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F. Guerra D.W. Robinson R. Stora

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Colloques Internationaux C.N.R.S.
N° 248 – Les méthodes mathématiques de la théorie quantique des champs

Quasi Invariant Measures, Symmetric Diffusion Processes and Quantum Fields *

by

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RESUME

On dérive plusieurs propriétés des mesures de probabilité quasi-invariantes, en particulier on étudie les propriétés de fermabilité, ergodicité, et des perturbations des systèmes associés. Ces résultats sont appliqués aux champs quantiques et on montre que pour les interactions polynomes à deux dimensions, le vide physique restreint aux champ initial est une mesure de la classe supérieure.

ABSTRACT

We show that for a large class of quasi invariant probability measures on a separable Hilbert space with a nuclear rigging the Dirichlet form $\int \nabla\overline{f}.\nabla g \; d\mu$ in $L_2(d\mu)$ is closable and its closure defines a positive self-adjoint operator H in $L_2(d\mu)$, with zero as an eigenvalue to the eigenfunction 1 . The connection with the hamiltonian formalism and canonical commutation relations is also studied. We show moreover that H is the infinitesimal generator of a symmetric time homogeneous Markov process on the rigged Hilbert space, with invariant measure μ . For strictly positive μ this process is ergodic if and only if μ is ergodic, which is the case if and only if zero is a simple eigenvalue of H .
Moreover we study perturbations of H and μ as well as weak limits of quasi invariant measures and their associates Markov processes. Finally we apply our results to quantum fiels. In particular we show that for polynomial interactions in two space-time dimensions the physical vacuum restricted to the σ-algebra generated by the time zero fields is a measure μ in the above class of quasi invariant measures and the physical Hamiltonian coincides on a dense domain of $L_2(\mu)$ with the generator of the Markov process given by the Dirichlet form determined by μ .

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1. Introduction

Within the general theory of Markov stochastic processes with continuous time parameter and finite dimensional state space the class of diffusion processes is of special importance due to its connection with second order partial differential equations. Since moreover every such Markov process is the solution of a stochastic differential equation, one has a beautiful interplay of the theory of partial differential equations, diffusion processes and stochastic differential equations. For this we refer to [1], [2], [3]. This paper introduces our study of the extension of these subjects, and in particular of the theory of Markov diffusion processes, to the infinite dimensional case. For a more detailed account and further results see [4].

We first mention shortly some previous work. A whole direction of early studies, mainly connected with the names Friedrichs, Gelfand and Segal, arose in connection with problems of quantum fields and in particular of the representations of canonical commutation relations, see e.g. [5]. A related stimulating influence came from Feynman's path integral formulation of quantum dynamics, see the references in [6]. Some studies dealing with differential and stochastic differential equations in infinite dimensional spaces are in [7]-[11] and references therein. Results from constructive field theory which are most related to our subject will be mentioned below. Let us now summarize the content of our paper.

In section 2 we start by assembling some facts about Gelfand's representation of Weyl's canonical commutation relations by means of probability measures on N', quasi invariant with respect to translations by elements in N, where $N \subset K \subset N'$ is a real separable Hilbert space with a nuclear rigging. References to previous work on this representation are [5] and [12]. We then isolate a class of quasi invariant measures, which we call measures with first order regular derivatives and which in the finite dimensional case correspond to the density functions having L_2 derivatives. This class is suitable for the construction of the self-adjoint positive operator H associated with the Dirichlet form $\int \nabla \overline{f} \cdot \nabla g \, d\mu$ and acting in the representation space $L_2(d\mu)$ for the canonical commutation relations.

The relation of Dirichlet forms * with the canonical formalism has been discussed, modulo some domain questions, by Araki, in his algebraic approach to the Hamiltonian formalism and canonical commutation relations [13]. Some of our results in this section can be looked upon as providing analytic versions of algebraic derivations of Araki.

The self-adjoint operator H mentioned above is the Friedrichs operator given by the closure of the Dirichlet form, first defined on a dense set F^2 of finitely based C^2 functions. H is non negative and has the eigenvalue zero with the eigenfunction identically equal to 1 in $L_2(d\mu)$. e^{-tH} is a Markov semigroup so that H is the infinitesimal generator of a time homogeneous Markov process on N' with invariant measure μ . Accordingly we call H the diffusion operator associated with μ . A (possibly strict) self adjoint extension of H is the operator $\hat{H} = \nabla^* \nabla$ with domain equal to the domain of the closed gradient operator ∇ in $L_2(d\mu)$. We have $H = \hat{H}$ on F^2 . $e^{-t\hat{H}}$ is also a Markov semigroup and in this case we have both time ergodic and N-ergodic decompositions of μ, $L_2(d\mu)$, $\hat{H}$ and the representation of canonical commutation relations, which coincide for a class of measures μ containing the so called strictly positive measures. For such measures zero is a simple eigenvalue of $\hat{H}$ if and only if μ is ergodic, which in turn is equivalent with the representation of the canonical commutation relations given by μ being irreducible.

In section 3 we study perturbations of quasi invariant measures μ with regular first order derivatives and of the associated diffusion operators. We also find sufficient conditions for the stability under weak limits of the correspondence between quasi invariant measures with regular first order deri-

*Dirichlet forms have also been considered in related contexts in [31] and [32].

vatives and the associated diffusion processes.

In section 4 we apply the general results of the preceding sections to the case of quantum fields. Stochastic methods in constructive quantum field theory are of course not new. It suffices to recall the intervention of Doob's processes in Nelson's early work and the development of Euclidean field theory. For general references to constructive quantum field theory see [14] as well as other contributions to this Colloquium. Particular results with direct probabilistic implications are in [15], [16] and [17]. Recently connections between problems of quantum fields and the theory of stochastic processes have been emphasized particularly by Klauder [18]. Coming now to our present applications of the methods of sections 2 and 3 to the quantum fields, we first remark that the diffusion operator associated by the procedure of section 2 with the Dirichlet form given by the Gaussian measure μ_0 of the unit process on $S(R^d) \subset L_2(R^d) \subset S'(R^d)$ coincides with the infinitesimal generator of the Markov process of the free Markov time zero field.

Finally we consider the interacting fields in two space-time dimensions, where the interaction is given by a polynomial of even degree with sufficiently small coefficients. We first show that the measure μ, given by the physical vacuum, restricted to the σ-algebra generated by the time zero fields has regular first order derivatives hence belongs to the class of quasi invariant measures discussed in Section 2. By means of the perturbation theory given in Section 3 and direct estimates, we then show that the corresponding diffusion operator coincides on a dense domain with the physical Hamiltonian.

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2. Symmetric diffusion processes.

2.1 The finite dimensional case.

Let us first consider the Schrödinger operator in R^n

$$-\Delta + V , \tag{2.1}$$

where V is the operation of multiplication by the potential energy $V(x)$ and Δ is the Laplacian in R^n. Under some well known mild regularity conditions on V (see for instance [19]) the operator (2.1) is a self-adjoint operator H on $L_2(R^n,dx)$. The corresponding unitary group e^{-itH} gives then the solution of the initial value problem for Schrödinger's equation in $L_2(R^n,dx)$.

Since we are interested in the infinite dimensional case, where R^n is replaced by a real separable Hilbert space K and $L_2(R^n,dx)$ has no obvious counterpart, it is better to look for a realization of the operator (2.1) in a more suitable space. This is possible if we assume that H has at least one eigenfunction in $L_2(R^n,dx)$ and that H is, as a self-adjoint operator in $L_2(R^n,dx)$, bounded from below. Again under some quite general regularity conditions on V, the bottom of the spectrum of H will then be an eigenvalue E, so that $H \geq E$, and the corresponding eigenfunction $\Omega(x)$ will be positive almost everywhere. This follows from the ergodicity of the Markov semigroup generated by the Laplacian and for details about these known results we refer to [20]. We can always normalize the eigenfunction Ω so that

$$(\Omega,\Omega) = \int_{R^n} \Omega(x)^2 dx = 1 . \tag{2.2}$$

Setting $\rho(x) = \Omega(x)^2$ we then have that $\rho(x)$ is the density of the probability measure $d\mu(x) = \rho(x)dx$ on R^n.

Since Ω is in the domain of H it must (again under slight regularity conditions on V) have locally integrable derivatives up to second order. Now if $f(x)$ is a smooth function of compact support an easy computation, using $\Delta\Omega = (V-E)\Omega$, shows that

$$\int \overline{\nabla f}\, \nabla f \, d\mu = (f\Omega,(H-E)f\Omega) \;, \tag{2.3}$$

where ∇ is the gradient in R^n . Hence the correspondence $f \leftrightarrow f\Omega$, which is a unitary equivalence between $L_2(R^n,dx)$ and $L_2(R^n,d\mu)$, takes the quadratic form $(f,(H-E)f)$ into the quadratic form $(f\Omega,(H-E)f\Omega) = \int \overline{\nabla f}\cdot\nabla f \, d\mu$. Thus we see that the operator H-E , looked upon as an operator in $L_2(R^n,d\mu)$, is actually the unique self-adjoint operator associated with the closure of the Dirichlet form

$$\int \overline{\nabla f}\cdot\nabla g \, d\mu \tag{2.4}$$

defined by the probability measure μ . We recall that the relation between the operator H-E and the measure $d\mu = \rho \, dx$ is given by

$$H-E = -\Delta + (V-E) \;, \tag{2.5}$$

where V is related to ρ through

$$V-E = \frac{\Delta\rho^{\frac{1}{2}}}{\rho^{\frac{1}{2}}} \;. \tag{2.6}$$

Note also that H-E is non negative and that 1 is the eigenfunction of H-E in $L_2(R^n,d\mu)$ to the eigenvalue zero. Under smoothness assumptions on ρ , e.g. such that each component of

$$\beta(x) = \frac{\nabla\rho(x)}{\rho(x)} \tag{2.7}$$

is in $L_2(R^n,d\mu)$, it is possible to write H-E as a differential operator in $L_2(R^n,d\mu)$, namely as

$$-\Delta-\beta(x)\cdot\nabla \ . \tag{2.8}$$

If we define for $x,a \in R^n$

$$\alpha(x,a) \equiv \frac{\rho(x+a)}{\rho(x)} = \frac{d\mu(x+a)}{d\mu(x)} \tag{2.9}$$

we have that

$$\beta(x) = \nabla_a \alpha(x,a)|_{a=o} \ . \tag{2.10}$$

Moreover

$$V(x)-E = \frac{1}{4}\beta(x)\cdot\beta(x)+\frac{1}{2}\nabla_x\cdot\beta(x) = \Delta_a\alpha^{\frac{1}{2}}(x,a)|_{a=o} \ . \tag{2.11}$$

Viceversa, let us now suppose we are given an arbitrary probability measure μ on R^n, quasi invariant with respect to translations. Then, as well known, μ is equivalent to Lebesgue measure, hence $d\mu(x) = \rho(x)dx$ with $\rho(x) > 0$ almost everywhere and (2.9) holds again.

Consider now the Dirichlet form associated with μ,

$$\int \overline{\nabla f} \, \nabla g \, d\mu \ , \tag{2.12}$$

defined first on smooth f and g over R^n. If this form turns out to be closable, then its closure is the form of a unique self-adjoint positive operator H_μ, so that, with $(,)$ being the inner product in $L_2(R^n,d\mu)$,

$$(f,H_\mu f) = \int \overline{\nabla f}\cdot\nabla f \, d\mu \ . \tag{2.13}$$

We have thus defined a self-adjoint positive operator H_μ starting from the quasi-invariant probability measure μ. H_μ is an operator in $L_2(R^n,d\mu)$ and has zero as an eigenvalue to the eigenfunction 1. Let us remark that from H_μ, by the unitary equivalence of $L_2(R^n,dx)$ and $L_2(R^n,d\mu)$, we get also in $L_2(R^n,dx)$ a self-adjoint operator. However only in the case that ρ is smooth enough we have that this operator is of the

2

form $-\Delta + V_\mu(x)$, with a measurable function V_μ (given by $\rho^{-\frac{1}{2}}\Delta\rho^{\frac{1}{2}}$ or, equivalently, by the right hand sides of (2.11)). Thus, whereas H_μ in $L_2(R^n,d\mu)$ is always defined as a positive self adjoint operator, whenever the Dirichlet form associated with μ is closable, the expressions (2.7), (2.8), (2.10), (2.11) and in particular the potential V_μ may or may not make sense as measurable functions.

Example

Let $n = 3$ and take $d\mu$ to be the probability measure in R^3 given by

$$d\mu(x) = \frac{m}{2\pi} \frac{e^{-2m|x|}}{|x|^2} dx .$$

We may verify that the form (2.4) is closable in $L_2(d\mu)$, so that H_μ is well defined. In this case

$$\alpha^{\frac{1}{2}}(x,a) = \frac{|x|}{|x+a|} e^{-m|x+a|}.e^{m|x|} .$$

For $x \neq 0$ we see that $\Delta_a\alpha^{\frac{1}{2}}(x,0) = m^2$. In fact we may easely prove that H_μ is a self adjoint operator such that, when restricted to smooth functions f which are zero at zero, then $H_\mu f = (-\Delta+m^2)f$. However, $H_\mu-m^2$, when represented in $L_2(dx)$, has $\frac{e^{-m|x|}}{|x|}$ as an eigenfunction with eigenvalue $-m^2$, so $H_\mu - m^2 \neq -\Delta$. In fact $H_\mu-m^2$ form a one parametric family of self adjoint extensions of the restriction of $-\Delta$ to functions $f \in D(\Delta)$ such that $f(0) = 0$.

One can now ask the question when is the operator H_μ the infinitesimal generator of a Markov semigroup i.e. when does e^{-tH_μ} have a positive kernel. In this finite dimensional case a very weak regularity condition on μ is actually sufficient, see Section 2.6. A well known simple situation is the one in which H_μ can be given in $L_2(R^n,dx)$ in the form $-\Delta + V_\mu(x)$ with a smooth function $V_\mu(x)$. The stationary symmetric Markov process $\xi(t)$ in R^n, given by the Markov semigroup e^{-tH_μ} and its invariant measure $d\mu$, is then the unique solution of the stochastic differential equation

$$d\xi(t) = \beta(\xi(t))dt + dw(t) \tag{2.14}$$

where $w(t)$ is the standard Wiener process in R^n and the drift $\beta(\xi)$ is given by

$$\beta(\xi) = \nabla \ln \rho(\xi) \tag{2.15}$$

where

$$d\mu(x) = \rho(x)dx\ . \tag{2.16}$$

$\beta(\xi)$ is actually the osmotic velocity of Nelson's stochastic mechanics and (2.14) is a case of the equation of Nelson's stochastic mechanics, equivalent with the Schrödinger equation. For more details on this we refer to Ref. [21] and the references contained therein. It follows from the methods in Ref. [21] that the stochastic process $\xi(t)$ is always a solution of (2.14), although one can prove that this solution is unique only under some conditions (e.g. boundedness or Lipschitz continuity and at most linear growth at infinity) on the osmotic velocity, see [21].

We would like now to extend these results to the infinite dimensional case of a real separable Hilbert space K instead of the finite dimensional Euclidean space R^n. From our arguments above it is visible that it is convenient to start from a quasi invariant measure μ and then construct the associated Dirichlet forms and in this way get a self adjoint operator H_μ on the relevant space $L_2(d\mu)$.

2.2 Recalling the relation between quasi invariant measures on rigged Hilbert spaces and canonical commutation relations.

The setting which we shall always use in the rest of this section is given by a nuclear rigging, in the sense of [5],

$$N \subset K \subset N' ,$$

where N is a real nuclear space densely contained in K and N' is the dual of N. Moreover the inner product (x,y) in K when restricted to N coincides with the dualization between N and N'.

Definition 2.1

We shall say that a probability measure μ on N' is quasi invariant if it is quasi invariant under translations by elements in N, i.e. for any $x \in N$ the Radon-Nikodym derivative

$$\alpha(\xi,x) = \frac{d\mu(\xi+x)}{d\mu(\xi)}$$

exists. Then we have $\alpha(\xi,x) > 0$ for μ-almost every $\xi \in N'$ and

$$\int \alpha(\xi,x)d\mu(\xi) = 1$$

$$\alpha(\xi,x+y) = \alpha(\xi+x,y)\alpha(\xi,x) .$$

We recall in a Theorem the following well known results:

Theorem 2.1 Let K be a real separable Hilbert space and $N \subset K \subset N'$ be a nuclear rigging of K. Then:

1) Any quasi invariant probability measure μ on N' defines two strongly continuous unitary representations $U(x)$ and $V(x)$ of the additive group N in the separable Hilbert space $\mathcal{H} = L_2(d\mu)$, by

$$(U(x)f)(\xi) = e^{i\langle x,\xi\rangle} f(\xi)$$

and

$$(V(x)f)(\xi) = \alpha^{\frac{1}{2}}(\xi,x)f(\xi+x) .$$

$U(x)$, $V(x)$ satisfy the Weyl commutation relations

$$V(x)U(y) = e^{i(x,y)} U(y)V(x)$$

for any x and y in N.

The function $\Omega(\xi) \equiv 1$ in $L_2(d\mu)$ is a cyclic element for the representation $U(x)$.

2) Conversely suppose we are given two representations $U(x)$ and $V(x)$ of N by unitary operators on a separable Hilbert space $\mathcal{H}$ such that $x \to U(x)$ is weakly continuous from N into the set of all bounded operators on $\mathcal{H}$ and such that there exists a cyclic element Ω for the representation $U(x)$. Suppose moreover that U and V satisfy the Weyl commutation relations

$$V(x)U(y) = e^{i(x,y)} U(y)V(x) .$$

Then there exists a probability measure μ on N' such that μ is quasi invariant, one has

$$(\Omega,U(x)\Omega) = \int_{N'} e^{i\langle x,\xi\rangle} d\mu(\xi) ,$$

and the map $U(x)\Omega \Leftrightarrow e^{i\langle x,\xi\rangle}$ gives an isomorphism of $\mathcal{H}$ with $L_2(d\mu)$. By this isomorphism $U(x),V(x)$ are unitary equivalent hence identified with the operators $(U(x)f)(\xi) = e^{i\langle x,\xi\rangle} f(\xi)$ and $(V(x)f)(\xi) = z(\xi,x)\alpha^{\frac{1}{2}}(\xi,x)f(\xi+x)$, where $z(\xi,x)$ is a measurable function on N' such that, for almost every ξ, $|z(\xi,x)| = 1$ and

$$z(\xi,x+y) = z(\xi+x,y)z(\xi,x)$$

and $z(\xi,0) = 1$.

In particular $U(x)$, $V(x)$ are strongly continuous.

Proof: Except for the strong continuity of $V(x)$ this theorem is first proven in [5]. The strong continuity of $V(x)$ was proven in Ref. [12],3), Theorem 3.3. □

We shall now use the following:

Definition 2.2 We say that a unitary representation (U,V) of the Weyl canonical commutation relations is irreducible iff the only bounded operators that commute with all $U(x)$, $V(x)$, for all $x \in N$, are the constants.

Remark: U has a cyclic element e.g. if (U,V) is irreducible. See e.g. Theorem 6.2.6 and its Corollary in Ref. [12],2).

Definition 2.3 We shall call a quasi invariant probability measure μ on N' ergodic if the only functions in $L_\infty(d\mu)$ which are invariant under all translations by elements of N are the constant functions.

The following results are well known:

Theorem 2.2 The following propositions are equivalent:

1) The quasi invariant measure μ is ergodic
2) all N-invariant measurable subsets of N' have either μ-measure zero or one
3) the representation (U,V) of the Weyl canonical commutation relations determined by μ as in Theorem 2.1, 1) is irreducible. □

Remark: Any quasi-invariant measure μ, not necessarily ergodic, has an ergodic decomposition, in the sense that there exists a standard Borel space Z, a finite measure dz on Z and for each z an ergodic measure $\mu_z(.)$ on N' such that

$$\mu(.) = \int \mu_z(.)dz \ .$$

Moreover, with $\mathcal{H} = L_2(d\mu)$, $\mathcal{H}_z = L_2(d\mu_z)$, one has the integral decompositions

$$\mathcal{H} = \int \mathcal{H}_z \, dz$$

$$(U,V)_\mu = \int_Z (U,V)_{\mu_z} \, dz \ ,$$

where $(U,V)_\mu$ and $(U,V)_{\mu_z}$ are the unitary representations of Weyl's canonical commutation relations given by μ resp. μ_z, according to Theorem 2.1,1).

For more details see [12],3).

2.3 A suitable subclass of quasi invariant measures.

Let us start with an arbitrary quasi invariant probability measure μ on N'.

From Theorem 2.1,1) we have in particular that, for each real t, $V(tx)$ is, for fixed $x \in N$, a strongly continuous one parameter unitary group acting in $L_2(d\mu)$. Let iPx be its infinitesimal generator, so that

$$i\,Px = s - \lim_{t \to o} t^{-1}[V(tx) - 1] \ , \qquad (2.17)$$

where the limes is the strong limes in $L_2(d\mu)$.

For each $x \in N$, Px is thus a densely defined self-adjoint operator in $L_2(d\mu)$.

Definition 2.4 A quasi invariant probability measure μ on N' is said to have regular first order derivatives if the function $\Omega(\xi) \equiv 1$ is in the domain of Px for all x in N. Equivalently, μ has regular first order derivatives iff the strong $L_2(d\mu)$ limit of $t^{-1}[\alpha^{\frac{1}{2}}(\xi,tx)-1]$ as $t \downarrow 0$ exists for all x in N. We denote by $\mathcal{P}_1(N')$ the set of all such measures μ.

Remark: For μ to have regular first order derivatives it is sufficient that the weak $L_2(d\mu)$-limit of $t^{-1}[\alpha(\xi,tx)-1]$ as $t \downarrow 0$ exists.

From now on we shall always consider probability measures μ on N' with regular first order derivatives.

Proposition 2.3 For each u in the domain $D(Px)$ of Px, the map $x \to Pxu$ is a linear continuous map from N into $L_2(d\mu)$.

Corollary Let $|x|_p$, $p = 1,2,\ldots$ be the countable set of norms that defines the topology of N. Then there is a p such that $x \to Pxu$ is continuous in the norm $|x|_p$, i.e. as a map fom K_p into $L_2(d\mu)$, where K_p is the Hilbert space with norm $|\ |_p$ and $N = \bigcap_p K_p$.

Proof: Setting $\eta(x) = \|(V(x)-1)u\|$ we get that η is sublinear i.e. $\eta(x+y) \leq \eta(x) + \eta(y)$. Since $u \in D(Px)$, we have

$$\lim_{t\downarrow 0} \frac{1}{t}\,\eta(tx) = p(x) = \|Px\cdot u\| . \tag{2.18}$$

By the linearity of Px in x, $p(x)$ is a seminorm on N. By the sublinearity of η we get $\eta(2x) \leq 2\eta(x)$ hence $2^{n+1}\,\eta\,(2^{-n-1}x) \geq 2^n\,\eta\,(2^{-n}x)$, so that

$$p(x) = \sup_n 2^n\,\eta(2^{-n}x) . \tag{2.19}$$

Now η is continuous on N, so that $p(x)$ is lower semicontinuous and, being a seminorm, it is then bounded in some neighborhood of 0 ([5], Ch.I,Sect.1, Th.1). This implies that $x \to Px\,u$ is continuous from N into $\mathcal{H}$. This proves the Proposition. The corollary follows from the proposition and theorem 5 of Ch I, section 3.5 of ref. [5]. □

It is now useful to exhibit the relation between the infinitesimal generator $i\,Px$ of translations in the direction x and a quantity $\beta(x)$ which, in the finite dimensional case $K = R^n$, reduces, as we shall see below, to the osmotic velocity or drift coefficient of the stochastic equation (2.14). In order to do this we introduce the following

Definition 2.5 A quasi invariant probability measure μ on N' is said to be strongly L_1 differentiable if the strong $L_1(d\mu)$ limit of $t^{-1}[\alpha(\xi,tx)-1]$ as $t\downarrow 0$ exists, for any given $x \in N$. We call then $\beta(\xi)x$ this limit, i.e.

$$\beta(\xi)x = s - L_1 - \lim_{t\downarrow 0} t^{-1}[\alpha(\xi,tx)-1] .$$

We have

Proposition 2.4 Let μ be any quasi invariant probability measure on N'. If μ has regular first order derivatives, then it is strongly L_1 differentiable and one has that the function $\Omega(\xi) \equiv 1$ is in the domain of Px, that $\beta(\xi)x$ exists and

$$2\,i\,Px\,\Omega(\xi) = \beta(\xi)x$$

for all $x \in N$.

Remark: This implies in particular that $x \to \beta(\xi)x$ is a linear map from N into $L_2(d\mu)$. It is natural to denote this map itself by $\beta(\xi)$.

Proof: That μ has regular first order derivatives is obviously equivalent with the condition that $\frac{1}{t}(\alpha^{\frac{1}{2}}(\xi,tx)-1)$ converges in $L_2(d\mu)$ as $t \to 0$. Now we have that

$$\frac{1}{t}(\alpha(\xi,tx)-1) = \frac{1}{t}(\alpha^{\frac{1}{2}}(\xi,tx)-1)(\alpha^{\frac{1}{2}}(\xi,tx)+1) \qquad (2.20)$$

and, by an easy consequence of the strong continuity of $V(tx)$, $\alpha^{\frac{1}{2}}(\xi,tx)$ converges to 1 in $L_2(d\mu)$ as $t \downarrow 0$. This then gives that the right hand side converges in L_1 . We observe from (2.20) that $i(Px\,\Omega)(\xi) = \frac{1}{2}\beta(\xi)x$, and this proves the proposition. □

Another observation on the quantity $\beta(\xi)x$ is the following

Proposition 2.5 If μ is a probability measure on N' with regular first order derivatives, then, for any fixed $\xi \in N'$, the map $x \to \beta(\xi)x$ is a linear continuous functional on N , hence there exists an element $\hat{\beta}(\xi)$ in N' such that

$$\langle \hat{\beta}(\xi), x \rangle = \beta(\xi)x$$

for all x in N . $\hat{\beta}(\xi)$ is thus a measurable map of N' into N'.

Remark: Since by the Remark following Proposition 2.4 the map $x \to \beta(\xi)x$ is a linear map from N into $L_2(d\mu)$, it is natural to identify $\beta(\xi)$ and $\hat{\beta}(\xi)$, and we shall do so in the following.

Proof: By the Corollary to Prop. 2.3 and by Prop. 2.4, $x \to \beta(\xi)x$ is continuous from K_p into $L_2(d\mu)$. Use then the Abstract Kernel Theorem (Theor.3, Ch.I, Sect.3 of ref. [5]). □

Remark: In the finite dimensional case $K = R^n$ we have

$\beta(\xi)x = \sum_{j=1}^{n} \beta_j(\xi)x_j$, where x_j , $i = 1,\ldots,n$ are the components of x and $\beta_j(\xi)$ is defined as $\beta(\xi)$ but with the translation tx in $\alpha(\xi,tx)$ replaced by the translation te_j , where e_j is the unit vector in the direction of the j-th axis. $\beta(\xi)$ is thus in this finite dimensional case, for fixed ξ , a vector with components $\beta_j(\xi)$, namely the vector $\frac{\nabla\rho(\xi)}{\rho(\xi)}$, where $\rho(\xi)$ is the density of the quasi invariant measure μ , i.e. $d\mu(\xi) = \rho(\xi)d\xi$. Since in the finite dimensional case $\beta(\xi)$ is, as remarked above, the osmotic velocity of the process of equation (2.14), we shall call $\beta(\xi)$ the osmotic velocity also in the infinite dimensional case. Also in this case our aim is to study the stochastic differential equation

$$d\xi(t) = \beta(\xi(t))dt + dw(t) , \tag{2.21}$$

where $\xi(t)$ is a stochastic process with values in N' . In this case $w(t)$ is understood as the Wiener process on N' given by the nuclear rigging $N \subset K \subset N'$. By this we mean that $w(t)$ is the time homogeneous Markov process with state space N' and with transition probability function $P_t(\xi,d\eta)$ determined by prescribing that the Fourier transform

$$\int_{N'} e^{i\langle x,\eta\rangle} P_t(0,d\eta)$$

be equal to $e^{-\frac{t}{2}(x,x)}$, for all $x \in N$. That this defines indeed a Markov process is easily verified, since the Chapman-Kolmogorov equations for $P_t(\xi,d\eta)$ are satisfied. Note that $w(t)$ is in fact the well known Wiener process studied by Gross, but for the fact that Gross prefers to study it relative to a Banach rigging $B \subset K \subset B'$. This is possible since $w(t)$ actually takes values in a dual Banach space B' such that $K \subset B' \subset N'$. For the work of Gross see the reference [9].

We shall now examine the possibility of constructing the process $\xi(t)$ of equation (2.21).

2.4 Dirichlet forms associated with quasi invariant measures.

We shall consider subspaces of the Banach space $C(N')$ of continuous bounded functions on N'. Let E be an orthogonal projection from K onto a finite dimensional subspace EK of N and let $e_1, e_2, \ldots, e_m$ be an orthonormal base in EK, where m is the dimension of EK. Then for any $u \in K$ we have

$$Eu = \sum_{i=1}^{m} (e_i, u) e_i \ . \qquad (2.22)$$

Since $e_i \in N$ we have from (2.22) that E extends by continuity to a continuous projection $N' \to N$ given by $\xi \to \sum_{i=1}^{m} \langle e_i, \xi \rangle e_i$. We shall denote this projection again by E. We see thus that any orthogonal projection on K with finite dimensional range in N extends continuously to a projection from N' into N. We shall say that a complex valued measurable function f defined on N' is finitely based (on EN') if there exists a finite dimensional subspace of N such that $f(\xi) = f(E\xi)$ for all $\xi \in N'$, where E is the projection from N' onto the finite dimensional subspace. Let $F^n(N')$ be the set for all functions on N' which are finitely based and n-times continuously differentiable on their base. For $f \in F^1$ we define the gradient ∇f in the obvious way, i.e. if E is such that $f(\xi) = f(E\xi)$ and EN' is finite dimensional in N, then $(\nabla f)(\xi)$ is a continuous map from N' into $(EK)^*$ and, since EK is self dual, we may consider ∇f as a map from N' into EK. For f and g in F^1 there is a common projection E of finite dimensional range in N such that f and g are both based on EN', and we then denote by $(\nabla\bar{f}\cdot\nabla g)(\xi)$ the inner product of $\nabla\bar{f}(\xi)$ and $\nabla g(\xi)$ in the natural complexification of EK. For any $f \in F^2$ with base E, Δf is defined in the natural way, so that $(\Delta f)(\xi) = \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2} \tilde{f}(x)$, where $\tilde{f}$ is the restriction of f to the m-dimensional space EN', $\{e_i\}$ is a base in $EN' \subset N$, and $x = \{x_i\}$, $x_i = \langle\xi,e_i\rangle$, $i = 1,\ldots,m$. Finally if $\beta(\xi)$ is the osmotic velocity to $\mu \in \mathcal{P}_1(N')$, then for any $f \in F^1$ with base EN', $\beta(\xi)\cdot\nabla f(\xi) = \sum_{i=1}^{m}\beta_i(\xi)\frac{\partial\tilde{f}}{\partial x_j}(x)$. We thus see that the operator H_{F^2} given by

$$(H_{F^2}f)(\xi) = -\Delta f(\xi) - \beta(\xi)\cdot\nabla f(\xi) \tag{2.23}$$

is well defined for all $f \in F^2$. We have the following

<u>Theorem 2.6</u> Let $\mu \in \mathcal{P}_1(N')$, then the Dirichlet form $D_\mu(f,g) = \int \nabla\bar{f}\cdot\nabla g \, d\mu$ is defined for all f,g in F^1 as a sesquilinear, non negative and closable form. The closure of the Dirichlet form is the form of a well defined self-adjoint nonnegative operator H in $L_2(d\mu)$, which coincides with H_{F^2} on F^2 and has the eigenvalue zero as the infimum of its spectrum, with eigenfunction $\Omega(\xi) \equiv 1$. H is the Friedrichs extension of H_{F^2}, thus the domain of $H^{\frac{1}{2}}$ is the domain of $\bar{D}_\mu$, where $\bar{D}_\mu$ is the closure of the form D_μ.

<u>Proof</u>: By the observations preceding the theorem we know that $D_\mu(f,g)$ for $f,g \in F^1$ and $H_{F^2}g$ for $g \in F^2$ are well defined. Since f,g are finitely based there exists a projection E with finite dimensional $EK \subset N$, such that f,g are based on it. Let $\{e_i\}$ be an orthonormal base in EK and set

$x_i = \langle \xi, e_i \rangle$ for all $\xi \in N'$, $i = 1,\ldots,m$. We have

$$\int_{N'} \nabla\overline{f}\cdot\nabla g \, d\mu = \sum_{i=1}^{m} \int \lim_{t\downarrow 0} \frac{1}{t}(\overline{f}(\xi + te_i) - \overline{f}(\xi))\cdot \frac{\partial g}{\partial x_i}(\xi) d\mu \, ,$$

which by dominated convergence and the quasi invariance of μ is equal to

$$\sum_{i=1}^{m} \lim_{t\downarrow 0} \int \overline{f}(\xi) \frac{1}{t}(\frac{\partial g}{\partial x_i}(\xi - te_i) - \frac{\partial g}{\partial x_i}(\xi))\alpha(\xi, -te_i) d\mu +$$

$$+ \sum_{i=1}^{m} \lim_{t\downarrow 0} \int \overline{f}(\xi)\frac{\partial g}{\partial x_i}(\xi) \frac{1}{t}(\alpha(\xi, -te_i) - 1) d\mu \, .$$

As $t\downarrow 0$, the first term converges to $-\int \overline{f}\cdot\Delta g \, d\mu$, by dominated convergence, and $\frac{1}{t}(\alpha(\xi, te_i) - 1)$ converges to $\beta(\xi)\cdot e_i$ strongly in L_1, by Prop. 2.4, hence the second term converges to $-\int \overline{f}\beta\cdot\nabla g \, d\mu$. We have thus proven that $D_\mu(f,g) = (f, H_{F^2} g)$ for all $f \in F^1$, $g \in F^2$, which shows in particular that the form D_μ is closable. The rest follows easily. □

Theorem 2.6 permits to associate to any quasi invariant probability measure on N', only restricted to have regular first order derivatives, a self-adjoint contraction semigroup e^{-tH}, where H is the unique self-adjoint operator associated with the closure of the Dirichlet form D_μ given by μ. In the next subsection we shall see that e^{-tH} is a Markov semigroup i.e. $e^{-tH} f \geq 0$ for all $f \geq 0$, $f \in L_2(d\mu)$, i.e. e^{-tH} is positivity preserving.

2.5. The Markov semigroups generated by the Dirichlet forms.

Theorem 2.7 Any quasi invariant probability measure on N' with regular first order derivatives gives rise to a Markov semigroup e^{-tH}, where H is the self-adjoint operator associated with the Dirichlet form of Theorem 2.6. H is the infinitesimal generator for a symmetric Markov process, with invariant measure μ. We call H the diffusion operator given by μ.

<u>Proof</u>: Let M be a finite dimensional subspace of N, with orthonormal base $e_1,\dots,e_m$ and let E be the orthogonal projection from K onto M. Define for any $f \in F^2$ based on M

$$H_M f = -\Delta_M f - \beta \cdot \nabla_M f \,, \qquad (2.24)$$

where $\nabla_M = E\nabla f$ and $\Delta_M = \nabla_M \cdot \nabla_M$. Then for any $f,g \in F^2$

$$(f, H_M g) = \sum_{i=1}^{m} \int e_i \cdot \nabla \bar{f} \; e_i \cdot \nabla g \; d\mu \,. \qquad (2.25)$$

The positive form on the right hand side is thus given by a symmetric operator, hence it is closable and its closure is the Friedrichs extension of H_M, which we denote again by H_M. Obviously $0 \le H_M \uparrow H$ as forms. Using then the theorem on convergence from below of symmetric semibounded forms (Theorem 3.13, Ch. VIII, Ref. [19],1)) we get strong resolvent convergence of H_M to H, hence by the semigroup convergence theorem (Theorem 2.16, Ch. IX, Ref. [19],1)) we have strong convergence of e^{-tH_M} to e^{-tH}, uniformly on finite t intervals, when $M \uparrow N$ through the net of finite dimensional subspaces. Thus to prove that e^{-tH} is positivity preserving it suffices to prove that e^{-tH_M} is positivity preserving. We have the direct decomposition $N' = M \oplus M^\perp$, where $M^\perp$ is the annihilator of M in N'. By the continuity of E, the map $(x,\eta) \to x \oplus \eta$, $x \in M$, $\eta \in M^\perp$ is one-to-one and bicontinuous, hence we may consider $d\mu$ as a measure $d\mu(x,\eta)$ on the product measure space $M \times M^\perp$. The quasi invariance of μ and the finite dimensionality of M yield $d\mu(\xi) = \rho(x|\eta)dx \, d\nu(\eta)$, where

$\rho(x|\eta)$ is, for almost every η, the quasi invariant measure on M obtained by conditioning μ with respect to $M^{\perp}$, and ν is the measure induced on $M^{\perp}$ by μ. The correspondence

$$f(\xi) \leftrightarrow \rho(x|\eta)^{\frac{1}{2}} f(x,\eta) \qquad (2.26)$$

gives a unitary correspondence between $L_2(d\mu)$ and $L_2(dx\times d\nu)$ so that, in the sense of direct integrals of Hilbert spaces,

$$\mathcal{H} = \int_{M^{\perp}} \mathcal{H}_\eta \, d\nu(\eta) \ . \qquad (2.27)$$

where $\mathcal{H}_\eta = L_2(R^n, \rho(x|\eta\, dx)$. The correspondence (2.26) takes the form $(f,H_M g)$ into a direct integrals of Dirichlet forms $D_{\mu(\cdot|\eta)}(f,g)$, where $d\mu(x|\eta) = \rho(x|\eta)dx$. Hence

$$H_M = \int_{M^{\perp}} H_\eta d\nu(\eta) \ ,$$

where H_η is the self-adjoint operator associated with the closure of the form $D_{\mu(\cdot|\eta)}(f,g)$. So we see that H_M generates a Markov semigroup if H_η does so, where H_η operates on $L_2(d\mu(\cdot|\eta))$, which is a L_2 space over the <u>finite</u> dimensional space R^n. We can now use results of Fukushima [22] to prove that e^{-tH_η} is indeed a Markov semigroup. By Theorem 3.2 of [22], p. 56 (points a),c)) it is sufficient to prove that $\bar{D}_{\mu(\cdot|\eta)}$ is a Markov symmetric form and by Theorem 3.3 of [22], p. 58, this is so whenever $D_{\mu(\cdot|\eta)}$ is a Markov symmetric form on $C^1(M)$. The latter is however easily verified.*

* For any $f \in C^1(M)$, $\delta > 0$ choose a $\varphi_\delta \in C^1(R)$, with $|\varphi_\delta(t)| \le t$, $-\delta \le \varphi_\delta(t) \le 1+\delta$ and $|\frac{d}{dt}\varphi_\delta(t)| \le 1$ for all $t \in R$ and $\varphi_\delta(t) = t$ for all $t \in [0,1]$. Then $\varphi_\delta \circ f \in C^1(R)$ and $D_{\eta(\cdot|\eta)}(\varphi_\delta(f),\varphi_\delta(f)) \le D_{\mu(\cdot|\eta)}(f,f)$.

Hence we have that H_η generates a Markov semigroup. Since (N',μ) is a regular probability space, an adaptation of Kolmogorov construction associates to the Markov semigroup e^{-tH} an homogeneous Markov process with state space N' and invariant measure μ having H as the infinitesimal generator of its transition probabilities. □

<u>Remark:</u> In [4] we show that the Markov process $\xi(t)$ given by the Markov semigroup e^{-tH} of Theorem 2.7 solves the stochastic differential equation

$$d\xi(t) = \beta(\xi(t))dt + dw(t) ,$$

in the sense of weak processes on N' , where $w(t)$ is the standard weak Wiener process on K .

2.6 Some remarks on ergodicity.

<u>Definition 2.7</u> A homogeneous Markov process $\eta(t)$ with state space some measure space (X,ν) is called ergodic if for any measurable sets A,B with $\nu(A) > 0$, $\nu(B) > 0$ there exists some $t \geq 0$ such that $\Pr\{\eta(0) \in A$ and $\eta(t) \in B\} > 0$.

<u>Theorem 2.8</u> Let $\eta(t)$ be a homogeneous Markov process with state space (X,ν) and invariant measure ν and suppose $\eta(t)$ is symmetric (i.e. such that the adjoint process $\eta(-t)$ is equivalent to it). Let H be the self-adjoint infinitesimal generator of the Markov semigroup e^{-tH}, $t \geq 0$, in $L_2(X,\nu)$ giving the transition probabilities of the process. Then the following statements are equivalent

1. η is ergodic.

2. e^{-tH} has ergodic kernel (i.e. for any $f,g \geq 0$, $f,g \not\equiv 0$ in $L_2(X,d\nu)$, we have $(f,e^{-tH}g) > 0$ for some $t \geq 0$).

3. e^{-tH} is positivity improving (i.e. $e^{-tH}f > 0$ for any $f \geq 0$, $f \in L_2(X,d\nu)$ and some $t \geq 0$).

4. there do not exist any bounded multiplication operators commuting with e^{-tH}.

5. Zero is a single eigenvalue at the infimum of the spectrum of H, and the corresponding eigenfunction may be taken to be identically equal to 1.

Proof: $2 \rightarrow 1$ is evident. We prove $1 \rightarrow 2$ ad absurdum. Suppose for some $f,g \geq 0$, $f,g \not\equiv 0$ we have $(f,e^{-tH}g) = 0$ for all $t \geq 0$. Since e^{-tH} is Markov, this implies $(\chi_{A_n}, e^{-tH}\chi_{B_m}) = 0$ for all positive integers n,m and all $t \geq 0$, where χ_{A_n}, χ_{B_m} are the characteristic functions of the sets $A_n \equiv \{x \in X | f(x) \geq \frac{1}{n}\}$, $B_m \equiv \{x \in X | g(x) \geq \frac{1}{m}\}$. But since $\nu(A_n) > 0$ and $\nu(B_m) > 0$ for some n,m, the vanishing of the scalar product contradicts 1, which then shows $1 \rightarrow 2$. The equivalence of 2 with 3 is proven in [23]. For the equivalence of 4 and 5 see e.g. [19] 2), Theorem 10.3 and for the one of 2 and 4 see e.g. [24]. □

The Theorem applies in particular to the case where η and H are the Markov process and its infinitesimal generator associated with a measure $\mu \in \mathcal{P}_1(N')$ and given by Theorems 2.6, 2,7. Note also that $F^1_{inv} \subset \mathcal{H}_0$, F^1_{inv} denoting the set of functions in F^1 which are invariant under translations by elements of N and $\mathcal{H}_0$ is the eigenspace to the eigenvalue zero of H.

Remark: Stronger results hold for a related operator $\hat{H}$ which coincides with H on the dense subset F^2 of $L_2(d\mu)$.* We shall here briefly mention some of these and for more details we refer to [4]. Let $C^1(N')$ be the subset of all measurable functions on N' which are such that for any $\xi \in N'$ and any $x \in N$ one has $f(\xi + tx)$ in C^1 as a function of t. For $f \in C^1(N')$ we define the gradient in the x direction, $(x\cdot\nabla f)(\cdot)$, by $(x\cdot\nabla f)(\xi) \equiv \frac{d}{dt} f(\xi + tx)|_{t=0}$. Let $\mu \in \mathcal{P}_1(N')$, then the adjoint of $x\cdot\nabla$ is $-x\nabla - \beta\cdot x$, whose domain contains $C^1(N')$, since $\beta\cdot x$ is also defined on $C^1(N')$, as one sees from $-\frac{1}{2}i\beta(\xi)\cdot xf(\xi) = (P\cdot xf)(\xi) + ix\cdot\nabla f(\xi)$. Thus $(x\cdot\nabla)$ is closable and we let now $x\cdot\nabla$ denote its closure. Let $\{e_i\}$, $i = 1,2,\dots,$ be an orthonormal base in K with $e_i \in N$. Define $(f,f)_\mu \equiv \sum_{i=1}^{\infty} \|e_i\cdot\nabla f\|_2^2$ and if $(f,f)_\mu < \infty$ denote by ∇f the element in $K \otimes L_2(d\mu)$ such that $\int \nabla\bar{f}\cdot\nabla f \, d\mu = (f,f)_\mu$. We easily see that the adjoint ∇^* of ∇ is densely defined, hence ∇ is closable and its closure, denoted by the same symbol, is a map from $L_2(d\mu)$ into $K \otimes L_2(d\mu)$ with domain $W_\mu^1 \equiv \{f \in L_2(d\mu) \mid (f,f)_\mu < \infty\}$.

Theorem 2.9 Let $\mu \in \mathcal{P}_1(N')$. Then $\hat{H} = \nabla^*\nabla$ is the unique self-adjoint operator associated with the closed form $(f,f)_\mu = \int \nabla\bar{f}\cdot\nabla f \, d\mu$ and for any $f \in F^2$ we have $\hat{H}f = Hf = -\Delta f - \beta\cdot\nabla f$ where H is the operator of Theorems 2.6, 2.7. $\hat{H}$ is non negative and the infimum of its spectrum is the eigenvalue zero with eigenfunction 1. $e^{-t\hat{H}}$, $t \geq 0$ is a Markov semigroup giving rise to a Markov process $\hat{\xi}$ on N' with invariant measure μ.

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* At the moment of writing it is an open question whether $\hat{H}$ coincides on its domain as a self-adjoint operator with H, i.e. whether F^2 is a core for $\hat{H}$ and H, for all $\mu \in \mathcal{P}_1(N')$.

Proof: $\hat{H}f = Hf$ for $f \in F^2$ follows from $(f,\hat{H}f) = (f,f)_\mu$. The Markov property of $e^{-t\hat{H}}$ is proven in the same way as for e^{-tH} in Theorem 2.7. □

Clearly Theorem 2.8 applies to the case where the process is $\hat{\xi}$ and the infinitesimal generator is $\hat{H}$. In addition we have here easily the following

Theorem 2.10 Let $\mu \in \mathcal{P}_1(N')$. Then $L_2^{inv}(d\mu) \subset \hat{\mathcal{H}}_0$, where $L_2^{inv}(d\mu) \equiv \{f \in L_2(d\mu) | f(\xi + x) = f(\xi) \text{ for all } x \in N\}$ and $\hat{\mathcal{H}}_0$ is the eigensubspace of $\mathcal{H} \equiv L_2(d\mu)$ to the eigenvalue zero of $\hat{H}$. In particular if zero is a simple eigenvalue of $\hat{H}$, then μ is ergodic and the representation of the Weyl commutation relations (U,V) given by μ (Theor. 2.1, 2.2) is irreducible. The N-ergodic decomposition of the Remark following Theor. 2.2 gives also the direct decomposition $\hat{H} = \int_Z \hat{H}_z \, dz$, where $\hat{H}_z$ is the self-adjoint operator associated with the closed form $(f,f)_{\mu_z}$. Moreover one has the time-ergodic decompositions

$$L_2(d\mu) = \int_V L_2(d\mu(\cdot|v))dv$$

$$\hat{H} = \int_V \hat{H}_v \, dv ,$$

where V is the Gelfand spectrum of the commutative C^* algebra of multiplication operators on $L_2(d\mu)$ which commute with all $e^{-t\hat{H}}$, dv the measure induced on V by μ. The measure $\mu(\cdot|v)$ is μ conditioned with respect to the σ-algebra generated by $L_\infty(V)$ and $\hat{H}_v$ is the self-adjoint operator associated with the closed form $(f,f)_{\mu(\cdot|v)}$. Zero is a simple eigenvalue of $\hat{H}_v$, the corresponding eigenfunction is positive almost everywhere. $\hat{\mathcal{H}}_0$ is the closure in $L_2(d\mu)$ of $L_\infty(V)$.

The proof is given in [4]. Moreover it is shown in [4] that the time ergodic decomposition is in general strictly finer than the N-ergodic one. An important case however where both decompositions are equivalent is the one where $\mu \in \mathcal{P}_1(N')$ is strictly positive in the sense that the densities $\rho(x|\eta)$ appearing in the proof of Theorem 2.7 are, for ν almost all η, bounded away from zero whenever x is in any compact of one dimensional subspaces M. A simple condition for this is given in the following

<u>Theorem 2.11</u> Let μ be in $\mathcal{P}_1(N')$ and be strictly positive. Then the N-ergodic and time ergodic decompositions of Theorem 2.10 coincide. In particular the Markov process of Theorem 2.9 is ergodic if and only if the measure μ is ergodic in the sense of Definitions 2.7 resp. 2.3. A sufficient condition for μ to be strictly positive is that 1 be an analytic vector for $P \cdot x$ for all $x \in N$.

<u>Proof</u>: See [4]. □

<u>Remark</u>: In [4] other sufficient conditions for μ to be strictly positive are given. Note that from Theorem 2.11 and Theorems 2.8 and 2.2 we have e.g. $\hat{\xi}$ ergodic <-> μ ergodic <-> unique ground state for $\hat{H}$ <-> (U,V) irreducible.

3. Perturbations of symmetric diffusion processes

Let $N \subset K \subset N'$ be a real nuclear rigging of the real separable Hilbert space K and let $\mu \in \mathcal{P}_1(N')$ in the notation of Definition 2.4. Let H be the infinitesimal generator in $L_2(d\mu)$ for the corresponding diffusion process, given by Theorems 2.6, 2.7. Let $V(\xi)$ be a real measurable function on N' such that

$$H_1 = H + V \tag{3.1}$$

is essentially self adjoint and bounded below. Consider now for $k < l$

$$H_1^{k,l} = H + V^{k,l} \tag{3.2}$$

where

$$V^{k,l}(\xi) = \begin{cases} k & \text{if } V(\xi) < k \\ V(\xi) & \text{if } k \leq V(\xi) \leq l \\ l & \text{if } V(\xi) > l . \end{cases} \tag{3.3}$$

Using well known theorems on monotone convergence of symmetric semibounded forms (Theor. 3.13, 3.11, ch. VIII, Ref. [19],1)) we get

$$\underset{k\to-\infty}{\text{s-lim}} \ \underset{l\to+\infty}{\text{s-lim}} \ e^{-tH_1^{k,l}} = e^{-tH_1} . \tag{3.4}$$

But e^{-tH} is positivity preserving and by Trotter's product formula

$$e^{-tH_1^{k,l}} = \text{st} \lim_{n\to\infty} \left[e^{-\frac{t}{n}H} e^{-\frac{t}{n}V^{k,l}} \right]^n \tag{3.5}$$

we get that $e^{-tH_1^{k,l}}$ is also positivity preserving. Hence, by (3.4) and (3.5), e^{-tH_1} is positivity preserving. We have thus the following theorem.

Theorem 3.1

Let H be the diffusion operator given by μ according to Theorem 2.7 and let V be a measurable real function on N', where $N \subset K \subset N'$ is the nuclear rigging. If

$$H_1 = H + V$$

is essentially self adjoint and bounded from below, then e^{-tH_1} is positivity preserving. □

Note that H_1 is assumed to be bounded below, but contrary to H it need not have any eigenvectors.

Remark 3.1 Theorem 3.1 holds in the same way for the case where H is replaced by $\hat{H}$, $\hat{H}$ being the self-adjoint operator given by the form $(f,f)_\mu$ in Theorem 2.9 and the assumption being that $\hat{H}_1 = \hat{H} + V$ is essentially self-adjoint and bounded from below. Under the N-ergodic decomposition of Theorem 2.10 V decomposes directly as $V = \int V_z dz$ and we have then $\hat{H}_1 = \int \hat{H}_{1z}\, dz$ with $\hat{H}_{1z} = \hat{H}_z + V_z$ essentially self-adjoint. Moreover if μ is strictly positive, then by Theorem 2.11 we have that the N-ergodic decomposition coincides with the time-ergodic decomposition induced by $\hat{\xi}$. In particular the eigenvalue zero of $\hat{H}$ has multiplicity equal to the number of irreducible components in the representation of the canonical commutation relations given by μ and this multiplicity is also the same as the one of the infimum of the spectrum of $\hat{H}_1$, if this is an eigenvalue.

Theorem 3.2 Let H, H_1, V be as in Theorem 3.1. Assume that zero is a simple eigenvalue of H. If there is an eigenvalue E_1 of H_1 such that $H_1 \geq E_1$ and $H_1 - V$ is essentially self-adjoint, then E_1 is a simple eigenvalue of H_1. Moreover we may take the correspondent eigenfunction to be positive almost everywhere.

Proof: By Theorem 3.1 $e^{-t(H_1-E_1)}$ is a Markov semigroup. By Theorem 2.8, points 1 and 5, we have that if, ad absurdum, zero were not a simple eigenvalue of H_1-E_1, then the Markov process generated by H_1-E_1 would not be ergodic, hence $(\chi_A, e^{-t(H_1-E_1)}\chi_B) = 0$ for all $t \geq 0$ and the characteristic functions χ of some sets A,B with $0 < \mu(A) < 1$, $0 < \mu(B) < 1$. Since $e^{-t(H_1-E_1)}$ is Markov, χ_A, χ_B are projections in $L_2(d\mu)$ onto orthogonal subspaces which reduce H_1-E_1. In particular χ_A commutes with e^{itH_1} so by Trotter's product formula χ_A commutes with e^{itH}, hence χ_A is an eigenfunction to the eigenvalue zero of H and since $\chi_A \not\equiv 1$ this contradicts the assumption that zero is a single eigenvalue of H. Hence the ad absurdum assumption is untenable, which proves the Theorem. □

Remark 3.2 The same Theorems holds for the case where H, H_1 are replaced by $\hat{H}$, $\hat{H}_1$. Moreover if μ is strictly positive, we have from the preceding Remark 3.1 that the general case where zero is not a simple eigenvalue of $\hat{H}$ can be reduced to the case where it is a simple eigenvalue by using the ergodic decomposition.

Consider now the operator H of Theorem 2.6. We have for any $f \in F^2$

$$(Hf)(\xi) = -(\Delta f)(\xi) - \beta(\xi)\cdot\nabla f(\xi) \tag{3.6}$$

Assume again that zero is a simple eigenvalue of H and that $H_1 = H+V$ has an eigenvalue E_1 such that $H_1 \geq E_1$, then the corresponding eigenfunction φ of H_1 satisfies the equation

$$(\varphi,(V-E_1)f) = (\varphi,\Delta f + \beta\cdot\nabla f) \tag{3.7}$$

for all $f \in F^2$, where β is the osmotic velocity for H.
Let us normalize φ such that $\varphi > 0$ and $\int\varphi^2 d\mu = 1$. Since φ is positive almost everywhere we also have that

$$V-E_1 = \frac{\Delta\varphi}{\varphi} + \beta\cdot\frac{\nabla\varphi}{\varphi}, \tag{3.8}$$

which gives the relation in the weak sense between the function $V-E_1$ and the eigenfunction φ. Since V is a multiplication by a measureable function in $L_2(d\mu)$ we have $F^2\Omega \subset D(H)\cap D(V) \subset D(H_1)$ and for any $F \in F^2$,

$$[H_1,f] = [H,f] = -2\nabla f\cdot\nabla - \Delta f \tag{3.9}$$

on the domain $F^2\Omega$. Let us now assume that $H_1 = H+V$ is essentially self adjoint and that $\Omega_1 = \varphi\Omega$ is in $D(H)$ as well as in $D(V)$, and that the measure $d\mu_1 = \varphi^2 d\mu$ has regular first order derivatives with corresponding osmotic velocity β_1.
Let $f \in F^2$ then $f\Omega_1$ is in $D(V)$ since $f\Omega_1 \in D(V)$ is equivalent with $Vf\varphi \in L_2(d\mu)$. Now f is μ-essentially bounded and by assumption $V\varphi \in L_2(d\mu)$ so that $f\Omega_1 \in D(V)$. Moreover by (3.9) we have

$$[H,f]\Omega_1 = -\Delta f\Omega_1 - 2\nabla f\cdot\nabla\Omega_1 . \tag{3.10}$$

That is

$$[H,f]\Omega_1 = -\Delta f\Omega_1 - \beta_1\cdot\nabla f\Omega_1 \tag{3.11}$$

and $-\Delta f - \beta_1\cdot\nabla f \in L_2(d\mu_1)$, since the components of β_1 are in $L_2(d\mu_1)$ by assumption. Hence, since $\Omega_1 \in D(H)$ so that $fH\Omega_1$ is well defined, we have that $f\Omega_1 \in D(H)$. Because of $D(H_1) \supset D(H) \cap D(V)$ we have therefore that $f\Omega_1 \in D(H_1)$. But then again by (3.9) and (3.11) we have

$$H_1 f\Omega_1 = [H_1, f]\Omega_1 + E_1 f\Omega_1 \tag{3.12}$$

i.e.

$$H_1 f\Omega_1 = (-\Delta f + E_1 f)\Omega_1 - \beta_1 \cdot \nabla f\Omega_1 \ . \tag{3.13}$$

Hence $H_1 - E_1$ coincides on $F^2\Omega_1$ with the unique diffusion operator given by μ_1 . We have therefore the following

<u>Theorem 3.4</u>

Let $\mu \in \mathcal{P}_1(N')$ and let H have zero as a simple eigenvalue. Let V be measurable and in $L_2(d\mu)$ such that $H_1 = H+V$ is essentially self adjoint with an eigenvalue E_1 such that $H_1 \geq E_1$. Then the corresponding eigenfunction φ is positive μ - a.e. and $d\mu_1 = \varphi^2 d\mu$ is quasi invariant. If moreover μ_1 is in $\mathcal{P}_1(N')$ and $\Omega_1 = \varphi\Omega$ is in $D(H) \cap D(V)$, then $F^2\Omega_1 \subset D(H) \cap D(V)$ and on $F^2\Omega_1$ we have that $H_1 - E_1$ coincides with the diffusion operator given by μ_1 . □

<u>Remark 3.3</u> The same Theorem holds also for the case where the operator H is replaced by $\hat{H}$, as given by Theorem 2.9, so that accordingly H_1 is replaced by $\hat{H}_1 = \hat{H} + V$. In this case by "the diffusion operator given by μ_1" we have to understand the self-adjoint operator $\hat{H}_1 - E_1$ given by the form $(f,f)_{\mu_1}$ according to Theorem 2.9. We recall that then $e^{-t(\hat{H}_1 - E_1)}$ is also a Markov semigroup.

We have also the following

<u>Theorem 3.5</u> Let the assumptions be as in the previous theorem. If in addition $H_1 = H + V$ is self-adjoint i.e. $D(H_1) = D(H) \cap D(V)$, then $H_1 - E_1$ is the diffusion operator given by μ_1 .

Proof: By the previous theorem we have that if H' is the diffusion operator generated by μ_1, then H' coincides with H_1-E_1 on $F^2\Omega_1$. Hence $H' = H+V-E_1$ on $F^2\Omega_1$, so by definition H' is the Friedrichs extension of $H+V-E_1$ on $F^2\Omega_1$. Hence the domain of $H'^{\frac{1}{2}}$ is exactly the elements for which the form $(f\Omega_1,(H+V-E_1)f\Omega_1)$ makes sense as continued from $F^2\Omega_1$. From this it follows that

$$D(H'^{\frac{1}{2}}) \supseteq D(H) \cap D(V) . \qquad (3.14)$$

Now if $H_1 = H+V$ is self adjoint we have that

$$D(H_1) = D(H) \cap D(V) . \qquad (3.15)$$

Therefore

$$D(H_1) \subseteq D(H'^{\frac{1}{2}}) . \qquad (3.16)$$

Now by a well known theorem ([19],1), Ch.VI.Th.2.11) we have that among all lower bounded self adjoint extensions of the operator H_1-E_1 restricted to $F^2\Omega_1$ only the Friedrichs extension has domain contained in the domain of the form i.e. in $D(H'^{\frac{1}{2}})$. Hence by (3.16) H_1 is the Friedrichs extension. This proves the theorem. □

Remark 3.4 The above Theorems and Remarks lead us to consider another type of perturbation of symmetric diffusion processes. Let $\mu \in \mathcal{P}_1(N')$ and let $\rho(\xi) > 0$ be a measurable function that is positive μ-almost everywhere such that $d\mu' = \rho d\mu$ is a probability measure. Then μ' is obviously quasi invariant, and let us now further assume that μ' is in $\mathcal{P}_1(N')$. We get then that the osmotic velocity β' for μ' is given in terms of the osmotic velocity β of μ by

$$\beta'(\xi)x = x\cdot\nabla \ln\rho + \beta(\xi)\cdot x , \qquad (3.17)$$

and the assumption $\mu' \in \mathcal{P}_1(N')$ means then $\beta'(\xi)x \in L_2(d\mu')$.

We see this is the case if for instance $\nabla\rho^{\frac{1}{2}}$ as well as $\rho^{\frac{1}{2}}\beta(\xi)\cdot x$ are both in $L_2(d\mu)$. For such perturbations we have the following theorem.

Theorem 3.6

Let μ and μ' be two equivalent quasi invariant measures in $\mathcal{P}_1(N')$ which are strictly positive. Let $\hat{H}$ and $\hat{H}'$ be the corresponding self-adjoint operators given by Theorem 2.9, then zero is an eigenvector of the same multiplicity for both operators. In fact there is a natural one-to-one isomorphism of the respective eigenspaces corresponding to the eigenvalue zero.

Proof: By Theorem 2.11 we have that the eigenspace for the eigenvalue zero is in one-to-one correspondence with the set of functions in $L_2(d\mu)$ which are invariant under translations by elements in N . Since by assumption μ and μ' are equivalent, there is a natural one-to-one isomorphism between $L_2(d\mu)$ and $L_2(d\mu')$, which takes N-invariant functions of $L_2(d\mu)$ into N-invariant functions of $L_2(d\mu')$. This isomorphism then induces a one-to-one isomorphism of the eigenspaces of $\hat{H}$ and $\hat{H}'$ to the eigenvalue zero. This proves the theorem. □

Let now μ, μ_n , n = 1,2,..., be arbitrary quasi invariant measures in $\mathcal{P}_1(N')$ and let H_μ, H_{μ_n} be the corresponding diffusion operators , and $\xi_\mu(t)$, $\xi_{\mu_n}(t)$ the corresponding diffusion processes, as given by Theorems 2.6, 2.7. If the $\mu_n \in \mathcal{P}_1(N')$ converge weakly to some measure μ , then $H_{\mu_n} \to H_\mu$ in the sense that for any f and $g \in F^2$ we have that $(f\Omega_n, H_{\mu_n} g\Omega_n) \to (f\Omega, H_\mu g\Omega)$. We do not know however whether the $\xi_{\mu_n}(t)$ converge weakly to $\xi_\mu(t)$, but we shall see that if the $\xi_{\mu_n}(t)$ converge weakly to some Markov process, then under slight regularity conditions the infinitesimal generator of this Markov process coincides on a dense domain with the infinitesimal generator H_μ. In fact we have the following theorem.

<u>Theorem 3.7</u> Let $\mu_n \in \mathcal{P}_1(N')$ and suppose that, as $n \to \infty$, μ_n converges weakly to a measure $\mu \in \mathcal{P}_1(N')$. Then for all f and g in F^2 we have that

$$(f\Omega_{\mu_n}, H_{\mu_n} g\Omega_{\mu_n}) \to (f\Omega_\mu, H_\mu g\Omega_\mu) .$$

If moreover the osmotic velocities $\beta_n(\xi)$ of μ_n have components uniformly bounded in L_2, i.e. for any $x \in N$ there is a $c_x > 0$ independent of n such that

$$\int |\beta_n(\xi)\cdot x|^2 d\mu_n(\xi) \le c_x^2$$

then, for any f and g in F^2, the expectation $E(f(\xi_{\mu_n}(0))g(\xi_{\mu_n}(t)))$ has a uniformly bounded second derivative with respect to t. If moreover the process $\xi_{\mu_n}(t)$ converges weakly to some process $\eta(t)$, in the sense that the joint distribution measure of $\{\xi_{\mu_n}(t_1),\ldots,\xi_{\mu_n}(t_k)\}$ converges weakly to that of $\{\eta(t_1),\ldots,\eta(t_k)\}$ for any k and any $t_1 \le \ldots \le t_k$, then $E[(f(\eta(0))g(\eta(t))]$ is a twice differentiable function of t and

$$-\frac{d}{dt} E[f(\eta(0))g(\eta(t))/_{t=0} = (f\Omega_\mu, H_\mu g\Omega_\mu)$$

for any f and g in $F^2(N')$. In particular if $\eta(t)$ is a Markov process, then $H_\eta = H_\mu$ on F^2, where H_η is the infinitesimal generator of η.

Proof. Since

$$(f\Omega_{\mu_n}, H_{\mu_n} g\Omega_{\mu_n}) = \int_{N'} \nabla\bar{f}\cdot\nabla g \, d\mu_n \tag{3.18}$$

the first convergence is obvious. Now, for $f \in F^2$ we have that $f\Omega_n$ is in $D(H_{\mu_n})$ and

$$H_{\mu_n} f\Omega_{\mu_n} = (-\Delta f - \beta_n \nabla f)\Omega_{\mu_n} \; . \tag{3.19}$$

By the assumption on f there is a orthogonal projection P_E of finite dimensional range $E \subset N$ such that $f(\xi) = f(P_E\xi)$. We then have

$$\|H_{\mu_n} f\Omega_{\mu_n}\| \leq \|f\|_2 + \sum_i c_i \|f\|_1 \tag{3.20}$$

where $c_i = c_{e_i}$ are constants and $e_1, \ldots, e_k$ is an orthonormal base of E in K and $\|f\|_2$ and $\|f\|_1$ are the $\dot{F}^2$ and F^1 norms of f respectively, the norm in F^n being the natural one induced by the C^n norm on the bases. We recall that F^n has been defined in Subsect. in 2.4. We see that the estimate (3.20) is independent of n , so that

$$(f\Omega_{\mu_n}, e^{-tH_{\mu_n}} g\Omega_{\mu_n}) = E[f(\xi_{\mu_n}(0))g(\xi_{\mu_n}(t))] \tag{3.21}$$

is continuously twice differentiable with uniformly bounded second derivative

$$(H_{\mu_n} f\Omega_{\mu_n}, e^{-tH_{\mu_n}} H_{\mu_n} g\Omega_{\mu_n}) \; . \tag{3.22}$$

If $\xi_{\mu_n}(t)$ converges weakly, we have in particular that (3.21) converges and the limit is $E[f(\eta(0))g(\eta(t))]$. Since the second derivatives are uniformly bounded the first derivatives

$$- (H_{\mu_n} f\Omega_{\mu_n}, e^{-tH_{\mu_n}} g\Omega_{\mu_n}) \tag{3.23}$$

converge uniformly to the first derivative of the limit. This gives us then that

$$- \frac{d}{dt} E[f(\eta(0))g(\eta(t))]/_{t=0} = (f\Omega_\mu, H_\mu g\Omega_\mu) \ . \tag{3.24}$$

Now assume that $\eta(t)$ is a Markov process. Then by the convergence of the processes $\xi_{\mu_n} \to \eta$ and their invariant measures $\mu_n \to \mu$ we see that $\eta(t)$ is homogeneous with invariant measure μ, and since the ξ_{μ_n} are symmetric under time reflection so is η. Hence the infinitesimal generator H_η of η is a positive self-adjoint operator in $L_2(d\mu)$ with $\Omega(\cdot) = 1$ as an eigenfunction of eigenvalue zero for H_η. Thus

$$E(f(\eta(0))g(\eta(t))) = (f\Omega, e^{-tH_\eta} g\Omega) \ . \tag{3.25}$$

From (3.24) we then get that $H_\eta = H_\mu$ on F^2. □

4. The Euclidean Markov fields as diffusion processes

The free Euclidean Markov field in $d+1$ dimensions is the generalized random field $\xi(x)$ on R^{d+1} such that

$$E\left[e^{i\int \xi(x)\psi(x)dx}\right] = e^{-\frac{1}{2}(\psi,\psi)_{-1}} \tag{4.1}$$

where

$$(\psi,\psi)_{-1} = \int_{R^{d+1}} (p^2+m^2)^{-1}|\hat{\psi}(p)|^2 dp \tag{4.2}$$

and

$$\hat{\psi}(p) = (2\pi)^{-\frac{d+1}{2}}\int e^{-ipx}\,\psi(x)dx\ , \tag{4.3}$$

and $m \geq 0$ is a constant called the mass of the free Euclidean Markov field. If $d = 0$ or 1 we have to take $m > 0$ in order for (4.2) to be well defined. The right hand side of (4.1) is obviously a continuous positive definite function on the real nuclear Schwartz space $S(R^{d+1})$ so that (4.1) defines a measure on its dual $S'(R^{d+1})$, i.e. the space of tempered distributions on R^{d+1}. Hence the generalized random field $\xi(x)$ is a random field of tempered distributions. It is well known that $\xi(x)$ is a Markov field, but we shall not need this property here.

Let $\varphi \in S(R^d)$, then $(\varphi\otimes\delta_\tau)(\vec{x},t) = \varphi(\vec{x})\cdot\delta(t-\tau)$ is in the Sobolev space $\mathcal{H}_{-1}$, in fact

$$(\varphi\otimes\delta_\tau,\varphi\otimes\delta_\tau)_{-1} = \tfrac{1}{2}(\varphi,\varphi)_{-\frac{1}{2}}\ , \tag{4.4}$$

where

$$(\varphi,\varphi)_{-\frac{1}{2}} = \int_{R^d} (\vec{p}^2+m^2)^{-\frac{1}{2}}\ |\hat{\varphi}(\vec{p})|^2\ d\vec{p} \tag{4.5}$$

with

$$\hat{\varphi}(\vec{p}) = (2\pi)^{-\frac{d}{2}}\int e^{-i\vec{p}\vec{x}}\ \varphi(\vec{x})d\vec{x}\ .$$

From (4.1) we get that

$$E\left[e^{i\int \xi(\vec{x},t)\varphi(\vec{x})d\vec{x}}\right] = e^{-\frac{1}{4}(\varphi,\varphi)_{-\frac{1}{2}}} . \tag{4.6}$$

Hence since the right hand side of (4.6) is a positive definite continuous function on the real nuclear space $S(R^d)$ we have that the conditional expectation of the measure with respect to the σ-algebra generated by functions of the form $\langle \xi, \varphi \otimes \delta_\tau \rangle$ exists and defines a measure on $S'(R^d)$. The corresponding random variable with values in $S'(R^d)$ we have already denoted by $\xi(\vec{x},t)$. Hence $t \to \xi(\vec{x},t)$ is a stochastic process with values in $S'(R^d)$. Let now μ_o be the probability measure on $S'(R^d)$ whose Fourier transform is given by (4.6), i.e.,

$$\int e^{i\langle \xi,\varphi\rangle} d\mu_o(\xi) = e^{-\frac{1}{4}(\varphi,\varphi)_{-\frac{1}{2}}} \tag{4.7}$$

where $\langle \xi,\varphi\rangle$ is the dualization between $S'(R^d)$ and $S(R^d)$. μ_o is then a Gaussian measure on $S'(R^d)$ and we see easely that it is quasi invariant with respect to translations from $S(R^d)$, in fact if

$$\alpha(\xi,\varphi) = \frac{d\mu_o(\xi+\varphi)}{d\mu_o(\xi)} \tag{4.8}$$

then

$$\alpha(\xi,\varphi) = e^{-(\varphi,\varphi)_{\frac{1}{2}}} e^{-2\langle \omega\varphi,\xi\rangle} \tag{4.9}$$

where

$$\widehat{\omega\varphi}(\vec{p}) = (\vec{p}^2+m^2)^{\frac{1}{2}} \hat{\varphi}(\vec{p}) \tag{4.10}$$

and

$$(\varphi,\varphi)_{\frac{1}{2}} = \langle \varphi, \omega\varphi\rangle .$$

From (4.9) it easely follows that μ_o has regular first order derivatives and that the osmotic velocity $\beta(\xi)$ is given by

$$\beta(\xi)\cdot\varphi = 2\langle \omega\varphi, \xi\rangle . \tag{4.11}$$

which is obviously in $L_2(d\mu)$.

