}} \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_{\ell} + \frac{\bar{k}\cdot\bar{\ell}}{m} - 2\tilde{M}} \tag{A.65}$$

where $\omega_{\ell}\Big|_{\ell=0} = 0$. Then, by using (A.45),

$$\begin{aligned} 1 + Q^{12}(\ell,E) &= (E+\omega_{\ell})A(\ell,E) \\ 1 + Q^{21}(\ell,E) &= (E-\omega_{\ell})B(\ell,E) \end{aligned} \tag{A.66}$$

where

$$\begin{aligned} A(\ell,E) &= -B(-\ell,-E) = \\ &= \lambda\int_{\tilde{Q}_{\downarrow}^{+}}^{\tilde{Q}_{\uparrow}^{-}} \frac{d^3k}{(2\pi)^3} \frac{1}{(-\omega_{\ell} - \frac{\bar{k}\cdot\bar{\ell}}{m} + 2\tilde{M})} \frac{1}{(E - \frac{\bar{k}\cdot\bar{\ell}}{m} + 2\tilde{M})} \end{aligned} \tag{A.67}$$

Use of (A.43) and (A.66) shows that $\delta\vec{j}^{(1)}(x)$, $\delta\vec{j}^{(2)}(x)$, $\delta\rho^{(1)}(x)$ and $\delta\rho^{(2)}(x)$ have poles at $E = \pm\omega_{\ell}$ (cf. (A.55) and (A.56)).

By solving (A.65) with respect to ω_{ℓ} we find $\omega_{\ell} \propto \ell^2$ for small ℓ; indeed

$$\omega_{\ell} = -\left[\frac{1}{5\tilde{M}m^2}(Q_{\uparrow}^5 - Q_{\downarrow}^5) - \frac{1}{m}(Q_{\uparrow}^3 + Q_{\downarrow}^3)\right]\frac{\ell^2}{2(Q_{\uparrow}^3 - Q_{\downarrow}^3)} \tag{A.68}$$

for small ℓ.

The fact that spin currents and densities, for $i = 1$ and 2, have singularities at $E = \pm\omega_{\ell}$, suggests to us that there exists a boson of energy ω_{ℓ}. We have then to modify the set of the quasiparticles by introducing the magnon field $B(x)$. Consequently, we must add terms containing the field $B(x)$ to the dynamical map (A.20). The terms we add to the right hand side of (A.20) are

$$\int d^3\ell\ \tilde{T}^{(1)}(\ell,x,y)B^{\dagger}_{\vec{\ell}} + \int d^3\ell\ \tilde{T}^{(2)}(\ell,x,y)B_{\vec{\ell}} + \ldots , \qquad (A.69)$$

where $B_{\vec{\ell}}$ is the annihilation operator of the boson quantum (the magnon) and dots mean higher order normal product terms. We are going to prove the existence of this boson. On the basis of the arguments presented in Chapter I, the boson quantum is a composite particle, i.e. a bound state of electrons.

Let us study the B-S amplitude $G(x_1,x_2)$ obtained by considering the states $|j\rangle = |0\rangle$ and the one-magnon state $|i\rangle = |B_\ell\rangle$:

$$G(x,y) = \tilde{T}^{(2)}(\ell,x,y) \qquad (A.70)$$

By a computation similar to the one for $G(x,y)$ defined in (A.34), we find

$$G(x,y) = -i\int d^4\xi\ S(x-\xi)F(\xi)S(\xi-y) \qquad (A.71)$$

with $F(\xi)$ defined in (A.37). We can write

$$G(x,x) = \frac{1}{(2\pi)^3}\ G(\ell,E)\ e^{i\vec{\ell}\cdot\vec{x}-iEt} \qquad (A.72)$$

and obtain

$$
\begin{aligned}
G_{11}(\ell,E) &= Q^{11}(\ell,E)G_{22}(\ell,E) \\
G_{12}(\ell,E) &= -Q^{12}(\ell,E)G_{12}(\ell,E) = G^{(1)} \\
G_{21}(\ell,E) &= -Q^{21}(\ell,E)G_{21}(\ell,E) = -G^{(2)} \\
G_{22}(\ell,E) &= Q^{22}(\ell,E)G_{22}(\ell,E)
\end{aligned}
\tag{A.73}
$$

11

which are the B-S equations corresponding to (A.43) (in (A.73) there are no inhomogeneous terms). (A.73) together with (A.66) gives

$$
\begin{aligned}
(E+\omega_\ell)G^{(1)}(\ell,E) &= 0 \\
(E-\omega_\ell)G^{(2)}(\ell,E) &= 0
\end{aligned}
\tag{A.74}
$$

which are the wave equations for the magnon and show that the magnon energy is $\pm\omega_\ell$.

