SUPERSTRINGS :
TOPOLOGY, GEOMETRY and PHENOMENOLOGY
&
ASTROPHYSICAL IMPLICATIONS OF SUPERSYMMETRIC MODELS

VOLUME I
(2 vol. bound together)

Brian Randolph Greene
Magdalen College

Department of Theoretical Physics
University of Oxford

Thesis submitted for the degree of Doctor of Philosophy
in the University of Oxford

July 1986

SUPERSTRINGS :
TOPOLOGY, GEOMETRY AND PHENOMENOLOGY
&
ASTROPHYSICAL IMPLICATIONS OF SUPERSYMMETRIC MODELS

Brian Randolph Greene
Magdalen College

ABSTRACT

Much of the low energy phenomenology which can be extracted from the field theory limit of the intrinsically ten dimensional $E_8 \otimes E_8$ heterotic superstring depends upon the topological and geometrical properties of the six dimensional compactified component of spacetime. After briefly reviewing the topological constraints on the latter manifold which ensure the survival of N=1 four dimensional supersymmetry, we present and apply the mathematics necessary for the rigorous construction of vacuum solutions and the determination of the four dimensional massless field content.

Two phenomenologically attractive classes of solutions, with unbroken $E_8 \otimes SU(5)$ and $E_8 \otimes SO(10)$ gauge groups, arise if the vacuum configuration contains a Ricci flat Kähler manifold with SU(3) holonomy (Calabi–Yau manifold), which admits certain SU(5) or SU(4) vector bundles. Further reduction of the gauge group and emergence of naturally light weak Higgs doublets may also occur by flux breaking if the Calabi–Yau manifold is multiply connected. We analyse the feasibility of such scenarios for Calabi–Yau manifolds with any possible fundamental group. Phenomenological considerations place severe constraints on the dimensions and transformation properties of certain cohomology groups and thereby lead to a highly restricted class of acceptable models.

We then present the mathematical analysis of a three generation heterotic superstring–inspired model, with $E_8 \otimes E_6$ gauge symmetry. A detailed description of the manifold of compactification is given, along with a determination of its Hodge numbers and of the associated light supermultiplet structure. For a particular choice of vacuum moduli we derive this manifold's symmetry group, and determine its action on the massless fields in the theory. Preliminary investigation indicates that these transformation properties give rise to a remarkably realistic model.

In the second volume we derive cosmological constraints on a supersymmetric extension of the standard model in which weak gauge symmery breaking is triggered at the tree–level by a Higgs singlet superfield. The fermionic component of this gauge singlet (the "nino") is shown to be the lightest supersymmetric particle with a relic abundance near the critical closure density for a surprisingly wide range of the unconstrained parameters.

The previously favoured photino dark matter scenario has been eliminated by the non–observation of high energy solar neutrinos. After briefly reviewing this argument, we extend the analysis to eliminate Higgsino dark matter scenarios with $\langle H_1{}^0 \rangle \neq \langle H_2{}^0 \rangle$. We show that the nino produces an *acceptably low level* of solar neutrinos and that it may also account for the anomalously high level of cosmic ray antiproton flux.

Thesis submitted for the degree of Doctor of Philosophy
in the University of Oxford

July 1986

ACKNOWLEDGEMENTS

I would like to express my gratitude to Dr. Robin Fletcher and the Rhodes Scholarship Trust for the generous financial support I have received during my two years at Oxford.

It has been a great privilege to work with Dr. Graham Ross and I owe him many thanks for guidance and encouragement, and for his contagious enthusiasm which has made for an immensely enjoyable research environment. I would also like to thank my initial supervisor, Dr. James Binney for many interesting and enlightening conversations during the last two years.

I am very grateful to Dr. John Wheater for many patient and invaluable discussions pertaining to both this work and physics in general.

Conversations with Professor Simon Donaldson regarding the material presented in the last chapter were very helpful and are greatly appreciated.

I am indebted to Paul Miron and Kelley Kirklin for sharing late night meals from Vegetable Man and vegetable samozas from Abu Dhabi, for protection from ravenous lizards in Koh Samui and overzealous natives in Suko–Thai, and for all the productive discussions and arguments which have made for a successful collaboration.

I would especially like to thank Nicholas Boles, Jane Hanna, and Clare Marshall, who by befriending a 'Yank' have helped to make my stay in England a unique and memorable experience.

For their constant support and ineffable friendship, I am forever indebted to Anne Coyle, Eero Simoncelli and my dear siblings Wendy, Susan and Joshua.

I owe a special word of thanks to Dr. John Cocke for his invaluable guidance and friendship. I have enjoyed and learned a great deal from our many conversations in which he has shared his wealth of clever and inventive ideas.

Both as a teacher and friend, Neil Bellinson has had a profound influence on my development. My gratitude is inestimable.

The boundless warmth and insatiable curiosity of Daliah Shapiro has been a great inspiration; her friendship shall always be cherished.

Finally, I owe great thanks to Mrs. Margot Long for her heroic efforts in typing this manuscript.

To my Parents
whose selfless love and support make all things possible.

When I heard the learn'd astronomer,
When the proofs, the figures, were ranged in columns before me,
When I was shown the charts and diagrams, to add, divide, and
    measure them,
When I sitting heard the astronomer where he lectured with much
    applause in the lecture-room,
How soon unaccountable I became tired and sick,
Till rising and gliding out I wander'd off by myself,
In the mystical moist night-air, and from time to time,
Look'd up in perfect silence at the stars.

Walt Whitman

From *Leaves of Grass*

VOLUME I

CONTENTS

VOLUME II

CONTENTS

CHAPTER I

INTRODUCTION

I.1 INTRODUCTORY REMARKS

Man has been part of the universe for but a flicker of cosmic time. Thrust on to a small yet habitable planet, hurtling through space about a nondescript star in the corner of an equally nondescript galaxy, it is no wonder that man's evanescent moment has been imbued with the existential dilemma. It is a tribute to man's benevolent audacity that he has not cowered when faced with his own purposelessness — rather, throughout recorded human history, he has addressed the issue with great passion. These attempts are often divided into the twin pursuits of religion and science, whose respective paradigms only differ in one essential point — the latter conveniently ignores the eternal interrogative which any 'final' solution will necessarily fail to address, while the former is forthright in acquiescing to and subsequently revelling in the cosmic mystery. This aspect of the scientific paradigm, however, serves a great purpose. By refusing to appeal to an unquestionable and hence unfalsifiable answer, one is led to contemplate, probe, and confront the intricacies of the surrounding universe. A deep familiarity with a mystery can at least partially satiate a desire for complete understanding.

It would seem unlikely, however, that the feeble slice of the cosmos discerned by our five meagre senses supplemented recently by devices of our making, would submit to a coherent understanding. Nature, though, has been remarkably accommodating. As Einstein once remarked [1], "The most incomprehensible thing about the world is that it is comprehensible."

The last century has witnessed the advent of great leaps in our understanding of both the large-scale structure of the universe through general relativity, as well as the microscopic properties of matter through quantum mechanics and gauge field theories. The standard model of elementary particle physics based on the gauge group $SU(3) \otimes SU(2) \otimes U(1)$ is able to successfully address all experimental results.

Paying homage to the deity of simplicity through unification, many attempts have been made to realize the standard model as part of a 'grand unified' theory based on a single underlying gauge theory. Furthermore, one might optimistically hope to incorporate gravity into this unification. Such attempts are generally plagued by unsatisfactory characteristics such as the non-renormalizability of quantum gravaity, the gauge hierarchy problem, and an abundance of arbitrary features which must be introduced in an *ad hoc* fashion. Examples of the latter are the unifying gauge group, fermion representations, family replication, Higgs representations, and the dimension of spacetime. (One might be experimentally convinced that the universe has four extended dimensions, but as we shall see this is not in confict with the possibility that the actual number of dimensions is larger).

Superstrings — supersymmetric theories based upon the supposition that elementary particles are one-dimensional instead of point-like — appear to be quite adept at dealing with these issues. Quantum consistency of string theory cannot only accommodate gravity, but requires it; the supersymmetric aspect of the theory provides a means of addressing the gauge hierarchy problem. Furthermore, quantum consistency appears to restrict the gauge group to only two possibilities and uniquely determines the dimension of the universe to be ten. Finally, six of these dimensions must compactify into a space of unobservably small radius for the theory to be realistic. The topology and geometry of this compactified component of the universe can then determine the surviving low energy gauge group, matter content, and even effective Yukawa couplings.

This latter element of compactification is what allows us to test our faith in superstring unification by making contact with low-energy phenomenology, and to which this thesis is devoted.

I.2 OUTLINE AND SCOPE

One of the very appealing aspects of compactification in superstring theory is that it provides a physical context for many powerful and beautiful results from algebraic geometry and topology. As the latter formalism may not be completely familiar, and plays a prominent role in the original work presented, we have had to make a pedagogical choice as to presentation. On the well founded belief [2] that mathematics in a vacuum is boring, we introduce a number of mathematical concepts in a physical context before attempting a more formal treatment.

In particular, Chapter II reviews the form which the compactified component of the universe must assume in order to preserve $N=1$ four dimensional supersymmetry [3,4,5]. This requires some notions regarding the topological classification of holomorphic vector bundles, as well as some geometric properties of connections on holomorphic bundles.

Having motivated these mathematical concepts by showing their physical relevance, we briefly formalise them in the first section of Chapter III. In this section we also introduce some properties of holomorphic bundle valued Dolbeault cohomology over compact Kähler manifolds as this proves to be instrumental in the second section where we determine the massless sector of our compactified theories. Although the basic physical ideas in these two sections are not new, their mathematical justification has not appeared in the superstring literature. We have attempted to fill this gap by presenting and applying the mathematics necessary for a reasonably rigorous treatment of the physical properties of interest. In the course of this formalisation, we generalise some previously known results and find an additional topological restriction on certain proposed vacuum configurations.

In the remaining sections of Chapter III, we discuss the flux mechanism for breaking gauge symmetry and give a brief presentation of the origin and use of discrete symmetries and the topological computation of Yukawa couplings.

Chapter IV consists of the material contained in ref. [6] which was done in collaboration with K.H. Kirklin and P.J. Miron. This work investigates the phenomenological prospects of Calabi–Yau compactifications which admit certain holomorphic SU(5) or SU(4) vector bundles that can yield phenomenologically pleasing SU(5) or SO(10) effective superstring unifying groups. After classifying all possible flux breaking scenarios in terms of the fundamental group of the compact manifold, we show that phenomenological considerations place severe constraints on the respective dimensions and discrete transformation properties of certain cohomology groups and thereby lead to a highly restricted class of acceptable models.

Chapter V consists of the work presented in ref. [7] which was done in collaboration with K.H. Kirklin, P.J. Miron and G.G. Ross. In this chapter the topological and geometrical analysis of one of only three known three–generation Calabi–Yau manifolds is performed. After giving a detailed description of this manifold of compactification, its Hodge numbers are computed (correcting ref. [8]) along with the associated light supermultiplet structure. For a particular choice of vacuum moduli we derive the manifold's symmetry group and determine its action on the massless fields in the theory. We briefly indicate that these transformation properties lead to a model with remarkably realistic phenomenological characteristics. In the first draft of ref. [7], I wrote sections 1, 4 and 5. The original results presented represent a collaborative effort.

To allow both of these chapters to be read independently of the rest of the thesis, each contains a brief introduction which contains the essential features discussed in Chapters II and III.

Finally, in the remainder of this chapter we provide a brief introduction to string theory. We do not attempt to be rigorous or complete as this material is not directly pertinent to the main emphasis of this thesis. Rather, we attempt to briefly convey the way in which string theory singles out particular gauge symmetries and critical spacetime dimensions.

I.3 BOSONIC STRINGS, SUPERSTRINGS, AND THE DIMENSION OF THE UNIVERSE

In this section we briefly describe the classical bosonic string, its quantization and indicate the determination of a unique spacetime dimension for quantum consistency. By formulating a supersymmetric extension of the bosonic string we construct the superstring and give a similar discussion of its essential features.

The classical mechanics describing the motion of a free point particle is embodied in the action

$$S = -m \int d\tau \left[\frac{dx^{\mu}}{d\tau} \frac{dx_{\mu}}{d\tau}\right] \qquad (I.1)$$

where $x^{\mu} = x^{\mu}(\tau)$ is the spacetime position of the particle as a function of the parameter τ. Geometrically (I.1) is the length of the world line followed by the particle. This geometric interpretation is made manifest by the invariance of this action under reparameterizations

$$x^{\mu}(\tau) \longrightarrow x^{\mu}(\tilde{\tau}(\tau)). \qquad (I.2)$$

The same reasoning may be used to formulate the classical mechanics describing the motion of a free object with one extended dimension : a string [9]. The spacetime position of the string, in addition to being a function of τ, also depends on σ — a coordinate labelling the intrinsic points which compose the string. The notation $\zeta^0 = \tau$ and $\zeta^1 = \sigma$ shall be of convenience. In analogy with (I.1) in which the action for a point particle is given by the invariant geometric length of its world line, the action for a free string is given by the geometric area swept out by its world sheet, $X^{\mu}(\sigma,\tau)$, $\mu = 0,1,\ldots,D-1$, according to [9,10,11]

$$S = -\frac{1}{4\pi\alpha}\int d\sigma\, d\tau(-\det g)^{\frac{1}{2}} \quad . \tag{I.3}$$

In this expression, D is the dimension of the ambient spacetime, α has the dimension of area and is related to the string tension T by $T = 1/(2\pi\alpha)$, and (det g) is the determinant of the world sheet metric which is determined from

$$\begin{aligned} dX^\mu dX_\mu &= \partial_\alpha X^\mu(\sigma,\tau)\partial_\beta X_\mu(\sigma,\tau) \\ &\equiv g_{\alpha\beta}(\sigma,\tau)d\zeta^\alpha d\zeta^\beta \end{aligned} \tag{I.4}$$

with $\alpha,\beta = 0,1$.

The geometric interpretation of (I.3) as an area is made manifest by the invariance of this action under world sheet reparameterizations:

$$\zeta_\alpha \longrightarrow \tilde{\zeta}_\alpha(\sigma,\tau)$$

$$g_{\alpha\beta} \longrightarrow \left[\frac{\partial\zeta^\delta}{\partial\tilde{\zeta}^\alpha}\frac{\partial\zeta^\kappa}{\partial\tilde{\zeta}^\beta}\right] g_{\delta\kappa} \quad . \tag{I.5}$$

Using this invariance, as shown in [9,10,11], the world sheet metric

$$g_{\alpha\beta} = \begin{bmatrix} \dot{X}^2 & \dot{X}\cdot X' \\ \dot{X}\cdot X' & X'^2 \end{bmatrix} \tag{I.6}$$

where $\dot{X} = \partial X^\mu/\partial\tau$ and $X' = \partial X^\mu/\partial\sigma$ may be transformed to

$$g_{\alpha\beta} = \dot{X}^2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} . \tag{I.7}$$

This form is obtained by using the reparameterization freedom to impose the Lorentz gauge:

$$\dot{X}^{\mu} X'_{\mu} = 0 \tag{I.8a}$$

$$\dot{X}^2 + X'^2 = 0 . \tag{I.8b}$$

In this gauge, the equations of motion for $X^{\mu}(\sigma,\tau)$ obtained from extremizing the action (I.3) are found to be

$$\ddot{X}^{\mu} - X''^{\mu} = 0 \tag{I.9}$$

which is recognized as the two dimensional wave equation.

Imposing closed string periodic boundary conditions

$$X^{\mu}(\tau, \sigma + \pi) = X^{\mu}(\tau, \sigma) \tag{I.10}$$

the solution to (I.9) may be expanded in a Fourier series [9,10,11]

$$X^{\mu} = X_0^{\mu} + P^{\mu}\tau + \sum_{n\neq 0} \frac{i}{2n} \left[\alpha_n{}^{\mu} e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n{}^{\mu} e^{-2in(\tau+\sigma)} \right] \tag{I.11}$$

where $X_0{}^{\mu}$ is the initial centre of mass coordinate for the string and P^{μ} is the conserved momentum.

Note, however, that we still have a reparameterization invariance [9,10,11] that allows us to transform to coordinates in which one of the X^μ is trivial. In particular, if we pass to light cone coordinates defined by

$$X^{\pm} = \frac{X^0 \pm X^{D-1}}{\sqrt{2}} \tag{I.12}$$

we may choose to set

$$X^+(\sigma,\tau) = X_0^+ + P^+\tau \tag{I.13}$$

where X_0^+ and P^+ are constants. In other words, the timelike variable τ parameterizing the world sheet is set to measure the spacetime 'temporal' evolution of the string. (This allows us to avoid questions concerning the appearance of two time variables). With this choice, the differential equations (I.8) may be used to solve for $X^-(\sigma,\tau)$ in terms of $X^i(\sigma,\tau)$, i = 1,2,...,.D−2. In light cone coordinates, then, the latter D−2 variables are the only propagating coordinates. The string action, in terms of these coordinates, is

$$S_{LC} = -\frac{1}{4\pi\alpha}\int d\sigma\ d\tau\ \eta^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^i\ . \tag{I.14}$$

The transverse coordinates may be Fourier analyzed to yield

$$X^i(\tau-\sigma) = \tfrac{1}{2}X_0{}^i + \tfrac{1}{2}P^i(\tau-\sigma) + \tfrac{1}{2}i\sum_{n\neq 0}\frac{\alpha_n{}^i}{n}\ e^{-2in(\tau-\sigma)}$$

$$X^i(\tau+\sigma) = \tfrac{1}{2}X_0{}^i + \tfrac{1}{2}P^i(\tau+\sigma) + \tfrac{1}{2}i\sum_{n\neq 0}\frac{\tilde{\alpha}_n{}^i}{n}\ e^{-2in(\tau+\sigma)} \tag{I.15}$$

where we have split the solution into a right moving $X_i(\tau-\sigma)$ and a left moving

$X^i(\tau+\sigma)$ sector.

Having described the classical bosonic string in the light cone gauge, we now quantize the theory.

Using (I.14), the conjugate momenta $P^i(\sigma,\tau) = \delta L/\delta\dot{X}^i$ are found to be

$$P^i(\sigma,\tau) = 1/\pi\ \dot{X}^i(\sigma,\tau) \ . \tag{I.16}$$

To quantize, we impose

$$[P^i(\sigma,\tau), X^j(\sigma',\tau)] = -i\delta^{ij}\delta(\sigma-\sigma') \ . \tag{I.17}$$

Using (I.15) this implies:

$$[\alpha_m{}^i, \alpha_n{}^j] = [\tilde{\alpha}_m{}^i, \tilde{\alpha}_n{}^i] = m\delta_{m+n,o}\delta^{ij} \ . \tag{I.18}$$

The excitations of the string are thus seen to be equivalent to an infinite number of harmonic oscillators, each described by raising operators (with negative indices) and lowering operators (with positive indices).