It is well known that $t \to \xi(x,t)$ is a Markov process in $S'(R^d)$: We shall see now that this process is the diffusion process given by the nuclear rigging

$$S(R^d) \subset L_2(R^d) \subset S'(R^d) \tag{4.12}$$

and the quasi invariant measure μ_o with regular first order derivatives in the sense of theorem 2.7. We formulate this in the following theorem

Theorem 4.0

Consider the nuclear rigging

$$S(R^d) \subset L_2(R^d) \subset S'(R^d)$$

and the measure μ_o on $S'(R^d)$ given by

$$\int e^{i\langle\xi,\varphi\rangle} d\mu_o(\xi) = e^{-\frac{1}{4}(\varphi,\varphi)_{-\frac{1}{2}}} .$$

Then μ_o is quasi invariant with regular first order derivatives and the diffusion process given by μ_o and the nuclear rigging by theorem 2.7 is the free Euclidean Markov field in d+1 dimensions.

Proof:

Since the free Euclidean Markov field induces a Markov process $t \to \xi(\vec{x},t)$ on $S'(R^d)$, we have only to show that this process has as infinitesimal generator the diffusion operator given by theorem 2.6. By (4.1) we have that

$$E\left[e^{-i\int \xi(\vec{x},0)\varphi_1(\vec{x})d\vec{x}+i\int \xi(\vec{x},t)\varphi_2(\vec{x})d\vec{x}}\right]$$
$$= e^{-\frac{1}{4}[(\varphi_1,\varphi_1)_{-\frac{1}{2}}+(\varphi_2,\varphi_2)_{-\frac{1}{2}}]}\cdot e^{\frac{1}{2}(\varphi_1,e^{-t\omega}\varphi_2)_{-\frac{1}{2}}} \tag{4.13}$$

where

$$(\varphi_1, e^{-t\omega}\ \varphi_2)_{-\frac{1}{2}} = \int \frac{e^{-t\omega}}{\omega}\ \bar{\hat{\varphi}}_1(\vec{p})\hat{\varphi}_2(\vec{p})d\vec{p} \tag{4.14}$$

and $\omega(\vec{p}) = \sqrt{\vec{p}^2+m^2}$. Taking the derivative of (4.13) with respect to t at $t = 0$ we get

$$-\frac{d}{dt}\int e^{-i\langle\xi(0),\varphi_1\rangle}\ e^{i\langle\xi(t),\varphi_2\rangle}\ d\mu(\xi)\ \Big|\ t = 0$$
$$= \langle\varphi_1,\varphi_2\rangle \int e^{-i\langle\xi(0),\varphi_1\rangle}\ e^{i\langle\xi(0),\varphi_2\rangle}\ d\mu(\xi)\ . \tag{4.15}$$

From this it follows that for f and g in $F^1(S'(R^d))$ we have that

$$-\frac{d}{dt}\int \bar{f}(\xi(0))g(\xi(t))d\mu(\xi) = \int \overline{\nabla f(\xi(0))}\cdot\nabla g(\xi(0))d\mu(\xi), \tag{4.16}$$

which proves that the infinitesimal generators coincide on F^2 . Moreover let e^{-tH_o} be the semigroup generated by the free Euclidean Markov field, then we have the following well known formula

$$e^{-tH_o}{:}\,e^{i\langle\xi,\varphi\rangle}{:} = {:}\,e^{\,i\langle\xi,e^{-t\omega}\varphi\rangle}{:} \tag{4.17}$$

where

$${:}\,e^{i\langle\xi,\varphi\rangle}{:} = e^{\frac{1}{2}(\varphi,\varphi)_{-\frac{1}{2}}}\ e^{i\langle\xi,\varphi\rangle}\ . \tag{4.18}$$

Hence the linear span of $e^{i\langle\xi,\varphi\rangle}$ for $\varphi \in S(R^d)$ is invariant under the semigroup e^{-tH_o} and it is obviously dense in $L_2(d\mu)$, therefore it is a core for the infinitesimal generator H_o. Now this core is obviously contained in F^2 which proves the theorem. □

We shall now consider the case of one space dimension, i.e. $d=1$, where the perturbations of the free Euclidean Markov field by local interactions of different types have been intensively studied. For simplicity we shall here restrict our attention to the polynomial interactions. So let $p(s)$ be a real polynomial for one real variable s such that $p(s)$ is bounded from below. We recall (see e.g. [25]) that the Wick power $:\xi^n:(h)=\int_R :\xi(x)^n: h(x)dx$, with $h \in L_2(R)$, is defined as the unique element in $P^{(n)}$ such that for all $\varphi_i \in S(R)$,

$$(:\xi^n:(h), \langle\varphi_1,\xi\rangle\ldots\langle\varphi_n\xi\rangle) = n! \int\ldots\int \prod_{i=1}^{n} G(y_i-x)\varphi_i(y_i)h(x)dy_i dx \,, \tag{4.19}$$

where

$$G(x) = \tfrac{1}{2}\int_R (p^2+m^2)^{-\frac{1}{2}} e^{ipx}\, dp \tag{4.20}$$

and $P^{(n)} = P^{(\le n)} \ominus P^{(\le n-1)}$, $P^{(\le k)}$ being the closed subspace of $L_2(d\mu_o)$ generated by polynomials of degree at most k on $S'(R)$.

Now if

$$p(s) = \sum_{k=o}^{2n} a_k s^k \tag{4.21}$$

we define as usual

$$:p:(h) = \sum_{k=o}^{2n} a_k :\xi^k:(h) \,. \tag{4.22}$$

Since the sum is an orthogonal one, we easily compute

$$\|:p:(h)\|_2^2 = \sum_{k=o}^{2n} a_k^2\, k! \iint G(x-y)^k h(x)h(y)dxdy \,. \tag{4.23}$$

Let now H_o be the diffusion operator generated by μ_o and the real rigging $S(R) \subset L_2(R) \subset S'(R)$. We have seen that H_o is the infinitesimal generator of the Markov process given by the free Euclidean Markov field. Let $V_1(\xi)$ be the real function in $L_2(d\mu_o)$ given by

$$V_1 = :p:(\chi_1) , \tag{4.24}$$

where χ_1 is the characteristic function for $[-1,1]$. It is well known, see for instance ref. [14] 3), that

$$H_1 = H_o + V_1 \tag{4.25}$$

is essentially self adjoint and bounded below and has an isolated simple eigenvalue E_1 such that $H_1 \geq E_1$. The corresponding eigenfunction $g_1(\xi)$ may be chosen positive μ_o - almost everywhere. The measure

$$d\mu_1 = g_1^2 \, d\mu_o \tag{4.26}$$

is therefore equivalent with μ_o, hence quasi invariant with respect to translation in S. Now let $\Omega_1 = g_1\Omega_o$, where $\Omega_o(\cdot) \equiv 1$ in $L_2(d\mu_o)$.

<u>Lemma 4.1</u>

Let $\varphi \in S(R)$ and $P\varphi$ the infinitesimal generator for the one parameter unitary group of translations by $t\varphi$ in $L_2(d\mu_o)$. Then $i[P\varphi,H_1]$ is a densely defined operator whose closure is given by

$$\overline{i[P\varphi,H_1]} = :p':(\chi_1\cdot\varphi) + \langle \xi,(-\Delta+m^2)\varphi\rangle ,$$

where p' is the derivative of p and χ_1 the characteristic function for $[-1,1]$.

<u>Proof</u>: The proof follows immediately from the fact that $i[P\varphi,H_1]$ is the derivative at $t = 0$ of $e^{itP\varphi}H_1e^{-itP\varphi} = H_1^{t\varphi}$, where

$$H_1^{\varphi} = H_o + \langle \xi,(-\Delta+m^2)\varphi\rangle + \tfrac{1}{2}\langle \varphi,(-\Delta+m^2)\varphi\rangle + :p_\varphi:(\chi_1) \tag{4.27}$$

and

$$:p_\varphi:(\chi_1) = \sum_{k=1}^{2n} a_k \int_{-1}^{1} :(\xi+\varphi)^k(x):dx , \tag{4.28}$$

with

$$\int_{-1}^{1} :(\xi+\varphi)^n(x): \, dx = \sum_{j=1}^{n} \binom{n}{j} :\xi^j:(\varphi^{n-j}\chi_1) . \quad \square \tag{4.29}$$

Theorem 4.1

μ_l is a quasi invariant measure on $S'(R)$ which has regular first order derivatives. Moreover the components of the corresponding osmotic velocity β_l have $L_2(d\mu_l)$ norms which are bounded uniformly in l if the coefficients of p are small enough.

Proof: Let $\varphi \in S(R)$, then $\beta_l \cdot \varphi$ is equal to twice the derivative of $e^{itP\varphi}\Omega_l$ at $t = 0$, if it exists, so that $\beta_l \varphi$ is in $L_2(d\mu_l)$ iff $\Omega_l \in D(P\varphi)$ and

$$\beta_l \cdot \varphi = 2iP\varphi\Omega_l \ . \qquad (4.30)$$

Now

$$P\varphi\Omega_l = -(H_l-E_l)^{-1}[P\varphi, H_l-E_l]\Omega_l \qquad (4.31)$$

so that

$$P\varphi\Omega_l = -\frac{H_l-E_l+C}{H_l-E_l}(H_l-E_l+C)^{-1}[P\varphi, H_l]\Omega_l \ . \qquad (4.32)$$

But $(H_l-E_l+C)^{-1}[P\varphi, H_l-E_l]\Omega_l$ is in the range of H_l-E_l, hence orthogonal to Ω_l. Now, for fixed $C > 0$, $(H_l-E_l+C)(H_l-E_l)^{-1}$ is bounded in norm on the complement of Ω_l by a constant that depends only on the distance m_l from E_l to the rest of the spectrum of H_l. This distance m_l is called the mass gap for H_l and it is well known (see [26]) that if all the coefficients of p are small enough this distance is bounded from below by a positive constant. Hence in that case $(H_l-E_l+C)(H_l-E_l)^{-1}$ is bounded in norm uniformly in l. Therefore

$$\|P\varphi\Omega_l\| \leq C_1 \|(H_l-E_l+C)^{-1}[P\varphi, H_l](H_l-E_l+C)^{-1}\Omega_l\|, \qquad (4.33)$$

where C_1 is a constant that depends only on p and C. By lemma 4.1 it is therefore enough to prove that, if $:p_1:(h) = :p':(h) + \langle \xi, (-\Delta+m^2)\varphi \rangle$, then

$$(H_l-E_l+C)^{-1} :p_1:(\chi_l\varphi)(H_l-E_l+C)^{-1} \qquad (4.34)$$

is norm bounded uniformly in l . But this follows from

$$\pm :p_l:(\chi_l\varphi) \leq C_2(H_l-E_l+C) \,, \tag{4.35}$$

where C_2 is independent of l . This is proved by resolution of the identity from Ref. [27]. We also remark that recently Glimm and Jaffe have proved similar inequalities for the polynomially interacting fields with Dirichlet boundary conditions [28].

From (4.34) we have that

$$(H_l-E_l+C)^{-\frac{1}{2}}:p_l:(\chi_l\varphi)(H_l-E_l+C)^{-\frac{1}{2}} \tag{4.36}$$

is a bounded operator with norm independent of l . Hence (4.33) is bounded with norm independent of l . This proves the theorem. □

Now it follows from ref. [26] that if the coefficients of p are small enough, then the process $\xi_l(t)$ converges weakly to a process $\xi(t)$, however it is not known whether $\xi(t)$ is a Markov process. Consider now for f and g in $F^2(S')$

$$(f\Omega_l, e^{-t(H_l-E_l)} g\Omega_l) = E[\overline{f}(\xi_l(0))g(\xi_l(t))] \tag{4.37}$$

which by the results of Ref. [26] converge to

$$(f\Omega, e^{-tH} g\Omega) = E[\overline{f}(\xi(0))g(\xi(t))] \,, \tag{4.38}$$

where H is the physical Hamiltonian.

By theorem 4.1 and theorem 3.7 we have that (4.37) is twice differentiable with respect to t and the first derivative converges uniformly to the first derivative of (4.38). Hence we have in particular that μ_l converges weakly to a measure μ which is actually the physical vacuum Ω restricted to the time zero fields i.e.

$$\int e^{i\langle\xi,\varphi\rangle}\, d\mu(\xi) = (\Omega, e^{i\langle\varphi,\xi(0)\rangle}\Omega) \ . \tag{4.39}$$

Now from (4.34) it follows by standard methods [29], [14],3)

$$\pm :p_1:(\varphi) \leq C_2(H+C) \tag{4.40}$$

and from lemma 4.1 that

$$i\overline{[P\varphi,H]} = :p_1:(\varphi) \ . \tag{4.41}$$

Hence in the same way as for μ_1 we get that μ has regular first order derivatives in particular that μ is quasi invariant. Therefore we have the following theorem.

Theorem 4.2

Let μ be the physical vacuum restricted to the σ-algebra generated by the time zero fields as defined by (4.39). Then μ is a quasi invariant measure with regular first order derivatives. Moreover the physical Hamiltonian H restricted to $F^2\Omega$ coincides with the diffusion operator generated by μ, by theorem 2.7.

Proof: This follows by what is said above and theorem 3.6.

Remark: Bounds of the form (4.35) and (4.40) have been recently proved also for the Dirichlet boundary conditions on the fields by Glimm and Jaffe [24]. Hence theorem 4.1 and theorem 4.2 will also hold for the Dirichlet boundary conditions and their infinite volume limits, which also exist, by the method of Nelson [30], e.g. for arbitrary even polynomial p. In this case there is no smallness condition on the coefficients of p.

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RELATIVE ENTROPY AND ITS APPLICATION

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ABSTRACT

The properties of relative entropy are elaborated for a general Von Neumann algebra. This concept is applied to establish that large class of 1 - dimensional quantum spin systems has a unique KMS state.

RESUME

Les propriétés de l'entropie relative sont élaborées pour une algèbre de Von Neumann générale. Ce concept est utilisé pour établir l'unicité de l'état KMS pour une grande classe de système quantique de spin dans une dimension.

Relative Entropy and Its Applications

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§1. Relative entropy

The purpose of this review is to show usefulness of relative entropy by two examples of its applications to satistical mechanics.

For finite matrices, the relative entropy of non-negative matrices σ and ρ with unit trace (i.e. density matrices) is defined by

$$S(\sigma/\rho) = tr(\rho\log\rho) - tr(\rho\log\sigma) \qquad (1.1)$$

which takes either non-negative real value or $+\infty$. It has the following nice properties:

(1) Positivity: $S(\sigma/\rho) \geq 0$.

$S(\sigma/\rho) = 0$ if and only if $\rho = \sigma$.

(2) Convexity: $S(\lambda\sigma_1+(1-\lambda)\sigma_2/\lambda\rho_1+(1-\lambda)\rho_2)$

$\leq \lambda S(\sigma_1/\rho_1) + (1-\lambda)S(\sigma_2/\rho_2)$.

(3) Lower Semicontinuity: If $\lim \|\rho_n-\rho\| = \lim\|\sigma_n-\sigma\| = 0$, $\underline{\lim}\, S(\sigma_n/\rho_n) \geq S(\sigma/\rho)$.

(4) Monotonicity: $S(E_N(\sigma)/E_N(\rho)) \leq S(\sigma/\rho)$.

Here E_N is a conditional expectation to a subalgebra N with respect to the trace. (If the original matrices belongs to the tensor product $N \otimes N'$, then E_N is the partial trace $tr_{N'}$.)

For application, it is necessary to generalize this definition to a von Neumann algebra M, which does not necessarily have a trace. Let φ and ψ be faithful normal states, Φ and Ψ be their cyclic vector representatives. Then the <u>relative entropy</u> is defined by

$$S(\varphi/\psi) = -(\Psi, \{\log \Delta_{\Phi,\Psi}\}\Psi) \qquad (1.2)$$

in terms of the <u>relative modular operator</u> $\Delta_{\Phi,\Psi}$ —— a positive self-adjoint operator given as the absolute square

$$\Delta_{\Phi,\Psi} = (S_{\Phi,\Psi})^* \bar{S}_{\Phi,\Psi} \qquad (1.3)$$

of an antilinear operator $S_{\Phi,\Psi}$, defined on $M\Phi$ by

$$S_{\Phi,\Psi}\, x\Psi = x^*\Phi \ . \qquad (1.4)$$

The definition (1.2) does not depend on the choice of representative vectors of states.

The relative entropy so generalized enjoys the nice properties (1) ∿ (4), where E_N is now simply the restriction of the state to a von Neumann subalgebra N of M. The monotonicity (4) is at the moment proved only for a certain class of sub-algebras N, which include any finite dimensional subalgebra for arbitrary M. This will be sufficient for our applications.

The connection of the generalized definition (1.2) with the original definition (1.1) is simple: For finite matrix algebra M, a state φ is expressible through a density matrix ρ_φ by $\varphi(x) = \mathrm{tr}\, \rho_\varphi x$, $x \in M$. Then $S(\varphi/\psi)$ of (1.2) is the same as $S(\rho_\varphi/\rho_\psi)$ given by (1.1). (The relevant Hilbert space is M itself with the inner product (x,y) =

$tr(x^*y)$, Φ and Ψ are, for example, vectors $(\rho_{\varphi})^{1/2}$ and $(\rho_{\psi})^{1/2}$, and $\Delta_{\Phi,\Psi}$ is the left multiplication of ρ_{φ} times the right multiplication of $(\rho_{\psi})^{-1}$.)

§2. KMS condition

Our applications are centered on KMS condition, which characterizes equilibrium states in statistical mechanics and at the same time plays an important role in purely mathematical Tomita-Takesaki theory in theory of von Neumann algebras. This link enables us to apply the relative entropy defined above in the language of Tomita-Takesaki theory to statistical mechanics.

For a $*$-algebra M of finite matrices, the time translation for a given Hamiltonian H is the following one-parameter group of (inner) automorphisms:

$$\alpha_t(x) = e^{itH}xe^{-itH} \tag{2.1}$$

The corresponding equilibrium state proposed by Gibbs is

$$\varphi_{\beta}^{G}(x) = tr(e^{-\beta H}x)/tr(e^{-\beta H}) \tag{2.2}$$

where β is a real parameter (the inverse temperature). It is easy to verify that a state φ is the Gibbs state φ^G_β if and only if it satisfies the following KMS condition at β with respect to α_t: For each x and y in M, there exists a function $F(z)$ on the strip $0 \leq \mathrm{Im}\, z \leq \beta$ ($0 \geq \mathrm{Im}\, z \geq \beta$ if $\beta < 0$), which is holomorphic inside the strip, continuous and bounded on the strip and satisfies

$$F(t) = \varphi(x\alpha_t(y)),\ F(t+i\beta) = \varphi(\alpha_t(y)x). \qquad (2.3)$$

One of main ingredients of Tomita-Takesaki theory is that the modular operator $\Delta_\Phi \equiv \Delta_{\Phi,\Phi}$ has the property that

$$\sigma^\varphi_t(x) \equiv e^{itH_\Phi} x e^{-itH_\Phi},\quad H_\Phi = \log \Delta_\Phi \qquad (2.4)$$

is in M if $x \in M$. Hence σ^φ_t is a continuous one-parameter group of automorphisms of M, called the modular automorphism. (It is independent of the choice of the vector representative of the state φ.)

It is the crucial link between Tomita-Takesaki theory and statistical mechanics that φ satisfies the KMS condition at $\beta = -1$ with respect to σ^φ_t, and this property characterizes the uniformly bounded mapping $t \to \sigma^\varphi_t(x) \in M$. This implies, in particular, that for an equilibrium state φ at inverse temperature β for a time-translation α_t, the mathematical modular automorphism σ^φ_t coincides with $\alpha_{-\beta t}$:

$$\sigma^\varphi_t(\pi_\varphi(Q)) = \pi_\varphi(\alpha_{-\beta t}(Q)) \qquad (2.5)$$

where π_φ is the cyclic representation associated with φ.

§3. Perturbation of states

The perturbation in physics is an addition of some operator h to the Hamiltonian H. The resulting perturbation in the Gibbs state is what we want to formulate. For finite matrices, $\varphi(x) = tr(e^H x)$ is perturbed to

$$\varphi^h(x) = tr(e^{H+h}x) \tag{3.1}$$

where we have set $\beta = -1$ and omitted normalization factor for computational simplicity. If necessity arises, we consider

$$\psi = \varphi^{h-c1}, \quad c = \log\varphi^h(1) \tag{3.2}$$

which is automatically normalized.

For normal faithful positive linear functional φ of a von Neumann algebra M, the <u>perturbed functional</u> φ^h for $h = h^* \in M$ is defined by

$$\varphi^h(x) = (\Phi(h), x\Phi(h)), \; x \in M, \tag{3.3}$$

$$\Phi(h) = \exp\{(\log \Delta_\Phi + h)/2\}\Phi . \tag{3.4}$$

Here Φ is always in the domain of the operator preceding it in (3.4). This definition reduces to (3.1) for finite matrices and inherits properties obvious for (3.1):

(a) $\varphi^h = \varphi^k$ if and only if $h = k$.

(b) $(\varphi^h)^k = \varphi^{h+k}$, $\varphi = (\varphi^h)^{-h}$.

(c) $\log \Delta_{\Phi(h),\Phi} = \log \Delta_\Phi + h$.

Because of $\Delta_\Phi \Phi = \Phi$, the property (c) implies

$$S(\varphi^h/\varphi) = -\varphi(h) . \tag{3.5}$$

This formula yields an important estimate. Let φ be a normal faithful state of M and $h = h^* \in M$. Let ψ be defined by (3.2) in terms of φ^h. Then (3.5) implies

$$S(\psi/\varphi) = -\varphi(h) + \log \varphi^h(1). \tag{3.6}$$

By the property $\varphi = (\varphi^h)^{-h}$ and (3.5), we also have

$$S(\varphi/\psi) = \psi(h) - \log \varphi^h(1). \tag{3.7}$$

Hence

$$S(\psi/\varphi) + S(\varphi/\psi) = \psi(h) - \varphi(h) \leq 2\|h\|. \tag{3.8}$$

Since both $S(\psi/\varphi)$ and $S(\varphi/\psi)$ are non-negative, each is bounded by $2\|h\|$.

The infinitesimal generator of the modular automorphism σ_t^{φ} is something like Hamiltonian and should be changed by ih under perturbation of the positive linear functional This is actually the case in the following sense:

In general

$$(D\varphi : D\psi)_t \equiv (\Delta_{\Phi,\Psi})^{it}\Delta_{\Psi}^{-it} \tag{3.9}$$

is a unitary element of M and intertwines the modular automorphisms for φ and ψ:

$$(D\varphi : D\psi)_t \sigma_t^{\psi}(x)(D\varphi : D\psi)_t^{-1} = \sigma_t^{\varphi}(x) . \tag{3.10}$$

In terms of relative modular operator, this is the same as

$$(\Delta_{\Phi,\Psi})^{it}x(\Delta_{\Phi,\Psi})^{-it} = \sigma_t^{\varphi}(x) . \tag{3.11}$$

By (c) we obtain

$$(d/dt)\{\sigma_t^{\varphi^h}(x) - \sigma_t^{\varphi}(x)\}_{t=0} = i[h,x], \qquad (3.12)$$

$$(d/dt)(D\varphi^h:D\varphi)_t = i(D\varphi^h:D\varphi)_t\sigma_t^{\varphi}(h). \qquad (3.13)$$

The equation (3.13) at $t = 0$ actually characterizes φ^h.

§4. Quasi-equivalence

Let π_1 and π_2 be two representations of a C*-algebra $\mathfrak{A}$. The quasi-equivalence of π_1 and π_2 is one criterion of how close they are and is defined by the following conditions: the kernels (elements represented by 0) of π_1 and π_2 are the same and the *-isomorphism $\pi_1(Q) \to \pi_2(Q)$, $Q \in \mathfrak{A}$, extends to a *-isomorphism of weak closures $\pi_1(\mathfrak{A})''$ and $\pi_2(\mathfrak{A})''$. It turns out that this is the same as unitary equivalence up to multiplicity. (For commutative $\mathfrak{A}$, it coincides with the notion of equivalence of measures.)

If a representation π has an invariant subspace $\mathfrak{H}_0$ (i.e. $\pi(\mathfrak{A})\mathfrak{H}_0 \subset \mathfrak{H}_0$), then the restriction of the representation to the invariant subspace is called a subrepresentation of π. If π_2 is quasi-equivalent to a subrepresentation of π_1, then π_1 is said to quasi-contain π_2. If π_1 and π_2 quasi-contain each other, then they are quasi-equivalent.

A representation π which is minimal under the ordering of quasi-containment (i.e. which quasi-contains only representations quasi-equivalent to itself) is called primary. A representation π is primary if and only if $\pi(\mathfrak{A})''$ is a factor (i.e. has a trivial center).

The relative entropy can be used under certain circumstances as a computational tool for judging quasi-containment of representations through the following Lemma.

Lemma Let $\mathfrak{A}$ be a C*-algebra with an increasing sequence of finite dimensional *-subalgebras $\mathfrak{A}_n$ whose union is dense in $\mathfrak{A}$. For two states φ and ψ of $\mathfrak{A}$, the cyclic representation π_φ associated with φ quasi-contains the cyclic representation π_ψ associated with ψ if $S(\varphi_n/\psi_n)$ is uniformly bounded for restrictions φ_n and ψ_n of φ and ψ to $\mathfrak{A}_n$ for all n.

Corollary If $S(\psi_n/\varphi_n) + S(\varphi_n/\psi_n)$ is uniformly bounded, then π_φ and π_ψ are quasi-equivalent.

We note that $S(\psi_n/\varphi_n)$ and $S(\varphi_n/\psi_n)$ are monotonously increasing.

One can define relative entropy $S(\varphi/\psi)$ of two states φ and ψ of a C*-algebra $\mathfrak{A}$ in Theorem 1 by

$$S(\varphi/\psi) = \sup_N S(E_N(\varphi)/E_N(\psi)) \qquad (4.1)$$

where N runs over all finite dimensional *-subalgebras of $\mathfrak{A}$ and $E_N(\varphi)$ and $E_N(\psi)$ are restrictions of φ and ψ to N.

§5 Gibbs condition

By using the concept of perturbed functionals, we can derive a useful property of KMS states (i.e. states satisfying the KMS condition) under the following circumstances: Let $\mathfrak{A}$ be a C*-algebra generated by an increasing sequence of finite dimensional *-subalgebras $\mathfrak{A}_n$ and α_t be a one-parameter group of *-automorphisms of $\mathfrak{A}$ such that there exists $h_n = h_n^* \in \mathfrak{A}$ for each n satisfying in norm topology

$$(d/dt)\alpha_t(Q)|_{t=0} = i[h_n,Q] \qquad (5.1)$$

for all $Q \in \mathfrak{A}_n$ and decomposing as

$$h_n = u_n + w_n \qquad (5.2)$$

with $u_n = u_n^* \in \mathfrak{A}_n$.

The Gibbs condition at β for a state φ of $\mathfrak{A}$ is as follows:

(i) The normal extention of φ to the weak closure

$\pi_\varphi(\mathfrak{A})''$ <u>is faithful</u>.

(ii) <u>The perturbation by</u> βw_n <u>yields a product functional relative to the tensor product decomposition</u> $\mathfrak{A} = \mathfrak{A}_n \otimes (\mathfrak{A} \cap \mathfrak{A}_n')$:

$$\varphi^{\beta w_n} = \varphi^G_{n,\beta} \otimes \varphi' , \tag{5.3}$$

$$\varphi^G_{n,\beta}(Q) \equiv \mathrm{tr}(e^{-\beta u_n} Q)/\mathrm{tr}\, e^{-\beta u_n} , \quad Q \in \mathfrak{A}_n. \tag{5.4}$$

The second condition (restricted to a maximal abelian subalgebra of $\mathfrak{A}$) coincides with the condition given by Dobrushin and Lanford and Ruelle for equilibrium states of classical spin lattice systems (after an appropriate identification). The condition (i) is needed to define $\varphi^{\beta w_n}$, which is, precisely speaking, the restriction to $\pi_\varphi(\mathfrak{A})$ of the perturbed state $\tilde{\varphi}^{\beta \pi_\varphi(w_n)}$ which is obtained from the normal extension $\tilde{\varphi}$ of φ to the weak closure $\pi_\varphi(\mathfrak{A})''$.

<u>Theorem 1</u> <u>For a state</u> φ <u>of</u> $\mathfrak{A}$, <u>the KMS condition at</u> β <u>implies the Gibbs condition at</u> β .

It is well-known that the condition (i) follows from the KMS condition. The proof of condition (ii) is based on

$$(d/dt)\sigma_t^{\varphi_n}\{\pi_\varphi(e^{i\beta t u_n} Q e^{-i\beta t u_n})\} = 0 \tag{5.5}$$

for $Q \in \mathfrak{A}_n$ and $\varphi_n = \varphi^{\beta u_n}$, which can be verified for $t = 0$ from (2.5) implied by the KMS condition, (5.1) and (3.12) and extended to general t by the group property and by

$$\alpha_t^{(n)}(Q) \equiv e^{i\beta t u_n} Q e^{-i\beta t u_n} \in \mathfrak{A}_n \quad (Q \in \mathfrak{A}_n). \tag{5.6}$$

The equation (5.5) implies $\sigma_t^{\varphi n}\pi_\varphi = \pi_\varphi \alpha_t^{(n)}$, which easily leads to the condition (ii). Q.E.D.

While we need only the direction of Theorem 2 in our application, the converse also holds under some additional assumptions on h_n. Let D be a subset of $\mathfrak{A}$ consisting of $x = \lim x_m$ where $\{x_m\}$ is such that each x_m belongs to $\mathfrak{A}_{n(m)}$ for some $n(m)$ and $[h_{n(m)}, x_m]$ is convergent in $\mathfrak{A}$. (It is the domain of the closure of the derivation, say δ, defined on $\bigcup_n \mathfrak{A}_n$ by (5.1).)

<u>Theorem 2</u> <u>If</u> D <u>contains</u> $\alpha_t(\mathfrak{A}_n)$ <u>for all</u> n <u>and</u> $|t| < \varepsilon$ <u>for some</u> $\varepsilon > 0$ <u>independent of</u> n, <u>then the Gibbs condition at</u> β <u>implies the KMS condition at</u> β.

By a computation similar to the above proof of Theorem 1, it follows from the Gibbs condition that

$$(d/dt)\{\sigma_t^\varphi\,[\pi_\varphi\,(\alpha_{\beta t}(Q))]\}_{t=0} = 0 \tag{5.7}$$

for $Q \in \bigcup_n \mathfrak{A}_n$. If we have this for all Q in $\alpha_t(\mathfrak{A}_n)$ for $|t| < \varepsilon$ and every n, then we have $\sigma_t^\varphi\,(\pi_\varphi\,(Q)) = \pi_\varphi\,(\alpha_{-\beta t}(Q))$ for $Q \in \bigcup_n \mathfrak{A}_n$ and $|t| < \varepsilon$, which suffices to show $\sigma_t^\varphi \pi_\varphi = \pi_\varphi \alpha_{-\beta t}$ and hence the KMS condition for φ at β with respect to α_t.

For the passage from (5.7) for $Q \in \bigcup_n \mathfrak{A}_n$ to (5.7) for $Q \in \alpha_t(\mathfrak{A}_n)$, we use the assumption that $Q \in D$ if $Q \in \alpha_t(\mathfrak{A}_n)$, namely there exists $Q_m \in \mathfrak{A}_{n(m)}$ such that $\lim Q_m = Q$ and $\lim i\,[h_{n(m)}, Q_n] \equiv \dot{Q} \in \mathfrak{A}$. Since (5.7) is equivalent to

$$(d/dt)\{\sigma_t^{\varphi}(\pi_{\varphi}(Q)) - \pi_{\varphi}(\alpha_{-\beta t}(Q))\}_{t=0} = 0 . \qquad (5.8)$$

we obtain (5.7) for $Q \in \alpha_t(\mathfrak{A}_n)$ by the following computation

$$(d/dt)\sigma_t^{\varphi}(\pi_{\varphi}(Q))_{t=0} = \lim_m\{(d/dt)\sigma_t^{\varphi}(\pi_{\varphi}(Q_m))_{t=0}\}$$

$$= \lim\{(d/dt)\pi_{\varphi}(\alpha_{-\beta t}(Q_m))_{t=0}\}$$

$$= (d/dt)\pi_{\varphi}(\alpha_{-\beta t}(Q))_{t=0}$$

where the limit exists (and is $\dot{Q}$) by the choice of Q_m, the second equality is (5.8) for $Q_m \in \cup \mathfrak{A}_n$ and the two exchanges of the limit and differentiation is justified by the convergence of the derivative uniform in t because automorphisms are isometric. (It is the closability of generators of automorphisms.) Q.E.D.

The condition of Theorem 3 is satisfied in the case of quantum lattice system for which α_t has been shown to exist (interaction exponentially decreasing at higher-body interaction, which includes general finite-body interactions). The verification can be done by a direct computation or by the analyticity of $\alpha_t(Q)$, $Q \in \mathfrak{A}_n$, in t.

§6 First application —— Uniqueness of KMS states

We consider the same situation as section 5, where $\mathfrak{A}$, $\mathfrak{A}_n$, α_t, h_n, u_n and w_n are introduced.

Theorem 3 *If $\|w_n\|$ is uniformly bounded, then KMS state at any β is unique if it exists.*

Remark The assumption of the Theorem is satisfied in one dimensional spin lattice system when the total interaction across a point is finite. Since w_n is a surface energy essentially proportional to surface area, it is not uniformly bounded in higher dimensional case.

Proof consists of several steps, first aiming at the quasi-equivalence of all KMS states.

Step 1 Let φ be any KMS state at β and φ_0 be a weak accumulation point of the sequence of states

$$\chi_n = \varphi^G_{n,\beta} \otimes \tau$$

where τ is the tracial state on $\mathfrak{A} \wedge \mathfrak{A}_n'$ (any fixed state for each n will do). By definition, there exists a subsequence $n(m)$ for each p such that

$$\lim \|(\chi_{n(m)} - \varphi_0)|\mathfrak{A}_p\| = 0 .$$

Hence

$$\underline{\lim} \{S(E_p(\varphi^G_{n(m),\beta})/E_p(\varphi)) + S(E_p(\varphi)/E_p(\varphi^G_{n(m),\beta}))\}$$

$$= \underline{\lim} \{S(E_p(\chi_{n(m)})/E_p(\varphi)) + S(E_p(\varphi)/E_p(\chi_{n(m)}))\}$$

$$\geq S(E_p(\varphi_0)/E_p(\varphi)) + S(E_p(\varphi)/E_p(\varphi_0))$$

where E_p denotes the restriction of states to $\mathfrak{A}_p$.

Step 2 By the Gibbs condition for φ, the monotonicity and the estimate (3.8), we obtain for $n \geq p$ the following bound, which is uniform in n by the assumption:

$$S(E_p(\varphi^G_{n,\beta})/E_p(\varphi)) + S(E_p(\varphi)/E_p(\varphi^G_{n,\beta}))$$

$$= S(E_p(\psi_n)/E_p(\varphi)) + S(E_p(\varphi)/E_p(\psi_n))$$

$$\leq S(\psi_n/\varphi) + S(\varphi/\psi_n) \leq 2\|\beta w_n\|$$

where $\psi_n = \{\varphi^{\beta w_n}(1)\}^{-1}\varphi^{\beta w_n}$ given by (3.2) with $h = \beta w_n$.

Step 3 By Steps 1 and 2, the condition for Corollary in Section 5 is satisfied. Hence the cyclic representation associated with any KMS state at β is quasi-equivalent to the fixed cyclic representation associated with φ_0 and thus mutually quasi-equivalent.

Step 4 The set of KMS states forms a compact convex set. A KMS state φ is extremal in this convex set if and only if the associated representation is primary. Since a compact convex set has an extremal point, all KMS state φ must be extremal. Since a convex set consisting solely of extremal points is either empty or one-point set, we have the uniqueness of KMS state. Q.E.D.

§7 Second application —— Variational principle.

The Gibbs state (2.2) can be characterized also by the following variational principle: Let ρ_φ be the density matrix of a state φ on a finite dimensional algebra M.

Define

$$S(\varphi) = -\mathrm{tr}(\rho_\varphi \log \rho_\varphi) \quad \text{(entropy)}, \tag{7.1}$$

$$E(\varphi) = \varphi(H) \quad \text{(energy)}. \tag{7.2}$$

Then φ_β^G of (2.2) is the unique state φ maximizing $S(\varphi)-\beta E(\varphi)$, the maximum value being given by

$$P_\beta(H) \equiv \log \mathrm{tr}\, e^{-\beta H} \tag{7.3}$$

In other word, the variational inequality

$$P_\beta(H) \geqq S(\varphi) - \beta E(\varphi) \tag{7.4}$$

holds for all states φ and the equality is satisfied if and only if $\varphi = \varphi_\beta^G$.

In terms of relative entropy, this variational principle is nothing but the positivity:

$$S(\varphi_\beta^G/\varphi) \geqq 0 \tag{7.5}$$

where the equality holds if and only if $\varphi = \varphi_\beta^G$.

For infinitely extended lattice systems, (7.1) and (7.2) can not be defined. However the corresponding density can be defined:

$$s(\varphi) = \lim S(\varphi_n)/V_n\ , \tag{7.6}$$

$$e(\varphi) = \lim E(\varphi_n)/V_n\ , \tag{7.7}$$

$$p_\beta = \lim (\mathrm{tr}\, e^{-\beta u_n})/V_n\ : \tag{7.8}$$

where φ_n is the restriction of φ to $\mathfrak{A}_n$, $E(\varphi_n) = \varphi(u_n)$, V_n is proportional to $\log(\dim \mathfrak{A}_n)$ (the volume) and the limit is known to exists for any translationally invariant state φ for an appropriate sequence $\mathfrak{A}_n$. The variational principle is formulated as

$$p_\beta \geqq s(\varphi) - \beta e(\varphi) \qquad (7.9)$$

where a state φ_{eq} is a solution of the variational principle if the equality holds in (7.9) for $\varphi = \varphi_{eq}$.

For translationally invariant states, it has been known for some time that any solution of the variational principle satisfies the KMS condition at β. The converse holds:

Theorem 4 A translationally invariant KMS state φ at β is a solution of the variational principle.

Proof By computation (same as the equivalence of (7.4) and (7.5)), we have

$$p_\beta - s(\varphi) + \beta e(\varphi) = \lim S(\varphi^G_{n,\beta}/\varphi_n)/V_n \qquad (7.10)$$

where $\varphi^G_{n,\beta}$ is given by (5.4). By Gibbs condition for φ and (3.8),

$$|S(\varphi^G_{n,\beta}/\varphi_n)| \leqq \|\beta w_n\| . \qquad (7.11)$$

It can be shown that

$$\lim \|w_n\|/V_n = 0 \qquad (7.12)$$

essentially because $\|w_n\|$ is proportional to surface area and the ratio of surface area to volume tends to zero in any dimension as the volume grows in a nice manner. This then implies

$$\lim S(\varphi^{G}_{n,\beta}/\varphi_n)/V_n = 0 \qquad (7.13)$$

and hence the variational equality for φ . Q.E.D.

It is to be noted that the translational invariance of φ is not needed up to (7.13). The equation (7.13) can be interpreted as the vanishing of the relative entropy density.

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On the equivalence problem between Wightman and Schwinger functionals.

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RÉSUMÉ

Les relations entre les fonctions de Wightman et les fonctions de Schwinger sont discutées.

ABSTRACT

The relations between Wightman functions and Schwinger functions are discussed.

Contents :

I. Introduction
II. Review of previous results
III. The algebra of functions on $\mathcal{S}'(\mathbb{R}^d)$ and definition of a topology
IV. Restriction of this topology to the tensor algebra.
V. Other interpretations of this topology.
VI. Continuity requirements for Wightman functions on Schwinger points.

I. Introduction

In this seminar I will talk about joint work with J. Yngvason [1, 2, 3] which will appear in the Communications of Mathematical Physics.

Starting from a Wightman functional it is possible to get by analytic continuation to the Schwinger-points, i.e. the points with real space- and pure imaginary time-coordinals, and where no two points coincide. On these Schwinger points the Wightman functions are analytic and symmetric under permutation of the coordinates.

K. Symanzik [4] has introduced the idea of viewing these functions as expectation values of a commutative field, the so-called Schwinger field. In recent years these ideas have become standard tool in constructive field theory, so that it is interesting to ask whether this concept can be derived from Wightman's axioms.

Having a set of Wightman functions which is a positive linear functional over some test-function algebra, for example $\underline{\mathcal{S}}$, we can compute the Wightman function on the Schwinger points by analytic continuation. Because of symmetry we might look at this set of functions as a linear functional defined on a linear subspace of the symmetric tensor algebra over $\mathcal{S}(\mathbb{R}^d)$ which I will denote by $S(\mathcal{S})$. That this functional is only given on a linear subspace is due to the fact that these functions are only defined for non-coinciding points. Therefore an extension of this functional is needed in order to get a functional on the whole abelian algebra $S(\mathcal{S})$.

A simple inspection of the properties of the Wightman functions on the Schwinger-points shows that there is enough continuity to guarantee such an extension by the Hahn-Banach theorem. But, at this point the problem enters, namely, if we ask for extensions of a linear functional then we should look for nice extensions. This leads us to the following question:

What are nice functionals on the algebra $S(\mathcal{S})$?

On an abelian algebra the nicest linear functional are characters i.e. a linear functional χ with the property

$$\chi(xy) = \chi(x)\ \chi(y) \quad ,$$

where x, y are elements of the algebra in question. Characters are very special and give rise only to a one-dimensional representation of the algebra. If a functional shall contain more information, than what one can expect from a character, one gets a nice family of linear functionals by integrating over characters, i.e. by investigating expressions of the form

$$T(x) = \int_{\Lambda} \chi(x)\, d\mu(\lambda)$$

where (Λ, μ) is some measure-space.

Looking at the case of abelian C^*-algebras then one knows that here every linear functional is of this form. But, in the case of general abelian $*$-algebras this is no longer true. The new feature which enters are pathologies which are associated with commuting unbounded operators. If we restrict ourselves to those functionals which are decomposable into characters, this is nothing else than looking for functionals which are free of such pathologies.

Originally Symanzik was thinking of Schwinger fields, this means, besides being decomposable into characters he wanted the functional to be positive and decomposable into sums of characters with positive coefficients. But we will not ask for this property for two reasons. First, from the positivity in Minkowski-space one gets only the so-called Wightman-positivity (see [5]) and the positivity for the Schwinger field cannot be derived from it. The Schwinger-positivity is an additional hypothesis which we cannot expect to be shared by all field theories. If we believe in the existance of non time-reversal invariant theories then we have to get acquainted with the idea that the measure in $\int \chi_\lambda \, d\mu(\lambda)$ can be a complex measure because reality of μ guarantees already time reversal invariance. The second reason is purely technical. As we will see in the next section functionals which are decomposable into characters with a positive measure fulfill some additional requirement which cannot be characterized purely algebraically and topologically. Therefore the extension-problem requires additional tools which are beyond the Hahn-Banach theorem. At least to me it seems completely hopeless to deal with this problem in the near future.

What I will explain here can also be carried through for other test-function algebras like for instance $\mathcal{D}$ or the Jaffe space $\mathcal{J}$. Important is only the nuclearity of the space.

This presentation will not contain any proofs, but, I hope I succeed in explaining some of the underlying ideas of the subject and the difficulties we had to overcome.

II. Review of previous results

In this section I will give a short report on the papers [1, 2] which are the basis of the following investigations (see also the lectures in Strasbourg [6]). The part which is relevant for our purpose is dealing with the following question: Let A be an abelian $*$-algebra containing the identity and ω be a state on A, this means ω is a positive normalized linear functional on A, what are the conditions on ω such that it can be represented as an integral over characters with respect to a positive measure. Looking into this problem one finds out that this question has two different aspects which are not related to eachother and which can be solved with different level of generality.

The first problem is the following: Given the state ω then we associated to it, by the G.N.S. construction, a representation π_ω of the algebra A defined on some common domain $\mathfrak{D}_\omega$ which is dense in the representation Hilbert-space . If x is a symmetric element of A then the operator $\pi(x)$ is generally only a densely defined symmetric operator. Therefore one would like to find a common extension $\hat{\pi}(x)$ of all the operators $\pi(x)$ in such a way that

α) all the operators $\hat{\pi}(x)$ are defined on the same extended domain $\hat{\mathcal{D}}$.

β) If x is a symmetric element of A then $\hat{\pi}(x)$ should be essentially self-adjoint on $\hat{\mathcal{D}}$.

γ) If β holds then we can construct the spectral projections of every $\hat{\pi}(x)$ and we would like to have that all these spectral projections commute with each other.

This problem can be solved in full generality without using any topological requirement. This is due to the fact that one understands the obstacle. This obstruction is known from the finite dimensional moment-problem. Our result says there is no other difficulty in the case of an arbitrary number of operators than those which are known from the case of finitely many operators. The obstacle which appears is the following fact discovered by Hilbert [7]: In two or more dimensions there exist real polynomials $P(x_1 \ldots x_n) \geq o$ without being the sum over squares of other polynomials, i.e.

$$o \leq P(x_1 \ldots x_n) \neq \sum_i |Q_i(x_1 \ldots x_n)|^2$$

where Q_i are also polynomials. Since characters are associated to points of the spectrum, a character has to be positive also on such elements. The result dealing with this situation is the following

II. 1. Theorem

Let A be an abelian $*$-algebra with identity and let ω be a state on A, then the following two statements are equivalent.

1) The representation π_ω of A defined by the G. N. S. construction has an extension $\hat{\pi}$ on a common domain $\hat{\mathcal{D}}$ such that

   a) All $\hat{\pi}(x)$ are essentially self-adjoint on $\hat{\mathcal{D}}$ for $x = x^* \in A$

   b) If $x = x^*$ and $y = y^*$ then their spectral projections commute with each other.

2) ω is positive on every positve polynomial, this means that if $P(\lambda_1 \ldots \lambda_n) \geq o$ for $\lambda_i \in \mathbb{R}$ and $x_1 = x_1^*, \ldots x_n = x_n^* \in A$ then $(P(x_1, \ldots x_n)) \geq o$.

The second problem is a measure-theoretical one. Assume the conditins of Theorem II. 1. are fulfilled then the von Neumann algebra $\mathcal{M}$ generated by the spectral projections of all the $\mathbb{T}(x)$ will be a maximal abelian von Neumann algebra having the vector Ω as cyclic and separating vector. The operators $\hat{\mathbb{T}}(x)$ are then linear operators affiliated to this von Neumann algebra. Assume that the Hilbert space abtained by applying $\mathcal{M}$ to Ω is separable then one can represent ω on $\mathcal{M}$ by a measure on the spectrum S of $\mathcal{M}$ this means $\mathcal{M}$ is identified with $\mathcal{L}^{\infty}(S,\mu)$. But you would also like to identify the algebra A with some algebra of functions on S. For every single operator $\hat{\mathbb{T}}(x)$ it can be done, but in general one can not do it simultaneously for all of A wothout some additional continuity requirement on A. One technique which gives nice results is the so-called nuclear spectral theorem [8, 9]. Since nuclearity is fulfilled in all examples of interest we restricted ourselves to this situation. Before giving the result I have to make some

II. 2. Remarks

a) The algebra in question will be the symmetric tensor algebra over $\mathcal{S}$, $\mathcal{D}$ or any other nuclear test-function space. For simplicity I will restrict my attention on $S(\mathcal{S})$.

b) On this algebra there exists a natural topology $\mathcal{T}$ defined by the topology of the base space $\mathcal{S}$. It is convenient to assume that the algebra $S(\mathcal{S})$ is complete in this topology. The reason for this is that every positive linear functional is automatically continuous if the algebra is complete [10, 11].

c) If we have a character χ on $S(\mathcal{S})$ then under the hypothesis of b) it is automatically continuous. Since $\mathcal{S}$ is a linear subspace of $S(\mathcal{S})$ we can restrict χ to $\mathcal{S}$. Therefore it defines an element $\omega \in \mathcal{S}'$ and it turns out that the characters are in one to one correspondence to elements of $\mathcal{S}'$

via the formula

$$\chi_\omega(P(x_1, \dots x_n)) = P(\omega(x_1), \dots \omega(x_n))$$

where P is a polynomial and $x_i \in \mathcal{S}$

d) If one wants to represent states subject to theorem II. 1. by means of integrals over characters then there are two possibilities, one method is using the spectrum of the maximal abelian algebra $\mathcal{M}$ I talked about earlier, the second is the space $\mathcal{S}'$ by means of the correspondence mentioned under In this case one generally gets cylindric measures on $\mathcal{S}'$ or the corresponding dual spaces in question.

e) The assumption of nuclearity has some other advantage, namely the continuous image of any nuclear space into a Banach space is automatically separable, so that we don't have to worry about the separability of the representation space.

Under these assumptions we get the following result:

II. 3. Theorem: Let T be a linear functional on $S(\mathcal{S})$ then the following statements are equivalent

1) T is positive and fulfills the conditions of Theorem II. 1. This means T is positive on all positive polynomials.

2) T has a weak integral decomposition

$$T(x) = \int_\Lambda \chi_\lambda(x)\, d\mu(\lambda)$$

where

a) (Λ, μ) is a standard measure space, $\mu \geq 0$ and $\mu(\Lambda) = 1$

b) χ_λ are characters on $S(\mathcal{S})$ μ - a · e

c) there exists a function $c(\lambda) \geq 0$,

$c(\lambda) \in \mathcal{L}^2(\Lambda, \mu)$ and continuous seminorms p_n on $\mathcal{S}$

such that for $x \in \mathcal{S}$

$$|\chi_\lambda(x)| \leq C(\lambda)^{1/n} \, p_n(x) \qquad , \; n = 1, 2, \ldots$$

3) There exists a cylinder-measure ν_ω on $\mathcal{S}_h'$ such that

$$T(P(x_1, \ldots x_n)) = \int_{\mathcal{S}_h'} P(\omega(x_1), \ldots \omega(x_n)) \, d\nu_\omega ,$$

with the property, that for every continuous polynomially bounded function f on $\mathbb{R}^n$ the integral

$$\int f(\omega(x_1), \ldots \omega(x_n)) \, d\nu_\omega$$

exist and is jointly continuous in $x_1 \ldots x_n \in \mathcal{S}$

Who is interested in more details about abelian algebras can find them in reference [2] and the papers of R. Powers [12, 13].

III. The algebra of functions on $\mathcal{S}'(\mathbb{R}^d)$ and definition of a topology

Remember that we want to characterize functionals which have an integral decomposition into characters. If such an integral exists then in view of theorem II. 3. it is also defined for a wider class of functions defined on $\mathcal{S}'$. So we are looking for an algebra of functions $\mathcal{F}$ on $\mathcal{S}'$ such that the following is fulfilled

III. 1. Conditions on $\mathcal{F}$

a) $\mathcal{F}$ shall be a $*$-algebra

b) $\mathcal{F}$ shall be a lattice in the natural order, so that we can decompose linear functionals into positive ones.

c) If possible $\mathcal{F}$ should be such that positive functionals are automatically continuous.

d) Every functional on $\mathcal{F}$ should be representable by a measure on $\mathcal{S}'$ via the formular discussed in theorem II. 3.

e) $\mathcal{F}$ shall be generated by $S(\mathcal{S})$ and bounded functions on $\mathcal{S}'$.

Looking at all these conditions there seems to be one natural choice of namely:

III. 2. Definition

We put $\mathcal{F} = \{ f : \mathcal{S}'_h \to \mathbb{C}$; there exists a continuous function $g(\lambda_1, \cdots \lambda_n)$ on $\mathbb{R}^n$ which is polynomially bounded and $x_1 \ldots x_n \in S(\mathcal{S})$ such that $f(\omega) = g(\chi_\omega(x_1), \cdots \chi_\omega(x_n)) \}$

By $\mathcal{F}^+$ we denote the functions such that $f(\omega) \geq 0$.

In order to see that the other conditions are fulfilled we need some topology on $\mathcal{F}$.

Since the functions $f(\omega)$ are polynomially bounded one cannot use supremum norms or seminorms. Natural candidates for norms or seminorms are objects of the form

$$\|f\|_F = \sup_{\omega} \frac{|f(\omega)|}{F(\omega)}$$

for a suitable family of functions $F(\omega)$. This family of functions shold also reflect the original topology $\mathcal{T}$ defined on the abelian algebra $S(\mathcal{S})$.

Our choice is the following: If p is a seminorm on $S(\mathcal{S})$ we put

$$F_p(\omega) = \sup \left\{ x(\chi_\omega) \; ; \; x \in S(\mathcal{S}) \text{ and } p(x) \leq 1 \right\}$$

The topology defined by these two formulas we denote by $\hat{\mathcal{T}}$. With these notations we get :

III.3. Lemma: If we restrict the topology $\hat{\mathcal{T}}$ to $S(\mathcal{S})$ then every $\hat{\mathcal{T}}$ continuous seminorm is also $\mathcal{T}$ continuous.

III.4. Theorem: For a linear functional $\hat{T}$ on $\mathcal{F}$ the following statements are equivalent

1) $\hat{T}$ is $\hat{\mathcal{T}}$ -continuous

2) There exists a unique complex measure on $\mathcal{S}_h'$ such that $\hat{T}(f) = \int f(\omega)\, d\nu_\omega$

This theorem tells us that we have fulfilled exactly all requirements listed under III. 1. (To see that III. 1. c. holds one has to use theorem III. 3. and remark II. 2. b). Therefore we have the following

III. 5. Conclusion:

Let $\mathcal{L}$ be a linear subspace of $S(\mathcal{S})$ and t a linear functional on $\mathcal{L}$. Then t has an extension T to all of $S(\mathcal{S})$ such that T is representable by a measure on $\mathcal{S}'$ if and only if t is continuous with respect to the topology $\hat{\mathcal{T}} \restriction \mathcal{L}$.

IV. Restriction of this topology to the tensor algebra

Since we did know what we were aiming at it was not very hard to find the algebra $\mathcal{F}$ and the topology $\hat{\mathcal{T}}$ on it. But the real problem starts by trying to characterize its restriction to $S(\mathcal{S})$. Remember that $S(\mathcal{S})$ is the symmetric tensor algebra over $\mathcal{S}$ which is a graded algebra equipped with a topology well adapted to the grading. But in the algebra $\mathcal{F}$ the grading has completely disappeared and essentially also the same is true for the topology $\hat{\mathcal{T}}$. Now we have to restrict $\hat{\mathcal{T}}$ to $S(\mathcal{S})$ and we can hope that we can handle problems only if there exists an equivalent formulation of $\hat{\mathcal{T}} \upharpoonright S(\mathcal{S})$ which uses the terminology of the original topology $\mathcal{T}$, in particular the graded structure of $S(\mathcal{S})$.

The first result in this direction is the following

IV. 1. Lemma: Let $x = \{x_o, x_1 \dots x_i, o, o \dots\} \in S(\mathcal{S})$ and let $\hat{\mathcal{T}}$ be the restriction of $\hat{\mathcal{T}}$ to $S(\mathcal{S})$ then the family of the seminorms

$$x \longrightarrow \sum_{i=0}^{\infty} \| x_i \|_F$$

form a basis of this topology.

If we want to investigate this topology any further then we have to say how the functions $F(\omega)$ look like.

The algebra $S(\mathcal{S})$ consists of sequences $\{x_o, x_1 \dots\}$ where $x_i \in \mathcal{S}(\mathbb{R}^{id})$ and if p are seminorms on $\mathcal{S}^d$ then $p^{\otimes_\pi^{id}}$ are seminorms on $\mathcal{S}(\mathbb{R}^{id})$. From this one reads of:

IV. 2. Lemma: Functions of either one of the two forms define a basis for the topology $\hat{\mathcal{T}}$:

1) $F(\omega) = \sum_{\nu=0}^{\infty} (p_\nu^\circ(\omega))^\nu$

2) $F(\omega) = \prod_{\nu=1}^{\infty} (1 + p_\nu^\circ(\omega))$

where $\{p_\nu\}$ is a sequence of continuous seminorms on $\mathcal{S}(\mathbb{R}^d)$ with $p_\nu \leq p_{\nu+1}$ and $p_\nu^\circ(\omega) = \sup \{ |\omega(x)| ; p_\nu(x) \leq 1, x \in \mathcal{S} \}$.

In the following we will take the second form for the functions $F(\omega)$. Then we get for the n th component:

IV. 3. Lemma: There exists $\varepsilon_n > 0$ independent of the seminorms $\{p_\nu\}$ such that for a homogenious element x_n we get

$$\sup_{\omega} \frac{|x_n(\omega)|}{\prod_{\nu=1}^{n} p_\nu^o(\omega)} \leq \sup_{\omega} \frac{|x_n(\omega)|}{\prod_{\nu=1}^{\infty} (1+\varepsilon_\nu^{-1} p_\nu^o(\omega))} = \|x_n\|_{F(\{\varepsilon_\nu^{-1} p_\nu^o\})}$$

Remarks: 1) Since x_n is a homogenious element of degree n we also might write the left hand side as

$$\sup_{\omega} \Big(\prod_{\nu=1}^{n} \otimes \frac{\omega}{p_\nu^o(\omega)} \Big)(x_n) .$$

2) In order to give an even nicer characterization of this topology we have to define the algebra $S(\mathcal{S})$ carefully. Since we are looking for symmetric functionals we can define it also on the usual tensor algebra $\underline{\mathcal{S}}$ where it anihilates the ideal J generated by all commutators. Therefore we put

$$S(\mathcal{S}) = \underline{\mathcal{S}}/J$$

also as topological space.

3) Since J is of a very special structure it is easy to see that graded structure survives the passage to the quotient, this means

$$S(\mathcal{S}) = \sum_{0}^{\infty} \oplus S_n(\mathcal{S}) \quad \text{with}$$

$$S_n(\mathcal{S}) = \mathcal{S}^{\hat{\otimes} n} / J \cap \mathcal{S}^{\hat{\otimes} n} .$$

4) If $p_1 \dots p_u$ are seminorms on $\mathcal{S}$ then we define seminorms on $S_u(\mathcal{S})$ by

$$(p_1 \otimes_\pi \dots \otimes_\pi p_n)_{symm} (x_n) = \inf_{y_n \in J \cap \mathcal{S}^{\hat{\otimes} n}} (p_1 \otimes \dots \otimes p_n)(x_n + y_n)$$

With this notation we get :

IV.4. Theorem: The $\hat{\tau}$ topology on $S(\mathcal{S})$ is given by the collection of the following seminorms:

$$p(x) = \sum_{n=0}^{\infty} (p_1 \otimes_\pi \cdots \otimes_\pi p_n)_{symm}(x_n) \qquad (*)$$

where $\{p_\nu\}$ is a sequence of continuous seminorms on $\mathcal{S}$.

Remark: As I will indicate in the next section the topology defined by the seminorms $(*)$ of the last theorem is constructed in such a way that the product in $S(\mathcal{S})$ becomes a continuous operation. The topology defined by these seminorms and the topology are in the same relation as the original norm and the enveloping C^*-norm for Banach $*$-algebras. Therefore the result of this theorem is not true for arbitrary graded algebras. The nuclearity was an essential ingredient for the proof of the theorem.

From this theorem one gets:

IV.5. Corollary: Let $T = (T_0, T_1, \ldots)$ be a sequence of tempered distributions $T_n \in \mathcal{S}'(\mathbb{R}^{dn})$ with $T_n(x_1 \otimes x_2 \cdots \otimes x_n) = T_n(x_{\pi 1} \otimes x_{\pi 2} \cdots \otimes x_{\pi n})$ for any permutation π and $x_i \in \mathcal{S}(\mathbb{R}^d)$ then the following conditions are equivalent

1) There are continuous seminorms p_ν on $\mathcal{S}(\mathbb{R}^d)$ such that

$$|T_n(x_1 \otimes \cdots \otimes x_n)| \leq p_1(x_1) \cdots p_n(x_n)$$

2) There exists a complex measure ν_ω on $\mathcal{S}'(\mathbb{R}^d)_h$ such that

$$T_n = \int \underbrace{\omega \otimes \omega \otimes \cdots \otimes \omega}_{n\text{ - factors}} \, d\nu_\omega$$

exist as a weak integral.

V. Other interpretations of this topology

Before going to the application I will list for completeness sake other characterizations of this topology $\hat{\tau}$.

V.1. Theorem: On the algebra $S(\mathscr{S})$ the following topologies coincide

1) The topology $\hat{\tau}$.

2) The strongest topology τ_m which is weaker than τ such that the multiplication is jointly continuous.

3) The strongest topology such that $S(\mathscr{S}) \cap \mathcal{T}^+$ is a normal cone .

4) The strongest topology such that $S^+(\mathscr{S})$ = closure of $\{ \sum_i x_i^* x_i \; , \; x_i \in S(\mathscr{S}) \}$ is a normal cone.

Further properties are:

V.2. Lemma:

S( ) is a complete nuclear vector space but it is neither barnological nor barrelled.

Remarks: 1) For an arbitrary locally convex topological algebra it is possible to construct the finest topology coarser than the given one such that the product becomes jointly continuous. It is also possible to characterize the neighbourhood of this topology in terms of the original ones. If the algebra is abelian, then the formulas become simple and they are those which we used in theorem IV.4.

2) The equivalence of the two topologies described by 3) and 4) of the last theorem came as a surprise to us. From the treatment of the "infinite dimensional" moment problem we did know that not every positive linear functional can be represented by an integral over characters with a positive measure. But, since any positive linear functional fulfills the condition of theorem IV.4. automatically (iterated use

of Schwarz' inequality) it follows that it is an integral over characters but with a signed measure.

3) In the case of non-commutative algebras the multiplicative and the normal topology (2 and 4 of the last theorem are generally not identical (see e.g. J. Yngvason [14])

VI. Continuity requirements for Wightman-functions on Schwinger points

Now it is easy to translate the conditions of corollary IV. 5. into the language of Wightman functions. In the following y will denote points of the Euclidean space R^d and $S_n(y_1 \dots y_n)$ the analytic continuation of the n-th Wightman-function to the Schwinger points. Note that these functions are only defined for non-coinciding points and that they are real analytic functions on these points. Therefore the necessary estimates are only concerned with the coinciding points and the points at infinity. The result is the following

VI. 1. Theorem: Let $S_n(y_1, \dots y_n)$, n = o, 1, ... be the Wightman functions on Schwinger points of a given quantum field theory, then the following are equivalent

1) There exists a (complex) measure ν_ω on $\mathcal{S}'(\mathbb{R}^d)_h$ such that

$$\int_{\mathbb{R}^{dn}} S_n(y_1, \dots y_n)\, f(y_1, \dots y_n)\, dy = \int_{\mathcal{S}'_h} (\omega \otimes \omega \dots \otimes \omega)\, d\nu_\omega$$

for all test functions with an infinite zero at coinciding points.

2) There exist constants $C_n > o$, $k_n \geq o$, $L_n \geq o$ such that

$$|S_n(y_1, \dots y_n)| \leq C_n \cdot \sum_{\nu=2}^{n} \sum_{i_1 < i_2 \dots < i_\nu} \left\{ d(y_{i_1}, \dots y_{i_\nu})^{-k_\nu} + r(y_{i_1}, \dots y_{i_\nu})^{L_\nu} \right\}$$

where

$$d(y_1, \dots y_n) = \max_{i \neq j} |y_i - y_j|$$

and

$$r(y_1, \dots y_n) = \min_{i \neq j} |y_i - y_j|$$

I want to conclude my lecture with some final

VI. 2. Remarks:

1) What the estimate says is the following: If k points are coming together, then the singularity they produce shall be independent of the number of the other $n - k$ points as long as they stay apart.

2) If we forget for a moment the behaviour at infinity then these estimates are fulfilled if there exists a Wilson-Zimmermann expansion [15] for operator products. On the other hand if this estimates are fulfilled it seems to me very likely that one can derive from it the Wilson-Zimmermann expansion.

3) Very important is the fact the coefficients C_n in front of these estimates are allowed to grow arbitrarily fast. In deed this freedom is necessary since one can construct trivial Wightman fields where the coefficients increase as fast as you want and which are representable by measures.

4) Looking at the behaviour of k points, it follows from Schwarz inequality that the conditions for these points are fulfilled as long as they appear at one end of

the Schwinger n-point function. Therefore it is tempting to try a general proof for these estimates. The difficulty consists in proving some kind of crossing symmetry for estimates.

5) The converse problem namely going back from the Schwinger points to the Minkowski-space ist still an open one. There exist some sufficient conditions due to Osterwalder and Schrader [16]. But since there a strong restriction on the coefficients C_n is needed, I believe that this problem is not well understood up to now.

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Colloques Internationaux C.N.R.S.
N° 248 – Les méthodes mathématiques de la théorie quantique des champs

THE CONSTRUCTION OF $\lambda\varphi_3^4$ QUANTUM FIELD MODELS[1]

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RESUME : Nous résumons les progrès récents en construction des modèles quantiques $\lambda\varphi_3^4$. En particulier, nous discutons le problème d'un champ externe fort et le problème de grand couplage.

Abstract : We summarize recent progress in the construction of $\lambda\varphi_3^4$ quantum field models. In particular we discuss the strong external field problem, and the strong coupling problem.

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This talk will deal with some recent developments in the construction of $\lambda\varphi_3^4$ models. These models are superrenormalizable. However, in addition to the Wick ordering renormalizations which are sufficient for the $P(\varphi)_2$ models, the Euclidean action requires

1) a linearly divergent, second order counterterm (corresponding partly to a vacuum energy renormalization and partly to a wave function renormalization)

2) a logarithmically divergent third order scalar counterterm

3) a logarithmically divergent mass counterterm

Hence the cutoff Euclidean Green's functions are given by

$$S^n_{\kappa,\Lambda}(f_1,\cdots,f_n) = Z^{-1}_{\kappa,\Lambda}\langle \Phi(f_1)\cdots\Phi(f_n)e^{-V_{\kappa,\Lambda}}\rangle_{m_0}$$

$$Z_{\kappa,\Lambda} = \langle e^{-V_{\kappa,\Lambda}}\rangle_{m_0}$$

where

$$V_{\kappa,\Lambda} = \lambda\int_\Lambda d^3x{:}\Phi_\kappa^4(x){:} - \mu\int_\Lambda d^3x\Phi_\kappa(x) + V_c$$

$$V_c = \tfrac{1}{2}\langle(\lambda\int_\Lambda d^3x{:}\Phi_\kappa^4(x){:})^2\rangle_{m_0} - \tfrac{1}{6}\langle(\lambda\int_\Lambda d^3x{:}\Phi_\kappa^4(x){:})^3\rangle_{m_0} + \frac{\lambda^2}{2}\delta m^2\int_\Lambda d^3x{:}\Phi_\kappa^2(x){:}\,.$$

The expectation $\langle\cdot\rangle_{m_0}$ is taken with respect to the Gaussian measure on $\mathcal{S}'_R(R^3)$ with mean 0 and covariance $-\Delta + m_0^2$. In other words, formally

$$\langle \cdot \rangle_{m_0} = \text{normalization const.} \int \cdot \, e^{-\frac{1}{2}\int \nabla\Phi(x)^2 + m_0^2 \Phi^2(x)} \prod_{x \in R^3} d\Phi(x) \, .$$

As for $P(\varphi)_2$ models we introduce boundary conditions into the $\{S^n_{\kappa,\Lambda}\}$ by replacing $-\Delta$ with the Laplacian $-\Delta_{\partial\Lambda}$ having some boundary conditions (e.g. periodic, zero Dirichlet data) on $\partial\Lambda$. In this event we always keep the Wick ordering matched to the covariance but we hold the coefficient δm^2 of the mass counterterm fixed. (Say, for example, we always use the δm^2 appropriate to free boundary conditions.)

There are four different circumstances under which we have existence theorems.

	Coupling Restrictions	Boundary Conditions
W1	$\lambda \geq 0$ sufficiently small $m_0 > 0$ sufficiently large	free, periodic, Dirichlet
W2	$\lambda \geq 0$ $m_0 > 0$ $\lvert\mu\rvert$ sufficiently large	periodic
S1	$\lambda \geq 0$ $m_0 > 0$ $\mu = 0$	Dirichlet
S2	$\lambda \geq 0$ $m_0 > 0$ $\mu \neq 0$	Dirichlet

Theorem I. [case W1 - MS, FeO 1; cases W2, S1, S2 - FeO 2]

The no-cutoff limit

$$S^n_{\kappa,\Lambda}(f_1 \cdots f_n) = \lim_{\Lambda \to R^3} \lim_{\kappa \to \infty} S^n_{\kappa,\Lambda}(f_1 \cdots f_n)$$

exists in all the above cases. The $\{S^n\}$ satisfy the axioms for Euclidean Green's functions of [OS 1, 2] (with the possible exception, in case S1, of clustering). Hence they are the Euclidean Green's functions of a uniquely determined Wightman theory. (Again, in case S1, the vacuum need not be unique.) They are also the moments of a unique probability measure on $\mathcal{S}'_R(R^3)$.

Theorem II. [W1 - MS, FeO 1; W2, S2, FeO 2]

This Wightman theory has a non-zero mass gap in cases W1, W2, and S2.

Theorem III. [W1 - MS, FeO 1; W2, FeO 2]

The $\{S^n\}$ are C^∞ in λ and analytic in μ. Perturbation theory provides an asymptotic expansion for the Euclidean Green's functions.

Three tools are used in the proofs of Theorems I, II, and III: the cluster expansion, correlation inequalities, and the phase space cell expansion.

The Cluster Expansion [GJS], [S] is a weak coupling (high temperature) expansion used to control the infinite volume limit in the weak coupling cases W1 and W2. It is based on the observation that if there were no coupling between different unit cubes in a partition of space-time, the infinite volume limit would be trivial. In particular, suppose we partition space-time into disjoint unit cubes and use boundary conditions having zero Dirichlet data on all surfaces of these unit cubes. Then if supt $\{f_1, \cdots f_n\} \subset \Lambda_1$ where Λ_1 is a union of unit cubes

$$S^n(f_1 \cdots f_n) = \lim_{\Lambda \to R^3} S^n_\Lambda(f_1 \cdots f_n)$$

$$= \lim_{\Lambda \to R^3} Z_\Lambda^{-1} \langle \Phi(f_1) \cdots \Phi(f_n) e^{-V_\Lambda} \rangle_{m_0}$$

formally (since $V_{\Lambda, \kappa=\infty}$ does not exist)

$$= \lim_{\substack{\Lambda \to R^3 \\ \Lambda_1 \subset \Lambda}} Z_{\Lambda_1}^{-1} Z_{\Lambda \sim \Lambda_1}^{-1} \langle \Phi(f_1) \cdots \Phi(f_n) e^{-V_{\Lambda_1}} \rangle_{m_0} \langle e^{-V_{\Lambda \sim \Lambda_1}} \rangle$$

formally

$$= S^n_{\Lambda_1}(f_1 \cdots f_n)$$

Of course we do not have this complete decoupling in practice, but in the weak coupling cases W1 and W2 different cubes are exponentially decoupled, i.e. the coupling between x_1 and x_2 is roughly $e^{-m_0 |x_1 - x_2|}$ with m_0 large and this suffices.

In case W2, while m_0 need not, a priori, be large, the Goldstone picture, in which the mass is the curvature at the minimum of the classical potential $V(x) = \lambda x^4 + \frac{1}{2}m_0^2 x^2 - \mu x$, suggests that the mass grows with μ. By translating the field $\Phi \to \Phi + f$ and scaling we can transform our original action having a large external field into an action having zero external field, large bare mass and small bare coupling constant. (These transformations are most conveniently executed when we use periodic boundary conditions.) We can then apply the old cluster expansion to the transformed theory to control the infinite volume limit.