The computation of the magnon current $\vec{j}_B^{(i)}(x)$ and of the "magnon density" $\rho_B^{(i)}(x)$ is analogous to the one for the spin current and density. We find

$$
\begin{aligned}
&\langle 0|\vec{j}_B^{(1)}(x)|B_\ell\rangle = \\
&\frac{1}{(2\pi)^3}\,\frac{\vec{\ell}}{\ell^2}\,e^{i\vec{\ell}\cdot\vec{x}-i\omega_\ell t}\;\omega_\ell\left(G^{(2)}(\ell,\omega_\ell)-G^{(1)}(\ell,\omega_\ell)\right)
\end{aligned}
\tag{A.75}
$$

$$<0|\vec{j}_B^{(2)}(x)|B_\ell> =$$

$$- \frac{i}{(2\pi)^3} \frac{\vec{\ell}}{\ell^2} e^{i\vec{\ell}\cdot\vec{x}-i\omega_\ell t} \omega_\ell (G^{(2)}(\ell,\omega_\ell)+G^{(1)}(\ell,\omega_\ell)) \qquad (A.76)$$

$$<0|\rho_B^{(1)}(x)|B_\ell> =$$

$$\frac{1}{(2\pi)^3} e^{i\vec{\ell}\cdot\vec{x}-i\omega_\ell t} (G^{(2)}(\ell,\omega_\ell)-G^{(1)}(\ell,\omega_\ell)) \qquad (A.77)$$

$$<0|\rho_B^{(2)}(x)|B_\ell> =$$

$$- \frac{i}{(2\pi)^3} e^{i\vec{\ell}\cdot\vec{x}-i\omega_\ell t} (G^{(2)}(\ell,\omega_\ell)+G^{(1)}(\ell,\omega_\ell)) \ , \qquad (A.78)$$

with the conservation law

$$\vec{\nabla}\vec{j}_B^{(i)}(x) + \frac{\partial}{\partial t} \rho_B^{(i)}(x) = 0 \qquad i = 1,2 \qquad (A.79)$$

We write now the field B(x) as

$$B(x) = \int \frac{d^3\ell}{(2\pi)^{3/2}} B_\ell \, e^{i\vec{\ell}\cdot\vec{x}-i\omega_\ell t} \ , \qquad (A.80)$$

with the commutation relation

$$[B(x),B^\dagger(y)]_{t_x=t_y} = \delta(\vec{x}-\vec{y}) \ . \qquad (A.81)$$

The free field equations for $B(x)$ and $B^\dagger(x)$ are then (cf. (A.74))

$$(i \frac{\partial}{\partial t} - \omega_\ell) B(x) = 0, \quad (i \frac{\partial}{\partial t} + \omega_\ell) B^\dagger(x) = 0 \ . \tag{A.82}$$

From (A.75-78) and (A.80) we see that

$$\vec{j}_B^{(1)}(x) = - \frac{i}{(2\pi)^{3/2}} \frac{\omega_\ell}{\ell^2} g(i\vec{\nabla}) [\vec{\nabla}B(x) - \vec{\nabla}B^\dagger(x)] \tag{A.83}$$

$$\vec{j}_B^{(2)}(x) = - \frac{1}{(2\pi)^{3/2}} \frac{\omega_\ell}{\ell^2} g(i\vec{\nabla}) [\vec{\nabla}B(x) + \vec{\nabla}B^\dagger(x)] \tag{A.84}$$

$$\rho_B^{(1)}(x) = \frac{1}{(2\pi)^{3/2}} g(i\vec{\nabla}) [B(x) + B^\dagger(x)] \tag{A.85}$$

$$\rho_B^{(2)}(x) = - \frac{i}{(2\pi)^{3/2}} g(i\vec{\nabla}) [B(x) - B^\dagger(x)] \tag{A.86}$$

where $g(i\vec{\nabla})$ is defined by

$$g(i\vec{\nabla}) e^{i\vec{\ell}\cdot\vec{x} - i\omega_\ell t} \equiv g(\ell) \ e^{i\vec{\ell}\cdot\vec{x} - i\omega_\ell t} \tag{A.87}$$

with

$$|g(\ell)|^2 = (2\pi)^3 |\langle 0|\rho^{(i)}(x)|B_\ell\rangle|^2 \ , \quad i = 1,2 \ . \tag{A.88}$$

It can be shown [72] that

$$g(\ell) = \left[\frac{1}{\lambda A(-\ell, -\omega_\ell)}\right]^{\frac{1}{2}} \underset{\ell\to 0}{\to} \left(\frac{2\tilde{M}}{\lambda}\right)^{\frac{1}{2}} \tag{A.89}$$

with $A(\ell, \omega_\ell)$ given in (A.67). Note that (A.83-86) satisfy the conservation (A.79).