The mass operator for the string is given in terms of the harmonic oscillator operators by [9,10,11]

$$M^2 = \frac{1}{4\alpha^2}\sum_{n>0}\sum_{i=1}^{24}\alpha_{-n}{}^i\alpha_n{}^i - \alpha_0 = \frac{1}{4\alpha^2}\sum_{n>0}\sum_{i=1}^{24}\tilde{\alpha}_{-n}{}^i\tilde{\alpha}_n{}^i - \alpha_o \tag{I.19}$$

where α_0 carries the zero point oscillation effects and can be shown to equal $(D-2)/24$.

By passing to the light cone gauge and hence treating the transverse coordinates and light cone coordinates differently, we have sacrificed manifest Lorentz invariance. To explicitly verify that the theory is Lorentz invariant, we construct the Lorentz generators

$$M^{\mu\nu} = \frac{1}{2\pi} \int_0^\pi d\sigma (X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu) \tag{I.20}$$

and determine the algebra which they generate. A long and tedious calculation [9] shows that the commutator $[M^{i-}, M^{j-}]$ causes problems: the Lorentz algebra is only satisfied for $D = 26$. Lorentz invariance thus singles out a *unique* spacetime dimension. A consistent relativistic quantum mechanical treatment of the bosonic string requires the dimension of spacetime to be twenty-six.

The bosonic string theory, however, is still sick. As can be seen from (I.19) the lowest state is a tachyon. This problem motivates the introduction of additional anticommuting coordinates to supplement the X^μ used above. These anticommuting variables, if equal in number to the X-variables, can cancel the zero point fluctuations associated with the X-modes. The anticommuting nature of these coordinates naturally leads us to assume that they transform according to a spinor representation of the relevant spacetime group, SO(D-2).

In order to have an equal number of bosonic and fermionic degrees of freedom, D must be chosen so that the vector and spinor representations of SO(D-2) have the same number of components. This is the case for $D = 3,4,6$ and 10 [10]. For later use, we note that in $D = 10$ the spinor representation must be restricted to be Majorana-Weyl. Bearing these numbers in mind, we augment the light-cone action of the pure bosonic string (I.3) to include the anticommuting coordinates $S^{Aa}(\sigma,\tau)$ where $A = 1,2$ and a is a D-2 component spinor index:

$$S_{L.C.} = \frac{1}{4\pi\alpha} \int d\tau \int_0^\pi d\sigma \, [\partial_\alpha X^i \partial^\alpha X_i + i\bar{S}\gamma^- \rho^\alpha \partial_\alpha S] \qquad (I.21)$$

where

$$\rho^0 = \begin{bmatrix} o & -i \\ i & o \end{bmatrix}, \quad \rho^1 = \begin{bmatrix} o & i \\ i & o \end{bmatrix},$$

$$\gamma^- = \frac{1}{\sqrt{2}} (\gamma^o - \gamma^9), \quad \text{and}$$

$$\bar{S}^{Aa} = S^{+Bb} (\gamma^o)^{ba} (\rho^o)^{BA} .$$

After invoking the quantization procedure followed earlier, we must again investigate Lorentz invariance. The Lorentz generators are different from the bosonic case as they now act on the fermionic coordinates as well. Similar analysis again reveals a dimension dependent anomaly in the Lorentz algebra [10,11] which only vanishes for D = 10. Thus, although it appears possible to construct supersymmetric classical string theory in D = 3,4,6 and 10, quantum consistency demands that the dimension of spacetime be ten.

Having indicated the way in which string theories single out a unique spacetime dimension for quantum consistency, we now examine their implications for the gauge symmetry of the universe.

I.4 SUPERSTRINGS AND THE GAUGE SYMMETRY OF THE UNIVERSE : HETEROTIC REALIZATION

Gauge groups arise in open string theory by assigning gauge labels to the ends of the string according to the procedure of Chan and Paton [12]. This procedure can yield SO(N) and Sp(2N) as possible gauge groups [12,10]. At first sight our desire for uniqueness is unfulfilled. However, as shown by Green and Schwarz [13], absence of gauge and gravitational anomalies constrains the gauge group to be SO(32) or $E_8 \otimes E_8$. The SO(32) gauge group is readily accommodated by the Chan–Paton formulation; the $E_8 \otimes E_8$ gauge group, however, is not.

The latter situation motivated Gross *et al.* [14] to search for a string theory based upon an $E_8 \otimes E_8$ gauge group. Rather than attempt to use a Chan–Paton like mechanism, they sought to generate the gauge group via a Kaluza–Klein procedure. In particular, the authors noticed that both SO(32) and $E_8 \otimes E_8$ have rank 16, which by standard numerology arguments is 26–10. Kaluza–Klein compactification of a 26 dimensional bosonic string to the 10 dimensions required by the superstring has the potential to generate these 16 symmetries. An inherently string dependent mechanism may then account for the remaining gauge bosons required to fill out the adjoint of SO(32) or $E_8 \otimes E_8$. This, in fact, is the basic idea behind the heterotic superstring as we now briefly discuss.

For closed strings, as shown by (I.15), the physical degrees of freedom may be classified as right–moving (depending on $\tau - \sigma$) or left–moving (depending on $\tau + \sigma$). The heterotic superstring exploits this decomposition by taking the right–moving sector to consist of the degrees of freedom associated with a D = 10 superstring, and the left–moving sector to consist of those associated with a D = 26 bosonic string.

The light–cone action for the heterotic string is obtained from (I.21) by only retaining the right-moving component of the anticommuting sector, and augmenting the action with sixteen additional left-moving commuting coordinates, $X^I(\tau+\sigma)$, I = 1,...,16. The light-cone action is therefore [14]

$$S_{L.C.} = - \int d\tau \int_0^\pi d\sigma \frac{1}{4\pi\alpha} \Big[\partial_\alpha X^i \partial^\alpha X^i + \partial_\alpha X^I \partial^\alpha X^I + i\bar{S}\gamma^-(\partial_\tau+\partial_\sigma)S\Big] \qquad (I.22)$$

where S is a ten-dimensional Majorana-Weyl spinor.

In [14] it is suggested that the additional sixteen left moving coordinates should be viewed as parameters for a compact internal space T (internal with respect to the ten dimensional superstring spacetime). The authors restrict themselves to a flat internal space, and hence take T to be a hypertorus. Ordinary Kaluza-Klein reasoning then suggests that T should give rise to the gauge bosons of its isometry group $[U(1)]^{16}$. However, something inherently 'stringy' can happen in the compactification which changes this result: the string may 'wind' around the compact space forming solitons which can act as gauge bosons. Consistency of the heterotic string, in fact, requires [14] that the hypertorus T take a particular form which generates 480 of these massless solitons. Together with the 16 gauge bosons from the isometries of T, these remarkably compose the 496 dimensional adjoint representation of a group with rank 16. In other words, the gauge group must be SO(32) or $E_8 \otimes E_8$! (For completeness, we note that Narain [15] has recently developed new compactification schemes for the heterotic string which yield other possible gauge groups. The details of these schemes are under investigation and we do not consider them further.)

The Fock space of physical states arising from the heterotic string may be expressed as the direct product of the respective Fock spaces for the right and left moving sectors, subject to certain consistency constraints [14]. As shown in [14], the ground state of the heterotic string is massless and consists of the field content of $N = 1$, $D = 10$ supergravity coupled to super-Yang-Mills theory with gauge group SO(32) or $E_8 \otimes E_8$.

Explicitly, the massless states are:

(1) $(g_{MN}, \Psi_M, B_{MN}, \lambda, \varphi)$ where g_{MN} is the 10 space metric, Ψ_M is a spin-$^3/_2$ right-handed Majorana-Weyl spinor, λ is a spin-$^1/_2$ right-handed Majorana-Weyl spinor, B_{MN} is an antisymmetric two form, and φ is a scalar field, which comprise an $N = 1$, $D = 10$ supergravity multiplet.

(2) (A_M, χ) where A_M is a gauge field in the adjoint representation of SO(32) or $E_8 \otimes E_8$ and χ is a left-handed Majorana-Weyl spinor in the same representation, which form an $N=1$, $D=10$ super-Yang-Mills multiplet.

In all that follows, our analysis will concentrate on this $N = 1$, $D = 10$ massless supergravity/super-Yang-Mills sector. As shown by Green and Schwarz [13], anomaly freedom requires this theory to be the super-Yang-Mills theory constructed by Chapline and Manton [16], with one crucial modification which has important ramifications, as shall be discussed in Chapter II. Furthermore, we shall only emphasize the theory with an $E_8 \otimes E_8$ gauge group as the SO(32) scenario suffers from phenomenological problems [17]. It is clear, then, that the only rôle played by string theory in our analysis is that of motivating this particular supergravity theory. In all that follows, therefore, any occurrence of the word 'superstring' should really be interpreted as 'superstring-inspired'.

Following this philosophy, we shall now investigate compactification of $N = 1$, $D = 10$ supergravity coupled to $E_8 \otimes E_8$ super-Yang-Mills to a four spacetime dimensional theory. For completeness we note that this subjet has been studied from an inherently string-based vantage point [3,18], but we shall not explore this area.

REFERENCES

CHAPTER I

1. A. Einstein as quoted in P.A. Schlipp, Albert Einstein : Philosopher– Scientist, ed. 1949.
2. P.J. Miron, private confirmation (1986).
3. P. Candelas, G.T. Horowitz, A. Strominger and E. Witten, Nucl. Phys., B258 (1985) 46.
4. A. Strominger, ITP Preprint, NSF– ITP– 86– 16 (1986).
5. E. Witten, Nucl. Phys., B268 (1986) 79.
6. B.R Greene, K.H. Kirklin and P.J. Miron, Nucl. Phys., B274 (1986) 574.
7. B.R. Greene, K.H. Kirklin, P.J. Miron and G.G. Ross, to appear in Nucl. Phys.B.
8. S– T. Yau in Proceedings of the Argonne Symposium on Anomalies, Geometry and Topology (World Scientific, 1985).
9. J. Scherk, Rev. Mod. Phys., 47 (1975) 123.
10. J.M. Schwarz, Phys. Rep., 89 (1982) 223; in "Superstrings and Supergravity", Proc. 28th Scottish Universities Summer School in Physics (1985) 301.
11. L. Brink, CERN Preprint TH4006 (1984); C. Bachas, SLAC Preprint SLAC– PUB– 3902 (1986); J.W. Moffat, University of Toronto Preprint (1985).
12. J. Paton and Chan Hong– Mo, Nucl. Phys., B10 (1969) 519.
13. M.B. Green and J.H. Schwarz, Phys. Lett., 149B (1984) 117.
14. D.J. Gross, J.A. Harvey, E. Martinec and R. Rohm, Nucl. Phys., B265 (1985) 252.
15. K.S. Narain, Phys. Lett., 169B (1986) 41.
16. G.F. Chapline and N.S. Manton, Phys. Lett., 120B (1983) 105.

17. E. Witten, Phys. Lett., 149B (1985) 351; J. Ellis, Proc. of 28th Scottish Universities Summer School in Physics (1985) 399.

18. C. Hull, DAMTP Preprints (1985, 1986).

CHAPTER II

COMPACTIFICATION AND SUPERSYMMETRY

II.1 INTRODUCTION

Superstring theories, as discussed, require a ten dimensional universe. Succumbing to the burden of experiment, we demand that six of these degrees of freedom arise from a compact manifold K with unobservedly small radius, probably on the order of the Planck length 10^{-33} cm [1]. The universe, then, is assumed to take the form $M_4 \, x \, K$ where M_4 is a maximally symmetric four-space identified as the universe of common experience. Although not directly observable because of its small size, the topological and geometrical properties of K have a profound influence on the laws of physics manifest in M_4.

The form of the compact manifold K (along with the topology of certain vector bundles which can be constructed with K as the base manifold) may be used to determine the surviving low energy gauge group [2,3], the low energy matter spectra [3], discrete symmetries which the theory must respect [3], and even Yukawa couplings [3,4]. This is an extremely appealing situation : physical properties of the observable universe are determined by its underlying geometry.

Ultimately one hopes that the topology and geometry of the vacuum will be determined dynamically. Lacking the ability to presently achieve this goal, one works backwards guided by phenomenology.

Supersymmetry, besides having the intriguing property of being the last hitherto unused symmetry of the S-matrix, seems to be the most viable means of addressing the gauge hierarchy problem. We therefore require that four dimensional N=1 supersymmetry survives the process of compactification. Furthermore, the form of the compactification must result in a surviving gauge group and low energy field content which are able to accommodate observed phenomenology as it is successfully described by the standard model.

These phenomenological constraints, although not sufficiently powerful to pick out a unique vacuum configuration, are highly restrictive. In this chapter we shall examine their implications.

In section 2 we shall derive the topological and geometrical properties which necessarily characterize any compactification which ensures the survival of four dimensional N=1 supersymmetry. Our treatment follows ref. [5] and ref. [6]. In section 3 we will make use of theorems due to Calabi and Yau [7], and Ulhenbeck and Yau [8], from the mathematical literature, in order to demonstrate the existence of compactifictions meeting the constraints derived in section 2. In applying these theorems, we shall make an assumption regarding the torsion of the connection on the compact space. In section 4 we will briefly analyse the implications of relaxing this assumption.

II.2 COMPACTIFICATIONS ENSURING N=1 SPACETIME SUPERSYMMETRY

A vacuum configuration ensures the existence of N=1 supersymmetry if the background fields are invariant under supersymmetry transformations. The supposition that the vacuum assumes the form M_4 *x* K with M_4 maximally symmetric immediately constrains the background fermion field components in M_4 to vanish. Since spinors in ten dimensions may be decomposed as the direct product of spinors in M_4 with spinors on K, this implies that the background fermion fields must vanish. Since supersymmetry transformations map bosons into fermions (and vice versa), we see that the Bose fields are invariant under supersymmetry transformations. We now examine the supersymmetry transformation properties of the fermion fields which consist of the spin 3/2 field ψ_M, the spin 1/2 field λ (both from the N=1 supergravity multiplet which also contains the metric g_{MN}, the two-form potential B_{MN}, and a scalar field φ) and the spin 1/2 fields χ^a (from the super Yang-Mills multiplet, which also contains the Yang-Mills field $F_{MN}{}^a$ with a being a gauge group index). From Chapline and Manton [9] (as adapted by Candelas *et. al.* [5]), we have the transformations laws:

$$\delta\psi_\mu = \nabla_\mu\varepsilon + (\tfrac{1}{32}\sqrt{2})e^{2\varphi}(\gamma_\mu\gamma_5 \otimes H)\varepsilon$$

$$\delta\psi_m = \nabla_m\varepsilon + (\tfrac{1}{32}\sqrt{2})e^{2\varphi}(I_{4x4} \otimes \gamma_m H - 12I_{4x4} \otimes H_m)\varepsilon$$

$$\delta\lambda = \sqrt{2}(I_{4x4} \otimes \gamma^m\nabla_m\varphi)\varepsilon + (\tfrac{1}{8}\, e^{2\varphi}I_{4x4} \otimes H)\varepsilon$$

$$\delta\chi^a = -\frac{1}{4}\, e^{\varphi}(I_{4x4} \otimes F^a_{mn}\gamma^{mn})\varepsilon \qquad \text{(II.1)}$$

where ε is a ten dimensional Majorana-Weyl spinor parameterizing supersymmetry transformations.

There are a number of conventions which we must make clear. Lower case

Latin indices refer to K and take values 4, ..., 9, while Greek indices refer to M_4 and take values 0, ..., 3. The covariant derivative operator ∇ is constructed with the Christoffel connection. The ten dimensional Γ matrices satisfy the usual relations $\{\Gamma^M,\Gamma^N\} = 2g^{MN}$ (with upper case Latin indices representing all ten dimensions) for a metric with signature (1,9). This representation of the Dirac algebra may be built up from the respective algebras on M_4 and K according to

$$\Gamma^\mu = \gamma^\mu \otimes I \quad \Gamma^m = \gamma_5 \otimes \gamma^m \tag{II.2}$$

where I and γ^m are the Dirac matrices appropriate to the six dimensional representation on K, while γ^μ and γ_5 are the familiar 4 x 4 Dirac matrices. The chirality projection operator for the six dimensional representation is

$$\gamma = \frac{i}{6!}\, g^{\frac{1}{2}} \epsilon_{mnpqrs} \gamma^{mnpqrs} \tag{II.3}$$

where multiple indices on a gamma matrix indicate a completely antisymmetric product. Choosing a Majorana representation for the Γ matrices (Γ^M real and Hermitian except for Γ^o which is real and antihermitian) we thus see that the γ^μ may be chosen to constitute a Majorana representation while γ^m, γ^5, and γ are imaginary and Hermitian. The field H is defined according to the anomaly cancellation prescription of Green and Schwarz [10]

$$H = dB - \omega_{3Y} + \omega_{3L} \tag{II.4}$$

where ω_{3Y} and ω_{3L} are the Chern–Simons three forms from the Yang–Mills and local Lorentz gauge structures. Explicitly,

$$\omega_{3Y} = \frac{1}{30}\,\mathrm{Tr}(A \wedge F - \frac{1}{3} A \wedge A \wedge A)$$

and

$$\omega_{3L} = \mathrm{tr}(\omega \wedge R - \frac{1}{3}\,\omega \wedge \omega \wedge \omega) \qquad \text{(II.5)}$$

where A and F (ω and R) are the Yang–Mills (tangent bundle) connection and curvature. Our interest focuses on the gauge group $E_8 \otimes E_8$ for which Tr denotes the trace in the adjoint representation while tr denotes the trace in the vector representation of SO(1,9). In components we have

$$\begin{aligned} \mathcal{H}_{MNQ} &= \partial_{[M}B_{NQ]} \\ &\quad - 1/30\ \mathrm{Tr}\ (A_{[M}F_{NQ]} - 1/3 A_{[M}A_N A_{Q]}) \\ &\quad + \mathrm{tr}\ (\omega_{[M}R_{NQ]} - 1/3\ \omega_{[M}\omega_N\omega_{Q]}). \end{aligned} \qquad \text{(II.6)}$$

The remaining fields in (II.1) are defined according to

$$H = \mathcal{H}_{pqr}\gamma^{pqr} \text{ and } H_m = \mathcal{H}_{mqr}\gamma^{qr}. \qquad \text{(II.7)}$$

The identity matrices explicitly written in (II.1) will be suppressed when their presence is obvious from context.