Correlation Inequalities [N], [GRS] are used to give the infinite volume limit in the strong coupling cases S1 and S2. Firstly, when we have zero Dirichlet data on $\partial\Lambda$, Nelson's monotonicity says

$$S^n_\Lambda(f_1 \cdots f_n)_{\text{Dirichlet}} \le S^n_{\Lambda'}(f_1 \cdots f_n)_{\text{Dirichlet}}$$

if $\Lambda \subset \Lambda'$, $f_i \ge 0$, and $\mu \ge 0$. It is important that, since we have a φ^4 theory, this inequality is true for full Dirichlet boundary conditions. (We need to keep the Wick ordering matched to the covariance to do the renormalization properly.) We now only need to get an upper bound. This follows in three steps:

1) $S^n_\Lambda(f_1 \cdots f_n)_{\text{Dirichlet}} \le S^n_\Lambda(f_1 \cdots f_n)_{\text{periodic}}$

   if $f_i \ge 0$, $\mu \ge 0$,

2) $S^n_\Lambda(f_1 \cdots f_n)_{\text{periodic},\mu} \le S^n_\Lambda(f_1 \cdots f_n)_{\text{periodic},\mu'}$

   if $f_i \ge 0$, $0 \le \mu \le \mu'$,

and

3) $|S^n_\Lambda(f_1 \cdots f_n)_{periodic, \mu'}| \le (n!)^{\frac{1}{2}} |f_1| \cdots |f_n|$

by the cluster expansion in case W2 provided we choose μ' large enough.

All three correlation inequalities used above follow from the second Griffiths inequality and the observation that the system on the right hand side of the inequality is more ferromagnetic than that on the left.

The Phase Space Cell Expansion [GJ] is used to control the ultraviolet limit. The extensive use of complicated boundary conditions in the cluster expansion suggests that we use an ultraviolet cutoff based on the representation

$$(-\Delta + m_0^2)^{-1}(x, y) = \int_0^\infty dt e^{-m_0^2 t} \int P^t_{xy}(d\omega) B(\omega)$$

$$P^t_{x,y}(d\omega) = \text{conditional Wiener measure}$$

$B(\omega)$ determines boundary conditions of the bare two point function. Furthermore in the PSCE it is obligatory that we be able to introduce different momentum cutoffs in different regions of (Euclidean) space. To this end we define an auxilliary Gaussian field

$$\psi(t, x) \quad t \in (0, \infty) \quad x \in R^3$$

whose two point function is given by

$$\langle \psi(t,x)\psi(s,y)\rangle = \delta(t-s)e^{-m_0^2 t}\int P^t_{x,y}(d\omega)B(\omega)$$

Then our familiar field $\Phi(x)$ is given by

$$\Phi(x) = \int_0^\infty \psi(t,x)dt$$

and an ultraviolet cutoff field $\Phi_\kappa(x)$ is given by

$$\Phi_\kappa(x) = \int_{\kappa^{-2}}^\infty dt\psi(t,x) .$$

The latter statement is justified by the calculation

$$\langle \Phi_\kappa(x)\Phi_\kappa(y)\rangle \le \int_{\kappa^{-2}}^\infty dt\, e^{-m_0^2 t} e^{-\frac{(x-y)^2}{4\epsilon}} (4\pi t)^{-3/2}$$

$$= \int_{\kappa^{-2}}^\infty dt \int d^3p\, e^{ip\cdot(x-y)} e^{-t(p^2+m_0^2)}$$

$$= \int d^3p\, e^{ip\cdot(x-y)} \frac{e^{-\kappa^{-2}(p^2+m_0^2)}}{p^2+m_0^2}$$

(Euclidean) momenta obeying $p^2 > \kappa^2$ is suppressed. Notice that even in the presence of these cutoffs and with non-free boundary conditions,

1) the bare propagator decays exponentially with mass m_0

2) the action is local.

We use these cutoffs in the PSCE to derive ultraviolet uniform estimates which can, in turn, be plugged into the cluster expansion to yield estimates that are independent of both the ultraviolet and volume cutoffs. For details see [FeO 1, 2].

Outlook. The mathematical techniques developed so far seem to be sufficient to allow us to carry over to $(\varphi^4)_3$ models all the detailed analysis of the $P(\varphi)_2$ models. But $(\varphi^4)_3$ models also allow us to study some aspects of quantum field theory not present in $P(\varphi)_2$ models, e.g. the problem of ultraviolet divergences or the presence of Goldstone bosons in $(\vec{\varphi}^2)_3^2$ models.

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N° 248 – Les méthodes mathématiques de la théorie quantique des champs

Poetic Phenomena in (Two Dimensional) Quantum Field Theory : Non-Uniqueness of the Vacuum, the Solitons and All That

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ABSTRACT

Various aspects of the construction and analysis of new Bose Quantum field models are described. Aspects of convergence, long range order, super-selection rules, and the quantum soliton are emphasized.

RESUME

Plusiseurs aspects de la construction et de l'analyse des nouveaux modèles des champs quantiques sont décrits. On insiste sur les aspects de convergence, l'ordre à longue portée, les règles de super sélection et le soliton quantique.

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Poetic Phenomena in (Two Dimensional) Quantum Field Theory: Non-Uniqueness of the Vacuum, the Solitons and All That

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I. Introduction: The programm and the framework

I. 1 Outline of too big a programm:
My contribution to these proceedings is centered around the construction and analysis of new Bose quantum field models which are interesting from the following points of view:
1) For these models the construction of a vacuum sector theory in the sense of Wightman requires some new methods which may be of more general interest for quantum field theory and statistical mechanics.

2) They are a fascinating laboratory for testing old and new field theoretic concepts and programms such as:

a) Ultraviolet renormalizations.
b) Accuracy (e.g. convergence or Borel summability) of perturbation theory; [E1, F1, 2].
c) Long range forces and non-uniqueness of the vacuum; [C1, F3]
d) Long range order and spontaneous symmetry breaking; [G1]
e) Non-translation invariant vacuum states; [F4]
f) Super-selection rules: The quantum soliton: [D1, 2 ; F4, 5 ; G2 ; C2]
g) Goldstone bosons ; [C3, E2, L1].

Each circle of problems mentioned here may be considerably clarified and sharpened by a concentrated effort of what one calls mathematical physicists. Hopfully it will result in a

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disproof of "T ....'s Theorem : One can never learn anything from the axiomatists".

Here I can of course at best formulate some of these problems in precise terms and report results which may be relevant steps towards their solution. Nevertheless I hope my contribution is adventurous enough to deserve being part of these proceedings.

Even if I restrict my analysis to (1) and (2)(b), (d), (f) the problems involved have a degree of complexity which makes it impossible to present detailed arguments or proofs at this place. I shall however strictly follow the convention that whatever I state as a result (e.g.a theorem or lemma) has a written proof which has been discussed with at least three different mathematical physicists who have found no mistakes.

In the following I mostly study quantum field theories in two space-time dimensions. Therefore, as statistics is a matter of convention in two dimensions, I may consider only Bose field theories. Some of the models can however be regarded, more naturally, as Fermi field theories, [C4].

I.2 Notations:

The number of space-time dimensions is denoted by d ; $\xi = (x, t)$ is a point in $\mathbb{R}^d$. We will be concerned with the construction and analysis of models of a self interacting, relativistic Bose quantum field

$$\vec{\phi}(\xi) = (\phi_1(\xi), \dots, \phi_N(\xi))$$

over $\mathbb{R}^d$, with real, scalar components $\{\phi_\alpha(\xi)\}_{\alpha=1}^{N}$. We set $q = (\alpha, \xi)$.

In general N = 1,2 or 3 and d = 2. Occasionally we will mention results for d=3.

The Schwartz space over $\mathbb{R}^d$ is denoted by $\mathcal{S}(\mathbb{R}^d)$; $\mathcal{S}_C = [\mathcal{S}(\mathbb{R}^d)]^{\times N}$, $\mathcal{S}$ is the real part of $\mathcal{S}_C$, $\mathcal{S}_C'$ and $\mathcal{S}'$ are the corresponding spaces of N-component, tempered distributions. Finally $\mathscr{S}$ is the tensor algebra over $\mathscr{S}_{\mathbb{C}}$ in the sense of Borchers; [B1, O1].

A relativistic model for $\vec{\phi}$ is a state on $\mathscr{S}$ (a sequence of Wightman distributions) $\omega^{\vec{\phi}} = \{ W_n(q_1, \ldots, q_n) \}_{n=0}^{\infty}$ satisfying the Wightman axioms with the possible exception of the cluster property (uniqueness of the vacuum); $\omega^{\vec{\phi}}$ is then called a Wightman state. It uniqualy determines a sequence

$$\underline{S}^{\vec{\phi}} = \{ S_n(q_1, \ldots, q_n) \}_{n=0}^{\infty}$$

of Euclidean Green's functions (EGF) which are the restrictions of the Wightman functions to the Euclidean region. Osterwalder and Schrader have found sufficient conditions (Axioms (E0)-(E4) of [O1]) for $\underline{S}^{\vec{\phi}}$ to be the EGF's of a unique Wightman state $\omega^{\vec{\phi}}$

I.3 Euclidean field theory :

A field theory (a Wightman state) is called Symanzik-Nelson (S-N) positive if and only if

(1) the EGF's $S_n(q_1, \ldots, q_n)$ are locally integrable functions on $\mathbb{R}^{dn}$, for all n;

(2) $\underline{S}^{\vec{\phi}}$ is a state on $\mathscr{S}$, i.e. the EGF's are the vacuum expectation values (v.e.v.) of commutative, Euclidean covariant fields (also denoted by $\vec{\phi}$), whence the name: Euclidean field theory.

Assuming ((1) and) (2) Borchers et al. have isolated necessary and sufficient conditions for the EGF's to be the moments of a Euclidean invariant probability measure $d\nu(\vec{\phi})$ on $\mathscr{S}'$ [B2]. It is called a physical measure; see also [N1, F6]. Euclidean field theory establishes a famous connection between q.f.t. and classical statistical mechanics, [G3].

Theorem 1 : (Decomposition into pure phases)

Let $d\nu$ be a physical measure. Then "almost all" components of $d\nu$ ergodic under the time-translation group are physical measures associated with a unique Wightman state satisfying all axioms <u>including the cluster property</u>; (i.e. Poincaré covariance and S-N positivity are stable under the decomposition into pure phases).

A detailed version of this theorem including "stability of estimates" results have been proved in [F6] , where we also derived sufficient conditions for the spontaneous breaking of an internal symmetry assuming long range order (of relevance for Section IV) and for $d\nu$ to be a physical measure.

I.4 The simplest S-N positive theory: The free field.

The free field $\vec{\phi}$ is described by a Gaussian physical measure $d\nu^0$ with mean 0 and covariance $(-\Delta + m^2)^{-1}$. Here Δ is the Laplacian and m the bare mass. Let Λ be a rectangle in $\mathbb{R}^d$. For the construction of interacting fields we must consider "freefields" with periodic, Dirichlet and Neumann boundary conditions (b c) at $\partial\Lambda$; [G3, Q2]. The space of real, N-component periodic C^∞ functions and the space $C_0^\infty(\Lambda)^{\times N}$ are also denoted by $\mathcal{S}$, their topological dual by $\mathcal{S}'$ (without danger of confusion).

The "free field" with periodic,.... b c at $\partial\Lambda$ is described by the Gaussian measure on $\mathcal{S}'$ with mean 0 and covariance $(-\Delta_\Lambda + m^2)^{-1}$ where Δ_Λ is the Laplacian with periodic, 0-Dirichlet.... bc at $\partial\Lambda$.

I.5 Interacting fields :

Let $\mathcal{L}_I(\vec{\phi})$ be a formal interaction Lagrangian without derivative coupling. For the construction of a relativistic field $\vec{\phi}$ with interaction $\mathcal{L}_I$ Nelson has developped the method of multiplicative functionals; [N 1] : One defines the Euclidean action by

$$U_\Lambda(\vec{\phi}) = \int_\Lambda d\xi :\mathcal{L}_I(\vec{\phi}):(\xi),$$

where the colons denote Wick ordering with respect to $d\nu^0$ Let $\langle \cdot \rangle_{0,\Lambda}$ denote expectation with respect to $d\nu^0_\Lambda$. A model for a relativistic field $\vec{\phi}$ with interaction $\mathcal{L}_I$ is presently usually constructed in three steps, [N1, Q2,3] :

Step 1 : Define

$$d\nu_\Lambda(\vec{\phi}) \equiv \langle e^{-V_\Lambda(\vec{\phi})} \rangle_{0,\Lambda}^{-1} e^{-V_\Lambda(\vec{\phi})} d\nu_\Lambda^\circ(\vec{\phi})$$

rigorously as a measure on $\mathcal{S}'$. For d = 3 the definition of $d\nu_\Lambda$ requires ultraviolet renormalizations [F7, M1] and as a consequence the measures $d\nu_\Lambda$ and $d\nu_\Lambda^\circ$ are presumably mutually singular.

Step 2 : Show that the characteristic functionals

$$\langle e^{i\vec{\phi}(\vec{f})} \rangle_\Lambda \equiv \int_{\mathcal{S}'} d\nu_\Lambda(\vec{\phi}) e^{i\vec{\phi}(\vec{f})}$$

converge, as $\Lambda \nearrow \mathbb{R}^d$, for all $\vec{f} \in \mathcal{S}$.

Step 3 : Show that the measure $d\nu(\vec{\phi})$ obtained from the limiting functional $\langle e^{i\vec{\phi}(\vec{f})} \rangle \equiv \langle e^{i\vec{\phi}(\vec{f})} \rangle_{\mathbb{R}^d}$ is a physical measure; e.g. that its moments satisfy the Osterwalder-Schrader axioms [O1]. (See also [F6]).

After having completed Steps 1 - 3 for a given interaction one can start to investigate the physically more interesting problems described in (2), (b) - (g).

II An illustration of programms (1) and (2)(a)(b)(d)(g) : Construction of a field $\vec{\phi}$ with a $(\vec{\phi}\cdot\vec{\phi})^2$ – interaction

We first sketch a slightly non-conventional version of Steps 1 - 3 for the construction of an interacting field $\vec{\phi}$ with interaction Lagrangian

$$\mathcal{L}_I(\vec{\phi}) = g(\vec{\phi}\cdot\vec{\phi})^2 - \sigma\phi_1^2 - \mu\phi_1 \qquad (2.1)$$

with $g > 0$, $\sigma \geq 0$, μ real, N=1, 2 or 3 and d = 2 or 3. Our approach is best summarized as follows :"Controlling the vacuum energy density (pressure) means controlling the theory". It illustrates (1) and (2)(b). Details of results and proofs presented here can be found in [F2,6].

Let Λ be a rectangle in $\mathbb{R}^d$, Γ_Λ the momentum lattice dual to

$$\Lambda\,,\ \underline{k} = \{k_i\}_{i=1}^{n+1},\ k_i \in \Gamma_\Lambda\,,\ i=1,\dots,n;\quad k_{n+1} = -\sum_{i=1}^{n} k_i\,, \tag{2.2}$$
$$\text{and}\quad \underline{\varepsilon}(\underline{k}) = \{\varepsilon_i\, e^{ik_i\xi}\}_{i=1}^{n+1}.$$

We introduce the "funny" Euclidean action

$$U_\Lambda^{(n)}(g,\sigma,\mu,\underline{\varepsilon}(\underline{k})) = \int_\Lambda d\xi\, \{ g:(\vec{\phi}\,\vec{\phi})^2:(\xi) - \sigma:\phi_1^2:(\xi) - \mu\phi_1(\xi) - \sum_{i=1}^{n+1} \varepsilon_i\, e^{ik_i\xi} \phi_{\alpha_i}(\xi)\}\,, \tag{2.3}$$

where $\alpha_i = 1,2,\dots,N$, for all i.

In three dimensions the correct definition of $U_\Lambda^{(n)}(\cdots)$ requires introducing counterterms which are however <u>indepen-dent</u> of σ,μ and $\underline{\varepsilon}(\underline{k})$; [F7, M1].

II.1 Step 1 of the construction :

We define the "funny" pressure

$$p_\Lambda^{(n)}(g,\sigma,\mu,\underline{\varepsilon}(\underline{k})) = |\Lambda|^{-1} \log \langle e^{-U_\Lambda^{(n)}(g,\sigma,\mu,\underline{\varepsilon}(\underline{k}))} \rangle_{0,\Lambda}\,, \tag{2.4}$$

where $\langle\ \cdot\ \rangle_{0,\Lambda}$ is the free field expectation with periodic bc at $\partial\Lambda$. For d=3 the definition of $p_\Lambda^{(n)}$ involves Ultraviolet renormalizations: (2)(a)! ; [F7, M1].

Then Step 1 of I.5 can be reduced to proving bounds on $p_\Lambda^{(n)}(g,\sigma,\mu,\underline{\varepsilon}(\underline{k}))$ which are uniform in Λ. For N=1 such bounds follow from [N1, G3, Q2] $(d=2)$, and from [F7, S4], $(d=3)$. The extension to N>1 is straightforward; [F2].

The Fourier transform of the truncated $(n+1)$-point EGF associated with the cutoff action $U_\Lambda^{(0)}(g,\sigma,\mu)$ is then given by

$$\tilde{S}^{\Lambda}_{n+1}(\alpha_1,k_1,\dots,\alpha_n,k_n,\alpha_{n+1},k) = \delta_\Lambda\Big(\sum_{i=1}^{n} k_i + k\Big)\, s^{\Lambda}_{n+1}(\alpha_1,k_1,\dots,\alpha_n,k_n,\alpha_{n+1})\,, \tag{2.5}$$

where

$$s^{\Lambda}_{n+1}(\alpha_1,k_1,\cdots) = \frac{\partial^{n+1}}{\partial\varepsilon_1\cdots\partial\varepsilon_{n+1}}\, p_\Lambda^{(n)}(g,\sigma,\mu,\underline{\varepsilon}(\underline{k}))\Big|_{\underline{\varepsilon}=\underline{0}}\,,$$

$$\text{and}\quad \delta_\Lambda(k) = |\Lambda|\,\delta_{k,0}\,. \tag{2.6}$$

Relations similar to (2.5) and (2.6) were first used in [L2] in a statistical mechanics context and rediscovered in [F6] in the context of the ϕ^4 - model.

II.2 Step 2 of the construction : Large μ - expansion and Lee-Yang theorem

For d=2, N=1 Spencer has proved the following result; [S1]:

Theorem 2 :

There exist constants c_1, c_2 such that for
$|Im\,\mu| < c_1$, $|Re\,\mu| > c_2$, $g > 0$, $\sigma \geq 0$, and
$|\varepsilon_i| < \frac{1}{n+1}$, $i = 1, \cdots, n+1$

$$\lim_{\Lambda \to \mathbb{R}^2} p_\Lambda^{(n)}(g,\sigma,\mu,\underline{\varepsilon}(\underline{k})) = p^{(n)}(g,\sigma,\mu,\underline{\varepsilon}(\underline{k}))$$

exists and is rotation invariant. In [F7] an extension of this theorem to d=3, N=1 is announced. Whereas in [S1, F7] $\underline{\varepsilon}(\underline{k}) = \underline{0}$, modefications of these results accounting for $\underline{\varepsilon}(\underline{k}) \neq \underline{0}$ and the extension to N > 1 can be found in [F2] .

The intuitive argument leading to Theorem 2 is as follows: For simpicity we let $\sigma = 0$ and we set

$$\phi_1' = \phi_1 + \left(\frac{\mu}{4g}\right)^{1/3}, \quad \phi_\alpha' = \phi_\alpha, \; \alpha = 2, \cdots, N.$$

Expressing the Euclidean action $U_\Lambda^{(0)}(g,\sigma,\mu)$ - see (2.3)- as a functional of $\vec{\phi}'$ and absorbing all quadratic terms in the free Langrangian the dimension-less coupling constants of the terms cubic and quartic in $\vec{\phi}'$ are $O\left(\left(\frac{4g}{\mu}\right)^{1/3}\right) \ll 1$ for $\mu \gg 1$. Therefore the cluster expansion [Q3, S1, F7] converges.

Theorem 3 :

For $g > 0$, $\sigma \geq 0$ and for all $\Lambda \subseteq \mathbb{R}^2$ and $\underline{k}$ as in (2.2) $p_\Lambda^{(n)}(g,\sigma,\mu,\underline{\varepsilon}(\underline{k}))$ is holomorphic in μ and $\varepsilon_1, \cdots, \varepsilon_{n+1}$ for Re $\mu \neq 0$ and $|\varepsilon_i| < r(g,\sigma,\mu)$, $i = 1, \cdots, n+1$, where $r(g,\sigma,\mu)$ is positive for $Re\,\mu \neq 0$ and is independent of Λ.

Concerning the proof of Theorem 3 we remark: A combination of the Lee-Yang theorem of Suzuki and Fisher [S2] with the convergence of the rotator approximation of the $(\vec{\phi}\cdot\vec{\phi})^2$ -

theory [D3] yields a Lee-Yang theorem for $p_\Lambda(g,\sigma,\mu, Re(Im)\underline{\varepsilon}(\underline{k}))$: Analyticity in μ for $Re\,\mu \neq 0$. Combining this with analyticity properties of $p_\Lambda(g,\sigma,\mu,\underline{\varepsilon}(\underline{k}))$ in $\varepsilon_1, \dots, \varepsilon_{n+1}$ and μ for $|Im\,\mu| < c_1$, $|Re\,\mu| > c_2$ (see Theorem 2) and applying the Malgrange-Zerner (generalized tube) theorem we obtain as a result Theorem 3. Details can be found in [F2].

If we now apply equations (2.5) and (2.6) and Theorems 2, 3 we observe that Step 2 is complete for $\mu \neq 0$.

The existence of the limiting theories as $\mu \nearrow 0$ and $\mu \searrow 0$ follows for N=1,2 from correlation inequalities [D3] and for N=3 from uniform bounds by a compactness argument : see [F2]. Step 3 is routine ; (our approach makes the verfication of the Osterwalder-Schrader axioms [O1] particularly easy).

II.3 The final result, an illustration of (1) and (2)(a)(b): We summarize these and other finings in

Theorem 4 :

For $g > 0$, $\sigma \geqslant 0$ and μ real the Lagrangion (2.1) determines a Wightman state $\omega^{\vec{\phi}}_{\mu}$ satisfying the axioms with the possible exeception of uniqueness of the vacuum for $\mu = 0$. For $\mu \neq 0$ the vacuum is unique, and the energy-momentum spectrum has a mass gap. For N= 1,2

$$\lim_{\mu \searrow 0} \omega^{\vec{\phi}}_{\mu} \equiv \omega^{\vec{\phi}}_{+} \quad \text{and} \quad \lim_{\mu \nearrow 0} \omega^{\vec{\phi}}_{\mu} \equiv \omega^{\vec{\phi}}_{-}$$

exist and are unique.

For N=1,2,3 d=2 and $\mu \neq 0$ the theory is uniquely determined by its perturbation expansion; (Borel summability [E1] and analyticity in the bare parameters [F2]).

Remarks : The ultraviolet renormalizations required in Theorem 4 for d=3 are taken from [G4, F7, M1, S4].
Further results - concerning equivalence of boundary conditions, $\vec{\phi}$ - and: $\vec{\phi}^2$: - bounds (for d=3!) can be found in [F2], [S4], respectively.

II.4 Spontaneous symmetry breaking [G1] :

For the constuction of soliton states for d=2 (Section IV) we need

Theorem 5 :

Let $d=2$, $N=1$ or 2, $g \ll 1$, $\sigma \gg 1$; [G1].

Then $\omega_+^{\vec{\phi}} \neq \omega_-^{\vec{\phi}}$ and

$$\varphi_c \equiv \omega_+^{\vec{\phi}}(\phi_1(x,t)) = -\omega_-^{\vec{\phi}}(\phi_1(x,t)) > 0$$

whence spontaneous $\phi_1 \mapsto -\phi_1$ symmetry breaking.

For N=1 this theorem is due to Glimm, Jaffe and Spencer who have proved the phase transition for ϕ_2^4 in an admirable paper [G1] .

The extension to N=2 requires only one new estimate which is given in [F2] .

II.5 Concerning (2)(g) : Remarks about the Goldstone boson.

As a consequence of the Goldstone theorem [E2] there is no spontaneous $\vec{\phi} \mapsto -\vec{\phi}$ symmetry breaking for d=2, N=2 or 3, and $\sigma = \mu = 0$!

For d=3, N=2 or 3 and $\sigma = \mu = 0$, $g \gg 1$ the 1/N - expansion [C3] predicts that the 0 (N) symmetry is spontaneously broken and $\phi_{transverse}$ couples the vacuum to the Goldstone boson one particle states. If this prediction is valid for N=2 then, as a consequence of a deep observation of [B3] and of $\vec{\phi}$-bounds settling domain problems [S4, F2], there exists a <u>scattering theory</u> for Goldstone bosons! Interesting results for the two point function of the $(\vec{\phi}\cdot\vec{\phi})_3^2$ -theory on a lattice are proved in [L1] . We conclude Section II with two problems:

1) Define "k-loop" contributions to the Vertex functions in a non-perturbative way and solve the relation

$$\frac{\text{"Goldstone expansion"}}{\text{1/N-expansion [C3]}} = \frac{\text{Cluster expansion [Q3]}}{\text{Perturbation-expansion}}$$

2) Derive monotonicity properties for φ_c as a function of the number N of components of $\vec{\phi}$.

III. An illustration of programms (1) and (2)(a)(b)(c) :

The quantum sine - Gordon equation

This section might be the most interesting one would we not suppress all details. The methods involved here are too numerous and complex to be even only sketched. The model we consider is defined by a field equation :

$$(\Box + m^2)\,\phi(x,t) = \varepsilon\lambda \; :\sin(\varepsilon\phi(x,t)+\theta):_1 \;, \qquad (3.1)$$

where d=2, N=1, $m^2 \geq 0$, λ real, $0 < \varepsilon^2 (\leq) 4\pi$, and $\theta \in [0, 2\pi)$. Wick odering is done with respect to a fixed bare mass 1. The interaction Lagrangian is

$$\mathcal{L}_I(\phi) = -\lambda \; :\cos(\varepsilon\phi+\theta):_1 \qquad (3.2)$$

The following three equivalence theorems are not only amusing but basic for the analysis of the s-G equation:
The Euclidean s-G field theory determined by (3.1) and (3.2) is equivalent to :

(A) A generalized, continuous spin ferromagnetic Ising model [F3] , so that most of the statistical mechanics methods of [G3, Q2,3] apply.

(B) The theory of the two dimensional, classical, two-component Yukawa - $(m>0)$, Coulomb gas $(m=0)$, respectively, in the grand canonial ensemble [F1,3] , so that all the results about these gases (see [F3]) apply. The two theories are equivalent if one identifies m with the exponential decay rate of the Yukawa potential, λ with the fugacity and ε with the charge of the point particles; (inverse temperature $\beta = 1$).
Equivalence (B) is interesting for the hydrodynamics of vortices of incompressible fluids in two dimensions which is equivalent to the theory of the Coulomb gas, [L3] .

(C) The theory of a two component Dirac field ψ $(d=2)$ with interaction Lagrangian

$$\mathcal{L}_I(\psi) = g/2 :j^\mu j_\mu: + M :\bar{\psi}\psi: + e^2 :j^0 V_c * j^0:$$

where $V_c(x) = \frac{\pi}{2}|x|$ is the Coulomb potential in one dimension. The free Lagrangian has no mass term. The equivalence identifies

$j^\mu = :\bar{\psi}\gamma^\mu\psi:$ with $-\frac{\varepsilon}{2\pi}\varepsilon^{\mu\nu}\partial_\nu\phi$

$:\bar{\psi}\psi:$ with $:\cos(\varepsilon\phi+\theta):_1$, $:j^0 V_c * j^0:$ with $:\phi^2:$ (Schwinger mechanism !) and sets

$$\frac{\varepsilon^2}{4\pi} = \frac{1}{1+g/\pi}, \quad m^2 = \frac{e^2}{1+g/\pi}, \quad \lambda = M .$$

For $e^2 = 0$ equivalence (C) has been clearly explained in [C4]. Results which are modified, simplified and generalized relative to [C4] have been found by many authors. For d=2 the fermion-boson equivalence is a completely general fact : It can be extended to the Yukawa model, fermion models with non-trivial internal symmetries (yielding examples for confined quantumnumbers [F3]), etc.

The sine-Gordon theory illustrates the programms

I.1 - (1) : The crucial step: We use equivalence (B) to prove that the vacuum energy denstiy is finite, (stability) [F3]. {(A) + [G3, Q2]} yield existence of Wightman states $\{\omega_\theta \mid \theta \in [0,2\pi)\}$ for $\varepsilon^2 < \frac{16}{\pi}$, [F1,3], and, by (B), of Gibbs equilibrium states for the classical gases, [F3] .

I.1 - (2)(a) : Let $m^2 = 0$. For $4\pi < \varepsilon^2 < 8\pi$ the theory has (non-super-) renormalizable UV divergencies, [F3] . For $\varepsilon^2 = 4\pi$ it describes the free, massive Diracfield [C4, F1] . For $\varepsilon^2 \geq 8\pi$ the theory seems to be meaningless, [C4, F3] .

I.1 - (2)(b) : If $\varepsilon^2 < 4\pi$ and $m^2 = \frac{e^2}{1+g/\pi} > 0$ there exists a $\lambda_0(\varepsilon, m) > 0$ such that for $|\lambda| < \lambda_0(\varepsilon,m)$ the pertur bation series for the EGF's converges yielding a Wightman state ω_θ. The energy - momentum spectrum has a mass gap and for $\varepsilon^2 < \varepsilon_0^2 < 4\pi$ an isolated one particle shell. This result is proved in [F1] and is based on [Q3,S3].

I.1. - (2)(c): Equivalence (C) leaves the value of the angle θ in the Lagrangian (3.2) undetermined; θ determines the value of a universal, constant electric field which affects the dynamics of the field in a non-trivial way so that we obtain infinitely many inequivalent Wightman states $\{\omega_\theta \mid \theta \in [0,2\pi)\}$. As $M \to 0$ all these states become equivalent: The limiting theory is chirally invariant.

On a formal level all these and other phenomena have been discovered and analyzed in [C1]. Rigorous proofs ("mathematische Klugscheisseleien") are given in [F3].
Fascinating heuristic results on the mass spectrum of the sine-Gordon equation which have the flavour of being exact can be found in [D2]. The results of [F5 , D2] and [F3] (where the vacuum energy density and the anomalous dimension of ψ are calculated explicitly for $e^2 = 0$) suggest the
Conjecture: The mass spectrum and other quantities of the sine-Gordon theory are explicitly calcuable.

IV. An illustration of program (2)(f): The quantum soliton

IV. 1 What is a quantum soliton?

Every theory of a canonical scalar field $\vec{\phi} = (\phi_1, \ldots, \phi_N)$ in two dimensions has the conserved currents

$$\vec{j}^{\,\mu}(\xi) = \{j^\mu_\alpha(\xi)\}^N_{\alpha=1} \qquad j^0_\alpha(\xi) = \operatorname{grad}\phi_\alpha(\xi)$$

$$j^1_\alpha(\xi) = \partial_t \phi_\alpha(\xi) \equiv \pi_\alpha(\xi) ,$$

and hence the conserved charges $\vec{Q} = \{Q_\alpha = \int dx \operatorname{grad}\phi_\alpha(x,t)\}^N_{\alpha=1}$.

Any vacuum sector in the sense of Wightman of such a theory is an eigenspace of $\vec{Q}$ with eigenvalue 0. Until recently it was never conceived that such a theory may have super-selection sectors on which

(A) the space-time translations are unitarily implemented, forming a continuous unitary group which satisfies the relativistic spectrum condition, and

(B) $\vec{Q} \neq 0$, (i.e. it was not conceived that $\vec{Q}$ can be non-trivial).

In the following a sector satisfying (A) and (B) is called a soliton-sector. A quantum soliton is a one particle state in a soliton-sector.

We now know that theories possessing soliton-sectors exist;[F4].

In order to be precise we must distinguish:

(α) Soliton states = the vectors in a soliton-sector.

In general only two dimensional theories may have soliton states.

(β) Non-space-translation invariant vacuum states: Ground states of the Hamiltonian which are eigenstates of $\vec{Q}$ with eigenvalue $\neq (0,\cdots 0)$. Space translations are in general not unitarily implemented on the sector reconstructed from such a vacuum state. Presumably such states only exist for $d \geq 3$.

A rigorous analysis of (α) and a preliminary discussion of (β) are given in [F4].

IV. 2 General results about soliton-sectors:

For d=2 a general theory of soliton-sectors has been developped in [F4]. The main results are :

1. Let $\omega^{\vec{\phi}}$ be a Wightman state on $\mathscr{S}$ and d=2 . The theory reconstructed from $\omega^{\vec{\phi}}$ has soliton-sectors if and only if it has at least two pure phases χ_1, χ_2 (see Theorem 1,I.3) with

$$\omega^{\vec{\phi}}_{\chi_1}(\phi_\alpha(\xi)) \neq \omega^{\vec{\phi}}_{\chi_2}(\phi_\alpha(\xi))$$, for some

$\alpha \in \{1, \cdots, N\}$, + technical conditions specified in [F4] (which are met in the models disucssed below).

This result emphasizes the connection between the existence of soliton-sectors and the non-uniqueness of the vacuum, (i.e. the existence of a phase transition).

2. For $d \geq 3$ soliton-sectors do in general not exist, except possibly in a theory with a dynamically broken, non-abelian gauge symmetry.

3. Under natural assumptions (see [F4]) which are met in all models analyzed so far the vacuum - and the soliton-sectors are labelled by the elements of a group which (as a consequence of the Goldstone theorem [E2]) is in general discrete. This group is called the soliton-group. Using this group structure one can construct field bundles [D1] which have non-vanishing matrix elements between a vacuum - and a soliton-sector.

4. Let ψ be a soliton-state. Then

$$\lim_{x\to\infty} \langle \psi, \phi_\alpha(x,t)\psi \rangle - \lim_{x\to-\infty} \langle \psi, \phi_\alpha(x,t)\psi \rangle = \langle \psi, Q_\alpha \psi \rangle$$

and hence, by (B), there is an α such that the function $\langle \psi, \phi_\alpha(x,t)\psi \rangle$ has a kink. From this we conclude that

5. Parity is spontaneously broken in a soliton-sector. The spectrum of the space-translation group is purely continuous.

6. If a soliton-sector contains one particle states, i.e. quantum solitons, then the conjugate sector obtained by space-reflection contains one-particle states, too, which are called anti-solitons. A soliton-sector and its conjugate sector have opposite $\vec{Q}$-charge. Any numbers of pairs of solitons and anti-solitons form a vector in a vacuum sector of the theory.

For detailed statements and proofs of these and other results see [F4].

IV. 3 Applications to models

1) The $[g(\vec{\phi}\cdot\vec{\phi})^2 - \sigma\phi_1^2]_2$ - models with $N = 1$ or 2:

Theorem 6: Under the conditions of Theorem 5, Section II.4, there exist (at least) two pure phase vacuum states $\omega_+^{\vec{\phi}}$ and $\omega_-^{\vec{\phi}}$ and associated with these one soliton - and one conjugate anti-soliton-sector satisfying conditions (A) and (B) of IV.1. If ψ is a vector in a (anti-) soliton-sector which is in the quadratic form domain of $\phi_1(\xi)$ then

$$\lim_{x\to+\infty} \langle \psi, \phi_1(x,t)\psi \rangle = \omega_{\pm \atop (-)}^{\vec{\phi}}(\phi_1(x,t)) = {+ \atop (-)}\varphi_c$$

but $$\lim_{x\to-\infty} \langle \psi, \phi_1(x,t)\psi \rangle = \omega_{- \atop (+)}^{\vec{\phi}}(\phi_1(x,t)) = {- \atop (+)}\varphi_c$$

$$Q_1 \psi = (\pm) 2 \varphi_c \; , \; (Q_2 \psi = 0).$$

The soliton group has four elements $\{e, s, \bar{s}, i\}$. We label $\omega_+^{\vec{\phi}}$ by the identity element e, $\omega_-^{\vec{\phi}}$ by i , the soliton-sector by s and the anti-soliton-sector by $\bar{s}$. With this labelling the soliton group has the multiplication table:

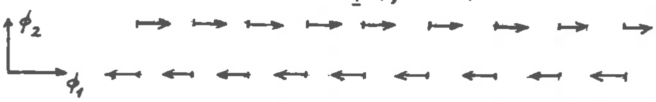
	e	s	$\bar{s}$	i
e	e	s	$\bar{s}$	i
s	s	e	i	$\bar{s}$
$\bar{s}$	$\bar{s}$	i	e	s
i	i	$\bar{s}$	s	e

As a consequence of the structure of the soliton-group we obtain the following result: Assume that there exists a quantum (anti-) soliton (one-particle state). Let N_s be the number of solitons and $N_{\bar{s}}$ the number of anti-solitons in a scattering state. Then on a vacuum sector of the $[g(\vec{\phi}\cdot\vec{\phi})^2 - \sigma\phi_1^2]_2$- theory $N_s - N_{\bar{s}}$ is even, whereas on the (anti-) soliton-sector $N_s - N_{\bar{s}}$ is odd.

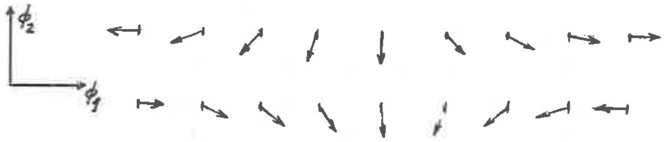
As an example: A two-soliton state is in a vacuum sector and hence has $\vec{Q}$-charge 0 rather than $4\varphi_c$!

These and other results are proven in [F4] where we give an explicit construction for the soliton-states of these models using * automorphisms. Our construction must be placed within the algebraic framework of [D1] .

For N = 2 or 3 the physical interpretation of the soliton-states of the $[g(\vec{\phi}\cdot\vec{\phi})^2 - \sigma\phi_1^2]_2$ – theory is as follows: For simplicity consider this theory on a space lattice and interpret $\vec{\phi}$ as a polarization field. It then describes an anisotropic, anharmonic dielectric chain which has two groundstates - we plot the vector fields $\omega_\pm^{\vec{\phi}}(\vec{\phi}(x,t))$-:



The soliton-states are obtained by twisting a ground-state by a total angle π :



For N = 1 the construction and interpretation of soliton-states are more difficult; but see [F4].

2) The sine - Gordon equation with $m^2 = 0$:
Let $m^2 = 0$ $(\varepsilon^2 < 16/\pi$, λ real and $\theta \in [0, 2\pi))$.
Then the Lagrangian of the sine - Gordon theory studied in Section III is invariant under the substitutions

$$\phi \mapsto \phi + \frac{2n\pi}{\varepsilon}, \quad n \in \mathbb{Z}. \tag{4.1}$$

It is plain that the symmetry (4.1) of the dynamics is spontaneously broken and that the theory has therefore infinitely many Wightman states $\{\omega_n^\phi\}_{n=-\infty}^{\infty}$ labelled by the vacuum expectation value of the field: $\omega_n^\phi(\phi(x,t)) = \frac{2n\pi}{\varepsilon} + const.$

These states coincide however on the physical observables generated by the fields: grad ϕ, π, $\{:\cos(\varepsilon\phi + \vartheta):_1 \mid \vartheta \in [0, 2\pi)\}$, where π is the momentum canonically conjugate to ϕ. A soliton-state of Q - charge n coindides with ω_m^ϕ on all functions of ϕ localized in $\{x \ll -1\}$ and with ω_{m+n}^ϕ on the ones localized in $\{x \gg +1\}$, for some $m \in \mathbb{Z}$.
Let g_n be a function with
$\lim_{x\to-\infty} g_n(x) = 0$, $\lim_{x\to+\infty} g_n(x) = \frac{2n\pi}{\varepsilon}$, $\text{grad}\, g_n \in L^2(\mathbb{R})$.
A soliton-state of Q - charge n is obtained by applying the "operator" $e^{i\pi(g_n)}$ to on arbitrary vector ψ in the vacuum sector reconstructed from ω_m^ϕ, any m.

Theorem 7 : Under the above assumption there exist infinitely many soliton-sectors satisfying conditions (A) and (B) of IV. 1. They are eigenspaces of the charge Q with eigenvalues $\frac{2n\pi}{\varepsilon}$, $n \in \mathbb{Z}$. The soliton group is equal to $\mathbb{Z}$.

Remarks :

1. The isomorphism (C) of section III suggests that there exist local spinor fields (rather than just field bundles) with non - vanishing matrix elements between the vacuum sector and the Q - charge ± 1 (anti -) soliton - sectors. See [C4] .

2. The hard part in the proof of Theorems 6 and 7 is the verification of condition (A). See [F4] .

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Euclidean Fermion Fields

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RESUME

Le problème de la construction d'une théorie euclidienne des fermions qui contient les idées de probabilité analogues à la théorie pure des bosons, est posée. On propose une solution qui est basée sur une interprétation mathématique spécifique de la formule euclidienne de Mathews-Salam em terme des algèbres de Clifford.

ABSTRACT

The problem of constructing a Euclidean theory for Fermions which suitably incorporates probabilistic ideas analogous to pure Boson theory is posed. A solution is proposed which is based on a particular mathematical interpretation of the Euclidean Matthews-Salam formula in terms of Clifford algebras.

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1. Introduction. The problem of constructing a Euclidean theory for Fermions, which suitably incorporates probabilistic ideas analogous to the pure Boson theory [9, 10], may be posed as follows. Consider the Schwinger functions for a Dirac field interacting with a Boson field. Let us suppress the dependence of the Schwinger functions on the Boson field variables and moreover ignore, for simplicity, the problem of coincident Schwinger points. Then the Schwinger functions, which are skew symmetric in their arguments, define a linear functional

$$S: \Lambda(\mathcal{S}(R^4;V)) \to \mathbb{C}$$

where V is an eight dimensional complex vector space (4 for ψ and 4 for $\bar{\psi}$), $\mathcal{S}(R^4;V)$ denotes the Schwartz space with values in V, and $\Lambda(\mathcal{S})$ is the algebraic exterior algebra over $\mathcal{S}$. We wish to find a probability gage space $\mathcal{M}$, (see [16] for a definition) with expectation function E, and a linear map

$$\eta: \Lambda(\mathcal{S}(R^4;V)) \to \mathcal{M}$$

such that

$$S = E \circ \eta \quad . \tag{1.1}$$

In the corresponding pure Boson theory the map η is determined by the Euclidean Boson field φ by the requirement that $\eta = \varphi$ on $\mathcal{S}(R^4)$ and η extends as a homomorphism from the symmetric algebra over $\mathcal{S}(R^4)$ to random variables.

Of course one wants a solution to 1.1) which is formally Euclidean invariant and which, in its cutoff versions, is useful for removing cutoffs. The presence of a probability gage space provides a setting within which one may hope to establish correlation inequalities.

The solution to 1) which we propose here is based on giving a particular mathematical interpretation of the Euclidean Mathews-Salam formula [5] in terms of Clifford algebras [17, 3].

In the Euclidean region the Mathews-Salam formula for a scalar Yukawa interaction is informally given by

1.2) $S(x_1,\ldots,x_{2n},y_1,\ldots,y_m) =$

$$\frac{1}{N}\int \bar{\psi}(x_1)\ldots\bar{\psi}(x_n)\psi(x_{n+1})\ldots\psi(x_{2n})\varphi(y_1)\ldots\varphi(y_m)$$

$$\cdot\{e^{-\int\bar{\psi}(x)(\not{p}+m+\varphi(x))\psi(x)d^4x}\, d\psi d\bar{\psi}\}$$

$$\cdot e^{-\int(m^2\varphi^2+|\nabla\varphi|^2)d^4x - \int\mathcal{P}(\varphi)d^4x}\, d\phi$$

where φ is a Euclidean neutral scalar Boson field, and the fields ψ and $\bar{\psi}$ are anti-commuting 4 component Euclidean Fermion fields. That is, for all α,β,x and y,

$$[\psi_\alpha(x),\psi_\beta(y)]_+ = [\psi_\alpha(x),\bar{\psi}_\beta(y)]_+ = [\bar{\psi}_\alpha(x),\bar{\psi}_\beta(y)]_+ = 0 \quad .$$

The polynomial $\mathcal{P}(\varphi)$ may include renormalization terms as well as a Boson self-interaction. The meaning of the infinite dimensional φ integral as a Gaussian integral is well understood (at least with suitable cutoffs in the $\mathcal{P}(\varphi)$ term). Our purpose in this note is to describe how one can give a meaning to the factor in braces, i.e. to the $d\psi\, d\bar{\psi}$ integration, in such a way as to bring to the foreground an underlying probability gage space and thereby find a solution to 1.1).

If one informally carries out the $d\psi\, d\bar{\psi}$ integration in 1.2) one obtains expressions which can be justified in another way by using the formalism developed by Osterwalder and Schrader [13]. These expressions are the basis for fundamental advances made in the Y_2 theory by E. Seiler [18], O. McBryan [7,8] and Seiler and Simon [19, 20] .

In section 2 we discuss the meaning of $\int \ldots d\psi\, d\bar{\psi}$ for finitely many degrees of freedom, establish its connection with Clifford algebras and then discuss its meaning in infinite dimensions. In section 3 we return to equation 1.2) and show how to apply Section 2 to it.

We cannot say that equation 1.1) is the only way to formulate the problem of Euclidean Fermion fields cum probability theory. Other attempts, [2, 12, 13, 21], to find a Euclidean Fermi theory seem not to have sought a solution to equation 1.1). For example, Frohlich and Osterwalder [2] have described a number of natural approaches to a free Euclidean Dirac field, but although they have constructed a gage space (c.f. [2], Section V.1) there is no linear connection between the Schwinger functions and expectations of polynomials in the gage fields given, directly or indirectly, in their work, even in the case of a free field, and consequently no solution to 1.1) .

2. The Berezin trace formula.

Let K be a finite dimensional complex vector space. We denote by $\Lambda(K)$ the exterior algebra over K. As is well known, if $x_1,\dots,x_m$ is a basis for K, then the products $x_{i_1}\wedge x_{i_2}\wedge\dots\wedge x_{i_j}$, $i_1 < i_2 < \dots < i_j$ form a basis of $\Lambda(K)$ where the product with $j = 0$ is to be interpreted as 1. $\Lambda^j(K)$ will denote the linear span of those products having exactly j factors. Then

2.1) $$\Lambda(K) = \Sigma_{j=0}^{m} \Lambda^j(K)$$

represents $\Lambda(K)$ as a direct sum of subspaces. $\Lambda^m(K)$ has dimension one. Choose an element $\alpha \neq 0$ in $\Lambda^m(K)$. Now any element u in $\Lambda(K)$ is uniquely of the form

2.2) $$u = c\alpha + v$$

where c is a complex number and v is of degree less or equal to $m-1$. That is, $v \in \Sigma_{j=0}^{m-1} \Lambda^j(K)$. We define

2.3) $$E_\alpha(u) = c$$

when u is given by 2.2).

We assert that the linear functional E_α is an analog of m dimensional Lebesgue measure, and, in its infinite dimensional version to be described below, is the appropriate linear functional for the role of $\int\dots d\phi\, d\overline{\psi}$ in the Mathews-Salam formula, equation 1.2).

First, E_α is translation invariant in the sense that if u is represented on the above basis, i.e. as a noncommuting polynomial in the generators $x_1,\dots,x_m$, and if we replace each x_j in this expression by $x_j + a_j$, where a_j is a complex number (times the identity element of $\Lambda(K)$) then we obtain an element u_a in $\Lambda(K)$ which deserves to be called the translate of u by $a \equiv (a_1, \ldots, a_m)$. For if the x_j were coordinate functions on R^n and u an ordinary polynomial in the x_j s then the above prescription would indeed yield the translated function. But it is clear upon expanding the resulting products that the highest degree terms in u_a are the same as those of u. Consequently $E_\alpha(u) = E_\alpha(u_a)$, which shows that E_α is translation invariant. Moreover, E_α behaves under a linear change of variables in similar way to Lebesgue measure. Specifically, if $A: K \to K$ is linear and $\Gamma(A): \Lambda(K) \to \Lambda(K)$ is the usual (unique) homomorphism that extends A (thus $\Gamma(A)x_1 \wedge x_2 = Ax_1 \wedge Ax_2$, etc.) then it follows immediately from the definition of E_α and well known properties of the determinant that

$$E_\alpha(\Gamma(A)u) = (\det A)E_\alpha(u). \tag{2.4}$$

A suitably suggestive notation for E_α is as follows. Choose $\alpha = x_m \wedge \ldots \wedge x_1$. Define $\int 1dx_j = 0$ and $\int x_j dx_j = 1$. Then one verifies easily that

$$E_\alpha(u) = \int \ldots \int u \, dx_1 dx_2 \ldots dx_m \quad . \tag{2.5}$$

The preceeding discussion is largely contained in Berezin's book [1] along with many other exterior algebra analogs of Lebesgue measure notions. (E.g. Fourier transforms are discussed.)

In our application to the Mathews-Salam formula the space K will be an infinite dimensional space of test functions for the Euclidean Fermi fields ψ and $\bar{\psi}$. There is additional algebraic structure on K which arises from the fact that the Dirac field is charged. The fact that the Fermi field is charged seems essential for our methods. Schwinger has pointed out [14, 15] that successful Euclideanization for Fermions depends on the field being charged.

We assume henceforth that K has an inner product $(\ , \ )$ (linear on the right) and a conjugation J (i.e. an anti-unitary operator with $J^2 = 1$) and that a Hermitian operator q is given on K which anti-commutes with J and satisfies $q^2 = 1$. In the applications J will interchange Euclidean test functions for ψ with those for $\bar{\psi}$. q will be 1 on ψ test functions and -1 on $\bar{\psi}$ test functions.

With this structure there is a natural two form $\omega \in \Lambda^2(K)$ which may be constructed as follows. We let

2.6) $$\langle x,y \rangle = (Jx,y) .$$

Then one sees easily that $\langle \ , \ \rangle$ is a symmetric, non-degenerate, bilinear form on K while $\langle qx,y \rangle$ is skew symmetric. The latter determines an element ω of $\Lambda^2(K)$ in a standard way, but we shall be explicit about this. The eigenspaces K_+ and K_- of q for eigenvalues 1 and -1, respectively, span K and are orthogonal. Moreover $JK_{\pm} = K_{\mp}$ because J anti-commutes with q. Let $e_1, \ldots, e_m$ be a basis (not necessarily orthonormal) of K_+. We may identify K_- with the dual space of K_+ by means of the bilinear pairing 2.6), which is non degenerate on $K_+ \times K_-$ because $JK_+ = K_-$. Let $f_1, \ldots, f_n$ be the basis of K_- which is dual to $e_1, \ldots, e_n$. That is, $\langle e_i, f_j \rangle = \delta_{ij}$. Define

$$\omega = \Sigma_{j=1}^{n} e_j \wedge f_j \ .$$

We assert that ω is independent of the choice of basis $e_1, \ldots, e_n$. In fact one can compute easily that

$$2.7) \quad \langle \omega, x \wedge y \rangle \equiv \frac{1}{2} \Sigma_{j=1}^{n} (\langle e_j, x \rangle \langle f_j, y \rangle - \langle f_j, x \rangle \langle e_j, y \rangle)$$

$$= -\frac{1}{2} \langle qx, y \rangle$$

and this establishes the independence of basis. Let

$$2.8) \qquad \alpha = \omega^n / n!$$

Then α is a non zero element of maximal degree (namely $2n$) in $\Lambda(K)$. The form ω is an analog of the fundamental 2-form, $\Sigma_i dp_i \wedge dq_i$, of classical mechanics and E_α is consequently the corresponding analog of the Liouville measure $\Pi_i (dp_i \wedge dq_i)$.

We shall not actually give a meaning to E_α in the infinite dimensional case but, just as in the Boson case we must throw in a "density" before passing to infinite dimensions. We replace the Gaussian density $e^{-|x|^2}$ where $|x|^2$ is the "unit form" on K by $e^{-\omega}$ where ω is the above "unit 2 form" on K. Thus we shall give a meaning in infinite dimensions only to the formal expression

$$2.9) \qquad u \to E_\alpha (u e^{-\omega}) \ .$$

Notation. If K is a complex Hilbert space and J is a conjugation on K then $\mathcal{C}(K)$ will denote the Von Neumann algebra generated by the operators $\{C_x + A_{Jx}: x \in K\}$ where C_x and A_x are the creation and annihilation operators on the skew-symmetric Fock space over K. $\mathcal{C}(K)$ is the Clifford algebra over K and the function

$$\text{trace}(A) = (A\Omega,\Omega)$$

defines a trace function on $\mathcal{C}(K)$, with respect to which one can discuss the usual notions of non-commutative integration theory. See [3, 11, 16, 17] .

Lemma. Let K be a complex Hilbert space, J a conjugation on K and q a Hermitian operator which anti-commutes with J and satisfies $q^2 = 1$. Let $K_\pm$ be the two eigenspaces for q and let $\Lambda(K)$ be the algebraic exterior algebra over K. Then there is a unique linear map

$$\theta: \Lambda(K) \to \mathcal{C}(K)$$

such that

a.) $\theta(1) = 1$, $\theta(x) = C_x + A_{Jx}$ for x in K.

and b.) $\theta(u\wedge v) = \theta(u)\theta(v)$ if $u \in \Lambda(K_-)$ or $v \in \Lambda(K_+)$.

We omit the proof of this lemma (see [4]) but we note that θ is just a "charge ordering" map. Thus for example if x is in K_+ and $y \in K_-$ then $\theta(x\wedge y)$ is determined by a) and b) by means of $\theta(x\wedge y) = -\theta(y\wedge x) = -\theta(y)\theta(x) = -(C_y + A_{Jy})(C_x + A_{Jx})$. In terms of the Euclidean fields ψ and $\bar{\psi}$ the map θ will put all factors of ψ to the right and change sign in accordance with Fermi statistics, as we shall see in our application. In order to emphasize the naturality of this map we mention that it is a standard map in the theory of Hopf algebras.

The following theorem in finite dimensions provides the basis for the transition to infinite dimensions. The theorem is a variant of Berezin's trace formula [1, Page 85].

Theorem. Let K be a finite dimensional inner product space with J, q, ω, θ as above and $\alpha = \omega^n/n!$ Then

2.10) $$E_\alpha(ue^{-\omega}) = \mathrm{trace}[\theta(u)] .$$

We refer the reader to [4] for a proof.

With the motivation of 2.10) we are now justified in defining the left side of 2.10) in infinite dimensions simply to be equal to the right side, which as we have seen is meaningful even if $\dim K = \infty$. For one can reasonably expect that with this definition, heuristic calculations based on any other (informal) interpretation of $E_\alpha(ue^{-\omega})$ in infinite dimensions will have a meaningful and correct statement in terms of the trace composed with θ. We shall see that this is the case for the Mathews-Salam formula.

In closing this section I wish to emphasize the analogy with the Boson case. The Gauss integral $\int_{R^n} f(x)(2\pi)^{-n/2}e^{-|x|^2/2}d^nK$ has a simple meaning for $n = \infty$ if one first lumps together the last three factors $(2\pi)^{-n/2}e^{-|x|^2/2}d^nx$ to form Gauss measure $d\nu$. The first and third of these factors by themselves are zero and meaningless respectively when $n = \infty$. Similarly 2.10) shows that the expression $E_\alpha(ue^{-\omega}) = \int \dots \int ue^{-\omega}dx_1 \dots dx_n$ remains meaningful when $n = \infty$ if one first lumps together the factors $e^{-\omega}dx_1 \dots dx_m$ to get trace$\circ\theta$.

3. The Mathews-Salam formula.

Although the integral, $\int \ldots d\psi d\bar{\psi}$, which Mathews and Salam intended for use in 1.2) was a Gaussian integral (i.e. each "coordinate" of ψ runs over $(-\infty,\infty)$), (c.f. [5, page 564] and [6, page 127] nevertheless, informal calculations show that such integrals (of polynomials times a Gaussian "function of anti-commuting variables") yield the same moments as integrals defined by 2.2) and 2.3). This happens partly because the transformation property 2.4) is similar to that for the Lebesgue integral; $\int f(A^{-1}x)dx = (\det A)\int f(x)dx$, and partly because of the form of the integrand, a polynomial times exponential of a quadratic form.

We shall make some informal manipulations with the right side of 1.2) for heuristic purposes, treating it as an infinite dimensional version of 2.3). To begin with we note that in 1.2) the Boson factor $\exp[-\int(m^2\varphi^2+|\nabla\varphi|^2)d^4x]$ is the exponential of the "unit form" of the Sobolev space $\mathcal{H}_1$ and consequently the Boson integral is the integral with respect to the normal distribution over the dual space (with respect to L^2 inner product) $\mathcal{H}_{-1}$. In order to apply the preceding section it is necessary to have the Fermion exponential factor playing the analogous role - the "unit 2-form" for the appropriate one particle space of test functions. To this end we make a change of variables. We put

3.1) $$\bar{\psi} = (\not{p} + m + \varphi)\chi$$

in 1.2), obtaining on the right (we suppress the Boson integration henceforth.)

$$\frac{1}{N}\int [\{(\not p+m+\varphi(x_1))\chi(x_1)\} \cdots \{(\not p+m+\varphi(x_n))\chi(x_n)\}\psi(x_{n+1})\cdots\psi(x_{2n})]$$

$$e^{-\int\{(\not p+m+\varphi(x))\chi(x)\}\cdot\{(\not p+m+\varphi(x))\psi(x)\}d^4x}\ d\psi d(\not p+m+\varphi)\chi \qquad 3.2)$$

The exponent in 3.2) is now the "unit 2-form" of a Hilbert space whose dual space K_φ "over" which we integrate is the following. If S denotes 4-dimensional Dirac spin space then K_φ consists of those generalized functions f from R^4 to $S\oplus S$ such that

$$\|f\|_\varphi^2 = \|(\not p+m+\varphi)^{-1}f\|^2_{L^2(R^4;S\oplus S)} \qquad 3.3)$$

is finite. Here φ is a classical time dependent field (whose exact behavior can be chosen to reflect a momentum cutoff on the Boson field, if necessary, and space time cutoff on the interaction). The two summands of S in K_φ are for χ test functions and ψ test functions. Thus the Euclidean theory is an eight component theory just as the relativistic Mathews-Salam theory is. The invertibility of $(\not p+m+\varphi)$ that is required in 3.3) has been proven by Seiler [18] for all φ in the pseudo-scalar Y_2 theory. When the Boson field is quantized we need it only for almost all φ.

Applying 2.10) and choosing the constant N to be equal to $\det(\not p+m)$ (informally) 3.2) becomes

$$\det(1+(\not p+m)^{-1}\varphi)\ \ \mathrm{trace}_\varphi(\theta_\varphi[\ldots]) \qquad 3.4)$$

where trace_φ represents the trace in the Clifford algebra over K_φ and θ_φ is the corresponding map (which, properly put, maps $\Lambda(\mathcal{S}(R^4;S\oplus S))$ into $\mathcal{C}(K_\varphi))$. The expression $[\ldots]$ in 3.4) denotes the similarly denoted expression from 3.2).

Making the above transition from 1.2) to 3.2) to 3.4) leaves us, after smoothing with test functions, a well defined quantity on the right of 1.2) as well as on the left. Our main theorem asserts that these are equal (in the presence of momentum and space time cutoffs.) The proof first establishes equality for a fixed time dependent external field φ, by showing both sides satisfy the same (Schwinger) differential equations and boundary conditions.

We have chosen in this note only one of several possible ways to apply the general formalism of Section 2 to the validation of the Euclidean Mathews-Salam formula, 1.2). By including the interaction $\bar{\psi}\varphi\psi$ into the norm of the "local" (in Q space) one particle space K_{φ} we arrive at a probability Hilbert algebra for the total Fermion-Boson system which is a Clifford algebra bundle over Q space. It is also possible to exclude the interaction term from 3.1) and 3.3) so that K does not depend on φ, and instead include the interaction in a density back in (a completion of) $\Lambda(\mathscr{S}(R^4;S\oplus S))$. This would be a step closer to the formalism of Osterwalder and Schrader [13]. We mention finally that we have explored, but only superficially, the possibility of replacing the change of variables 3.1) by one which is more symmetric between ψ and $\bar{\psi}$.

Such variations of the above method of applying the general theory of Section 2 may conceivably prove necessary for the utility of the formalism in constructive quantum field theory.

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ϕ^j BOUNDS IN $P(\phi)_2$ QUANTUM FIELD MODELS

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RESUME On discute la ϕ^j-borne fondamentale $\pm :\phi^j: (h) \leq \|h\|_\infty D(h)\, C(H+I)$ et on donne certaines extensions nouvelles de la borne fondamentale.

ABSTRACT The basic ϕ^j- bound $\pm :\phi^j: (h) \leq \|h\|_\infty D(h)\, C(H+I)$ is discussed and certain new extensions of the original bound are noted.

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The basic estimate we prove is a ϕ^j bound

(1) $$\pm:\phi^j:(h) \le \|h\|_\infty D(h)\, C\,(H+I),$$

where h is real and H is a $P(\phi)_2$ Hamiltonian. The constant C depends only on the coefficients in P, while

(2) $$D(h) = 1 + \text{diam. suppt } h.$$

In [1], we proved this estimate for an arbitrary semibounded polynomial P, in the case of free boundary conditions. Here we establish (1) for the Hamiltonian $H = H_\ell - E(H_\ell)$. H_ℓ denotes the $P(\phi)_2$ Hamiltonian with Dirichlet boundary conditions at $x = \pm\ell$ and $E(H_\ell)$ is its vacuum energy. We prove (1) in a finite volume using the method of [6,4]. The transfer of this estimate to the infinite volume limit $\ell = \infty$ is achieved by standard methods [2]. Certain extensions of the bound (1) are noted in the concluding remarks. Results of this nature are used in [3], and should have other applications as well.

Theorem 1. Let $j \le \deg P$. There exists a sequence $\ell_\nu \to \infty$ (depending on $D(h)$) and there exists a constant C, depending only on j and P, such that (1) holds for all $h \in L_\infty$ with compact support, and for $H = H_{\ell_\nu} - E(H_{\ell_\nu})$.

For an operator A, let $E(A) = \inf \sigma(A)$ be its vacuum energy. For simplicity we assume that $\text{suppt. } h \subset [-1,1]$ and $\|h\|_\infty \le 1$. In case $j = \deg P$, we also suppose that $\|h\|_\infty$ is strictly smaller than the leading coefficient in P. To prove the theorem, it is sufficient to show that

(3) $$-E(H_\ell \pm :\phi^j(h):) \le -E(H_\ell) + \text{const.},$$

where the constant is independent of h and ℓ. Rewriting (3) gives

(4) $$0 \le H_\ell - E(H_\ell) \pm :\phi^j(h): + \text{const.},$$

from which (1) follows with H replaced by $H_\ell - E(H_\ell)$.

We remark that it is sufficient to prove

(5) $$-E(H_\ell \pm :\phi^j(h):) \le -E(H_{\ell-1}) + \text{const.}$$

Let Ω_ℓ^o be the free vacuum with Dirichlet data at $x = \pm\ell$. Then we have the linear upper bound

$$\begin{aligned} E(H_\ell) &= \inf_\psi \langle \psi H_\ell \psi \rangle / \|\psi\|^2 \\ &\le \langle \Omega_\ell^o H_\ell \Omega_\ell^o \rangle = \int_{-\ell}^{\ell} \langle \Omega_\ell^o : P(\phi(x)) : \Omega_\ell^o \rangle dx \\ &\le \int_{-\ell}^{\ell} O(1)(1 + |\ln(\ell - x)| + |\ln(x+\ell)|) dx \\ &= O(\ell) \end{aligned}$$

on the vacuum energy. It follows that $E(H_\ell)$ has at most bounded growth over some infinite sequence $[\ell_\nu - 1, \ell_\nu]$ of disjoint unit intervals, and so

(6) $$-E(H_{\ell_\nu - 1}) \le -E(H_{\ell_\nu}) + \text{const.}$$

Substituting (6) in (5) yields (3); (5) in turn follows from

(7) $$\langle \Omega_\ell^o, e^{-t(H_\ell \pm :\phi^j(h):)} \Omega_\ell^o \rangle \le e^{ct} \langle \Omega_{\ell-1}^o, e^{-tH_{\ell-1}} \Omega_{\ell-1}^o \rangle$$

as in [2, Lemma 5.4].

The desired inequality (7) is rewritten as a Euclidean integral. The Euclidean integral is bounded as follows: The contribution for $x \leq -1$ is bounded by the L_2 norm of its conditional expectation onto the line $x = -1$. The contribution of the region $x \geq 1$ is similarly bounded by the L_2 norm of its conditional expectation onto the line $x = 1$. The contribution of the region $-1 \leq x \leq 1$ is bounded by its norm as an operator acting between the two L_2 spaces $x = \pm 1$. We assert that the product of the first two contributions is exactly $\langle \Omega^o_{\ell-1}, e^{-tH_{\ell-1}} \Omega^o_{\ell-1} \rangle$, and that the third contribution is bounded by $e^{\text{const}(t+1)}$. Before proving these assertions, we digress on conditional expectations, covariance operators, and Dirichlet boundary conditions.

Lemma 2. Consider a Gaussian integral, with covariance operator C and a single particle space K. Let K_o be a subspace of K and let $\mathcal{M}(K_o)$ be the algebra of observables measurable over K_o. Let Π_o be the conditional expectation onto $\mathcal{M}(K_o)$. Then Π_o is the L_2 projection of $\mathcal{M}(K)$ onto $\mathcal{M}(K_o)$ and it is the second quantization of the projection of K onto K_o. In other words, its Fock space representation maps the n-particle states $\theta(x_1, \ldots, x_n)$ onto the state $\prod_{i=1}^{n} E^{(i)} \theta$, where $E^{(i)}$ acts on the i^{th} variable of θ, as the projection of K onto K_o.

Proof. Write the orthogonal decomposition $K = K_o + K_1$. Then the assertions follow by the canonical isomorphism of Fock spaces $\mathcal{F}(K) \cong \mathcal{F}(K_o) \otimes_s \mathcal{F}(K_1)$.

In the following we let $K_o \subset K$ denote functions with support in a given set $X \subset R^d$. We regard $C(x,y)$ as the kernel of an integral operator C on $L_2(R^d)$, and let C_{K_o,K_o} be the integral operator on $L_2(X)$ formed by restriction of both variables of $C(x,y)$ to K_o. (E.g., if E denotes the characteristic function of X and if $L_2(x) = EL_2(R^d)$, then $C_{K_o,K_o} = ECE$.) Let C_{K_o} denote C acting on $L_2(R^d)$, followed by restriction to $L_2(X)$. (In the above example $C_{K_o} = EC$.)

Theorem 3. Let $K_o \subset K$ as above, and let P denote the projection of K onto K_o. Then

$$P = C_{K_o,K_o}{}^{-1} C_{K_o} \,. \tag{8}$$

Corollary 4. Let K_o be the set of distributions in K with support on the line $x = a$. Let C be time translation invariant, and let $D_p(x,y)$ be the Fourier transform of $C(x,y,t-t')$ with respect to $t-t'$. Then taking Fourier transforms in t, the kernel of P is

$$P = D_p(a,a)^{-1} D_p(a,x) \,. \tag{9}$$

<u>Proof</u>. Clearly P defined by (8) maps functions on R^d into functions supported in X. Furthermore

$$\begin{aligned} P^2 &= C_{K_o,K_o}{}^{-1}(C_{K_o} C_{K_o,K_o}{}^{-1}) C_{K_o} \\ &= P \end{aligned}$$

since the range of $C_{K_o,K_o}{}^{-1}$ lies in K_o and $C_{K_o} \restriction K_o = C_{K_o,K_o}$. Thus $C_{K_o} C_{K_o,K_o}{}^{-1} \restriction K_o = I \restriction K_o$. Finally P is self adjoint, since for $f,g \in K$, and with an L_2 inner product.

$$\langle g, CPf\rangle = \langle g, CC_{K_o K_o}^{-1} C_{K_o} f\rangle = \langle g, C^*_{K_o} C_{K_o K_o}^{-1} C_{K_o} f\rangle = \langle Pg, C_{K_o} f\rangle = \langle Pg, Cf\rangle$$

Thus with the inner product given by C,

$$\langle g, Pf\rangle = \langle Pg, f\rangle .$$

This completes the proof of the theorem. The corollary is a special case, since in this case C_{K_o, K_o} is a multiplication operator after Fourier transformation in t.

Lemma 5. Let $C_{\ell_1 \ell_2} = (-\Delta_\ell + m_o^2)^{-1}$, where Δ has Dirichlet data on two parallel lines $x = \ell_1$, $x = \ell_2$. Let p be the Fourier transform variable dual to t, and let $\mu = (p^2 + m_o^2)^{1/2}$. Taking Fourier transformation in the t variable, we have

$$C \equiv C_{\ell_1 \ell_2}(x,y) = \begin{cases} \dfrac{\sinh(\mu(\ell_2 - x))}{\sinh(\mu(\ell_2 - y))} \; \dfrac{1}{\mu} \, \dfrac{1}{\coth(\mu(\ell_2 - y)) + \coth(\mu(y - \ell_1))} \\ \\ \dfrac{\sinh(\mu(x - \ell_1))}{\sinh(\mu(y - \ell_1))} \; \dfrac{1}{\mu} \, \dfrac{1}{\coth(\mu(\ell_2 - y)) + \coth(\mu(y - \ell_1))} \end{cases}$$

and the top line holds for $\ell_1 < y < x < \ell_2$, the bottom for $\ell_1 < x < y < \ell_2$.

Proof. Direct calculation.

We note that Π_{K_o} has the kernel

$$k(x,p) = \begin{cases} \dfrac{\sinh(\mu(\ell_2 - x))}{\sinh(\mu(\ell_2 - a))} , & a < x < \ell_2 \\ \\ \dfrac{\sinh(\mu(x - \ell_1))}{\sinh(\mu(a - \ell_1))} , & \ell_1 < x < a \end{cases}$$

In particular, when acting on functions supported in $a < x < \ell_2$, Π_{K_o} is is independent of ℓ_1.

We write the Euclidean integrand $\exp(-\int_o^t \int_{-\ell}^{\ell} :P(\phi): + :\phi^j(h):)$ as a product F_-GF_+, where G is supported in the strip $-1 < x < 1$, and $F_\pm$ is supported in the strip between ± 1 and $\pm\ell$. For an interval $I = \{x : x_1 < x < x_2\}$, let Π_I denote conditional expectation onto the strip $x \in I$. Then

$$\Pi_{[-1,1]}F_-GF_+ = G\Pi_{[-1,1]}F_-F_+ = G\Pi_{(-\infty,1]}\Pi_{[-1,\infty)}F_-F_+$$

$$= G\Pi_{(-\infty,1]}(F_+\Pi_{[-1,\infty)}F_-) \,.$$

By the Markov property, $\Pi_{[-1,\infty)}F_- = \Pi_{\{-1\}}F_-$. Let $f_\pm = \Pi_{\{\pm 1\}}F_\pm$. Thus

$$\Pi_{[-1,1]}F_-GF_+ = G(\Pi_{(-\infty,1]}F_+f_-) = Gf_-\Pi_{(-\infty,1]}F_+$$

$$= Gf_-f_+ \,.$$

From the above formulas, we see that the Euclidean integral (4) is equivalent to the integral of Gf_-f_+, with covariance $C_{-\ell\ell}(x,y)$, $-1 \leq x;\ y \leq 1$. Let $F(\pm 1)$ denote L_2 of the Gaussian measure space defined over

$$K_o = \{f = \delta_{\pm 1}\, f_1(p)\},$$

with covariance

$$A_+ = C_{-\ell+2,\ell}(1,1), \text{ resp. } A_- = C_{-\ell,\ell-2}(-1,-1) \,.$$

We note that the Dirichlet data in the definition of $A_{\pm}$ is symmetric about the line $x = \pm 1$. By our definitions,

$$\|f_-\|^2_{F(-1)} = \|f_+\|^2_{F(+1)} = \langle \Omega^o_{\ell-1}, e^{-tH_{\ell-1}}\Omega_{\ell-1}\rangle ,$$

so the proof of Theorem 1 reduces to showing that $G : F(1) \to F(-1)$ is a bounded operator.