We also have

$$\langle 0|\vec{j}_B^{(3)}(x)|B_\ell\rangle =$$

$$- \frac{1}{(2\pi)^3} \frac{\vec{\ell}}{\ell^2} e^{i\vec{\ell}\cdot\vec{x}-i\omega_\ell t} \omega_\ell(G_{11}(\ell,\omega_\ell)-G_{22}(\ell,\omega_\ell)) \tag{A.90}$$

$$\langle 0|\rho_B^{(3)}(x)|B_\ell\rangle =$$

$$- \frac{1}{(2\pi)^3} e^{i\vec{\ell}\cdot\vec{x}-i\omega_\ell t} (G_{11}(\ell,\omega_\ell)-G_{22}(\ell,\omega_\ell)) \tag{A.91}$$

However, since

$$\begin{aligned} &1-Q^{22}(\ell,E) \neq 0 \quad \text{at } E = \pm\omega_\ell, \\ &1-Q^{11}(\ell,E) \neq 0 \quad \text{at } E = \pm\omega_\ell, \end{aligned} \tag{A.92}$$

from (A.73) we have that

$$G_{11}(\ell,\omega_\ell) = G_{22}(\ell,\omega_\ell) = 0 \tag{A.93}$$

which means that the right hand sides of (A.90) and (A.91) are zero. This tells us that there are no linear terms in B or $B^\dagger$ in $\vec{j}_B^{(3)}$ and $\rho_B^{(3)}$, i.e. $\vec{j}_B^{(3)}$ and $\rho_B^{(3)}$ have terms at least bilinear in B and $B^\dagger$. In the pair approximation, however, such terms cannot be computed since B is a bound state of electrons: terms bilinear in B and $B^\dagger$ represent thus at least two pairs.

Let us now note that the hamiltonian (A.1) is invariant under the spin rotation

$$\psi(x) \to e^{i\theta_i \lambda_i} \psi(x) \ , \qquad i = 1,2,3 \tag{A.94}$$

where $\lambda_i = \frac{1}{2}\sigma_i$. Generator of this transformation is[19]

$$S^{(i)} = \frac{1}{2}\int d^3x \ \rho^{(i)}_{tot}(x) = \frac{1}{2}\int d^3x \ \psi^\dagger(x)\sigma_i\psi(x) \ . \tag{A.95}$$

When $\rho^{(i)}_{tot}(x)$ is written in terms of quasielectrons and magnons, we can use the notation

$$\rho^{(i)}_{tot}(x) = \rho^{(i)}_F(x) + \rho^{(i)}_B(x) \tag{A.96}$$

where the subscript F refers to quasielectrons.

At $\ell = 0$, (A.64) (in which a subscript F should be understood) tells us that

$$\begin{aligned} \rho^{(i)}_{tot}(x) &= \rho^{(i)}_B(x) \quad \text{for } i = 1,2, \text{ at } \ell = 0 \\ \rho^{(3)}_{tot}(x) &= \rho^{(3)}_o(x) + \rho^{(3)}_B(x) \qquad \text{at } \ell = 0 \end{aligned} \tag{A.97}$$

This results (cf. also (A.85) and (A.86)) agree with the ones obtained in Sec. 2.5 if one assumes that in $\rho^{(3)}_B(x)$ only a term like $B^\dagger(x)B(x)$ appears (cf. eq. (II.158)). Note also that in the present notation $\frac{\tilde{M}}{\lambda}$ corresponds to M in Sec. 2.5. Once more we find then that due to dynamical effects the original symmetry is rearranged and the

---

[19]Here the prescription (II.10) is understood.

original spin rotation symmetry group is changed into the E(2) group. The difference of the energy $2\tilde{M}$ between spin-up and spin-down quasielectrons forbids any mixing of them: quasielectrons are frozen under spin rotation around 1 and 2 directions. The spin property is instead carried by the magnon which undergoes a boson transformation: magnons (i.e. spin wave quanta) are condensed in the ground state which thus acquires the characteristic structure of ferromagnetism. The original invariance under spin rotation manifests itself in the invariance of the free field equations under the quasiparticles transformations.