The requirement of surviving d=4 supersymmetry forces all of the variations in (II.1) to vanish for arbitrary ϵ_4 and possibly constrained ϵ_6, where $\epsilon = \epsilon_4 \otimes \epsilon_6$. In the following we drop these subscripts as they are clear from context. Before analyzing the constraints which arise from the vanishing of these variations, we note that H contains both Yang–Mills and spacetime curvatures. This is the first sign that the geometry of the Yang–Mills bundle may be related to that of the tangent bundle. To make this relation explicit, note that from (II.4) we find

$$dH = \mathrm{tr}\ R \wedge R - 1/30\ \mathrm{Tr}\ F \wedge F \quad . \tag{II.8}$$

For H to be globally well defined, as it must be since it appears in expressions for physical observables like the energy [11], we must have

$$0 = \int_K dH = \int_K (\mathrm{tr}\ R \wedge R - 1/30\ \mathrm{Tr}\ F \wedge F) \quad . \tag{II.9}$$

We shall shortly re–express this relation in terms of topological invariants of K and the Yang–Mills bundle built upon it.

(a) Require $\delta\psi_\mu = 0$:

Squaring the operator in the first line of equation (II.1) and taking the antisymmetric part, we find

$$[\nabla_\mu, \nabla_\nu]\varepsilon - \frac{1}{(16)^2} e^{4\varphi}(\gamma_{\mu\nu} \otimes H^2)\varepsilon = 0 \quad . \tag{II.10}$$

The first term on the left hand side yields the curvature tensor on M_4 which we have assumed takes the maximally symmetric form

$$R_{\mu\nu\rho\sigma} = \kappa(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \tag{II.11}$$

and hence

$$[\nabla_\mu, \nabla_\nu]\varepsilon = 1/4 R_{\mu\nu\rho\sigma}\gamma^{\rho\sigma}\varepsilon = 1/2\kappa\gamma_{\mu\nu}\varepsilon \tag{II.12}$$

where we have used

$$\nabla_\mu \varepsilon = (\partial_\mu + \frac{1}{2}\, \omega_{\mu\delta\zeta}\gamma^{\delta\zeta})\varepsilon \tag{II.13}$$

as the action of the covariant derivative on spinors. Substituting (II.12) in (II.10) we find

$$\gamma_{\mu\nu} \otimes (\frac{1}{2}\,\kappa I - \frac{1}{(16)^2}\, e^{4\varphi}H^2)\varepsilon = 0 \tag{II.14}$$

Inverting $\gamma_{\mu\nu}$ ($\mu \neq \nu$) leads to

$$H^2\varepsilon = (16)^2\, e^{-4\varphi}\, \frac{1}{2}\, \kappa\varepsilon. \tag{II.15}$$

(b) Require $\delta\lambda = 0$:

Squaring the operator in the third line of (II.1) we find

$$(2\gamma^m\nabla_m\varphi\gamma^n\nabla_n\varphi + 1/(64)e^{4\varphi}H^2 + \frac{\surd 2}{8}\, e^{2\varphi}\{\gamma^m\nabla_m\varphi, H\})\varepsilon = 0\,. \tag{II.16}$$

Using (II.15) and $\gamma^m\nabla_m\varphi\gamma^n\nabla_n\varphi = \nabla_m\varphi\nabla^m\varphi$ this gives

$$1/8\surd 2\;\; e^{2\varphi}\{\gamma^m\nabla_m\varphi, H\}\varepsilon = -2(\nabla_m\varphi\nabla^m\varphi + \kappa)I\varepsilon. \tag{II.17}$$

Notice that the H, due to its definition in terms of the antisymmetric product of hermitian gamma matrices, is an antihermitian matrix. Therefore, $\{\gamma^m\nabla_m\varphi, H\}$ is also antihermitian and hence only has imaginary eigenvalues. For (II.17) to hold, then, we must have

$$\nabla_m\varphi\nabla^m\varphi = -\kappa. \tag{II.18}$$

This implies that the length of the vector $\nabla_m\varphi$ on K is constant. However, since K is assumed to be compact φ necessarily has a maximum where $\nabla_m\varphi\nabla^m\varphi = 0$, and hence

$$\kappa = 0 \tag{II.19}$$

implying that M_4 is Minkowski spacetime. Substituting (II.19) into (II.15) we find

$$H\epsilon = 0 \tag{II.20}$$

(c) Require $\delta\psi_m = 0$:

Using (II.20) in the second line of (II.1), this constraint implies

$$\nabla_m\epsilon - 3/8\surd 2\ e^{2\varphi}H_m\epsilon = 0\ . \tag{II.21}$$

Defining $\beta = 3/8\surd 2\ e^{2\varphi}$ and using (II.13) we see that (II.21) is just

$$(\partial_m + (\tfrac{1}{2}\ \omega_{m\delta\zeta} - \beta H_{m\delta\zeta})\gamma^{\delta\zeta})\epsilon = 0. \tag{II.22}$$

As H is antisymmetric in all of its indices, the operator in (II.22) is effectively a covariant derivative based upon a non-symmetric connection : a covariant derivative with torsion. Denoting this operator by $\tilde{\nabla}_m$, we rewrite (II.22) as

$$\tilde{\nabla}_m\epsilon = 0 \tag{II.23}$$

From (II.23) we also have

$$[\tilde{\nabla}_m, \tilde{\nabla}_n]\,\varepsilon = 1/4\tilde{R}_{mnpq}\gamma^{pq}\,\varepsilon = 0 \qquad \text{(II.24)}$$

Using these results, we now show that K is a complex manifold.

Recall that ε is Majorana–Weyl and is therefore real and satisfies

$$\frac{1}{2}(I + \Gamma_{11})\varepsilon = 0 \qquad \text{(II.25)}$$

which is just

$$(\gamma_5 \otimes \gamma)\varepsilon = -\varepsilon \quad . \qquad \text{(II.26)}$$

The Majorana and Weyl constraints can not be imposed simultaneously in four or six dimensions as the chirality projection operators γ_5 and γ are imaginary in a Majorana representation. Since all of the analysis to this point trivially holds for $-\varepsilon = \Gamma_{11}\varepsilon$, the constraints pertaining to K hold true for $\gamma\varepsilon_6$. We may therefore form the definite chirality solution $\eta_\pm = \frac{1}{2}(\varepsilon_6 \pm \gamma\varepsilon_6)$. Since $\eta_\pm$ are covariantly constant (with respect to the connection with torsion) we may impose the normalization $\eta_\pm{}^+ \eta_\pm = 1$. We may now construct a complex structure for K.

Consider the antisymmetric tensor

$$J_m{}^n = i\,\eta_+{}^+\Gamma_m{}^n\eta_+ \quad . \qquad \text{(II.27)}$$

By Fierz rearranging one can show

$$J_m{}^n J_n{}^p = -\,\delta_m{}^p \qquad \text{(II.28)}$$

and hence $J_m{}^n$ defines an almost complex structure. Locally, then, we may use J to define complex tangent spaces

$$J \,:\, T_p(K) \to T_p(K) \qquad (II.29)$$

with

$$J^2 = -1 \qquad (II.30)$$

and hence define local complex coordinates. A necessary and sufficient condition for this procedure to extend globally over K and yield holomorphic transition functions (and hence define a complex structure for K) is that the Nijenhuis tensor

$$N_{mn}{}^{p} = J_m{}^{q}\nabla_{[q}J_{n]}{}^{p} - J_n{}^{q}\nabla_{[q}J_{m]}{}^{p} \qquad (II.31)$$

be identically zero. From the definition of $J_m{}^n$ in terms of the spinor η_+ which is covariantly constant with respect to the connection with torsion, we see that $J_m{}^n$ is also covariantly constant:

$$\tilde{\nabla}_p J_{mn} = \nabla_p J_{mn} + 8\beta H_{sp[m}J_{n]}{}^{s} = 0 \quad . \qquad (II.32)$$

Using (II.32) in (II.31) we find

$$N_{mn}{}^{p} = \eta_+{}^{\dagger}\{H, \gamma_{mn}{}^{p}\}\eta \quad . \qquad (II.33)$$

The right hand side vanishes by virtue of

$$H\eta_+ = 0 \qquad (II.34)$$

and therefore K is a complex manifold. Furthermore, notice that the tensor

$$T_{mn} = J_m{}^p J_n{}^q g_{pq} \tag{II.35}$$

is symmetric and covariantly constant. We conclude, therefore, that this tensor is proportional to the metric, and the normalization on the $\eta_\pm$ spinors enforces equality. Thus

$$J_m{}^p J_n{}^q g_{pq} = g_{mn} \tag{II.36}$$

This last relation implies that the metric tensor g, initially expressed in real coordinates as

$$g = g_{mn} dx^m \otimes dx^n \tag{II.37}$$

may be re-expressed in terms of local complex coordinates as

$$g = g_{a\bar{b}} dz^a \otimes d\bar{z}^{\bar{b}} \tag{II.38}$$

with $(g_{a\bar{b}})$ a Hermitian matrix, z $(\bar{z})$ holomorphic (antiholomorphic) coordinates and a $(\bar{b})$ holomorphic (antiholomorphic) indices. We have thus shown that K is a complex manifold with a Hermitian metric (with respect to the complex structure). From the Hermitian metric g we define the real (1,1) form

$$J = i g_{a\bar{b}} dz^a \wedge d\bar{z}^{\bar{b}} \tag{II.39}$$

which is known as the fundamental form. (Note: A (p,q) form is the tensor product of a p-form $\epsilon\ \Lambda^p T'^*$ and a q-form $\Lambda^q T''^*$). If J is closed, that is, $dJ = 0$ where

$$dJ = (\partial + \bar{\partial})J = \sum(\frac{\partial J}{\partial z_K})dz_K + \sum(\frac{\partial J}{\partial \bar{z}_K})d\bar{z}_K), \qquad (II.40)$$

then the manifold K is a Kähler manifold with Kähler form J.

Not surprisingly, given a complex structure $J_m{}^n$ and a hermitian metric $g_{a\bar{b}}$ on a manifold K, the torsion H_{mnp} for which the fundamental constraints (II.20) and (II.23) hold is determined. In fact

$$H_{mnp} = (-1/16\beta)g^{\frac{1}{2}}\epsilon_{mnprst}\nabla^r J^{st} \propto (\bar{\partial}-\partial)J \qquad (II.41)$$

where the last proportionality is proven in [6] (the derivation of which would lead us astray). We see from this result that if the complex manifold K is Kähler then the torsion necessarily vanishes.

Before analysing the last constraint in (II.1), we summarize three more piece of topological data which may be extracted from the above relations [6].

First, recall that inequivalent fibre bundles E are distinguished by invoking characteristic classes. For complex vector bundles of dimension k the relevant characteristic classes are the Chern classes $C_1, \ldots, C_k$ with $C_p \in H^{2p}(M)$ where $H^{2p}(M)$ denotes the $2p^{th}$ cohomology group of the base manifold M. These Chern classes may be extracted from the total Chern form according to

$$C(E) \equiv \mathrm{Det}(I + i/2\pi\ \Omega) = 1 + C_1(\Omega) + C_2(\Omega) + \ldots\ldots \qquad (II.42)$$

where Ω is the pullback of the curvature 2-form for the bundle.

The first few Chern classes are then explicitly given by [12]

$$\begin{aligned} C_0(\Omega) &= 1 \\ C_1(\Omega) &= i/2\pi \operatorname{Tr} \Omega \\ C_2(\Omega) &= 1/8\pi^2\{\operatorname{Tr} \Omega\wedge\Omega - \operatorname{Tr} \Omega \wedge \operatorname{Tr} \Omega\}, \end{aligned} \tag{II.43}$$

etc.

As the Chern classes reflect the topological structure of the vector bundle being characterized, they are insensitive to the choice of connection used in computing the curvature form.

We now apply these notions to show the vanishing first Chern class of the tangent bundle of K (in short: the first Chern class of K).

To prove this, we make use of the connection with torsion which gives rise to the covariant derivative operator defined by (II.22) and (II.23). From (II.23) we recall that we must have a covariantly constant spinor (with respect to this connection) on K. This places a severe constraint on the holonomy group of the connection. (Recall that the holonomy group of a connection on a tangent bundle is the group of transformations which map any tangent vector (tangent spinor) into its image under parallel transport around any closed loop). From (II.32) we see that the holonomy group is contained in U(3) as the full covariant derivative annihilates $J_m{}^n$ and hence parallel transport preserves the complex structure. Equation (II.23) further requires the tangent spinor space to contain a holonomy invariant component. This restricts the holonomy group to be contained in SU(3) [5,3,6] under which the tangent spinors decompose as

$$\begin{aligned} & 4 \oplus \bar{4} \text{ (under } O(6) \cong SU(4)) \\ & \quad \cong 3 \oplus \bar{3} \oplus 1 \oplus 1 \text{ (under } SU(3)) \end{aligned} \tag{II.44}$$

which contains the required singlet. Since the holonomy group is contained in

SU(3), the curvature 2-form takes values in this traceless Lie Algebra and hence $C_1 \propto \mathrm{Tr}\,\Omega$ vanishes.

The second topological property that K necessarily possesses is $h^{n,0} = 1$ where $h^{p,q}$ denotes the complex dimension of the cohomology group $H^{p,q}(K)$. We will convey the flavour of the argument; for complete details consult ref. [6],

It can be shown that the manifold K admits an everywhere non-vanishing holomorphic (3,0) form, ω, with everywhere non-zero, positive real and finite norm:

$$||\omega|| \equiv (\omega_{a_1a_2a_3}\bar{\omega}_{\bar{b}_1\bar{b}_2\bar{b}_3}g^{a_1\bar{b}_1}g^{a_2\bar{b}_2}g^{a_3\bar{b}_3})^{\frac{1}{2}} \tag{II.45}$$

Since ω is trivially ∂ closed, we must prove it to be non-exact to show $h^{3,0} \geqslant 1$.

This is accomplished by considering the integral

$$\int_K \omega\wedge\bar{\omega} = \int_K d^3z d^3\bar{z}\sqrt{g}\; ||\omega||^2 > 0 \tag{II.46}$$

which would vanish if ω were exact.

Consider now another holomorphic (3,0) form ω'. Since ω is everywhere non-vanishing, ω' may be written $\omega' = f\cdot\omega$ where f is a holomorphic function on K. Holomorphic functions on a compact complex manifold, however, are constant. Thus $h^{3,0}$ is exactly equal to one, and in fact ω can be expressed as $\omega_{mnp} = \eta_+{}^{\dagger}\gamma_m\gamma_n\gamma_p\eta_+$, where $\omega = \omega_{mnp}\, dz^m\wedge dz^n\wedge dz^p$.

The third topological property is merely a restatement of (II.9), the requirement that H be a globally defined field. With the use of (II.42) we see that (II.9) becomes

$$C_2(Y) - \tfrac{1}{2}C_1(Y) \wedge C_1(Y) = C_2(T) - \tfrac{1}{2}C_1(T) \wedge C_1(T) \tag{II.47}$$

where Y is the $E_8 \times E_8$ Yang–Mills adjoint (= fundamental) vector bundle, and T is the tangent bundle of K. We have just seen that $C_1(T) = 0$, and since E_8 is semisimple, $C_1(Y)$ also vanishes. Thus (II.47) becomes the topological constraint

$$C_2(Y) = C_2(T) \tag{II.48}$$

We now examine the final constraint of (II.1).

(d) Require $\delta\chi^a = 0$:

To analyze this constraint, it is useful to decompose the gamma matrices on K, which transforms as a 6 of SO(6), into two sets transforming as a $\underline{3}$ and $\underline{\bar{3}}$ of SU(3). The defining relation

$$\{\gamma^m, \gamma^n\} = 2g^{mn} \tag{II.49}$$

becomes

$$\begin{aligned} \{\gamma_i, \gamma_j\} &= \{\gamma_{\bar{i}}, \gamma_{\bar{j}}\} = 0 \\ \{\gamma_{\bar{i}}, \gamma_j\} &= g_{\bar{i}j} \; . \end{aligned} \tag{II.50}$$

In other words, the three operators γ_i may be thought of as annihilation operators while the three operators $\gamma^{\bar{j}}$ may correspondingly be viewed as creation operators. This acting on the tangent spinor Fock space eight dimensional tangent spinor space (which forms a representation of the above Clifford algebra) may be built up by the creation operators $a^{*\bar{i}} \simeq \gamma^{\bar{i}}$ according to

$$|\Omega\rangle,\ |\Omega^{\bar{i}}\rangle = a^{*\bar{i}}|\Omega\rangle,$$

$$|\Omega_i\rangle = \tfrac{1}{2}\epsilon_{ijk}a^{*\bar{j}}a^{*\bar{k}}|\Omega\rangle,\ |\bar{\Omega}\rangle = 1/6\epsilon_{ijk}a^{*\bar{i}}a^{*\bar{j}}a^{*\bar{k}}|\Omega\rangle. \qquad \text{(II.51)}$$

In this expression, $|\Omega\rangle$ is a Fock vacuum which satisfies $a_i|\Omega\rangle = 0$. The decomposition in (II.44) is thus explicitly realized with $|\Omega\rangle$ and $|\bar{\Omega}\rangle$ being the two SU(3) invariants.

The constraint

$$\delta\chi^a = F_{mn}{}^a\gamma^{mn}\epsilon = 0 \qquad \text{(II.52)}$$

where ϵ is a linear combination of the chirality eigenstates $|\Omega\rangle$ and $|\bar{\Omega}\rangle$ may be recast as

$$F^a_{ij} = F^a_{\bar{i}\bar{j}} = 0 \qquad \text{(II.53a)}$$

$$g^{i\bar{j}}F^a_{i\bar{j}} = 0 \qquad \text{(II.53b)}$$

by expressing (II.52) in Fock space.

The first line in (II.53) expresses the fact that the Yang–Mills gauge field must be a holomorphic connection on a holomorphic vector bundle. This statement is further explained and justified in Chapter III.

The second line of (II.53), together with (II.48) are highly restrictive constraints on the Yang–Mills bundles. Constructing base manifolds K and Yang–Mills bundles which meet all of the above requirements will be the subject of the next section.

II.3 CANDIDATE VACUUM SOLUTIONS

Having derived the topological and geometrical properties which characterize any compactification that purports to maintain an N=1 supersymmetry in four dimensions, we now seek solutions. A powerful theorem of Uhlenbeck and Yau [8] which generalizes an earlier result of Donaldson [13] will greatly facilitate this search. To motivate this theorem (a full presentation would double the length of this thesis!) we take a short excursion into Yang–Mills theory on a Kähler manifold K [14].