Let $F(0)$ be the L_2 space of Gaussian measure space defined over the $x = 0$ subspace $\{f = \delta_0 f_1(\rho)\}$, with covariance $C_{-\ell,\ell}(0,0)$. In the bound on G, an important fact is the inequality

$$0 \leq C_{-\ell,\ell}(0,0)^{-1} C_{-\ell,\ell}(0,\pm 1) C_{-\ell,\ell}(\pm 1,\pm 1)^{1/2} A_{\pm}^{-1/2} \leq 1-\varepsilon < 1 ,$$

valid for ℓ large, as a consequence of Lemma 5. Let $B_{\pm}$ be the above operator. The first two factors on the right in $B_{\pm}$ change the metric in the $x = \pm 1$ subspace from that given by $A_{\pm}$ to the metric given by the covariance $C_{-\ell,\ell}(\pm 1,\pm 1)$, while the next two factors in $B_{\pm}$ give the projection of the $x = \pm 1$ subspace onto the $x = 0$ subspace. By Lemma 2, $B_{\pm}$ is the single particle operator which yields the conditional expectation $\Pi_{\{0\}} : F(\pm 1) \to F(0)$, and by the above bound, $\Pi_{\{0\}} : F(\pm 1) \to F(0)$ is hypercontractive. This means that there is a $q > 1$, with $\Pi_{\{0\}} |h_{\pm}|^q \in F(0)$ for any $h_{\pm} \in F(\pm 1)$, and $\|\Pi_{\{0\}} |h_{\pm}|^q\|_{F(0)} \leq \|h_{\pm}\|_{F(\pm 1)}$. Thus with some $r < \infty$,

$$\left| \int h_- G h_+ dq \right| \leq \left(\int |G|^r\right)^{1/r} \left(\int |h_-|^q |h_+|^q\right)^{1/q} .$$

By standard bounds,

$$\left(\int |G|^r\right)^{1/r} \leq e^{\text{const}(t+1)} ,$$

and by hypercontractivity,

$$(\int|h_-|^q|h_+|^q)^{1/q} \leq \|\Pi_{\{0\}}|h_-|^q\|_{F(0)}^{1/q}\|\Pi_{\{0\}}|h_+|^q\|_{F(0)}^{1/q}$$

$$\leq \|h_-\|_{F(-1)}\|h_+\|_{F(1)} .$$

This completes the proof of Theorem 1.

Remark 5. The condition $h \in L_\infty$ is stronger than is required. In fact the bound

$$\int G^r dq \leq e^{\mathrm{const}(t+1)}$$

is satisfied if h has compact support and belongs to $L_{d/d-j}(R)$.

Remark 6. The theorem may also be generalized by allowing a momentum cutoff $\varkappa$ in the perturbation. With bounds uniform in κ,

$$:\phi_\kappa^j(h): \leq \|h\|_\infty D(h)\, C\,(H+I).$$

Remark 7. One may also show that

$$:\phi_{\kappa_1}^j(h): - :\phi_{\kappa_2}^j(h): \leq \|h\|_\infty D(h) o(1)(H+I)$$

where $o(1) \to 0$ as $\kappa_1, \kappa_2 \to \infty$, uniformly in h and ℓ.

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CRITICAL PROBLEMS IN QUANTUM FIELDS*

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RESUME Les liaisons entre le problème de la construction des champs quantiques non triviaux à quatre dimensions et le problème du comportement au point critique à quatre dimensions sont expliqués.

ABSTRACT The connections between the problem of constructing non trivial quantum fields in four dimensions and the problem of critical point behaviour in four dimensions are explained.

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The last two years have seen considerable progress in our understanding of the mathematical structure of quantum fields. In two areas, the progress has been close to definitive, and the problems may be largely resolved in the near future. These areas are (a) the construction of more singular superrenormalizable models: Yukawa_2 and ϕ^4_3 and (b) the detailed structure of $P(\phi)_2$ models which are close to free theories, namely particles, bound states, analyticity, unitarity in subspaces of bounded energy, and phase transitions. In two other areas there has been progress, but the progress is far from being definitive. These areas are (c) the structure away from the neighborhood of free theories, and especially near a critical point and (d) results which pertain indirectly to the construction of four dimensional models. For the results in areas (a) and (b), we merely list recent references, and we then turn to the open problems, including (c) and (d).

Yukawa_2-Euclidean methods [Br 1, Sei, McB 1,2, Br 2, Sei-Si]

ϕ^4_3 - weak coupling expansions [Fe-Os, Ma-Sen]

$P(\phi)_2$ - scattering [Sp 1, Sp-Zi]

$P(\phi)_2$ - analyticity [EMS]

$P(\phi)_2$ - phase transitions [GJS 2,3]

The central problem of constructive quantum field theory has not changed over many years (cf. [St-Wi; p. 168]): the construction of nontrivial quantum fields in four dimensions. We explain how this problem is related to critical point theory in four dimensions, and how a number of simpler problems (of independent interest, and involving two or three dimensional quantum fields) are related to this central problem.

The simplest four dimensional interactions, ϕ^4_4 and Yukawa_4 are renormalizable, but not superrenormalizable. This means that the bare and physical coupling constants are dimensionless. In addition to this dimensionless constant, the field theory is parametrized by two or more parameters with dimension of $(\text{length})^{-1}$. Namely, there are one or more masses and an ultraviolet cutoff κ. To make the exposition explicit, we choose the ultraviolet cutoff as a lattice, and then $\kappa^{-1} = \varepsilon$ is the lattice spacing.

The goal of the construction is to take the limit $\kappa\to\infty$, i.e. $\varepsilon\to 0$. Because scaling is a unitary transformation, and because scaling multiplies all lengths by an arbitrary parameter s, the theory with ε small and mass $m=1$ is equivalent to the theory with $\varepsilon=1$ and mass small. In this equivalence, the test functions also scale, and so if we choose $\varepsilon=1$, a typical test function will have support on a set of large diameter $O(m^{-1})$. Thus if we choose $\varepsilon=1$, we must focus on the long distance behavior, i.e. on the distance scale $O(m^{-1})$ in a theory with small mass. It follows that the limit $\kappa\to\infty$, $\varepsilon\to 0$ which removes the ultraviolet cutoff is equivalent to the limit $m\to 0$ with $\varepsilon=1$, if in this latter limit we consider the behavior on the distance scale $O(m^{-1})$. This latter limit (correlation length $=m^{-1}\to\infty$) and distance scale is traditionally considered in critical point theories, namely the "scaling limit" in statistical mechanics. Thus we see that the critical point limit, with fixed lattice spacing $\varepsilon=1$, is equivalent to the removal of the ultraviolet cutoff and to the construction of a (continuum) quantum field ($\varepsilon=0$). Since the long distance (infrared) singularities are worse in two and three dimensions, we see that critical point theories in two and three dimensions provide a very realistic test for the mathematical difficulties presented by four dimensions. Indeed the two and three dimensional infrared behavior is typical of nonrenormalizable field theories. A simplification of the two and three dimensional problem (and one which we hope will prove to be minor) is that the critical point can be approached by Lorentz covariant fields satisfying Wightman axioms, in place of the lattice theories introduced above, see [GJ5]. For this reason, in two and three dimensions, the spectral representation of the two point function and (presumably) the particle structure and S-matrix theory can be used as tools to study the theories which are approaching the critical point.

To construct the critical point limit, there are four essential steps:

(i) mass renormalization

(ii) field strength renormalization

(iii) uniform estimates up to the critical point

(iv) nontriviality of the limit.

The first three steps concern existence for this question we would be happy to allow a compactness principle and selection of a convergent subsequence, while hoping that the full sequence converged also. This follows principles well accepted in other branches of mathematics (e.g. partial differential equations) where questions of existence and uniqueness are often studied by separate methods. The last step (nontriviality) depends upon the correct choice of charge renormalization. We will see below that for the ϕ^4 interaction each step can be studied independently of the others.

We now examine each of these four steps in turn. We will see which portions have been solved, which portions seem feasible for study at the present time, which steps are highly interesting in their own right, independently of their role in a possible construction of ϕ_4^4, and which portions seem to present essential difficulties and whose resolution will presumably require essentially new ideas.

The first step, mass renormalization, is the step nearest to completion. For a $\lambda\phi^4 + \frac{1}{2} m_0^2\phi^2$ theory (or more generally for an even $P(\phi)$ theory), the physical mass m is a monotonic function of m_0 for a single phase theory [GRS]. This statement also pertains to a lattice theory (as required for the four dimensional program) if the mass is defined as the exponential decay rate of the two point function. For a ϕ_2^4 theory at least, the mass $m(m_0^2)$ is differentiable [GJ 2] for $m > 0$. The analysis of [Ba] suggests that $m(m_0^2)$ is continuous for $m = 0$; in the lattice case this has been rigorously established [J. Ro 2].

Assuming that the ϕ_2^4 mass is continuous for $m \geq 0$ in a single phase $\lambda\phi^4 + \frac{1}{2} m_0^2\phi^2$ theory, then the mass renormalization is defined as the inverse function

$$m_0^2 = m_0^2(m,\lambda).$$

To see that m may take on all values, $0 \leq m < \infty$, we argue by continuity. For $m_0^2 \to \infty$, $m \to \infty$ also [GJS 1], and so we require a critical theory $(m = 0)$ at the end $m_0^2 = m_{0,c}^2$ of the single phase region, with

$$m(m_0^2) \searrow 0 \quad \text{as} \quad m_0^2 \searrow m_{0,c}^2 .$$

To be explicit, we define

$$M(m_0^2) = \lim_{|x-y|\to\infty} \langle\phi(x)\phi(y)\rangle^{1/2}$$

and

$$m_{0,c}^2 = \sup_{m_0^2}\{m_0^2 \mid M(m_0^2) > 0 \quad \text{or} \quad m(m_0^2) = 0\} .$$

The existence of a phase transition for a ϕ^4 lattice theory [Nel] and for ϕ_2^4 [GJS 2,3] shows that $m_{0,c}^2$ is finite. Combining this fact with the method of [GJ 2, Ba], it can be shown that $m \searrow 0$ as $m_0^2 \searrow m_{0,c}^2$ [J.Ro 2], at least in the lattice case. We summarize the problems of this section under the name: existence of the critical point.

For ϕ_3^4, one expects a similar structure for phase transitions. Assuming this conjecture and using the decay at infinity of the zero mass free field, it follows that $M(m_{0,c}^2) = 0$, but the question of whether $m(m_{0,c}^2) = 0$ remains open. For ϕ_2^4 and for a lattice theory the reasoning concerning M does not apply. In two dimensions the zero mass free field two point function does not decay at infinity, and in a lattice theory, the absence of a Lehmann spectral formula means that the free field is not known to bound the lattice two point function.

For the Yukawa interaction, none of the above results have been obtained. Major steps for the ϕ^4 interaction depend on correlation inequalities, which are presumably not valid for the Yukawa interaction. For the pseudoscalar Yukawa interaction, a phase transition associated with a breaking of the $\phi \to -\phi$ symmetry may be expected on formal grounds. For the scalar Yukawa theory, should one expect an absence of phase transitions and $m_{0,c}^2 = -\infty$? What about cases closer to strong interaction physics, such as one charged and one neutral fermion coupled to three mesons (charged $\pm 1, 0$) ? In general the problem here is: to locate the critical point. This problem is important because renormalizable fields (e.g. ϕ_4^4, Y_4) are equivalent to lattice or ultraviolet cutoff fields studied in the critical point limit. From this point of view, one reason for studying phase transitions in field theory is as an aid in locating the critical points.

The second step is to introduce the renormalized field $\phi_{ren} = Z^{-\frac{1}{2}}\phi$, where Z is defined in terms of the spectral representation for the two point function:

$$\langle\phi(x)\phi(y)\rangle^{\sim} = \frac{Z}{p^2+m^2} + \int_{a>m^2} \frac{d\rho(a)}{p^2+a} .$$

The essential problem is to show that $Z \neq 0$ for $m > 0$, or in other words to show that for each field in a noncritical theory, there is a corresponding elementary particle. Furthermore we expect only delta function contributions to $d\rho(a)$ (bound states) below the two particle threshold $a = (2m)^2$, and for an even $P(\phi)$ theory, the same should be true below the three particle threshold, because of the $\phi \to -\phi$ symmetry. In this more general form, the problem could be called Hunziker's theorem for field theory.

We now split the discussion of Hunziker's theorem into two independent paths, which we call the repulsive route and the general route. The repulsive route seeks to make maximum use of the special features of the ϕ^4 interaction, in particular of the presumably repulsive forces in this field theory. Since the ϕ^4 terms should dominate for a $P(\phi)$ critical point which is not near a tricritical point, we expect that the main results obtained in the repulsive route should be valid for general $P(\phi)$ theories near critical points which are not tri, or multicritical.

The repulsive route makes essential use of correlation inequalities. An example is the absence of even bound states for single phase ϕ^4 theories [GJS 1, Fe1, Sp 2]. However correlation inequalities by themselves cannot control the critical point behavior, because the ferromagnetic spin ½ Ising model on the Caley tree lattice has anomalous approach to the critical point [Zi]. Thus correlation inequalities must be used in conjunction with the lattice structure, as reflected in the Euclidean invariance and Hamiltonian structure of ϕ_2^4 and ϕ_3^4. A proposed correlation inequality, $\Gamma^{(6)} \leq 0$, implies that $d\rho(a)$ is supported above the three particle threshold, in the interval $[(3m)^2, \infty)$, and that $Z \gneq 0$ [GJ 5]. Thus $\Gamma^{(6)} \leq 0$ would completely settle step two, following the repulsive route. Lowest order perturbation theory suggests that $\Gamma^{(6)} \leq 0$ for weak coupling, and $\Gamma^{(6)} \leq 0$ has been checked for the one dimensional Ising model [J. Ro 1, Ro-Sy]. In view of its importance here and in step three below, further investigation of this inequality would be very desirable. Numerical calculations in some simple cases support the conjecture $\Gamma^{(6)} \leq 0$ [Is - Mar].

The general route is contained in and is substantially equivalent to the problem of asymptotic completeness. In fact the problem of step two -- the existence of a (discrete mass) particle at the bottom of the energy spectrum -- is equivalent at higher energies to the absence of continuous mass spectrum beyond that associated with multiparticle states. For weak coupling and bounded energies, the problem has been solved using cluster expansions. (In [Sp - Zi], energies up to the three particle threshold are allowed for even $P(\phi)$ interactions). For other regions of convergence of the cluster expansion, large external field [Sp 1] or low temperature [GJS 2,3], the situation is expected to be the same. For weak coupling but arbitrary energies, the present methods do not apply.

Outside of the region of convergence of the cluster expansion, the problem seems to involve all major elements of structure of the field theory, including the structure of the vacuum, bound states and superselection sectors [DHR]. The relation of bound states and superselection sectors to asymptotic completeness is well known, since extra bound states as well as extra elementary particles in some extra charge one superselection sector give rise to extra multiparticle continuous spectrum. The relation of the vacuum structure of phase transitions to solitons and superselection rules in two dimensions is contained in [Go-Ja, DHN, Fr 3]. It is an old question to ask whether the Goldstone picture provides a qualitatively correct picture of phase transitions, and now we ask whether the ideas of [Go-Ja, DHN, Fr 3] are sufficient to describe all superselection sectors for $P(\phi)_2$ fields. Can the reasoning be reversed in the sense that $Z=0$ and $\text{suppt}\ \rho(a) = [m^2, \infty)$ would imply existence of a new superselection sector?

On a less etherial level, we ask whether each pure phase for a $P(\phi)$ interaction can be obtained by an appropriate choice of boundary conditions, as is the case in statistical mechanics. Can the

FKG inequalities be used to prove convergence and Euclidean invariance of the infinite volume limit, for general $P(\phi)$ theories? See the introduction to [GJS 3] for a further discussion of the Goldstone picture of phase transitions.

The third step is a bound, uniform as the critical point is approached, on the renormalized n-point Schwinger functions. We follow the repulsive route in our reliance on correlation inequalities. For the ϕ^4 interaction, a correlation inequality reduces this problem to a bound on the renormalized two point Schwinger function

$$S^{(2)} = <\phi_{ren}(x)\phi_{ren}(y)> = Z^{-1}<\phi(x)\phi(y)> ,$$

[GJ 1]. The reduction applies to all cases (ϕ_2^4, ϕ_3^4 and lattice ϕ_4^4) considered here. It is convenient to choose the scale parameters so that m=1 and $\varepsilon\to 0$, and then a sufficient bound on $S^{(2)}$ is e.g.

$$\int S^{(2)}(x)dx \leq \text{const.}$$

with a constant independent of ε, or more generally, $|S^{(2)}|_{\mathcal{S}'} \leq$ const. for some $\mathcal{S}'$-norm$|\cdot|_{\mathcal{S}'}$ independent of ε. In the absence of level crossings, the required bound on $S^{(2)}$ is equivalent to a bound on CDD zeros [GJ 5]. The conjectured inequality, $\Gamma^{(6)} \leq 0$, would imply an absence of bound states (and thus of bound state level crossings), and of CDD zeros below the three particle threshold [GJ 5]. Thus $\Gamma^{(6)} \leq 0$ would bound $S^{(2)}$ and complete the third step. This application of the $\Gamma^{(6)} \leq 0$ inequality was derived in the context of the ϕ_2^4 interaction. The methods extend without change to the ϕ_3^4 interaction. The adaptations of these methods to the lattice ϕ_4^4 interaction is an open problem.

As a concluding remark on the repulsive route, we mention that considerable progress has been made in deriving new correlation inequalities and in finding interrelations between, and simplified proofs of, old ones. See [Sy, New 1,2, El-New, Du-New, El-Mo].

Unfortunately, we have little to say about a possible general route for the third step. In particular there is no argument for believing (or disbelieving) that the phenomena considered above -- absence of bound states and of CDD zeros below the three particle threshold -- occur in the scaling limit approach to a ϕ^6 tricritical point. An absence of level crossings between bound states and the elementary particle and a bound on CDD zeros away from the elementary particle mass might be a general picture, and would yield a bound on the two point function, but not on the general n point functions. In the approach to a general $P(\phi)$ critical point which is not near a tricritical point, the ϕ^4 (repulsive) critical behavior should dominate. Is there any argument (even heuristic) which can be used to discuss bound states, CDD zeros and/or bounds on the renormalized two point function in this region, other than the proposed $\Gamma^{(6)}$ inequality?

The fourth step is the nontriviality of the limit. We hold m fixed (for example $m=1$) throughout the discussion and consider first $d=2,3$ dimensions, then $d=1$ and finally $d=4$. Starting with a lattice field theory with $\lambda<\infty$ and $\varepsilon>0$, we have the definition $(d\leq 3)$

$$\phi^4\text{-field theory} = \lim_{\varepsilon\to 0} .$$

We believe that

$$\text{Ising model} = \lim_{\lambda\to\infty} \quad \text{(lattice spacing } \varepsilon)$$

$$\text{scaling limit } \phi^4 \text{ field theory} = \lim_{\lambda\to\infty}\ \lim_{\varepsilon\to 0}$$

$$\text{scaling limit Ising model} = \lim_{\varepsilon\to 0}\ \lim_{\lambda\to\infty} .$$

It is reasonable to conjecture that the ε and λ limits above can be interchanged and thus that the scaling limits of the ϕ^4 field theory and the Ising model coincide. This conjecture is a variant of the universality principle for critical exponents in statistical mechanics. Because the Ising critical exponents are known to be nontrivial for $d=2,3$, we can expect the scaling limit for ϕ_2^4, ϕ_3^4 to be nontrivial.

For $d=1$, all steps one-four have been completed [Is], including interchange of the ε-λ limits. The one dimensional Ising model is already scale invariant, and so the $\varepsilon \to 0$ limit has a trivial form. Control over the $\lambda \to \infty$ limit is obtained from an analysis of anharmonic oscillator eigenvalues and eigenfunctions in a neighborhood of the critical point $m_0^2 = -\infty$.

In $d=4$ dimensions the situation is somewhat different from $d < 4$. In terms of the Callan-Symanzik equations, the sign of the crucial function $\beta(\lambda)$ is reversed. This change in the sign of β has its origin in the fact that λ is dimensionless (and thus scale invariant). In terms of the above constructions, the scale invariance means that λ is not taken to infinity by an infinite scale transformation. Rather λ, the bare charge, must be chosen (renormalized) to yield some desired value λ_{phys} of the physical charge. We define the physical charge by

$$\lambda_{phys} = \lambda_{phys}(\lambda,\varepsilon m) = -\varepsilon^3 Z^{-2} \chi^{-4} \sum_{x_1,x_2,x_3} \langle\phi(x_1)\cdots\phi(x_4)\rangle^T ,$$

where $\langle \quad \rangle^T$ denotes the connected, Euclidean Green's function (Ursell function). By Lebowitz' inequality,

$$0 \leq \lambda_{phys} .$$

We have taken advantage of the scale invariance of λ_{phys} to write it as a function of the scale invariant parameters λ and εm. Recalling that $\lambda = \infty$ is the Ising model, we define

$$\lambda_I(\varepsilon m) = \lambda_{phys}(\infty,\varepsilon m) .$$

Also note that $\lambda = 0$ is a free lattice field, and

$$0 = \lambda_{phys}(0,\varepsilon m) .$$

To simplify the discussion of renormalization, we suppose that $\lambda_{phys}(\lambda,\varepsilon m)$ is monotone increasing as a function of λ for fixed εm. (However, we have no argument to support such an hypothesis.)

We claim that λ_{phys} should be continuous in λ and εm. We assume that $G^{(4)} \equiv \langle\phi(x_1)\cdots\phi(x_4)\rangle^T Z^{-2}$ is continuous in λ and εm. Then upper bounds on the two point function, suggested by perturbation theory, substituted in the inequalities of [GJ 3] yield an integrable upper bound for $|G^{(4)}| = -G^{(4)}$, independent of λ and εm, for m fixed, $m > 0$. Continuity of λ_{phys} follows from the Lebesgue bounded convergence theorem.

By definition, charge renormalization is the inverse function,

$$\lambda = \lambda(\lambda_{phys}, \varepsilon m),$$

and by continuity, we can choose $\lambda = \lambda(\varepsilon m)$ so that $\lambda_{phys} = \lambda_{phys}(\lambda(\varepsilon m), \varepsilon m))$ approaches any desired value in the interval

$$[0, \lambda_I(0)],$$

as $\varepsilon m \to 0$, $m \neq 0$. Nontriviality of the Ising model (in its critical point limit) is the statement that $\lambda_I(0) \neq 0$. We conclude that the ϕ_4^4 fields constructed here should be nontrivial if and only if the critical behavior of the Ising model is.

According to conventional ideas, $\lambda(\lambda_{phys}, \varepsilon m) \to \infty$ as $\varepsilon m \to 0$ in order to ensure $\lambda_{phys} \neq 0$ (infinite charge renormalization). In order to discuss the long and short distance scaling limits of the ϕ_4^4 field, we also suppose

$$\lambda(\lambda_{phys}, \varepsilon m) \nearrow \infty$$

as $\varepsilon m \searrow 0$.

In the context of the Callan-Symanzik equations, one changes m_0^2, followed by a scale transformation to keep m fixed. The decrease of

m_0^2 is called long distance scaling; the increase of m_0^2 is called short distance scaling. According to conventional ideas, there are two fixed points to this transformation, the points $\lambda_{phys} = 0$, $\lambda_{phys} = \lambda_I(0)$. The zero mass theories associated with these fixed points are scale invariant.

At the endpoint $\lambda_{phys} = 0$ (assuming χ is finite), the field is Gaussian [Newman]. Presumably it is the free field, invariant under the above transformation group (the renormalization group). At the endpoint $\lambda = \lambda_I(0)$, we expect the field theory to coincide with the long distance scaling limit of the Ising model.

We now consider λ_{phys} lying in the interval $(0, \lambda_I(0))$. For such a theory, according to conventional ideas, the short distance behavior is governed by the fixed point $\lambda_{phys} = \lambda_I(0)$, while the long distance behavior is governed by the fixed point $\lambda_{phys} = 0$. We show that λ_{phys} is monotone increasing in its dependence on m_0^2. Since λ_{phys} is dimensionless, and hence unchanged under scale transformations, this also shows that λ_{phys} decreases under long distance renormalization group transformations and increases under short distance transformations, i.e. $\beta \geq 0$.

Consider two values of the bare mass, m_0, m_0^* satisfying $(m_0^*)^2 < m_0^2$. Let $m^* < m$ be the corresponding masses. By definition

$$\lambda_{phys} = \lim_{\varepsilon \to 0} \lambda_{phys}(\lambda(\lambda_{phys}, \varepsilon m), \varepsilon m)$$

$$\lambda_{phys}^* = \lim_{\varepsilon \to 0} \lambda_{phys}(\lambda(\lambda_{phys}, \varepsilon m), \varepsilon m^*) .$$

Since ε is a dummy variable, we replace it in λ_{phys}^* by $\varepsilon m/m^*$

$$\lambda_{phys}^* = \lim_{\varepsilon \to 0} \lambda_{phys}(\lambda(\lambda_{phys}, \varepsilon m^2/m^*), \varepsilon m) .$$

Since $m/m^* > 1$, we have by monotonicity of λ_{phys} in λ and monotonicity of λ in εm that

$$\lambda_{phys}^{*} \le \lambda_{phys} .$$

This completes the proof.

The statement that the charge renormalization is infinite is equivalent to the statement that the lattice ϕ_4^4 field is free in its critical point behavior (e.g. $Z \to 1$ as $\varepsilon \to 0$, with $\lambda = \text{const.} < \infty$). This is, of course, an open problem.

The existence of ϕ_3^4 and $P(\phi)_2$ fields suggests that the critical point scaling limit exists for the corresponding lattice fields; in the $P(\phi)_2$ case, tri- and multi-critical point limits should also exist. More generally, we summarize the discussion up to this point by asserting that a Euclidean quantum field is the critical point scaling limit of a corresponding lattice field. In the limit of strong physical coupling, the lattice field is replaced by an Ising model.

An alternate approach to nontriviality of the ϕ_4^4 field theory could be based on existence of the classical limit $\hbar \to 0$. Scattering for $\hbar = 0$ is known to be nontrivial. We thank Raczka for this comment.

Turning away from the construction of ϕ_4^4 via critical point theory, we note that recent work [Co, Fr 1] solves the (renormalizable but not superrenormalizable) massive Thirring model in two dimensions. Does this solution provide insight into the problems of charge and wave-function renormalization? Can other solvable two dimensional models be used as a starting point to prove existence of fields, for interactions of the form (explicitly solvable) + (superrenormalizable)?

Most thinking in contemporary particle physics uses nonabelian gauge fields and a Higgs mechanism as an ingredient. There are many problems here, including a proof of the existence of a Higgs mechanism, even in the lattice case.

We conclude by mentioning two other problems in mathematical physics which may be related to the critical and nonrenormalizable infrared problems considered above. First, the approach of Kolmogoroff to a statistical theory of turbulence uses scaling arguments to deduce exponents governing (short distance) asymptotic behavior. The second problem is the divergence of the virial expansion for the transport coefficients. Here the problem is infrared (slow decay of large time correlations) and infrared. In some cases, the leading divergences can be resummed, and the leading nonanalytic dependence on the density explicitly determined, on a formal level, cf.[Ha - Co].

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Colloques Internationaux C.N.R.S.
N° 248 – Les méthodes mathématiques de la théorie quantique des champs

Existence of Phase Transitions for φ_2^4 Quantum Fields

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RESUME L'existence des transitions de phase pour les champs quantiques $\lambda\varphi_2^4$ dans la région $\lambda \gg 1$ de couplage nu est établie. La brisure de symétrie pour l'interaction $\lim \mu \to \pm 0$ $(\varphi^4 - \mu\varphi)$ est aussi démontrée. On fait la distinction entre les transitions de phase et la brisure de symétrie.

ABSTRACT The existence of phase transitions for $\lambda\varphi_2^4$ quamtum fields in the region $\lambda \gg 1$ of bare coupling is established. Symmetry breaking for the interaction $\mu \to \pm 0$ $(\varphi^4 - \mu\varphi)$ is alsoproved and the distinction between phase transitions and symmetry breaking is emphasized.

1 - New Results

We prove the existence of phase transitions for $\lambda\varphi_2^4$ quantum fields in the region $\lambda \gg 1$ of bare coupling. The same methods apply in principle to even $\lambda P(\varphi)_2$ models. We demonstrate the existence of long range order in the (even) $P(\varphi)_2$ theory defined with zero Dirichlet boundary data. (However, we restrict attention in this talk to φ_2^4.) We also prove the existence of symmetry breaking for the interaction

$$\lim_{\mu \to \pm 0} (\varphi^4 - \mu\varphi) .$$

As in statistical mechanics, where phase transitions may occur without symmetry breaking [4], we expect phase transitions in certain quantum field models which do not possess a symmetry group, such as the interaction

$$(\varphi^2 - \sigma^2)^4 + \epsilon\varphi^3 - \mu\varphi ,$$

with $\sigma >> 1$, $\epsilon << 1$, $\mu = \mu(\epsilon, \sigma)$. Thus we emphasize this distinction between phase transitions and symmetry breaking.

In a separate article [5], we give a cluster expansion for strong (bare) coupling of even φ_2^4 models. This expansion allows us to construct two pure phases, each satisfying the Wightman and Osterwalder-Schrader axioms, with a unique vacuum and with a mass gap.

In contrast to our detailed study based on the cluster expansion [5], we present at this conference a simple, direct proof that phase transitions occur. The details of this talk will be published separately [6]. An alternative approach to the problem of phase transitions has been announced in [7], but the proof has not appeared.

<u>Theorem 1</u>. Consider the $\lambda{:}\varphi_2^4{:}_{m_0^2} + \frac{1}{2}m_0^2{:}\varphi^2{:}_{m_0^2}$ theory with Wick ordering mass m_0, bare mass m_0, and zero Dirichlet boundary conditions. For λ/m_0^2 sufficiently large, there is long range order (lack of clustering).

<u>Theorem 2</u>. Consider the model

$$\lim_{\mu\to 0+}(\lambda{:}\varphi_2^4{:}_{m_0^2} + \tfrac{1}{2}m_0^2{:}\varphi^2{:}_{m_0^2} - \mu\varphi)$$

with Wick ordering mass m_0 and bare mass m_0. For λ/m_0^2 sufficiently large, there is symmetry breaking, i.e.

$$\lim_{\mu\searrow 0}\langle\varphi\rangle > 0 ,$$

where $\langle\cdot\rangle$ denotes the vacuum expectation value. Likewise the model defined by $\mu \to 0-$ has $\langle \varphi \rangle < 0$.

Our proof of these theorems is based on a Peierls argument, similar to the proof of phase transitions in statistical mechanics. The basic idea is to study the average field

$$\varphi(\Delta) = \int_{\Delta} \varphi(x)dx$$

where the average is taken over a unit square Δ in Euclidean space-time. The average (low momentum) field dominates the description of phase transitions, while the error

$$\delta\varphi(x) = \varphi(x) - \varphi(\Delta), \qquad x \in \Delta\,,$$

the "fluctuating field" is estimated in terms of the kinetic part of the action, $\frac{1}{2}(\nabla\varphi)^2$. Technically, we use φ^j bounds to establish the estimates which give the convergent Peierls expansion, and show the probability of "flipping" values of $\varphi(\Delta)$ is small.

In place of repeating the material in [6], we explain the classical (mean field) approximation to the φ^4 theory. This classical picture is the basis for our convergent expansions about the mean field.

2. Classical Approximation

Consider a quantum field defined by the Euclidean action density

$$\tfrac{1}{2}{:}(\nabla\varphi)^2{:}_{a^2} + {:}\mathcal{V}(\varphi){:}_{a^2} = {:}\tfrac{1}{2}(\nabla\varphi)^2 + \tfrac{1}{2}a^2\varphi^2{:}_{a^2} + {:}P(\varphi){:}_{a^2}$$

Here $:\ :_{a^2}$ denotes Wick ordering with respect to mass a, and by convention we include a bare mass a in the free part of the action, $\frac{1}{2}:(\nabla\varphi)^2 + a^2\varphi^2:$. The classical approximation for the ground state of the field φ is obtained by regarding $\frac{1}{2}(\nabla\varphi)^2$ as a kinetic term and $\mathcal{V} \equiv \frac{1}{2}a^2\varphi^2 + P(\varphi)$ as a potential term. Then in the classical approximation the vacuum expectation (mean) $\langle\varphi\rangle$ of φ equals φ_c, a value of φ which minimizes $\mathcal{V}$. The classical mass m_c is given by

$$m_c^2 = \mathcal{V}''(\varphi_c) = a^2 + P''(\varphi_c) .$$

In other words the classical low mass states of φ are those of a free field with action density

$$\mathcal{V}_c = :\tfrac{1}{2}(\nabla\varphi)^2 + \tfrac{1}{2}m_c^2(\varphi - \varphi_c)^2:_{m_c^2} .$$

For convenience, we choose the constant in P so that $P(0) = 0$. (The same then holds for $\mathcal{V}$.)

We expect the classical approximation to be accurate (up to higher order quantum corrections) for those interaction polynomials $\mathcal{V}$ such that

(i) $\mathcal{V} - \mathcal{V}_c$ is small for $\varphi - \varphi_c$ small,

and

(ii) $a^2 = m_c^2$.

We say that an interaction $:P:_{a^2}$ satisfying (i) and (ii) is classical.

To understand the conditions (i) and (ii) concretely, we write $\mathcal{V}(\varphi)$ in terms of its Taylor series about $\varphi = \varphi_c$, namely

$$\mathcal{V}(\varphi) = \mathcal{V}(\varphi_c) + \tfrac{1}{2}m_c^2(\varphi - \varphi_c)^2 + \sum_{i\geq 3} \lambda_i(\varphi - \varphi_c)^i ,$$

where

$$\lambda_i = (i!)^{-1}\mathcal{V}^{(i)}(\varphi_c)$$

$$= (i!)^{-1}P^{(i)}(\varphi_c) , \qquad i \geq 3 .$$

In particular, condition (i) is satisfied if

(1) $$|\lambda_i/m_c^2| \ll 1 ,$$

where $i \geq 3$.

To achieve (ii) will normally require Wick reordering, and in preparation, we calculate the α dependence of the Wick constant

$$c(\alpha^2) = \frac{1}{(2\pi)^2}\int \frac{d^2p}{p^2 + \alpha^2} .$$

Then

$$-\frac{d}{d\alpha^2}c(\alpha^2) = \frac{1}{(2\pi)^2}\int \frac{d^2p}{(p^2+\alpha^2)^2} = (4\pi\alpha^2)^{-1} ,$$

where we interpret this formula as a $\kappa \to \infty$ limit of cutoff equations in which $p^2 \leq \kappa^2$. We expect that (ii) will be satisfied after Wick reordering if

(2) $$(\lambda_i/m_c^2)\ln^{i/2}(m_c^2/a^2) \ll 1\,, \quad i \geq 3\,.$$

In the following section we carry out this choice for the φ_2^4 model. The classical approximation is also referred to as the Goldstone approximation or the mean field approximation.

3. The φ^4 Interaction

The conventional definition of the φ^4 interaction is

(3) $$:\mathcal{V}:_{m_0^2} = \lambda :\varphi^4:_{m_0^2} + \tfrac{1}{2}m_0^2 :\varphi^2:_{m_0^2}$$

$$= :P(\varphi):_{m_0^2} + \tfrac{1}{2}m_0^2 :\varphi^2:_{m_0^2}\,.$$

The weak coupling region $\lambda/m_0^2 \ll 1$ satisfies (1) and (2), and hence is also a classical region. In this region $\varphi_c = 0$, $m_c = m_0$. Thus the classical picture of weakly coupled φ^4 is a field with mean zero, and particles of mass $m_c = m_0$. The $\varphi \to -\varphi$ symmetry preserves $\langle\varphi\rangle = 0$ as an exact identity, but we expect quantum corrections to give a physical mass

(4) $$m = m_c(1 + O(\lambda/m_c^2)), \quad \lambda/m_c^2 \to 0\,.$$

In fact the weak coupling region is well understood from the cluster expansion [8], which yields a Wightman-Osterwalder-Schrader theory for $\lambda/m_0^2 \ll 1$, and $m - m_c = o(\lambda/m_c^2)$.

We now turn our attention to the region $\lambda/m_0^2 \gg 1$. In this region $:\mathcal{V}:_{m_0^2}$ given by (3) is clearly not classical, since both (1) and (2) fail. In order to obtain a classical interpretation, we rewrite (3) in terms of a new Wick ordering mass a satisfying

$$a^2 \gg \lambda \gg m_0^2. \tag{5}$$

Then we write (3) as a new polynomial $:\mathcal{V}_1(\varphi):_{a^2}$ satisfying $\mathcal{V}_1(0) = 0$. Thus

$$\begin{aligned} :\mathcal{V}(\varphi):_{m_0^2} + \text{const.} &= :\mathcal{V}_1(\varphi):_{a^2} \\ &= :\lambda\varphi^4 - \left(\frac{6\lambda}{4\pi} \ell n \frac{a^2}{m_0^2} - \tfrac{1}{2}m_0^2\right)\varphi^2:_{a^2} \end{aligned} \tag{6}$$

Here

$$\frac{1}{4\pi} \ell n \frac{a^2}{m_0^2} = c(m_0^2) - c(a^2) .$$

Likewise

$$:\mathcal{P}_1(\varphi):_{a^2} = :\lambda\varphi^4 - \left(\frac{6\lambda}{4\pi} \ell n \frac{a^2}{m_0^2} + \tfrac{1}{2}a^2 - \tfrac{1}{2}m_0^2\right)\varphi^2:_{a^2} \tag{7}$$

and

$$\mathcal{V}_1 = \mathcal{P}_1 + \tfrac{1}{2}a^2\varphi^2 .$$

By (5), the coefficient of φ^2 in $\mathcal{V}_1$ is negative, so $\mathcal{V}_1$ has a double minimum at $\varphi = \varphi_c = \pm\sigma$. Here m_c and σ are related by

$$m_c^2 = 8\lambda\sigma^2 = \frac{6\lambda}{\pi} \ln \frac{a^2}{m_0^2} - 2m_0^2 . \tag{8}$$

We now choose a so that $m = a$, as can be achieved by letting x solve the equation

$$x = \frac{\lambda}{m_0^2} \frac{6}{\pi} \ln x - 2 .$$

For λ/m_0^2 sufficiently large, this equation has exactly two solutions. The larger solution determines a by the relation $x = (a/m_0)^2$. The smaller solution is spurious in the sense that is gives an interaction satisfying (ii) but not (i).

Next we perform a scale transformation so the classical mass becomes one. Since the Wick ordering mass transforms similarly, it also becomes 1. Thus after the scale transformation we obtain an interaction polynomial $:\mathcal{V}_2:_1$ given by

$$:\mathcal{V}_2:_1 = :\frac{1}{8\sigma^2}\varphi^4 - \frac{3}{4}\varphi^2:_1 + \tfrac{1}{2}:\varphi^2:_1 \tag{9}$$

$$= :P_2:_1 + \tfrac{1}{2}:\varphi^2:_1 .$$

By (8), we see that $\sigma \gg 1$. Thus the interaction (3), in the strong coupling region $\lambda/m_0^2 \gg 1$, is equivalent to the weakly coupled φ^4 interaction (9), with a negative quadratic term, with bare mass 1 and with Wick mass 1. For the interaction (9), we find that

(10) $$\varphi_c = \pm\sigma, \qquad m_c = 1,$$

$$\lambda_3 = \pm(2\sigma)^{-1}, \quad \lambda_4 = (8\sigma^2)^{-1}.$$

Thus for σ large, both (1) and (2) are satisfied and (9) is classical. It exhibits the two phase classical approximation to strongly coupled φ_2^4, since φ_c has two possible mean field values. In our second paper [5], we present a systematic expansion about the classical field φ_c, combined with a Peierls argument to select a given phase. We find that in each of two pure phases, the physical mass is positive.

References and Footnotes

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Colloques Internationaux C.N.R.S.
N° 248 – Les méthodes mathématiques de la théorie quantique des champs

Exponential Bounds in Lattice Field Theory * X

by

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RESUME Un résumé des bornes exponentielles des modèles de mécanique statistique, suggéré par l'approximation du réseau pour la théorie euclidienne des champs est donné.
Le but principal est de formuler ces bornes dans une forme qui est convenable pour la discussion de la limite lorsque l'échelle du réseau tend vers zero pour les interactions quantiques à trois et quatre dimensions.

ABSTRACT A report on the exponential bounds for models of statistical mechanics, suggested by the lattice approximation to Euclidean field theory, is given. The principal concern is to phrase these bounds in a form which is convenient for the limit as the lattice spacing goes to zero for quantic interactions in 3- and 4-dimensional space-time.

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1. INTRODUCTION

In the constructive approach to quantum field theory, advanced by Glimm and Jaffe and their followers [7,38,37,12,19,22,34,11,39], a very important role is played by the so called φ-bounds, firstly established by Glimm and Jaffe for the $P(\varphi)_2$ theory in [9], see also [18,34]. These bounds allow uniform estimates on the Wightman functions of a volume cut off theory as the size of the volume goes to infinity. In the Euclidean strategy to constructive quantum field theory, based on the development of ideas of Symanzik [36] and Nelson [26], the Glimm-Jaffe φ-bounds have a very important consequence in the form of exponential bounds for the Euclidean-Markov fields. These exponential bounds were established by Fröhlich [5] and are the neatest way to complete the program of construction of the Schwinger functions, using the Nelson monotonicity argument [26], which relies on the correlation inequalities established in [19].

The purpose of this note is to report on the exponential bounds for models of statistical mechanics, suggested by the lattice approximation to Euclidean quantum field theory [19]. Our main concern is to put these bounds in a form which may be convenient for the limit as the lattice spacing goes to zero for quartic interaction in three-and four-dimensional space-time. We follow mainly the methods of references [15,18] and give some detailed results for the three-dimensional case. The expression of the exponential bounds derived by these methods involves in an essential way the infinite volume energy density of the theory. As a consequence all Schwinger functions can be dominated by the one point Schwinger function in some external field. It is expected that this can be useful for the control of the lattice spacing going to zero for three and four-dimensional quartic theories. The organization of the report is as follows. In Section 2 we introduce the lattice field models as suggested by renormalized quantum field theory in the Euclidean formulation. These models are represented by an array of continuous spins with polynomial selfinteraction and ferromagnetic nearest neighbor mutual interaction. We introduce also, in Section 3, the transfer matrix formalism using a symmetrization trick which will allow us to exploit the Euclidean symmetry of the theory (Nelson symmetry) as in [15,18]. This trick could be used also for other models of statistical mechanics, for example the Ising model, in order to get a transfer matrix formalism without the need of periodic boundary conditions or boundary terms.

In Section 4 we consider the energy density (or pressure) and investigate its behavior in the infinite volume limit, extending the results of [15,18,19,20] to the present situation.

Section 5 contains the main results of this report in the form of bounds for the perturbed energy, which are the lattice analogs of the Glimm-Jaffe φ-bounds [9], and bounds for the exponentials of the lattice fields, which are the extension of Fröhlich exponential bounds [5] to the present situation.

In Section 6 we give some applications. In particular we establish a very simple inequality for the long range order in function of the derivatives of the pressure with respect to an external field, and we discuss the role of field strength renormalization for quartic coupling in four dimensions.

We will give only a brief sketch of all proofs, a more detailed report will be published elsewhere.

In conclusion the author would like to thank the "U.E.R. Scientifique de Luminy" and the "Centre de Physique Theorique du CNRS", and in particular Daniel Kastler and Raymond Stora, for the kind hospitality extended to him in Marseille, where part of the present work was performed.

2. THE LATTICE FIELD MODELS

In the d-dimensional Euclidean space, $\mathbb{R}^d$, we consider the lattice of spacing ε, $\varepsilon\,\mathbb{Z}^d$, associated to the unit lattice, $\mathbb{Z}^d$, with the notations

$$\mathbb{R}^d \ni x \equiv \{x_{(1)},\dots,x_{(d)}\}\ ,\quad x_{(i)}\in\mathbb{R}\ ,$$
$$\mathbb{Z}^d \ni n \equiv \{n_{(1)},\dots,n_{(d)}\}\ ,\quad n_{(i)} = 0,\pm1,\pm2,\dots\ ,$$
$$\mathbb{R}^d \supset \varepsilon\,\mathbb{Z}^d \ni x_n \equiv \varepsilon n\ ,\quad (x_n)_{(i)} = \varepsilon n_{(i)}\ ,\quad i=1,\dots,d\ ,\quad \varepsilon>0 .$$

The number d plays the role of space-time dimension, so the physical case is $d=4$, but we consider also the cases $d=1,2,3$, according to the practice of constructive quantum field theory and statistical mechanics.

Following [19], see also [17], we introduce the free lattice field $\varphi_\varepsilon(x_n)$ as the real Gaussian random process indexed by the lattice $\varepsilon\,\mathbb{Z}^d$, with mean zero and covariance given by

$$\langle \varphi_\varepsilon(x_n)\,\varphi_\varepsilon(x_{n'})\rangle = Z_\varepsilon^{-1}\, S_\varepsilon(x_n - x_{n'})\ ,\quad x_n, x_{n'} \in \varepsilon\,\mathbb{Z}^d .$$

Here the free two point function is defined by

$$S_\varepsilon(x_n - x_{n'}) = (2\pi)^{-d}\int_{\mathbb{C}_\varepsilon^d} e^{ik\cdot(x_n - x_{n'})}\,\mu_\varepsilon^{-2}(k)\,dk\ ,$$

where $\mathbb{C}_\varepsilon^d \equiv \{k \mid k\in\mathbb{R}^d,\ |k_{(i)}|\le \pi/\varepsilon\ ,\ i=1,\dots,d\}$ and

$$\mu_\varepsilon^2(k) = m^2 + 4\varepsilon^{-2}\sum_{i=1}^{d} \sin^2(\varepsilon k_{(i)}/2)$$
$$\simeq m^2 + k^2 \quad \text{as} \quad \varepsilon\to 0 .$$

For the field strength renormalization constant we have $Z_\varepsilon = 1$ for $d=1,2,3$, and $0<Z_\varepsilon\le 1$ for φ_4^4, depending on the interaction. It is expected that $Z_\varepsilon \to 0$ as $\varepsilon\to 0$ in φ_4^4 in order to prevent that all Schwinger functions vanish in the limit $\varepsilon\to 0$. We find convenient to introduce the field strength renormalization directly in the "bare" field. It will be clear in the following the equivalence

of this procedure with the customary one which introduces field strength renormalization in the counterterms.

It is immediately seen [19] that the free lattice field can be realized through an array of continuous spins q_n, sitting at the lattice points x_n, with Gaussian single spin distributions

$$\exp\left[-\frac{1}{2} Z_\varepsilon \, \varepsilon^d \left(m^2 + 2d\,\varepsilon^{-2}\right) q_n^2\right]$$

and ferromagnetic nearest neighbor coupling of the type

$$\exp Z_\varepsilon \, \varepsilon^{d-2} q_n q_{n'}$$

where (n,n') is each couple of nearest neighbors on $\varepsilon \mathbb{Z}^d$. Each random field $\varphi_\varepsilon(x_n)$ is represented through q_n.

One easily recognizes that the single spin distributions and the ferromagnetic couplings are the lattice approximation of the (formal) expression

$$e^{-\frac{1}{2} Z \int \left[(\nabla\varphi)^2 + m^2\varphi^2 \right] dx}$$

well familiar in the functional formulation of quantum field theory (in the Euclidean region). This makes also clear the connection between our way of introducing the field strength renormalization and the customary way, which is based on the introduction (among others) of the counterterm

$$\frac{1}{2}(Z-1)\left[(\nabla\varphi)^2 + m^2\varphi^2 \right]$$

to the Euclidean free action density $\frac{1}{2}\left[(\nabla\varphi)^2 + m^2\varphi^2\right]$. In fact we have $1 + (Z-1) = Z$.

The interaction $P(\varphi)$ modifies the single spin distribution by a factor $\exp\left[-\varepsilon^d \left(P(q) + R_\varepsilon(q)\right)\right]$,
where P is the bounded below interaction polynomial and $R_\varepsilon(q)$ is a polynomial containing the renormalization counterterms.

For the $P(\varphi)_2$ theory only the counterterms coming from Wick subtractions in P are necessary [12,19]. For the φ_3^4 theory

[6,10,2,28,3,25] one should introduce also a mass renormalization counterterm necessary to cancel the divergent part of the selfenergy graph  . For the φ_4^4 theory the renormalized interaction is expected to be of the form $g Z_{1,\varepsilon} :\varphi^4: + \frac{1}{2} Z_\varepsilon \delta m_\varepsilon^2 \varphi^2$. Unfortunately at present the expression of the vertex constant Z_1 , the field strength Z and the mass counterterm δm^2 , is known only by pertubation theory [21]. This prevents the actual study of the ultraviolet limit $\varepsilon \to 0$, see however the interesting proposal of Schrader for the renormalization of φ_4^4 in these proceedings [31]. From now on we will consider general quartic theories on a lattice of fixed spacing $\varepsilon > 0$. We find convenient to add also a linear interaction of the type $-\lambda\varphi$. Is should be remarked that according to general wisdom, relying on symmetry and power counting, the constant λ is neither renormalized nor affects the ultraviolet divergent contributions to the renormalization constants. In the following we will suppose that the renormalization constants are determined for $\lambda = 0$, then the term $-\lambda\varphi$ is added to the interaction, which results therefore linear in λ.

For this kind of interactions we can exploit all machinery of statistical mechanics of ferromagnetic systems,

in particular the correlation inequalities of GKS and FKG type [13,23,4] . For quartic interactions, using the Simon-Griffiths results [35], we have also the possibility to exploit the correlation inequalities of GHS [4], Lebowitz [24], u_6 [1] and Newmann [27] type. Also the powerful Lee-Yang theorem [30] is available [35] .

In particular the infinite volume limit can be obtained using monotonicity arguments like in the Ising model theory. Since here the interacting spins are unbounded it is necessary to have some a priori bounds on the correlation functions. We will establish them in the following in a form which looks promising for the limit $\varepsilon \to 0$.

3. THE TRANSFER MATRIX FOR LATTICE THEORIES

Firstly we consider the free lattice field $\varphi_\varepsilon(x_n)$. Call (Q,Σ,μ) the underlying probability space and let Σ_t, for $t \in \varepsilon\mathbb{Z}$, be the sub-$\sigma$-algebra of Σ generated by the fields $\varphi_\varepsilon(x_n)$ supported by the hyperplane $(x_n)_{(d)} = t$.

All spaces $L^p(Q,\Sigma_t,\mu)$, $t \in \varepsilon\mathbb{Z}$, are isomorphic to the same $L^p(\bar{Q},\bar{\Sigma},\bar{\mu})$, $1 \le p \le \infty$, so we can consider the natural injections

$$J_t : L^p(\bar{Q},\bar{\Sigma},\bar{\mu}) \to L^p(Q,\Sigma,\mu) \quad , \text{ for } t \in \varepsilon\mathbb{Z} .$$

Exploiting the Markov property of the lattice theory [19,17], we introduce the semigroup $e^{-tH_0} = J_t^+ J_0$, $t \in \varepsilon\mathbb{Z}$, which acts on the $L^p(\bar{Q},\bar{\Sigma},\bar{\mu})$ spaces as "transfer matrix" from one hyperplane to the other. The selfadjoint operator H_0 can be understood as "Hamiltonian" of the theory.

We now introduce a useful form of the transfer matrix for the interacting theory through the following symmetrization trick. In order to be able to draw pictures, we limit ourselves to the threedimensional case in the following, but it will be clear how to extend our considerations to the more general d-dimensional case (in particular the physically interesting d=4).

Let us therefore consider $\mathbb{R}^3$ and the lattice points $x_n \in \varepsilon\mathbb{Z}^3$. We call $y_{n'}$ the center of the generic cube $\Delta_{n'}$, of side ε, having as vertexes nearest neighbors in $\varepsilon\mathbb{Z}^3$. Let x_n, $n \in C(n')$, be the eight vertexes of $\Delta_{n'}$. Then for any $h \in C_0^\infty(\mathbb{R}^3)$ we introduce the smeared field

$$\varphi_\varepsilon(h) = \sum_{n'} \varepsilon^3 \sum_{n\in C(n')} \frac{1}{8}\varphi_\varepsilon(x_n) h(y_{n'}) .$$

Notice that for each n' the sum $\sum_{n\in C(n')}$ is a random variable measurable with respect to the sub-σ-algebra $\Sigma_{n'}$ generated by the fields $\varphi_\varepsilon(x_n)$ with $n \in C(n')$, Moreover there is complete symmetry for the group which leaves $\Delta_{n'}$ invariant, because only the value of h at the center $y_{n'}$ enters in each expression.

The factor $1/8$ is introduced in order to avoid overcounting because each point x_n of the lattice belongs to 8 cubes. In general, in d dimensions, the factor must be 2^{-d}.

Following the methods of [19] one can see that, in the limit $\varepsilon \to 0$, theright expression $\varphi(f) = \int \varphi(x) f(x) dx$ is obtained,where $\varphi(f)$ is the Euclidean-Markov field for the continuum theory on $\mathbb{R}^d$.

In general if P is a polynomial we put

$$U_{n'} = \varepsilon^3 \sum_{n \in C(n')} \frac{1}{8} P(\varphi_\varepsilon(x_n)) ,$$

and if Λ is a region of $\mathbb{R}^3$ union of cubes $\Delta_{n'}$ we define

$$U_\Lambda = \sum_{n'} U_{n'}$$

Now we are ready for the introduction of the transfer matrix in the interacting case.

Let $\Delta_{11}, \ldots, \Delta_{ss}$ be cubes contained between the planes $x_3=0$ and $x_3=\varepsilon$, and consider the collection $\{P\} \equiv \{P_{11}, \ldots, P_{ss}\}$ of associated polynomials. Put

$$U(\{P\}) = \sum_{ij} U_{ij}$$

then the transfer matrix associated to the collection $\{P\}$ is given by

$$e^{-\varepsilon H(\{P\})} = J_\varepsilon^+ e^{-U(\{P\})} J_0 .$$

Piling up t/ε layers, $t \in \varepsilon \mathbb{Z}^+$, and exploiting Markov property we have also

$$e^{-\varepsilon H(\{P_{t/\varepsilon}\})} \cdot \ldots \cdot e^{-\varepsilon H(\{P_1\})} = J_t^+ e^{-\sum_{ijk} U_{ijk}} J_0 ,$$

where $\{P_k\}$ is the collection $\{P_{ijk}\}$, k fixed, $k = 1, \ldots, t/\varepsilon$. Finally if we take the vacuum averages we have

$$\langle e^{-\sum_{ijk} U_{ijk}} \rangle = \langle \Omega_0, e^{-\varepsilon H(\{P_1\})} e^{-\varepsilon H(\{P_2\})} \ldots \Omega_0 \rangle ,$$

where, by Euclidean symmetry (Nelson symmetry) we may take

$\{P_1\} \equiv \{P_{ij1}\}$, $\{P_2\} \equiv \{P_{ij2}\}$, etc., or

$\{P_1\} \equiv \{P_{i1k}\}$, $\{P_2\} \equiv \{P_{i2k}\}$, etc., or

$\{P_1\} \equiv \{P_{1jk}\}$, $\{P_2\} = \{P_{2jk}\}$, etc.,

according to the direction of transfer we are interested in.

Notice that in our notation $\{P\}$ is always a collection (of polynomials) depending on two parameters, associated to a layer of cubes Δ_{ij}.

4. THE ENERGY DENSITY OR PRESSURE

Given $a, b, c \in \varepsilon \mathbb{Z}^+$, we consider the parallepiped of sides a,b,c. To each elementary cube Δ_{ijk}, $i=1, \ldots, a/\varepsilon$, $j=1, \ldots, b/\varepsilon$, $k=1, \ldots, c/\varepsilon$, we associate the same polynomial P. By transfer in the direction k, using the methods of the previous Section, we can define the semigroup $e^{-cH_{ab}}$. In particular we have for the partition function, defined by

$$Z(a,b,c) = \langle e^{-\sum_{ijk} U_{ijk}} \rangle ,$$

the following expression $Z(a,b,c) = \langle \Omega_0, e^{-cH_{ab}} \Omega_0 \rangle$.

We call $\Omega(a,b)$ the ground state of H_{ab} and $E(a,b)$ the corresponding energy $H_{ab}\,\Omega(a,b) = E(a,b)\,\Omega(a,b)$.

Therefore we have $\| e^{-cH_{ab}} \| = e^{-c\,E(a,b)}$

We define also the overlap function $\eta(a,b)$ through

$$\| \Omega(a,b) \|_1^2 = \exp\left[-a\,b\,\eta(a,b)\right] ,$$

where $\| \ \|_1$ is the norm in $L^1(\bar{Q}, \bar{\Sigma}, \bar{\mu})$. By standard arguments [8,34], we have $\Omega(a,b) > 0$ and $\eta(a,b) > 0$.

Lemma 1. The partition function Z(a,b,c) is symmetric in $a,b,c \in \varepsilon \mathbb{Z}^+$. This symmetry follows easily from the considerations of the end of the previous Section.

Theorem 2. The function $\log Z(a,b,c)$ is convex in a,b,c separately.
Proof.- The expression $\langle \Omega_0, e^{-cH_{ab}} \Omega_0 \rangle$, originally defined for $c \in \varepsilon \mathbb{Z}^+$, can be easily extended to all $c \in \mathbb{R}^+$ through the spectral theorem. Then the convexity of the logarithm is obvious. By restriction again to $c \in \varepsilon \mathbb{Z}^+$ and using Lemma 1 the theorem is proven.

Theorem 3 . The expression $(a,b,c)^{-1}\log Z(a,b,c)$ is uniformly bounded in a,b,c.

This can be proven easily, using Nelson symmetry, as in the two-dimensional case, see [18,16] . For the φ_3^4 theory, with the right counterterms, it can also be proven that the bound is uniform with respect to the lattice spacing ε , using arguments of Glimm and Jaffe [10] and Park [28] .

Since we have $Z(a,b,c) = 1$ if one among the parameters a,b,c is zero, then the following theorems are simple consequences of theorems 2 and 3.

Theorem 4 , The following bounds and monotone convergence results hold:

a) $(abc)^{-1}\log Z(a,b,c) \le \alpha_\infty \equiv \sup_{abc} (abc)^{-1}\log Z(a,b,c)$,

b) $c^{-1}\log Z(a,b,c) \uparrow -E(a,b)$ as $c \to \infty$,

c) $-b^{-1}E(a,b) \uparrow \alpha(a) \equiv \sup_b -b^{-1}E(a,b)$ as $b \to \infty$,

d) $a^{-1}\alpha(a) \uparrow \alpha_\infty$ as $a \to \infty$,

e) $-E(a,b) \le ab\alpha_\infty$, $-E(a,b) \le b\alpha(a)$, $\alpha(a) \le a\alpha_\infty$.

Theorem 5 . For $a' \ge a$, $a, a' \in \mathbb{Z}^+$, we have

$$-E(a',b) \le -E(a,b) + (a'-a)\,\alpha(b),$$

$$\alpha(a') \le \alpha(a) + (a'-a)\,\alpha_\infty .$$

By the same methods as in [18,16] we can also prove

Theorem 6 . There is a function $\eta_2(a)$ and a constant η_2 such that for the overlap function $\eta(a,b)$ we have

$\eta(a,b) \le \eta_2(a)$ for b large enough, and

$\eta_2(a) \le \eta_2$ for a large enough.

If we define $\beta(a;b)$ and $\beta(a)$ through

$$-E(a,b) = a\,\alpha(b) + \beta(a;b) \;, \qquad \alpha(a) = a\,\alpha_\infty + \beta(a)$$

then from theorems 4,5,6 it follows

Theorem 7 . The non positive functions $\beta(a;b)$ and $\beta(a)$ are convex and decreasing in a and bounded from below by

$$\beta(a,b) \geq -b\,\eta_2(b) \quad \text{and} \quad \beta(a) \geq -\eta_2 .$$

Let us define

$$\beta_\infty(b) = \lim_{a\to\infty} \beta(a;b) \quad , \quad \beta_\infty = \lim_{a\to\infty} \beta(a) .$$

Since $\lim_{b\to\infty} b^{-1}\beta(a;b) = \beta(a)$, we have also

$$\beta(a;b) \geq \beta_\infty(b) \, , \; -b\,\eta_2(b) \leq \beta_\infty(b) \leq 0 \, , \; \beta(a) \geq \beta_\infty ,$$

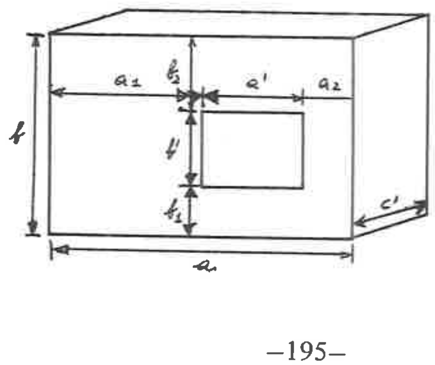
$$-\eta_2 \leq \beta_\infty \leq 0 \, , \; \overline{\lim_{b\to\infty}} \, b^{-1}\beta_\infty(b) \leq \beta_\infty .$$

Let us remark that Theorems 2 - 7 are the natural extension to the threedimensional case of the analogous results for the $P(\varphi)_2$ theory, as presented for example in [18,16,34] . It is clear the central role played by Nelson symmetry and Markov property. The extension of these results to general d-dimensional lattices is straight-forward.

5. EXPONENTIAL BOUNDS

This Section contains the main results of this report. Detailed proofs will appear in a forthcoming paper.

The first theorem refers to a bound on the ground state energy of a locally perturbed Hamiltonian. In Section 3, through the transfer matrix method, we have defined the Hamiltonian $H(\{P\})$ associated to the two-parameter family of polynomials $\{P\} \equiv \{P_{ij}\}$. Consider the following geometric situation, with $a, b, a', b', c', a_1, a_2, b_1, b_2 \in \varepsilon \mathbb{Z}^+$, $a = a_1 + a' + a_2$, $b = b_1 + b' + b_2$,



Consider $c' = \varepsilon$ and let $\{f\} \equiv \{f_{ij}\}$ be constants associated to the cubes contained in the parallepiped $\Lambda(a', b', \varepsilon)$ of sides (a', b', ε) . Given the interaction polynomial P consider the family of polynomials $\{P_{ij}\}$ associated to $\Lambda(a,b,\varepsilon)$ defined as follows

$P_{ij}(X) = P(X) - f_{ij} X$ for cubes in $\Lambda(a', b', \varepsilon)$,

$P_{ij}(X) = P(X)$ for cubes in $\Lambda(a,b,\varepsilon)$ but not in $\Lambda(a', b', \varepsilon)$.

Let $H(a,b;\{f\})$ be the associated Hamiltonian, then we have

Theorem 8 . For the ground state energy of $H(a,b;\{f\})$ we have the following estimate

$$- E(a,b;\{f\}) \leq \frac{1}{2}\left[E(a, 2b_1) + E(a, 2b_2)\right] +$$
$$+ b' \frac{1}{2}\left[\alpha(2a_1) + \alpha(2a_2)\right] + \sum_{ij} \varepsilon^2 \alpha_\infty (P - f_{ij} X) .$$

Here $-E(a,b)$ and $\alpha(a)$ are like those defined in Section 4, for the interaction polynomial P , and $\alpha_\infty(Q)$ is the pressure for the polynomial Q . The sum $\sum_{ij}$ extends to all cubes in $\Lambda(a', b', \varepsilon)$.

The proof of this theorem is not complicated. It is based on the repeated application of the rotation method, like in the proof of the Glimm-Jaffe φ-bounds for $P(\varphi)_2$ given in [18].

Finally let us consider the exponential bounds, which follow from theorem 8. Let $h \in C_0^\infty(\mathbb{R}^3)$ with $\operatorname{supp} h \in \Lambda(a', b', c')$ and $h \geq 0$ and consider the smeared field $\varphi_\varepsilon(h)$ defined in Section 3 with the symmetrization trick.

Consider the volume cut off expectation value for the exponential of the field

$$\langle e^{\varphi_\varepsilon(h)} \rangle_{(a,b)} = \langle J_0(\Omega(a,b)) J_{c'/\varepsilon}(\Omega(a,b)) e^{-\sum_{ijk} U_{ijk}} e^{\varphi_\varepsilon(h)} \rangle e^{c' E(a,b)},$$

where $\sum_{ijk}$ extends to all cubes in $\Lambda(a,b,c')$ and U_{ijk} is defined like in Section 3 by means of the interaction polynomial P .

Then we have

Theorem 9 . The following estimate holds

$$< e^{\varphi_\varepsilon(h)} >_{(a,b)} \leq e^{\rho(a;b)} e^{\sum_{n'} \varepsilon^3 [\alpha_\infty(P - h(y_{n'})X) - \alpha_\infty(P)]}$$

where $\alpha_\infty(Q)$ is the lattice pressure associated to the interaction polynomial Q(X). For the function $\rho(a;b)$ we have

$$\lim_{a\to\infty} \lim_{b\to\infty} \rho(a;b) = 0 ,$$

where $b\to\infty$ means both b_1 and $b_2 \to\infty$, the same for $a\to\infty$.

Using Griffiths correlation inequalities Theorem 9 gives us bounds in the infinite volume limit for all lattice Schwinger functions.

6. APPLICATIONS

In the $P(\varphi)_2$ case no lattice cutoff is necessary. If we consider an interaction of the type $P=Q-\lambda X$ with Q even and bounded below, then the infinite volume limit for the half-Dirichlet Schwinger functions [19] can be obtained through Nelson monotonicity theorem [26] . Then the exponential bound in the form analogous to Theorem 9 is expressed as follows.

Theorem 10 . Call < > the half-Dirichlet infinite volume limit for the interaction $P=Q-\lambda X$. For $h \in C_0^\infty(\mathbb{R}^2)$, $h \geq 0$, we have

$$< e^{\varphi(h)} > \leq \exp \int [\alpha_\infty(P - h(x)X) - \alpha_\infty(P)] dx$$

In the following we will need some properties of the pressure α_∞ as a function of the external field λ .

Proposition 11 . Let $\alpha_\infty(\lambda)$ be the pressure associated to the interaction $Q-\lambda X$ with Q even bounded below. Then

a) $\alpha_\infty(\lambda)$ is convex and continuous in λ and increasing for $\lambda \geq 0$.

b) The left and right derivatives, defined by

$$M^{(-)}(\lambda) = \lim_{\varepsilon\to 0^+} \varepsilon^{-1}[\alpha_\infty(\lambda) - \alpha_\infty(\lambda-\varepsilon)] , \quad M^{(+)}(\lambda) = \lim_{\varepsilon\to 0^+} \varepsilon^{-1}[\alpha_\infty(\lambda+\varepsilon) - \alpha_\infty(\lambda)] ,$$

exist for any λ and are equal almost everywhere.

c) For $\lambda \geq 0$ we have

$$M^{(+)}(\lambda) = \inf_{\varepsilon > 0} \varepsilon^{-1} \left[\alpha_\infty(\lambda+\varepsilon) - \alpha_\infty(\lambda) \right] ,$$

therefore $M^{(+)}(\lambda)$ is upper semicontinuous in λ

$$M^{(+)}(\lambda) = M^{(+)}(\lambda+0) , \text{ for } \lambda \geq 0 .$$

d) For $\lambda > 0$ we have

$$M^{(-)}(\lambda) = \sup_{\varepsilon > 0} \varepsilon^{-1} \left[\alpha_\infty(\lambda) - \alpha_\infty(\lambda-\varepsilon) \right] ,$$

therefore $M^{(-)}(\lambda)$ is lower semicontinuous in λ

$$M^{(-)}(\lambda) = M^{(-)}(\lambda-0) , \text{ for } \lambda > 0 .$$

e) $M^{(+)}(\lambda)$ and $M^{(-)}(\lambda)$ are increasing in λ for $\lambda \geq 0$, moreover

$$M^{(+)}(\lambda) \geq 0 \text{ for } \lambda \geq 0 , \quad M^{(-)}(\lambda) \geq 0 \text{ for } \lambda > 0 .$$

f) We have also

$$M^{(-)}(\lambda) \leq M^{(+)}(\lambda) \leq M^{(-)}(\lambda') \quad \text{for } \lambda < \lambda' , \text{ and } M^{(+)}(\lambda) = M^{(-)}(\lambda+0).$$

g) $\alpha_\infty(\lambda) = \alpha_\infty(-\lambda)$, $M^{(+)}(\lambda) = - M^{(-)}(-\lambda)$.

h) If Q is a fourth order polynomial then $M^{(+)}(\lambda) = M^{(-)}(\lambda) \equiv M(\lambda)$ for $\lambda \neq 0$, and $M(\lambda)$ is concave in λ for $\lambda \geq 0$.

Properties a),...,g) are standard statistical mechanics results [30], property h) follows from the classical Ising approximation of Simon and Griffiths [35].

Like in statistical mechanics, see also [34], we have

<u>Proposition 12</u> . For the infinite volume half-Dirichlet state $\langle \rangle$ we have

$$M(\lambda) \equiv \langle \varphi(x) \rangle = M^{(-)}(\lambda) \quad \text{for } \lambda > 0 .$$

<u>Proposition 13</u> . (Simon [32])

a) The truncated two point function $S_2^T(x-y) = \langle \varphi(x)\varphi(y) \rangle - M(\lambda)^2$, is decreasing to a positive constant c^2 as $|x-y| \to \infty$.

b) The Wightman theory associated to the Euclidean state $\langle \rangle$ has a unique vacuum if and only if c=0 .

Now we can state our first main result of this Section

Theorem 14 . The long range order c^2 is related to the right and left derivatives of the pressure through

$$c^2 \leq M^{(+)}(\lambda)^2 - M^{(-)}(\lambda)^2 \qquad \text{for } \lambda > 0 ,$$
$$c^2 \leq M^{(+)}(0)^2 \qquad \text{for } \lambda = 0 .$$

Remarks. $M^{(+)}(0) = \lim_{\lambda \to 0^+} M(\lambda)$ is the spontaneous magnetization [30] .

We believe that the equalities hold, like in the Ising model, but it seems very difficult to prove it in the quantum field theory case.

Let us sketch the proof of Theorem 14.

From the convexity of the pressure (Proposition 11) and Proposition 12, we have

$$\alpha_\infty(\lambda + h) - \alpha_\infty(\lambda) \leq h\left[\alpha_\infty(\lambda + \bar{h}) - \alpha_\infty(\lambda)\right]\bar{h}^{-1} \leq h\, M(\lambda + \bar{h}) ,$$

for $0 \leq h \leq \bar{h}$, $\bar{h} > 0$.

Put $h(x) = \bar{h}\,\chi_\Lambda(x)$, $\bar{h} > 0$, in Theorem 10, where χ_Λ is characteristic function of the region Λ in $\mathbb{R}^2$. By Griffiths inequalities we have in general

$$\frac{1}{2} \exp \langle \varphi^2(h) \rangle^{\frac{1}{2}} \leq \langle \exp \varphi(h) \rangle .$$

From Proposition 13, we have

$$\langle \varphi^2(\chi_\Lambda) \rangle \geq \left(c^2(\lambda) + M^2(\lambda)\right) |\Lambda|^2$$

therefore collecting all results

$$\frac{1}{2} \exp\left[\bar{h}\,|\Lambda|\,\left(c^2(\lambda) + M^2(\lambda)\right)^{\frac{1}{2}}\right] \leq \exp\left[\bar{h}\,|\Lambda|\, M(\lambda + \bar{h})\right]$$

If we take the logarithm, divide by $|\Lambda|$, let $|\Lambda| \to \infty$ and then $\bar{h} \to 0$ we obtain immediately the results of the theorem.