BIBLIOGRAPHY

1. W. Heisenberg, Zeits. f. Phys. 77, 1 (1932).
2. H. Umezawa, Acta Phys. 19, 9 (1965).
3. H. Umezawa, Lectures on Dynamical Effects in Quantum Field Theory, Helsinki, 1966;
   L. Leplae, R.N. Sen and H. Umezawa, Progr. Theor. Phys. Suppl. Extra Number 637 (1965).
4. H. Umezawa, Nuovo Cimento 40, 450 (1965).
5. L. Van Hove, Physica 18, 145 (1952);
   K.O. Friedrichs, Mathematical Aspects of Quantum Theory of Fields (Interscience, N.Y. 1953);
   S.S. Schweber and A.S. Wightman, Phys. Rev. 98, 812 (1955);
   R. Haag, Mat. Fys. Medd. Dan. Vid. Selsk. 29, No. 12 (1955);
   L. Gärding and A.S. Wightman, Proc. Nat. Acad. Sci. USA 40, 617 (1956);
   I.E. Segal, Trans. Am. Math. Soc. 88, 12 (1958);
   H. Fukutome, Progr. Theor. Phys. 23, 989 (1960).
6. J. Von Neumann, Ann. Math. 104, 570 (1931).
7. Y. Takahashi and H. Umezawa, Nucl. Phys. 51, 193 (1964);
   Y. Takahashi, Introduction to Field Quantization (Pergamon Press, 1969).
8. R. Haag, Phys. Rev. 112, 669 (1958);
   K. Nishijima, Phys. Rev. 111, 994 (1958);
   W. Zimmerman, Nuovo Cimento 10, 597 (1958).
9. J. Rest, V. Srinivasan, and H. Umezawa, Phys. Rev. D3, 1890 (1971).

10. L. Leplae and H. Umezawa, Nuovo Cimento 33, 372 (1964).
11. K. Nakagawa, R.N. Sen and H. Umezawa, Nuovo Cimento 42, 565 (1966).
12. H. Lehmann, K. Symanzik and W. Zimmermann, Nuovo Cimento 11, 205 (1955).
13. H. Ezawa, K. Kikkawa and H. Umezawa, Nuovo Cimento 25, 1141 (1962).
14. H. Ezawa, T. Muta and H. Umezawa, Progr. Theor. Phys. 29, 877 (1963).
15. S. Weinberg, Phys. Rev. B137, 672 (1965).
16. H. Umezawa, Progr. Theor. Phys. Suppl. No. 37, 38, 585 (1966).
17. R.N. Sen and H. Umezawa, Nuovo Cimento 50, 53 (1967).
18. L. Leplae, R.N. Sen and H. Umezawa, Nuovo Cimento 49, 1 (1967).
19. K. Nakagawa, Nuovo Cimento 50, 37 (1967).
20. L. Leplae and H. Umezawa, Nuovo Cimento 44, 410 (1966).
21. S. Coleman, Lectures given at the 1973 Inter. Summer School of Physics E. Majorana.
22. Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122, 345 (1961); 124, 246 (1961).
23. W. Heisenberg, Zeits. Naturf. 14a, 441 (1959).
24. H. Matsumoto, H. Umezawa, G. Vitiello and J.K. Wyly, Phys. Rev. D9, 2806 (1974).
25. M.N. Shah, H. Umezawa and G. Vitiello, Phys. Rev. B, to be published.
26. J. Goldstone, Nuovo Cimento 19, 15 (1961);

J. Goldstone, A. Salam and S. Weinberg, Phys. Rev. 127, 965 (1962).

27. Y. Nambu, Phys. Rev. Letters 4, 380 (1960).

28. R. Ferrari and L.E. Picasso, Nuclear Phys. B20, 553 (1970);

E. Fabri and L.E. Picasso, Phys. Rev. Letters 16, 408 (1966);

S. Coleman, Phys. Letters 19, 144 (1965);

R.F. Streater, Proc. Roy. Soc. 287A, 510 (1965).

29. H. Umezawa, Y. Takahashi and S. Kamefuchi, Ann. of Phys. 26, 336 (1964).

30. H. Umezawa, Lectures given at the Nato School on Renormalization and Invariance in Quantum Field Theory, Capri 1973.

31. L. Leplae, F. Mancini and H. Umezawa, Phys. Rev. B2, 3594 (1970).

32. L. Leplae, F. Mancini and H. Umezawa, Phys. Reports 10, No. 4 (1974).

33. H. Matsumoto, N.J. Papastamatiou and H. Umezawa, Nuclear Phys. B, in press.

34. R.P. Feynman, Rev. Modern Phys. 20, 367 (1947); Phys. Rev. 74, 1430 (1948);

J. Schwinger, Proc. Nat. Acad. of Sciences 37, 452 (1951);

H. Umezawa and A. Visconti, Nuovo Cimento 1, 1079 (1955);

K. Symanzik, Zeit. Naturf. 9, 809 (1954);

D. Lurie, Particles and Fields (Interscience, New York, 1968);

I.M. Gel'Fand and A.M. Yaglom, J. of Math. Phys. 1, 48 (1960).