We write the Yang–Mills action

$$I = \int_K \mathrm{tr}\ F^{\mu\nu}F_{\mu\nu} \tag{II.54}$$

where $F_{\mu\nu}$ is the field strength (curvature) for a connection A on a vector bundle V with structure group G. To facilitate minimization, we rewrite this action as

$$I = I_1 + I_2 = \int_K \mathrm{tr}\ F_{i\bar{j}}F_{\bar{i}j}g^{i\bar{i}}g^{\bar{j}j} + \int_K \mathrm{tr}\ F_{ij}F_{\bar{i}\bar{j}}g^{i\bar{i}}g^{j\bar{j}} \tag{II.55}$$

The minimum will be attained if the absolute minimum of I_1 is obtained while I_2 vanishes, so assume

$$F_{ij}{}^a = F_{\bar{i}\bar{j}}{}^a = 0. \tag{II.56}$$

To minimize I_1, we first prove that it has a lower bound given by the second Chern number of the vector bundle V

$$[C_2] = -\int_K J \wedge \mathrm{tr}(F \wedge F) \tag{II.57}$$

Recall that $F = F^a\Lambda^a$, where Λ^a are the matrix generators of the Lie Algebra of

G, is a matrix of two forms. By use of the splitting principle [15] we may approximate F as

$$F = \begin{bmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_k \end{bmatrix}. \qquad \text{(II.58)}$$

that is $\qquad F_{i\bar{j}} = x_i \delta_{i\bar{j}} \qquad$ (II.59)

Now, we may write

$$g^{i\bar{j}} F_{i\bar{j}} = \sum_i x_i \qquad \text{(II.60)}$$

and

$$tr(F \wedge F) = tr \sum_{i \neq j} x_i x_j \qquad \text{(II.61)}$$

(Recall, this trace is over the Lie Algebra indices ℓ,m in $(F_{i\bar{j}})_{\ell,m}$). Furthermore

$$I_1 = \int tr(\Sigma x_i^2) = \int tr\ [(\Sigma x_i)^2 - \sum_{i \neq j} x_i x_j$$

$$= \int tr[(\Sigma x_i)^2] + [C_2]. \qquad \text{(II.62)}$$

Thus $I_1 \geqslant [C_2]$ and the minimum is achieved only if

$$\Sigma x_i = g^{i\bar{j}} F_{i\bar{j}} = 0 \qquad \text{(II.63)}$$

As a check the minimization conditions, (II.56) and (II.63) should imply the Yang–Mills equation

$$d^*F + A \wedge {}^*F - {}^*F \wedge A = 0 \qquad \text{(II.64)}$$

$$d^*F + A\wedge{}^*F - {}^*F\wedge A = 0 \qquad \text{(II.64)}$$

where * is the Hodge star operator which maps p– forms on a (real) n– dimensional manifold to (n– p) forms, according to

$$*(dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_p}) = \frac{|g|^{\frac{1}{2}}}{(n-p)!}\ \epsilon^{\mu_1\cdots\mu_p}{}_{\mu_{p+1}\cdots\mu_p}\ dx^{\mu_{p+1}}\wedge\ldots\wedge dx^{\mu_p} \qquad \text{(II.65)}$$

The Yang– Mills equation (II.64) when written in local coordinates is

$$g^{\mu\nu}\nabla_\mu F_{\nu\lambda} = 0 \qquad \text{(II.66)}$$

Rewriting this in terms of (anti) holomorphic coordinates and using (II.9) we find

$$g^{i\bar{J}}\nabla_i F_{k\bar{J}} = 0 \qquad \text{(II.67)}$$

The Bianchi identity gives [14]

$$\nabla_i F_{k\bar{J}} = \nabla_k F_{i\bar{J}} \qquad \text{(II.68)}$$

which, with (II.67) implies that the Yang– Mills equations are

$$\nabla_k(g^{i\bar{J}}F_{i\bar{J}}) = 0 \qquad \text{(II.69)}$$

and the complex conjugate of (II.69). Clearly, then, this follows from (II.63) so our minimization does yield the Yang– Mills equation. Comparing (II.56) and (II.63) (the minimization conditions for pure Yang– Mills theory on a Kähler manifold) with (II.53) (conditions on Yang– Mills fields for unbroken N=1 supersymmetry) we see that they coincide. The problem of finding solutions to these equations has recently been considered, and solved, from the former point of view by Donaldson [13], and Uhlenback and Yau [8]. The above analysis shows that we may adopt their result which states that for any holomorphic vector bundle V over a Kähler manifold K there is a unique connection satisfying (II.56) and (II.63). We shall elaborate upon this in Chapter III. (There is an additional technical constraint that the vector bundle must be 'stable'; a condition which we shall assume to be satisfied. Also, the uniqueness is with respect to the fixed cohomology class defined by the Kähler form).

We thus have an *existence* theorem for solutions to the constraint on the Yang–Mills equation derived in (II.53), at least in the case when K is Kähler. Beyond satisfying (II.53), as seen in the last section, there are a number of other requirements which any supersymmetry preserving compactification must meet. To aid in our search for solutions, we will concentrate on the torsion–free case, i.e. we set $H \equiv 0$. (See section 4 for a discussion of the torsion–full case.)

From (II.41) we see that the fundamental form will now satisfy

$$dJ = 0 \tag{II.70}$$

and hence K is a Kähler manifold. With vanishing torsion the covariant derivative $\tilde{\nabla}$ becomes the covariant derivative ∇ (with respect to the Christoffel connection) and hence the curvature tensor in (II.24) becomes the ordinary Riemann tensor and thus

$$R_{mnpq}\gamma^{pq}\eta = 0 \tag{II.71}$$

Contracting with γ^{n}, we find

$$R_{mn}\gamma^n\eta = 0 \tag{II.72}$$

Now, the matrix $R_{mn}\gamma^n$ only has zero eigenvalues if

$$R_{mn} = 0 \tag{II.73}$$

and hence K is Ricci flat.

We thus seek a Kähler metric on a Kähler manifold with vanishing first Chern class such that the Christoffel connection has SU(3) holonomy and the Ricci tensor vanishes. In 1957 Calabi [7] conjectured that on any Kähler manifold with the topological property of vanishing first Chern class such a metric could be found. This conjecture was proven by Yau in 1977 [7], and we therefore have an *existence* theorem for a solution to (II.23) and (II.73). These Kähler manifolds with vanishing first Chern class equipped with Ricci flat, Kähler metrics of precisely SU(3) holonomy are known as Calabi–Yau manifolds. No metrics on Calabi–Yau manifolds are explicitly known although Calabi has given a laborious (even for a supercomputer!) algorithm to calculate them numerically.

The two existence theorems have implicitly provided Yang–Mills and tangent bundle connections which meet the requirements for preserving N=1 supersymmetry in four dimensions. Equation (II.48) places one topological constraint on the Yang–Mills bundle which necessitates a conspiracy between the two bundles on which these connections are defined: the second Chern class of the Yang–Mills bundle must equal that of the tangent bundle. In the case of vanishing torsion, we see from (II.8), this equality between cohomology classes becomes the somewhat stronger requirement of the *equality* between the respective differential forms,

$$\mathrm{tr}\ R \wedge R = 1/30\ \mathrm{Tr}\ F \wedge F \tag{II.74}$$

In the superstring literature, there has appeared only one means of satisfying (II.74) which we now describe [5].

The Yang–Mills bundle Y has a structure group which is contained in $E_8' \otimes E_8$. The tangent bundle T with SU(3) structure group may be identified with a sub–bundle of Y with structure group $I \otimes SU(3)$. If we then demand that this is the only non–trivial structure in Y, the topological properties of the Yang–Mills bundle will essentially be the same as those of the tangent bundle. This is explicitly accomplished by setting the vacuum expectation value of the Yang–Mills connection (on K) equal to the spin connection ω of T

$$\langle A_m \rangle = \omega \tag{II.75}$$

where the connection ω is embedded totally in one E_8, being identified with the second factor in the group theoretic embedding

$$E_8 \supset E_6 \otimes SU(3) \tag{II.76}$$

To verify that (II.74) is satisfied, recall that the adjoint $\underline{248}$ of E_8 decomposes under $E_6 \otimes SU(3)$ according to

$$\underline{248} = (\underline{1},\underline{8}) \oplus (\underline{3},\underline{27}) \oplus (\overline{\underline{3}},\overline{\underline{27}}) \oplus (\underline{78},\underline{1}) \tag{II.77}$$

and note that the quadratic trace of SU(3) in the $\underline{8}$ representation is three times the trace in the $\underline{3} \oplus \overline{\underline{3}}$ representation. The adjoint trace of SU(3) embedded in E_8 is thus 3 + 27 = 30 times the trace in the $\underline{3} \oplus \overline{\underline{3}}$ representation. The left hand side of (II.74) is a trace in the $\underline{6}$ of O(6) which is the $\underline{3} \oplus \bar{\underline{3}}$ of SU(3), and hence the embedding (II.75) leads to the exact equality (II.74)

We see from the above that this solution breaks the $E_8' \otimes E_8$ gauge group of the heterotic superstring to $E_8' \otimes E_6$, as E_6 is the maximal subgroup of E_8 which commutes with SU(3). Relegating the E_8' factor to a "shadow" world, (possibly responsible for supersymmetry breaking, and only communicating with the observed sector through gravitational interactions) the observed sector is described by a potentially viable E_6 grand unifying group. In Chapter III we will analyze the low energy four dimensional fields which arise through this solution, and in Chapter V we will apply these results to decribe an exciting model based on a particular Calabi–Yau manifold which yields three generations of standard quarks and leptons [16]. Although it appears that one may be able to construct phenomenologically viable models based upon invoking (II.75) to meet the constraint (II.74), it is worthwhile to explore the feasibility of other solutions which might yield, for example, SO(10) or SU(5) unifying groups instead of E_6. (No such solutions appear in the literature, but apparently at least one solution has been found [17]).

Imagine that we construct a Yang–Mills vector bundle Y over a Calabi–Yau manifold K which satisfies (II.74) and has structure group S contained in $E_8' \otimes E_8$. (In this context, it would certainly facilitate the search for solutions if we were to relax (II.74) to the torsion–full constraint (II.48) and hence only require the equality between the second Chern classes of the Yang–Mills and the tangent bundles. Non–vanishing torsion, however, implies some other unattractive features, and we therefore concentrate on the torsion free case. The torsion full case will be briefly discussed in the next section.) The surviving four space gauge group is then the maximal subgroup T of $E_8' \otimes E_8$ which commutes with S. When S is SU(3), and we embed in one E_8 factor, we recover the solution just described with unifying gauge group $E_8' \otimes E_6$. If, however, we find a solution to (II.74) with S being SU(4) or SU(5) and again embed in one E_8 factor, the unifying group will be $E_8' \otimes SO(10)$ or $E_8' \otimes SU(5)$, respectively. It is worthwhile to express this notion in a mathematically precise language.

As noted in [14], the previous solution of embedding the spin connection in the gauge group is best described as the construction of an $E_8' \otimes E_8$ structure group vector bundle out of the SU(3) structure group holomorphic tangent bundle. This idea readily accommodates the more general solutions, as we now discuss.

Let Y be a principle bundle contained within the $E_8' \otimes E_8$ principle Yang–Mills bundle. That is, the structure group S of Y is contained in $E_8' \otimes E_8$. In order to satisfy the anomaly cancellation condition (II.74) we must necessarily have the equality

$$C_2(Y) = C_2(T(K)) \qquad (II.48)$$

where Y is the associated vector bundle to Y based upon the fundamental representation of S. The unbroken gauge group is then the maximal subgroup of $E_8' \otimes E_8$ which commutes with S. The study of candidate vacuum configuration is thus equivalent to the study of principle fibre sub–bundles contained in the $E_8' \otimes E_8$ Yang–Mills bundle, (with the above stated constraint on the second Chern class) along with their associated vector bundles. The classification of vacuum configurations therefore amounts to the classification of $E_8' \otimes E_8$ principle fibre bundles, over Calabi–Yau manifolds, with the one topological constraint dictated by anomaly cancellation.

The classification of topologically inequivalent fibre bundles with a given structure group S over a fixed base manifold is a well studied mathematical problem [18]. The central idea involved in such a classification is straightforward.

A vector bundle V over a base manifold K induces a vector bundle V_k on k–simplices in a triangulation of K. We inductively discern the non–triviality of V by investigating the topological obstructions to extending a given trivialization (if one exists) of V_k to V_{k+1}. This is a well posed inductive framework since V_0 clearly does admit a trivialization. As shown in [19], the topological obstruction one encounters involves $\pi_k(S)$. For $S = E_8$, the homotopy groups satisfy [19] $\pi_k(E_8) = 0$ for $1 \leq k \leq 14$, $k \neq 3$ and $\pi_3(E_8) = \mathbb{Z}$. The only topological obstruction to trivializing an E_8 vector bundle over a base manifold of dimension less than fifteen occurs in the passage from three simplices to four simplices. Such obstructions are labelled by the elements in the fourth simplicial cohomology group of K with integral coefficients, $H^4(K,\mathbb{Z})$. Thus, since there are no other topological obstructions, we see that for every element in $H^4(K,\mathbb{Z})$, there is one isomorphism class of E_8 bundles over K.

Recall that in passing from integral coefficient cohomology to real or complex coefficient cohomology, the torsion component (not to be confused with the torsion of a connection!) of the cohomology group is lost. Thus, we see that the requirement (II.48) constrains the torsion free part of the only parameter labelling inequivalent E_8 bundles. In fact, it is argued in [20] that the constraint (II.48) should probably be augmented to

$$\gamma(Y,\mathbb{Z}) = \gamma(T(K),\mathbb{Z}) \qquad (II.78)$$

where $\gamma \in H^4(K,\mathbb{Z})$ chacterizes the E_8 bundle.

If we momentarily consider the full $E_8' \otimes E_8$ gauge group, then Y may be realised as the direct sum of two E_8 vector bundles, $Y = V_1 \oplus V_2$ and hence we have

$$\gamma(V_1, Z) + \gamma(V_2, Z) = \gamma(T(K), Z) \qquad (II.79)$$

as the most general solution. Our previously followed ansatze of relegating an unbroken E_8' to the hidden sector is equivalent to making V_1 trivial and yielding (II.78) with $Y = V_2$. We therefore conclude that with the above ansatze, there is a topologically *unique* Yang–Mills vacuum configuration for a given base manifold. Of course, the geometrical structure is not unique, as evidenced by the previously mentioned SU(4) and SU(5) holomorphic Yang–Mills connection. (The bundle V_2 also has topologically inequivalent sub–bundles).

In Chapter III we will further discuss these geometrically inequivalent solutions and derive their low energy four dimensional field content. In Chapter IV, we shall place additional and highly restrictive topological constraints on their holomorphic bundle valued cohomology, based upon phenomenological considerations [29].

To summarize thus far, heterotic superstring compactifications on Calabi–Yau manifolds ensure the survival of N=1 spacetime supersymmetry. Consistency places one topological constraint on the $E_8' \otimes E_8$ Yang–Mills bundle, which with the unbroken E_8' hidden sector ansatze, has a unique topological solution. If the structure group of this solution fills out a subgroup $S \subset E_8$, the unbroken gauge group is the maximal subgroup T of E_8 which commutes with S. The most commonly studied scenario involves S = SU(3) and hence T = E_6. In Chapter IV we will study the alternate and phenomenologically pleasing cases of S = SU(4) and S = SU(5) which yield T = SO(10) and T = SU(5) unifying groups, respectively.

II.4 THE CASE OF NON-VANISHING TORSION

In constructing candidate vacuum solutions in the last section, we assumed that the torsion H_{mnp} of the connection $\tilde{\nabla}$ vanished. In this section we briefly consider the case of non-vanishing torsion. As this material is not directly pertinent to the original work presented in Chapters IV and V, and is included for completeness, our discussion will be concise. For a more detailed presentation see [6,21,22].

We first discuss the conditions under which four dimensional N=1 supersymmetry can be maintained in the presence of non-vanishing torsion [21].

From equation (II.22) we see that the connection $\tilde{\nabla}$ has torsion given by

$$T_{mnp} = 4\beta H_{mnp} \tag{II.80}$$

It is a short and unenlightening calculation to show that (II.20) and (II.23) imply the following useful relation between the scalar curvature based on the torsion free Christoffel connection and the torsion tensor:

$$R = \frac{16}{3} \beta^2 H_{mnp} H^{mnp}. \tag{II.81}$$

From the intrinsic definition of the Riemann tensor [23]

$$R(A,B)C = [\nabla_A, \nabla_B]C + \nabla_{[A,B]}C \tag{II.82}$$

where the first bracket denotes antisymmetrization and the second denotes, Lie bracket, it is straightforward to express the generalized Riemann tensor built on a torsion full covariant derivative in terms of the Christoffel-based Riemann tensor.

In components this relation is

$$\tilde{R}_{mnpq} = R_{mnpq} + \nabla_p T_{mnq} - \nabla_q T_{mnp} + T_{rmp} T^r{}_{qn} - T_{rmq} T^r{}_{pn} \qquad (II.83)$$

From this expression and the fact that the torsion is completely anti-symmetric we find the relation between the generalized and ordinary Ricci tensors to be

$$\tilde{R}_{mn} \equiv \tilde{R}^p{}_{mpn} = R_{mn} - T^{pq}{}_m T_{pqn} + \nabla^p T_{pmn} \qquad (II.84)$$

Now following precisely the same reasoning used in section (II.3) to show the vanishing of R_{mn} in the case of vanishing torsion, it is clear that the corresponding statement in the torsion-full case is the vanishing of $\tilde{R}_{mn}$. Using this in (II.84) and contracting over the remaining indices to form the scalar curvature, we find

$$\begin{aligned} R &= T^{mnp} T_{mnp} \\ &= 16\beta^2 H^{mnp} H_{mnp} \end{aligned} \qquad (II.85)$$

Comparing this result with (II.81) shows that consistency demands

$$H_{mnp} = 0 \qquad (II.86)$$

Thus, the vanishing of $\tilde{R}$ implies the vanishing of H. It seems, therefore, that referring to trivial torsion as an assumption is a misnomer — apparently it follows from the requirement of maintaining unbroken N=1 supersymmetry in four dimensions. This statement is true, however, only in the context of full space metrics of the form

$$g_{MN}(x,y) = \begin{bmatrix} g_{\mu\nu}(x) & 0 \\ 0 & g_{mn}(y) \end{bmatrix} \tag{II.87}$$

which has been used in the analysis of the preceeding sections.