A simple consequence of Theorem 14 and Proposition 13 is the following

Theorem 15 . If the pressure is differentiable in λ , i.e. $M_{(+)} = M_{(-)}$, then the Wightman theory, associated to the half-Dirichlet $Q - \lambda X$, has a unique vacuum.

In particular, since for quartic interaction Q and $\lambda \neq 0$ the pressure is differentiable by a result of Simon and Griffiths [35], see

Proposition 11.h, we can have a simple proof of the following result of Simon [34], for which we do not use the Lee-Yang theorem but only Theorem 15.

Theorem 16 . (Simon [34]) The Wightman theory, associated to the half-Dirichlet $Q-\lambda X$, for Q of fourth order, has a unique vacuum for $\lambda \neq 0$.

This kind of proof is very similar to analogous results in statistical mechanics [29].

Finally let us make few remarks about the role of field strength renormalization in φ_4^4 .

Let us consider the infinite volume limit of the lattice cut off half-Dirichlet theory. The using the analog of Theorem 9 in four dimensions and exploiting the convexity of the pressure, like in the proof of Theorem 14, we have

Theorem 17. Call $\langle \ \rangle_\lambda$ the infinite volume half-Dirichlet state for the interaction $Q-\lambda X$, then the following estimate hold

$$\langle e^{\varphi_\varepsilon(h)} \rangle_\lambda \leq e^{\|h\|_1 M(\lambda+\bar{h})} ,$$

where

$$h \in C_0^\infty(\mathbb{R}^4),\ 0 \leq h(x) \leq \bar{h},\ \|h\|_1 = \sum_{n'} \varepsilon^4 h(y_{n'}) ,\ M(\lambda+\bar{h}) = \langle \varphi_\varepsilon(x_n) \rangle_{\lambda+\bar{h}} .$$

This theorem shows that in order to get uniform bounds on the lattice cutoff infinite volume Schwinger functions it is enough to control the one point function (the magnetization) in the external field λ. We explained in Section 2 that the renormalization constants Z_1, Z and δm^2 can be chosen to be independent of λ , therefore by GHS inequalities $M(\lambda)$ is concave in λ for $\lambda \geq 0$. But $M(\lambda)$ must be also increasing therefore by combining convexity and increase of $M(\lambda)$ with Theorem 17, we have

Theorem 18 . If for some fixed value $\lambda_0 > 0$ the magnetization $M(\lambda_0)$

goes to zero or infinity, as the lattice spacing $\varepsilon \to 0$, then $M(\lambda)$ must go to zero or infinity for all values of $\lambda > 0$. As a consequence all Schwinger functions go to zero or infinity.

The last part of the theorem follows from the fact that $\langle \exp \varphi_\varepsilon(h) \rangle_\lambda \to 1$ as $\varepsilon \to 0$ if $M(\lambda_0) \to 0$, or from Griffiths inequalities if $M(\lambda_0) \to \infty$.

This shows that it is enough to control the magnetization for one value $\lambda_0 > 0$ of λ.

As far as the uniform bounds are concerned the case $M(\lambda_0) \to \infty$ can be easily cured by an additional field strength renormalization, which is equivalent to a change in the renormalization constants $(Z_1, Z, \delta m^2)$.

In fact let us suppose $\langle \varphi_\varepsilon(x_n) \rangle_{\lambda_0} \to \infty$ as $\varepsilon \to 0$, then define $\varphi_{ren} = Z'^{\frac{1}{2}}_\varepsilon \varphi_\varepsilon$,

where the additional renormalization constant Z'_ε is defined in such a way that

$$M_0 \equiv \langle \varphi_{ren}(x_n) \rangle_{\lambda_0} = Z'^{\frac{1}{2}}_\varepsilon \langle \varphi_\varepsilon(x_n) \rangle_{\lambda_0} ,$$

for a fixed $M_0 > 0$. Then clearly $Z'_\varepsilon \to 0$ as $\varepsilon \to 0$.

Therefore we have

$$\langle e^{\varphi_{ren}(h)} \rangle_\lambda = \langle e^{Z'^{\frac{1}{2}}_\varepsilon \varphi_\varepsilon(h)} \rangle_\lambda \leq \langle e^{\varphi_\varepsilon(h)} \rangle^{Z'^{\frac{1}{2}}_\varepsilon} \leq e^{\|h\| \langle \varphi_{ren}(x_n) \rangle_{\lambda + \bar{h}}} .$$

Since $\langle \varphi_{ren}(x_n) \rangle_{\lambda + \bar{h}}$ stays finite as $\varepsilon \to 0$ we still have bounds for φ_{ren} of the type given in Theorem 17.

As a matter of fact the real danger comes from the eventuality $\langle \varphi_\varepsilon(x_n) \rangle_{\lambda_0} \to 0$ as $\varepsilon \to 0$, and the choice of the field strength renormalization Z_ε, with $Z_\varepsilon \to 0$ as $\varepsilon \to 0$, should prevent this disaster. Unfortunately at present we are unable to prove in general that for given Z_1 and δm^2, there is a choice of Z such that $M(\lambda_0)$ assumes a fixed value $M_0 > 0$. Nevertheless the form of the bounds given by Theorem 17 suggests that this should be a relevant question in the future investigation of the theory.

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Colloques Internationaux C.N.R.S.
N° 248 – Les méthodes mathématiques de la théorie quantique des champs

Critical Behaviour in Terms of Probabilistic Concepts

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- Résumé

Le comportement des phénomènes critiques est habituellement défini en terme de singularité des fonctions thermodynamiques. D'un point de vue microscopique, il est naturel d'aborder les phénomènes critiques directement en terme de propriétés du procédé stochastique. Le but de cet article est de fournir une caractérisation du phénomène critique à l'aide de concepts et de méthodes appartenant aux différentes branches de la théorie de la probabilité.

- Abstract

Critical behaviour is usually defined in terms of singularities of thermodynamic functions. However from a microscopic stand point it is natural to approach criticality directly in terms of properties of the stochastic process underlying the statistical description. The purpose of this paper is to provide a characterization of criticality using concepts and methods typical of various branches of probability theory.

1. - Introduction

In recent years considerable progress has been made towards a theoretical understanding of critical phenomena. The concept of renormalization group has provided the key ideas on which a very interesting qualitative and to a certain extent quantitative picture of critical behaviour has been built. From the mathematical stand point the techniques employed are entirely heuristic and, at first look, rather esoteric.

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However it has been recently recognized that the renormalization group approach is closely related to well known problems in probability theory and that in fact implies a generalization of some probabilistic methods.[1] What is involved is a systematic extension of limit theorems for stationary random fields. Once this connection with limit theorems is established it becomes very natural to exploit further the probabilistic point of view. Behind any characterization of critical behaviour there is the idea of long range correlations among the fluctuating variables. Since the decay of correlations is related to the ergodic and mixing properties of a stationary process (considered as a dynamical system with respect to the translation operator) it is natural to inquire whether it is possible to describe criticality in terms of those properties. A question in this direction was raised some time ago in the context of Ising models by Di Liberto, Gallavotti and Russo[2] in a paper dealing with the Bernoullicity of such systems. Since they were able to prove Bernoullicity only away from the critical point they considered the possibility of distinguishing critical from non critical systems in terms of failure of the isomorphism with a Bernoulli scheme. However it is now known that the two-dimensional Ising model is Bernoulli also at the critical point.[3] It is therefore necessary to look for a different property. In the following we shall argue that failure of strong mixing is what discriminates critical from non critical behaviour. The strong mixing property is interesting because it is strictly connected with limit theorems for the underlying random field. Actually it is through limit theorems that one obtains a definite indication that strong mixing is the relevant concept in volved in critical systems. In this way the probabilistic interpretation of the renormalization group becomes part of a more articulate description in which criticality is characterized directly in terms of properties of the stochastic processes underlying the microscopic description of thermodynamic phenomena.

The plan of the paper is as follows: in Section 2 we review the probabilistic significance of the renormalization group and we introduce the concept of stable random field which corresponds to the usual notion of fixed point Hamiltonian. In Section 3, we discuss the notions of complete regularity and strong mixing. In Section 4, we present the argument showing that violation of strong mixing provides a demarcation line between critical and non critical behaviour. In Section 5, we give two concrete examples to illustrate various aspects of the problem.

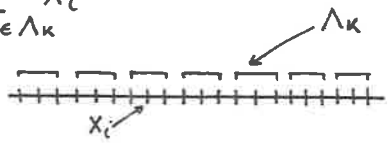
2. - Stable Random Fields

To visualize the problem in a simple way, consider a one-dimensional lattice at each point of which is associated a real random variable X_i. This system will be described by a translational invariant measure μ in the space $K = \prod_{i \in Z} R$, where R is the real line.

Construct now a new lattice by dividing the original system into blocks containing L original variables and associating with each block the random variable

$$\varphi_K^L = \frac{M_K^L - E(M_K^L)}{L^{\rho/2}} \qquad 1 \leq \rho < 2$$

where $M_K^L = \sum_{i\in\Lambda_K} X_i$



To the new lattice will be associated a new measure μ_L^ρ which can be written symbolically

$$\mu_L^\rho = H_L^\rho \mu$$

H_L^ρ represents the transformation we have just described. If ρ is properly chosen, it may happen that by repeating the same operation indefinitely, we obtain convergence to a limit measure

$$\mu_\infty^\rho = \lim_{L\to\infty} H_L^\rho \mu \tag{2.1}$$

Clearly

$$\mu_\infty^\rho = H_L^\rho \mu_\infty^\rho \tag{2.2}$$

These equations have to be interpreted in the sense of weak convergence. Eq.(2.2) can be written also in analytic form. For L = 2 for example we obtain that it is equivalent to the following system of equations for joint distributions

$$P_m(\xi_1,\dots,\xi_m)=\frac{1}{a^m}\int\prod_{i=1}^{m} d\xi'_{2i}\cdot P_{2m}\left(\frac{\xi_1}{a}-\xi'_2,\xi'_2,\dots,\frac{\xi_m}{a}-\xi'_{2m},\xi'_{2m}\right) \tag{2.3}$$

where $a = 2^{-\rho/2}$

(2.3) can be viewed as an analog for dependent variables of the usual convolution operation on distribution functions.

A measure satisfying Eq.(2.2) will be called a Stable Random Field. The reason for this terminology is that in many respects (2.1) and (2.2) represent the natural generalization to random fields of the classical problem of probability theory which consists in determining the limit distributions for sums of identically distributed independent random variables. It is easy to verify that if μ_∞^ρ reduces to a product of identical factors, i.e. it describes a set of independent variables, Eq.(2.2) reduces to the usual definition of a Stable Distribution i.e.

$$\nu(ax) = \frac{a_1 a_2}{a} \int dx' \, \nu(a_1(x-x')) \, \nu(a_2 x')$$

where a_1, a_2, a are positive real numbers. The solutions of this equation are easily found in terms of characteristic functions

$$f(t) = e^{-c|t|^\alpha} \qquad 1 < \alpha \leq 2$$

For $\alpha = 2$ ($\rho = 1$) the solution is the normal distribution while for $\alpha \neq 2$ ($\rho = \frac{2}{\alpha}$) we obtain distributions with infinite variance. This property will be important later. All the distributions which have a given ν_α as limit distribution, define the domain of attraction of ν_α.

In the present context the interesting case is when μ_∞^ρ is not a product. In such a case μ_∞^ρ corresponds to a scaling invariant system and this is the reason why Stable Random Fields are relevant to critical phenomena. The aim of a theory of Stable Random Fields is to classify the possible solutions of Eq.(2.2) and to calculate their domain of attraction, i.e. to determine for each ρ all the measures μ which under the limit process (2.1) tend to μ_∞^ρ. The domain of attraction to a large extent corresponds to the physicist's notion of universality class. To develop such a theory is therefore a task of great practical significance.

3. - The Lattice as a Dynamical System[4][5]

To develop further our analysis we need an auxiliary concept: this is the notion of abstract dynamical system. An abstract dynamical system is a triple (K, μ, T) where (K, μ) is a measure space and T a measure preserving transformation. For our lattice system the space K was introduced in the previous section. A point x in K is an infinite sequence of real numbers

$$x = \dots, X_{-1}, X_0, X_1, \dots$$

T will be the shift operator whose action on K is defined by

$$Tx = x'$$

$$x' = \dots, X'_{-1}, X'_0, X'_1, \dots$$

$$X'_i = X_{i-1}$$

We shall use the notation Σ_a^b for the σ-algebra generated by sets of the form

$$\{X_{i_1} \in A_1, \dots, X_{i_m} \in A_m\} \quad a \leq i_1 \leq \dots \leq i_m \leq b$$

where A_j is a Borel set on the real line.

The description of the lattice as a dynamical system is interesting within our context because it leads to a very natural characterization of criticality. In the physicist's conception the critical point is associated with the idea of long range correlations which in the language of stochastic processes means a stronger dependence of the random variables X_i if compared with non critical situations. In the theory of stochastic processes it is customary to introduce a hierarchy of degrees of dependence for the X_i in terms of mixing properties of the dynamical system (K, μ, T). For our purpose the following properties appear as the relevant ones

a) Mixing

$$\lim_{\substack{n \to \infty \\ A \in \Sigma_{-\infty}^{0} \\ B \in \Sigma_{M}^{\infty}}} \mu(A \cap B) - \mu(A)\mu(B) = 0$$

b) Strong Mixing

$$\lim_{n \to \infty} \sup_{\substack{A \in \Sigma_{-\infty}^{0} \\ B \in \Sigma_{M}^{\infty}}} |\mu(A \cap B) - \mu(A)\mu(B)| =$$

$$= \lim_{n \to \infty} \alpha(n) = 0$$

$\alpha(n)$ is called the mixing coefficient.

c) Complete Regularity

Let $\eta^{(1)}$ be a function measurable with respect to $\Sigma_{-\infty}^{0}$ and $\eta^{(2)}$ a function measurable with respect Σ_{M}^{∞}. Assume that

$$E(\eta^{(1)}) = E(\eta^{(2)}) = 0$$

$$E(|\eta^{(1)}|^2) = E(|\eta^{(2)}|^2) = 1$$

where E denotes the expectation value. Functions having these properties clearly define Hilbert spaces with respect to the scalar product induced by E . With an obvious notation we indicate these spaces $H^0_{-\infty}$ and H^{∞}_{n} . A system is then completely regular if

$$\lim_{n\to\infty} \sup_{\substack{\eta^{(1)}\in H^0_{-\infty}\\ \eta^{(2)}\in H^{\infty}_{n}}} \left| E(\eta^{(1)}\eta^{(2)})\right| =$$

$$= \lim_{n\to\infty} \rho(n) = 0$$

$\rho(n)$ is also called the maximal correlation coefficient. a), b), c) describe situations of increasing statistical independence.(*)

As we will show in the next section our previous description of the critical point in terms of Stable Random Fields implies a violation of property b) (and therefore of property c)) i.e. at the critical point $\lim_{n\to\infty} \alpha(n) \neq 0$.

4. - Violation of Strong Mixing

We begin by collecting some facts which are immediate consequences of the description presented in section 2. According to our point of view a measure μ is critical if $\mu^{\rho}_{\infty} = \lim_{L\to\infty} H^{\rho}_{L}\mu$ is non trivial in the sense that it is not a product measure.

(*) In the literature the term "complete regularity" is sometimes used for different properties. Here we have followed the terminology of ref.(5).

So we expect μ^{ρ}_{∞} to have a non trivial covariance function. This implies that

$$\lim_{L\to\infty} E(\varphi^L_K \varphi^L_{K'}) = \text{Finite} \qquad K \neq K' \qquad (4.1)$$

A straightforward argument then shows that for this equation to hold it is necessary that

$$\sum_j \left[E(X_i \cdot X_j) - E(X_i)^2\right] = \text{Infinite} \qquad (4.2)$$

Therefore the existence of μ^{ρ}_{∞} implies the usual physicist's picture of long range correlations.

Furthermore if $E(|X_i - E(X_i)|^2)$ is finite one also obtains that

$$\lim_{L\to\infty} E\left(|\varphi^L_K|^2\right) = \text{Finite} \qquad (4.3)$$

Therefore the limit distribution for a one-block variable has finite variance.

We now outline the argument leading to (4.2) and (4.3). From the definition of the block-variable φ^L we have

$$E(\varphi^L_0 \varphi^L_M) = E\left(\frac{\sum_{i=1}^{L} X_i - E(\sum_{i=1}^{L} X_i)}{L^{\rho/2}} \; \frac{\sum_{j=ML+1}^{(M+1)L} X_j - E(\sum_{j=ML+1}^{(M+1)L} X_j)}{L^{\rho/2}}\right)$$

$$= \frac{1}{L^{\rho}} \sum_{i=1}^{L} \sum_{j=ML+1}^{(M+1)L} R(i-j)$$

where $R(i - j) = E(x_i x_j) - E(x_i)^2$

Therefore

$$E(\varphi_0^L \varphi_n^L) = \frac{1}{L^{\rho-1}} \sum_{\ell=(n-1)L+1}^{(n+1)L-1} R(\ell) \qquad (4.4)$$

Since $\rho > 1$ Eq.(4.2) follows immediately by requiring that the limit $L \to \infty$ be finite.

Similarly

$$E(|\varphi_0^L|^2) = E\left(\frac{\sum_{i=1}^{L} X_i - E(\sum_{i=1}^{L} X_i)}{L^{\rho/2}} \frac{\sum_{j=1}^{L} X_j - E(\sum_{j=1}^{L} X_j)}{L^{\rho/2}}\right)$$

$$= \frac{1}{L^{\rho}}\left[L\,E(|X_i - E(X_i)|^2) + 2\sum_{\ell=1}^{L-1}(L-\ell)R(\ell)\right] \qquad (4.5)$$

From the previous discussion the second term in brackets divided by L^{ρ} is finite for large L. Thus from the finiteness of the variance of x_i it follows that also the variance of φ^L is finite as $L \to \infty$.

To show that Eq.s.(4.2) and (4.3) imply violation of strong mixing we need some connection between mixing properties, long range behavior of correlations and limit theorems. We first notice that from the scaling properties[6] of $E(\varphi_0^{\infty} \varphi_n^{\infty})$ for large n it follows that asymptotically $R(\ell) \xrightarrow[\ell\to\infty]{} \ell^{\rho-2}$.

From the analysis of complete regularity carried out in ref.(5) we can easily see that the following proposition holds

I - If $R(\ell) \xrightarrow[\ell\to\infty]{} \ell^{-a}$ with $a < 1$, then (K, μ, T) violates complete regularity.

A critical system with $\rho > 1$ therefore violates complete regularity. The limiting case $\rho = 1$ cannot be decided in general.

The connection with strong mixing is now given by the two propositions[4][5]

II - <u>If (K, μ, T) is a Gaussian process, then $\alpha(n) \leq \rho(n) \leq 2\pi\alpha(n)$</u>

This means that for a Gaussian process strong mixing and complete regularity coincide and violation of the latter implies violation of the first.

III - <u>If (K, μ, T) is strongly mixing then the limit distribution of the one-block variable ξ^L is a stable distribution. If the latter distribution has exponent α, then the normalization factor is $L^{1/\alpha} h(L)$ where $h(L)$ is a slowly varying function as $L \to \infty$</u>

This theorem takes care of non-Gaussian critical processes. Consider in fact a general process with a non-Gaussian one-block limit distribution. Since, as we have seen in section 2, all the stable distributions except the Gaussian, have infinite variance, Eq. (4.3) and the above theorem imply violation of strong mixing. Of course the finiteness of $E(|X_i - E(X_i)|^2)$ is physically quite natural. Some comments are in order. The last case is usually considered as the most interesting for physics. This belief is based on experience with the renormalization group. A paradigmatic case is the two-dimensional Ising model where one has indications that the one-block limit distribution is non Gaussian and non stable.[7] Although our analysis was carried out in the one-dimensional case there seems to be no reason of principle forbidding its extension to higher dimensional systems.

As a further comment we would like to point out the interest of the second part of theorem III. It states that for strong mixing systems the normalization factors for the block-variables are essentially the same as for the independent variable case. This means that even if the one-block distribution is Gaussian, an unconventional normalization is already a sufficient sign of violation of strong mixing.

For non critical systems we expect strong mixing to hold. Considering again Ising models, it has been shown in (7) that the central limit theorem is valid away from the critical point and that μ_{∞}^{ρ} reduces to a product measure for $\rho = 1$.

5. - Examples

In this section we discuss two explicit examples (A. and B. below) of non Gaussian processes. The first, as pointed out elsewhere,[8] provides a nice example of a process leading to a non Gaussian Stable Random Field. The second example shows that it is possible to violate complete regularity without violating strong mixing. However in this case a pathology arises because it will not be possible to obtain a non singular one-block distribution function and a finite two-block correlation function using the same normalization for the block variables. Therefore it does not exist a ρ leading to a sensible μ_{∞}^{ρ}.

A. This is a process well known to probabilists[4]. One starts from a sequence of independent variables

$$\ldots, \xi_{-1}, \xi_0, \xi_1, \ldots$$

normally distributed with unit variance. Then one constructs the Gaussian process

$$Y_j = \sum_{k=-\infty}^{-1} |k|^{-a} \xi_{k+j}$$

which is characterized by the correlation function

$$R(i - j) = E(y_i y_j) \underset{|i - j| \to \infty}{\longrightarrow} |i - j|^{1-2a}$$

The next step consists in considering the sequence

$$x_j = y_j^2 - E(y_j^2) \tag{5.1}$$

The correlation function of (5.1) behaves asymptotically as

$$R(\ell)^2 \underset{|\ell| \to \infty}{\longrightarrow} |\ell|^{2-4a}$$

and for $a < \frac{3}{4}$ is not integrable. It can be shown that the normalized sums

$$\varsigma^L = \frac{\sum_{i=1}^{L} X_i}{\sigma^L}$$

where $(\sigma^L)^2 = E\left(|\sum_{i=1}^{L} X_i|^2\right)$, for $\frac{1}{2} < a < \frac{3}{4}$ do not satisfy the central limit theorem as $L \to \infty$. The characteristic function of the limit distribution of ς^L can be calculated explicitely and it turns out to be non Gaussian and with finite variance. The sequence (5.1) therefore violates strong mixing. For the details of the calculation the reader is referred to (4) pag.384.

Since all the expectations of the form $E(x_1 x_2 \dots x_n)$, $n \geqslant 2$, can be calculated explicitely in terms of $R(\ell)$ it is not difficult to obtain also $E(\varsigma^L_{K_1}, \varsigma^L_{K_2}, \dots \varsigma^L_{K_n})$ as $L \to \infty$, i.e. it is possible to know all the moments of μ^{ρ}_{∞} where in this case $\rho = 4(1-a)$.

B. The following model is due to Davydov.[9] Consider a Markov chain whose states are the integers. The transition matrix is defined as follows

$$p_{m,m+1} = p_{-m,-m-1} = \alpha_m \qquad m \geqslant 0$$

$$p_{m,0} = p_{-m,0} = 1 - \alpha_m \qquad m > 0$$

$$p_{0,0} = 0 \qquad \alpha_0 = \frac{1}{2} \quad 0 < \alpha_m < 1$$

The probability of obtaining the state 0 after exactly n steps is

$$f_{00}^{1} = 0$$

$$f_{00}^{m} = \beta_{m-1} - \beta_m \qquad m \geqslant 2$$

with $\beta_0 = \beta_1 = 1$, $\beta_m = \alpha_1 \alpha_2 \dots \alpha_{m-1}$. This chain has a stationary distribution if $\sum_0^\infty \beta_m < \infty$. The model is specified further by chosing

$$f_{00}^{m} = A\, m^{-2-\delta} \qquad 0 < \delta < 1 \tag{5.2}$$

If ξ_i is the state of the chain at time i, consider the new process

$$x_i = g(\xi_i) \tag{5.3}$$

where $\quad g(k) = -g(-k) = 1 + \frac{1}{k} \qquad\qquad g(0) = 0$

Clearly $|x_i| \leqslant 2$. It can be shown that the correlation function for the sequence (5.3) behaves asymptotically as $R(\ell) \underset{|\ell| \to \infty}{\longrightarrow} |\ell|^{-\delta}$. However strong mixing is not violated. The distribution of the normalized sums

$$\varphi^L = \frac{\sum_{i=1}^{L} X_i}{B^L}$$

converge to a stable law with exponent $\alpha = 1 + \delta$ and $B^L \xrightarrow[L\to\infty]{} L^{\frac{1}{1+\delta}}$.
On the other hand the variance of $\sum_{i=1}^{L} X_i$ has the asymptotic form
$\sigma^L \xrightarrow[L\to\infty]{} L^{1-\delta/2}$.
Therefore if σ^L is taken as the normalization factor we obtain a finite two-block correlation but the one-block limit distribution becomes singular.

In a subsequent paper in collaboration with M.Cassandro we shall consider multidimensional processes and give additional examples.

I wish to thank G.Gallavotti for discussing on several occasions the ideas presented here.

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Colloques Internationaux C.N.R.S.
N° 248 – Les méthodes mathématiques de la théorie quantique des champs

The $:\varphi^2:$ Field: Infinite Volume Limit and Bounds on the Physical Mass*

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RESUME On construit la limite de volume infini des fonctions de Schwinger et des fonctions caractéristiques d'un champ $:\varphi^2:$ pour les modèles $P(\varphi)_2$ pairs. On établit les bornes pour $P(\varphi) = \lambda\varphi^4 + \sigma\varphi^2 - \mu\varphi$. On utilise les bornes pour déduire une borne supérieure pour la masse physique de $(\lambda\varphi^4 + \sigma\varphi^2)_2$ dans la région d'une phase unique et une relation entre la masse physique et la moyenne $\langle\varphi\rangle$ pour chaque phase pure d'une théorie $(\lambda\varphi^4)_2$.

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In this article we report results involving the $:\varphi^2:$ field. We construct the infinite volume limit of Schwinger functions and characteristic functions involving the $:\varphi^2:$ field in even $P(\varphi)_2$ models. We then establish bounds on $<:\varphi(x)^2:>$ for $P(\varphi) = \lambda\varphi^4+\sigma\varphi^2-\mu\varphi$. These bounds are used to derive an upper bound on the physical mass of $(\lambda\varphi^4+\sigma\varphi^2)_2$ in the single phase region and a relation between the physical mass and the field expectation $<\varphi>$ in each pure phase of a $(\lambda\varphi^4)_2$ theory.

1. The Infinite Volume Limit for the $:\varphi^2:$ Field in Even $P(\varphi)_2$ Models.

Wick powers involve polynomials containing infinite negative coefficients. For this reason correlation inequalities for Wick powers appear to be considerably more difficult (see Guerra, Rosen, Simon [4]; for small coupling Wick powers have been studied by Schrader [9] by means of the cluster expansion). We study the properties of the simplest Wick power, $:\varphi^2:$. As $:\varphi^2:$ differs from φ^2 only by an "infinite constant", we explore this fact to obtain the infinite volume limit of Schwinger functions and characteristic functions involving the $:\varphi^2:$ field in even $P(\varphi)_2$ models (with maybe a linear term, as in [4]).

By φ we denote the free Euclidean field with mass $m_o > 0$ in 2 space-time dimensions, $\varphi(f) = \int\varphi(x)f(x)d^2x$, $:\varphi^2:(g) = \int:\varphi^2(x):g(x)dx$.

We write
$$< F(\varphi)>_\Lambda = \frac{\int F(\varphi)\exp\left\{-\int_\Lambda :P(\varphi(x)):d^2x\right\}d\mu_o}{\int\exp\left\{-\int_\Lambda :P(\varphi(x)):d^2x\right\}d\mu_o} ,$$

where P is an even polynomial bounded from below, $\Lambda \subset R^2$, μ_0 is the Gaussian measure associated with φ, and $F(\varphi)$ is a function of φ. We are mostly going to work with Half- or Full-Dirichlet boundary conditions [4] and we will denote the respective finite volume expectations by $< >_\Lambda^{HD}$, $< >_\Lambda^{D}$. $\mathcal{D}$ will denote the set of infinitely differentiable functions on R^2 with compact support, $\mathcal{D}(\Lambda)$ the subset of functions in $\mathcal{D}$ with support in Λ, and $\mathcal{S}$ the set of infinitely differentiable functions in R^2 which decrease faster than any inverse power at infinity.

An application of Griffiths inequalities [4] yields:

<u>Theorem 1</u>: Let $g \in \mathcal{D}(\Lambda)$. Then

a) $< e^{:\varphi^2:(g)} >_\Lambda^{HD} \leq < e^{:\varphi^2:(g)} >_\Lambda$ if $g \geq 0$

$< e^{:\varphi^2:(g)} >_\Lambda^{HD} \geq < e^{:\varphi^2:(g)} >_\Lambda$ if $g \leq 0$

b) If $\Lambda \subset \Lambda'$,

$< e^{:\varphi^2:(g)} >_\Lambda^{HD} \leq < e^{:\varphi^2:(g)} >_{\Lambda'}^{HD}$ if $g \geq 0$

$< e^{:\varphi^2:(g)} >_\Lambda^{HD} \geq < e^{:\varphi^2:(g)} >_{\Lambda'}^{HD}$ if $g \leq 0$.

We also study the Full-Dirichlet expectation $< e^{:\varphi^2:_D(g)} >_\Lambda^{D}$ in the case $P(\varphi) = \lambda\varphi^4 + \sigma\varphi^2$ (with maybe a linear term, as in [4]), where $: :_D$ denotes Wick ordering with respect to the Dirichlet covariance $S_{\Lambda, D}$. We will use S for the free covariance. As

$:\varphi^2:_D(g) = :\varphi^2:(g) + \int_\Lambda [S(x-x) - S_{\Lambda, D}(x,x)]g(x)d^2x$, and $[S(x-x) - S_{\Lambda,D}(x,x)] \in L_p(\Lambda, d^2x)$ for all $p < \infty$

with $[S(x-x) - S_{\Lambda,D}(x,x)] \searrow 0$ as $\Lambda \nearrow R^2$ [4],

it follows that

$$\lim_{\Lambda \nearrow R^2} \langle e^{:\varphi^2:_D(g)} \rangle_\Lambda^D = \lim_{\Lambda \nearrow R^2} \langle e^{:\varphi^2:(g)} \rangle_\Lambda^D$$

if the limits exist (and if one of them exists so does the other). It suffices therefore to consider $\langle e^{:\varphi^2:(g)} \rangle_\Lambda^D$. We then have

<u>Theorem 1'</u>: Let $P(\varphi) = \lambda\varphi^4 + \sigma\varphi^2$, and $g \in \mathcal{D}(\Lambda)$. Then

a) $\langle e^{:\varphi^2:(g)} \rangle_\Lambda^D \le \langle e^{:\varphi^2:(g)} \rangle_\Lambda^{HD} \le \langle e^{:\varphi^2:(g)} \rangle_\Lambda$ if $g \ge 0$

$\langle e^{:\varphi^2:(g)} \rangle_\Lambda^D \ge \langle e^{:\varphi^2:(g)} \rangle_\Lambda^{HD} \ge \langle e^{:\varphi^2:(g)} \rangle_\Lambda$ if $g \le 0$

b) If $\Lambda \subset \Lambda'$,

$\langle e^{:\varphi^2:(g)} \rangle_\Lambda^D \le \langle e^{:\varphi^2:(g)} \rangle_{\Lambda'}^D$ if $g \ge 0$

$\langle e^{:\varphi^2:(g)} \rangle_\Lambda^D \ge \langle e^{:\varphi^2:(g)} \rangle_{\Lambda'}^D$ if $g \le 0$

According to Theorem 1b (1'b), $\langle e^{:\varphi^2:(g)} \rangle_\Lambda^{HD}$ ($\langle e^{:\varphi^2:(g)} \rangle_\Lambda^D$, $P(\varphi) = \lambda\varphi^4 + \sigma\varphi^2$) is monotone increasing in Λ for $g \ge 0$ and monotone decreasing in Λ for $g \le 0$, Λ containing the support of g. The study of the infinite volume limit is completed by obtaining bounds uniform in Λ.

<u>Theorem 2:</u> Let $g \in \mathcal{D}(\Lambda_o)$, where Λ_o is a finite union of unit squares,

and let $\Lambda \supset \Lambda_o$. There exists a continuous norm $\| \ \|_s$ on $\mathscr{S}$ such that given $0 < c < \infty$ there exists $0 < d < \infty$ such that $\|g\|_s \le c$ implies $|< e^{:\varphi^2:(g)} >_\Lambda^{HD}| \le d$. Moreover d depends only on c and is independent of Λ and Λ_o .

To prove the theorem we use

$$|< e^{:\varphi^2:(g)} >_\Lambda^{HD}| \le (< e^{:\varphi^2:(2g_+)} >_\Lambda^{HD})^{\frac{1}{2}} (< e^{-:\varphi^2:(2g_-)} >_\Lambda^{HD})^{\frac{1}{2}}$$

where g_+ , g_- are the positive and negative parts of Re g , the real part of g (Re $g = g_+ - g_-$, g_+ , $g_- \ge 0$, $g_+ g_- = 0$) . The first term is estimated by an argument of Frohlich [1] . To estimate the second term we use the fact that

$$< e^{-:\varphi^2:(2g_-)} >_\Lambda^{HD} \le < e^{-:\varphi^2:(2g_-)} >_{\Lambda_o}^{HD} .$$

for all $\Lambda \supset \Lambda_o$ (Theorem 1b) .

Theorems 1 and 2 lead to the infinite volume Schwinger functions via Vitali's theorem, as discussed by Frohlich [1].

Theorem 3: Let $f_1, \ldots, f_n, g_1, \ldots, g_m \in \mathscr{D}$. Then $<\varphi(f_1), \ldots . \varphi(f_n):\varphi^2:(g_1) \ldots :\varphi^2:(g_m)>_\Lambda^{HD}$ converges as $\Lambda \nearrow R^2$, and there exists a continuous norm $\| \ \|_s$ on $\mathscr{S}$ such that the infinite volume Schwinger functions satisfy the bound

$$|<\varphi(f_1)\ldots\varphi(f_n):\varphi^2:(g_1)\ldots:\varphi^2:(g_m)>^{HD}| \le c^n n! \, \|f_1\|_s \ldots \|f_n\|_s \|g_1\|_s \ldots \|g_m\|_s$$

for some constant c . Moreover, they satisfy the Osterwalder-Schrader axioms and thus can be analytically continued to Lorentz invariant Wightman functions.

Remark: Theorems 2 and 3 are also true for the Full-Dirichlet expectations when $P(\varphi) = \lambda\varphi^4 + \sigma\varphi^2$.

For further details see Klein and Landau [5]

2. Bounds on $<:\varphi(x)^2:>$

From now on we will denote by $< > = \lim_{\Lambda \nearrow R^2} < >_\Lambda$.

From Theorem 3, $<:\varphi^2:(g)> = \lim_{\Lambda \nearrow R^2} <:\varphi^2:(g)>_\Lambda$ exists and by translation invariance $<:\varphi(x)^2:>$ is a finite number $\gamma = <:\varphi(0)^2:>$. The integration by parts formula [3] gives us: *

$$<:\varphi(x)^2:>_\Lambda + <\int S_\Lambda(x,y)^2 :P''(\varphi(y)):d^2y>_\Lambda = <(\int S_\Lambda(x,y)^2 :P'(\varphi(y)):d^2y)^2>_\Lambda \geq 0$$

With $P(\varphi) = \lambda\varphi^4 + \sigma\varphi^2 - \mu\varphi$ we get

$$<:\varphi(x)^2:>_\Lambda + 12\lambda <\int S_\Lambda(x,y):\varphi^2(y):d^2y>_\Lambda + 2\sigma\int S_\Lambda(x,y)^2 d^2y \geq 0$$

Taking the limit as $\Lambda \nearrow R^2$ and using translation invariance we have

Lemma 1: In a $(\lambda\varphi^4 + \sigma\varphi^2 - \mu\varphi)_2$ infinite volume theory with either Half or Full-Dirichlet boundary conditions,

$$<:\varphi(x)^2:> \geq -\frac{1}{2\pi}\frac{\sigma}{m_o^2 + (3\lambda/\pi)}$$

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* Here we use Full-Dirichlet boundary conditions; $S_\Lambda(x,y)$ is the Dirichlet covariance and the Wick ordering : : is with respect to $S_\Lambda(x,y)$. In the Half-Dirichlet case there is a slight modification due to the fact that the Wick ordering is done with respect to the free covariance.

Remark: Our proof of Lemma 1 depends crucially on the translation invariance of the infinite volume expectation. The same proof gives a lower bound on $<:\varphi(x)^2:>$ for a finite volume theory with Full-periodic boundary conditions (i.e. the Wick ordering is with respect to the periodic covariance) since one again has the required "translation invariance".

Similar methods also prove the following two lemmas. We will write

$\tilde{g}(p) = \int e^{ipx} g(x) d^2x$; $:\varphi^2(g): = \varphi(g)^2 - <\varphi(g)^2>_o$, where $<>_o$ denotes the free (no interaction) expectation, i.e. $P = 0$.

Lemma 2: Let $g \in \mathcal{D}$. In a $(\lambda\varphi^4 - \mu\varphi)_2$ infinite volume theory with either Half or Full-Dirichlet boundary conditions.

$$<:\varphi(g)^2:> + \frac{3\lambda}{\pi^2}\int d^2p \frac{|\tilde{g}(p)|^2}{(p^2+m_o^2)^2} <:\varphi(0)^2:> \geq \tilde{g}(0)^2 <\varphi(0)>^2$$

Lemma 3: Let $0 \leq g \in \mathcal{D}$. In a $(\lambda\varphi^4 + \sigma\varphi^2 - \mu\varphi)_2$ infinite volume limit theory with either Half or Full-Dirichlet boundary conditions,

$$|<:\varphi(g)^2:>| \leq \tilde{g}(0)^2 \left\{ (1 + (3\lambda/\pi m_o^2)) <:\varphi(0)^2:> + (\sigma/2\pi m_o^2) \right\}$$

$$+ \frac{1}{2\pi^2} \int d^2p \frac{|\hat{g}(p)|^2}{(p^2+m_o^2)^2} |6\lambda <:\varphi(0)^2:> + \sigma| \quad .$$

In particular $|<:\varphi(g)^2:>|$ remains bounded as $g \to \delta$.

For further details see Klein and Landau [6, 5].

3. An Upper Bound on the Physical Mass of $(\lambda\varphi^4+\sigma\varphi^2)_2$ in the Single Phase Region.

We use a small distance limiting procedure $\lim_{|x-y|\to 0} <:\varphi(x)\varphi(y):>$ rather than the more usual large distance behavior $\lim_{|x-y|\to\infty} <\varphi(x)\varphi(y)>$ to obtain information on the physical mass. Lemma 3 tells us that $|<:\varphi(x)\varphi(y):>| = |S(x-y) - S_{m_o}(x-y)|$ remains bounded as $|x-y|\to 0$, where $S(x-y)$ is the covariance of the infinite volume theory and $S_m(x-y)$ denotes the covariance of the free field with mass m. The most general such covariance $S(x-y)$ is given by the analytic continuation of the Kallen-Lehmann representation:

$$S(x-y) = \int_0^\infty d\rho(m)\, \frac{(1+\Delta)^N}{(1+m^2)^N}\, S_m(x-y) + <\varphi(0)>^2 + \sum_{j=0}^{J} a_j(-\Delta)^j \delta(x-y) ,$$

where $\int_0^\infty \frac{d\rho(m)}{(1+m^2)^{N+1}} < \infty$ and $\int_0^1 d\rho(m)\ln\frac{1}{m} < \infty$ (restriction particular to two space-time dimensions). The bound of Lemma 3 imposes further restriction on the form of S.

<u>Lemma 4</u>: Let $0 \le g \in \mathcal{D}$, $\int g(x)d^2x = 1$, $g_n(x) = n^2 g(nx)$.

Suppose $|<:\varphi(g_n)^2:>| \le c$ for all n. Then $S(x-y) = \int_0^\infty d\rho(m) S_m(x-y) + <\varphi(0)>^2 + a$,

where $\int_0^\infty d\rho(m) = 1$, $\int_0^\infty d\rho(m)\, |\ln m| < \infty$. It follows that $\lim_{n\to\infty} <:\varphi(g_n)^2:>$ exists and equals $\frac{1}{2\pi}\int_0^\infty d\rho(m)\ln\frac{m_o}{m} + <\varphi(0)>^2 + a$.

In the case of a $(\lambda\varphi^4 + \sigma\varphi^2)_2$ theory in the single phase region, $<\varphi(0)> = 0$ and $a = 0$. We then have

Lemma 5: Let $S(x-y) = \int_{m_1}^{\infty} d\rho(m) S_m(x)$, $m_1 \geq 0$, $\int_{m_1}^{\infty} d\rho(m) = 1$.

Let $0 \leq g \in \mathcal{D}$, $\int g(x) d^2x = 1$, $g_n(x) = n^2 g(nx)$. If $\lim_{n\to\infty} <:\varphi(g_n)^2:> = \gamma$, then

$\frac{m_1}{m_o} \leq e^{-2\pi\gamma}$ with equality holding if and only if $\rho = \delta(m-m_1)$.

Let us prove this lemma. We assume $m_1 > 0$ (otherwise there is nothing to prove). Let

$$S(g_n) = \int S(x-y) g_n(x) g_n(y) d^2x d^2y = <\varphi(g_n)^2>$$

Since $\int_{m_1}^{\infty} d\rho(m) = 1$, $S(g_n) \leq S_{m_1}(g_n)$ with equality holding only if $\rho = \delta(m-m_1)$.

Thus $\gamma \leq \lim [S_{m_1}(g_n) - S_{m_o}(g_n)] = \frac{1}{2\pi} \ln \frac{m_o}{m_1}$. This finishes the proof.

We now want to combine Lemma 5 with the lower bound of Lemma 1. The technical complication that arises is the identification of $<:\varphi(x)^2:>$ with $\lim_{n\to\infty} <:\varphi(g_n)^2:>$. One may formulate the problem as follows: Is it true that

$$\lim_{n\to\infty} \lim_{\Lambda \nearrow R^2} <:\varphi(g_n)^2:> = \lim_{\Lambda\nearrow R^2} \lim_{n\to\infty} <:\varphi(g_n)^2:> (= <:\varphi(0)^2:>) ?$$

It is interesting to notice that we don't need the equality, it will suffice to show:

Lemma 6: Let $0 \leq g \in \mathcal{D}$, $\int g(x) d^2x = 1$, $g_n(x) = n^2 g(nx)$. In a $(\lambda\varphi^4 + \sigma\varphi^2 - \mu\varphi)_2$ infinite volume theory with either Half or Full-Dirichlet boundary conditions,

$$\lim_{n\to\infty} <:\varphi(g_n)^2:> \geq <:\varphi(0)^2:>$$

The proof uses the existence of the infinite volume limits, Griffiths inequalities and the translation invariance of the infinite volume theory. We would like to remark that using the methods in Klein and Landau [5] plus the results of Glimm and Jaffe [2] we can prove equality.

We are now ready for

Theorem 4: Consider the infinite volume $(\lambda\varphi^4+\sigma\varphi^2)_2$ theory in the single phase region, obtained with either Half or Full-Dirichlet boundary conditions. Let m_{phys} denote the physical mass and m_o the bare mass. Then

i) $\frac{m_{phys}}{m_o} < \exp\left\{\frac{\sigma}{m_o^2+(3\lambda/\pi)}\right\}$

In particular $m_{phys} < m_o$ for a $(\lambda\varphi^4)_2$ theory in the single phase region.

ii) In the case of Half-Dirichlet boundary conditions and $\sigma \geq 0$, one also has

$$\frac{m_{phys}}{m_o} < \sqrt{\frac{1+2\sigma}{m_o^2}}$$

The proof of part i) follows from Lemmas 1, 3, 4, 5 and 6. To prove part ii) we use the equivalence $(\lambda, \sigma, m_o) \sim (\lambda, 0, m_\sigma)$ if $\sigma = \frac{m_\sigma^2 - m_o^2}{2} + \frac{3\lambda}{2\pi} \ln \frac{m_\sigma^2}{m_o^2}$ which is valid for Half-Dirichlet boundary conditions [4].

For further details see Klein and Landau [6].

4. A Relation between m_{phys} and $<\varphi>$ in each pure phase of a $(\lambda\varphi^4)_2$ theory.

We now use the techniques of sections 2 and 3 to obtain information relating the physical mass m_{phys} and the field expectation $<\varphi>$ in each pure phase of $(\lambda\varphi^4)_2$. More precisely, we consider first a $\lambda\varphi^4 - \mu\varphi$ interaction with $\mu > 0$. Then it is known (Simon [7]) that the ground state is unique. Let m_μ be the physical mass and $<\varphi>_\mu$ the field expectation. Then

$$m_{phys} = \lim_{\mu \searrow 0} m_\mu \,, \quad <\varphi>_+ = \lim_{\mu \searrow 0} <\varphi>_\mu$$

The limits exist since both m_μ and $<\varphi>_\mu$ are non-negative and monotone decreasing as $\mu \searrow 0$. $<\varphi>_\mu$ decreases by the Griffiths inequalities and m_μ decreases by the GHS inequalities (Simon [8]). Our result is

<u>Theorem 5:</u> $$\frac{m_\mu}{m_o} < \exp\left\{\frac{2\pi <\varphi>_\mu^2}{1+(\pi m_o^2/3\lambda)}\right\}$$

and consequently

$$\frac{m_{phys}}{m_o} < \exp\left\{\frac{2\pi <\varphi>_+^2}{1+(\pi m_o^2/3\lambda)}\right\}$$

<u>Remark</u>: Note that in contrast to the result $\frac{m_{phys}}{m_o} < 1$ for $(\lambda\varphi^4)_2$ in the single phase region, we see that it is possible for $\frac{m_{phys}}{m_o}$ to be > 1 provided $<\varphi>_+$ is sufficiently large.

To prove Theorem 6, we recall that the uniqueness of the ground state implies $a = 0$ and thus

$$S(x-y) = \int_{m_\mu}^{\infty} d\rho(m) S_m(x) + <\varphi(0)>_\mu^2$$

in the representation of Lemma 4. We may assume $m_\mu > 0$.

From Lemma 2

$$\lim_{g\to\delta} <:\varphi(g)^2:>_\mu + \frac{3\lambda}{\pi m_o^2} <:\varphi(0)^2:>_\mu \geq <\varphi(0)>_\mu^2$$

Thus from Lemma 6

$$\left(1+\frac{3\lambda}{\pi m_o^2}\right) \lim_{g\to\delta} <:\varphi(g)^2:>_\mu \geq <\varphi(0)>_\mu^2$$

It follows that

$$\lim_{g\to\delta} \int_{m_\mu}^{\infty} d\rho(m)\,[s_m(g) - s_{m_o}(g)] \geq - \frac{<\varphi(0)>_\mu^2}{1+\frac{\pi m_o^2}{3\lambda}}$$

An application of Lemma 5 now completes the proof.

For further details see Klein and Landau [6].

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Colloques Internationaux C.N.R.S.
N° 248 – Les méthodes mathématiques de la théorie quantique des champs

CONVERGENCE OF THE VACUUM ENERGY DENSITY, -BOUNDS AND EXISTENCE OF WIGHTMAN FUNCTIONS FOR THE YUKAWA$_2$ MODEL

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RESUME : On étudie la théorie de Yukawa à deux dimensions. On obtient la convergence de la densité d'énergie du vide E_ℓ/ℓ. On montre les bornes -φ:

$$\pm\psi_\alpha(f)| \;,\; \pm\phi(f) \;\leq\; c(f)\,(H_\ell - E_\ell + 1)$$

et les bornes de Fröhlich pour les fonctions génératrices. Dans la limite du volume infini, les champs satisfont les bornes correspondantes, les champs de Boson sont auto-adjoints pour φ réel et les fonctions de Wightman existent comme distributions tempérées.

Convergence of the Vacuum Energy Density, ϕ-bounds and

Existence of Wightman Functions for the Yukawa_2 Model†

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Abstract: We study the Yukawa quantum field theory in two dimensional space-time. Denote by H_ℓ the Hamiltonian for volume ℓ suggested by perturbation theory (to second order) and by E_ℓ the corresponding vacuum energy, E_ℓ = inf spectrum H_ℓ. For each finite ℓ we show that the subspace of vacuum vector(s) Ω_ℓ for H_ℓ is not orthogonal to the free vacuum Ω_0. As a consequence we obtain convergence of the vacuum energy density E_ℓ/ℓ (and of the euclidean pressure) as the volume ℓ tends to infinity. We also prove ϕ-bounds, uniform in the volume, dominating time-zero fields by the Hamiltonian:

$$\pm|\psi_\alpha(f)|,\ \pm\phi(f)\ \leq\ c(f)(H_\ell-E_\ell+1),$$

as well as corresponding euclidean statements — Fröhlich bounds for the generating functions. In the infinite volume limit, the fields satisfy the corresponding bounds, the boson fields are self-adjoint for real f and the Wightman functions for the theory exist as tempered distributions.

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I. Introduction

We study the Yukawa model for boson-fermion interactions in two space-time dimensions. Formally the finite volume Hamiltonian is given by:

$$H_\ell = H_0 + \lambda H_I(\ell) - \frac{1}{2}\lambda^2 \delta m_2^2 \int_{-\ell/2}^{\ell/2} dx :\phi^2(x): - \lambda^2 \delta E_2(\ell),$$

$$(1.1) \qquad H_I(\ell) = \int_{-\ell/2}^{\ell/2} dx :\bar{\psi}\psi\phi:(x),$$

$$\delta m_2^2 = -(2\pi)^{-1}\int dp\,(p^2+1)^{-\frac{1}{2}}, \qquad \delta E_2(\ell) = -\langle H_I(\ell) H_0^{-1} H_I(\ell)\rangle_0,$$

where ϕ,ψ describe free boson, fermion fields with masses μ_0, m_0 respectively. The logarithmically divergent renormalization constants δm_2^2, $\delta E_2(\ell)$ are required because of ultraviolet divergences in the vacuum energy and boson mass. All of our results apply equally to a pseudoscalar Yukawa interaction. The Hamiltonian (1.1) has been studied by Glimm [1,2] and by Glimm and Jaffe [3,4] who showed that H_ℓ may be defined as a limit (in the resolvent sense) of corresponding momentum cutoff Hamiltonians $H_{\ell;\kappa}$. They show that the resulting operator H_ℓ is self-adjoint and bounded below and that the vacuum energy $E_\ell \equiv \inf \operatorname{spec} H_\ell$ is an isolated eigenvalue of finite multiplicity. For further details on the Hamiltonian formulation we refer to Glimm and Jaffe [5]. Schrader [6] has shown that $|E_\ell|$ is bounded by $\text{const} \times \ell$ and thus $\alpha_\ell \equiv -E_\ell/\ell$ is bounded uniformly in ℓ. Other proofs of this result, using euclidean or semi-euclidean techniques, have been given by Brydges [7], McBryan [8], by Simon and Seiler [9] and by Magnen and Seneor [10]. The principal results of this paper are to prove convergence of α_ℓ as $\ell \to \infty$ and to show that the time-zero boson and fermion fields are bounded by $\hat{H}_\ell \equiv H_\ell - E_\ell$, uniformly in the volume.

<u>Theorem</u> 1: The energy density $\alpha_\ell = -E_\ell/\ell$ converges to a finite limit α_∞ as $\ell \to \infty$.

Theorem 2: Let $f \in C_0^\infty(R^1)$, suppt $f \subset (-\ell/2, \ell/2)$, $\ell \geq 1$. Then for a constant C and a suitable Schwartz space norm $|||f|||$, both independent of ℓ and f,

$$\pm|\psi_\alpha(f)|,\ \pm\phi(f) \leq C|||f|||(\hat{H}_\ell+1)\ . \tag{1.2}$$

For the fermi fields the bound (1.2) is trivial since $\psi_\alpha(f)$ is a bounded operator for $f \in L_2(R^1)$. An immediate consequence of Theorem 2 is the existence of infinite volume Wightman functions as tempered distributions and existence of infinite volume Green's functions, see Glimm and Jaffe [11] and Nelson [12]. We also obtain self-adjointness of the boson fields $\phi(f)$, $f \in \mathcal{S}(R^2)$ real, on the physical Hilbert space, [11]. The proofs given in [11] require only a slight modification because our bounds (1.2) involve the Schwartz space norm $|||\cdot|||$ rather than the L_1 norm.

There are corresponding euclidean versions of Theorems 1 and 2. For the euclidean pressure $\alpha_{t,\ell} \equiv (t\ell)^{-1}Z_{t,\ell}$, $Z_{t,\ell} \equiv \langle\Omega_0, e^{-V_{t,\ell}}\Omega_0\rangle$, we obtain convergence as the space-time volume $t\ell$ goes to infinity (for the definition of $Z_{t,\ell}$ we refer to McBryan, [8], and below):

Theorem 3: The euclidean pressure $\alpha_{t,\ell} = (t\ell)^{-1}Z_{t,\ell}$ converges to α_∞ as $t,\ell \to \infty$ in any direction.

The euclidean version of the ϕ-bounds are Fröhlich bounds, [13], on the generating functional for boson Schwinger functions:

$$S_\ell(e^{\phi(f)}) \equiv \lim_{t\to\infty} \langle\Omega_0, e^{-V_{t,\ell}}e^{\phi(f)}\Omega_0\rangle / \langle\Omega_0, e^{-V_{t,\ell}}\Omega_0\rangle$$

Theorem 4: For each $f \in C_0^\infty(R^2)$ there is a constant $c(f)$ with $S_\ell(e^{\phi(f)}) \leq e^{c(f)}$, uniformly in ℓ.

More general bounds of the same form apply to generating functionals which include also fermion operators. The bounds in Theorem 4 follow directly from Theorem 2 and the Feynman-Kac formula in the same way as for $P(\phi)_2$ models.

In the case of the $P(\phi)_2$ model, Theorem 1 has been proved by Guerra [14], using Nelson's symmetry, and Theorem 2 by Glimm and Jaffe [11]; an alternative proof of Theorem 2, using Nelson's symmetry, has been given by Guerra, Rosen and Simon [15]. Our proofs for the Yukawa_2 model also use Nelson's symmetry, but there are two complications. It has not been shown that the finite volume vacuum(s) Ω_ℓ for H_ℓ are not orthogonal to the free vacuum Ω_0. The requirement that $\langle\Omega_0,\Omega_\ell\rangle \neq 0$ seems to be important for the proof of Theorems 1,2 using a Feynman-Kac formula connecting the free euclidean and relativistic Fock spaces. A second complication in the Yukawa case is the presence of finite second order wave-function renormalization terms in the Feynman-Kac formula which necessitate some care in the use of Nelson's symmetry. Both problems would presumably be avoided by a more natural Feynman-Kac formula for the Hilbert space defined by the Wightman functions for H_ℓ, which already incorporates the wave-function renormalization.

The principal technical estimate in this paper is a proof that $P_\ell\Omega_0 \neq 0$ where P_ℓ denotes the projection operator onto the vacuum states of H_ℓ:

<u>Theorem</u> 5: The vacuum overlap $P_\ell\Omega_0$ is nonzero for each finite ℓ.

Our proof of Theorem 5 applies also to the $P(\phi)_2$ model and in that case provides an alternate to the original proof of Glimm and Jaffe [16] which used positivity preserving properties of e^{-tH_ℓ}. Our proof requires only the Feynman-Kac formula and Euclidean invariance.

We prove Theorem 5 in section II and Theorems 1-4 in section III. For further details on the euclidean formulation of the Yukawa model in the Matthews-Salam representation we refer to Seiler [17] and to McBryan [8,18], while for the definition of euclidean fermi fields and of the cutoff Feynman-Kac formula we refer to Osterwalder and Schrader [19].

II. The Vacuum Overlap

We begin this section with a brief discussion of the Feynman-Kac formula for the Yukawa$_2$ model, see [19]. The renormalized momentum cutoff euclidean action for this model is given by

$$V_{t,\ell;\kappa} = \lambda{:}V_{I,t,\ell;\kappa}{:} - \frac{\lambda^2}{2}\,\delta m_\kappa^2 \int_{-t/2}^{t/2}\int_{-\ell/2}^{\ell/2} d^2x{:}\Phi_\kappa(x)^2{:} + \frac{\lambda^2}{2}\langle{:}V_{I,t,\ell;\kappa}{:}^2\rangle,$$

$$V_{I,t,\ell;\kappa} = \int_{-t/2}^{t/2}\int_{-\ell/2}^{\ell/2} d^2x\,\Psi_\kappa^{(2)}(x)\Psi_\kappa^{(1)}(x)\Phi_\kappa(x), \quad \delta m^2 = -(2\pi^2)^{-1}\int d^2p\,(p^2+1)^{-1}.$$

Here $\Phi, \Psi^{(i)}$ are the momentum-cutoff euclidean fields defined in [19], and we have denoted by κ a momentum cutoff only in the space direction. Let X and X' denote vectors in the positive time subspace $\mathcal{E}_+$ of the euclidean Fock space and let W be the unitary map defined in [19] from $\mathcal{E}_+$ into the relativistic Fock space $\mathcal{F}$. The Feynman-Kac formula takes the form:

$$\text{(2.1)}\quad (WX, e^{-tH_{\ell;\kappa}}WX') = (\Theta U^{t/2}X, e^{-V_{t,\ell;\kappa}}U^{t/2}X')e^{-W_\kappa(t,\ell)}, \quad t > 0,$$

where $W_\kappa(t,\ell)$, the wavefunction renormalization to second order, denotes the difference between the Hamiltonian and Euclidean vacuum energy renormalization constants

$$\text{(2.2)}\quad \begin{aligned} W_\kappa(t,\ell) &\equiv \lambda^2\langle H_{I,\ell;\kappa}H_0^{-1}H_{I,\ell;\kappa}\rangle t - \frac{\lambda^2}{2}\langle{:}V_{I,t,\ell;\kappa}{:}^2\rangle \\ &= \lambda^2\langle H_{I,\ell;\kappa}H_0^{-2}(1-e^{-tH_0})H_{I,\ell;\kappa}\rangle, \end{aligned}$$

U^t is the unitary euclidean time translation by t and Θ is the unitary involution defined in [19]. In order to exploit euclidean covariance, we will need a form of (2.1) in which the momentum cutoff κ has been removed. The right-hand side of (2.1) is well-defined as $\kappa \to \infty$ and equals $(WX, e^{-tH_\ell}WX')$. Also from (2.2) we see that $W_\kappa(t,\ell)$ converges to the finite constant $W(t,\ell) \equiv \lambda^2 \langle H_{I,\ell} H_0^{-2}(1-e^{-tH_0}) H_{I,\ell} \rangle$. To treat the euclidean scalar product in (2.1), we first integrate out the fermions, using the Matthews-Salam representation, and then we may pass to the limit $\kappa \to \infty$ explicitly, [17]. First we introduce some notation.

Let $\mathcal{T}_+$ denote the union for $0 \le n < \infty$ of the sets of all sequences $F = \{F_1, \ldots, F_r, \ldots, F_n\}$ with elements which are ordered pairs $F_r = (i_r, f_r)$, where $i_r \in \{0,1,2\}$ and where $f_r \in \mathcal{H}_{-1,+}^{(\mu_0)}$ if $i_r = 0$ or $f_r \in \mathcal{H}_{-\frac{1}{2},+}^{(m_0)} \otimes C^2$ if $i_r = 1,2$ ($\mathcal{H}_{s,+}^{(m)} = \{f \in L_2(R^2, (k^2+m^2)^s d^2k) : f(x) = 0$ for $x_0 < 0\}$), and where if $f_r = 0$ for some r_o then $f_r = 0$ for all $r \ge r_0$. For any pair $F_r = (i_r, f_r)$ we define euclidean fields:

$$\textstyle\sum(F_r) = \int d^2x \Phi(x) f(x), \quad i_r = 0, \qquad \text{or} \qquad \sum(F_r) = \sum_\alpha \int d^2x \Psi_\alpha^{(i_r)}(x) f_\alpha(x), \quad i_r = 1,2,$$

and to each sequence F we assign the number $n(F) \equiv \sup\{r : f_r \neq 0\}$, a vector $X(F)$ in $\mathcal{E}_+$ by:

$$X(F) = \textstyle\sum(F)\Omega_0, \qquad \sum(F) = \prod_{r=1}^{n(F)} \sum(F_r),$$

and a Schwinger function for the action $V_{t,\ell;\kappa}$ by:

$$S_{t,\ell;\kappa}(F) = \langle \Omega_0, \textstyle\sum(F) e^{-V_{t,\ell;\kappa}} \Omega_0 \rangle .$$

It is also convenient to define charge conjugation of a sequence F by $C : F \to F_C$ where $F_{C,n(F)+1-r} = F_r$, if $i_r = 0$, and $F_{C,n(F)+1-r} = (3-i_r, \gamma_0 f_r)$, if $i_r \neq 0$. We allow the inhomogeneous Lorentz group $L(a,\Lambda)$ to act on F by defining $L : F \to F_L$ where $F_{L,r} = (i_r, f_{L,r})$ and $f_{L,r}(x) = f_r(L^{-1}x)$, $i_r = 0$, $f_{L,r}(x) = S(\Lambda)^T f_r(L^{-1}x)$, $i_r = 1$, or $f_{L,r}(x) = S(\Lambda)^* f_r(L^{-1}x)$, $i_r = 2$, where $S(\Lambda)$ are the unitary rotation matrices defined in [19]. For convenience we let θ denote time reversal, and we denote by F_t, $F_\perp$, a time translation by t and a rotation by $\frac{\pi}{2}$, respectively.

Using the definition of the unitary involution θ we can rewrite (2.1) as

$$(WX(F), e^{-tH_{\ell;\kappa}} WX(F')) = S_{t,\ell;\kappa}(F_{t/2,C,\theta}, F'_{t/2}) e^{-W_\kappa(t,\ell)} .$$

Seiler, [17], has shown that euclidean Schwinger functions converge as $\kappa \to \infty$ and thus we obtain the limiting Feynman-Kac formula

$$(2.3) \qquad (WX(F), e^{-tH_\ell} WX(F')) = S_{t,\ell}(F_{t/2,C,\theta}, F'_{t/2}) e^{-W(t,\ell)} .$$

We now restrict the choice of allowed test functions to the subset $\mathcal{T}_{+,\ell} \subset \mathcal{T}_+$ of sequences such that each $f_r(x)$ has support in $\{x : x_0 \geq 0,\ x_1 > \ell/2\}$. By an elementary generalization of the Reeh-Schlieder theorem, [20], vectors of the form $X(F)$, $F \in \mathcal{T}_{+,\ell}$ are total in $\mathcal{E}_+$. Theorem 5 will follow easily from the following lemma, in which we have used the notation θ_x to denote a space reflection.

<u>Lemma</u> 6: For fixed ℓ, and $F \in \mathcal{T}_{+,\ell}$, then for all $t \geq 0$:

$$(2.4) \qquad \| e^{-\frac{1}{2}tH_\ell} WX(F) \|^2 \leq \| e^{-\frac{1}{2}tH_\ell} \Omega_0 \| \| e^{-\frac{1}{2}tH_\ell} WX(F_{C,\theta_x} F) \| .$$

<u>Proof of Theorem</u> 5: From the existence of vacuum vectors for H_ℓ, we know that $P_\ell \neq 0$. Thus since $\{X(F) : F \in \mathcal{T}_{+,\ell}\}$ are total in $\mathcal{E}_+$ and since W maps $\mathcal{E}_+$ into a dense subset of $\mathcal{F}$, [19], it follows that there is an $F^{(\ell)} \in \mathcal{T}_{+,\ell}$ such that $P_\ell WX(F^{(\ell)}) \neq 0$. From the functional calculus we have

$$P_\ell = \operatorname{st-lim}_{t \to \infty} e^{-t\hat{H}_\ell},$$

and thus multiplying both sides in Lemma 6 by e^{tE_ℓ} and letting $t \to \infty$:

$$0 < \| P_\ell WX(F^{(\ell)}) \|^2 \leq \| P_\ell \Omega_0 \| \| P_\ell WX(F^{(\ell)}_{C,\theta_x} F^{(\ell)}) \|$$

proving that the vacuum overlap $P_\ell \Omega_0$ is nonzero.

Proof of Lemma 6: We apply the Feynman-Kac formula (2.3) to the left-hand side of (2.4), rotate by $\pi/2$ and apply (2.3) again in reverse to obtain:

$$\begin{aligned}(WX(F), e^{-tH_\ell} WX(F)) &= S_{\ell,t}(F_{t/2,C,\theta,\perp}, F_{t/2,\perp})\, e^{-W(t,\ell)} \\ &= (\Omega_0, e^{-\ell H_t} WX(F_{t/2,C,\theta,\perp,-\ell/2}, F_{t/2,\perp,-\ell/2}))\, e^{-W(t,\ell)+W(\ell,t)} .\end{aligned}$$

Using the Schwartz inequality to bound the scalar product, followed by applying the Feynman-Kac formula (2.3) to each term, a further rotation by $-\pi/2$, and finally application of (2.3) once more to each term leads to

$$\begin{aligned}(WX(F), e^{-tH_\ell} WX(F)) &\le \| e^{-\frac{1}{2}\ell H_t}\Omega_0 \| \, \| e^{-\frac{1}{2}\ell H_t} WX(F_{t/2,C,\theta,\perp,-\ell/2}, F_{t/2,\perp,-\ell/2}) \| \\ &\qquad \times e^{-W(t,\ell)+W(\ell,t)} \\ &= Z_{\ell,t}^{\frac{1}{2}} S_{\ell,t}(F_{t/2,C,\theta,\perp,C,\theta}, F_{t/2,\perp,C,\theta}, F_{t/2,C,\theta,\perp}, F_{t/2,\perp})^{\frac{1}{2}} e^{-W(t,\ell)} \\ &= Z_{t,\ell}^{\frac{1}{2}} S_{t,\ell}(F_{t/2,\theta,\theta_x}, F_{t/2,C,\theta_x}, F_{t/2,C,\theta}, F_{t/2})^{\frac{1}{2}} e^{-W(t,\ell)} \\ &= (\Omega_0, e^{-tH_\ell}\Omega_0)^{\frac{1}{2}} (WX(F_{C,\theta_x}, F), e^{-tH_\ell} WX(F_{C,\theta_x}, F))^{\frac{1}{2}}\end{aligned}$$

where in the third line we have used the fact that C commutes with translations and rotations; $Z_{t,\ell}$ represents the partition function $(\Omega_0, e^{-V_{t,\ell}}\Omega_0)$.

III. Convergence of α_ℓ and the ϕ-bounds

The linear upper and lower bounds [6-10] for E_ℓ ensure that the sequence $\alpha_\ell = -E_\ell/\ell$ is bounded. Thus to prove Theorem 1 it suffices to show:

Lemma 7: There is a positive constant c such that $\alpha'_\ell \equiv \alpha_\ell - c/\ell$ is a monotone increasing function of ℓ.

Proof: Because of the nonzero vacuum overlap we have

$$-E_\ell = \lim_{t\to\infty} \frac{1}{t} \ln(\Omega_0, e^{-tH_\ell}\Omega_0) .$$

Now let $0<a<1$. Then using the Feynman-Kac formula (2.3)

$$\begin{aligned}
(\Omega_0, e^{-tH_{a\ell}}\Omega_0) &= Z_{t,a\ell}e^{-W(t,a\ell)} \\
&= Z_{a\ell,t}e^{-W(t,a\ell)} \\
&= (\Omega_0, e^{-a\ell H_t}\Omega_0)e^{-W(t,a\ell)+W(a\ell,t)} \\
&= (\Omega_0, e^{-\ell H_t}\Omega_0)^a e^{-W(t,a\ell)+W(a\ell,t)} \\
&= Z^a_{\ell,t}e^{-W(t,a\ell)+W(a\ell,t)-aW(\ell,t)} \\
&= Z_{t,\ell}{}^a e^{-W(t,a\ell)+W(a\ell,t)-aW(\ell,t)} \\
&= (\Omega_0, e^{-tH_\ell}\Omega_0)^a e^{-W(t,a\ell)+aW(t,\ell)+W(a\ell,t)-aW(\ell,t)}
\end{aligned}$$

where we have used the inequality $(\theta, A^a\theta) \le (\theta, A\theta)^a$, $0<a<1$. Thus

$$-E_{a\ell} \le -aE_\ell + \lim_{t\to\infty} t^{-1}\{-W(t,a\ell)+aW(t,\ell)+W(a\ell,t)-aW(\ell,t)\} .$$

Since $W(t,\ell)$ is bounded uniformly in t, it follows that the first two terms in the bracket do not contribute. For the other terms note that

$$W(a\ell,t) - aW(\ell,t) = \langle H_{I,t} H_0^{-2}(1 - a + ae^{-\ell H_0} - e^{-a\ell H_0}) H_{I,t}\rangle$$

$$= (1-a)\langle H_{I,t} H_0^{-2} H_{I,t}\rangle$$

$$= (1-a)ct$$

for some constant c. Thus

$$-E_{a\ell} \leq -aE_\ell + (1-a)c\ ,$$

from which we conclude that $\alpha'_\ell \equiv -(E_\ell + c)/\ell$ satisfies

$$\alpha'_{a\ell} \leq \alpha'_\ell\ , \quad 0 < a < 1, \quad \text{all } \ell,$$

$$\alpha'_\ell = \alpha_\ell - \frac{c}{\ell} \leq \sup_\ell \alpha_\ell < \infty\ .$$

It follows that α'_ℓ is monotonically convergent to a finite limit α_∞ and clearly α_ℓ also converges to this limit.

The proof of Theorem 3 is now immediate, for from the previous argument we have for $0 < a,b < 1$:

$$Z_{at,b\ell} \leq Z^{ab}_{t,\ell}\, e^{W(b\ell,at) - bW(\ell,at) + bW(at,\ell) - abW(t,\ell)}\ ,$$

$$\alpha_{at,b\ell} \leq \alpha_{t,\ell} + (abt\ell)^{-1}\{W(b\ell,at) - bW(\ell,at) + bW(at,\ell) - abW(t,\ell)\}$$

$$\leq \alpha_{t,\ell} + (abt\ell)^{-1}\{(1-b)cat + (1-a)cb\ell\}.$$

Thus $\alpha''_{t,\ell} \equiv \alpha_{t,\ell} - \frac{c}{\ell} - \frac{c}{t}$ is monotone increasing in each variable separately and since $\alpha_{t,\ell}$ is bounded in t,ℓ, both $\alpha''_{t,\ell}$ and $\alpha_{t,\ell}$ converge to a limit as $t,\ell \to \infty$. By inspection of the equations above it is clear that this limit is α_∞.

The ϕ-bound of Theorem 2 is a special case of a more general result for polynomial perturbations of H_ℓ. Our proof is similar to that of

Guerra, Rosen and Simon, [15,21], for the $P(\phi)_2$ case, but requires a somewhat different treatment because we do not have convergence of $\beta_\ell \equiv \ell(\alpha_\ell - \alpha_\infty)$ to a finite limit.

<u>Theorem</u> 8: Let $W(f) = \int dx f(x):Q(\phi(x)):$ where (i) f is measurable non-negative with $\|f\|_\infty \leq 1$ and suppt $f = [a,b]$ (ii) $H_\ell + Q_\ell$ is semi-bounded for all ℓ, where $Q_\ell = W(\chi_{[-\ell/2,\ell/2]})$. Then for $[a,b] \subset (-\ell/2,\ell/2)$, $\ell \geq 1$,

$$(3.1) \qquad -W(f) \subseteq \hat{H}_\ell + \text{const}(|a| + |b| + 1),$$

with the constant independent of ℓ or f. Whenever $0 \in [a,b]$, $|a| + |b|$ may be replaced in (3.1) by $|b-a|$.