35. G. t'Hooft, Nuclear Phys. B35, 167 (1971);
A. Salam and J. Strathdee, Nuovo Cimento 11A, 397 (1972);
B.W. Lee, Phys. Rev. D5, 823 (1972);
B.W. Lee and J.Z. Zinn-Justin, Phys. Rev. D5, 3121, 3137, 3155 (1972).

36. H. Matsumoto, N.J. Papastamatiou and H. Umezawa, Nuclear Phys. B68, 236 (1974; Phys. Letters 46B, 73 (1973).

37. J.C. Ward, Phys. Rev. 78, 1824 (1950);
Y. Takahashi, Nuovo Cimento 6, 370 (1957).

38. N. Nakanishi, Progr. Theor. Phys. 51, 1183 (1974).

39. N.J. Papastamatiou and H. Umezawa, Phys. Rev. D7, 571 (1973);
A. Aurilia, Y. Takahashi, N.J. Papastamatiou and H. Umezawa, Phys. Rev. D5, 3066 (1972);
A. Aurilia, Y. Takahashi and H. Umezawa, Phys. Rev. D5, 851 (1972).

40. R. Gilmore, Lie Groups, Lie Algebras, and Some of Their Applications, John Wiley & Sons, New York 1974, p. 640.

41. Y. Chow and C.J. Liu, J. Math. Phys. 12, 2144 (1971); Intern. J. Quantum Chem. Symp. No. 7, 521 (1973).

42. F.J. Dyson, Phys. Rev. 102, 1217 (1956).

43. T. Holstein and H. Primakoff, Phys. Rev. 58, 1098 (1940).

44. P.W. Anderson, Phys. Rev. 130, 433 (1963).

45. P.W. Higgs, Phys. Letters 12, 132 (1964); Phys. Rev.

145, 1156 (1966);

F. Englert and R. Brout, Phys. Rev. Letters 13, 321 (1964);

T.W. Kibble, Phys. Rev. 155, 1554 (1967).

46. S. Weinberg, Phys. Rev. Letters 19, 1264 (1967);

H. Georgi and S.L. Glashow, Phys. Rev. Letters 28, 1494 (1972).

47. G.S. Guralnik, C.R. Hagen and T.W. Kibble, Phys. Rev. Letters 13, 585 (1964);

R. Lange, Phys. Rev. Letters 14, 3 (1965).

48. Y. Freundlich and D. Lurie, Nucl. Phys. B19, 557 (1970).

49. A. Aurilia, Y. Takahashi and H. Umezawa, Progr. Theor. Phys. 48, 290 (1972); Lettere al Nuovo Cimento 1, 603 (1971);

A. Aurilia, Syracuse preprint 1974;

Y. Takahashi and H. Umezawa, UWM Report 1973.

50. L. Leplae, F. Mancini and V. Srinivasan, Report UWM-4867-71-72 (1971).

51. N. Nakanishi, Progr. Theor. Phys. 50, 1388 (1973).

52. H. Matsumoto, N.J. Papastamatiou, H. Umezawa and G. Vitiello, UWM Preprint (1974).

53. K. Johnson, Nucl. Phys. 25, 435 (1961);

N. Nakanishi, Progr. Theor. Phys. 49, 640 (1973).

54. L.D. Faddeev and V.N. Popov, Phys. Letters 25B, 29 (1967); Report NAL-THY-57, May 1972.

55. C.N. Yang and R.L. Mills, Phys. Rev. 96, 191 (1954).

56. R. Haag, Phys. Rev. 112, 669 (1958);

D. Ruelle, Helv. Phys. Acta 35, 147 (1962);

H. Araki, J. of Math. Phys. 1, 492 (1960);

F. Coester and R. Haag, Phys. Rev. 117, 1137 (1960);
E. Ezawa, J. Math. Phys. 5, 1078 (1964); J. Math. Phys. 6, 380 (1965);
J.R. Klauder, J. McKenna and E.J. Woods, J. Math. Phys. 7, 822 (1966).

57. S. Kamefuchi and H. Umezawa, Nuovo Cimento 31, 429 (1963).

58. D. Hall and A.S. Wightman, Mat. Fys. Medd. Dan. Vid. Selsk. 31, No. 5 (1957);
O.W. Greenberg, Phys. Rev. 115, 706 (1959).