By generalizing (II.87) to

$$g_{MN}(x,y) = e^{2\varphi(y)} \begin{bmatrix} g_{\mu\nu}(x) & 0 \\ 0 & g_{mn}(y) \end{bmatrix} \tag{II.88}$$

where φ is the dilation field from the supergravity multiplet (and often referred to in this context as a warp factor), it can be shown that survival of four-dimensional N=1 supersymmetry does <u>not</u> necessarily imply the vanishing of the generalized Riemann tensor if φ has a non-vanishing vacuum expectation value [6]. We therefore have the result that if the background fields only include the metric and torsion, then survival of four dimensional N=1 supersymmetry implies vanishing torsion. With the inclusion of non-trivial background dilation field this conclusion no longer holds.

A complete analysis, analogous to that presented in (II.1), for the more general situation of (II.88) has been carried out in [6]. To give the flavour of the new possibilities (which we shall not discuss further) we quote the results of [6].

The survival of four-dimensional N=1 supersymmetry is ensured if

1. The compactified component of the universe is a complex manifold.
2. The fundamental form J (defined in (II.39)) must satisfy

$$\partial\bar{\partial}J = 1/30\ i\ \mathrm{Tr}\ F\wedge F - i\ \mathrm{tr}\ R\wedge R \tag{II.89}$$

$$d^{+}J = i(\partial-\bar{\partial})\ell n\ ||\omega|| \tag{II.90}$$

where ω is defined in (II.45).

3. The Yang–Mills field strength must obey

$$g^{a\bar{b}}F_{a\bar{b}} = 0 \tag{II.91a}$$

$$F_{ab} = F_{\bar{a}\bar{b}} = 0 \tag{II.91b}$$

We note that (II.89) when integrated over a four dimensional surface implies the familiar relation (II.48) and that (II.90) is not well defined unless the first Chern class of the base manifold vanishes (c.f. text after (II.44)) and $h^{n,o} = 1$ (c.f. text after (II.46)), as these conditions ensure that ω has no zeroes or poles [6].

The background values of the torsion and dilation are then given by

$$H = i/2(\bar{\partial}-\partial)J$$

$$\varphi = 1/8\ \ell n\ ||\omega|| + \varphi_o \tag{II.92}$$

where φ_o is a constant.

Although potentially viable, these solutions suffer from two potential drawbacks. The first is related to the fact that the supersymmetry transformations of Chapline and Manton [9] used in (II.1) are only valid for a field strength H defined without a Lorentz Chern–Simons three form (c.f. (II.4)) as this is the situation studied in [9]. Inclusion of the Lorentz Chern–Simons three form, as dictated by the anomaly concellation mechanism of Green and Schwarz [10], changes the supersymmetry transformations derived in [9], and thereby apparently invalidates <u>all</u> of the analysis presented, as (II.1) has played a crucial role. However, Romans and Warner [22] have studied this problem and tentatively concluded that in the case of vanishing torsion one is justified in using the Chapline and Manton supersymmetry transformations. For non-zero torsion the case is not as clear. In [6] it is claimed that this is not a problem, but a great deal of care is certainly warranted.

The second, and more definitive problem, is that all of the existence theorems invoked in section (II.3) are based on the assumption that the base manifold is Kähler. With non-vanishing torsion this is not the case. Sufficiently powerful generalizations of these theorems probably exist [24], but as yet no one has explicitly shown this to be true.

In all that follows, we shall avoid the case of non-vanishing torsion and therefore only consider compactifications on Calabi-Yau manifolds. As a final comment, we mention that there has been some question as to whether Calabi-Yau manifolds are viable compactifications based upon arguments stemming from the study of corresponding non-linear sigma models [25] and from analysis of the superstring effective action [26]. However, it appears that Calabi-Yau manifolds are at least the starting point for a perturbatively constructed compactification [27,28]; we shall therefore concentrate on Calabi-Yau compactifications.

REFERENCES

CHAPTER II

1. V.S. Kaplunovsky, Phys. Lett., 155B (1985) 151; J.H. Schwarz in Proc. 28th Scottish Universities Summer School in Physics (1985) 301.
2. Y. Hosotani, Phys. Lett., 126B (1983) 309; 129B (1983) 193.
3. E. Witten, Nucl. Phys., B258 (1985) 75.
4. A. Strominger and E. Witten, Princeton University Preprint (to appear in Comm. Math. Phys.) 1985; A. Strominger ITP Preprint, ITP−105 (1985).
5. P. Candelas, G.T. Horowitz, A. Strominger and E. Witten, Nucl. Phys. B258 (1985) 75.
6. A. Strominger, ITP Preprint NSF−ITP−86−16 (1986).
7. E. Calabi in Algebraic Geometry and Topology, a Symposium in Honour of S. Lefshetz, Princeton University Press (1957) 78; S.T. Yau, Proc. Natl. Acad. Sci. 74 (1977) 1798.
8. K. Uhlenbeck and S.T. Yau, On the Existence of Hermitian−Yang−Mills Connections in Stable Vector Bundles, to appear.
9. G.F. Chapline and N.S. Manton, Phys. Lett., 120B (1983) 105.
10. M.B. Green and J.H. Schwarz, Phys. Lett. 149B (1985) 117.
11. E. Witten, Phys. Lett., 149B (1984) 351.
12. T. Eguchi, P.B. Gilkey and A.J. Hanson, Phys. Rep. 66 (1980) 213.
13. S.K. Donaldson, J. Diff. Geometry, to be published.
14. E. Witten, Nucl. Phys., B268 (1986) 79.
15. F. Hirzebruch, Topological Method in Algebraic Geometry (Springer Verlag 1966).

16. B.R. Greene, K.H. Kirklin, P.J. Miron and G.G. Ross, Nucl. Phys.B, to appear.

17. E. Witten, private communication (1986).

18. R. Bott and L. Tu, Differential Forms in Algebraic Topology (Springer, 1982).

19. E. Witten, in Proceedings of Santa Barbara Conference on Unified String Theories (1985).

20. E. Witten, in Proceedings of Argonne Symposium on Anomalies, Geometry and Topology (World Scientific, 1985).

21. I. Bars, D. Nemeschansky and S. Yankielowicz, SLAC Preprint 3758 (1985).

22. L.J. Romans and N.P. Warner, Caltech Preprint 68-1291 (1985).

23. C.W. Misner, K.S. Thorne, J.A. Wheeler, Gravitation (W.H. Freeman and Co., 1970).

24. S.K. Donaldson, private communication (1986).

25. D. Zanon, Harvard Preprint, HUTP-85/A064 (1985); M.T. Grisaru, A.E.M. van de Ven and D. Zanon, preprints HUTP-86/A020, HUTP-86/A026.

26. D.J. Gross and E. Witten, Princeton Preprint 1986; M.D. Freeman and C.N. Pope, Imperial Preprint TP/85-86/17 to appear.

27. D. Nemeschansky and A. Sen, SLAC preprint, SLAC-PUB-3925.

28. P. Candelas, M.D. Freeman, C.N. Pope, M.F. Sohnius and K.S. Stelle, Imperial Preprint, TP/85-86/26.

29. B. R. Greene, K.H. Kirklin, P.J. Miron, Nucl. Phys., 274B (1986) 574.

CHAPTER III

COMPACTIFICATION AND SPACETIME PHENOMENOLOGY

III.1 TOPOLOGICAL AND GEOMETRICAL PRELIMINARIES

In sections III.2 – III.4 we shall elucidate the deep and mathematically pleasing relation between the topology of the compactified component of the universe and the matter fields, gauge group, and Yukawa couplings which determine the effective low energy theory. A number of mathematical concepts [1], which we have tersely introduced in previous sections, will play a prominent role. With the aim of not disturbing the logical flow in the last three sections with disruptive mathematical digressions, and formalising some notions presented earlier, we devote this section to some relevant mathematical concepts.

On a complex n–dimensional manifold, M, there are four different notions of the tangent space at a point p. The first is the familiar real tangent space in which the complex structure on M is ignored and it is viewed as a real 2n–dimensional manifold. For our purposes, we will find the following two tangent space formulations, which do reflect the complex structure, far more useful. The holomorphic tangent space is defined to have the basis $\{\partial/\partial z_i\}$ over the complex field $\mathbb{C}$. That is

$$T_p'(M) = \mathbb{C}\{\partial/\partial z_i\} \tag{III.1}$$

Similarly, the antiholomorphic tangent space is given by

$$T_p''(M) = \mathbb{C}\{\partial/\partial \bar{z}^i\} \tag{III.2}$$

We note that the complexified tangent space is defined by

$$T_{\mathbb{C},p}(M) = T_{\mathbb{R},p}(M) \otimes \mathbb{C} = T_p'(M) \oplus T_p''(M), \tag{III.3}$$

although we shall concentrate on (III.1) and (III.2). In later chapters we will be concerned with holomorphic symmetries of complex manifolds: f: M → M. Notice that because f is holomorphic it induces a map

$$f_* : T_{\mathbb{C},p}(M) \to T_{\mathbb{C},f(p)}(M) \tag{III.4}$$

which respects the holomorphic structure

$$f_*(T_p{}'(M)) \subset T'_{f(p)}(M) \tag{III.5}$$

In our quest to make contact with four dimensional physics through the compactification procedure discussed previously, we will find cohomology theory of primary importance (due to the close link with harmonic analysis). De Rham cohomology is defined in terms of the quotient groups

$$H_d{}^p(M,\mathbb{C}) = Z^p(M,\mathbb{C})/dA^{p-1}(M,\mathbb{C}) \tag{III.6}$$

where $A^p(M,\mathbb{C})$ denotes the space of complex valued differential p–forms, $Z^p(M,\mathbb{C})$ denotes the subspace of closed differential p–forms, and d is the familiar nilpotent exterior derivative operator of order two. As M is a complex manifold, the decomposition (III.3) allows us to similarly decompose the cotangent space

$$T^*_{\mathbb{C},z}(M) = T^{*\prime}_z(M) \oplus T^{*\prime\prime}_z(M) \tag{III.7}$$

and its exterior products

$$\Lambda^n T^*_{\mathbb{C},z}(M) = \bigoplus_{p+q=n} (\Lambda^p T^{*\prime}_z(M) \otimes \Lambda^q T^{*\prime\prime}_z(M)) \qquad \text{(III.8)}$$

The space of differential forms of degree n thus decomposes into a sum of subspaces identified by their respective holomorphic and antiholomorphic degrees

$$A^n(M,\mathbb{C}) = \bigoplus_{p+q=n} A^{p,q}(M,\mathbb{C}) \qquad \text{(III.9)}$$

Notice from (III.8) that a (p,q) form is a *tensor* product of antisymmetric holomorphic and antiholomorphic wedge products.

Similarly, as indicated in (II.40), the exterior derivative $d : A^n(M) \to A^{n+1}(M)$ may be decomposed according to

$$d = \partial + \bar{\partial} \qquad \text{(III.10)}$$

with
$$\partial : A^{p,q}(M,\mathbb{C}) \to A^{p+1,q}(M,\mathbb{C})$$
$$\bar{\partial} : A^{p,q}(M,\mathbb{C}) \to A^{p,q+1}(M,\mathbb{C}) \qquad \text{(III.11)}$$

A holomorphic map $f : M \to M$ clearly preserves (III.7) since it preserves (III.3). We therefore find that the decomposition of arbitrary forms into those with definite holomorphic and antiholomorphic degrees as in (III.9) is also preserved

$$f^*(A^{p,q}(M)) \subseteq A^{p,q}(M) \qquad \text{(III.12)}$$

Using the $\bar{\partial}$ operator, which like d is nilpotent of order two, we may define another class of cohomology groups. In place of $Z^p(M,\mathbb{C})$, let $Z_{\bar{\partial}}{}^{p,q}(M,\mathbb{C})$ denote the space of $\bar{\partial}$ closed (p,q) forms. The Dolbeault cohomology groups are then defined by

$$H_{\bar\partial}^{p,q}(M,\mathbb{C}) = Z_{\bar\partial}^{p,q}(M,\mathbb{C}) \; / \; \bar\partial(A^{p,q-1}(M,\mathbb{C})). \qquad \text{(III.13)}$$

Notice that the holomorphic map f, as manifest by f^* in (III.12), commutes with the $\bar\partial$ operator and we therefore have the induced map

$$f^* : H_{\bar\partial}^{p,q}(M,\mathbb{C}) \to H_{\bar\partial}^{p,q}(M,\mathbb{C})$$

In section two we will describe an identification between massless matter fields and elements of cohomology classes. The relation (III.14), and a minor generalization described later, will thus translate into symmetry transformations of low energy fields appearing in an effective Lagrangian, under the action of holomorphic isometries of the base manifold. As the theory must respect these isometries, the transformation properties of matter fields, as manifest in (III.14) will place strong restrictions on allowed interactions.

In the presence of gauge fields, we are naturally led to consider complex vector bundles built on M. We start with C^∞ complex vector bundles, but show that our interest lies in the subset of holomorphic vector bundles.

A C^∞ complex vector bundle E on M is composed of the union of complex vector spaces E_x associated with each point $x \epsilon M$, upon which a C^∞ manifold structure is defined such that

1. The projection map is C^∞. That is $\pi : E = \bigcup_{x \epsilon M} E_x \longrightarrow M$ which acts according to $\pi : E_x \longrightarrow x$ is C^∞.

2. There exists local trivializations which are diffeomorphisms. That is, every point $x \epsilon M$ lies in an open set $U \subset M$ for which there exists a differomorphism : Φ_u: $\pi^{-1}(U) \longrightarrow U \otimes \mathbb{C}^k$ which isomorphically maps $E_x \longrightarrow \{x\} \otimes \mathbb{C}^k$.

Clearly, the transition functions

$$g_{UV}(x) = (\Phi_u \cdot \Phi_V^{-1}) \Big|_{\{x\} \otimes \mathbb{C}^k}$$

are C^∞ $GL(k,\mathbb{C})$ maps.

A section of a vector bundle is a map which picks out a unique point on each fibre, in a continuous manner. Specifically, a section σ of a vector bundle E over a subset $U \subseteq M$ is a map

$$\sigma : U \to E$$

such that

$$\pi\sigma(x) = x$$

In other words $\sigma(x) \in E_x$. On C^∞ vector bundles, there is no natural frame independent definition for an exterior derivation on the space of sections, $\Gamma(E)$. This motivates us to consider holomorphic vector bundles for which the local trivializations

$$\Phi_U : \pi^{-1}(U) \to U \otimes \mathbb{C}^k$$

are biholomorphic maps of complex manifolds. This implies, of course, that the transition functions are holomorphic maps as well.

We now show that the bundle of E valued (p,q) forms $A^{p,q}(M,E)$ (which we shall often write as $A^{p,q}(E)$) has a natural frame independent exterior derivation

$$\bar{\partial} : A^{p,q}(E) \to A^{p,q+1}(E) .$$

Consider a section $\sigma \in \Gamma(A^{p,q}(E))$ and let $\{e_1,\ldots,e_k\}$ be a local holomorphic frame for the vector bundle E over $U \subseteq M$. (Note: when clear from content, we shall omit the Γ and write $\sigma \in A^{p,q}(E)$). In terms of this local basis we express the section σ as

$$\sigma = \omega_i \otimes e_i \tag{III.15}$$

(where we employ the summation convention) with $\omega_i \in A^{p,q}(U)$. We define $\bar{\partial}$ by

$$\bar{\partial}\sigma = \bar{\partial}\omega_i \otimes e_i \ . \tag{III.16}$$

To show that this is well defined, we must prove that $\bar{\partial}\sigma$ is frame independent. To this end, let $\{e'_1,\ldots,e'_k\}$ be another holomorphic frame, related to the original frame by the holomorphic functions g_{ij} according to

$$e_i = g_{ij}e'_j \ . \tag{III.17}$$

We find

$$\sigma = g_{ij}\omega_i \otimes e'_j \tag{III.18}$$

and

$$\bar{\partial}\sigma = \bar{\partial}(g_{ij}\omega_i) \otimes e'_j = g_{ij}\bar{\partial}\omega_i \otimes e'_j = \bar{\partial}\omega_i \otimes e_i \ . \tag{III.19}$$

Thus, $\bar{\partial}\sigma$ is frame independent and hence well defined. This result is highly relevant because we shall shortly demonstrate that low energy matter fields after compactification arise from elements of holomorphic bundle valued cohomology groups defined in terms of this $\bar{\partial}$ operator.

In section II.2 we discussed the constraints which the preservation of N=1 spacetime supersymmetry places on the Yang–Mills bundle. In particular we have from (II.53a) and (II.56) the requirement that the (2,0) and (0,2) components of the curvature two–form F vanish. To understand the significance of this requirement, it is useful to recall that a connection D on a complex vector bundle E is a map

$$D : \Gamma(E) \longrightarrow \Gamma(T^*(M) \otimes E) \qquad \text{(III.20)}$$

which satisfies the Leibnitz rule

$$D(f \cdot s) = df \otimes s + f \cdot D(s) \qquad \text{(III.21)}$$

for all $s \in \Gamma(E)$, $f \in C^\infty(M)$.

We note that in general there is no preferred connection on a vector bundle E. However, if E is a holomorphic vector bundle and is endowed with a fibre–wise Hermitian metric h, there are two requirements which uniquely determine D.

(i) Compatability with the hermitian metric h:

$$d(h(s,t)) = h(Ds,t) + h(s,Dt) \ . \qquad \text{(III.22)}$$

This is simply the familiar requirement that the metric be covariantly constant with respect to the connection D.

(ii) Compatability with the complex structure:

From (III.7) we may decompose D in the form

$$D = D' + D'' \tag{III.23}$$

with

$$D' : \Gamma(E) \longrightarrow \Gamma(T^{*\prime}(M) \otimes E) \tag{III.24a}$$

and

$$D'' : \Gamma(E) \longrightarrow \Gamma(T^{*\prime\prime}(M) \otimes E) \tag{III.24b}$$

Recall that on a holomorphic vector bundle we may consistently define $\bar{\partial}$ which has the same domain and range as D when acting on $A^{0,0}(E)$. (Of course, using the Leibnitz rule we can extend the domain of D). A connection D is said to be compatible with the complex structure if $D'' = \bar{\partial}$.