<u>Proof</u>: We first suppose f to be a sum of step functions:

$$f(x) = \sum_{i=1}^{m} f_i \chi_{[a_{i-1},a_i]}(x), \quad -\ell/2 < a = a_0 < \ldots < a_m = b < \ell/2 ,$$

with $0 \leq f_i \leq 1$. Defining $H_\ell(\lambda) \equiv H_\ell + \lambda Q_\ell$ we obtain from a euclidean rotation of the Feynman-Kac formula:

$$\langle\Omega_0, e^{-t(H_\ell + W(f))}\Omega_0\rangle = \langle\Omega_0, e^{-\frac{1}{2}(\ell+2a)H_t} \prod_{i=1}^{m} e^{-(a_i - a_{i-1})H_t(f_i)} e^{-\frac{1}{2}(\ell-2b)H_t}\Omega_0\rangle e^{W(\ell,t) - W(t,\ell)} .$$

Taking logarithms, dividing by t and allowing $t \to \infty$ we conclude:

$$(3.2) \qquad -E(H_\ell + W(f)) \leq -\tfrac{1}{2}E_{\ell+2a} - \tfrac{1}{2}E_{\ell-2b} + \sum_{i=1}^{m}(a_i - a_{i-1})\alpha_\infty(f_i) + c ,$$

where $\alpha_\infty(\lambda) = -\lim_{\ell\to\infty} \ell^{-1}E(H_\ell(\lambda))$, $c = \limsup_{\ell\to\infty} \ell^{-1}\langle H_{I,\ell} H_0^{-2} H_{I,\ell}\rangle$. We now consider $E_{\ell\pm r}$, $r \geq 0$. From equation (3.2) with $f=0$ we have:

$$(3.3) \qquad \begin{aligned} -E_{\ell+r} &\leq -\tfrac{1}{2}E_\ell - \tfrac{1}{2}E_\ell + r\alpha_\infty + c \\ &= -E_\ell + r\alpha_\infty + c , \end{aligned}$$

while from the monotonicity of α'_ℓ we obtain:

$$-E_{\ell-r} = -E_\ell + (\ell-r)\alpha_{\ell-r} - \ell\alpha_\ell$$

$$(3.4) \qquad = -E_\ell + (\ell-r)(\alpha'_{\ell-r} - \alpha'_\ell + c/(\ell-r) - c/\ell) - r(\alpha'_\ell + c/\ell)$$

$$\leq -r\alpha'_\ell .$$

Applying the estimates (3.3), (3.4) to (3.2):

$$-E(H_\ell + W(f)) \leq -E_\ell + \sum_{i=1}^{m}(a_i - a_{i-1})(\alpha_\infty(f_i) - \alpha_\infty) + \frac{3}{2}c$$

$$(3.5) \qquad + (b + |b| - a + |a|)(\alpha_\infty - \alpha'_\ell)/2 .$$

Note that $\alpha_\infty(\lambda) - \alpha_\infty$ is concave in λ and vanishes for $\lambda = 0$ so that $\alpha_\infty(f_i) \leq f_i\alpha_\infty(1)$. Also by the monotonicity of $\alpha_{\ell'}$, $0 \leq (\alpha_\infty - \alpha_{\ell'})/2 \leq (\alpha_\infty - \alpha'_1)/2$, $\ell \geq 1$. Thus for a suitable constant independent of ℓ or f:

$$-E(H_\ell + W(f)) \leq -E_\ell + \mathrm{const}(\|f\|_1 + |a| + |b| + 1),$$

and since $\|f\|_1 \leq |b-a|$ (for $\|f\|_\infty \leq 1$), Theorem 8 follows by a limiting argument. In the case that $0 \in [a,b]$ we obtain $|b-a|$ in place of $|a| + |b|$ in the last equation as is evident from the last term in (3.5).

<u>Corollary</u> 9: Let f be measurable, non-negative, $\|f\|_\infty < \infty$ and with $\mathrm{suppt}\ f \subset (-\ell/2, \ell/2)$, $\ell \geq 1$. Then

$$-W(f) \leq C|||f|||(\hat{H}_\ell + 1),$$

where C is independent of ℓ or f, and $|||f||| = \|(1+|x|)^3 f(x)\|_\infty$.

<u>Proof</u>: Applying Theorem 8 for $\frac{f}{\|f\|_\infty}$ we obtain, with a constant independent of ℓ, f:

$$-W(f) \leq \mathrm{const}(|a| + |b| + 1)\|f\|_\infty(\hat{H}_\ell + 1) .$$

Now decompose $f(x) = \sum_{-\infty}^{\infty} f_i(x)$, $f_i(x) = f(x)\chi_{[i-1,i]}(x)$ so that

$$-W(f_i) \leq \text{const}(2+2|i|)\|f\|_\infty (\hat{H}_\ell+1)$$

$$\leq \text{const}(1+|i|)^{-2}\|(2+|x|)^3 f_i(x)\|_\infty (\hat{H}_\ell+1).$$

Summing over i gives

$$-W(f) \leq \text{const}(\sum_i (1+|i|)^{-2})\sup_j \|(1+|x|)^3 f_j\|_\infty (\hat{H}_\ell+1)$$

$$= \text{const}|||f|||(\hat{H}_\ell+1).$$

<u>Remark</u>: In addition to Theorem 1, where we have taken $Q(\phi) = \pm\phi$ in Corollary 9, we may also take the case of $Q(\phi) = \pm{:}\phi^2(x){:}$ by results of McBryan [22] and of Simon and Seiler [23], obtaining

$$\pm{:}\phi^2{:}(f) \leq \text{const}|||f|||(\hat{H}_\ell+1).$$

<u>Note added in preparation</u>: Seiler and Simon have also announced these results at the Colloquium on Quantum Field Theory, CNRS Marseille, June 1975.

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SEMICLASSICAL QUANTIZATION METHODS IN FIELD THEORY

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RESUME On donne un résumé de la méthode WKB semi-classique. La méthode est appliquée pour trouver des solutions non-perturbatives des équations de mouvement non-linéaires classiques avec interactions de plusieurs modèles. En particulier les solutions avec des propriétés de particule.

This is an overview of the semiclassical WKB method which has been developed by R. DASHEN, B. HASSLACHER and myself, in Ref. 1 and which can be applied to finding solutions to field theories which are inaccessible to perturbation techniques.

In particular, it is possible to find solutions to the full non-linear interacting classical equations of motion of various models, which behave like bound, stable field configurations in space-time, with particle properties. The question arises as to whether these solutions survive the process of second quantization. In Ref. 1, we give a method for answering that question, the accuracy of which depends both on how much one knows about the classical problem, and the strength of the coupling constant, in direct proportion.

Our method is based on the works of KELLER, GUTZWILLER and MASLOV, who developed a general semi-classical formalism for use in atomic physics. These techniques are directed toward the computation of energy levels, or particle masses in field theory. We approach the problem through the quantum action principle in the Feynman path integral representations, since this provides the most natural connection between the classical problem and its second quantized analogue. Also, since we start from a Lagrangian formalism, any divergences that emerge can be handled by standard renormalization techniques.

For weak coupling, it was found that time-independent classical solutions are interesting [1,2]. In the weak coupling limit, our WKB quantization of static solutions to classical field equations is equivalent to a number of other schemes. The difference comes when one contemplates classical motions which cannot be reduced to a time independent field. That such solutions are interesting should be obvious from the fact that the Bohr orbits of hydrogen are not time-independent solutions to classical equations of motion but rather are motions which are periodic in time. The real power of WKB method is the quantization of motions analogous to Bohr orbits. To find an example of how the semi-classical method works in field theory, we have studied the sine-Gordon equation in one space and one time dimension.[3] It is defined by the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu \varphi\right)^2 + \frac{m^4}{\lambda}\left[\cos\left(\varphi \frac{\sqrt{\lambda}}{m}\right) - 1\right] \qquad (1)$$

and is completely solvable at the classical level : there exists an algorithm [3] from which all solutions to the Lagrange equations for φ can be constructed. In particular, to apply our quantization method, we look for classical solutions which become particles when quantized. There are two types of these :

First, there is the soliton (and the antisoliton) which is a solution that is time independent in its rest frame. The other, which we call the doublet, is a soliton-antisoliton bound state. In its rest frame, the doublet field oscillates periodically in time. Doublet solutions exist for a continuous range of classical energies. The WKB method will quantize the doublet energies, yielding a discrete spectrum of particle masses.

The particle spectrum of the sine-Gordon Hamiltonian turns out to be the following. The soliton and antisoliton have a mass $M = 8m/\gamma'$, where $\gamma' = \left[\lambda/m^2\right] / \left[1 - \lambda/8\pi m^2\right]$
The doublet produces the remaining series of states at masses

$$M_n = \frac{16m}{\gamma'} \sin \frac{n\gamma'}{16}$$

$$n = 1,2,3 \ldots\ldots < 8\pi/\gamma' \qquad (2)$$

The original "elementary particle" of the theory is the n = 1 state in eq. (2). As $\lambda \to 0$, γ' vanishes, and one easily sees that M, approaches the weak coupling mass, $m + O(\lambda^2)$, of the elementary particle. Notice that, according to eq. (1,2) there is a finite number of doublet states. As the coupling γ' increases, the states disappear one by one. What happens is that they decay into soliton-antisoliton pairs. This may be seen by observing that when the $n-$ th state disappears, M_n is just $16\, m/\gamma'$, or twice the soliton mass. At $\gamma' = 8\pi$, the $n = 1$, or "elémentary particle" state itself breaks up and disappears from the spectrum ; only solitons and antisolitons remain.

The weak coupling behaviour of M_n is quite interesting. Expanding, one finds :

$$M_n = nM_1 - \frac{M_1}{6} \left(\frac{\lambda}{16m^2} \right)^2 (n^3 - n) + O(\lambda^3)$$

$$M_1 = \frac{m}{16\gamma'} \sin \frac{\gamma'}{16} = m \left[1 - \frac{1}{6} \left(\frac{\lambda}{16m^2} \right)^2 \right] + O(\lambda^3) \qquad (3)$$

which corresponds to a non-relativistic n-body bound state made up of n particles with physical mass M .

This is the same as one finds upon solving the n-body Schrödinger equation with δ - function potential obtained from the φ^4 term in the interaction Lagrangian. Thus, for weak coupling, the doublet states can be thought of as bound states of n "elementary particles". Of course, n cannot be too big. When $\gamma' n$ is greater than 8π , the state breaks up into a soliton-antisoliton pair. In fact, for $\gamma' n$ large (but less than 8π), the states are probably best thought of as soliton-antisoliton bound states.

The semi-classical calculation suggests that all states with $\gamma' n$ less than 8π are stable. The mass ratio as given by eq. (2) and the symmetry of the Lagrangian under $\varphi \to -\varphi$ account for the stability of the $n = 1,2,3$ states. It takes further symmetry to keep the $n = 4$ state from decaying into two $n = 1$ states. At a classical level, the sine-Gordon equation has an infinite number of non-trivial conserved quantities [3]. If as conjectured, these survive in the quantum theory, they would provide enough quantum numbers to stabilize all the bound states : the S-matrix, as conjectured in Ref. 4, would be pure phases.

We have also extended our work on the φ^4 theory in two dimensions. This system is not exactly solvable. For small coupling, however, one can find the analogue of the sine-Gordon doublet states. We obtain a formula like (3) with a different coefficient of $n^3 - n$. The interpretation is the same except that we no longer know what happens for strong coupling. It is a reasonable speculation, however, that for large λn the states break up into a kink - antikink pair. Although our results for the φ^4 theory are neither as complete nor as elegant as those for the sine-Gordon case, we regard this calculation as important. It shows that the method is not restricted to special, classically solvable equations like the sine-Gordon system.

Coleman [5] has obtained the remarkable result that the sine-Gordon system can be mapped into the massive Thirring model. The relationship between the sine-Gordon coupling λ and the four-fermion coupling g of the Thirring model is $\frac{\lambda}{4\pi m^2} = \frac{1}{1+g/\pi}$, or $\gamma' = 8\pi/(1 + 2g/\pi)$. What are the fermions ? They are almost certainly the solitons. To see this, we observe that at $\gamma' = 8\pi$, the Thirring model coupling g vanishes. This is just the point where the $n = 1$ state unbinds. For γ' slightly less than 8π, the four-fermi coupling is weak and attractive. There will then be one non-relativistic fermion-antifermion bound state. Summing diagrams in the Thirring model, one finds that through order g^3, the mass M_B of the bound state is given in terms of the fermion mass M_f by

$$\frac{2 M_f - M_B}{M_f} = g^2 - \frac{4g^3}{\pi} + O(g^4) \tag{4}$$

Identifying M_B with M_1, and M_f with the soliton mass $8 m / \gamma'$, we compare this to

$$\frac{2M\,(\text{soliton}) - M_1}{M\,(\text{soliton})} = 2\left(1 - \sin\frac{\gamma'}{16}\right) = g^2 - \frac{4g^3}{\pi} + O\,(g^4) \tag{5}$$

where we have used Coleman's identification of the coupling constants. It is remarkable that both the g^2 and g^3 terms agree. We have not computed beyond order g^3 in the Thirring model. For $\gamma' > 8\pi$, the four-fermion coupling is repulsive and there is no bound state.

Coleman also finds that the theory is singular at $\lambda / m^2 = 8\pi$. At this point, γ' goes to infinity and it is evident that our semi-classical solution is also singular.

The agreement between our approximation and Coleman's precise results suggests that WKB may be exact for the mass spectrum of the sine-Gordon equation. This is not beyond the realm of possibility. Recall that the Bohr-Sommerfeld quantization conditions give the energy levels of hydrogen exactly. To investigate this question, we have gone to the weak coupling regime, and carried out an exact calculation of M_2/ M_1 through order $(\lambda / m^2)^4$. This is done by summing Feynman diagrams in a way which is equivalent to solving the Bethe-Salpeter equation.

The exact result is

$$\frac{2M_1 - M_2}{M_1} = \left(\frac{\lambda}{16m^2}\right)^2 + \frac{4}{\pi}\left(\frac{\lambda}{16m^2}\right)^3 + \left(\frac{12}{\pi^2} - \frac{1}{12}\right)\left(\frac{\lambda}{16m^2}\right)^4 + O(\lambda^5) \quad (6)$$

One can easily calculate the same quantity using eq. (2) for M_1 and M_2. Expanding, one finds that the coefficients of λ^2, λ^3 and λ^4 are identical. This is a highly non trivial result : to get the exact order λ^4 term, one has to keep two-loop diagrams in the kernel of the Bethe-Salpeter equation. We can show that the agreement in order λ^4 is special to the sine-Gordon equation, and will not occur in the generic case.

As argued above, we conjecture that eq. (2) gives the mass ratios of Lagrangian (1) exactly to all orders of perturbation theory. It does not, however, give the absolute masses exactly as a function of the bare mass, as can already be seen in lowest non-trivial order (order $\bar{\lambda}^2$).

We have also investigated a model which contains fermions and developed a general method for handling them in semi-classical calculations.

Specifically, we have used a WKB method to compute the particle spectrum of the Gross-Neveu model. It is in two dimensional space-time and is defined by the Lagrangian.

$$\mathcal{L} = \sum_{k=1}^{N} i\,\bar\psi^{(k)} \not\partial\, \psi^{(k)} + \frac{g^2}{2}\left(\sum_{k=1}^{N} \bar\psi^{(k)} \psi^{(k)}\right)^2 \tag{7}$$

The model thus contains N fermions coupled symmetrically through a scalar-scalar interaction. We will generally surpress the particle type indices k and use the notation

$$\begin{aligned} i\,\bar\psi \not\partial\, \psi &= \sum_k i\,\bar\psi^{(k)} \not\partial\, \psi^{(k)} \\ \bar\psi \psi &= \sum_k \bar\psi^{(k)} \psi^{(k)} \end{aligned} \tag{8}$$

The model is renormalizable (g is dimensionless), γ_5 invariant and formally scale invariant. For large N one can sum the leading sets of diagrams and establish that in this limit the model is asymptotically free. Gross and Neveu[6] also found that $\bar\psi\psi$ develops a vacuum expectation value so that γ_5 invariance is spontaneously broken. In the process the dimensionless coupling constant g is traded for an arbitrary dimensional parameter $g\,\langle\bar\psi\psi\rangle$ and disappears from the theory. The end result is that the theory contains no dimensionless parameter other than the number of fermions N. Consequently, any physical dimensionless quantity such as the ratio of two particle masses can depend only on N. This rather striking phenomenon, whose ultimate origin is the renormalization group, will be present in our WKB calculations. We can take this as an indication that semi-classical methods are compatible with renormalization group ideas.

Following Gross and Neveu, we find it useful to replace (7) by

$$\mathcal{L} = \bar\psi\, i \not\partial \psi + g\sigma\bar\psi\psi - \frac{\sigma^2}{2} \tag{9}$$

where we have used the notation of (8) and have introduced a neutral scalar field σ. Using the equation of motion

$$\sigma = g \bar{\psi} \psi \qquad (10)$$

the Lagrangian in (9) becomes equivalent to that in (1.1). Our WKB method is based upon the evaluation of certain functional integrals by a stationary phase approximation. It is not obvious how to use a stationary phase method when there are integrations over anticommuting fermion fields. The advantage of the Lagrangian in 9 is that the fermi fields enter bilinearly and can be integrated out of the problem leaving an effective action containing only the boson field σ. We then do the σ-integration by stationary phase. To do this we must find space-time dependent fields σ around which the effective action is stationary. This effective action is non-local and highly non-linear but it turns out to be possible to find stationary points. The first such example was found by Callan, Coleman, Cross and Zee.[7] It is analogous to the kink in the φ^4 theory or the soliton in the sine-Gordon equation, i.e., it is a particle-like solution which is time-independent in its rest frame and which has a peculiar topology. We have found a large number of further stationary points of the effective action. In particular, we find solutions which are particle-like but have a non-trivial time dependence in the rest frame. The WKB method then quantizes these classical solutions producing a spectrum of particle masses.

The kink-like solutions produce an exotic sort of particle which probably has no counterpart in four dimensions. However, the vast majority of our solutions are not kinks. They correspond to less exotic objects such as the original fermion, fermion-antifermion bound states or multifermion bound states. Such states surely exist in four-dimensional theories and we would conjecture that in four, as well as in two dimensions, there is a correspondence between classical field configurations and particle states. Assuming this to be so, it remains to be seen if such a correspondence can be effectively exploited.

Below we will describe the particle spectrum of the model as given by our WKB calculation. To interpret this spectrum we will need to know something about the symmetries of the model. The Gross-Neveu model has an obvious U(N) internal symmetry. Actually it has an O(2N) symmetry of which U(N) is a subgroup. This may be seen as follows. Choose a Majorana representation for the γ matrices $\gamma^0 = \sigma^y$, $\gamma^1 = i\,\sigma^x$ and write

$$\psi^{(k)} = \psi_1^{(k)} + i\,\psi_2^{(k)} \tag{11}$$

where $\psi_1^{(k)}$ and $\psi_2^{(k)}$ are hermitian two component spinors. The Lagrangian then takes the form

$$\mathcal{L} = \sum_k i\left(\psi_1^{(k)} \frac{\partial}{\partial t}\psi_1^{(k)} + \psi_2^{(k)}\frac{\partial}{\partial t}\psi_2^{(k)} + \psi_1^{(k)}\sigma_z \frac{\partial}{\partial x}\psi_1^{(k)} + \psi_2^{(k)}\sigma_z\frac{\partial}{\partial x}\psi_2^{(k)}\right)$$
$$- g\,\sigma\sum_k\left(\psi_1^{(k)}\sigma_y\psi_1^{(k)} + \psi_2^{(k)}\sigma_y\psi_2^{(k)}\right) - \frac{\sigma^2}{2} \tag{12}$$

which is hermitian and non-vanishing because the ψ's anticommute. When written in the form 12 , it is clear that the Lagrangian is invariant under orthogonal transformations on the 2N component vector $\psi_j^{(k)}$, $k = 1, 2, \ldots N$,

$j = 1, 2$. The fermion number operator $Q \equiv \int \psi^{+} \psi\, dx$ has non-trivial commutation relations with other generators of O(2N). Therefore a non-trivial representation of O(2N) will contain states with more than one value of Q. Hence we may expect, for example, that some fermion-antifermion states will be degenerate with fermion-fermion states. The σ field is an O(2N) scalar while ψ is an O(2N) vector. The only other O(2N) representations which we will encounter are the totally antisymmetric O(2N) tensors of rank $n_0 < N$. The number of states in a multiplet corresponding to such a tensor is $n_0!(2N - n_0)!/(2N)!$. Scalars and O(2N) vectors are special cases of completely antisymmetrical tensors of rank $n_0 = 0$ and $n_0 = 1$ respectively.

Because of our inability to evaluate certain Gaussian functional integrals we have not been able to carry through a complete WKB calculation in the Gross-Neveu model. What we have been able to do is a sort of zeroth order calculation which, in ordinary potential theory, is analogous to using the quantization rule $\oint p dq = 2n\pi$ rather than the more accurate $\oint p dq = (2n+1)\pi$. [In the sine-Gordon equation the analogous approximation is equivalent to setting $\gamma' = \frac{\lambda}{m^2}\left[1 - \frac{\lambda}{8\pi m^2}\right]^{-1} \approx \frac{\lambda}{m^2}$]. Even with this approximation our results should become exact in the limit of large N and are probably qualitatively correct for any N greater than 2 or 3.

We find the particle spectrum shown in Fig. (1). There is a large, unexpected degeneracy beyond that required by O(2N) symmetry. This degeneracy might be real or it may be an artifact of our zeroth order calculation. There are supermultiplets listed by a "principle quantum number" $n = 1, 2, \ldots < N$. The common mass of the states in the n^{th} supermultiplet is

$$M_n = g\sigma_0 \frac{2N}{\pi} \sin\left(\frac{n}{N}\frac{\pi}{2}\right)$$

$$n = 1, 2, \ldots < N \tag{13}$$

where σ_0 is the vacuum expectation value of σ. We see that ratios of masses are independent of g as they should be. If n is odd the supermultiplet is composed of fermions and contains $O(2N)$ representations corresponding to all completely antisymmetrical tensors of rank $n_0 = 1, 3, 5 \ldots \leq n$. For example, the $n = 1$ state is a fermion belonging to a vector representation of $O(2N)$. This is the "elementary particle" of the theory. For large N,

$$M_1 \approx g\sigma_0 \tag{14}$$

which agrees with the result of Gross and Neveu. The $n=3$ supermultiplet contains an $O(2N)$ vector which is some kind of excited state of the elementary particle and a completely antisymmetrical $O(2N)$ tensor of rank 3. The latter is a bound state of three fermions and/or antifermions. The supermultiplets with n even are composed of bosons and contain $O(2N)$ antisymmetrical tensors of rank $n_0 = 0, 2, 4 \ldots \leq n$. For example, $n = 2$ contains an $O(2N)$ scalar and an antisymmetric tensor of second rank. The tensor is a set of two body bound states with fermion-fermion, antifermion-antifermion and fermion-antifermion quantum numbers. The $O(2N)$ scalar is a different sort of object. It may be thought of as a particle associated with the σ field. At the n = 4 level there is an excited σ; a state which can be thought of as an excitation of the second rank tensor at n = 2 and a new state corresponding to a completely antisymmetrical tensor of rank 4. This new object is a bound state of 4 fermions and/or antifermions analogous to the 2 and 3 particle states found at levels n = 2 and 3. The pattern continues in the same way for n = 5, 6... on up to N.

The quantum numbers of the states in our spectrum are not unexpected. In the limit of large N the leading exchange is a sum of bubbles.

In the non-relativistic limit, this exchange produces an attractive δ-function potential. Such a potential will produce bound states only in channels where the spacial wave function is completely symmetrical. For fermions this means that the O(2N) wave function must be completely antisymmetrical, i.e., an O(2N) antisymmetric tensor.

For large N the bubble exchange is weak[6] and a Schroedinger equation calculation is valid. In this way one finds a binding energy which agrees with that computed from (13)

$$|E_B| = nM_1 - M_n = M_1 \left(\frac{\pi}{2N}\right)^2 (n^3 - n) + O\left(\frac{1}{N^3}\right) \quad (15)$$

to the indicated order in N^{-1}. These non-relativistic bound states correspond to the states with $n_0 = n$. They are the lowest states with given O(2N) quantum numbers and are consequently stable. Eq. (15) is valid only if n/N is small. For n and N both large the binding energy per particle is, in units of M_1

$$\frac{n M_1 - M_n}{n M_1} = 1 - \frac{2N}{\pi n} \sin\left(\frac{n}{N}\frac{\pi}{2}\right) \tag{16}$$

which for $n/N \sim 1$ shows binding by a finite fraction of the rest mass. Thus strong binding can occur even for large N.

The bubble exchanges are not the only important interaction for large N. For fermion-antifermion interactions in an $O(2N)$ single state the annihilation bubbles are dominant. The sum of these bubbles leads to an interaction which is marginally attractive. In leading order in N, Gross and Neveu[6] found a σ bound state at the fermion-antifermion threshold. It is presumably the $n=2$, $O(2N)$ singlet state discussed above. We find that it is bound in the next order in N^{-2}. This disagrees with a detailed diagrammatic calculation by Schoenfeld[8] who finds that the bound state remains at threshold to this order. We do not understand the origin of this discrepancy. In any case there is a weak attraction between fermion-antifermion pairs in an $O(2N)$ singlet state. One might therefore imagine that the particles in the model will be made up of a number of fermions and antifermions paired into $O(2N)$ singlet states plus further "valence" fermions and antifermions in an antisymmetrical tensor state. Our particle spectrum is consistent with such a picture.

The particle spectrum ends at $n = N$ where the mass is $M_N = 2Ng\sigma_0/\pi$. The mass of the Callan-Coleman-Gross-Zee kink is (in our zeroth approximation) $M_{kink} = N g \sigma_0 / \pi$. Thus the N^{th} state is just at the kink-antikink threshold. Higher mass states would be unstable against decay into kink-antikink pairs.

There is a striking similarity between the sine-Gordon equation and the Gross-Neveu model. In the zeroth order WKB approximation the particle spectrum of the sine-Gordon theory is given by $M_n \approx [m\, 2\xi/\pi] \sin(\pi n / 2\xi)$ where $\xi = 8\pi m^2/\lambda$, plus a soliton at mass $M_{soliton} \approx m\xi\pi$. With the identification $N \to \xi$ the energy levels are identical to those of the Gross-Neveu model. The particle content of the levels is of course very different in the two theories. There is no doubt an underlying reason for this correspondence between the theories but we do not know what it is. However, we can use this correspondence to try and guess what would happen if we could do a complete WKB calculation. In the sine-Gordon equation the result of the complete calculation is simply to replace λ/m^2 in the zeroth order formula by $\frac{\lambda}{m^2}\left[1 - \frac{\lambda}{8\pi m^2}\right]^{-1}$ which is equivalent to making the replacement $\xi \to \xi - 1$. The analogous replacement in the present model would be to replace N by $N-1$ in Eq. (1.7) and in the formula for the kink mass. The theory would then be singular at $N=1$. One expects such a singularity since at $N=1$ the Gross-Neveu model can be Fierz transformed to the usual Thirring model which contains a single massless fermion. Our zeroth order calculation is certainly not valid for N as small as 1.

If it were to turn out that a full WKB calculation differs from the present one only by changing N to $N-1$ then the extra degeneracy in the mass spectrum would presumably be real and a consequence of some underlying dynamical symmetry. Another possibility is that in a complete WKB calculation the masses within a supermultiplet will be split by terms of order N^{-2}. If this happens, the $n = 2$ singlet state might remain at threshold to order N^{-2} in agreement with Schoenfeld.

While the finer details of our approximate semiclassical spectrum are clearly not to be taken too seriously, the qualitative picture of a rich spectrum organized into some kind of supermultiplets is almost certainly correct. This unexpected wealth of particle states seems to be a consequence of the asymptotic freedom of the theory. The detailed form of the classical σ field which corresponds to a quantum bound state suggests that the binding mechanism is not a direct interaction between the bound fermions but rather is some kind of vacuum polarization effect. The fact that the theory is unstable in the infrared is most likely the reason for this.

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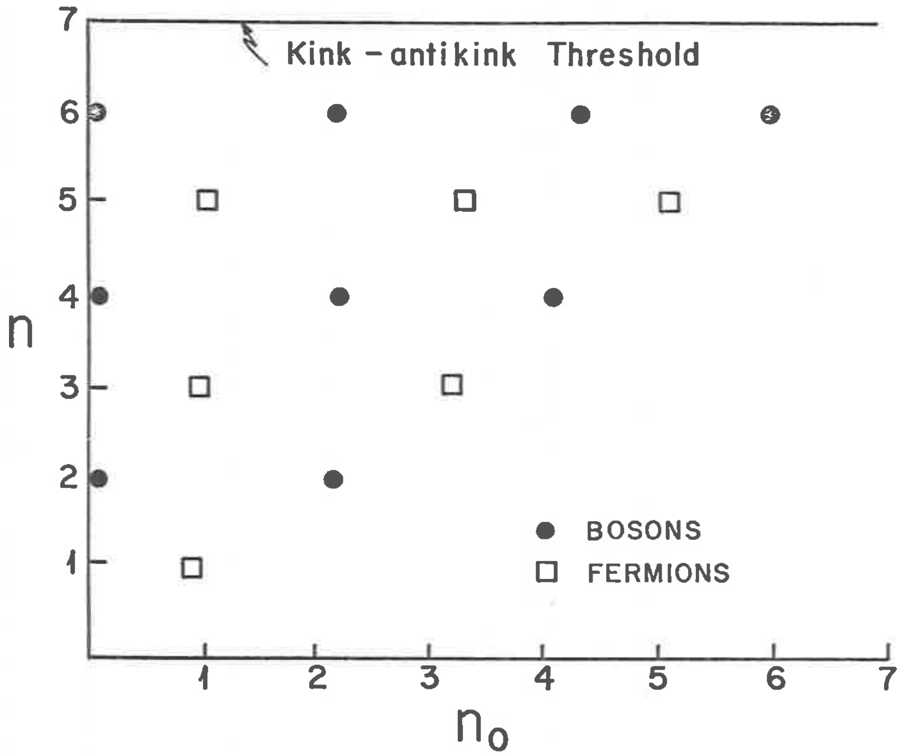

Kink - antikink Threshold
7
6
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4
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n
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3
4
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n_0
BOSONS
FERMIONS

FIG. 1

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Classifying General Ising Models + //

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ABSTRACT : A review of various correlation inequalities is given together with a survey of their applicability to various spin models. The Lee - Young theorem is discussed in the same context.

RESUME : On donne un résumé de plusieurs inégalités de corrélation ainsi qu'un aperçu de leur application à des modèles de spin divers. Le théorème de Lee - Young est discuté de la même façon.

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Let $\{X_j : j = 1,\ldots,N\}$ be random variables with joint distribution

$$(1) \quad \frac{1}{Z(h_1,\ldots,h_N)} \exp\left(\sum_{j\leq k=1}^{N} J_{jk}x_jx_k + \sum_{j=1}^{N} h_jx_j\right) \prod_{j=1}^{N} \rho_j(x_j) \quad .$$

The measures ρ_j are assumed to belong to the set $\mathcal{E}$ of even probability measures which satisfy $\int \exp(bx^2)d\rho_j(x) < \infty$ for some $b > 0$; we consider only the ferromagnetic case ($J_{jk} \geq 0$ for all j,k) and assume that the J_{jk}'s are small enough so that

$$(2) \quad Z(h_1,\ldots,h_N) \equiv \int_{R^N} \exp\left(\sum_{j\leq k=1}^{N} J_{jk}x_jx_k + \sum_{j=1}^{N} h_jx_j\right) \prod_{j=1}^{N} d\rho_j(x_j)$$

is finite for all real $h_1,\ldots,h_N$. Such a family of random variables constitutes a (finite) general Ising model with pair ferromagnetic interactions (J_{jk}) in an external magnetic field (h_j); spin-1/2 Ising models correspond to the choice of each ρ_j as the Bernoulli measure,

$$(3) \qquad b(x) = (\delta(x-1) + \delta(x+1))/2 \quad .$$

Most results about Ising models were first proved in the spin-1/2 case. Some of these results, such as the Griffiths-Kelly-Sherman inequalities [Gr1,KS] and the Fortuin-Kastelyn-Ginibre inequality [FKG], extend to all general Ising models

[Gi,GRS]. Most results however, such as the Lee-Yang Theorem [LY], the Griffiths-Hurst-Sherman inequality [GHS], the "Gaussian" inequalities of [N3], and other inequalities [L,Pe,Syl], do not apply to arbitrary general Ising models but rather rely heavily for their validity on the specific character of the ρ_j's in (1).

There is one class of general Ising models to which basically all spin-1/2 results extend; this class consists of those models in which each ρ_j can itself be constructed from spin-1/2 models. Based on Griffiths' method of analogue spin systems [Gr2] and its extension in [SiGr], we follow [N3] and define a measure ρ in $\mathcal{E}$ to be ferromagnetic if there is a sequence of spin-1/2 Ising models $\{X_j(n)\ ;\ j=1,\dots,N(n)\}$ (as defined by (1)) with $h_j \equiv 0$, and some choice of $\lambda_j(n) \geq 0$ so that

(4) $$E\left(\exp\left\{ir\sum_{j=1}^{N(n)} \lambda_j(n)X_j(n)\right\}\right) \to \int \exp(irx)d\rho(x)$$

as $n \to \infty$ for all real r while

(5) $$E\left(\left[\sum_{j=1}^{N(n)} \lambda_j(n)X_j(n)\right]^2\right) < K$$

for some K independent of n. We then have [N3]:

THEOREM 1. Let $\{X_j : j = 1, \ldots, N\}$ be a general Ising model defined by (1) with each ρ_j ferromagnetic; then

(6) $\quad \text{Re } h_j > 0 \text{ for all } j \Rightarrow Z(h_1, \ldots, h_n) \neq 0$

(this is the Lee-Yang property),

(7) $\quad h_j \geq 0 \text{ for all } j \Rightarrow \partial^3 \log Z(h_1, \ldots, h_N)/\partial h_i \partial h_j \partial h_k \leq 0$

for any i,j,k (this is the GHS inequality), and

(8) $\quad h_j = 0 \text{ for all } j \Rightarrow E(X_{j_1} \cdots X_{j_n}) \leq E(Z_{j_1} \cdots Z_{j_n})$

for any $n, j_1, \ldots, j_n$, where $\{Z_j\}$ is a jointly Gaussian family of random variables with zero means and covariance identical to that of $\{X_j\}$ (these are the Gaussian inequalities of [N3]); the inequalities of [L,Pe,Syl] are also valid.

Some important known examples of ferromagnetic distributions are [Gr2,SiGr]:

(9) $\quad \nu(x) = [\delta(x-n) + \delta(x-n+2) + \cdots + \delta(x+n)]/(n+1)$

(10) $$d\nu/dx = \begin{cases} (1/2A) & , \ |x| \leq A \\ 0 & , \ |x| > A \end{cases}$$

(11) $\quad d\nu/dx = \text{const.} \exp(-ax^4 + bx^2) \quad ; \quad a > 0 \, , \, b \in R \quad .$

It is an unsolved, and presumably difficult, problem to effectively characterize the class of ferromagnetic distributions. We can obtain as a corollary of Theorem 1 some necessary conditions for a measure to be ferromagnetic; the determination of a reasonable set of sufficient conditions would be of considerable interest.

COROLLARY 2. Suppose ρ in $\mathcal{E}$ is ferromagnetic and X is a random variable distributed by ρ; then

(12) $$E(\exp(hX)) = 0 \Rightarrow h = i\alpha \quad \text{for some} \quad \alpha \in R \ ,$$

(13) $$d^3 \log E(\exp(hX))/dh^3 \leq 0 \quad \text{for} \quad h \geq 0 \ , \quad \text{and}$$

(14) $$E(X^{2m}) \leq \frac{(2m)!}{2^m m!} [E(X^2)]^m \ .$$

We conjecture that the distribution

(15) $$d\nu/dx = \text{const.} \exp(-a \cosh x + bx^2) \ ; \ a > 0 \ , \ b \in R$$

is ferromagnetic; as we shall see below, this distribution is known to satisfy (12) for $b \geq 0$ and (13) for all real b. Inequality (14) is known as Khintchine's inequality and was first obtained for sums of independent Bernoulli random variables [K]; when ρ is in $\mathcal{E}$, Khintchine's inequality is a consequence of (12) [N2]. Another distribution which we suspect may be ferromagnetic is

$$(16)\quad d\nu/dx = \text{const.} \sum_{n=1}^{\infty} (4n^4\pi^2 e^{9x/2} - 6n^2\pi e^{5x/2})\exp(-n^2\pi e^{2x}) \quad ;$$

as explained in [N3], this distribution is related to the Riemann zeta function in a standard way and for it, (12) is equivalent to the Riemann Hypothesis. Some numerical calculations have been performed which indicate that this distribution satisfies (13) [A].

The study and classification of ferromagnetic distributions is not only useful for Euclidean quantum field theory (as in [SiGr]) but also has important applications to the theory of (block-spin) scaling limits of ferromagnetic Ising models. Consider, for example, a translation invariant spin-1/2 Ising model, $\{\sigma_\xi : \xi \in Z^d\}$, with pair ferromagnetic interactions in zero external field at its critical point and let us suppose that this infinite system was obtained as the thermodynamic limit of finite-volume systems with free boundary conditions so that $E(\sigma_\xi) \equiv 0$. The block spin variables of [GJ] are then given by

$$\nu_x(\ell) = \ell^{-dp/2} \sum_{\xi \in C_x} \sigma_\xi$$

where ℓ is an integer and $x \in Z^d$ labels some hypercube C_x of edge length ℓ in R^d. It follows from the GHS inequality [N3] or from the Lee-Yang Theorem [N2] that if p is chosen so that $E(\nu_x(\ell)^2) < K$ for some finite constant K independent of ℓ, then $\{\nu_x(\infty)\} = \lim_{k\to\infty} \{\nu_x(\ell_k)\}$ exists in an appropriate sense for some subsequence $\ell_k \to\infty$. Furthermore, the distribution of each $\nu_x(\infty)$ (or of any positive linear combination of them which has finite variance) is ferromagnetic and any finite subset of them, $\{X_j = \nu_{x_j}(\infty) : j = 1,\ldots,N\}$, satisfies (6), (7) and the inequality of (8) with $Z(h_1,\ldots,h_N)$ replaced by $E(\exp(\Sigma h_j X_j))$.

We now concentrate on giving conditions on the ρ_j's (other than being ferromagnetic) which yield Ising models for which (6) and/or (7) and/or (8) will be valid. For the Lee-Yang property, Theorem 3 below is "best possible"; for the GHS inequality, Theorems 4 and 5 seem fairly good but are probably not definitive; and for the Gaussian inequalities (8), there are presently only conjectures. We say a distribution ρ in $\mathcal{E}$ belongs to the class $\mathcal{L}$ if a random variable X distributed by ρ satisfies (12); the following theorem is from [N1].

<u>THEOREM 3</u>. Let $\{X_j : j = 1,\ldots,N\}$ be a general Ising model defined by (1) with each ρ_j in $\mathcal{L}$; then the Lee-Yang property (6) is valid.

Polya was particularly interested in distributions satisfying (12) because of their relation to the Riemann Hypothesis; he consequently obtained many examples of distributions in $\mathcal{L}$ including [Po; pp. 241,277] (15) for $b \geq 0$ and:

$$(17)\quad d\nu/dx = \begin{cases} \text{const. } (1-x^{2q})^{(a-1)} & , |x| \leq 1 \\ 0 & , |x| > 1 \end{cases} \; ; a > 0 \text{ , } q = 1,2,\ldots$$

(18) $d\nu/dx = \text{const.}\ \exp(-ax^{4q} + bx^{2q} + cx^2)$;

$a > 0\ ,\ b \in R\ ,\ c \geq 0\ ,\ q = 1,2,\ldots$.

With q = 1, (18) yields (11) while (17) is the one-dimensional marginal distribution of uniform surface measure on the unit sphere in R^{2a+1} (for 2a+1 an integer). As we shall see below, the distributions of (17) satisfy inequality (13) for $a \geq 1$ so that they are suitable candidates for being ferromagnetic. Polya also obtained various sorts of sufficient conditions for a distribution to belong to $\mathcal{L}$; for example, an absolutely continuous measure ν in $\mathcal{E}$ belongs to $\mathcal{L}$ if it satisfies either of the following conditions [Po; pp. 187,191]:

(19) $d\nu/dx$ is nondecreasing on $(0,1)$ and vanishes on $(1,\infty)$.

(20) $d\nu/dx$ is nonincreasing and concave on $(0,\infty)$.

Our final example of a distribution in $\mathcal{L}$ can be shown, by direct calculation, not to satisfy (13) and it is therefore not a ferromagnetic distribution;

(21) $d\nu/dx = \text{const.}\ (1+ax^2)\exp(-x^2/2)$; $a > 0$.

Conditions on the ρ_j's which are presently known to yield a GHS inequality are more complicated than those needed in Theorem 3 for the Lee-Yang property; in the following definition, we follow [E,EMN,Sy2]. Given ρ in $\mathcal{E}$, let X^α ($\alpha = 1,\dots,4$) be four independent random variables distributed by ρ, and let $Q^\alpha = \sum_{\beta=1}^{4} B_{\alpha\beta} X^\beta$, where

$$B = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{bmatrix}; \tag{22}$$

we will say ρ belongs to the class $\mathcal{G}_-$, if

$$E\left(\prod_{\alpha=1}^{4} [Q^\alpha]^{m^\alpha}\right) \geq 0 \tag{23}$$

for any choice of nonnegative integers $m^1,\dots,m^4$.

<u>THEOREM 4</u>. [SM,EMN,Sy2]. Let $\{X_j : j = 1,\dots,N\}$ be a general Ising model defined by (1) with each ρ_j in $\mathcal{G}_-$; then the GHS inequality (7) is valid.

The proof of Theorem 4 is based on first showing that the X_j's satisfy a multivariate version of (23) which in turn implies the validity of the GHS inequality; as shown in [SM,Sy2], these multivariate inequalities also imply the Lebowitz inequalities of [L]; it also follows from the multivariate inequalities that all ferromagnetic measures belong to $\mathcal{G}_-$. A class of distributions belonging to $\mathcal{G}_-$ was isolated in [E] and independently in [Sy2]:

$$(24)\quad d\nu/dx = \text{const.}\ \exp(-V(x))\ ;\ V = \sum_{k=1}^{\infty} a_k x^{2k}\ \text{ entire },$$

$$a_1 \in R\ ,\ a_k \geq 0\ (k \geq 2)\ .$$

These distributions are special cases of the following more general result [EN, EMN].

<u>THEOREM 5</u>. An absolutely continuous measure ν in $\mathcal{E}$ belongs to $\mathcal{G}_-$ if either of the following conditions is satisfied:

$$(25)\quad d\nu/dx = \text{const.}\ \exp(-V(x))\ ;\ V \in C^1(R), V'\ \text{ convex on } (0,\infty)$$

or

$$(26)\quad d\nu/dx = \begin{cases} \text{const.}\ \exp(-V(x)) , & |x| < A \\ 0 , & |x| \geq A \end{cases} ;$$

$$V \in C^1(-A,A), V'\ \text{ convex on } (0,A)\ .$$

The example of (21) shows that the class $\mathcal{L}$ is not contained in $\mathcal{G}_-$; we give below an example which shows that $\mathcal{G}_-$ is not contained in $\mathcal{L}$. Before going on to that example, we briefly consider the situation as to general Ising models satisfying the Gaussian inequalities (8). At the present time, there is no condition on the ρ_j's, other than their being ferromagnetic, which is known to imply (8); there are however several conjectures. It was conjectured in [EMN] for example that each ρ_j belonging to $\mathcal{G}_-$ is sufficient to imply (8); we further conjecture (based on the results of [N2]) that each ρ_j belonging to $\mathcal{L}$ is sufficient as well.

The next proposition is based on fairly straightforward calculations [EMN].

PROPOSITION 5. Let $\nu_a(x) = a\delta(x) + (1-a)(\delta(x-1) + \delta(x+1))/2$ for $0 \leq a < 1$ and let X be a random variable distributed by ν_a; then X satisfies (12) iff $0 \leq a \leq \frac{1}{2}$, X satisfies (13) iff $0 \leq a \leq \frac{2}{3}$, and X satisfies (14) iff $0 \leq a \leq \frac{2}{3}$; moreover ν_a is ferromagnetic iff $0 \leq a \leq \frac{1}{2}$, ν_a belongs to $\mathcal{L}$ iff $0 \leq a \leq \frac{1}{2}$, and ν_a belongs to $\mathcal{G}_-$ iff $0 \leq a \leq \frac{2}{3}$.

We then have as an immediate consequence of Theorems 1, 3 and 4.

THEOREM 6. Let $\{X_j : j = 1, \ldots, N\}$ be an Ising model defined by (1) with each $\rho_j(x) = a\delta(x) + (1-a)(\delta(x-1) + \delta(x+1))/2$; then for $0 \leq a \leq \frac{1}{2}$ and any choice of (nonnegative) J_{jk}'s, $\{X_j\}$ satisfies the Lee-Yang property (6), the GHS inequality (7), the Gaussian inequalities (8) and other spin-1/2 results as in [L,Pe,Syl]; for $\frac{1}{2} < a \leq \frac{2}{3}$, $\{X_j\}$ satisfies the GHS inequality and the Lebowitz inequalities of [L] for any J_{jk}'s but does not satisfy the Lee-Yang Theorem for an arbitrary choice of J_{jk}'s; for $\frac{2}{3} < a < 1$, $\{X_j\}$ satisfies neither the Lee-Yang property nor the GHS inequality nor the Guassian inequalities for arbitrary J_{jk}'s.

As explained above, we conjecture that the Gaussian inequalities remain valid for the model of Theorem 6 with $\frac{1}{2} < a \leq \frac{2}{3}$ based on the fact that the ρ_j's belong to $\mathcal{G}_-$. In any case, Theorem 6 is already an interesting result in that it gives, for $\frac{1}{2} < a \leq \frac{2}{3}$, a general Ising model whose individual spin distributions are not ferromagnetic but which nevertheless satisfies the GHS inequality. The validity of the GHS inequality here, can therefore not be established simply by using the spin-1/2 result together with Griffiths' analogue spin method; it is instead based on Theorem 4 in an essential way. The phase properties and critical phenomena associated with three valued spin Ising models should be much richer than those of spin-1/2 models; for example, they should exhibit (at least for a close to 1) phase transitions at non-zero external field with corresponding triple points and a tricritical point at some particular value a* of the parameter a [BEG]. As long as the GHS inequality is valid, a phase transition can occur only at zero external field [Pr]; thus by Proposition 5 and Theorem 6, a tricritical point can only occur with $a^* \geq \frac{2}{3}$. It is our hope that this preliminary result will prove useful in further investigations of such models. We note that without Proposition 5 it would follow from the Lee-Yang property only that $a^* \geq \frac{1}{2}$ and we further note that mean-field calculations actually give $a^* = \frac{2}{3}$ [BEG].

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Colloques Internationaux C.N.R.S.
N° 248 – Les méthodes mathématiques de la théorie quantique des champs

PRESENT STATUS OF CANONICAL QUANTUM FIELD THEORY *

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RESUME : On présente la quantisation canonique de la théorie des champs classique non linéaire relativiste. On montre d'abord que la théorie possède une structure algébrique, ensuite on donne une représentation par les opérateurs de cette structure. La construction des champs asymptotique $\hat{\Phi}_{in}$ et $\hat{\Phi}_{out}$ est donnée en même temps qu'une forme explicite de l'opérateur de diffusion.

Abstract

The canonical quantization of nonlinear relativistic classical field theory is presented. First it is shown that classical field theory possesses an algebraic structure precisely such as the quantum field theory in L.S.Z. formulation. Next an operator representation of this structure is given and the explicit form of a local interacting relativistic quantum field $\hat{\Phi}$ is obtained. The construction of asymptotic local relativistic quantum fields $\hat{\Phi}_{in}$ and $\hat{\Phi}_{out}$ as well as an explicit form of scattering operator is also given.

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Introduction

There is an impressive progress in $\lambda \Phi^4$ quantum field theory in two and threedimensional space time due to the recent results of Glimm and Jaffe [1] Feldman and Osterwarder [2] and Magnen and Seneour [3].

In case of $\lambda \Phi^4$ theory in four-dimensional space-time two distinct approaches were recently proposed: the first one of Glimm and Jaffe [4] and Guerra [5] is based on the use of statistical methods and four-dimensional Ising model; the second one, proposed by Schrader [6], is based on the Wilson-Zimmerman operator product expansion. In both approaches a series of fundamental mathematical problems must be solved before one is able to prove the existence and nontriviality of $\lambda \Phi^4_4$ theory.

It seems therefore useful to analyse other alternative approaches of constructing $\lambda \Phi^4_4$ quantum field theory. Particularly attractive is an approach based on the second quantization of classical $\lambda \Phi^4_4$ theory. This approach was proposed and developed by Segal in a series of papers [7]. The present work summarizes recent results obtained in this domain by the Warsaw group.

The recent progress in the domain of nonlinear relativistic wave equations , due mainly to Morawetz and Strauss [8], allows to prove that on the manifold of solutions of these equations there exists the algebraic structure, precisely such as that postulated on quantum level e.g. in L.S.Z. formalism [9]. Thus the problem of construction of relativistic interacting and asymptotic quantum fields is reduced to the problem of construction of operator representation of classical algebraic structure [10]. This problem is solved in Sec. III and IV. It is interesting that this method of quantization of classical field theory provides a certain form of generalized normal ordering for products of interacting fields which is local and relativistic [11]. Finally, the explicit form of quantum scattering operator is derived and its various properties, like relativistic covariance nontriviality, analyticity in coupling constant etc. are analysed. [12]

II. Symplectic Structure of Classical Nonlinear Field Theory.

Consider the nonlinear relativistic wave equation

$$(\Box + m^2)\,\Phi(x) = \lambda \Phi^3(x)\ , \quad \lambda < 0\ , \quad x = (t, \underset{\sim}{x}) \in R^4\ , \qquad /2.1/$$

with the initial conditions

$$\Phi(0, \underset{\sim}{x}) = \varphi(\underset{\sim}{x})\ , \quad \Pi(0, \underset{\sim}{x}) = \pi(\underset{\sim}{x})\ . \qquad /2.2/$$

It was shown by Morawetz and Strauss [8] that for every given Cauchy data (φ, π) from the Banach space $\mathcal{F}$ (defined in App. A) there exists the unique solution $\Phi(x)$ of Eq./2.1/ and the pair $\Phi_{in}(x)$ and $\Phi_{out}(x)$ of the solutions of the free Klein-Gordon equation such that

$$\Phi_{in}(t, \underset{\sim}{x}) \xleftarrow[t \to -\infty]{} \Phi(t, \underset{\sim}{x}) \xrightarrow[t \to \infty]{} \Phi_{out}(t, \underset{\sim}{x})\ , \qquad /2.3/$$

in the energy norm given by the formula

$$\| \Phi(t, \cdot) \|_E^2 = \int d^3\underset{\sim}{x} \left[\Pi^2(t, \underset{\sim}{x}) + |\nabla \Phi(t, \underset{\sim}{x})|^2 + m^2 \Phi^2(t, \underset{\sim}{x}) \right]. \qquad /2.4/$$

The Cauchy data $\zeta \equiv (\varphi, \pi)$ may be used as canonical variables. Let $F(\zeta)$ be a functional over the space $\mathcal{F}$. We say that the functional F possesses a Frechet differential at a point ζ if there exists a linear continuous map $DF[\zeta](\zeta_1)$ of the space $\mathcal{F}$ into R^1 such that

$$F(\zeta + \zeta_1) - F(\zeta) = DF[\zeta](\zeta_1) + r(\zeta : \zeta_1)\ , \qquad /2.5/$$

where

$$\lim_{\zeta_1 \to 0} \frac{|r(\zeta ; \zeta_1)|}{\| \zeta_1 \|_{\mathcal{F}}} = 0\ . \qquad /2.6/$$

The value of $DF[\zeta](\zeta_1)$ on a given $\zeta_1 \in \mathcal{F}$ is called the differential of the functional F and defines the Frechet derivative $\delta F / \delta \zeta$:

$$DF[\zeta](\zeta_1) = \left\langle \frac{\delta F}{\delta \zeta}, \zeta_1 \right\rangle = \left\langle \frac{\delta F}{\delta \varphi}, \varphi_1 \right\rangle + \left\langle \frac{\delta F}{\delta \pi}, \pi_1 \right\rangle. \qquad /2.7/$$

Hence the Frechet derivative $\delta F / \delta \zeta$ is in general an element of the dual space $\mathcal{F}'$ to $\mathcal{F}$.

The Poisson bracket $\{F, G\}$ of two functionals over the space $\mathcal{F}$ is formallly defined by the formula

$$\{F, G\} = \int_{R^3} d^3 z \left(\frac{\delta F}{\delta \varphi(z)} \frac{\delta G}{\delta \pi(z)} - \frac{\delta F}{\delta \pi(z)} \frac{\delta G}{\delta \varphi(z)} \right) \qquad /2.8/$$

If σ_2 is the Pauli matrix $\sigma_2 = \left\| \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right\|$ then $\sigma_2 \frac{\delta F}{\delta \zeta} = \left(-\frac{\delta F}{\delta \pi}, \frac{\delta F}{\delta \varphi} \right)$ and the Poisson bracket may be written in the form

$$\{F, G\} = DG[\zeta](\zeta_F) = -DF[\zeta](\zeta_G) \qquad /2.9/$$

where for a functional X we set $\zeta_X = \sigma_2 \frac{\delta X}{\delta \zeta}$. Hence we see that a Poisson bracket of two smooth functionals is well defined if either ζ_F or ζ_G is an element of the carrier Banach space $\mathcal{F}$.

A. Interacting Fields.

We now derive the explicit form of of commutator function for interacting classical fields. Denote by $\Delta^\lambda[x, y | \Phi]$ the Green function of the linear equation

$$(\Box + m^2) u(x) = V(x) u(x) \quad , \quad V(x) = 3\lambda \Phi^2(x) \quad , \qquad /2.10/$$

satisfying for $t_x = t_y = \tau$ the initial conditions :

$$\Delta^\lambda[\tau, \underline{x}, \tau, \underline{y} | \Phi] = 0 \quad , \quad (\partial_t \Delta^\lambda)[\tau, \underline{x}, \tau, \underline{y} | \Phi] = -\delta^3(\underline{x} - \underline{y}). \qquad /2.11/$$

The function Δ^λ can be written in the form of the series

$$\Delta^\lambda[x, y | \Phi] = \Delta(x - y) + \qquad /2.12/$$

$$+ \sum_{n=1}^{\infty} (3\lambda)^n \int\!\!\int_{y^0}^{x^0} \cdots \int_{y^0_{n-1}}^{x^0_{n-1}} \Delta(x - x_1) \Phi^2(x_1) \Delta(x_1 - x_2) \cdots \Phi^2(x_n) \Delta(x_n - y) \, d^4x_1 \cdots d^4x_n$$

This series, when smeared out with the function $\alpha(\underline{y}) \in S(R^3)$ is convergent in the energy norm: indeed

$$\| \Delta^\lambda[\cdot, \alpha | \Phi] \|_E \leqslant \| \alpha \|_{L^2} \exp \left[\frac{1}{m} \int_\tau^t \sup_{\underline{x}} | V(\underline{x}, t') | \, dt' \right]$$

Now we have

Theorem 2.1. The interacting classical field $\Phi(x)$ is a local field which satisfies the following commutation relations

$$\{ \Phi(x), \Phi(y) \} = \Delta^\lambda[x, y | \Phi]. \; \blacktriangledown \qquad /2.13/$$

Proof. The proof of Theorem 2.1 follows from Proposition 4 and Eq./3.5/ of [9] One may however give a direct proof which is very instructive. Indeed using the general expression /2.8/ for the Poisson bracket and Eq. /2.1/ one finds:

$$(\Box_x + m^2)\{\Phi(x), \Phi(y)\} = 3\lambda \Phi^2(x) \{\Phi(x), \Phi(y)\}$$

Moreover using the fact that $\Phi(0,\underline{x}) = \varphi(\underline{x})$ and $\Pi(0,\underline{x}) = \pi(\underline{x})$ we find

$$\{\Phi(0,\underline{x}), \Phi(0,\underline{y})\} = 0 \quad , \quad \partial_{t_x}\{\Phi(0,\underline{x}), \Phi(0,\underline{y})\} = -\delta^3(\underline{x}-\underline{y})$$

Hence, by Eqs /2.10/ and /2.11/ we obtain

$$\{\Phi(x), \Phi(y)\} = \Delta^\lambda[x, y \mid \Phi] \quad .$$

If x and y are space-like separated then by virtue of /2.12/ $\Delta^\lambda[x,y|\Phi] = 0$ Hence the interacting classical field Φ is local.

It also follows from the formula /2.12/ that $\Phi(x)$ is a canonical field. Indeed if $t_x = t_y$ then all terms on r.h.s. of /2.12/ vanish: hence

$$\{\Phi(t,\underline{x}), \Phi(t,\underline{y})\} = 0 \qquad /2.14/$$

If we take derivative ∂_{t_y} of both sides of Eq./2.13/ and set $t_x = t_y$ then only the first term on r.h.s. of Eq./2.12/ survives and we have

$$\{\Phi(t,\underline{x}), \Pi(t,\underline{y})\} = \delta^{(3)}(\underline{x}-\underline{y}) \qquad /2.15/$$

B. Asymptotic Fields.

For the analysis of properties of evolution and scattering operator it is useful to introduce a family Φ_τ, $-\infty \leq \tau \leq \infty$ of free fields. It was shown in [8] that for a fixed τ the functions $\Phi(\tau, \cdot)$ and $\Pi(t, \ )$ on a hyperplane $t = \tau$ belong to the Banach space of initial data and therefore the pair $(\Phi(\tau,\cdot), \Pi(\tau,\cdot))$ may be used for the construction of a new field $\Phi_\tau(t, x)$ by the formula

$$\Phi_\tau(t,\underline{x}) = -\int \Delta(t, \underline{x}-\underline{y}) \Pi(\tau,\underline{y}) d^3\underline{y} + \int (\partial_t \Delta)(t, \underline{x}-\underline{y}) \Phi(\tau,\underline{y}) d^3\underline{y}, \qquad /2.16/$$

We have in the energy norm [8]:

$$\lim_{\tau \to \mp\infty} \Phi_\tau = \Phi_{\substack{in \\ out}} \quad . \qquad /2.17/$$

Using Eqs. /2.14/ and /2.15/ and formula /2.16/ one obtains

$$\{\Phi_\tau(x), \Phi_\tau(y)\} = \Delta(x-y) \, , \qquad /2.18/$$

i.e. $\Phi_\tau(x)$ is a local canonical free field. Moreover we have:

Theorem 2.2 The fields $\Phi_{in}(x)$ and $\Phi_{out}(x)$ satisfy the following commutation relations

$$\{ \Phi_{\substack{in \\ out}}(x) , \Phi_{\substack{in \\ out}}(y) \} = \Delta(x-y; m) . \qquad /2.19/$$

/For the proof look[9]Theorem 3/

It follows from /2.19/ that asymptotic fields, similarly like interacting and Φ_τ fields are local canonical fields.

C. Relativistic Covariance.

The nonlinear equation /2.1/ may be derived from the following Lagrangian density:

$$\mathcal{L}(x) = 1/2 \, (\Phi_{,\mu} \Phi^{,\mu} + m^2 \Phi^2) - \lambda/4 \, \Phi^4 . \qquad /2.20/$$

Using the standard technique one derives the following form for the energy - momentum tensor associated with the density /2.20/

$$T_{\mu\nu}(x) = \Phi_{,\mu}(x) \Phi^{,\mu}(x) - g_{\mu\nu} \mathcal{L}(x) . \qquad /2.21/$$

Let σ be a space-like surface in the Minkowski space. Then the integrals

$$P_\mu(\sigma) = \int_\sigma d\sigma^\nu T_{\mu\nu} \quad , \quad M_{\mu\nu}(\sigma) = \int_\sigma d\sigma^\lambda (x_\mu T_{\nu\lambda} - x_\nu T_{\mu\lambda}), \qquad /2.22/$$

are constants of motion. One verifies, using Eqs. /2.14/ and /2.15/ that the quantities /2.22/ satisfy the standard commutation relations for generators of the Poincare Lie algebra and the scalar field. Using the free Lagrangians $\mathcal{L}_{in}(x)$ and $\mathcal{L}_{out}(x)$ one derives the corresponding expressions for generators of Poincare group for "in" and "out" fields. We have:

Proposition 2.3. The generators of Poincare group for interacting "in" and "out" fields, as functionals of initial data satisfy the following equalities

$$P_\mu^{in} = P_\mu = P_\mu^{out} , \quad M_{\mu\nu}^{in} = M_{\mu\nu} = M_{\mu\nu}^{out} , \quad \mu,\nu = 0,1,2,3. \qquad /2.23/$$

/For the proof cf. Bałaban and Rączka [10], Sec. 4/

It is instructive to derive the equality /2.23/ directly to see how the bilinear in fields generators for "in" or "out" representation can coincide with quatrilinear in fields generators for interacting representation. We show this in detail for the generator P_0. By virtue of Eq./2.22/ for a space-like surface $\sigma(t)$ perpendicular to the time axis, we have

$$P_0(\sigma(t)) = \frac{1}{2}\int_{\sigma(t)} d^3\underline{x}\left(\Pi^2 + |\nabla\Phi|^2 + m^2\Phi^2 - \frac{\lambda}{2}\Phi^4\right)(t,\underline{x}). \qquad /2.24/$$

We shall evaluate the expression /2.24/ for $t\to-\infty$. For the interaction term $\lambda\Phi^4$ utilizing the fact that $\int dt \max_{\underline{x}}|\phi|^2 \leq \|\Phi\|^2_{\mathcal{F}} < C$ and that $\int d^3\underline{x}\, \Phi^2(t,\underline{x})$ is smaller than the total energy E we have $\int d^3\underline{x}\,\phi^4 \leq \max_{\underline{x}}|\Phi^2|(t,\underline{x})\int d^3\underline{x}\,\Phi^2$ and $\int dt d^3\underline{x}\,\Phi^4 \leq E\int dt \max_{\underline{x}}|\phi|^2 \leq EC$.
Hence there exists a sequence $\{t_n\}$, $t_n\to-\infty$ such that

$$\int d^3\underline{x}\,\Phi^4(t_n,\underline{x}) \xrightarrow[t_n\to-\infty]{} 0$$

Hence by virtue of Eq./2.4/ we obtain:

$$\lim_{t\to-\infty} P_0(\sigma(t)) = \lim_{t\to-\infty} \tfrac{1}{2}\|\Phi(t,\cdot)\|^2_E = \frac{1}{2}\|\Phi_{in}(t,\cdot)\|^2_E = P_0^{in}.$$

Because P_0 is time independent we have $P_0 = P_0^{in}$ The derivation of Eq./2.23/ for remaining generators may be performed in a similar manner.

D. Scattering Operator and Analyticity in Coupling Constant.

It follows from Theorem 2.2 that the scattering operator defined in the space $\mathcal{F}$ by the formula $S:\Phi_{in}\to\Phi_{out}$ is canonical. In addition since

$$\|\Phi_{out}\|^2_E = \|S(\Phi_{in})\|^2_E = 2P_0^{out} = 2P_0^{in} = \|\Phi_{in}\|^2_E, \qquad /2.25/$$

the S-operator is isometric.

It was proven in [13] that the scattering operator is Poincare invariant on $\mathcal{F}$ i.e. for every $(a,\Lambda)\in\mathcal{P}$ we have:

$$U_{(a,\Lambda)}S = S\,U_{(a,\Lambda)}. \qquad /2.26/$$

For further applications of classical theory on quantum level it is crucial to know the analyticity properties of the scattering operator with respect to coupling constant λ and initial data (φ,π) For positive coupling constant the energy operator

$$P_0 = \int d^3\underset{\sim}{x} \left[\frac{1}{2}\left(\Pi^2 + |\nabla\Phi|^2 + m^2\Phi^2\right)(t,\underset{\sim}{x}) - \frac{\lambda}{4}\Phi^4\right]$$

consists of two parts with opposite signs. Hence the solution $\Phi(x)$ can increase arbitrarily and the asymptotic fields do not exist for all initial data from $\mathcal{F}$. Therefore one cannot expect the analyticity of S with respect to the coupling constant λ for all initial data from $\mathcal{F}$. However, if one restricts oneself to the space defined as the closure of the smooth free solutions in the norm

$$\|\Phi\|_Y = \sup_{-\infty<t<\infty} \left(\|\Phi(t)\|_E + (1+|t|^{3/2}) \sup_{\underset{\sim}{x}} |\Phi(t,\underset{\sim}{x})|\right) \qquad /2.27/$$

then $Y \subset \mathcal{F}$ and for small initial data we have

Theorem 2.4. Consider S as the operator $S:(\Phi_{in},\lambda) \to \Phi_{out}$ with the domain D

$$D = \left\{(\Phi_{in},\lambda) \mid \Phi_{in} \in Y,\ \lambda \in C^1,\ |\lambda|\,\|\Phi_{in}\|_Y < \eta\right\}$$

and with range Y. Then S is complex analytic on this domain ▼

/For the proof cf. Raczka and Strauss [14]/

It can be also shown that S cannot be analytic in λ for all initial data from $\mathcal{F}$. For details see [14].

It is very interesting that the inverse scattering problem can be solved in the nonlinear relativistic field theory for large class of interactions. In particular in $\lambda\Phi^4$ theory we have

Theorem 2.5. The coupling constant λ is determined by the scattering operator

$$\lambda = \lim_{\varepsilon \to 0_+} 1/6\varepsilon^4 \, W\left(S(2\varepsilon\Phi_{in}), S(\varepsilon\Phi_{in})\right) \qquad /2.28/$$

where W is the Wronskian

$$W(\Phi,\Psi) = \int d^3\underset{\sim}{x} \left(\Phi\partial_t\Psi - \partial_t\Phi\Psi\right)(t,\underset{\sim}{x}) \quad ▼$$

/for the proof cf. Morawetz and Strauss [13]/

The formula /2.28/ implies that for $\lambda \neq 0$ S is the nonlinear operator. In fact if S would be linear then

$$W(S(2\varepsilon\Phi_{in}), S(\varepsilon\Phi_{in})) = 2\varepsilon^2 W(\Phi_{in},\Phi_{in}) = 0$$

The nonlinearity of S implies that scattering is nontrivial in classical theory of self-interacting scalar fields. It should be stressed that in quantum case this problem is for the time being open. /see Wilson [15] and Rączka [16] for a discussion of this problem/

We conclude this section giving a very impressive result of classical field theory

Theorem 2.6. Let $F(\Phi)$ be the interacting term in Eq. /2.1/ given by an analytic function defined in the neighbourhood of the origin, which is odd and $F'(0)=0$ Then F is determined by S_{φ} /For the proof cf. Morawetz and Strauss [13] /

Thus the inverse scattering problem is solved completely in classical nonlinear field theory.

III. Operator Representation of Symplectic Structure.

We have shown in Sec.II that the algebraic structure of classical nonlinear relativistic field theory expressed in terms of Poison brackets is precisely such as that postulated in quantum field theory e.g. in formalism of Lehman, Symanzik and Zimmerman. In particular the asymptotic fields $\bar{\Phi}_{in}$ and $\bar{\Phi}_{out}$ are relativistic, local and canonical and interpolating field Φ is relativistic and local. Moreover the representation "in" interpolating and "out" of Poincare group coincide. Hence if one finds an operator representation of symplectic structure one will lift all desired properties of asymptotic and interacting fields onto operator level and one obtains a model of interacting quantum field theory. We shall now construct this operator representation of the Lie algebra of Poisson brackets.

It will be evident from the next considerations that in case of nonlinear field theory the most important role is played by a vector space Ω of functionals over the space $\mathcal{F}$, defined in the following manner.

Definition 1. A functional F over $\mathcal{F}$ belongs to Ω if i/ $F \in C^{\infty}(\mathcal{F})$

ii/ $D^k F[\mathfrak{z}](\mathfrak{z}_1, \mathfrak{z}_2, \dots, \mathfrak{z}_k)$ is bounded on bounded subsets of $\mathcal{F}^{k+1}$

iii/ $D^k\left(\sigma_2 \frac{\delta F}{\delta \mathfrak{z}}\right)(\mathfrak{z}_1, \mathfrak{z}_2, \dots, \mathfrak{z}_k) \in \mathcal{F}$, $k = 0, 1, 2, \dots$ ▼

If $F, G \in \Omega$ then $\{F, G\}$ is well defined and also belongs to Ω : indeed by virtue of /2.9/ one obtains

$$\sigma_2 \frac{\delta}{\delta \mathfrak{z}}\{F, G\} = \sigma_2 \frac{\delta}{\delta \mathfrak{z}} D G[\mathfrak{z}](\mathfrak{z}_F) = D \sigma_2 \frac{\delta}{\delta \mathfrak{z}} G[\mathfrak{z}](\mathfrak{z}_F) +$$

$$+ D G[\mathfrak{z}]\left(\sigma_2 \frac{\delta}{\delta \mathfrak{z}} \mathfrak{z}_F\right) = D \sigma_2 \frac{\delta}{\delta \mathfrak{z}} G[\mathfrak{z}](\mathfrak{z}_F) - D \sigma_2 \frac{\delta}{\delta \mathfrak{z}} F[\mathfrak{z}](\mathfrak{z}_G),$$

which is an element of $\mathcal{F}$. Similarly for $k=1,2,\dots$ we have :

$$D^k \sigma_2 \frac{\delta}{\delta \mathfrak{z}}\{F, G\}[\mathfrak{z}](\mathfrak{z}_1, \dots \mathfrak{z}_k) = D^{k+1} \sigma_2 \frac{\delta}{\delta \mathfrak{z}} G[\mathfrak{z}](\mathfrak{z}_F, \mathfrak{z}_1, \dots, \mathfrak{z}_k) - D^{k+1} \sigma_2 \frac{\delta}{\delta \mathfrak{z}} F[\mathfrak{z}](\mathfrak{z}_G, \mathfrak{z}_1, \dots, \mathfrak{z}_k),$$

which is also an element of $\mathcal{F}$. Therefore $\{F, G\} \in \Omega$. Similarly $\{\{F, G\}, H\}$ is in Ω if $F, G, H \in \Omega$ Consequently the vector space Ω is a Lie algebra under Poisson brackets.

We now construct two convenient carrier spaces. We take as the first carrier space a linear space of C^{∞} functionals $\Psi(\cdot)$ on $\mathcal{F}$. with the topology defined by the system of seminorms

$$\|\Psi\|_{B,m} = \sup_{\mathfrak{z} \in B} \sup_{\substack{\|\mathfrak{z}_i\|_F \leq 1 \\ i=1,\dots,m}} |D^m \Psi[\mathfrak{z}](\mathfrak{z}_1, \dots, \mathfrak{z}_m)|, \quad m = 0, 1, \dots, \qquad /3.1/$$

where B is an arbitrary bounded subset of $\mathcal{F}$. Since this space resembles Schwartz space $\mathcal{E}(R^n)$ we shall denote it by the symbol $\mathcal{E}(\mathcal{F})$.