59. A.K. Benson, Phys. Rev. B7, 4158 (1973).

60. G. Källen and W. Pauli, Kgl. Danske Vid. Selsk., Mat. Phys. 30, No. 7 (1955);
V. Glaser and G. Källen, Nucl. Phys. 2, 706 (1957);
R.E. Peierls, Proc. 1954 Glasgow Conf., London 1955;
M. Levy, Nuovo Cimento 13, 115 (1959); 14, 612 (1959);
P.T. Matthews and A. Salam, Phys. Rev. 112, 283 (1958); 115, 1079 (1959);
R.G. Newton, Scattering Theory of Waves and Particles, McGraw Hill, New York 1965;
G. Scarpetta and G. Vitiello, Nuovo Cimento 8A, 196 (1972);
J. Lukierski, ICPT Trieste Preprint IC/72/128;
F. Casagrande and L.A. Lugiato, Nuovo Cimento 17A, 464 (1973).

61. H. Umezawa, Theoretical Physics Seminar Notes, University of Wisconsin-Milwaukee, unpublished (1971);
Y. Takahashi and H. Umezawa, Collective Phenomena, in

press.

62. T. Matsubara, Prog. Theor. Phys. 14, 351 (1955).

63. A.A. Abrikosov, L. Gorkov and I. Dzyaloshinski, Methods of Quantum Field Theory in Statistical Physics, Pergamon, Oxford, 1965.

64. F. Mancini, H. Umezawa and G. Vitiello, Solid State Commun., 14, 1123 (1974).

65. J. Auer and H. Ullmaier, Phys. Rev. B7, 136 (1973).

66. L. Leplae, V. Srinivasan and H. Umezawa, Phys. Letters 45A, 177 (1973).

67. F. Mancini, Phys. Letters 45A, 179 (1973).

68. L. Leplae, V. Srinivasan and H. Umezawa, Preprint University of Wisconsin-Milwaukee (1973).

69. L. de Cesare and F. Mancini, Preprint University of Salerno (1973).

70. F. Mancini and H. Umezawa, Lettere al Nuovo Cimento 7, 125 (1973).

71. J. Bardeen, L.N.Cooper and J.R. Schrieffer, Phys. Rev. 108, 1175 (1957).

72. M.N. Shah and G. Vitiello, UWM Report (1974).

Dissertations Abstracts International 36/02,769-B (1975)

equation for bacterial chemotaxis. To achieve this end, we compare the DLK experimental distributions with those predicted by possible forms for the flux coefficient in the governing transport equation.

A review of the literature on both the experimental and theoretical aspects of bacterial chemotaxis is first presented. Most significant areas of research related to the bacterial chemotaxis transport equation are discussed in detail here.

The s-dependence of the Keller-Segel chemotactic flux coefficient is investigated by solving the governing partial differential equation for the motion of a population of chemotactic bacteria cells in a linear and steeply increasing gradient of attractant and comparing the theoretical distributions with the experimental data obtained by Dahlquist, Lovely and Koshland. Both an analytic asymptotic solution and a numerical solution are obtained for a simple s^{-1} form of dependence. Based on differences between the numerical solution and experimental data, a more complicated s-dependence for the chemotactic flux coefficient is indicated. A numerical solution of the resulting equation is obtained for a dependence of the Brown-Berg $K_D(K_D + s)^{-2}$ form and found to be in excellent agreement with the experimental data of Dahlquist, Lovely and Koshland for a linear and steeply increasing gradient of attractant. It is also shown that this form of the flux coefficient is consistent with observed travelling band phenomena. Thus, the Brown-Berg form of dependence appears to be indicated for such experimentally observed motions of a bacterial population. Order No. 75-16,999, 101 pages.

THE STRUCTURE OF GENERAL RELATIVITY WITH A NUMERICAL ILLUSTRATION: THE COLLISION OF TWO BLACK HOLES

SMARR, Larry Lee, Ph.D.
The University of Texas at Austin, 1975

Supervisor: Bryce S. DeWitt

The splitting of spacetime into space and time is analyzed by abstract methods, coordinate methods, and numerical computer methods. General relativity is assumed to be the correct theory of gravity and the Einstein field equations are studied in their canonical form. A comparison is made of the initial value and evolution equations for gravity in different formalisms (ADM, geometry of spacelike hypersurfaces, timelike congruences). These equations are then illustrated by a number of cosmological and black hole spacetimes. A study of coordinate conditions is undertaken.

The axisymmetric (non-stationary and non-spherical) Einstein equations are discussed, and it is shown how to set up a numerical computer program to integrate these equations starting with a given initial data set. Various applications of this computer approach are discussed: collapse of rotating non-spherical stars, "runaway" collapse, and headon collisions of black holes. All of these situations would involve generation of gravitational radiation from the formation of black holes.