The unique connection satisfying these two requirements on a holomorphic vector bundle is known as the metric connection. We will also refer to it as a holomorphic connection on a holomorphic bundle, although in the mathematical literature the more common name is a Hermitian connection on an Hermitian bundle. The connection, in a local frame $\{e_1, \ldots, e_k\}$ for E over U acts on e_i by

$$De_i = A_{ij} e_j \tag{III.25}$$

and hence on $\sigma \in \Gamma(E)$ by

$$D\sigma = d\sigma_i \otimes e_i + \sigma_i \otimes De_i \tag{III.26}$$

$$= (d\sigma_j + \sigma_i A_{ij}) \otimes e_j \tag{III.27}$$

where $A = (A_{ij})$ is a matrix of one forms. Notice that we find it convenient to suppress the one-form index and display the fibre indices, exactly the opposite of what was done in the previous sections.

The curvature two-form matrix F is obtained from

$$D^2 e_i = F_{ij} \otimes e_j \; . \tag{III.28}$$

In terms of the connection matrix we find

$$\begin{aligned} D^2 e_i &= D(A_{ij} \otimes e_j) \\ &= dA_{ij} \otimes e_j - A_{ij} \wedge A_{jk} \otimes e_k \\ &= (dA_{ij} - A_{ik} \wedge A_{kj}) \otimes e_j \; . \end{aligned} \tag{III.29}$$

The minus sign arises because A is a matrix of p=1 forms, and there is a $(-1)^p$ in the general Leibnitz rule for D. Now, the connection matrix A_{ij} associated with the metric connection D, only contains 1-forms of type (1,0) as $D''e_i = \bar{\partial} e_i = 0$, in a holomorphic frame.

In a unitary frame (that is, a frame in which the $\{e_1(x),\ldots,e_k(x)\}$ form an orthonormal basis for E_x for each x) we have $h(e_i,e_j) = \delta_{ij}$ and hence

$$\begin{aligned} 0 = dh(e_i,e_j) &= A_{ik}h(e_k,e_j) + \bar{A}_{jk}h(e_i,e_k) \\ &= A_{ij} + \bar{A}_{ji} \; . \end{aligned} \tag{III.30}$$

The connection matrix in a unitary frame is thus skew-hermitian.

From (III.29) we see that F_{ij} contains no (0,2) forms as these arise from $(D'')^2$ which vanishes as $D'' = \bar{\partial}$. In a unitary frame, though, F_{ij} by its relation to A_{ij} is also skew–hermitian and hence contains no (2,0) forms. Therefore, since this result is invariant under a change of frames, we find

<u>THEOREM</u>: The holomorphic connection on a holomorphic vector bundle has a curvature matrix which only contains (1,1) forms.

We may thus conclude that the constraint (II.53a) is the requirement that the gauge field A which gives rise to the field strength (curvature) F is a holomorphic connection on a holomorphic vector bundle. In section II.3 we showed that constraint (II.53b) is equivalent to this connection satisfying the Yang–Mills equations. We therefore seek holomorphic (hermitian) Yang–Mills connections over compact three dimensional complex manifolds. Fortunately, as mentioned earlier, precisely this issue has recently been addressed by the mathematicians K. Uhlenbeck and S.T. Yau [2] who have demonstrated

<u>THEOREM</u> (Uhlenbeck and Yau): There exists a holomorphic Yang–Mills connection for any stable holomorphic bundle over any compact Kähler manifold.

(In the case of non–zero torsion for which the manifold of compactification is not Kähler, a weaker version of this theorem may still apply [3]).

The above analysis indicates the rigorous mathematical justification for invoking this theorem as we did in Chapter II and shall again in Chapter IV.

In the next section we will show that the low energy spacetime field content is determined by the harmonic forms on the manifold of compactification. We therefore now provide a brief discussion of harmonic theory on compact complex manifolds. Many statements will be given without proof as these are often lengthy and easily found in the literature [1,4].

We first recall that on a real manifold we have the Hodge theorem [5].

THEOREM (Hodge): On a compact manifold M without boundary, any p-form section $\omega_p \epsilon \Gamma(\Lambda^p T^*(M))$ may be uniquely expressed as the sum of exact, coexact, and harmonic forms. (By Harmonic p-form we refer to a zero eigenmode of the Laplacian $\Delta_p = d_{p-1}d_p^* + d_{p+1}^* d_p$. We shall omit the p-subscript when it is clear from context). If, furthermore, ω_p is closed then it may uniquely be expressed as

$$\omega_p = d\alpha + \gamma \tag{III.31}$$

where γ is harmonic.

Any p-form ω_p' which differs from ω_p by an exact form, may similarly be expressed with the *same* harmonic form γ. We therefore find that to each de Rham cohomology class, we may assign a unique harmonic representative. This gives the important and useful isomorphism.

THEOREM: For M a compact manifold without boundary,

$$H_d{}^p(M,R) \cong H^p(M,R)$$

where $H_d{}^p$ is the p^{th} de Rham cohomology group, and H^p is the space of harmonic p-forms.

We would like to establish a similar result for complex manifolds, but immediately notice a new feature. On a complex manifold, there are three generalizations of the ordinary Laplacian. We have the usual

$$\Delta = dd^* + d^*d \tag{III.32a}$$

and also

$$\Delta_\partial = \partial\partial^* + \partial^*\partial \tag{III.32b}$$

$$\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial \tag{III.32c}$$

where *, in this context, denotes the adjoint operator.

In the following sections, as introduced in section II.3 and emphasized in section II.4, we will specialize to compactifications on Kähler manifolds. Remarkably, on Kähler manifolds, all of the Laplacians in (III.32) are identical (up to a constant factor). Furthermore, we have just established that our concern will be with holomorphic vector bundles over compact complex manifolds. For such a situation, we demonstrated in (III.19) that the natural, well defined exterior derivative is the $\bar\partial$ operator. Henceforth, we concentrate on the $\Delta_{\bar\partial}$ Laplacian, and for ease of presentation denote it by Δ.

We now state of number of theorems regarding harmonic theory on compact complex manifolds.

In analogy with the isomorphism between de Rham cohomology and harmonic forms just described, we have

<u>THEOREM</u>: For M a compact complex manifold without boundary, the space of harmonic (p,q) forms (under the $\Delta_{\bar\partial} \equiv \Delta$ operator) $H^{p,q}(M,\mathbb{C})$ is isomorphic to the (p,q) Dolbeault cohomology group $H_{\bar\partial}^{p,q},(M,\mathbb{C})$.

Thus, for each element of $H_{\bar\partial}^{p,q}(M,\mathbb{C})$ we have a unique harmonic representative. We also have

<u>THEOREM</u> : The dimension $H_{\bar\partial}^{p,q}(M,\mathbb{C})$ as a complex vector space is finite

and the dimension of $H_{\bar\partial}^{n,n}(M,\mathbb{C})$ is one.

THEOREM : (Kodaira–Serre duality). The space $H_{\bar\partial}^{p,q}(M,\mathbb{C})$ is isomorphic to $H_{\bar\partial}^{n-p,n-q}(M,\mathbb{C})$.

This latter theorem is proved by noting that the Hodge star operator

$$* : A^{p,q}(M) \longrightarrow A^{n-p,n-q}(M) \tag{III.33}$$

(in analogy with (II.58) and elaborated upon shortly) commutes with Δ which may be written

$$\Delta = -\,(\bar\partial*\bar\partial* + *\bar\partial*\bar\partial)\,, \tag{III.34}$$

where we use the fact that $\bar\partial^* = -*\bar\partial*$, (The first star denotes operator adjoint while the second is the Hodge star operator). All of these theorems hold regardless of whether M is a Kähler manifold. In the Kähler case, a remarkable and extremely useful level of symmetry arises in the harmonic structure. We find the following additional theorems when M is Kähler.

THEOREM : The dimension of $H_{\bar\partial}^{p,q,}(M,\mathbb{C})$ equals that of $H_{\bar\partial}^{q,p}(M,\mathbb{C})$.

This theorem is clear because complex conjugation exchanges (p,q) forms ω with (q,p) forms $\bar\omega$. Taking ω to be harmonic, and noting that $\Delta_\partial = \Delta_{\bar\partial}$, we see that $\bar\omega$ is necessarily harmonic as well.

THEOREM : The r^{th} de Rham cohomology group may be expressed as

$$H_d{}^r(M,\mathbb{C}) = \bigoplus_{p+q=r} H_{\bar\partial}{}^{p,q}(M,\mathbb{C}) \quad . \tag{III.35}$$

<u>COROLLARY</u>: If b_r denotes the dimension of $H_d{}^r(M)$ then

$$b_r = \sum_{p+q=r} \dim H_{\bar\partial}{}^{p,q}(M,\mathbb{C}) \quad .$$

The fact that the complex cohomology on a compact Kähler manifold splits into this direct sum, implies that harmonic forms may be taken to have definite (p,q) (as opposed to definite (p+q)).

The above relations will be useful in our treatment of the gauge singlet low energy field content of compactified supergravity theories. However, the Yang–Mills sector requires us to discuss harmonic theory for holomorphic vector bundles over compact complex manifolds.

From the discussion preceeding (III.19) we have that for any holomorphic vector bundle E over M

$$\bar\partial : A^{p,q}(M,E) \longrightarrow A^{p,q+1}(M,E) \tag{III.36}$$

is well defined for global C^∞ E–valued forms. Furthermore, since $\bar\partial$ is still nilpotent of order two, we may define E–valued Dolbeault cohomology by

$$H_{\bar\partial}{}^{p,q}(M,E) = \frac{Z_{\bar\partial}{}^{p,q}(M,E)}{\bar\partial A^{p,q-1}(M,E)} \tag{III.37}$$

(we will sometimes omit the M when clarity is not besmirched).

Many of the relations satisfied by the ordinary Dolbeault cohomology have direct analogues in the case of Dolbeault cohomology taking values in a holomorphic vector bundle. For example, note that a holomorphic map $f : M \to M$, still commutes with $\bar{\partial}$ and therefore we have the generalization of (III.14): A holomorphic map $f : M \to M$ induces a map f^* with

$$f^* : H_{\bar{\partial}}^{p,q}(M,E) \longrightarrow H_{\bar{\partial}}^{p,q}(M,E) \quad . \tag{III.14$'$}$$

The comments following (III.14) are relevant here as well, and shall be used in section III.4.

<u>THEOREM</u> : The space of E– valued harmonic (p,q) forms is isomorphic to the (p,q) E– valued Dolbeault cohomology group. That is, $H^{(p,q)}(M,E) \cong H_{\bar{\partial}}^{p,q}(M,E)$.

There is one subtlety that we have not mentioned. The Hodge star operator which enteres into the expression for the adjoint of an exterior derivative and hence into the definition of Laplace oprators, takes a slightly different form in the context of bundle valued differential forms. The (ordinary) Hodge star operator referred to in

(II.65) is defined by the requirement that for η, $\psi \in A^{p,q}(M)$

$$(\eta,\psi) = \int_M \eta \wedge * \psi \quad . \tag{III.38}$$

In this expression, (η,ψ) is defined by

$$(\eta,\psi) = \int_M (\eta(z),\ \psi(z))\Phi(z) \tag{III.39}$$

where $(\eta(z),\psi(z))$ is the induced inner product on $T_z^*(p,q)$ from the metric on M, and $\Phi(z)$ is the volume form for M. In the more general context of η, ψ ϵ $A^{p,q}(M,E)$, (III.38) and (III.39) are still used to define the Hodge star operator, except now the inner product on the right hand side of (III.39) is induced from the metric on M <u>and</u> the fibre metric on E. In particular, for (III.38) to be well defined in this context, we must have

$$*_E : A^{p,q}(M,E) \longrightarrow A^{n-p,n-q}(M,E^*) \qquad \text{(III.40)}$$

where the lattermost star denotes the dual bundle. By choosing a local unitary frame for E (unitary with respect to the Hermitian fibre matric) $\{e_\alpha\}$ and its dual frame in E^*, $\{e_\alpha^*\}$, the action of $*_E$ may explicitly be written

$$*_E\ \eta = *\eta_\alpha \otimes e_\alpha^* \qquad \text{(III.41)}$$

where we express

$$\eta = \eta_\alpha \otimes e_\alpha \qquad \text{(III.42)}$$

with η_α ϵ $A^{p,q}(M)$, and the first star on the right hand side of (III.41) is the ordinary Hodge star operator. It is this operator which must be used in constructing the adjoint of exterior derivatives acting on $A^{p,q}(M,E)$ and in the definition of the Laplacian. In particular

$$\Delta = -(\bar\partial *_E \bar\partial *_E + *_E \bar\partial *_E \bar\partial) \qquad \text{(III.43)}$$

$$= \bar\partial\bar\partial^* + \bar\partial^*\bar\partial, \qquad \text{(III.44)}$$

is the operator referred to in defining $H^{(p,q)}(M,E)$. The above discussion motivates the generalization of the Kodaira–Serre duality theorem given earlier. The difference between the two is simply a manifestation of the generalized action of the Hodge star operator. We have

THEOREM (Kodaira–Serre duality) : The Hodge star operator is a bijective map between $H_{\bar\partial}^{p,q}(M,E)$ and $H_{\bar\partial}^{n-p,n-q}(M,E^*)$. In fact, $H_{\bar\partial}^{n-p,n-q}(M,E^*)$ is the dual space of $H_{\bar\partial}^{p,q}(M,E)$.

This gives us the important corollary:

COROLLARY : Let M be a compact complex manifold of dimension n, and E a holomorphic vector bundle over M. Then, if E^* denotes the dual bundle of E, the dimension of the vector space $H_{\bar\partial}^{p,q}(M,E)$ is equal to that of $H_{\bar\partial}^{n-p,n-q}(M,E^*)$. That is

$$\dim H_{\bar\partial}^{p,q}(M,E) = \dim H_{\bar\partial}^{n-p,n-q}(M,E^*).$$

Our analysis in later sections relies heavily on this result.

Having identified some of the properties which characterize the cohomology of holomorphic vector bundles over compact complex manifolds, we now turn to index theory to gain further insight.

Index theory studies the relationship between the number of independent harmonic modes of certain generalized Laplacians and the topological properties of a given fibre bundle upon which they act. Although the index theorem of Atiyah and Singer [6] is a very powerful relation which may be used for general elliptic operators, we will only present its content for the case of our interest : the Dolbeault operator acting on $A^{p,q}(M,E)$.

Intuitively, the index theorem is a generalization of the familiar Gauss–Bonnet theorem. Recall that the latter relates the alternating sum of the number of zero modes of the de Rham Laplacian to the integral of the highest Chern form over the manifold. The index theorem is based on the same structure : an alternating sum of the number of zero modes of an arbitrary elliptic operator is related to an integral of certain cohomology classes over the base manifold. When this operator is chosen to be the de Rham operator acting on $A^p(M,\mathbb{C})$, the Gauss–Bonnet theorem arises. If, however, this operator is chosen to be the Dolbeault operator acting on $A^{0,q}(M,E)$ for $q=0,\ldots,n=\dim M$ (complex dimension) then the twisted Riemann–Roch theorem follows

$$\mathrm{index}(\bar{\partial}) \equiv \sum_{q=0}^{n} (-1)^q \dim H_{\bar{\partial}}{}^{0,q}(M,E) =$$

$$= \int_M \mathrm{td}\,(T_{\mathbb{C}}(M)) \wedge \mathrm{ch}(E) \qquad \text{(III.45)}$$

where $\mathrm{td}(T_{\mathbb{C}}(M))$ is the Todd class of the (complexified) tangent bundle and $\mathrm{ch}(E)$ is the Chern character of E. (Note: the term twisted is often used to denote bundle valued forms). To define these characteristic classes, it is useful to recall the definition of the total Chern form given in (II.42) and the splitting principle as used in (II.58). Using these relations we may write

$$C(E) = \mathrm{Det}\begin{bmatrix} 1+(i/2\pi)\Omega_1 & & \\ & \ddots & \\ & & 1+(i/2\pi)\Omega_k \end{bmatrix} \qquad \text{(III.46)}$$

where we have split over Lie Algebra indices as each Ω_j is a two–form, and k is the dimension of a typical fibre of E. Letting $x_j = (i/2\pi)\Omega_j$, we have

$$C(E) = \sum C_\ell(\Omega) = \prod_{j=1}^{k} (1 + x_j) \tag{III.47}$$

and hence $C_\ell(\Omega)$ is the ℓ^{th} elementary symmetric function of the curvature two form components x_j. In terms of these variables the Chern character ch(E) is given by

$$ch(E) = \sum_{j=1}^{k} e^{x_j} \tag{III.48}$$

and the Todd class is defined according to

$$td(E) = \prod_{j=1}^{k} \frac{x_j}{1-e^{-x_j}} \quad . \tag{III.49}$$

Since by using (III.47) we can express these characteristic classes in terms of the Chern forms, they have topologically invariant integrals as well.

As an example, we compute the index of the Dolbeault operator acting on the holomorphic tangent bundle over a three (complex) dimensional Kähler manifold with vanishing first Chern class. This result will be of great importance in the next section. To accomplish this, we first compute this index for an arbitrary holomorphic vector bundle — a result which we shall also make extensive use of in determining the number of (spacetime) generations of elementary particle multiplets.