The second space $\mathcal{K}(\mathcal{F})$ is the linear space $\Omega \subset \mathcal{E}(\mathcal{F})$ with a topology defined by seminorms

$$\|\Psi\|_{\mathcal{K}_{B,m}} = \sup_{\mathfrak{z} \in B} \sup_{\substack{\|\mathfrak{z}_i\|_F \leq 1 \\ i=1,\dots,m}} \left\|\sigma_2 \frac{\delta}{\delta \mathfrak{z}} D^m \Psi[\mathfrak{z}](\mathfrak{z}_1, \dots, \mathfrak{z}_m)\right\|_F . \qquad /3.2/$$

We now give the representation of Lie algebra Ω in these spaces. We denote for the sake of simlicity by D_F the first order differential operator given by the formula

$$D_F = \int_{R^3} d^3\underset{\sim}{z} \left(\frac{\delta F}{\delta \varphi(\underset{\sim}{z})} \frac{\delta}{\delta \pi(\underset{\sim}{z})} - \frac{\delta F}{\delta \pi(\underset{\sim}{z})} \frac{\delta}{\delta \varphi(\underset{\sim}{z})} \right) \qquad /3.3/$$

Theorem 3.1. Let F be in Ω Then the operator $\hat{F}$ associated with a given functional F by the formula

$$\hat{F}[\mathfrak{z}] = F[\mathfrak{z}] - \tfrac{1}{2} D F[\mathfrak{z}](\mathfrak{z}) + i D_F \qquad /3.4/$$

defines the continuous map of the spaces $\mathcal{E}(\mathcal{F})$ and $\mathcal{K}(\mathcal{F})$ into itself. If $F, G \in \Omega$ then for Ψ in $\mathcal{E}$ or $\mathcal{K}$ we have

$$[\hat{F},\hat{G}]\psi = i\{\hat{F},\hat{G}\}\psi. \blacktriangledown \qquad /3.5/$$

/For the proof cf. [10] Theorem 1/

One readily verifies using /3.4/ that if $F \in \Omega$ and $\varrho(\cdot)$ is C^∞ then

$$\hat{\varrho}(F) = \varrho(F) + \varrho'(F)[\hat{F} - F]. \qquad /3.6/$$

The formula /3.6/ implies that $\widehat{F^n} \neq (\hat{F})^n$ in general. Hence the quantization formula /3.4/ applied for a product Φ^n of fields gives some "renormalization" counter terms. /Cf. Sec. VI/

IV. Construction of Interacting and Asymptotic Local Quantum Fields.

Let $\Phi[x|\varphi,\pi]$ be a solution of the dynamical equation /2.1/. We begin the construction of a quantum field $\hat{\Phi}(t,\underline{x})$ by quantizing first the free field $\Phi_\tau(t,\underline{x})$. Let $\hat{\Phi}_\tau(t,\alpha)$ denote the operator field obtained from

$$\Phi_\tau[t,\alpha|\varphi,\pi] = \int d^3\underline{x}\; \alpha(\underline{x})\, \Phi_\tau[t,\underline{x}|\varphi,\pi], \quad \alpha \in S(R^3),$$

by formula /3.4/. Then we have

Theorem 4.1. The operator field $\hat{\Phi}_\tau(t,\alpha)$ for any $\tau \in (-\infty,\infty)$ and $\alpha \in S(R^3)$ is the continuous mapping of the spaces $\mathcal{E}(\mathcal{F})$ and $\mathcal{K}(\mathcal{F})$ into itself and satisfies on each of these spaces the commutation relations:

$$[\hat{\Phi}_\tau(t,\alpha), \hat{\Phi}_\tau(r,\beta)] = i\int d^3\underline{x}\, d^3\underline{y}\; \alpha(\underline{x})\,\Delta(t-r,\underline{x}-\underline{y})\,\beta(\underline{y}). \qquad /4.1/$$

The field $\hat{\Phi}_\tau(t,\alpha)$ is the strongly continuous function of τ and t. $\blacktriangledown$

/For the proof cf. [10] Theorem 2/

Remark 1. For simplicity of notation in the following we shall write formulae /4.1/ and similar formula in the unsmeared form:

$$[\hat{\Phi}_\tau(x), \hat{\Phi}_\tau(y)] = i\Delta(x-y).$$

The operator field $\hat{\Phi}_\tau$ plays the basic role in the determination of the quantum evolution operation $\hat{U}(\tau,\tau_0)$ and the quantum scattering operator $\hat{S}$. These problems will be considered in Section VII

We now describe the quantum interacting field $\hat{\Phi}$ associated with the classical field $\Phi[x,\varphi,\pi]$ by the formula /3.4/.

Theorem 4.2. The operator field $\hat{\Phi}(t,\alpha)$ is the continuous mapping of the spaces $\mathcal{E}(\mathcal{F})$ and $\mathcal{K}(\mathcal{F})$ into itself and satisfies in the distribution sense on each of these spaces the commutation relations

$$[\hat{\Phi}(x),\hat{\Phi}(y)] = i\,\widehat{\Delta^{\lambda}}[x,y\mid\Phi] \qquad /4.3/$$

where $\Delta^{\lambda}[x,y\mid\Phi]$ is given by formula /2.12/. The map $t\to\hat{\Phi}(t,\alpha)$ is strongly continuous. ▼

/For the proof cf. [10] Theorem 3/

Corollary 1. The field $\hat{\Phi}(x)$ is local i.e.

$$[\hat{\Phi}(x),\hat{\Phi}(y)] = 0 \quad \text{if } (x-y)^2 < 0, \qquad /4.4/$$

and satisfies on $\mathcal{E}(\mathcal{F})$ or $\mathcal{K}(\mathcal{F})$ the canonical commutation relations

$$[\hat{\Phi}(t,\underline{x}),\hat{\Pi}(t,\underline{y})] = i\,\delta^{(3)}(\underline{x}-\underline{y}),\quad [\hat{\Phi}(t,\underline{x}),\hat{\Phi}(t,\underline{y})] = [\Pi(t,\underline{x}),\Pi(t,\underline{y})] = 0 \qquad /4.5/$$

Proof.

If $(x-y)^2<0$ then by formula /2.12/ $\Delta^{\lambda}[x,y\mid\Phi]=0$ Similarly, if $t_x=t_y$ then $\partial_{t_y}\Delta^{\lambda}[x,y\mid\Phi]=\delta^3(\underline{x}-\underline{y})$

The formulae /4.5/, /4.3/ and /2.12/ show that the interacting field has the same distributional character sa the free field i.e. they represent the operator valued distributions of $S'(R^4)$ type. Let us note , however, that by Theorem 4.2 $\hat{\Phi}(t,\alpha)\psi$, $\alpha\in S(R^3)$ is the continuous function of t .

One can easily verify that the regularity properties of Φ Φ_τ, Φ_{in} and Φ_{out} fields will not change if we take initial conditions $\mathfrak{z}_{in}=(\varphi_{in},\pi_{in})$ at $t_0=-\infty$.

/Cf. [10] Remark 1 to Lemma 5 of App. A/

This implies that all assertions of Theorems /4.1/ and /4.2/ remain true also for this case.

We now find an equation of motion for the quantum field $\hat{\Phi}(x)$. Acting on the field $\hat{\Phi}(x)$ by the operator $\Box+m^2$ and using Eqs. /2.1/ and /3.4/ one finds that $\hat{\Phi}(x)$ satisfies the following dynamical equation

$$(\Box+m^2)\hat{\Phi}(x)=\lambda\widehat{\Phi^3}(x). \qquad /4.6/$$

By virtue of Eq. /3.6/, the interaction term in Eq. /4.6/ is automatically renormalized: consequently, Eq. /4.6/ represents a meaningful equality on the space $\mathcal{K}$. It should be stressed, however, that the dynamical equation /4.6/ losts its primary meaning as a tool for description of a dynamics of interacting quantum fields: in fact the quantum interacting field is not obtained by a solution of Eq. /4.6/ but is constructed independently from the classical solution $\Phi(x)$ by formula /3.4/.

It is instructive to apply the present quantization method in case of the free field equation $(\Box+m^2)\Phi_0(x)=0$ In this case the solution $\Phi_0[t,x|\varphi,\pi]$ is given by the formula

$$\Phi_0[t,x|\varphi,\pi]=-\int\Delta(t,x-y)\pi(y)d^3y+\int(\partial_t\Delta)(t,x-y)\varphi(y)d^3y$$

Applying the formula /3.4/ one obtains the quantum field $\hat{\Phi}_0(x)$ which satisfies the following commutation relations

$$[\hat{\Phi}_0(x),\hat{\Phi}_0(y)]=i\Delta(x-y)$$

Calculating in the standard manner the creation and annihilation operators one easily verifies that the equation $\hat{a}\Psi_0=0$ is satisfied by the Poincare invariant functional $\Psi_0(\varphi,\pi)=1$ and that the n-particle states are represented by polynomials in canonical variables. Restricting the field $\hat{\Phi}_0$ to the irreducible subspace generated from the vacuum by means of creation operators one obtains a realization which is identical with the conventional Bargmann-Segal representation.

Similarly, the quantization /3.4/ of external field problem

$$(\Box+m^2)\Phi(x)=V(x)\Phi(x) \qquad /4.7/$$

provides by restriction to irreducible vacuum sector of $\mathcal{E}(\mathcal{F})$ the conventional theory.

/For details see Raczka and Vladimirov [17] /.

Let $\hat{\Phi}_{in}(t,\alpha)$ and $\hat{\Phi}_{out}(t,\alpha)$ be the operator fields obtained from classical solutions $\Phi_{in}[t,\alpha|\varphi,\pi]$ and $\Phi_{out}[t,\alpha|\varphi,\pi]$ respectively by formula /3.4/. Then we have

Theorem 4.3. For every $\psi\in\mathcal{E}(\mathcal{F})$ or $\mathcal{K}(\mathcal{F})$ in the strong topology of these spaces we have

$$\lim_{\tau\to\mp\infty}\hat{\Phi}_{\tau}(t,\alpha)\psi=\hat{\Phi}_{\substack{in\\out}}(t,\alpha)\psi. \qquad /4.8/$$

The operator fields $\hat{\Phi}_{in}(t,\alpha)$ and $\hat{\Phi}_{out}(t,\alpha)$ represent the continuous mappings of the spaces $\mathcal{E}(\mathcal{F})$ and $\mathcal{K}(\mathcal{F})$ into themselves and satisfy on each of these spaces the commutation relations

$$[\hat{\Phi}_{\substack{in\\out}}(x),\hat{\Phi}_{\substack{in\\out}}(y)]=i\Delta(x-y) \qquad /4.9/$$

/For the proof cf. [10] Theorem 4 and 5/

V. Relativistic Covariance

Let $\varphi_{in}(\underline{x})=\Phi_{in}(0,\underline{x})$ and $\pi_{in}(\underline{x})=\Pi_{in}(0,\underline{x})$ be initial conditions for classical free field $\Phi_{in}(x)$. Let $\Phi_{in}(x)$ represent initial conditions at $t_0=-\infty$ for the interacting field $\Phi(x)$ which satisfies Eq. /2.1/ The map $(a,\Lambda)\to U_{(a,\Lambda)}$ in the Banach space $\mathcal{F}$ given by the formula $(U_{(a,\Lambda)}\Phi_{in})(x)=\Phi_{in}(\Lambda^{-1}(x-a))$ defines the continuous representation of the Poincare group in the space $\mathcal{F}$. The elements

$$(U_{(a,\Lambda)}\Phi_{in})(0,\underline{x}) \quad \text{and} \quad (U_{(a,\Lambda)}\Pi_{in})(0,\underline{x}) \qquad /5.1/$$

define the element $\zeta_{in}=(\varphi_{in},\pi_{in})$ after the transformation. We shall denote the transformed element /5.1/ by the symbol $U_{(a,\Lambda)}\zeta_{in}$

The map $(a,\Lambda)\to\hat{U}_{(a,\Lambda)}$ in the space $\mathcal{E}(\mathcal{F})$ given by the formula

$$(\hat{U}_{(a,\Lambda)}\psi)(\zeta_{in})=\psi(U^{-1}_{(a,\Lambda)}\zeta_{in}) \qquad /5.2/$$

defines the continuous representation of the Poincare group in $\mathcal{E}(\mathcal{F})$ or $\mathcal{K}(\mathcal{F})$.

We now show the covariance property of the quantum field $\hat{\Phi}$.

Proposition 5.1. The field $\hat{\Phi}(x)$ has the following transformation properties relative to the representation $(a,\Lambda) \to \hat{U}_{(a,\Lambda)}$ of the Poincare group

$$\left(\hat{U}_{(a,\Lambda)}\, \hat{\Phi}(x)\, \hat{U}^{-1}_{(a,\Lambda)}\, \psi\right)(\zeta_{in}) = \hat{\Phi}(\Lambda x + a)\, \psi\, (\zeta_{in})$$

/For the proof cf. [10], Proposition 6 /

By virtue of proposition 2.3 in the classical field theory we have $P^{in}_{\mu} = P_{\mu} = P^{out}_{\mu}$ and $M^{in}_{\mu\nu} = M_{\mu\nu} = M^{out}_{\mu\nu}$ hence by virtue of /3.4/ we obtain

$$\hat{P}^{in}_{\mu} = \hat{P}_{\mu} = \hat{P}^{out}_{\mu}\,, \qquad \hat{M}^{in}_{\mu\nu} = \hat{M}_{\mu\nu} = \hat{M}^{out}_{\mu\nu} \qquad /5.4/$$

By Eqs. /5.4/ and /3.4/ the quantum generators $\hat{P}_{\mu}$ and $\hat{M}_{\mu\nu}$ are represented by the first order differential operator only. Consequently, the vacuum state ψ_0 defined by the formula

$$\hat{P}_{\mu}\psi_0 = 0\,, \qquad \hat{M}_{\mu\nu}\psi_0 = 0\,, \qquad /5.5/$$

is given in $\mathcal{K}(\mathcal{F})$ by the functional $\psi_0(\varphi,\pi) = 1$ Hence by /5.5/ the interacting and the asymptotic quantum fields have the same vacuum ψ_0 in $\mathcal{K}(\mathcal{F})$. The elements of the Wightman domain given by the formula

$$\psi(f_1, \dots f_n) = \prod_{i=1}^{n} \hat{\Phi}(f_i)\, \psi_0\,, \qquad /5.6/$$

by virtue of Eq. /3.4/ are represented by the sums of products of Frechet derivatives of the classical field Φ . Hence the Wightman domain as well as the Fock space H_{in} of $\hat{\Phi}_{in}$ field are subspaces of the carrier space $\mathcal{K}(\mathfrak{F})$.

VI. Generalized Normal Ordering For Interacting Fields

The formula /4.6/ shows that the present quantization method provides a certain normal ordering which is given by the formula

$$N(\hat{\Phi}^n)(x) = \widehat{\Phi^n}(x). \qquad /6.1/$$

Using the formula /3.6/ we find that

$$\widehat{\Phi^n}(x) = (1-n)\,\Phi^n(x) + n\,\Phi^{n-1}(x)\,\hat{\Phi}(x) \qquad /6.2/$$

The powers $N(\hat{\Phi}^n)(x)$ are local with respect to the quantum field $\hat{\Phi}(x)$ and all other powers $N(\hat{\Phi}^m)$ n,m = 1,2,3,.... Indeed we have:

Theorem 6.1. The quantities $N(\hat{\Phi}^n)(t,\alpha)$, $\alpha \in S(R^3)$ are continuous maps of the spaces $\mathcal{E}(\mathfrak{F}$ and $\mathcal{K}(\mathfrak{F})$ into themselves and satisfy the following commutation relations

$$[N(\hat{\Phi}^n)(x), N(\hat{\Phi}^m)(y)] = i\,n\,m\,\Phi^{n-1}(x)\,\Phi^{m-1}(y)\,\Delta^{\lambda}[x,y\,|\,\Phi], \qquad /6.3/$$

where $\Delta^{\lambda}[x,y\,|\,\Phi]$ is the commutator function for interacting field given by the formula /2.12/. /For the proof cf. Bałaban and Rączka [11]/

If x and y are space-like separated then by virtue of /2.12/ $\Delta^{\lambda}[x,y\,|\,\Phi]=0$ and we have

$$[N(\hat{\Phi}^n)(x), N(\hat{\Phi}^m)(y)] = 0. \qquad /6.4/$$

The ordered powers $N(\hat{\Phi}^n)(x)$ are Poincare covariant. Indeed:

$$\hat{U}_{(a,\Lambda)} N(\hat{\Phi}^n)(x)\hat{U}^{-1}_{(a,\Lambda)}\Psi)(\zeta) = N(\hat{\Phi}^n)[x\,|\,U^{-1}_{(a,\Lambda)}\zeta]\,(\hat{U}^{-1}_{(a,\Lambda)}\Psi)(U^{-1}_{(a,\Lambda)}\zeta) = N(\hat{\Phi}^n)(\Lambda x+a)\Psi(\zeta). \qquad /6.5/$$

The formula /6.4/ and /6.5/ shows that the ordering /6.1/ satisfies the most important requirements which are usually imposed on a normal ordering for interacting fields in the axiomatic quantum field theory.

VII. Quantum Scattering Operator

Let $\Phi_\tau(t,\underline{x})$ be a free classical field, whose initial data for $t=\tau$ are determined by the interacting field i.e.

$$\Phi_\tau(\tau,\underline{x}) = \Phi(\tau,\underline{x}) \quad , \quad \Pi_\tau(\tau,\underline{x}) = \Pi(\tau,\underline{x}). \qquad /7.1/$$

The time evolution of $\Phi_\tau(t,\underline{x})$ is given by the one-parameter group U_t^τ which is generated by the free hamiltonian H^τ

$$H^\tau = \tfrac{1}{2}\int d^3\underline{x}\,\left[\Pi_\tau^2 + |\nabla\Phi_\tau|^2 + m^2\Phi_\tau^2\right](t,\underline{x}). \qquad /7.2/$$

Indeed by virtue of Eqs. /2.14/ and /2.15/ we have

$$\{\Phi_\tau, H^\tau\} = \partial_t\Phi_\tau. \qquad /7.3/$$

Let $\widehat{H^\tau}$ be the quantum operator corresponding to H^τ by virtue of formula /3.4/. Then we have:

Proposition 7.1. The global transformation $t \to \hat{U}_t^\tau$ generated by the operator $\hat{H}^\tau$ in the carrier space $\mathcal{E}(\mathcal{F})$ is given by the formula $(\mathfrak{z}=\mathfrak{z}_{in})$

$$\hat{U}_t^\tau\psi(\mathfrak{z}) = \exp\left\{i\int_0^t \hat{\mathring{H}}\left(\Phi[\tau,\cdot\,|\,((U_{t'}^\tau)^{-1}\mathfrak{z}]\right)dt'\right\}\psi[(U_t^\tau)^{-1}\mathfrak{z}], \qquad /7.4/$$

where $\hat{\mathring{H}}^\tau = H^\tau - \frac{1}{2}DH^\tau$ and U_t^τ is the classical transformation in the space $\mathcal{F}$ generated by the hamiltonian vector field associated with H^τ. ▼

/For the proof cf. Bałaban and Rączka [12]./

Let $\hat{\Phi}_\tau(x)$ be the quantum field associated with the classical field $\Phi_\tau(x)$. Since the map $\Phi_\tau \to \hat{\Phi}_\tau$ conserves Lie bracket structure of Poisson bracket Lie algebra, the field $\hat{\Phi}_\tau$ by virtue of Eq. /4.2/ is a free local relativistic quantum field. In particular, by virtue of Eq. /7.3/ the time evolution of $\hat{\Phi}_\tau$ is given by the operators $\hat{U}_t^\tau$ defined by Eq. /7.4/

$$\left(\hat{U}_{t'}^\tau\,\hat{\Phi}_\tau\left(\hat{U}_{t'}^\tau\right)^{-1}\right)(t,\underline{x}) = \hat{\Phi}_\tau(t+t',\underline{x}) \qquad /7.5/$$

Proposition 7.2. The fields $\hat{\Phi}_{\tau}(t,\underline{x})$ and $\hat{\Phi}_{\tau_0}(t,\underline{x})$ are connected by the transformation $\hat{V}(\tau,\tau_0)$ i.e.

$$\hat{\Phi}_{\tau}(t,\underline{x}) = \hat{V}(\tau,\tau_0)\, \hat{\Phi}_{\tau_0}(t,\underline{x})\, \hat{V}^{-1}(\tau,\tau_0), \qquad /7.6/$$

given by the formula

$$\hat{V}(\tau,\tau_0) = \hat{U}^{in}_{(\tau-\tau_0)}\, \hat{U}^{\tau_0}_{(\tau_0-\tau)} = \hat{U}^{\tau}_{(\tau-\tau_0)}\, \hat{U}^{in}_{(\tau_0-\tau)}\,. \qquad /7.7/$$

The operator $\hat{U}(\tau,\tau_0)\equiv\hat{V}(\tau_0,\tau)$ satisfies the following equation

$$\partial_\tau \hat{U}(\tau,\tau_0) = -i\, \hat{H}_{int}\left[\Phi_{\tau_0}(\tau,\cdot)\right] \hat{U}(\tau,\tau_0), \qquad /7.8/$$

where

$$\hat{H}_{int}\left[\Phi_{\tau_0}(\tau,\cdot)\right] = \frac{\lambda}{4}\int d^3\underline{x}\; \widehat{\Phi^4_{\tau_0}}(\tau,\underline{x})$$

Eqs. /7.6/ - /7.8/ hold in the sense of strong operator topology in $\mathcal{E}($
/For the proof cf. Bałaban and Rączka [12]./

Equation/7.8/ is the evolution equation for the evolution operator $\hat{U}(\tau,\tau_0)$, which in conventional quantum field theory is formally derived by the passage to the "interaction picture". It is usually solved by a formal construction of Louville-Neuman perturbation series, which in four-dimensional space-time is divergent. In the present approach the action of the evolution operator $\hat{U}(\tau,\tau_0)$ in the carrier space $\mathcal{E}(\mathcal{F})$ can be explicitly calculated. Indeed using the fac that the evolution operator is given as the product of two one-parameter groups of time translation $\hat{U}^{in}_{(\tau-\tau_0}$ and $\hat{U}^{\tau_0}_{(\tau_0-\tau)}$, by virtue of Eq. /7.7/ and /5.2/ one obtains

$$\hat{U}(\tau,\tau_0)\psi(\mathfrak{z}) = \exp\left\{i\int_0^{\tau-\tau_0} \hat{H}^{\tau_0}\left(\Phi\left[\tau_0,\cdot\,|\,(U^{\tau}_{t'})^{-1}\mathfrak{z}\right]\right)dt'\right\}\psi\left[U^{-1}(\tau,\tau_0)\mathfrak{z}\right]. \qquad /7.10/$$

We derive now the action in the carrier space of the quantum scattering operator. This operator is defined in the space $\mathcal{E}(\mathcal{F})$ by the formula

$$\hat{S} = \lim_{\substack{\tau\to\infty\\ \tau_0\to-\infty}} \hat{U}(\tau,\tau_0)\,. \qquad /7.11/$$

Theorem 7.3. The quantum scattering operator $\hat{S}$ is given by the formula:

$$(\hat{S}\psi)(\mathfrak{z}) = \psi(S^{-1}\mathfrak{z}) \qquad /7.12/$$

where S is the classical scattering operator. The operator $\hat{S}$ is invariant under the action of the Poincare group and satisfies the condition

$$\hat{S}^{-1}\,\hat{\Phi}_{in}\,\hat{S} = \hat{\Phi}_{out}. \qquad /7.13/$$

/For the proof cf. Bałaban and Rączka [12]/

It follows from formula /2.28/ that S is nontrivial in $\lambda\phi^4$ theory. This implies that the quantum scattering operator /7.12/ is also nontrivial.

It was shown is Section II.D that the classical scattering operator is nonanalytic in λ for all initial data from $\mathcal{F}$. Hence by virtue of formula /7.12/ the quantum scattering operator is also nonanalytic.

VIII. Discussion

A. The results of the present work can be extended to a large class of nonpolynomial analytic interactions $F(\Phi)$ satisfying the conditions [8]

i/ $F(z) \sim O(z^3)$ for $z \to 0$

ii/ $F(z) \sim O(z^{5-\varepsilon})$ for $|z| \to \infty$

iii/ $m^2 z^2 + z F(z) \geqslant 0$

It is interesting that there exist certain power interactions $F(\Phi) = |\Phi|^{p-1}\Phi$ $1 < p \leqslant 5/3$ for which the global solutions exist but asymptotic field Φ_{out} does not exist: consequently there is no scattering operator either [19 ii/]. Even worse can occur: if the function $G(\Phi) = \int^{\Phi} F(v)dv$ is negative somewhere and the initial data are sufficiently large, then the solution blows up in finite time [19 i/]; hence there exists only finite time dynamics. These facts illustrate the richness of non-linear field theories.

Since every quantum field theory in the limit $\hbar \to 0$ should give a classical field theory the nonexistence of classical scattering operator may serve as a test for admissible quantum interactions. It may also serve as a tool for proving the nontriviality of $\lambda \Phi^4_n$, $n=2$, and 4 quantum field theories. [16].

B. The present realization of interacting and asymptotic quantum fields is given in a topological vector space $\mathcal{E}(\mathcal{F})$ or $\mathcal{K}(\mathcal{F})$ of smooth functionals over the space $\mathcal{F}$ of initial data. The present approach will be completed if one would be able to introduce a scalar product of the form:

$$(\Psi_1, \Psi_2) = \int_{\mathcal{F}} \overline{\Psi}_1(\mathfrak{z})\, \Psi_2(\mathfrak{z})\, d\mu(\mathfrak{z})$$

where $\mu(\cdot)$ should be a measure invariant with respect to the action of Poincare transformations /5.2/ and the scattering operator /7.12/ This would guarrantee that scattering operator and Poincare group are represented unitarily. Since the manifold of solutions is parametrized by the Banach space $\mathcal{F}$ of initial data the integral represents in fact a Feynmann type integral over all histories.

There is one invariant measure with respect to Poincare group and scattering operator given by Dirac measure $\delta(\mathfrak{z})$: it leads however to a trivial physical theory. The existence of other invariant measures is, for the time being, open.

C. It is interesting that in cases when a given quantum field problem can be solved explicitly in Fock space and in the present formalism, the obtained results coincide. For instance consider the problem of a quantum scalar field in the external time-dependent potential

$$(\Box + m^2)\hat{\Phi}(x) = V(x)\,\hat{\Phi}(x) \qquad /8.3/$$

Solving the classical equation, performing the quantization /3.4/ of obtained solution $\Phi[x, \varphi, \pi]$ and restricting the obtained field $\hat{\Phi}$ to the vacuum sector one obtains precisely the conventional second quantized field and the unitary scattering operator /for details cf. Rączka and Vladimirov [17]./

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IX. Appendix

We summarize here the properties of the Banach space B_F which is a carrier space for solutions of Eq.(2.1). Let $\Phi_0(x)$ be a solution of the free Klein-Gordon equation, whose Cauchy data at $t = 0$ coincide with that of Φ. Define $\mathcal{F}_1$ as the space of free solutions such that $\Phi_0(0,x) = \varphi(x) = \Phi(0,x)$ has third derivatives in $L_1(R^3)$ and second derivatives in $L_2(R^3)$, while $\Pi_0(0,x) = \pi(x) = \Pi(0,x)$ has second derivatives in $L_1(R^3)$ and first derivatives in $L_2(R^3)$. Then every element of $\mathcal{F}_1$ is finite with respect to the following norm [8]:

$$\|\Phi\|^2 = \sup_t \|\Phi(t)\|_E^2 + \sup_x |\Phi(x)|^2 + \int_{-\infty}^{\infty} \sup_{\underline{x}} |\Phi(t,\underline{x})|^2 \tag{A.1}$$

The Banach space $\mathcal{F}$ of initial conditions used in this paper is the completion of $\mathcal{F}_1$ in the norm (A.1).

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PERTURBATIONS OF DYNAMICS AND GROUP COHOMOLOGY (*)

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RESUME : On explore le rôle de la cohomologie dans la description des perturbations dynamiques. On donne des exemples liés à des secteurs de supersélection, à la stabilité et aux perturbations bornées.

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In the course of this colloquium we have heard a great deal about perturbations of dynamics in one way or another but, up till now, nothing about group cohomology. I hope to redress the balance a little because I have gradually come to the conclusion, which is surely by no means original, that group cohomology is intimately related to perturbations of dynamics. In order to convince you that this conclusion is sound, I shall try and provide partial answers to the following three questions.

1) Why does cohomology arise in discussing perturbations of dynamics ?

2) Where does cohomology play a role ?

3) How does one solve the cohomology problems that arise here ?

None of the heavy machinery of cohomology will be required and for the most part it will suffice to consider the first cohomology of the additive group of the real line. Instead I shall illustrate my theme by referring to three concrete problems in quantum theory : superselection sectors, stability and bounded perturbations of dynamics.

But before turning to these questions, I have to say something about dynamics. A rather general way of looking at dynamical systems is to think in terms of the set of continuous homomorphisms between two topological groups, Hom(H,G) . Whilst there is little point in actually working at this level of generality, it helps to get things in perspective to talk a little in these terms. The group H is the dynamical group ; if you are a mathematician your favourite dynamical group is likely to be $\mathbb{Z}$, the group of integers, alghough, for a physicist, dynamics begins at $\mathbb{R}$, the real line, representing the time evolution of the system. There are, of course, a number of other candidates such as the group of space-time translations or the Poincaré group. However H is always some rather small group with a relatively well understood structure. G , by contrast, is a rather large flabby group whose structure is complicated and much less well understood. It might, for example, be the group of homeomorphisms of a topological space, the group of diffeomorphisms of a manifold, the group of measure-preserving transformations of a measure space, the group of unitary transformations of a Hilbert space or the group of automorphisms of a C^*-algebra. All these examples have a common feature : they may be thought of as the group of symmetries, or automorphisms,

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of some underlying space X : $G = \text{Aut}\, X$.

Perturbation theory means looking at the set Hom(H,G) from the point of view of a fixed basis element U of Hom(H,G) , the unperturbed dynamics. Group cohomology lies at the very root of perturbation theory. If $U' \in \text{Hom}(H,G)$ denotes the perturbed dynamics then the appropriate variable for perturbation theory is

(1) $$\Gamma_s = U'(s)\,U(s)^{-1}, \qquad s \in H .$$

Γ is a continuous function from H to G and satisfies

(2) $$\Gamma_s\, {}^s\Gamma_t = \Gamma_{st}, \qquad s,t \in H .$$

(3) $$\Gamma_e = 1 ,$$

here if $g \in G$, $s \in H$, then ${}^s g = U(s)\,g\,U(s)^{-1}$, e is the identity element of H and 1 the identity element of G . Eq.(2) and (3) define what is meant by a 1-cocycle on H with values in G . All cocycles arising here will be continuous, and I write $\Gamma \in Z^1_U(H, G)$, where the subscript U is written to remind one that the action of H on G determined by U appears in the definition of a cocycle. Two 1-cocycles Γ and Γ' are said to be cohomologous $\Gamma \sim \Gamma'$ if there is a $g \in G$ such that

(4) $$\Gamma'_s = g^{-1}\,\Gamma_s\, {}^s g , \qquad s \in H .$$

The set of cohomology classes is denoted by $H^1_U(H, G)$. $Z^1_U(H, G)$ has a base point $\mathring{\Gamma}$, $\mathring{\Gamma}_s = 1$, $s \in H$, corresponding to the absence of a perturbation and a cocycle cohomologous to $\mathring{\Gamma}$ is called a coboundary. The set of coboundaries is denoted by $B^1_U(H, G)$.

Of course all this is just a change of language ; $Z^1_U(H, G)$ is

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$Hom(H, G)$ in disguise and $H^1_u(H, G)$ merely identifies dynamics which are transforms of one another under some element of G, i.e. under some symmetry of the underlying space X.

As an example to illustrate these ideas, take $H = \mathbb{R}$ and $G = \mathcal{U}(\mathcal{H})$ the unitary group of a Hilbert space $\mathcal{H}$ in the weak operator topology. This is the setting for elementary quantum mechanics. Dynamics is given by a continuous 1-parameter unitary group U, $U(t) = e^{itH}$, H is the Hamiltonian. A perturbed dynamics is given by $U'(t) = e^{itH'}$ and the associated cocycle is

$$\Gamma_t = e^{itH'} e^{-itH}, \qquad t \in \mathbb{R}$$

Γ is a coboundary if and only if H and H' are unitarily equivalent. The cohomology classes are the unitary equivalence classes of self-adjoint operators and these are characterized by spectral multiplicity theory [1]. Scattering theory also gives us one sufficient condition for a cocycle to be a coboundary : if a wave operator $\Omega_\pm = \lim_{t\to\pm\infty} e^{itH'} e^{-itH}$ exists and is unitary then* $\Gamma_t = \Omega_\pm e^{itH} \Omega_\pm^{-1} e^{-itH} = \Omega_\pm ({}^t\Omega_\pm)^{-1}$.

Superselection Sectors

In considering perturbations of dynamics, one usually wishes to impose restrictions on the class of perturbations. In the theory of superselection sectors in elementary particle physics, the characteristic restriction on the perturbations is that they are localized in some sense because they correspond to the addition of localized "charges" to the system. If one looks for Poincaré covariant sectors, one deals with 1-cocycles Γ over the Poincaré group $\mathcal{P}$ with values in the unitary group $\mathcal{U}(\mathfrak{A})$ of the observable algebra $\mathfrak{A}$. In the description of superselection sectors given in [2 , 3] the locality restriction may be expressed as follows :

$$(5) \qquad \Gamma_L \in \bigcup_{\mathcal{O}} \{ \mathfrak{A}(\mathcal{O}') \cap \mathfrak{A}(L\mathcal{O}') \}', \qquad L \in \mathcal{P}.$$

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Here the union is taken over all double cones $\mathcal{O}$, $\mathcal{O}'$ denotes the spacelike complement of $\mathcal{O}$ and the observable net $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$ is supposed realized concretely on the Hilbert space $\mathcal{H}_o$ of the vacuum representation π_o. With such a cocycle Γ , there is an associated localized morphism ρ_Γ of the observable algebra given by a pointwise norm limit as L tends spacelike to infinity ([2 ; Lemma 3.1] and [3 ; Footnote 7]) :

$$(6) \qquad \rho_\Gamma(A) = \lim_{\mathcal{F}} \Gamma_L A \Gamma_L^* , \qquad A \in \mathfrak{A} .$$

Here $\mathcal{F}$ is the filter generated by the filter basis with elements $F(\mathcal{O}) = \{ L \in \mathcal{P} : L\mathcal{O} \subset \mathcal{O}' \}$. If $L \to U_o(L)$ implements the Poincaré automorphisms in the vacuum sector, $L \to \Gamma_L U_o(L)$ implements the Poincaré automorphisms in the representation $\pi_o \circ \rho_\Gamma$. Hence the perturbed dynamics can be reinterpreted as the original dynamics of states in some other sector [1] . Two such cocycles Γ and Γ' are cohomologous if there is a unitary $V \in \bigcup_{\mathcal{O}} \mathfrak{A}(\mathcal{O})$ such that $\Gamma'_L = V^{-1} \Gamma_L \alpha_L(V)$. This implies from (6) for the localized morphisms that $\rho_{\Gamma'}(A) = V^{-1} \rho_\Gamma(A) V, A \in \mathfrak{A}$. In fact $\Gamma \sim \Gamma'$ if and only if the representations $\pi_o \circ \rho_{\Gamma'}$ and $\pi_o \circ \rho_\Gamma$ are unitarily equivalent ([2 ; Lemma 1.3] and [3 ; Lemma 2.2]) so that the superselection sectors can be described in terms of cohomology classes.

This description of superselection sectors does not cover all cases of interest because one implicitly assumes

a) that charges can be strictly localized or equivalently that one can get away with strictly local fields in a Hilbert space with positive-definite metric. This is a result of taking the Γ_L strictly bilocalized in the sense of (5) .

b) that there are no spontaneously broken gauge symmetries. This is a consequence [4] of the duality assumption in the form $\mathfrak{A}(\mathcal{O}) = \mathfrak{A}(\mathcal{O}')'$.

A suggestion as to how to modify duality to allow for spontaneously broken gauge symmetries may be found in [4] . From the point of view of the

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1) To simplify the discussion, I am suppressing the fact that $\pi_o \circ \rho_\Gamma$ is, in general, reducible and corresponds to a mixture of sectors.

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1-cocycles what happens, roughly speaking, is that the cocycles Γ which reflect the spontaneously broken part of the gauge symmetry, although they are not coboundaries in $\mathcal{U}(\mathfrak{A})$, become coboundaries in the larger group $\mathcal{U}(\mathfrak{A}^-)$.

It is not yet clear how to modify things to take care of the charges which, as in the case of electrodynamics, cannot be strictly localized. An attractive possibility from a mathematical point of view is to look at cocycles which are asymptotically bilocalized in the sense that given $A \in \mathfrak{A}$ and $\varepsilon > 0$ there exists an $F \in \mathcal{F}$ such that

(7) $$\| \alpha_L(\Gamma_{L'}) A - A \alpha_L(\Gamma_{L'}) \| < \varepsilon \quad , \qquad L, LL' \in F$$

Much of the algebraic structure survives this generalization. Thus one can still use (6) to define "quasilocalized" morphisms ρ_Γ and the Poincaré automorphisms are still implemented by $L \to \Gamma_L U_o(L)$ in the representation $\pi_o \circ \rho_\Gamma$

As a final remark, if ad Γ_L denotes the inner automorphism generated by Γ_L, and we look at the cocycle γ , $\gamma_L = \mathrm{ad}\, \Gamma_L$, which takes values in Aut $\mathfrak{A}$, we see that (6) may be looked upon as an attempt to show that γ is a coboundary by taking a limit over the left-invariant filter $\mathcal{F}$ on the Poincaré group. This parallels the use of wave operators to show that cocycles are coboundaries which was discussed above.

Stability

We turn now to the question of the behaviour of fixed points under perturbations of dynamics. If $U \in \mathrm{Hom}(H, \mathrm{Aut}\, X)$ denotes the unperturbed dynamics and $x \in X$ with $U(h)x = x$, for $h \in H$ then x is said to be stable if there is a neighbourhood $\mathcal{N} \ni U$ and a smooth mapping $\phi . \mathcal{N} \to X$ such that

(8) $$V(h)\, \phi(V) = \phi(V) , \qquad h \in H , \quad V \in \mathcal{N}$$

and $\phi(U) = x$. To make this notion precise, we have to say what is meant by smooth. Very often smooth can be taken to imply continuously diffe-

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rentiable in some sense and then there is a weaker condition, infinitesimal stability, which, being a linear condition, is easier to analyse than stability. Under certain circumstances it turns out that infinitesimal stability is sufficient to imply stability. The point I wish to make here in general terms, without burdening the discussion with precise definitions, is that infinitesimal stability has a cohomological interpretation.

In treating infinitesimal stability one deals with infinitesimal perturbations of dynamics. This involves changing the coefficient group in the definition of cocycles from the infinite-dimensional Lie group $G = \text{Aut } X$ to its Lie algebra $\mathcal{L}G$ which has ar underlying structure of a vector space and hence of an Abelian group. An infinitesimal perturbation of dynamics is thus an element of $Z^1_u(H, \mathcal{L}G)$. $Z^1_u(H, \mathcal{L}G)$ has itself the structure of a vector space under pointwise operations. Since $G = \text{Aut } X$ there is a natural linear mapping ψ, say, of $\mathcal{L}G$ into $T_x X$, the tangent space of X at x. Since $U(h)x = x$ for $h \in H$, $T_x X$ carries an induced linear representation of H and ψ induces a linear mapping $\tilde{\psi}$, say, of $Z^1_u(H, \mathcal{L}G)$ into $Z^1_u(H, T_x X)$. $\tilde{\psi}$ maps coboundaries into coboundaries so that we have an induced linear map $\tilde{\psi}_*$, say, of $H^1_u(H, \mathcal{L}G)$ into $H^1_u(H, T_x X)$. The infinitesimal stability condition is that $\tilde{\psi}_* = 0$, in other words that the image of $Z^1_u(H, \mathcal{L}G)$ under $\tilde{\psi}$ is contained in $B^1_u(H, T_x X)$.

As an example of these considerations, I shall formulate an infinitesimal stability condition for an invariant state ω under a strongly continuous 1-parameter group α of automorphisms of a C^*-algebra $\mathfrak{A}$. Let $\text{Der}\,\mathfrak{A}$ denote the real Banach space of symmetric derivations of $\mathfrak{A}$, if $d \in \text{Der}\,\mathfrak{A}$, let ${}^s d \in \text{Der}\,\mathfrak{A}$ be defined by ${}^s d(A) = \alpha_s d \alpha_{-s}(A)$, $A \in \mathfrak{A}$. As an infinitesimal perturbation of dynamics, I take an element of $Z^1_\alpha(\mathbb{R}, \text{Der}\,\mathfrak{A})$, i.e. a continuous mapping d of $\mathbb{R}$ into $\text{Der}\,\mathfrak{A}$ such that

$$(9) \qquad d_s + {}^s d_t = d_{s+t}, \qquad s, t \in \mathbb{R}.$$

Let $[\omega]$ denote the real Banach space of Hermitian linear functionals which are normal functionals of the representation generated by ω. If $f \in [\omega]$

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then ${}^{s}f \in [\omega]$, where ${}^{s}f(A) = f\alpha_s(A)$. There is a linear mapping $\tilde{\Psi}$ of $Z^1_\alpha(\mathbb{R}, \mathrm{Der}\,\mathfrak{A})$ into $Z^1_\alpha(\mathbb{R}, [\omega])$ defined by

(10) $$\tilde{\Psi}(d)_s(A) = \omega d_s(A) \quad , \quad A \in \mathfrak{A} .$$

$\tilde{\Psi}$ maps coboundaries into coboundaries and one may define ω to be <u>infinitesimally stable</u> if $\tilde{\Psi}_*$, the induced mapping from $H^1_\alpha(\mathbb{R}, \mathrm{Der}\,\mathfrak{A})$ to $H^1_\alpha(\mathbb{R}, [\omega])$, is zero.

In order to see what is involved here, let me sketch one way of computing the first cohomology of the real line with coefficients in a Banach space. Let B be a Banach space and $s \to U(s)$ a strongly continuous representation of $\mathbb{R}$ by isometries of B .
Let δ denote the infinitesimal generator of U : $D(\delta) = \{ \Phi \in B : s \to U(s)\Phi$ is norm differentiable $\}$.
If $\Phi \in D(\delta)$ then $\delta\Phi = \lim_{s \to 0} s^{-1}(U(s)\Phi - \Phi)$.
One may compute the cohomology in three steps.

a) Given $\Psi \in Z^1_U(\mathbb{R}, B)$ define $\Phi = \tau^{-1}\int_0^\tau \Psi_s\, ds$ and set $\Psi'_s = \Psi_s + U(s)\Phi - \Phi$. Then Ψ' is a differentiable cocycle and one sees that every cohomology class contains a differentiable cocycle.

b) If Ψ is a differentiable cocycle then

(11) $$\Psi_s = \int_0^s U(t)\dot{\Psi}\, dt$$

where $\dot{\Psi}$ denotes the derivative of Ψ_s at $s = 0$.

c) A differentiable cocycle Ψ is a coboundary if and only if $\dot{\Psi} = \delta\Phi$ for some $\Phi \in D(\delta)$.

This establishes an isomorphism of $H^1_U(\mathbb{R}, B)$ and the quotient space $B/R(\delta)$, where $R(\delta)$ denotes the range of δ .

Note that the above steps are so simple that the proof holds in more general contexts where B is no longer a Banach space. In practice there is still

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the problem of giving a good characterization of $R(\delta)$. For examples of how to compute the first cohomology of more general Lie groups with coefficients in a Hilbert space, the reader may consult Araki [5] .

Using the above characterization of cohomology, one sees that an invariant state ω is infinitesimably stable if and only if for each $d \in \mathrm{Der}\,\mathfrak{A}$ such that $s \to {}^s d$ is norm continuous, there is a $\phi \in [\omega]$ such that $\| \omega d - s^{-1}(\phi\alpha_{-s} - \phi) \| \to 0$ as $s \to 0$. If one takes $\mathfrak{A}$ to be a simple C^*-algebra so that each derivation is inner and uses (11) one sees that ω is infinitesimally stable if and only if for each $h = h^* \in \mathfrak{A}$ there is a $\phi \in [\omega]$ such that

$$(12) \qquad i\int_0^s \omega([\alpha_t(h), A])\,dt = \phi\alpha_{-s}(A) - \phi(A), \quad A \in \mathfrak{A}.$$

Eq.(12) may be recognized as the stability condition used by Haag, Kastler and Trych-Pohlmeyer [6;Prop. 2] in deriving the K.M.S. condition.

Bounded Perturbations

In this section we consider a natural class of rather weak perturbations of 1-parameter groups of automorphisms of a C^*-algebra. If $\mathfrak{A}$ is a C^*-algebra with identity, $\mathcal{U}(\mathfrak{A})$ will from now on denote the unitary group of $\mathfrak{A}$ endowed with the norm topology. If α and α' are 1-parameter groups of automorphisms of $\mathfrak{A}$ then α' is said to be a bounded pertubation of α if $\|\alpha'_t - \alpha_t\| \to 0$ as $t \to 0$. This definition together with the results of this section are taken from [7]. Passing to the cocycle variable γ ', $\gamma_t = \alpha'_t\alpha_t^{-1}$, $t \in \mathbb{R}$, one sees that studying the bounded perturbations of α is equivalent to studying $Z^1_\alpha(\mathbb{R}, \mathrm{Aut}\,\mathfrak{A})$ where $\mathrm{Aut}\,\mathfrak{A}$ is given the norm topology. For simplicity, I shall only discuss here the case that $\mathfrak{A}$ is a von Neumann algebra $\mathfrak{M}$ and that α is a weakly continuous 1-parameter group of automorphisms of $\mathfrak{M}$. Since, by a result of Kadison and Ringrose [8;Thm.7], every automorphism γ of $\mathfrak{M}$ with $\|\gamma - \iota\| < 2$ is an inner automorphism $Z^1_\alpha(\mathbb{R}, \mathrm{Aut}\,\mathfrak{M}) = Z^1_\alpha(\mathbb{R}, \mathrm{In}\,\mathfrak{M})$, where $\mathrm{In}\,\mathfrak{M}$ denotes the subgroup of inner automorphisms. It turns out that it is actually the weakly dense C^*-algebra $\mathfrak{M}_0$ of elements of $\mathfrak{M}$ having a norm continuous orbit, which is important here :

$$\mathfrak{M}_o = \{ A \in \mathfrak{M} : t \to \alpha_t(A) \quad \text{is norm continuous} \}$$

The simplest way of expressing the main result is to suppose, as one may, that $\mathfrak{M}$ is realized on a Hilbert space $\mathcal{H}$, where α is implemented by a continuous 1-parameter unitary group : $\alpha_t(A) = e^{itH} A e^{-itH}$ $A \in \mathfrak{M}$. Then if α' is a bounded perturbation of α, $\alpha'_t(A) = e^{itH'} A e^{-itH'}$, where H' may be chosen to have the form

(13) $$H' = VHV^{-1} + h$$

with $V \in \mathcal{U}(\mathfrak{M}_o)$ and $h = h^* \in \mathfrak{M}_o$. Conversely if H' has the form (13) then $A \to e^{itH'} A e^{-itH'}$ defines a bounded perturbation of α. Of course one can also express this result in a way which makes no reference to an underlying Hilbert space [7;Thm 4.8].

Let me say a few words about how this result is obtained, stressing the cohomological parts of the argument. One has an exact sequence of groups:

$$1 \to \mathcal{U}(\mathfrak{Z}) \to \mathcal{U}(\mathfrak{M}) \to \mathrm{In}\,\mathfrak{M} \to 1$$

where $\mathfrak{Z}$ denotes the centre of $\mathfrak{M}$. The first step in the proof is to show that γ, which is a continuous map from $\mathbb{R}$ to $\mathrm{In}\,\mathfrak{M}$, may be lifted to a continuous map U from $\mathbb{R}$ to $\mathcal{U}(\mathfrak{M}_o)$:

$$\gamma_t = \mathrm{ad}\, U_t, \qquad t \in \mathbb{R}.$$

Now U is not necessarily a 1-cocyle but, and this illustrates a typical feature of cohomology, the deviation z of U from a 1-cocyle,

(14) $$z(s,t) = U_s \alpha_s(U_t) U_{s+t}^{-1}, \qquad s,t \in \mathbb{R}$$

is itself a 2-cocycle with values in $\mathcal{U}(\mathfrak{Z}_o)$, $\mathfrak{Z}_o = \mathfrak{Z} \cap \mathfrak{M}_o$,

(15) $$z(s,t)\, z(s+t,u) = \alpha_s(z(t,u))\, z(s,t+u), \qquad s,t,u \in \mathbb{R}.$$

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Now, one can show that if α is a strongly continuous 1-parameter group of automorphisms of an Abelian C^*-algebra $\mathfrak{A}$, then $H^2_\alpha(\mathbb{R}, \mathcal{U}(\mathfrak{A})) = 0$. This implies that there is a continuous function $\lambda \quad \mathbb{R} \to \mathcal{U}(\mathfrak{Z}_0)$ such that

(16) $$z(s,t) = \lambda_{s+t} \lambda_s^{-1} \alpha_s(\lambda_t^{-1}) , \qquad s, t \in \mathbb{R} .$$

Hence if one sets $\Gamma_s = \lambda_s U_s$, $s \in \mathbb{R}$ then $\Gamma \in Z^1_\alpha(\mathbb{R}, \mathcal{U}(\mathfrak{M}_0))$ and

(17) $$\gamma_t = \mathrm{ad}\, \Gamma_t , \qquad t \in \mathbb{R} .$$

The final step is to analyse $Z^1_\alpha(\mathbb{R}, \mathcal{U}(\mathfrak{A}))$ where α is a strongly continuous 1-parameter group of automorphisms of a C^*-algebra $\mathfrak{A}$. In the problem at hand one takes $\mathfrak{A} = \mathfrak{M}_0$. One finds that $Z^1_\alpha(\mathbb{R}, \mathcal{U}(\mathfrak{A}))$ may be regarded as a homogeneous space under the action of $\mathcal{IU}(\mathfrak{A})$, the inhomogeneous unitary group of $\mathfrak{A}$. $\mathcal{IU}(\mathfrak{A})$ is the group of pairs (h, V) with $V \in \mathcal{U}(\mathfrak{A})$ and $h = h^* \in \mathfrak{A}$ and the law of composition

(18) $$(h, V)(h', V') = (h + V h' V^{-1}, VV') .$$

To find a pair (h, V) which corresponds to a given $\Gamma \in Z^1_\alpha(\mathbb{R}, \mathcal{U}(\mathfrak{A}))$, one picks $V \in \mathcal{U}(\mathfrak{A})$ such that $\Gamma'_t = V^{-1} \Gamma_t \alpha_t(V)$, $t \in \mathbb{R}$, is a differentiable cocycle and then defines $h = -V i \frac{d}{dt} \Gamma'_t \big|_{t=0} V^{-1}$. The h and V which appear here, for $\mathfrak{A} = \mathfrak{M}_0$, are of course those which may be used in (13).

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N° 248 – Les méthodes mathématiques de la théorie quantique des champs

STATISTICAL MECHANICAL METHODS IN QUANTUM FIELD THEORY : CLASSICAL BOUNDARY CONDITIONS

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RESUME : On discute le rôle des conditions bornés pour le modèle $P(\phi)_2$.
En particulier on considère les conditions libres, Dirichlet, Neumann et périodiques. On montre que la pression est indépendante de ces conditions et on établit l'égalité variationnelle de Gibbs.

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Introduction

In Guerra-Rosen-Simon [16] we advanced the idea that the methods of statistical mechanics play a vital and natural role in boson quantum field theories. Indeed, the last three years have seen a tremendous infusion of techniques from statistical mechanics. In particular, I would mention the following areas:

a) Correlation inequalities: For a recent review, see Simon [32].

b) Expansion techniques of Glimm, Jaffe, and Spencer: See e.g., Glimm-Jaffe-Spencer [10] ("high temperature"), Spencer [34] ("low fugacity"), Glimm-Jaffe-Spencer [12] ("low temperature"), and, for applications to $(\phi^4)_3$, Feldman-Osterwalder [4] and Magnen-Seneor [23].

c) Lee-Yang Theorem (ϕ^4 theories): The original result of Simon-Griffiths [33] has been applied and extended by a number of authors (e.g., see Dunlop-Newman [3] for a Lee-Yang Theorem for multi-component fields).

d) Dynamical instability and phase transitions: We now know that the $(\phi^4)_2$ model exhibits a phase transition (see Jaffe [20] and Glimm-Jaffe-Spencer [11]). In general the results and techniques that have been developed in the investigation of the set of pure phases in the Ising model serve as a useful guide for ϕ^4_d models.

Although I shall have a little to say about a) and c) below, the main question that I shall discuss is the role of boundary conditions (B.C.) in the $P(\phi)_2$ model. For complete details I refer you to Guerra-Rosen-Simon [18]. Just as in statistical mechanics, one expects that B.C. are fundamental in the definition of equilibrium states, and are connected with the existence or nonexistence of phase transitions (I am referring to "± B.C."). Actually I shall not discuss these difficult questions but shall instead concentrate on "classical" B.C., namely free (F), Dirichlet (D), Neumann (N), and periodic (P).

More precisely, consider the free boson field ϕ with Gaussian measure $d\mu_0$ with covariance operator $G_0 = (-\Delta + m_0^2)^{-1}$ (for a discussion of Euclidean Q space consult e.g., [31]): we have

$$\int_Q \phi(f)\phi(g)\,d\mu_0 = (f, G_0 g) \tag{1}$$

for $f, g \in N = H_{-1}(\mathbb{R}^2)$, the Sobolev space with inner product (1). The standard strategy for constructing an interacting field theory is to restrict to a bounded region $\Lambda \subset \mathbb{R}^2$, to modify the measure $d\mu_0$ by a non-Gaussian factor depending on the fields in Λ,

$$d\nu_\Lambda = e^{-U_\Lambda} d\mu_0 \Big/ \int e^{-U_\Lambda} d\mu_0 , \tag{2}$$

and then to take the limit $\Lambda \to \mathbb{R}^2$. For the $P(\phi)_2$ model, $U_\Lambda = \int_\Lambda :P(\phi(x)):d^2x$. However there is no a priori reason to use "free" B.C. on the Gaussian part of the measure in (2) and we could replace $d\mu_0$ by a Gaussian measure $d\mu_\Lambda^X$ corresponding to the covariance

$$G_\Lambda^X = (-\Delta_\Lambda^X + m_0^2)^{-1} \tag{3}$$

where Δ_Λ^X is a self-adjoint extension of $\Delta \restriction C_0^\infty(\Lambda)$ with "X" B.C. on the boundary $\partial\Lambda$ of Λ. Indeed it was Nelson's discovery [24] that brand X B.C. (namely Dirichlet) lead to monotonicity properties of the interacting measure (2). Thus we generalize (2) to

$$d\nu_\Lambda^X = \text{const. } e^{-U_\Lambda^X} d\mu_\Lambda^X \tag{4a}$$

or to

$$d\nu_\Lambda^{HX} = \text{const. } e^{-U_\Lambda} d\mu_\Lambda^X \tag{4b}$$

where in (4a) Wick subtractions in U_Λ^X are made with respect to $d\mu_\Lambda^X$ and in (4b) (Half - X B.C.) Wick subtractions in $U_\Lambda = U_\Lambda^F$ are made with respect to $d\mu_\Lambda^F = d\mu_0 \restriction \Sigma_\Lambda$; here Σ_Λ is the σ - algebra generated by the fields in Λ.

A rather general class of covariances (3) is suitable for field theoretic purposes. It is natural to impose one regularity condition on this class: for some constant $c_\Lambda^X < \infty$

$$G_\Lambda^X \leq c_\Lambda^X G_0 \tag{5}$$

as operators on $L^2(\Lambda)$. One can prove [18, Theorem II.6] that if G_Λ^X satisfies (3) and (5), then for all $f, g \in C_0^\infty(\Lambda)$,

$$(f, G_\Lambda^X g) = (f, G_\Lambda^D g) + B_\Lambda^X(e_{\partial\Lambda} f, e_{\partial\Lambda} g)$$

where B_Λ^X is a bounded, positive definite quadratic form on $N_{\partial\Lambda}$, the subspace of $N = H_{-1}(\mathbb{R}^2)$ consisting of elements with support on $\partial\Lambda$; here $e_{\partial\Lambda}$ is the self-adjoint projection in N onto $N_{\partial\Lambda}$.

In addition to the choice of covariance, we may also wish to choose a free field with nonzero mean which is specified by a linear functional on $N_{\partial\Lambda}$ (see [18]). However, for the purposes of this lecture, I shall specialize to mean zero, Λ a rectangle, and $X = F, D, N, P$.

The Pressure is Independent of B.C.

The freedom to choose different B.C. provides considerable flexibility in the study of the thermodynamic limit. Certain operations and assertions are trivial with one choice of B.C. and inconvenient or impossible with others. Obviously, then, it is important to know what objects are independent of the choice of B.C. in the thermodynamic limit. Consider the pressure in the $P(\phi)_2$ model

$$\alpha_\Lambda^X = \frac{1}{|\Lambda|} \ln Z_\Lambda^X = \frac{1}{|\Lambda|} \ln \int e^{-U_\Lambda^X} d\mu_\Lambda^X \,, \tag{6a}$$

or the half - X pressure,

$$\alpha_\Lambda^{HX} = \frac{1}{|\Lambda|} \ln Z_\Lambda^{HX} = \frac{1}{|\Lambda|} \ln \int e^{-U_\Lambda} d\mu_\Lambda^X \,. \tag{6b}$$

The pressure should depend on the B.C. only through a surface effect in finite volume Λ and should be independent of the B.C. in the infinite volume limit. This is our first main result:

Theorem 1. Consider the $P(\phi)_2$ theory where P is any semibounded polynomial. For $\sigma = D, N, P, HD, HN$ or HP, the limits $\alpha_\infty^\sigma = \lim_{\Lambda\to\infty} \alpha_\Lambda^\sigma$ all exist and equal $\alpha_\infty = \lim_{\Lambda\to\infty} \alpha_\Lambda$.

Remarks. 1. The existence of the limit $\alpha_\infty = \lim_{\Lambda\to\infty} \alpha_\Lambda$ was established by Guerra [14].

2. The limit $\Lambda \to \infty$ may be taken in the sense of van Hove [30] (except of course for P and HP B.C.).

3. It is possible to prove a more general result of this sort for nonzero means (provided the means do not grow too quickly) and for the class of covariances specified by (3) and (5).

The corresponding result has been well studied in classical and quantum statistical mechanics. Our approach has been largely influenced by Robinson's work in quantum statistical mechanics [29]. Related results have been obtained by Novikov [26] and Ginibre [9]; see also Fisher-Lebowitz [5] and [30] for results of Fisher and

Ruelle. Baker [1] has proved the analogue of Theorem 1 for lattice boson models.

Conditioning Inequalities

First let me recall some basic facts about extensions of $(-\Delta + m_0^2) \upharpoonright C_0^\infty(\Lambda)$ with different B.C.:

Lemma 1. Let Λ be an open region in $\mathbb{R}^2$ (rectangular if P B.C. are considered). Then, as operator inequalities on $L^2(\Lambda)$,

$$G_\Lambda^D \le G_0 \le G_\Lambda^N , \tag{7}$$

$$G_\Lambda^D \le G_\Lambda^P \le G_\Lambda^N . \tag{8}$$

If Λ_1 and Λ_2 are disjoint open regions and $\Lambda = \mathrm{int}(\bar{\Lambda}_1 \cup \bar{\Lambda}_2)$ then

$$G_\Lambda^N \le G_{\Lambda_1}^N \oplus G_{\Lambda_2}^N , \tag{9}$$

$$G_\Lambda^D \ge G_{\Lambda_1}^D \oplus G_{\Lambda_2}^D \tag{10}$$

where $\oplus$ denotes direct sum according to the decomposition $L^2(\Lambda) = L^2(\Lambda_1) \oplus L^2(\Lambda_2)$.

This lemma is an immediate consequence of the theory of quadratic forms (see [21] or [18]). In fact, it is most convenient to *define* G_Λ^X via the quadratic form associated with $(-\Delta_\Lambda^X + m_0^2)$.

We now invoke the theory of conditioning [16, 31] which allows us to deduce inequalities between Q space expectations from operator inequalities among covariances such as (7) - (10). In particular we obtain the following transcription of the inequalities of Lemma 1:

$$Z_\Lambda^D \le \left\{ \begin{matrix} Z_\Lambda^P \\ Z_\Lambda \end{matrix} \right. \le Z_\Lambda^N . \tag{11}$$

If $\Lambda = \mathrm{int}(\bar{\Lambda}_1 \cup \bar{\Lambda}_2)$, $\Lambda_1 \cap \Lambda_2 = \phi$, then

$$Z_\Lambda^N \le Z_{\Lambda_1}^N \, Z_{\Lambda_2}^N \quad \text{(Submultiplicativity of N B.C.)} \tag{12}$$

$$Z_\Lambda^D \ge Z_{\Lambda_1}^D \, Z_{\Lambda_2}^D \quad \text{(Supermultiplicativity of D B.C.)} . \qquad (13)$$

The familiar relations (12) and (13) lead (together with appropriate bounds) to the convergence of $\alpha_\Lambda^N \to \alpha_\infty^N$ and $\alpha_\Lambda^D \to \alpha_\infty^D$ (see, e.g., [30]). The lattice of inequalities of (11)

$$D \begin{matrix} < & F & < \\ < & P & < \end{matrix} N \qquad (14)$$

is basic to our proof of Theorem 1: the strategy is to show (a) $\alpha_\infty^N \le \alpha_\infty$ and (b) $\alpha_\infty^D \ge \alpha_\infty$. Then the equalities $\alpha_\infty^D = \alpha_\infty^P = \alpha_\infty^F = \alpha_\infty^N$ follow by "bracketing". I shall sketch the proof of (a) below; the proof of (b), while similar, is a little more involved (see [18]).

Before going further, we should ask whether Z_Λ^N is finite $\left(\text{that } Z_\Lambda^D \le Z_\Lambda \le \exp(O(|\Lambda|)) \text{ is the well-known "linear lower bound"}\right)$. One way of showing that $Z_\Lambda^N < \infty$ is to establish the regularity condition (5) for N B.C. For then by conditioning $\left(c \text{ is the constant } c_\Lambda^N \text{ in (5)}\right)$

$$Z_\Lambda^N \le Z_\Lambda^{cF}$$

where the superscript cF indicates that the covariance in the measure (and in the Wick subtractions) is cG_0 . If we make a change of variables $\phi \to c^{\frac{1}{2}}\phi$ then we obtain

$$Z_\Lambda^{cF} = \int \exp\left(-\textstyle\int_\Lambda :\tilde{P}(\phi):\right) d\mu_0$$

where the new polynomial $\tilde{P}(y) \equiv P(c^{\frac{1}{2}}y)$. Hence the case of N B.C. reduces to that of F B.C. and we are done. The verification of (5) for N B.C. is non-trivial [18] and we omit the proof:

<u>Lemma 2</u>. If Λ is a rectangle, then $G_\Lambda^N \le c_\Lambda^N G_0$.

We remark that it is possible to prove this inequality for more general regions than rectangular Λ ; for instance, for circles or star-shaped regions with C^2 boundaries [18]. On the other hand, there are regions Λ for which the inequality fails.

Properties of the Classical B.C.

It is instructive to look at the lattice approximations [16, §IV] to the measures $d\mu_\Lambda^X$. With each site $n\delta \in \delta\mathbb{Z}^2$ we associate a spin q_n taking values in $\mathbb{R}$; the spacing parameter $\delta > 0$. The free lattice measure on the infinite lattice $\delta\mathbb{Z}^2$ is (formally)

$$d\mu_\infty^F = \text{const. } e^{-\frac{1}{2}q\cdot A_\infty^F q}\, dq$$

where the infinite matrix A_∞^F is defined by

$$q\cdot A_\infty^F q = m_0^2\delta^2 \sum_n q_n^2 + \frac{1}{2} \sum_{\langle n,n'\rangle} (q_n - q_{n'})^2 \tag{15a}$$

$$= (m_0^2\delta^2 + 4) \sum_n q_n^2 - \sum_{\langle n,n'\rangle} q_n q_{n'} \tag{15b}$$

where the notation $\langle n , n'\rangle$ indicates a sum over nearest neighbours. If we now restrict $d\mu_\infty^F$ to the set of lattice sites in the rectangle Λ we obtain the (rigorous) lattice measures $d\mu_{\Lambda,\delta}^X = \text{const. } e^{-\frac{1}{2}q\cdot A_\Lambda^X q}\, dq$ as follows. For D B.C. we drop the coupling terms $q_n q_{n'}$ across $\partial\Lambda$ (think of the sites $q_{n'}$ outside Λ being frozen to zero):

$$q\cdot A_\Lambda^D q = (m_0^2\delta^2 + 4) \sum_{n \text{ in } \Lambda} q_n^2 - \sum_{\langle n,n'\rangle \text{ in } \Lambda} q_n q_{n'} \tag{16}$$

where by n in Λ we mean $n\delta \in \Lambda$ and by $\langle n , n'\rangle$ in Λ we mean that the sum is over nearest neighbours with $n\delta , n'\delta$ in Λ . For N B.C. we drop the coupling terms $(q_n - q_{n'})^2$ across $\partial\Lambda$ since this simulates zero normal derivative:

$$q\cdot A_\Lambda^N q = m_0^2\delta^2 \sum_{n \text{ in } \Lambda} q_n^2 + \frac{1}{2} \sum_{\langle n,n'\rangle \text{ in } \Lambda} (q_n - q_{n'})^2 . \tag{17}$$

To obtain P B.C. we simply introduce couplings $q_n q_{n'}$ between boundary spins at opposite edges:

$$q \cdot A_\Lambda^P q = (m_0^2 \delta^2 + 4) \sum_{n \text{ in } \Lambda} q_n^2 - \sum_{\langle n,n' \rangle_P \text{ in } \Lambda} q_n q_{n'} \tag{18}$$

where the notation $\langle n, n' \rangle_P$ indicates actual nearest neighbours or sites at opposite edges of the rectangle Λ. The case of F B.C. on $\partial\Lambda$ is the most difficult to write down (see [16, §IV]). An examination of (16) - (18) shows that the measures $d\mu_{\Lambda,\delta}^X$ are all *ferromagnetic* in the sense that the off-diagonal entries of A_Λ^X are non-positive; however, some B.C. (F , P , N) are more ferromagnetic than others (D). (See Application 2 below for a discussion of the resulting correlation inequalities.)

In support of my contention that flexibility in the choice of B.C. is useful, let me list some of the advantages of each of the classical B.C.:

Free B.C. are simplest to calculate with since G_0 is diagonal in momentum space, $G_0(x, y) = (2\pi)^{-2} \int e^{ik\cdot(x-y)} (k^2 + m_0^2)^{-1} dk$.

Dirichlet B.C. provide the easiest way of introducing barriers between regions of space (as in the cluster expansion [10]); the pointwise inequality $G_\Lambda^D(x, y) \leq G_0(x, y)$ leads to particularly simple estimates. As can be seen from (16) D B.C. play the role of "free boundaries" in ferromagnetic spin systems; hence D B.C. are most suitable for correlation inequality arguments (e.g. Nelson's monotonicity theorem [24] and relations between D Schwinger functions and other B.C. Schwinger functions as described below).

Periodic B.C. are "closest" to the infinite volume theory in the sense that P states are translation invariant. P B.C. are best for implementing transformations of the field or measure such as a mass shift or $\phi(x) \to \phi(x) + c$ (see e.g. Spencer [34] and [18]). P B.C.

in the "time" direction of the free measure give a trace formula, as first pointed out by Hoegh-Krohn [19]; for instance, if Λ is the rectangle $(-\ell/2, \ell/2) \times (-t/2, t/2)$,

$$Z_\Lambda^{HP} = \int e^{-U_\Lambda} d\mu_\Lambda^P = \frac{\operatorname{Tr}(e^{-t H_\ell^{HP}})}{\operatorname{Tr}(e^{-t H_{0,\ell}^P})} .$$

Here $H_\ell^{HP} = H_{0,\ell}^P + H_{I,\ell}^{HP}$ where HP denotes free Wick subtractions.

Neumann B.C. The submultiplicativity of N B.C. ("repulsion" between regions) leads immediately to infinite volume estimates given a finite volume estimate. For example, suppose Λ_1 is a unit square and Λ a union of unit squares; then by (12) and (11)

$$\alpha_\Lambda^X \le \alpha_\Lambda^N \le \alpha_{\Lambda_1}^N \quad \text{(Linear lower bound).}$$

Proof that $\alpha_\infty^N \le \alpha_\infty$

We realize N B.C. in a passive picture by "changing coordinates". More precisely, we write

$$G_\Lambda^N = G_0 + \delta G$$

where, by (7), δG is a positive operator on $L^2(\Lambda)$; and we realize the N B.C. field as

$$\phi_\Lambda^N = \phi + \delta\phi . \qquad (19)$$

In (19), the right side is a sum of *independent* Gaussian processes, ϕ and $\delta\phi$, with zero means and covariances G_0 and δG respectively. We denote the expectation for the process ϕ_Λ^N by $\langle\cdot\rangle$.

If $R \subset \Lambda$ are rectangles, we introduce the "pressure" with interaction in R and N B.C. on $\partial\Lambda$ by

$$\alpha_R^{N,\Lambda}(\lambda) = \frac{1}{|R|} \log \langle e^{-\lambda U_R^{N,\Lambda}} \rangle$$

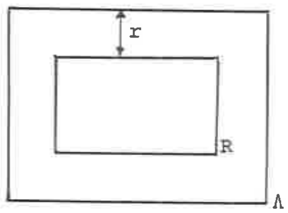
where $\lambda \geq 0$ and

$$U_R^{N,\Lambda} = \int_R :P(\phi_\Lambda^N):(x)\,dx$$

with Wick subtractions made with respect to G_Λ^N. The pressure in R with F B.C. is

$$\alpha_R(\lambda) = \frac{1}{|R|} \log \langle e^{-\lambda U_R} \rangle$$

where $U_R = \int_R :P(\phi):(x)\,dx$ with G_0 Wick subtractions.



Suppose that the sides of R are a distance $r = |\Lambda|^\eta$ from those of Λ, for some fixed η in $(0, \frac{1}{2})$; and suppose that $\Lambda \to \infty$ in the sense of Fisher (see [30]) so that in particular the diameter of Λ, $d(\Lambda)$, is of the order of $|\Lambda|^{\frac{1}{2}}$. It follows that the area of the corridor $|\Lambda \backslash R| = O(|\Lambda|^{\eta+\frac{1}{2}})$. We now prove that $\alpha_\infty^N \leq \alpha_\infty$ in two steps:

Step 1 (Strip removal) For any $\lambda > 1$ there is a constant c such that

$$\alpha_\Lambda^N \leq \alpha_R^{N,\Lambda}(\lambda)/\lambda + c|\Lambda|^{\eta-\frac{1}{2}} . \tag{20}$$

Step 2 $\lim_{\Lambda\to\infty} \alpha_R^{N,\Lambda}(\lambda) = \alpha_\infty(\lambda)$.

Combining these two steps we obtain

$$\lim_{\Lambda\to\infty} \alpha_\Lambda^N \leq \alpha_\infty(\lambda)/\lambda \tag{21}$$

for any $\lambda > 1$. But $\alpha_\infty(\lambda)$ is convex in λ and therefore continuous so that we may take $\lambda \to 1$ in (21) to deduce that

$\alpha_\infty^N \le \alpha_\infty$.

Proof of Step 1. By Holder's inequality $(1/\lambda + 1/\lambda' = 1)$

$$\langle e^{-U_\Lambda^{N,\Lambda}} \rangle \le \langle e^{-\lambda U_R^{N,\Lambda}} \rangle^{1/\lambda} \langle e^{-\lambda' U_{\Lambda\setminus R}^{N,\Lambda}} \rangle^{1/\lambda'} . \tag{22}$$

But by the linear lower bound

$$\langle e^{-\lambda' U_{\Lambda\setminus R}^{N,\Lambda}} \rangle \le e^{O(|\Lambda\setminus R|)} = e^{O(|\Lambda|^{\eta+\frac{1}{2}})} . \tag{23}$$

(20) follows from (22) and (23).