The particular case of two nonrotating black holes colliding headon is chosen as a test case for the computer. The initial value problem is reviewed and then a computer program is written to generate the spacetime. This generation involves solving four coupled quasilinear hyperbolic (evolution) equations and one linear elliptic equation (maximal slicing) on each spacelike sheet. The numerical difficulties are discussed and graphical methods are used to present the result of the computations. Order No. 75-16,738, 210 pages.

DYNAMICAL REARRANGEMENT OF SYMMETRY

VITIELLO, Giuseppe, Ph.D.
The University of Wisconsin - Milwaukee, 1974

Supervisor: Professor Hiroomi Umezawa

In Quantum Field Theory (Q.F.T.), the invariance of the theory is expressed as the invariance of field equations under certain transformations of Heisenberg fields. Since we are interested in physically relevant entities, we are faced with the problem of how the original invariance of the theory manifests itself at the level of observable (physical) particles. This is the problem studied in the present work.

In our analysis we start by the fundamental assumption that the set of physical field operators is an irreducible set of operators realized in the Fock space of physical particles. In Chapter I we give an outline of the self-consistent method in Q.F.T. In this method a mapping (the dynamical map) is introduced among Heisenberg fields and physical fields. The role of this mapping in the theory is fundamental since through it the dynamics described by the Heisenberg equations manifest its effects at the phenomenological level. Through the dynamical map we can thus express the Heisenberg field operators in terms of infields by collecting all the matrix elements of Heisenberg operators. In Chapter II, we show that this expression takes a simple form when we use the path-integral to express matrix elements. We then ask what kind of transformations of infields reproduce the original invariant transformation of Heisenberg operators. The invariance of the theory requires that these transformations of infields are such that they leave the free field equations invariant. When the invariant transformations of the Heisenberg operators appear in a different form from that of the infield operators, we say that a dynamical rearrangement of symmetry has taken place. It is found that there is dynamical rearrangement of symmetry when spontaneous breakdown occurs. The generating functional of the Green's functions is modified by the addition of an infinitesimal ε-term which fixes the direction of the breaking. Since the dynamical rearrangement of symmetry concerns the general structure of the theory, we generalize our study to relativistic as well as to non-relativistic problems. In each of them the Goldstone theorem is proved and the role of the massless Goldstone particles in recovering the invariance of the theory is analyzed. It is found that these particles undergo a transformation, the boson transformation, which leaves the free field equations invariant. A Goldstone-type model is studied as an example of a model with Abelian symmetry. The spontaneous breakdown of SU(2) symmetry is also investigated. We analyze a relativistic model (iso-triplet scalar field) and a non-relativistic one (a ferromagnetic system). In terms of the path-integral method the infield transformations which induce the original SU(2) transformations of Heisenberg operators are identified. It is found that the algebra of infield transformations is the E(2) symmetry group algebra. It is shown that the discrepancy of two algebras is caused by the local nature of the observation in which one misses the infrared contributions. When the total (integrated) infrared effects are considered, the original symmetry group algebra is recovered. Exact expressions of symmetry generators in terms of physical operators are given.

An analysis of the spontaneous breakdown of gauge theories is also presented. Massless unphysical modes are found. Their role in recovering the invariance of the theory is shown.

The dynamical rearrangement of symmetry is analyzed in the framework of infinite unitarily inequivalent representations of the canonical variables. The occurrence of such representations and their physical usefulness is studied in Chapter III. A Q.F.T. for finite temperature is presented as an example and an application to superconductors is given. In particular the temperature dependence of the critical value of the Ginzburg-Landau parameter which separates type-II/1 from type-II/2 is computed. Agreement with experimental data is satisfactory.

Finally, a self-consistent formulation of the itinerant electron ferromagnet is presented in the Appendix. The pair approximation is used and the Goldstone boson (magnon) is studied as bound state of fermions by means of the Bethe-Salpeter equation. Order No. 75-17,814, 208 pages.

AN APPLICATION OF CUMULANT TECHNIQUES TO IRREVERSIBLE PROCESSES

WESTERFIELD, Robert Estel, Ph.D.
Georgia Institute of Technology, 1975

Director: Dr. H. A. Gersch

In this work, we have developed a new approach to the problem of a simple system interacting with a quantum reservoir when the interaction can be described by $V = AB$, where A is an operator for the simple system, B is an operator for the reservoir. After defining a cumulant expansion of the reduced density operator $\rho_s(t)$ for the simple system, we show that the second cumulant is the only non-vanishing cumulant for a reservoir of non-interacting bosons. A scheme for obtaining successive approximations to the equations of motion for $\rho_s(t)$ is found in which the n^{th} approximation is obtained from the $(n-1)^{th}$ approximation by the addition of a term roughly proportional to the $(2n-1)^{th}$ power of the probability that the system makes a transition in a time interval of length t_c, the reservoir relaxation time. In these equations, reservoir variables occur only as correlation functions of bilinear combinations of reservoir operators in the interaction picture.