To this end, we first use (III.47), (III.48) and (III.49) to explicitly express the Chern character and Todd class in terms of the Chern form. As our interest focuses on a three dimensional base manifold, we perform this calculation to third order in the curvature 2-form. We begin with the Chern character. From (III.48)

$$\mathrm{ch}(E) = \sum_{j=1}^{k} (1 + x_j + \frac{1}{2} x_j \wedge x_j + \frac{1}{6} x_j \wedge x_j \wedge x_j + \ldots) \tag{III.50}$$

$$= k + \sum_{j=1}^{k} x_j + \frac{1}{2} \sum_{j=1}^{k} x_j \wedge x_j + \frac{1}{6} \sum_{j=1}^{k} x_j \wedge x_j \wedge x_j + \ldots \tag{III.51}$$

The first sum is just the trace of the curvature 2-form matrix and hence from (III.47) is equal to the first Chern class. To evaluate the second sum, note that

$$\sum_{j=1}^{k} (x_j \wedge x_j) = (\sum_{j=1}^{k} x_j) \wedge (\sum_{j=1}^{k} x_j) - 2 \sum_{j<\ell} x_j \wedge x_\ell \tag{III.52}$$

From (III.47)

$$\frac{1}{2} \sum_{j=1}^{k} x_j \wedge x_j = \frac{1}{2} \{C_1(E) \wedge C_1(E) - 2C_2(E)\} \tag{III.53}$$

Similarly, we evaluate the third sum by using

$$\sum_{j=1}^{k} x_j \wedge x_j \wedge x_j = \frac{1}{2}\Big\{- \Big[\sum_{j=1}^{k} x_j\Big] \wedge \Big[\sum_{j=1}^{k} x_j\Big] \wedge \Big[\sum_{j=1}^{k} x_j\Big] + 3\Big[\sum_{j=1}^{k} x_j\Big] \wedge \Big[\sum_{j=1}^{k} x_j \wedge x_j\Big]$$

$$+ 6 \sum_{i<j<\ell} x_i \wedge x_j \wedge x_\ell \Big\} \tag{III.54}$$

With (II.39), (III.47) and (III.52) we thus conclude that

$$\begin{aligned}\sum_{j=1}^{k} x_j \wedge x_j \wedge x_j &= \frac{1}{2} \Big\{-C_1(E) \wedge C_1(E) \wedge C_1(E) \\ &\qquad + 3C_1(E) \wedge (C_1(E) \wedge C_1(E) - 2C_2(E) \\ &\qquad + 6C_3(E)\Big\} \\ &= C_1(E) \wedge C_1(E) \wedge C_1(C) - 3C_1(E) \wedge C_2(E) \\ &\qquad + 3C_3(E)\end{aligned} \tag{III.55}$$

Thus we have

$$ch(E) = k + C_1(E) - \frac{1}{2}\left\{C_1(E)\wedge C_1(E) - 2C_2(E)\right\}$$
$$+ \frac{1}{6}\left\{C_1(E)\wedge C_1(E)\wedge C_1(E) - 3C_1(E)\wedge C_2(E) + 3C_3(E)\right\} \tag{III.56}$$

For the Todd class we use (III.49) to write

$$td(E) = \prod_{j=1}^{k} \frac{x_j}{1 - \left[\sum_{n=o}^{\infty} x_j{}^n/n!\right]} \tag{III.57}$$

$$= \prod_{j=1}^{k} \sum_{m=o}^{\infty} \left[\sum_{n=o}^{\infty} x_j{}^n/(n+1)!\ \right]^m \tag{III.58}$$

where multiplication is by the wedge product.

Truncating this generating function at third order,

$$td(E) = 1 + \frac{1}{2}\sum x_j + \frac{1}{12}\sum x_j\wedge x_j + \frac{1}{4}\sum_{\ell<j} x_\ell\wedge x_j$$
$$+ \frac{1}{24}\sum_{\ell\neq j} x_\ell\wedge x_j\wedge x_j + \frac{1}{8}\sum_{i<j<k} x_i\wedge x_j\wedge x_k + \dots \tag{III.59}$$

Following our analysis for the Chern character, we analyze each term, order by order. As before,

$$1 + \frac{1}{2}\sum x_j = 1 + \frac{1}{2}C_1(E)\ . \tag{III.60}$$

Now,

$$\sum_{\ell<j} x_\ell\wedge x_j = C_2(E) \tag{III.61}$$

and from (III.53)

$$\sum_j x_j \wedge x_j = C_1(E)\wedge C_1(E) - 2C_2(E). \tag{III.62}$$

Thus

$$\frac{1}{12}\sum x_j\wedge x_j + \frac{1}{4}\sum_{\ell<j} x_\ell\wedge x_j = \frac{1}{12}(C_1(E)\wedge C_1(E) - 2C_2(E))$$

$$+ \frac{1}{4}(C_2(E))$$

$$= \frac{1}{12}(C_1(E)\wedge C_1(E) + C_2(E)) \tag{III.63}$$

For the third order term notice that

$$\sum_{\ell\neq j} x_\ell\wedge x_j\wedge x_j = \sum_{\ell,j} x_\ell\wedge x_j\wedge x_j - \sum_j x_j\wedge x_j\wedge x_j \tag{III.64}$$

$$= (\sum_\ell x_\ell)\wedge(\sum_j x_j\wedge x_j) - \sum_j x_j\wedge x_j\wedge x_j \tag{III.65}$$

$$= C_1(E)\wedge(C_1(E)\wedge C_1(E) - 2C_2(E))$$

$$- \{C_1(E)\wedge C_1(E)\wedge C_1(E) - 3C_1(E)\wedge C_2(E) + 3C_3(E)\} \tag{III.66}$$

where we have made use of (III.52) and (III.55). Thus

$$\frac{1}{24}\sum_{\ell\neq j} x_j\wedge x_j\wedge x_j = \frac{1}{24}\{C_1(E)\wedge C_2(E) - 3C_3(E)\} \tag{III.67}$$

Putting (III.60), (III.63), (III.67) all together, we find to third order

$$td(E) = 1 + \frac{1}{2}C_1(E) + \frac{1}{12}(C_1(E)\wedge C_1(E) + C_2(E))$$

$$+ \frac{1}{24}(C_1(E)\wedge C_2(E)) . \tag{III.68}$$

We now use (III.56) and (III.68) in (III.45) to evaluate the index of the twisted Dolbeault operator over a three (complex) dimensional base manifold. The third order term in td $(T_{\mathbb{C}}(M)) \wedge ch(E)$ is seen to be

$$\frac{1}{6}\{C_1(E) \wedge C_1(E) \wedge C_1(E) - 3C_1(E) \wedge C_2(E) + 3C_3(E)\} +$$

$$+ \frac{1}{4} C_1(T_{\mathbb{C}}(M)) \wedge \{C_1(E) \wedge C_1(E) - 2C_2(E)\} +$$

$$+ \frac{1}{12} (C_1(T_{\mathbb{C}}(M) \wedge C_1(T_{\mathbb{C}}(M)) + C_2(T_{\mathbb{C}}(M)) \wedge C_1(E) +$$

$$+ \frac{k}{24} (C_1(T_{\mathbb{C}}(M)) \wedge C_2(T_{\mathbb{C}}(M))) \qquad \text{(III.69)}$$

From the last chapter we know that, in the case of interest, $C_1(T_{\mathbb{C}}(M)) = C_1(E) = 0$. Thus, (III.69) greatly simplifies to

$$\text{index}(\bar{\partial}) = \int_M \frac{1}{2} C_3(E) \equiv \frac{1}{2} [C_3(E)]. \qquad \text{(III.70)}$$

That is, the index of $\bar{\partial}$ is one-half of the third Chern number of the coefficient vector bundle. From (III.45) we have the explicit result

$$h^{0,0}(M,E) - h^{0,1}(M,E) + h^{0,2}(M,E) - h^{0,3}(M,E) = \frac{1}{2} \int C_3(E) \qquad \text{(III.71)}$$

where $h^{0,q}(M,E) = \dim H_{\bar{\partial}}^{0,q}(M,E)$.

Now from the Kodaira-Serre duality theorem, we have

$$h^{0,2}(M,E) = h^{3,1}(M,E^*) \qquad \text{(III.72)}$$

From the discussion in Chapter II, it is clear that our interest focuses on manifolds of compactification with vanishing first Chern class. As conveyed earlier, this implies the existence of a covariantly constant holomorphic (3,0) form, ω_{ijk}. This form provides an isomorphic map between $h^{0,q}(M,E)$ and $h^{3,q}(M.E)$. (As an aside, the mathematically alert reader will no doubt recognize this as a restatement of the triviality of the canonical bundle K_M which enters into an alternate statement of Kodaira–Serre duality: $H^q(M,\Omega^p(E)) \cong H^{n-q}(M,\Omega^{n-p}(E^*))^*$, which for vanishing p is just $H^q(M,\theta(E)) \cong H^{n-q}(M,\theta(E^* \otimes K_m))^*$, where $\theta(E)$ is the sheaf of holomorphic functions on E. For vanishing first Chern class, K_M is trival and hence $H^q(M,\theta(E)) = H^{n-q}(M,\theta(E^*))^*$, that is, $H_{\bar\partial}^{0,q}(M,E) = H_{\bar\partial}^{0,n-q}(M,E^*)^*$). Also, since the number of global holomorphic sections of E is equal to that of E^*, $h^{0,0}(M,E) = h^{0,0}(M,E^*)$. By the above we then have

$$h^{0,0}(M,E) = h^{0,3}(M,E) \quad . \tag{III.73}$$

We therefore have from (III.71)

$$h^{0,1}(M,E^*) - h^{0,1}(M,E) = \frac{1}{2}\int_M C_3(E). \tag{III.74}$$

In the next section we shall show that the difference on the left hand side is the net number of families of elementary particles in a given representation of the gauge group, and hence (III.74) shows that the number of generations is related to a topological invariant of the (Yang–Mills) bundle E.

As an important special case, take E to be the holomorphic tangent bundle $T'(M)$. The Dolbeault cohomology group $H_{\bar\partial}^{0,1}(M,T^{*\prime}(M))$ is just $H_{\bar\partial}^{1,1}(M)$. By invoking the vanishing of the first Chern class of M and hence the existence of a covariantly constant holomorphic (3,0) from ω, we find the isomorphism between $T'(M)$ and $A^{2,0}(M)$. Thus the Dolbeault cohomology group satisfies

$$H_{\bar\partial}^{0,1}(M,T'(M)) \cong H_{\bar\partial}^{0,1}(M,A^{2,0}(M)) \cong H_{\bar\partial}^{2,1}(M) \tag{III.75}$$

Furthermore, the third Chern class of holomorphic tangent bundle integrated over the three dimensional manifold M, is just the Euler characteristic of M, which we denote by $\chi(M)$. Hence, in this special case, (III.74) becomes

$$h^{1,1}(M) - h^{2,1}(M) = \frac{1}{2}\chi(M). \tag{III.76}$$

For completeness, (and for use in the next section) we compute the remaining Hodge numbers for a three dimensional Calabi–Yau manifold M. From the fact that M is Kähler with vanishing first Chern class, we have the symmetries in the harmonic sector $h^{p,q} = h^{3-p,3-q} = h^{q,p}$ and $h^{p,o} = h^{o,3-p}$. Thus, the independent Hodge numbers may be taken to be $h^{o,o}$, $h^{o,1}$, $h^{1,1}$ and $h^{2,1}$. The $h^{o,o}$ Hodge number measures the dimension of the space of constant functions on M. Assuming M is connected, we have $h^{o,o} = 1$. Furthermore, on a manifold with $\chi(M) \neq 0$, (which we will require phenomenologically as $|\frac{1}{2}\chi(M)|$, in this case, gives the number of generations) it can be shown [7] that $h^{o,1}$ vanishes. Thus, on a Calabi–Yau manifold with non–vanishing Euler characteristic, the cohomology (which as mentioned and shortly demonstrated determines the low energy field content) depends on the two independent Hodge numbers $h^{1,1}$ and $h^{2,1}$.

The final issue which we shall address in this section is the relation between harmonic spinors and harmonic differential forms on manifolds with vanishing first Chern class. From a purely field theoretic point of view, such a discussion is not essential as massless fermions (which, as we shall discuss, arise from harmonic spinors on K) may be determined by supersymmetry from massless bosons (which arise from harmonic bundle valued forms on K). For example, as shown in [8], the relation

$$\psi^a = A_{\bar{m}}{}^a \gamma^{\bar{m}} \eta_+ \qquad \text{(III.77)}$$

generates a zero mode of the Dirac equation, where η_+ is the covariantly constant spinor introduced in the last chapter and a is a gauge index. However, from a pedagogical and calculational point of view, it is quite useful to understand the fermionic sector in its own right.

Our enquiry into the nature of harmonic spinors is greatly facilitated by the fact that our interest focuses on Kähler base manifolds with vanishing first Chern class. From the mathematics literature we have [9].

<u>THEOREM</u> : For a Kähler manifold with vanishing first Chern class, the harmonic spinors and holomorphic forms are isomorphic.

Holomorphic forms are $\bar{\partial}$ closed and hence constitute Dolbeault cohomology. As emphasized earlier Dolbeault cohomology is isomorphic to the space of harmonic forms and we thus have the corollary that on Calabi–Yau manifolds the harmonic spinors and harmonic forms are isomorphic.

This mathematical statement of equivalence is made explicit by reasoning along the lines used in (II.49) – (II.51). From these equations it is clear that the gamma matrices satisfy the algebra associated with creation and annihilation operators. These operators do not create particles as in their more common field theoretic role, but rather rotate the eight basis states appropriate to the space on which a Clifford algebra on a six dimensional base manifold acts. Explicitly, the *antisymmetric* nature of these operators allows us to view the action of the operator $a^{*\bar{i}}$ as the creation of an antiholomorphic form index $\bar{i}$ (these indices label Fock space states so this action rotates the space) while the annihilation operator a^i destroys it. With this association between spinors and forms, it can then easily be shown [17] that a spinor satisfies the massless Dirac equation if and only if the corresponding form is harmonic.

The above explanation ignores the possible presence of non-trivial gauge transformation properties. However, holomorphic bundle valued harmonic forms clearly give rise to harmonic spinors taking values in the same coefficient bundle. That is, the gauge "indices" pass undaunted through the above analysis, and the correspondence thus holds in the more general cases of interest.

The usefulness of the isomorphism between harmonic spinors and forms is two-fold. First, we will be able to invoke all of the powerful mathematical machinery regarding Dolbeault cohomology, developed previously, in our treatment of the low energy spacetime fermion fields which descend from the ten dimensional theory. Second, this isomorphism gives a simple means of determining the chirality of a four dimensional spinor. In particular, Γ_{11} is the product of ten gamma matrices : four with spacetime indices and six with K-space tangent indices. A ten-space fermion with definite chirality, say $+1$, then necessarily has the same four and six-space chiralities. From the above, gamma matrices on K act as creation-annihilation operators of antiholomorphic indices. Furthermore, gamma matrices reverse the chirality of a state. Thus, we can measure the chirality of a massless spacetime fermion in terms of the degree q of its corresponding six-space harmonic form by computing $(-1)^q$, as this gives the eigenvalues of γ. We therefore find that chiral asymmetry in spacetime occurs if the Dirac operator on K (for a fixed set of quantum numbers) has an imbalance between the number of zero modes corresponding to even and odd values of q. This, of course, is precisely what the index theorem discussed earlier detects and quantifies.

III.2 TOPOLOGICAL DETERMINATION OF MASSLESS FIELD CONTENT

One of the profoundly appealing aspects of higher dimensional theories is the close link between the topology of the extra dimensions and the phenomenological properties of the observable four dimensional theory. We have demonstrated one important example of this phenomena: an N=1 anomaly free supersymmetric theory survives in the low energy four dimensional world only if the topology and geometry of the compactified six−manifold and the Yang−Mills vector bundle meet the highly non−trivial constraints elucidated in the previous sections. In this section we examine the relationship between the topology of the compactified component of the universe and the low energy four dimensional field content. By way of introduction and establishing the central concepts, we first describe some general notions from Kaluza−Klein theory [10]. We then specialize to the particular compactifications compatible with the survival of four dimensional N=1 supersymmetry.

An acute yet liberal observer of our ambient universe readily discerns the familiar four dimensions of common report, but grants the possibility of other dimensions which might be tightly curled intospaces of unobservedly small radius R. One would then expect to become directly aware of these additional dimensions only if they are probed in experiments with energy on the scale of R^{-1}. It is generally believed [11] that the compactification scale in superstring theories is on the order of the Planck scale : $R \approx 10^{-33}$ cm and $R^{-1} \approx 10^{19}$ GeV. It is reasonable to assume that it will be some time before we can attain such scales; this does not, as we shall now describe, make our interest in extra dimensional theories merely academic. Consider a generic higher dimensional field $\varphi^{mn\ldots}_{\mu\nu\ldots}(x,y)$ where μ,ν .. denote spacetime indices, m,n ... denote extra dimensional indices and the coordinates (x,y) are split in an analogous fashion. When φ is substituted into its full higher dimensional field equations, these may heuristically be cast in the form

$$[D_4 + D_6]\varphi = 0 \tag{III.78}$$

where D_4 (D_6) is a generic four (six) dimensional operator appropriate to the field φ. The field φ may now be decomposed as a sum of eigenmodes of D_6 on the compact six-manifold K

$$\varphi^{mn\ldots}_{\mu\nu\ldots}(x,y) = \sum_i \varphi^i_{\mu\nu\ldots}(x)\; Y^{mn\ldots}_i(y) \tag{III.79}$$

where

$$D_6\, Y_i = m_i{}^2\, Y_i\,. \tag{III.80}$$

The eigenvalues of these modes act as four dimensional mass terms in the field equations. The m_i are typically quantized in units of the compactification scale, R^{-1}. (e.g. the Klein-Gordon equation on $M_4 \otimes S^1$ has mass operator $\partial^2/\partial y^2$ and hence has eigenvalues $(n/R)^2$ where R is the radius of S^1.) These masses, then, are typically all quite large <u>except</u> for the zero eigenmode solution of (III.80). The low energy four dimensional fields are thus determined by the zero eigenmode solutions of appropriate mass operators on the compact space. (We note that in some scenarios it is possible for massive higher dimensional modes to descend as massless fields in the four dimensional theory. This 'space-invaders' mechanism is not relevant in our cases of interest [10]).

We now illustrate these ideas with the ten dimensional massless fields from the N=1 supergravity multiplet and the $E_8' \otimes E_8$ Yang-Mills multiplet. (Note that the use of 'massless' in this last context refers to fields arising as the zero slope limit of the Heterotic superstring theory. These ten dimensional massless fields give rise, as discussed above, to an infinite tower of massive four-dimensional fields, along with a finite number of massless fields). Since the analysis for gauge fields is slightly more involved, we begin with the supergravity multiplet.

Consider the fluctuations about the full space metric [10]

$$g_{MN}(x,y) = \langle g_{MN}(x,y)\rangle + h_{MN}(x,y) \tag{III.81}$$

where we are considering background product metrics of the form

$$\langle g_{MN}(x,y)\rangle = \begin{bmatrix} \langle g_{\mu\nu}(x)\rangle & 0 \\ 0 & \langle g_{mn}(y)\rangle \end{bmatrix} \tag{III.82}$$

To find the relevant mass operators we substitute the expansion (III.81) into the d=10 field equations and retain terms up to linear order in the fluctuations [10]. This has been done in [10,12] from which we have the harmonic expansions

$$\begin{aligned} h_{\mu\nu}(x,y) &= \sum_q \sum_i h^i_{\mu\nu}(x) Y^i_q(y) \\ h_{\mu n}(x,y) &= \sum_q \sum_i C^i_\mu(x) Y^i_{q,n}(y) \\ h_{mn}(x,y) &= \sum_q \sum_i S^i(x)\, X^i_{q,mn}(y) + \sum_q \sum_i (g_{mn}\, \hat{S}^i(x) Y^i_q(y)) \end{aligned} \tag{III.83}$$

where Y(y) represents eigenmodes of Δ_0, $Y_n(y)$ are eigenmodes of Δ_1, $X_{mn}(y)$ are eigenmodes of the Lichnerowicz operator Δ_L and q labels the eigenvalues. (In these expressions, the index i should really be written as i_q). For the massless sector we truncate to q=0.