It is possible to prove a stronger result than Step 2 (taking $\lambda = 1$ without loss of generality):

Step 2' (Principle of not feeling the boundary)

$$\lim_{R\to\infty} \frac{\langle e^{-U_R} \rangle}{\langle e^{-U_R^{N,\Lambda}} \rangle} = 1 . \tag{24}$$

Proof. Let $D = \langle e^{-U_R} \rangle / \langle e^{-U_R^{N,\Lambda}} \rangle - 1$. Since

$$e^{-x} - e^{-y} \le \tfrac{1}{2}|x-y|(e^{-x} + e^{-y})$$

we obtain

$$|D| \le \tfrac{1}{2} \langle |\delta U| (e^{-U_R^{N,\Lambda}} + e^{-U_R}) \rangle / \langle e^{-U_R^{N,\Lambda}} \rangle \tag{25}$$

where $\delta U = U_R^{N,\Lambda} - U_R$. Say that $P(y) = y^d$. An explicit calculation yields

$$\langle \delta U^2 \rangle = \sum_{j=1}^{d} c_{d,j} \int_R \int_R G_0(x-y)^{d-j} \, \delta G(x,y)^j \, dxdy \tag{26}$$

where the $c_{d,j}$ are combinatorial factors. It is easy to see by the method of images that for all x, y in R

$$|\delta G(x,y)| \le \text{const. } e^{-2m_0 r} . \tag{27}$$

Since $G_0(x-y)$ has only a logarithmic singularity we deduce from (26) and (27) that for some constant a

$$\langle \delta U^2 \rangle \le a^2 |R|^2 e^{-2m_0 r} .$$

By hypercontractivity (see e.g. [24]) we can extend this to an estimate on the L^p norm $\| \delta U \|_p = \langle | \delta U|^p \rangle^{1/p}$ for $p \ge 2$:

$$\begin{aligned} \| \delta U \|_p &\le (p-1)^{d/2} \| \delta U \|_2 \\ &\le a(p-1)^{d/2} |R| \, e^{-m_0 r} . \end{aligned} \tag{28}$$

Next we apply Holder to (25):

$$|D| \le a p^{d/2} |R| e^{-m_0 r} \| e^{-U_R^{N,\Lambda}} \|_{p'} / \| e^{-U_R^{N,\Lambda}} \|_1 \tag{29}$$

where we have used the fact that $\| e^{-U_R} \|_{p'} \le \| e^{-U_R^{N,\Lambda}} \|_{p'}$ by conditioning. (29) looks a little unfortunate since $\| e^{-U_R^{N,\Lambda}} \|_{p'} \approx e^{O(|R|)}$. However, we note that by interpolation for $1 \le p' \le 2$,

$$\| f \|_{p'} / \| f \|_1 \le (\| f \|_2 / \| f \|_1)^{2/p} .$$

Now $\| e^{-U_R^{N,\Lambda}} \|_2 \le e^{O(|R|)}$ by the linear lower bound and $\| e^{-U_R^{N,\Lambda}} \|_1 \ge e^{-O(|R|)}$ by Jensen's inequality. Thus if we choose $p = |R|$ we obtain

$$\| e^{-U_R^{N,\Lambda}} \|_{p'} / \| e^{-U_R^{N,\Lambda}} \|_1 \le e^{O(|R|) 2/p} = e^{O(1)}$$

so that by (29) $\quad |D| \le \text{const.} \, |R|^{d/2+1} e^{-2m_0 r}$

which goes to zero as $R \to \infty$.

<u>Convergence of the Lattice Approximation</u>

The main justification for saying that the lattice measures $d\mu^X_{\Lambda,\delta}$ (defined by (16) - (18)) correspond to X B.C. is the proof of convergence as $\delta \to 0$. In [16] we proved convergence for $X = F, D$. When Λ is a rectangle a similar proof for $X = N, P$ may be based on taking Fourier transforms with respect to the eigenfunctions of $-\Delta_\Lambda^X + m_0^2$ (see [18] for definitions and details):

Theorem 2. Let $\sigma = P, N, HP, HN$. Suppose $h_1, \ldots, h_r \in C_0^\infty(\Lambda)$. As $\delta \to 0$ the lattice Schwinger functions $S_{\Lambda,\delta}^\sigma(h_1, \ldots, h_r)$ converge to the continuum Schwinger functions

$$S_\Lambda^\sigma(h_1, \ldots, h_r) = \frac{\int \phi(h_1)\cdots\phi(h_r) e^{-U_\Lambda^\sigma} d\mu_\Lambda^\sigma}{\int e^{-U_\Lambda^\sigma} d\mu_\Lambda^\sigma} .$$

Remarks 1. For convenience we assume that $\delta \to 0$ through a sequence of values (e.g. $\delta_j = \ell/(2j+1)$) so that the sides of Λ lie midway between lattice points.

2. Thus we obtain correlation inequalities for X B.C. (see Applic. 2 below).

We now turn to some consequences of Theorems 1 and 2 (for further details see [17] and [18]).

Application 1. Gibbs Variational Equality

In [16] we were able to establish only the Gibbs variational *inequality*: for any weakly tempered translation invariant state f

$$s(f) - \rho(f, P) \le \alpha_\infty(P) \tag{30}$$

where $s(f)$ is the entropy density, $\rho(f, P)$ the mean interaction, and $\alpha_\infty(P)$ the pressure as a function of the semibounded polynomial P. Given Theorem 1, we can prove equality [18]:

$$\sup_f [s(f) - \rho(f, P)] = \alpha_\infty(P) \tag{31}$$

where the supremum takes place over all tempered, translation invariant states f.

The Gibbs variational equality was one of the main goals of Robinson's work [29] on the independence of the pressure on B.C. in quantum statistical mechanics. Our proof of (31) is patterned after his. In essence one considers states f^D constructed by carving up $\mathbb{R}^2$ into a union of large squares and defining f^D as the (averaged) product of Gibbs states for each square with Dirichlet B.C. Since such f^D factor, the calculations may be simply performed and one finds that

$$\sup\,[s(f^D) - \rho(f^D, P)] \geq \alpha_\infty^D(P) \quad . \tag{32}$$

Clearly the desired equality (31) follows from (30), (32) and the fact that $\alpha_\infty^D = \alpha_\infty$.

Application 2. Correlation Inequalities

The ferromagnetic nature of the lattice theories with X B.C. (see (16) - (18)) and the lattice convergence result of Theorem 2 imply immediately by the methods of [16, 33] that all the correlation inequalities known for free B.C. hold as well for X B.C. Thus we have for B.C. $\sigma = F, D, N, P, HD, HN, HP$:

a) Griffiths-Kelly-Sherman inequalities [8] for $P(x) = P_e(x) - \mu x$ where P_e is even and $\mu \geq 0$.

b) Fortuin-Kasteleyn-Ginibre inequality [6] for arbitrary (semi-bounded) P .

c) Griffiths-Hurst-Sherman [13], Lebowitz [22], and Newman [25] inequalities if $P(x) = ax^4 + bx^2 - \mu x$, $a > 0$, $\mu \geq 0$. For example [25],

$$|\langle e^{\phi(f)}\rangle| \leq \exp[S_1(|\mathrm{Re}\, f|) + \tfrac{1}{2}S_{2,T}(|\mathrm{Re}\, f|, |\mathrm{Re}\, f|)]$$

where S_j is the j-point Schwinger function and $S_{2,T}(x,y) = S_2(x,y) - S_1(x)S_1(y)$.

d) Ursell$_6$ (Cartier [2], Percus [27], Sylvester [35])

$$U_6(x_1, \ldots, x_6) \geq 0$$

if $P(x) = ax^4 + bx^2$.

In addition the Lee-Yang theorem holds for the above B.C. if $P(x) = ax^4 + bx^2 - \mu x$ [33].

What about the basic lattice (14)? It is natural to conjecture that the Schwinger functions for different B.C. are related as in (14). However it seems possible to relate only D B.C. to the others:

<u>Theorem 3</u>. [18] If $P(x) = P_e(x) - \mu x$, $\mu \geq 0$ then

$$S_\Lambda^{HD} \leq S_\Lambda^{HX} \qquad X = F, N, P . \tag{33}$$

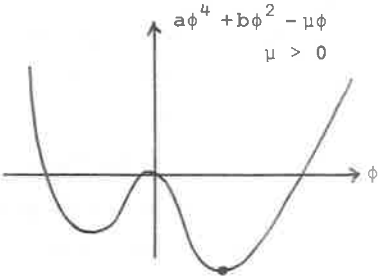
If in addition $\deg P \leq 4$, then

$$S_\Lambda^{D} \leq S_\Lambda^{X} \qquad X = F, N, P . \tag{34}$$

<u>Discussion</u>. As remarked above in the discussion of the lattice expressions (16)-(18), F, N and P B.C. are more ferromagnetic than D B.C.; hence (33) follows from the second GKS inequality (see e.g. [16, §V]). For X (as opposed to HX) B.C., a change in B.C. involves a change in Wick ordering; in the special case when $\deg P = 4$, this change involves only a quadratic term and can be controlled; hence (34) (see [16, Theorem V.20B]). As for inequalities among other B.C., consider S_Λ^{HP} and S_Λ^{HN} . An examination of (17) and (18) shows that for some n , $0 < (A_\Lambda^N)_{nn} < (A_\Lambda^P)_{nn}$, but, on the other hand, for some n, n' , $(A_\Lambda^P)_{nn'} < (A_\Lambda^N)_{nn'} \leq 0$. Thus neither of P or N is more ferromagnetic than the other and an inequality like $S_\Lambda^{HP} \leq S_\Lambda^{HN}$ seems doubtful. Similarly it is tempting to conjecture that S_Λ^{HN} is monotone decreasing in Λ (S_Λ^{HD} *is* monotone increasing); however such a result can at best be true only for some values of the coupling constants.

<u>Application 3</u>. <u>Mass Gap for Nonzero External Field</u>

Consider the $(a\phi^4 + b\phi^2 - \mu\phi)_2$ model (similar arguments hold in $d=3$ dimensions (see Frohlich [7])). For large $|\mu|$, Spencer [34] proved that the infinite volume theory with P B.C. exists and has a positive mass gap. According to the Goldstone picture, however, there should be a unique infinite volume state with positive mass gap for any $\mu \neq 0$. By adapting an argument of Penrose-Lebowitz [28] for finite range lattice spin systems that was based on the Lee-Yang theorem and the theory of subharmonic functions, Guerra-Rosen-Simon [17] proved this result:



unique minimum
with positive curvature

<u>Theorem 4</u>. Let $P(x) = ax^4 + bx^2 - \mu x$, $a > 0$, $\mu \neq 0$. Then the infinite volume (Dirichlet or half-Dirichlet) $P(\phi)_2$ theory has a positive mass gap.

<u>Remarks</u>. 1. Our methods also establish the following: Suppose $a > 0$ and b are such that $\mu = 0$ is not a limit point of the (purely imaginary) roots of

$$Z^X_{\ell,t}(\mu) = \int \exp\left[-\int_{\ell \times t} :a\phi^4 + b\phi^2 - \mu\phi: dx\right] d\mu^X_\ell = 0$$

for sufficiently large ℓ, t . Here $X = P$ or D and $d\mu^X_\ell$ is the free measure with X B.C. on the strip $[-\ell/2, \ell/2] \times \mathbb{R}$. Then the infinite volume (Dirichlet) $a\phi^4 + b\phi^2$ theory has a mass gap.

2. The reason that Theorem 4 may be considered an application of Theorem 1 is that (at the time of writing of [17]) we had to make a transition from P states (which we knew existed only for large $|\mu|$) to D states (which existed for all μ). To this end we used the inequality (34), $S^D_\Lambda \leq S^P_\Lambda$, and the equality of the one-point P and D functions in the infinite volume limit (a consequence of Theorem 1).

3. Subsequently Frohlich [7] has shown that the existence of a $a\phi^4 + b\phi^2 - \mu\phi$ theory for one value of μ implies existence for all μ (overcoming the difficulty referred to in the above Remark) and he has given a simple proof of Theorem 4.

Application 4. Identity of Certain States

Under the hypotheses of a uniformly positive mass gap in the space cutoff ℓ and the existence of the infinite volume limit, it is possible to prove the identity of HX states for $P(x) = P_e(x) - \mu x$ theories [18]. The proof is based on the idea that (in statistical mechanics language) the correlation functions are related to the tangent plane to the pressure functional; hence if the pressure is independent of B.C. the same should be true of the correlation functions, for almost all values of the thermodynamic variables (where the pressure has a unique tangent plane). The best result of this type is due to Frohlich who proves (among other things):

Theorem 5 (Frohlich [7]). Consider the $P(\phi)_2$ model with $P(\phi) = a\phi^4 + b\phi^2 - \mu\phi$ where $\mu \neq 0$. The infinite volume D, P, HD, HP Schwinger functions are identical.

One expects that this result remains true for the value $\mu = 0$, since, even if the infinite volume state is nonunique, the D, P, HD, HP theories will all be an exact average of the (presumed) two pure states in order to have $\langle\phi(0)\rangle = 0$.

Application 5. Covariances of and Bounds on α_∞

By working with the appropriate B.C. it is easy to establish certain covariances of the pressure under a) scaling, $x \to \lambda x$; b) field translations, $\phi(x) \to \phi(x) + c$; c) mass shifts, $m_0 \to m_0'$ (for details see [18]).

Using these covariances, we have shown that the bound of [15] (deg P = 2n),

$$\alpha_\infty(\lambda) \le \text{const. } \lambda(\log \lambda)^n$$

as the coupling constant $\lambda \to \infty$ is best possible in the sense that [18]

$$\alpha_\infty(\lambda) \ge \text{const. } \lambda(\log \lambda)^n .$$

The proof follows a suggestion by R. Baumel (private communication) and is based on a variational calculation with the mean of ϕ and the bare mass as parameters.

In [18] we also obtain a bound on α_∞ as the subdominant coupling constants go to ∞ : if $P(x) = a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_0$ with $a_{2n} > 0$ fixed, then

$$|\alpha_\infty(P)| \le \text{const.} \left(1 + \sum_{j=1}^{2n} |a_{2n-j}|^{2n/j}\right) . \tag{35}$$

Since the interaction polynomials for X and HX theories differ by lower order terms with only logarithmically divergent coefficients, (35) leads to the equality $\alpha_\infty^{HX} = \alpha_\infty^{X}$ of Theorem 1.

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A possible constructive approach to ϕ_4^4

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Abstract: We suggest a constructive (euclidean) approach to ϕ_4^4 using multiplicative renormalization.

Résumé : On suggère une approche constructive (euclidienne) pour ϕ_4^4 qui utilise la renormalisation multiplicative.

Talk presented at the International Colloquium on Mathematical Methods of Quantum Field Theory, Marseille, June 23-27, 1975.

Using euclidean methods's, constructive quantum field theory has obtained a considerable control over superrenormalizeable field theories (see e.g. [15] +)). Thus it seems to be consensus among constructivists that time has come to take a closer look at field models which are renormalizeable but not superrenormalizeable. The purpose of my talk is to propose an approach which is based on multiplicative renormalization.

Although Zimmermann [16] has added greatly to the understanding of multiplicative renormalization using ideas of Wilson on the short distance behaviour in operator product expansions, in my point of view multiplicative renormalization has up till now not been able to arouse the interest of the mathematically inclined physicist, at least not in proportion to its importance.

So my intention will also be to argue that multiplicative renormalization could be made into a powerful tool in constructive (euclidean) quantum field theory.

Since the content of my talk will not consist in presenting results as in outlining a possible constructive program for ϕ_4^4, I will make no effort in presenting the material with mathematical rigour. The listener, however, who feels uncomfortable, may translate everything down to two space-time dimensions, where most objects "defined" below will exist.

The ϕ_4^4 theory, the relativistic field theory of scalar, massive bosons in 4-space-time dimensions, is a good candidate of a renormalizeable but not superrenormalizeable theory. For two or three space-time dimensions the euclidean approach is well understood (see e.g. [3], [5], [9]) in terms of a measure

+) See also the contributions to these proceedings.

$$d\mu = \frac{e^{-V}\, d\mu^{0}}{\int e^{-V}\, d\mu^{0}}$$

where μ^{0} is a Gaussian measure with mean zero and covariance $(-\Delta + m^2)^{-1}$. V is the euclidean action determined by the interaction and contains the counterterms. The Taylor series expansion of the exponential in the moments of this measure leads to the Gell-Mann-Low series in the euclidean region.

The success of the euclidean formulation is roughly based on the fact that estimates on e^{-V} are possible which do not rely on perturbation theory.

Now for superrenormalizeable theories it is possible to incorporate the counterterms into V using additive renormalization. In the framework of additive renormalization, for theories which are renormalizeable but not superrenormalizeable an infinite series (in the coupling constant) of counterterms is necessary. Hence the definition of V even after the introduction of cut-offs becomes difficult. Now by the very definition of a renormalizeable field theory, there are (modulo numerical factors) only a finite number of operators appearing in the set of counter terms. Collecting the terms of the same operator form is the first step leading to multiplicative renormalization. This suggests the use of multiplicative renormalization in constructive quantum field theory. To see how this may be done we assume there exists a ϕ^4_4 theory which we control through multiplicative renormalization and we ask the question: If there is a measure μ on $\mathcal{S}'(\mathbb{R}^4)$ whose moments are the euclidean Green's functions, what should the measure look like? The answer is easy and may be obtained through the so called field equations. For the euclidean Green's functions they read as follows:

$$
\begin{aligned}
&Z_3(-\Delta+m^2)\langle\phi(x)\phi(y_1)\cdots\phi(y_N)\rangle\\
&-Z_3\,\delta m^2\langle\phi(x)\phi(y_1)\cdots\phi(y_N)\rangle\\
&+4\lambda\,Z_4\langle{:}\phi^3{:}(x)\phi(y_1)\cdots\phi(y_N)\rangle\\
&-\sum_{i=1}^{N}\delta(x-y_i)\langle\phi(y_1)\cdots\widehat{\phi(y_i)}\cdots\phi(y_N)\rangle=0
\end{aligned}
\tag{1}
$$

The notation is as follows: $\langle\cdot\rangle$ denotes the expectation w·r·t the measure μ, i.e $\langle\cdot\rangle=\int\cdot\,d\mu$. ϕ is the Euclidean field, i.e $\phi(f)=\int f(x)\phi(x)dx$ is the linear function

$$\phi(f)\;:\;f'\longrightarrow(f',f)$$

on $f'(\mathbb{R}^4)$. $: \;:$ denotes normal ordering w·r·t $d\mu$, i.e

$$
\begin{aligned}
{:}\phi^3{:}(x) &= \lim_{x_i\to x}\;{:}\phi(x_1)\phi(x_2)\phi(x_3){:}\\
&= \lim_{x_i\to x}\;\phi(x_1)\phi(x_2)\phi(x_3)-\phi(x_1)\langle\phi(x_2)\phi(x_3)\rangle\\
&\quad-\phi(x_2)\langle\phi(x_1)\phi(x_3)\rangle-\phi(x_3)\langle\phi(x_1)\phi(x_2)\rangle
\end{aligned}
$$

m is the physical mass, λ the renormalized coupling constant, Z_3 $(0\leq Z_3\leq 1)$ is the amplitude renormalization constant and $Z_4\geq 0$ the vertex function renormalization constant. δm^2 is the mass counter term. They are given explicitly (e.g. by Zimmermann [16]) in terms of (on-shell vacuum) expectation values of the relativistic fields. Also the two-point function is normalized on-shell in the usual way.

Zimmermann's analysis, however, easlily carries over to the so called intermediate renormalization [1] , where the two point function is normalized at zero momentum. This renormalization will of course be more convenient for the euclidean approach. In perturbation theory it differs from the standard on-shell normalization only by a finite multiplicative renormalization. The result is as follows

Let
$$\tilde{\phi}(p) = (2\pi)^{-2} \int e^{-ipx} \phi(x)\, dx$$
$$\tilde{\Delta}(p^2) = \int e^{-ipx} \langle \phi(0)\, \phi(x) \rangle\, dx$$
$$= (2\pi)^{2} \langle \phi(0)\, \tilde{\phi}(0) \rangle$$
$$\alpha(p^2) = (2\pi)^2 \langle :\phi^3:(0)\, \tilde{\phi}(p) \rangle\, \tilde{\Delta}(p^2)^{-1}$$
$$\gamma = \frac{(2\pi)^6}{6} \langle :\phi^3:(0)\, \tilde{\phi}(0)^3 \rangle^{IPI} \tag{2}$$

Here $\langle\,\rangle^{IPI}$ denotes the usual one-particle irreducible amplitude. Then

$$Z_4 = \gamma^{-1}$$
$$Z_3 = 1 - 4\lambda Z_4\, \alpha'(0) \tag{3}$$
$$\varepsilon = Z_3 \delta m^2 = 4\lambda Z_4 (\alpha(0) - m^2 \alpha'(0))$$

with

$$\alpha'(p^2) = \frac{\partial}{\partial p^2}\, \alpha(p^2)$$

In the intermediate renormalization the two point function is normalized by the condition

$$\tilde{\Delta}(p^2)^{-1} \approx p^2 + m^2 \tag{4}$$

for small p^2. Of course, in this renormalization m^2 will not be the physical mass (the physical mass will be smaller). Also Z_3 will not necessarily be smaller than 1. The measure μ leading to the field equations is then given by normalizing

$$d\mu' = e^{-\lambda Z_4 \int :\phi^4:(x)dx + \frac{\delta m^2 Z_3}{2} \int :\phi^2:(x) dx} \cdot d\mu^0_{(Z_3, m)} \tag{5}$$

Here $d\mu^0_{(Z_3,m)}$ is the Gaussian measure on $\mathcal{S}'(\mathbb{R}^4)$ with mean zero and covariance $[Z_3(-\Delta + m^2)]^{-1}$. Now due to the relations (4), μ is implicitly defined: The right hand side of (5) is given in terms of quantities depending on μ. In physical terms we may say the vacuum (i.e μ) is stable under the interaction to which it is associated. This has the smell of a Hartree-Fock self-consistency condition. We note that a similar point of view has been stressed and elaborated in a recent paper by Chang [2].

Mathematically this picture of course invites to a formulation as a fixed point problem. We now present such an approach: The set $\mathfrak{F}$ of all even, translation and rotation invariant ferromagnetic measures on $\mathcal{S}'(\mathbb{R}^4)$ is defined to consist of all probability measures ν of the form

$$d\nu = \left(\int d\nu'\right)^{-1} d\nu'$$

$$d\nu' = \exp\left\{-\lambda Z_4 \int :\phi^4:(x)\, dx + \frac{\varepsilon}{2}\int :\phi^2:\, dx\right\} \cdot d\mu^0(Z_3, m) \tag{6}$$

Here λ, m, Z_3, Z_4 are now arbitrary positive numbers and ε is real. $::$ denotes Wick-ordering w.r.t. $d\mu^0_{(Z_3,m)}$. For such ν the Griffiths [7] and Lebowitz [8] inequalities as well as the Lee-Yang theorem [14] hold.

Since our aim is to construct theories parametrized by λ and m, those parameters will stay fixed in what follows. Hence $\mathfrak{F}$ looks like $(\mathbb{R}^+)^2 \times \mathbb{R}$. In analogy to the quantities $\tilde{\Delta}(p)$, $\alpha(p^2)$, γ, Z_4 etc. obtained from μ (see (4)) to each ν we may associate corresponding quantities. We indicate the ν dependence by the suffix ν.

In addition we define

$$\lambda_{ren,\nu} = -\frac{(2\pi)^6}{4!} \langle \tilde{\phi}(0)^3 \phi(0) \rangle_\nu^{IPI} \tag{7}$$

By Griffiths first inequality

$$\langle \phi(x)\, \phi(y) \rangle_\nu \geq 0 \tag{8}$$

such that

$$\tilde{\Delta}_\nu(0) > 0 \tag{9}$$

Using in addition the Nelson-Symanzik positivity condition

$$0 \le \tilde{\Delta}_\nu(p^2) \le \tilde{\Delta}_\nu(0) \tag{10}$$

In particular

$$-\tilde{\Delta}_\nu'(0)\,\tilde{\Delta}_\nu(0)^{-2} = \frac{\partial}{\partial p^2}\tilde{\Delta}_\nu(p^2)^{-1}\Big|_{p^2=0} \ge 0 \tag{11}$$

Also by the Lebowitz inequality and a strong version of the Marcinkiewicz theorem [10]

$$\lambda_{ren,\nu} > 0 \tag{12}$$

Now if ν is of the form given by (6), then $\lambda \cdot Z_4$ and Z_3 may be recovered by the formulae

$$\lambda Z_4 = \lambda_{ren,\nu}\, Z_{4,\nu} \tag{13}$$

$$Z_3 = \frac{\partial}{\partial p^2}\tilde{\Delta}(p^2)^{-1}\Big|_{p^2=0} - 4\lambda Z_4\, \alpha_\nu'(0) \tag{14}$$

In particular we have

$$Z_{4,\nu} > 0 \tag{15}$$

We define for $\nu \in \tilde{S}$, $\tilde{\xi}$ real and $\tilde{Z}_3 > 0$ a new measure $\rho = \rho(\nu, \tilde{\xi}, \tilde{Z}_3)$ by setting

$$\rho = \left(\int d\rho'\right)^{-1} d\rho'$$

$$d\rho' = e^{-\lambda Z_{4,\nu}\int :\phi^4:_\nu(x)dx \;+\; \frac{\tilde{\xi}}{2}\int :\phi^2:_\nu(x)dx}$$

$$\cdot\, d\mu^0(\tilde{Z}_3, m) \tag{16}$$

By Griffiths second inequality

$$\frac{\partial}{\partial \tilde{\xi}}\tilde{\Delta}_\rho(0) \ge 0 \tag{17}$$

and

$$\frac{\partial}{\partial \tilde{\xi}}\tilde{\Delta}_\rho'(0) \le 0 \tag{18}$$

and we expect

$$\inf_{\tilde{\varepsilon},\tilde{Z}_3} \Delta_\rho(o)=o \ ; \quad \sup_{\tilde{\varepsilon},\tilde{Z}_3} \Delta_\rho(o) = \infty \tag{19}$$

$$\inf_{\tilde{\varepsilon},\tilde{Z}_3} \frac{\partial}{\partial p^2} \Delta_\rho(p^2)^{-1}\Big|_{p^2=o} = o \ ; \quad \sup_{\tilde{\varepsilon},\tilde{Z}_3} \frac{\partial}{\partial p^2} \Delta_\rho(p^2)^{-1}\Big|_{p^2=o} = \infty \tag{20}$$

This leads us immediately to our central conjecture: There are (unique) $\tilde{\varepsilon}=\tilde{\varepsilon}(\nu)$, $\tilde{Z}_3=\tilde{Z}_3(\nu)$ sucht that for $\bar{\rho} = \rho(\nu,\tilde{\varepsilon}(\nu), \tilde{Z}_3(\nu))$ we have

$$\Delta_\rho(o) = m^{-2} \tag{21}$$

$$\frac{\partial}{\partial p^2} \tilde{\Delta}_\rho(p^2)^{-1}\Big|_{p^2=o} = 1 \tag{22}$$

Thus $\tilde{\varepsilon}$ and $\tilde{Z}_3$ play a rôle similar to Lagrange multipliers. They serve to satisfy the subsidiary conditions (21)-(22). The map $T : \nu \to \bar{\rho}$ then maps $\mathfrak{F}$ into ifself.

Let μ be a fixed point, i.e. $\mu = T\mu$. Then by (21) and (22) we have the intermediate renormalization. Also by (11)

$$\lambda_{ren,\mu} = \lambda \tag{23}$$

and in particular we have a solution of the field equations (1)-(3) with $\varepsilon=\tilde{\varepsilon}(\mu)$, $Z_3=\tilde{Z}_3(\mu)$. The relations (21) (for $\bar{\rho}=\mu$) and (23) are of particular interest. (23) guarantees the non-triviality of the theory thus obtained. The result (21) allows an estimate of the two-point function as a tempered distribution. By Griffiths inequality:

$$|\langle\phi(f)\,\phi(g)\rangle_\mu| = \left|\int f(x)\,g(y)\,\langle\phi(x)\phi(y)\rangle_\mu\,dx\,dy\right|$$

$$\leq \sup|f(x)| \int |g(y)|\,\langle\phi(x)\,\phi(y)\rangle_\mu\,dx\,dy$$

$$= \sup|f(x)| \int |g(y)|\,dy \cdot \tilde{\Delta}_\mu(0) \tag{24}$$

By arguments due to Glimm and Jaffe[4] , the higher moments may then also be estimated thus establishing the temperedness axiom for the euclidean Green's functions [11] . Collecting our results we have established what we hoped for : Z_4 is responsible for the relation (21), i.e. λ is the renormalized coupling constant and Z_3 and $=Z_3 \delta m^2$ are responsible for the right renormalization of the two-point function.

Thus we have formulated the renormalization program (in the framework of multiplicative renormalization) as a combination of a fixed point problem and an implicit function theorem.

We note that the perturbative expansion of the solution (1)-(3) in λ gives the usual additive renormalization in the BPHZ framework.

How could this now be made rigorous? One could start with lattices on a 4-dimensional torus. This would preserve translation invariance which was essential in deriving (11). Then the corresponding maps T should constructed. This requires the verification of our central conjecture. We have already been able to prove the properties (19) and (20). A good analysis of T (such as continuity properties etc.) should then lead to fixed points. As in[4] , for the resulting moments one could then choose a convergent subsequence (qua districutions), when the torus becomes infinite and the lattice spacing zero. The limit distributions are then the moments of a measure by Minlos' theorem [4]. If in addition the relations (21)-(23) are preserved in the limit, the theory will be nontrivial. This point is rather subtle for the following reason: The method suggested here is not restricted to a particular space-time dimension. In particular one could look at the nonrenormalizeable $(\phi^4)_6$ theory. We expect that in the lattice approximation everything that may be done in 4 dimensions also may be done in higher dimensions. Hence the difference between $(\phi^4)_4$ and $(\phi)^4_6$ should show up in the fact for $(\phi^4)_6$ it is impossible to retain the relations (21)-(23) in the limit. In other words, the limit theory always becomes trivial. Returning to the 4-dimensional case, since the physical positivity condition holds on the infinite lattice (see appendix), presumeably only the rotation invariance and the mass gap remain to be verified in order to establish the Wightman axioms [11].

We note that at several places nonuiqueness might come in. First the solution $\tilde{\varepsilon}=\tilde{\varepsilon}\ (\nu)$, $Z_3=\tilde{Z}_3(\nu)$ of the normalization problem might not be unique, leading to different T. Secondly the fixed point (if it exists) might not be unique and thirdly convergent subsequences need not have the same limit. As a consequence the parameters λ and m would not be sufficient to characterize a theory.

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Appendix

In this appendix we prove the physical positivity condition (see [11]) for euclidean bose fields on the lattice $\mathbb{Z}^2$. The proof for arbitrary $\mathbb{Z}^d$ $(d \geq 1)$ is analogous. Let $\ell^2(\mathbb{Z}^2)$ be the Hilbert space of all sequences $f = \{f_j\}_{j\in\mathbb{Z}^2}, f_j \in \mathbb{C}$ such that $(f, f) = \sum_{j\in\mathbb{Z}^2} |f_j|^2 < \infty$. Let Δ be the selfadjoint operator

$$(\Delta f)_j = \sum_{|j-j'|=1} f_{j'} - 4 f_j \tag{A1}$$

such that $\Delta \leq 0$ and let $m^2 > 0$. We define $\mathcal{H}$ to be the completion of $\ell^2(\mathbb{Z}^2)$ with the norm

$$\langle f, f \rangle = (f, (-\Delta + m^2)^{-1} f) < \infty \tag{A2}$$

We denote by $\langle , \rangle$ the corresponding scalar product. We define a unitary involution ϑ on $\mathcal{H}$

$$(\vartheta f)_j = f_{\vartheta j} \tag{A3}$$

with

$$\vartheta j = (-j_0, j_1) \quad \text{for } j = (j_0, j_1)$$

Let furthermore $e_{\pm}$ be the orthogonal projection in $\mathcal{H}$ on the closed subspace in $\mathcal{H}$ spanned by all $f \in \mathcal{H}$ with $f_j \equiv 0$ whenever $\mp j_0 > 0$.

<u>Lemma</u> $e_- e_+$ and $e_+ e_-$ are positive operators on $\mathcal{H}$. Equivalently ϑ is positive on $e_+\mathcal{H}$ and $e_-\mathcal{H}$ respectively.

<u>Proof</u> : We give a direct proof. Define the Fourier transform

$$\tilde{f}(k_0, k_1) = (2\pi)^{-1} \sum_{j\in\mathbb{Z}^2} e^{i(j_0 k_0 + j_1 k_1)} f_j \tag{A4}$$

such that

$$\int_{-\pi}^{+\pi}\int_{-\pi}^{+\pi} |\tilde{f}(k_0,k_1)|^2 \, dk_0 \, dk_1 = \sum_{j\in\mathbb{Z}^2} |f_j|^2$$

Then $(-\Delta+m^2)^{-1}$ goes over into the multiplication operator A given by the function

$$\left(4-2\cos k_0-2\cos k_1+m^2\right)^{-1} \tag{A5}$$

We also define the partial Fourier transform by

$$\hat{f}(j_0,k_1)=(2\pi)^{-1}\sum_{j_1} e^{ik_1j_1} f_j$$

Let $f\in e_+\mathcal{H}$ such that $f\equiv 0$ for $j_0<0$. Then

$$\langle f,\vartheta f\rangle=\int_{-\pi}^{+\pi}\int_{-\pi}^{\pi}\sum_{j_0,j_0'}\overline{\hat{f}(j_0,k_1)}\,\hat{f}(j_0',k_1)\cdot$$

$$\frac{\cos(j_0+j_0')k_0}{4-2\cos k_0-2\cos k_1+m^2}\,dk_0\,dk_1 \tag{A6}$$

We now use the following formula (see [6] rel. (3.6.13)) for $\gamma<1$ and $n\in\mathbb{Z}$

$$\int_{-\pi}^{+\pi}\frac{\cos nx}{1-\gamma\cos x}\,dx=\frac{2\pi}{(1-\gamma^2)^{\frac{1}{2}}}\left(\frac{1-\sqrt{1-\gamma^2}}{\gamma}\right)^{|n|} \tag{A7}$$

Defining

$$\gamma(k_1)=\frac{2}{4-2\cos k_1+m^2}<1$$

$$\alpha(k_1)=\frac{\pi\,\gamma(k_1)}{(1-\gamma^2(k_1))^{\frac{1}{2}}}$$

$$\beta(k_1)=\frac{1-(1-\gamma^2(k_1))^{\frac{1}{2}}}{\gamma(k_1)}>0 \tag{A8}$$

we may continue (A6) as

$$\langle f,\vartheta f\rangle=\int_{-\pi}^{+\pi}\alpha(k_1)\,\beta(k_1)^{j_0+j_0'}\,\overline{\hat{f}(j_0,k_1)}\,\hat{f}(j_0',k_1)\,dk_1$$

$$=\int_{-\pi}^{+\pi}\alpha(k_1)\,\overline{\check{f}(k_1)}\,\check{f}(k_1)\,dk_1\geq 0 \tag{A9}$$

with

$$\check{f}(k_1)=\sum_{j_0\geq 0}\hat{f}(j_0,k_1)\,\beta(k_1)^{j_0}$$

This proves the lemma, since the discussion for e_- is analogous. Elaborating further on (A9) it is possible to relate it to the minimal dilation picture as in [13].

By second quantization (see e.g [7][12][13])

$$E_{\pm} = \Gamma(e_{\pm}); \quad \Theta = \Gamma(\vartheta)$$ satisfy

$$E_{\pm} = \Theta E_{\mp} \Theta^{-1}$$ and we have the

Corollary: $E_+ E_-$ and $E_- E_+$ are positive. Equivalently Θ is positive on the range of E_+ and E_- respectively.

This is the physical positivity condition for the free Bose field on the lattive $\mathbb{Z}^2$. For the interacting case the positivity follows as e.g in [12] or [13].

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Massive Thirring Model and Sine-Gordon Theory

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Abstract

The Hamiltonian of the massive Thirring field $\psi(\underline{x})$ is related to a Sine-Gordon Hamiltonian for the Boson field $\phi(\underline{x})$, which generates the Boson ghost $\psi^B(\underline{x})$ of the Fermion field. An analogous relation permits the calculation of all thermodynamic correlation functions of the Littinger model.

Résumé

On lie l'hamiltonien du modèle de Thirring avec l'hamiltonien Sine-Gordon. Une relation analogue permet de calculer toutes les fonctions thermodynamiques de corrélation du modèle de Littinger.

In this note we present results concerning the relation between the massive Thirring model (resp. the Luttinger model) and the Sine-Gordon Theory. Coleman [1] established equality of certain time ordered Greens functions of the two theories in the sense of Feynman perturbation theory in the mass (Thirring model) and the cosine term (Sine-Gordon theory). He recovered the fermion charge structure in the Sine-Gordon theory only through an ad-hoc zero mass limiting procedure. An operator identity between free massless Boson and Fermion fields - the Boson Fermion reciprocity-allows to connect the two theories (on the level of operator identities[2]. Similar methods have been applied in discussing the Luther-Emery model[3]

The following notation will be used. $\Psi(m;x)$ denotes the free massive Fermion field in two dimensional space time with periodic box cut-off L,

$$\Psi(m,\underline{x}) = \frac{1}{\sqrt{L}}\sum_{x}\left(\frac{m}{\omega}\right)^{1/2}\left(e^{-i\underline{x}\underline{x}}\, u(m,x)\, a(x) + e^{i\underline{x}\underline{x}}\, u_c(m,x)\, a_c^*(x)\right)$$

where

$$\omega(x) = x^0 = (m^2+x^2)^{1/2}, \quad x = \frac{2\pi}{L}n - \frac{\pi}{L}, \quad n \text{ integer.}$$

$$u(m,x) = [2m(\omega-x)]^{-1/2}\begin{pmatrix} m \\ \omega-x \end{pmatrix}, \quad u_c(m,x) = \gamma^5 u(m,x)$$

$$\gamma^5 = \gamma^0\gamma^1, \quad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The current is defined by

$$j^\mu(\underline{x}) = :\psi^+\gamma^\mu\psi:(\underline{x}), \quad \psi^+ = \psi^*\gamma^0.$$

with the charges

$$Q^\mu = \int_0^L dx\, j^\mu(x^0,x).$$

In case m=o we write $\Psi(\underline{x}) = \Psi(m=0, \underline{x})$,

$$\Psi(\underline{x}) = \frac{1}{\sqrt{L}} \sum_{x} \left(e^{-i\underline{x}\underline{x}} u(x) a(x) + e^{i\underline{x}\underline{x}} u_c(x) a_c^*(x) \right)$$

$$u(x) = \begin{pmatrix} \Theta(-x) \\ \Theta(x) \end{pmatrix} = \lim_{m \to 0} \left(\frac{m}{\omega} \right)^{1/2} u(m, x).$$

It is useful to introduce light cone variables,

$$u_\tau = x^0 - \tau x^1 , \tau = 1, -1.$$

$$j_\tau = \frac{1}{2}(j^0 + \tau j^1) , Q_\tau = \frac{1}{2}(Q^0 + \tau Q^1).$$

On the Hilbert space ℓ^2 of square- summable sequences the unitary shift operator U is defined by

$$(Ug)(n) = g(n+1) , g \in \ell^2.$$

The selfadjoint charge operator Q is defined by

$$(Qg)(n) = ng(n)$$

with natural domain D(Q). Let furthermore L^2 be the Hilbert space of square integrable functions on the circle,

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\alpha \, \overline{f(\alpha)} g(\alpha).$$

The Fourier transformation maps ℓ^2 unitarily on to L^2;

$$(Ff)(\alpha) = \sum_n e^{in\alpha} f(n) , (F^{-1}f)(\alpha) = \left(\frac{1}{2\pi}\right) \int_0^{2\pi} d\alpha \, e^{-in\alpha} f(\alpha).$$

The shift operator is diagonal on L^2 and the charge operator the differentiation,

$$(Uf)(\alpha) = e^{-i\alpha} f(\alpha), (Qf)(\alpha) = -i \frac{d}{d\alpha} f(\alpha).$$

The Boson-Fermion reciprocity can be formulated as follows: Let $\Psi(\underline{x})$ be the Fermion Fock field and $j^\mu(\underline{x})$ the corresponding current. There exists a Boson field

$$\varphi(\underline{x}) = \phi_+(\underline{x}) - \phi_-(\underline{x}) \quad (1)$$

$$\phi_\tau(\underline{x}) = \frac{1}{\sqrt{L}} \sum_{\tau k>0} |k|^{-1/2} \left(e^{-ik\underline{x}} c(k) + e^{ik\underline{x}} c^*(k) \right) \qquad (2)$$

with $k=\frac{2\pi}{L}n$,n integer, so that

$$j_\tau(\underline{x}) = \frac{1}{\sqrt{2\pi}} \frac{d}{dx^0} \phi_\tau(\underline{x}) + \frac{1}{L} Q_\tau , \quad \tau = 1,-1. \qquad (3)$$

The representation of φ in the Fermion Fock space $\mathcal{F}$ is reducible,$[\varphi, Q_\tau]$=o; in every charge sector the representation of φ is irreducible and Fock[4].

Conversely let φ be a free massless , real Boson field on Fock space $\mathcal{F}^B$ given by formulas (1) (2). Define the space $\mathcal{F} = (\ell^2 \otimes \ell^2) \otimes \mathcal{F}^B$ with the charge operators

$Q_- = (Q \otimes 1) \otimes 1 , \quad Q_+ = (1 \otimes Q) \otimes 1$

and the charge shift operators

$U_- = (U \otimes 1) \otimes 1 , \quad U_+ = (1 \otimes U) \otimes 1$

Then $\Psi(\underline{x}) = \Psi^F(\underline{x}) \otimes \Psi^B(\underline{x})$ defined by

$$\Psi^F_\tau(\underline{x}) = U_\tau \exp\left[-\frac{2\pi i}{L} u_\tau (Q_\tau - 1/2) + i\pi Q_+\right] \qquad (4)$$

$$\Psi^B_\tau(\underline{x}) = \frac{1}{\sqrt{L}} \vdots \exp - \sqrt{2\pi}\, i\, \phi_\tau \vdots (\underline{x}) \qquad (5)$$

is a free massless Fermion Fock field.

Instead of φ (x) we could have used the Boson field $\chi(\underline{x}) = \phi_+(\underline{x}) + \phi_-(\underline{x})$. The two fields have the same commutator. The Fermion ghost $\Psi^F(\underline{x})$ of $\Psi(\underline{x})$ gets formally simplyfied if one introduces the infinitesimal generator P_τ of U_τ ,

$$\Psi^F_\tau(\underline{x}) = \exp -i\left(\pi Q_+ + \frac{2\pi}{L} u_\tau Q_\tau + P_\tau\right). \qquad (6)$$

The identity of Ψ^F given by (4) and (6) is a non-trivial operator statement.

The operators H^F_0 and H^B_0 defined by

$$H_0^F = \int_0^L dx : \psi^* i\partial_0 \psi : (x)$$

$$H_0^B = \frac{1}{2}\int_0^L dx \vdots (i\partial_0\varphi)^*(i\partial_0\varphi) + (i\partial_1\varphi)^*(i\partial_1\varphi) \vdots$$

are selfadjoint on their natural domain and related by Kronig's identity [5,4],

$$H_0^F = H_0^B + \frac{\pi}{L}(Q_+^2 + Q_-^2). \tag{7}$$

This identity follows directly from (3) and from the relation

$$H_0^F = \pi \int_0^L dx \vdots j_+^2 + j_-^2 \vdots . \tag{8}$$

In Fermion Fockspace the charge shift operators can be expressed in terms of the Fermion field, the potential ϕ_τ and the charges Q_τ by just reading equations (4)(5) backwards. The result is

$$U_\tau = (-1)^{Q_+}\sqrt{L}\, \exp\left[\frac{i\pi}{L}u_\tau Q_\tau + i\sqrt{2\pi}\,\phi_\tau^+\right]\psi_\tau \exp\left[\frac{i\pi}{L}u_\tau Q_\tau + i\sqrt{2\pi}\,\phi_\tau^-\right] \tag{9}$$

where $\phi_\tau^+ (\phi_\tau^-)$ denotes the positive (negative) frequency part of ϕ_τ . This is the periodic box version of the σ -field as discussed by Lowenstein and Swieca [6,7] , which in turn is a particular solution of the Thirring model (Klaiber's solution for the parameters $\alpha = \beta = \sqrt{\pi}$ [8])

The Wightman functions for the field $\Psi(x)$ are

$$\langle \psi_\tau^*(x_1) \cdots \psi_\tau^*(x_n)\psi_\tau(y_1)\cdots\psi_\tau(y_n)\rangle = \frac{\prod_{i,k}\langle\psi_\tau^*(x_i)\,\psi_\tau(y_k)\rangle}{\prod_{m<n}\langle\psi_\tau^*(x_m)\,\psi_\tau(x_n)\rangle\prod_{r<s}\langle\psi_\tau^*(y_r)\,\psi_\tau(y_s)\rangle} \tag{10}$$

The Wightman functions of $\Psi(x)$

converge in $\mathcal{S}'$ towards those of the Fermion field $\Psi(\underline{x})$ with no cut-off. The Wightman functions not invariant under the two dimensional gauge group R^2, $\psi_{\xi}(\underline{x}) \rightarrow \exp -i\alpha_{\xi} \cdot \psi_{\xi}(\underline{x})$ vanish.

Now we discuss the massive free Fermion field and its Boson counterpart φ . The field $\Psi(m)$ and the massless Fermion field Ψ are related by a unitary transformation W(m) in Fockspace,

$$\Psi(m) = W(m)\,\Psi\,W(m)^{-1} \tag{11}$$

The definition of the mass perturbed Hamiltonian requires an additive renormalization

$$(H_o^F + M)_{ren} = \lim_{\Lambda \to \infty} (H_o^{F\Lambda} + M^{\Lambda} - E^{\Lambda}) \tag{12}$$

$$M = m \int_0^L dx : \psi^+ \psi : (x)$$

where Λ denotes an ultraviolet cutoff and the limit is in the norm resolvent sense. $W(m)^{-1}$ maps the Fock space vacuum into the ground state $\Omega(m) = W(m)^{-1}\Omega$ of $(H + M)_{ren}$ and $E^{\Lambda} =$ $= (\Omega(m), (H_o^{F\Lambda} + M^{\Lambda})\,\Omega(m))$. It turns out that

$$H_m = \int_0^L dx : \psi^+(m)\,(-i\partial_1\gamma^1 + m)\,\psi(m) : (x) = W(m)(H_o^F + M)_{ren} W(m)^{-1}. \tag{13}$$

The operator $(H_o^F + M)_{ren}$ can also be defined as follows

$$(H_o^F + M)_{ren} = \int_0^L dx \; N_m\,(\psi^+(-i\partial\gamma + m)\psi)(x), \tag{14}$$

where N_m denotes the normal product of the Fermion operators defined by

$$N_m(\psi^*(x)\,\psi(y)) = \psi^*(x)\,\psi(y) - (\Omega(m), \psi^*(x)\,\psi(y)\,\Omega(m)). \tag{15}$$

In terms of Boson operators the renormalized Hamiltonian takes the form

$$(H_0^F+M)_{ren} = \frac{1}{2}\int_0^L dx\, N_m\left((i\partial_0\varphi)^*(i\partial_0\varphi) + (i\partial_1\varphi)^*(i\partial_1\varphi)\right) + \frac{\pi}{L}\sum_\tau Q_\tau^2 + \frac{2m}{L}\int_0^L dx\, N_m \cos\left(\sqrt{2\pi}\,\varphi(x) + \sum \tau\left(P_\tau + \frac{2\pi}{L}u_\tau Q_\tau\right)\right). \tag{16}$$

The domain of definition is the $W^{-1}(m)$ image of the domain $D(H_m)$ of the selfadjoint operator H_m. The derivation of (16) makes use of a slight generalization of Kronig's identity,

$$\int_0^L dx\, N_m(\psi^* i\partial_0\psi)(x) = \frac{1}{2}\int_0^L dx\, N_m(\phi_-'^2 + \phi_+'^2)(x) + \frac{\pi}{L}\sum_\tau Q_\tau^2. \tag{17}$$

The Hamiltonian $(H_0^F+M)_{ren}$ is up to the term $\frac{2\pi}{L}u_\tau Q_\tau$ the direct integral of Sine-Gordon Hamiltonians and commutes with $Q=Q_+ + Q_-$.
It is remarkable that already the free massive Fermion field gives rise to the theory of a boson field which is rather complicated. As an operator in the Boson Fock space $(H_0^F+M)_{ren}$ is just outside the range of validity of the theory developped in ref.[9] The Fermion Wightman functions converge again for $L\to\infty$ (as elements in $\mathcal{S}'$) towards those of the massive Fermion field without cut-off.

The Hamiltonian of the Luttinger model (respectively the Thirring model with box cut-off in case of local interaction) is given by

$$H = H_0^F + H_1^F \ , \quad H_1^F = \int_0^L dx\, dy\, :j_\mu(x) j^\mu(y):\, V(x-y) \tag{18}$$

or in terms of the Boson field

$$H_1^F = H_1^B + \frac{2}{L}\tilde{V}(0)\, Q_+ Q_-$$
$$H_1^B = -\frac{1}{\pi}\int_0^L dx\, dy\, :\phi_-'(x)\phi_+'(y) + \phi_+'(x)\phi_-'(y):\, V(x-y) \tag{19}$$

where $\tilde{V}$ denotes the Fourier transform of $V(x)$. The Hamiltonian H can be diagonalized by a Bogoliubov transformation which can be implemented by a unitary transformation $W(V)$ in Fock space $\mathcal{F}^B$ if[4] $\sum |k\tilde{V}(k)|^2 < \infty$. The ground state of the interacting Hamiltonian is given by $\Omega(V) = W(V)^{-1}\Omega$ and the thermodynamic n-point functions by [10]

$$\mathrm{Tr}\left(e^{-\beta H}\prod_{j=1}^{n}\psi^{\sigma_j}_{\tau_j}(t_j,x_j)\right)/\mathrm{Tr}\, e^{-\beta H} = \delta\begin{bmatrix}\sigma_1 \cdots \sigma_n\\ \tau_1 \cdots \tau_n\end{bmatrix} \mathrm{sgn}[\tau_1,\ldots,\tau_n]\, L^{-n/2} \exp\{\ \},$$

$$\{\ \} = -\sum_{q\neq 0}\left|a_q\left(\begin{smallmatrix}\sigma_j\\ \tau_j\end{smallmatrix}, t_j, x_j\right)\right|^2 \frac{e^{-\beta\omega_q}}{1-e^{-\beta\omega_q}} - \sum_{j<j'}\sigma_j\sigma_{j'} B_{\tau_j\tau_{j'}}(t_j-t_{j'}, x_j-x_{j'}) \tag{20}$$

$$-n\frac{2\pi}{L}\sum_{p>0}\frac{q}{p}(hb^-_p)^2 - i\sum_{j\leq j'}\sigma_j\sigma_{j'}\,\delta_{\tau_j-\tau_{j'}}\cdot\frac{2\pi}{L}(t_{j'}-\tau_{j'}x_{j'})$$

$$+4\sum_{n=1}^{\infty}\frac{(-)^n}{n}\frac{e^{-\beta n\frac{\pi}{L}}}{1-e^{-\beta n\frac{2\pi}{L}}}\left[\sin^2\left(\frac{n\pi}{L}\sum_{\tau_j=+}\sigma_j(t_j-x_j)\right)+\sin^2\left(\frac{n\pi}{L}\sum_{\tau_j=-}\sigma_j(t_j+x_j)\right)\right.$$

Here we have used the abbreviations

$$\gamma_q = -\frac{2}{\pi}\tilde{V}(q), \quad \omega_q = q\cdot(1-\gamma_q^2)^{-1/2}$$

$$hb^\sigma_q = -(\mathrm{sgn}\,\gamma_q)^{\frac{1-\sigma}{2}}\cdot\left[\frac{1}{2}\left((1-\gamma_q^2)^{-1/2}+\sigma\right)\right]^{1/2}$$

$$a_q\left(\begin{smallmatrix}\sigma_j\\ \tau_j\end{smallmatrix}, t_j, x_j\right) = \left(\frac{2\pi}{L|q|}\right)^{1/2}\sum_{j=1}^{n}\sigma_j \exp(i(\omega_q t_j - q x_j))\cdot hb_q^{\tau_j\,\mathrm{sgn}\,q}$$

$$B_{\tau\tau'}(t,x) = \sum_{q\neq 0}\frac{2\pi}{L|q|}\exp(-i(\omega_q t - q x))\, hb_q^{\tau\,\mathrm{sgn}\,q}\cdot hb_q^{\tau'\,\mathrm{sgn}\,q}$$

If $k\tilde{V}(k)$ is not in ℓ^2 the Bogoliubov transformation is not implementable in Fock space $\mathcal{F}^B$. This is the case in the Thirring model with the Hamiltonian $(V(x) = g\,\delta(x),\ \gamma = -2g/\pi)$

$$H(\gamma) = H^B(\gamma) + \frac{\pi}{L}\left(Q_+^2 + Q_-^2 + \gamma Q_+ Q_-\right)$$
$$H^B(\gamma) = H_0^B - \tfrac{1}{2}\gamma \int_0^L dx : \phi_-' \phi_+' + \phi_+' \phi_-' : \tag{21}$$

However there is an incomplete tensor product space $\mathcal{H}(\gamma)$ contained in the complete tensor product space $\mathcal{H}$ over the Boson oscillator spaces so that the renormalized Hamiltonian $H^B(\gamma)_{ren}$ is selfadjoint on a domain $D(\gamma) \subset \mathcal{H}(\gamma)$ for $|\gamma|<1$. Furthermore there exists a unitary transformation $W(\gamma)$ which implements the Bogoliubov transformation and maps the ground state of $H^B(\gamma)_{ren}$ into the Fock space vacuum,

$$W(\gamma) : \mathcal{H}(\gamma) \longmapsto \mathcal{H}(0) = \mathcal{F}^B, \quad \Omega(\gamma) = W(\gamma)^{-1}\Omega \tag{22}$$

$W(\gamma)$ diagonalizes $H(\gamma)_{ren}$,

$$W(\gamma) H^B(\gamma)_{ren} W(\gamma)^{-1} = \sqrt{1-\gamma^2}\, H_0^B. \tag{23}$$

The representation of the canonical commutation relations in $H(\gamma)$ is non Fock. The Fermion field $\psi(\underline{x})$ is not well defined on the ground state $\Omega(\gamma)$. This calls for an infinite wave function renormalization if one wants to compute n-point functions. The (wave-function) renormalized field in $\mathcal{H}(\gamma)$ is defined by

$$\psi_{ren}(\underline{x}) = \psi^F(\underline{x}) \otimes \psi^B_{ren}(\underline{x})$$
$$\psi^B_{\tau\, ren}(\underline{x}) = \frac{1}{\sqrt{L}} N_\gamma \left(\exp -i\sqrt{2\pi}\, \phi_\tau\right)(\underline{x}). \tag{24}$$

N_γ denotes the Wick ordering with respect to the operators $W(\gamma)^{-1} c(k) W(\gamma)$ or else subtraction of the $\Omega(\gamma)$ -expectation values. The n-point function is given by the Fock space Wightman functions of the field

$$W(\gamma)\, \psi_{\tau\, ren}\, W(\gamma)^{-1}(x) = \frac{1}{\sqrt{L}} \psi^F(x) \otimes : \exp -i\sqrt{2\pi}\left[\tfrac{1}{2}\alpha^{1/2}\chi + \tfrac{1}{2}\alpha^{-1/2}\varphi\right] : , \tag{25}$$

where d denotes the scaling dimension of the exponentiated fields and is related to the coupling constant γ by $d^2(\gamma)=(1+\gamma)/(1-\gamma)$. They converge towards the corresponding Klaiber solutions of the Thirring model only if one drops the factor in front of H_o in (23). This amounts to an additional (finite) wave function renormalization of $\psi(\underline{x})$ [11] .

For simplicity we deal now with the addition of a mass to the Thirring model (rather than the Luttinger model). The massive Thirring model with periodic box cut off is formally given by the Hamiltonian $H(\gamma,m) = H(\gamma)+M$. For γ in the interval $-1<\gamma<o$ the renormalized Hamiltonian

$$H(\gamma,m)_{ren} = H(\gamma)_{ren} + M_{ren}$$

$$W(\gamma)\, M_{ren}\, W(\gamma)^{-1} = \frac{2m}{L}\int_0^L dx : \cos\left[\sqrt{2\pi d}\,\varphi(x) + \sum_\tau \tau\left(P_\tau + \frac{2\pi}{L} u_\tau Q_\tau\right)\right]: \tag{26}$$

can be shown to be a selfadjoint operator by using techniques which are analogous to those used in ref.[9]. For γ outside this interval different methods are necessary[12].

Now we return to the question of relating the massive Thirring model and the Sine-Gordon theory. The operator identity (4)(5) does not lead to an equivalence of the Thirring Hamiltonian and a Sine-Gordon Hamiltonian for the Boson field $\varphi(x)$ since in (26) there are terms present reflecting the charge structure of the Fermion Fock space. Neglecting the term $\frac{2\pi}{L} u_\tau Q_\tau$ -which one might expect to be a good approximation for L large-one remains with a Hamiltonian $\tilde{H}(\gamma,m)$ which can be represented by a direct integral over the spectrum of the operators U_τ,

$$\tilde{H}(\gamma,m) = \int_0^{2\pi \oplus} d\alpha_+ d\alpha_-\, \tilde{H}(\gamma,m;\alpha_+,\alpha_-). \tag{27}$$

Here $\tilde{H}(\gamma, m : \alpha_+, \alpha_-)$ is a Sine Gordon Hamiltonian for the Boson field $\varphi(x)$.

If one treats the mass term as a perturbation as in [1], one has to compute

$$\langle T \prod_i \varphi(x_i) \prod_\ell \frac{2m}{L} : \cos\left[\sqrt{2\pi\alpha}\, \varphi(y_\ell) + \sum \tau P_\tau\right] : \rangle \tag{28}$$

Due to the presence of the charge shift operators $U_\tau = \exp - i P_\tau$ only terms

$$\langle T \prod_i \varphi(x_i) \prod_\ell \frac{2m}{L} : \exp i \beta_\ell \sqrt{2\pi\alpha}\, \varphi(y_\ell) : \rangle \tag{29}$$

with $\sum \beta_\ell = 0$ contribute to (28). This is the same restriction as in ref.[1]. There it was the result of a zero mass limiting procedure. In a pure periodic box-cutoff Sine-Gordon theory other terms would contribute to the N-point functions.

Results on quasiclassical approximations to the Sine-Gordon theory [13] can be applied to the Hamiltonian $\tilde{H}(\gamma, m : \alpha_+, \alpha_-)$. The point spectrum in this approximation is given by the soliton-antisoliton and the bound state masses.

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Colloques Internationaux C.N.R.S.
N° 248 – Les méthodes mathématiques de la théorie quantique des champs

The Bethe-Salpeter Kernel in $P(\varphi)_2$ *

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RESUME Le but de cet exposé est d'expliquer comment étendre l'expansion du "cluster" pour obtenir les propriétés fortes de décroissance d'un et de deux noyaux P.I. (particule irréductible).
Pour comprendre l'importance de telles estimations on explique premièrement la relation entre les propriétés de décroissance des noyaux r - P.I. (r = 0, 1, 2) et le spectre de masse et la matrice S .

ABSTRACT

The purpose of this talk is to explain how to extend the cluster expansion [1] to obtain strong decay properties for one and two P. I. (particle irreducible) kernels. The details of these estimates are presented in [2]. See also [3] for related results. Since the proof of the estimates is somewhat complex we shall only sketch the main ideas behind the proof. To understand why such estimates are important we shall first explain the relation between decay properties of the r - P. I. (r = 0, 1, 2) kernels and the mass spectrum and the S-matrix. Throughout the talk P is a positive even polynomial and λ/m_0^2 is small.

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The Mass Spectrum

Let us first consider the case $r = 0$ and define $S(X)$ to be the Schwinger function of the points $X = \{x_1 \cdots x_n\}$, $x_i = (x_i^0, x_i^1)$. Then it is clear that

$$|S(X \cup Y) - S(X)S(Y)| \leq O(1)e^{-\sigma d(X, Y)} \qquad \sigma > 0$$

implies that the vacuum is unique and that there is a unique vacuum. Here

$$d(X, Y) = \min_{x \in X, y \in Y} |x - y| .$$

For the case $r = 1$ let $\Gamma(p)$ be the inverse of the two-point function in momentum space, i.e.

(1) $$S(p)\Gamma(p) = 1 .$$

The 1.P.I. kernels we consider are defined by

(2) $$k(x, y) = \Gamma_2(x - y) - m_0^2 + \Delta,$$

and for $n > 1$

(3) $$k(X, Y) = S(X \cup Y) - \int S(X, x)\Gamma(x - y)S(y, Y)dxdy - S(X)S(Y) .$$

In perturbation theory $k(X, Y)$ is the sum of all graphs which are one particle irreducible in the X - Y channel. This means that X and Y can not be disconnected by cutting a single internal line. Thus we omit graphs of the form

Let m be the physical mass and let $\epsilon(\lambda) \to 0$ as $\lambda \to 0$. It can be shown that a decay of the form

(4) $$|k(X, Y)| \leq O(1)e^{-2(m-\epsilon)d}$$

implies that

(5) $$\text{Spect } M \cap (0, 2m - \epsilon) = \{m\},$$

and in particular

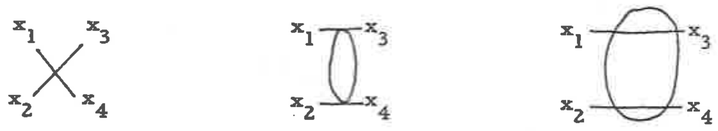
(6) $$S(p) = \frac{Z}{p^2 + m^2} + \int_{2(m-\epsilon)}^{\infty} \frac{d\rho(a)}{p^2 + a^2} .$$

Hence by the Haag-Ruelle theory there is an isometric S matrix.

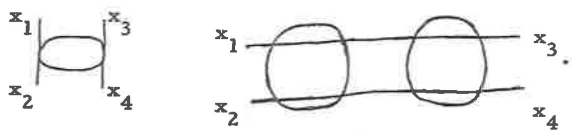
Let us establish this form of S(p) from (4). Note that (4) implies that $\Gamma(p)$ is analytic in $p^2 = p_0^2 + p_1^2$ for $|p| < 2(m - \epsilon)$. Hence S(p) is a meromorphic function without zeroes for $|p| < 2(m - \epsilon)$. But between any two poles of S there is at least one zero because S must vary continuously from $-\infty$ to $+\infty$. Hence there is at most one pole for $|p| < 2(m - \epsilon)$. We refer the reader to [4] for a complete proof of (5).

To obtain information about bound states and scattering we need to analyze the two particle irreducible Bethe-Salpeter kernel $K(x_1 \cdots x_4)$.

In perturbation theory K is the sum of all connected graphs which are two particle irreducible in the $\{x_1, x_2\}, \{x_3, x_4\}$ channel and whose external lines are amputated. Here are some typical graphs which contribute to the sum (for $\lambda\varphi^4$):



However these diagrams do not contribute



Now suppose

(7) $$|K(x_1 \cdots x_4)| \leq \Theta(x) e^{-(m-\epsilon)d_2(x)}$$

where

(8) $$d_2(x) = |x_1^0 - x_2^0| + |x_3^0 - x_4^0| + 2\,|x_1^0 + x_2^0 - x_3^0 - x_4^0|$$

and $\Theta(x)$ is a finite measure in x. This estimate together with the fact that

$$\Omega,\ e^{itH}\varphi(\Omega)\varphi(y)\Omega$$

span the even subspace of energy less than $4(m - \epsilon)$ [3] yield the following results:

(a) The mass spectrum is discrete below 2m.

(b) The mass operator restricted to the even subspace of energy less than $4(m - \epsilon)$ has no singular continuous spectrum.

(c) The S matrix is unitary on the even subspace up to energies less than $4(m - \epsilon)$.

Remarks: Bound states exist for the interaction $\varphi^6 - \varphi^4$ but there are no bound states of mass $\le 2m$ for $\lambda\varphi^4$, see [1, 5]. The presence of bound states for weak coupling is a result of the fact that we are in one space dimension. Quantum mechanics indicates that they would not be present in three space dimension for weak coupling.

We now describe briefly how results (a), (b), (c) follow from (7). This is work done in collaboration with F. Zirilli [6]. Many of our techniques appeared several years ago in a paper of Bros [7]. Let

$$D(x_1, x_2, x_3, x_4) = S(x_1 \cdots x_4) - S(x_1, x_2)S(x_3, x_4)$$

(9)

$$D_0(x_1 \cdots x_4) = S(x_1, x_3)X(x_2, x_4) + S(x_1, x_4)S(x_2, x_3)$$

and then by definition K satisfies

(10) $$D = D_0 + D_0 KD .$$

Here D (as well as D_0 and K) acts as an operator via

$$(Df)(x_1, x_2) = \int D(x_1, x_2, x_3, x_4) f(x_3, x_4) dx_3 dx_4 .$$

After Fourier transformation and analytic continuation (10) is the analogue of the resolvent equation in quantum mechanics. To see this fix

$$\xi = (x_2 - x_1)/2 \text{ and } \eta = (x_4 - x_3)/2$$

and define

$$\tau = (x_4 + x_3 - x_2 - x_1)/2 .$$

Then using the Feynman-Kac formula (with $z^1 = \eta^0 = \xi^0 = 0$)

$$\text{(11)} \quad \int D(\xi, \eta, \tau) e^{\tau \cdot z} d\tau$$

$$= \int \langle \Omega, \varphi_0(-\xi^1) \varphi_0(\xi^1) \left[\frac{1}{H - z^0} + \frac{1}{H + z^0} \right] \varphi_0(\tau^1 - \eta^1) \varphi_0(\tau^1 + \eta^1) \Omega \rangle d\tau^1 .$$

The τ^1 integration corresponds to restricting the Hamiltonian H to the zero momentum subspace. To obtain our results (a), (b), and (c) we study the behavior of D(z), $z = r + i\epsilon$ as $\epsilon \searrow 0$ with $0 \leq r \leq r(m - \epsilon)$. Estimate (7) implies that K(z) is a compact perturbation of D(z) and is analytic for $|z| \leq 4(m - \epsilon)^*$. We apply the analytic Fredholm theorem to the equation

$$D(z) = D_0(z) + D_0(z) K(z) D(z)$$

to show that $D_0(z)$ and $D(z)$ have the same singularities as $\epsilon \searrow 0$ apart from possible poles corresponding to bound states. $D_0(z)$ is easy to analyze using the form (6) for the two-point function and (a), (b), (c) follow in a straightforward manner.

The Expansion

The cluster expansion is a perturbation about a decoupled theory. We achieve our decoupling by inserting 0 Dirichlet boundary conditions in the covariance (or propagator). Let Γ be a union of lattice line segments and define Δ_Γ to be the Laplacian with Dirichlet boundary conditions on Γ. If X_i are the components of $R^2 - \Gamma$ then $(-\Delta_\Gamma + m_0^2)^{-1}$ leaves $L^2(X_i)$ invariant so that events localized in distinct X_i are independent. Equivalently

$$(-\Delta_\Gamma + m_0^2)^{-1}(x, y) = 0$$

whenever x and y belong to different components. For a subset of integers $I \subset Z$ let $\Delta_{(I\times R)}$ be abbreviated Δ_I. We define an interpolating set of covariances by the formula

$$C(t, x, y) = \sum_I \prod_{i\notin I} t_i \Pi(1 - t_i)(-\Delta_I + m_0^2)^{-1}(x, y) \tag{12}$$

Note that $C(t) = (-\Delta + m_0^2)^{-1}$ if $t_i = 1$ for all i. If $t_i = 0$ and $\ell_i = i \times R$ separates x and y we have

$$C(t, x, y) = 0 . \tag{13}$$

Hence t_i measures the coupling across the line ℓ_i.

We now explain how to extend the cluster expansion to the one-particle irreducible kernel k(x, y). The Bethe-Salpeter kernel can be analyzed similarly. Let $d\varphi(t)$ be the Gaussian measure of mean zero and covariance C(t). If Q is a polynomial in φ we define

(14) $$\langle Q\rangle_\Lambda(t) = \frac{\int e^{-V(\Lambda)} Q\,d\varphi(t)}{\int e^{-V(\Lambda)}\,d\varphi(t)},$$

and

$$\langle Q\rangle(t) = \lim_{\Lambda\uparrow R^2} \langle Q\rangle_\Lambda(t),$$

where

$$V(\Lambda) = \lambda\int_\Lambda :P(\varphi(x)):dx .$$

Similarly we define

(15) $$k(t, x, y) = [\Gamma(t) - C(t)^{-1}](x, y)$$

where

$$\int \Gamma(t, x, y)\langle\varphi(y)\varphi(x')\rangle(t)dy = \delta(x - x') .$$

The 2.P.I. property of k(x, y) is reflected in the following important observation

(16) $$\frac{d^r}{dt_i^{\,r}} k(t, x, y)\Big|_{t_i=0} = 0 \qquad r = 0, 1, 2$$

whenever the line $\ell_i = i \times R$ separates x and y. This identity is easy to see in perturbation theory, e.g. in first order

(17) $$\frac{d^r}{dt_i^{\,r}} k(t, x, y)\Big|_{t_i=0} \approx \frac{d^r}{dt_i^{\,r}} C(t, x, y)^3\Big|_{t_i=0} + \dots$$

$$\approx 0$$

It may be rigorously established by a straight computation in the lattice approximation and then taking the lattice limit, see [2]. With this identity and Taylor's formula

$$f(1) = f(0) + f'(0) + \dots \int_0^1 \frac{t^{n-1}}{(n-1)!} f^{(n)}(t)dt$$

we can express k in the form

(18) $$k(x, y) = \int_0^1 [\prod_{y^0 \le i \le x^0} \frac{t_i^2}{2} \frac{d^3}{dt_i^{\,3}}] k(t, x, y)dt.$$

At this point we turn to estimates on derivatives in t_i. Essentially we must show that each factor d/dt yields a factor of e^{-m}. It is difficult to estimate these derivatives directly.

Hence we illustrate our approach by considering an analogous problem. Consider parameters (μ_i) $i \in Z$ (similar to t_i) which measure the local strength of an external field localized in the strip $|x^0 - i| \le \frac{1}{2}$. Let

$$\langle \cdot \rangle(\mu)$$

denote the expectation with respect to the interaction

$$V + \sum_i \mu_i \int_{|x^0 - i| \le \frac{1}{2}} \varphi(x) dx .$$

For large m_0 the cluster expansion proves that $\langle Q \rangle(\mu)$ is bounded and analytic for $|\mu_i| \le M$. Thus if we apply the Cauchy formula as in [8]

(19) $$\left| \prod_{i \in I} \frac{1}{n_i!} \frac{\partial^{r_i}}{\partial \mu_i^{r_i}} \langle Q \rangle(\mu) \right|_{\mu=0} = \left| \int \prod_{i \in I} \mu_i^{-(r+1)} \langle Q \rangle(\mu) d\mu \right|$$

$$\le \prod_{i \in I} M^{-r} \text{const.}$$

We remark that if one were to compute the derivatives directly $(n|I|)!$ unconnected Schwinger functions would appear so that a term by term estimate would be useless. The analyticity method has the cancellation of these terms built into it. To apply this method to $k(\mu, x, y)$ we express $k(\mu)$ as a convergent Neumann series of Schwinger functions. This allows us to translate bounds on $S(\mu, X)$ to bounds on $k(\mu, x, y)$.

To extend this technique to the t variables we are forced to introduce an auxilliary parameter $h = (h(\alpha))$ because $\langle \ \rangle(t_i)$ is <u>not</u> analytic in t. We define an expectation

$$\langle \ \rangle(h, t)$$

which is analytic in h and such that

$$\prod_{i\in I} \frac{\partial}{\partial t_i}\langle \quad \rangle(h,t)\Big|_{h=0} = \delta_h^{(I)}\langle \quad \rangle(h,t)\Big|_{h=0}$$

Here δ_h is a differential operator in h. T e key estimate is to show that

$$|\langle Q\rangle(h(a)t)| \le \text{Const.}$$

$|h(a)| \le e^{+(m_0-\epsilon)d(a)}$. We refer the reader to [2] for details.

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