The first order approximation to the density operator equation of motion is applied to the case when the simple system is an harmonic oscillator. Two cases are treated, the damped harmonic oscillator and the oscillator with an external driving force. Comparison is made between these results and those obtained in other treatments.

Order No. 75-17,495, 136 pages.

FOURTH SOUND IN LIQUID HELIUM THREE

YANOF, Arnold William, Ph.D.
Cornell University, 1975

In this work we describe experimental proof of superfluidity in the recently discovered millikelvin phases of liquid ^3He. Using fourth sound, we have measured the superfluid density.

The experimental cell was pre-cooled to 15 millikelvin by a dilution refrigerator and cooled about .2 millikelvin below the second order phase transition temperature by demagnetizing finely powdered cerium magnesium nitrate. Fourth sound was generated in the liquid by applying an oscillating force to the suspended sample chamber. Resonances were detected by measuring the displacement of the cell.

In our analysis, we obtain the superfluid density in two distinct ways. The superfluid fraction as a function of temperature and pressure can be deduced from the fourth sound velocity, which we have measured. An independent value for the superfluid mass can also be obtained directly from the strength of the cell response to a fourth sound resonance. Comparison of the two measurements of superfluid density allows us to check the validity of a two-fluid model of fourth sound in ^3He. Our analysis takes careful account of the complications arising from the random, tortuous geometry in which the superfluid must flow. We also extract the quality factor of fourth sound resonances and the superfluid velocity at resonance.

We find that the superfluid fraction of liquid ^3He depends approximately linearly upon reduced temperature near the second order phase transition. No discontinuity appeared in the superfluid density as the cell crossed the temperature of the first order transition from "A" phase to "B" phase. The superfluid density measured directly from cell response is within ten percent of agreement with the values deduced from fourth sound velocity. The quality factor of fourth sound resonances is a weak function of temperature, and no values of Q greater than 65 were observed. We find that the maximum superfluid velocity reached in this experiment was about 1 mm-sec^{-1}.

Order No. 75-18,008, 169 pages.

PHYSICS, ACOUSTICS

ULTRASONIC INVESTIGATIONS OF THE JAHN-TELLER EFFECT

MULLEN, Michael Edward, Ph.D.
Rutgers University The State University of New Jersey, 1975

Director: Professor Bruno Lüthi

Several transition metal (i.e., $Ni_xZn_{1-x}Cr_2O_4$ and $CsCuCl_3$) and rare earth (lanthanum antimonides, TmCd, Pr metal) compounds were investigated ultrasonically, the changes in sound velocity being used as a probe. The coupling of the ultrasonically produced strain field to the energy levels of the transition metal and rare earth ions (a manifestation of the Jahn-Teller effect) causes readily discernable changes in the sound velocity.

It is found that three general types of behavior are noted as the temperature is changed, depending on the strength of the coupling and the crystal field parameters. First, if the coupling between the strain and the energy levels is vanishingly small, or if the ground state multiplet is a singlet or Kramer's doublet (which cannot be split by a strain) the usual behavior of a solid is seen, that is, the velocity increases as the temperature is lowered, and becomes constant at temperatures near absolute zero. Second, if the lowest level of the ground state multiplet is a singlet or Kramer's doublet, and the coupling constant is less than some critical value, there is a minimum in the velocity vs. temperature curve, at a temperature which is related to the energy of the lowest excited state. Finally, if the ground state is a singlet or Kramer's doublet, and the coupling is above the critical value, or if the lowest level is a multiplet other than a Kramer's doublet, there exists a temperature below which the crystal distorts.

Examples of all these types are experimentally seen, as well as variations of these types due to magnetic interactions and transitions.

For cubic structures, the coupling constants can be calculated from the crystallographic data of the compounds, using a point charge approximation. In the case of the lanthanide antimonides, the constants so calculated agree well with experimental values. In other cases, the agreement is not as well.

Studies of the velocity anisotropy in nematic and cholesteric liquid crystals, nematic-isotropic and cholesteric-nematic phase transitions were performed in conjunction with the above investigations, and are also reported here.

Order No. 75-17,464, 166 pages.