In these expressions we have chosen the transverse trace free gauge $\nabla^m h_{mn} = 0$, $g^{mn}h_{mn} = 0$. The Lichnerowicz operator enters in because unlike *antisymmetric* forms upon which the Laplacian acts, the metric tensor is symmetric. The Lichnerowicz operator acts according to

$$\Delta_L\, h_{mn} = -\Box h_{mn} - 2R_{mpnq}\, h^{pq} + 2R_{(m}{}^{p}h_{n)p} \qquad \text{(III.84)}$$

In our compactifications, we will generally require the vanishing of the Ricci tensor, so the last term in (III.84) is absent. On Ricci flat spaces, the eigenvalues of the operators are the four dimensional mass terms [10], so the low energy (massless) fields are the zero eigenmode solutions.

In the last section we discussed the Hodge decomposition theorem which we showed implies an isomorphism between the cohomology of K and the harmonic forms on K. We now have the first example of the use of this theorem as we may express the number of harmonic forms in terms of the topologically invariant dimensions of the corresponding cohomology groups. Thus, the ten dimensional metric gives rise to b_0 four dimensional spin-two metrics and b_1 four dimensional spin-one fields in addition to spin-zero fields arising as zero modes of the Lichnerowicz operator. From results for the harmonic structure of Calabi-Yau manifolds derived in the preceding section, we have $b_0 = h^{0,0} = 1$ and $b_1 = h^{1,0} + h^{0,1} = 0$. We therefore have (hearteningly) one four-space metric field. The harmonic modes of the Lichnerowicz operator Δ_L are not readily identifiable with cohomology groups However, it can be shown [10] that the number of zero modes of Δ_L is equal to the number of parameters in the metric on K. Now, since the unique Ricci flat Kähler metric guaranteed by the Calabi–Yau theorem discussed earlier depends on the complex structure and the cohomology class of the (real) Kähler form, we simply need to count these degrees of freedom. As we shall discuss in the following chapter, deformations of the complex structure may be identifed with elements of $H_{\bar{\partial}}{}^{0,1}(T)$ [4]. From (III.75), then, the number of real degrees of freedom due to deformations of the complex structure of K is given by $2\cdot h^{2,1}(K)$. The Kähler form determines a real (1,1) cohomology class (as the metric is hermitian). The number of degrees of freedom in choosing this topological class depends upon the dimension of $H^{1,1}(K)$ – thus, the Kähler form depends on $h^{1,1}(K)$ real parameters [10,13]. Putting these two observations together, the number of zero modes of the Lichnerowicz operator depends on $2\cdot h^{2,1} + h^{1,1}$ real parameters. We therefore have an equal number of real four–dimensional scalar fields originating from the ten dimensional metric.

The analysis just performed for the ten dimensional metric must also be done for the other fields in the supergravity multiplet. The antisymmetric two form potential $B_{MN}(x,y)$ may be expanded as

$$\begin{aligned} B_{\mu\nu}(x,y) &= \sum_q \sum_i B^i_{\mu\nu q}(x)\varphi^i_q(y) \\ B_{\mu n}(x,y) &= \sum_q \sum_i B^i_{\mu q}(x)\varphi^i_{nq}(y) \\ B_{mn}(x,y) &= \sum_q \sum_i B^i_q(x)\varphi^i_{mnq}(y) \end{aligned} \qquad (III.85)$$

where $\varphi^i(y)$, $\varphi^i_n(y)$ and $\varphi^i_{mn}(y)$ are harmonic modes on K of the Laplacians Δ_0, Δ_1, and Δ_2 respectively. Truncating to the massless q=0 sector, the ten–dimensional antisymmetric tensor field $B_{MN}(x,y)$ thus gives rise to $b_2 = h^{2,0} + h^{1,1} + h^{0,2} = h^{1,1}$ spin–zero four dimensional scalar fields $B^i(x)$, $b_1 = 0$

spin one gauge fields (as must occur since Calabi–Yau manifolds have no continuous symmetries so no gauge bosons arise as isometries of the compactification) and $b_0 = 1$ four dimensional spin–zero antisymmetric tensor $B_{\mu\nu}(x)$. (Recall that in d–dimensions an antisymmetric tensor field has $(d-3)(d-2)/2$ on–shell degrees of freedom.)

To complete this analysis for the bosonic sector of the supergravity multiplet, we note that the ten dimensional scalar field φ clearly gives rise to $b_0 = 1$ four dimensional real spin zero field, which we shall denote by the same symbol.

There are two ways to analyze the fermionic sector of the supergravity multiplet. One is simply to follow an analysis similar to the above for the appropriate Dirac and Rarita–Schwinger operators. The other makes use of the fact that in the case of interest, this analysis takes place on a Calabi–Yau manifold. In the previous section we described an isomorphism between the space of harmonic spinors and the space of harmonic forms which holds, in particular, on Calabi–Yau manifolds. It is therefore clear that in an expansion of the form

$$\psi(x,y) = \sum \psi^i(x) \otimes \psi^i(y) \qquad \text{(III.86)}$$

the number of harmonic spinors on K (which determines the massless spinor content of M_4) may be expressed in terms of the Hodge numbers computed earlier. It is simplest to recall the explicit map between spinors and forms given in (II.49–II.51). Since the gamma matrices (in the form of the operators a_i and $a^*_{\bar{\imath}}$) reverse chirality, spinors of positive chirality are begotten from (0,0) and (0,2) forms, while those of negative chirality from (0,1) and (0,3) forms. Since the spin $\frac{1}{2}$ field λ from the supergravity multiplet has definite chirality (and from our knowledge of the Hodge numbers) we thus see that λ gives rise to one massless four–dimensional spin $\frac{1}{2}$ field, which again will be referred to by the same symbol. We may now, in fact, invoke supersymmetry to complete the analysis of the massless four dimensional field content arising from the supergravity multiplet without need of further computation. By construction we know that an N=1 supersymmetry survives in four dimensions. Hence, the bosonic degrees of freedom computed above, must be matched by fermionic degrees of freedom. This implies that the spin 3/2 field ψ_m must give rise to 1 four-dimensional spin 3/2 field ψ_μ and $h^{1,1}$ + $h^{1,2}$ spin $\frac{1}{2}$ fields χ. To summarize, then, we have (generalizing the analysis of ref [10]) the four dimensional field content arising from the gravity sector:

d = 10	d = 4	Spin	No of Fields
g_{MN}	$g_{\mu\nu}$	2	1
	C_μ	1	0
	S	0	$h^{1,1} + 2h^{2,1}$
ψ_M	ψ_μ	$^3/_2$	1
	χ	$^1/_2$	$h^{1,1} + h^{2,1}$
B_{MN}	$B_{\mu\nu}$	0	1
	B_μ	1	0
	B	0	$h^{1,1}$
λ	λ	$^1/_2$	1
φ	φ	0	1

TABLE III.1

Massless spacetime fields arising from the ten-dimensional supergravity multiplet.

We now embark on analyzing the four dimensional fields which arise from the Yang–Mills sector. As discussed in Chapter II, the requirement (II.9) implies that the principle Yang–Mills bundle must contain a sub–bundle B (with structure group $S \subset E_8' \otimes E_8$) endowed with a connection having non–trivial vacuum value. In fact, we have shown that the second Chern class of the associated vector bundle B built on the fundamental representation of S must equal that of the tangent bundle. We confine our attention to the case of S lying completely within one E_8 factor and do not further discuss the 'hidden' E_8' factor. In this case, the unbroken gauge group is the maximal subgroup T of E_8 which commutes with S. In the ten dimensional Yang–Mills sector we have the spin 1 vector potential $A_M{}^i$, corresponding to the E_8 gauge bosons, where i = 1, ..., 248 is an adjoint E_8 index. We also have the spin ½ gaugino partners χ^i, i = 1, ..., 248. As the gauge group visible in four dimensions is $T \subset E_8$, it is useful to decompose the $\underline{248}$ under $T \otimes S$:

$$\underline{248} = \bigoplus_j (\underline{t}_j, \underline{s}_j) \tag{III.87}$$

where $\underline{t}_j$ and $\underline{s}_j$ denote T and S representations occuring in this decomposition. Correspondingly, the ten–dimensional Yang–Mills field decomposes into $T \otimes S$ subspaces:

$$A_M{}^i(x,y) = \bigoplus_j A_M{}^{(\underline{t}_j, \underline{s}_j)}(x,y) \tag{III.88}$$

Since $A_M{}^i$ lies in a real representation and because we have an unbroken gauge group T, we typically find that (adj T, 1) $\subset \oplus_j (\underline{t}_j, \underline{s}_j)$ and that the complex representations occur in conjugate pairs. The most familiar example of this is S = SU(3), the principle bundle associated with the holonomy of K. That is, the spin connection is embedded in the gauge group. This gives $T = E_6$ and

$$\bigoplus_j (\underline{t}_j, \underline{s}_j) = (\underline{78}, \underline{1}) \oplus (\underline{27}, \underline{3}) \oplus (\overline{\underline{27}}, \overline{\underline{3}}) \oplus (\underline{1}, \underline{8}). \tag{III.89}$$

Two other examples which we shall study in Chapter IV arise from taking S = SU(4) or S = SU(5). That is, we embed the holomorphic connection on the holomorphic Yang–Mills bundle with structure group S, whose existence is assured by the Uhlenbeck and Yau theorem discussed in the last section, in the E_8 sector of the original Yang–Mills bundle. This results in T = SO(10) or T = SU(5) respectively and

$$T = SO(10) \; : \; \bigoplus_j (\underline{t}_j, \underline{s}_j) = (\underline{45}, \underline{1}) \oplus (\underline{16}, \underline{4}) \oplus (\overline{\underline{16}}, \overline{\underline{4}}) \oplus (\underline{10}, \underline{6}) \oplus (\underline{1}, \underline{15}) \tag{III.90}$$

$$T = SU(5) \; : \; \bigoplus_j (\underline{t}_j, \underline{s}_j) = (\underline{24}, \underline{1}) \oplus (\underline{5}, \overline{\underline{10}}) \oplus (\overline{\underline{5}}, \underline{10}) \oplus (\underline{10}, \underline{5}) \oplus (\overline{\underline{10}}, \overline{\underline{5}}) \oplus (\underline{1}, \underline{24}) \tag{III.91}$$

To derive the low energy four–dimensional field content arising from the ten–dimensional vector potential $A_M{}^i$, we must harmonically expand the group theoretic decomposition given in (III.87). That is, we write

$$A_M{}^i(x,y) = \bigoplus_j A_M^{(\underline{t}_j, \underline{s}_j)}(x,y) = \bigoplus_j \left\{ \sum_q \sum_k A_{\mu q}^{k,\underline{t}_j}(x) \, \varphi_q^{k,\underline{s}_j}(y) + \right.$$

$$\left. + \sum_q \sum_k \varphi_q^{k,\underline{t}_j}(x) \, A_{mq}^{k,\underline{s}_j}(y) \right\} \tag{III.92}$$

where q labels the eignevalues of the relevant mass operators on K. As shown in [10], the mass operators for $A_M{}^i$ are just the Laplacian operators on K. For the massless fields, and hence the ones of interest, the eigenvalue labelled by q must vanish. We therefore truncate the sum in (III.92) to the zero eigenmode solutions and henceforth drop the q-dependence. Notice that the y-dependent modes on K typically carry a representation of S. This representation simply describes an associated vector bundle V to the principle bundle B with structure group S. As shown, B is a holomorphic vector bundle, and therefore so is V. In other words, the y-dependent modes occurring in (III.92) are harmonic modes taking values in holomorphic vector bundles. It is precisely for this reason that, in the last section, we studied harmonic theory on compact complex manifolds taking values in holomorphic vector bundles. From that discussion we know that the natural, well defined exterior derivative acting on V-valued (p,q) forms is the $\bar{\partial}$ operator and that the space of harmonic V-valued (p,q) forms $H^{p,q}(K,V)$ is isomorphic to the Dolbeault cohomology group $H_{\bar{\partial}}{}^{p,q}(K,V)$. This justifies our earlier references to the fact that massless four dimensional fields arise from certain elements of V-valued Dolbeault cohomology groups.

From (III.92) the number of massless spin-one fields transforming in the representation $\underline{t}_j$ of the gauge group T is given by the dimension of $H_{\bar{\partial}}{}^{0,0}(K,V[\underline{s}_j])$ where $V[\underline{s}_j]$ denotes the associated holomorphic vector bundle to the principle Yang-Mills sub-bundle B corresponding to the representation $\underline{s}_j$ of the structure group S. That is, the number of linearly independent holomorphic sections of the vector bundle $V[\underline{s}_j]$ gives the number of spin-one fields arising from the $(\underline{t}_j,\underline{s}_j)$ term in (III.92).

Similarly, from (III.92) we see that the massless four dimensional scalar fields arise from harmonic modes of the form

$$A_m^{\underline{s}j}(y) \ .$$

Now m, being a tangent index for K, may be holomorphic or antiholomorphic. In the former case, the massless spin zero fields are associated with elements of $H_{\bar{\partial}}^{1,0}(K,V[\underline{s}_j])$ and in the latter with elements of $H_{\bar{\partial}}^{0,1}(K,V[\underline{s}_j])$. It is important at this point to bear in mind the comment made earlier regarding the fact that the original adjoint of E_8 is a real representation. If the representation $(\underline{t}_j,\underline{s}_j)$ appears in (III.87) then so does $(\bar{\underline{t}}_j,\bar{\underline{s}}_j)$. We then find, employing the same analysis as above, that we have massless spin-zero fields associated with elements of $H_{\bar{\partial}}^{1,0}(K,V[\bar{\underline{s}}_j])$ and $H_{\bar{\partial}}^{0,1}(K,V[\bar{\underline{s}}_j])$. Note that $V[\bar{\underline{s}}_j]$ is the complex conjugate of $V[\underline{s}_j]$. The hermitian fibre metric h (needed for the Uhlenbeck and Yau theorem, see last section) allows us to consider $V[\bar{\underline{s}}_j]$ as the dual bundle $(V[\underline{s}_j])^*$ to $V[\underline{s}_j]$ which shows that it is manifestly holomorphic. Complex conjugation maps $H_{\bar{\partial}}^{1,0}(K,V[\underline{s}_j])$ to $H_{\bar{\partial}}^{0,1}(K,V[\bar{\underline{s}}_j])$ and $H_{\bar{\partial}}^{1,0}(K,V[\bar{\underline{s}}_j])$ to $H_{\bar{\partial}}^{0,1}(K,V[\underline{s}_j])$. (Recall that since K is Kähler all Laplace operators are equal, up to a constant.) We may therefore restrict our attention to $H_{\bar{\partial}}^{0,1}(K,V[\underline{s}_j])$ and $H_{\bar{\partial}}^{0,1}(K,V[\bar{\underline{s}}_j])$. As we shall now indicate this is equivalent to the familiar procedure of discussing fermions in terms of a definite chirality.

Recall from the discussion at the end of the last section that harmonic spinors may be identified with harmonic forms which may further be defined as elements of $H^{0,q}(K,V)$. The chirality of the corresponding four dimensional spinor is $(-1)^q$. Thus, concentrating on a particular chirality is equivalent to taking q to be odd or even. Choosing the former, we therefore have fermions arising from $H_{\bar{\partial}}^{0,1}(K,V)$ and $H_{\bar{\partial}}^{0,3}(K,V)$. We see that the latter supply fermionic partners to the spin one fields associated with $H_{\bar{\partial}}^{0,0}(K,V^*)$. (Note that dim $H_{\bar{\partial}}^{0,3}(K,V[\underline{s}_j])$ = dim $H_{\bar{\partial}}^{0,0}(K,V[\bar{\underline{s}}_j])$ by Kodaira–Serre duality and the fact that $C_1(K) = 0$).

The chiral fermions from $H_{\bar{\partial}}^{0,1}(K,V[\underline{s}_j])$ and $H_{\bar{\partial}}^{0,1}(K,V[\bar{\underline{s}}_j])$ are similarly the supersymmetric partners of the spin zero fields associated with $H_{\bar{\partial}}^{0,1}(K,V[\underline{s}_j])$ and $H_{\bar{\partial}}^{0,1}(K,V[\bar{\underline{s}}_j])$. As far as degrees of freedom are concerned, four dimensional fermions of definite chirality have two real on-shell degrees of freedom. Thus in the representation of $\underline{t}_j$ and $\bar{\underline{t}}_j$ we have $2\cdot(\dim H_{\bar{\partial}}^{0,1}(K,V[\underline{s}_j]) + \dim H_{\bar{\partial}}^{0,1}(K,V[\bar{\underline{s}}_j]))$ real fermionic degrees of freedom. For their scalar partners derived from the real gauge field, the conjugate of the (complex) harmonic forms in $H_{\bar{\partial}}^{1,0}(K,V[\bar{\underline{s}}_j])$ combine with those from $H_{\bar{\partial}}^{0,1}(K,V[\underline{s}_j])$ to form $2\cdot\dim H_{\bar{\partial}}^{0,1}(K,V[\underline{s}_j])$ real degrees of freedom. The counting for $H_{\bar{\partial}}^{0,1}(K,V[\bar{\underline{s}}_j])$ is the same, and we thus find $2\cdot(\dim H_{\bar{\partial}}^{0,1}(K,V[\underline{s}_j]) + \dim H_{\bar{\partial}}^{0,1}(K,V[\bar{\underline{s}}_j]))$ real scalar degrees of freedom, as expected.

To complete this general analysis, we note (and shortly illustrate with examples) that the chiral supermultiplets containing spin $\frac{1}{2}$ fermions and their scalar superpartners are responsible for the non-gauge field (and non-supergravity field) particle content. Known fermions (along with presently unobserved multiplet partners) will generally reside in representations $\underline{t}_j$ of the gauge group T. The fields occurring in the representations $\bar{\underline{t}}_j$ of T (which as we have shown above are independent fields and not simply complex conjugates of the formerly mentioned fields) act as mirror fields. The former are referred to as 'generations' and the latter, for obvious reasons, as 'antigenerations'. For the resulting theory to be chiral, there must be an imbalance between the two. Our analysis in the last section immediately gives us the machinery to detect and compute this imbalance. From (III.74) we have (for the representation $(\underline{t}_j,\underline{s}_j) \in \bigoplus_m (\underline{t}_m,\underline{s}_m)$)

$$|\dim H_{\bar{\partial}}^{0,1}(K,V[\underline{s}_j]) - \dim H_{\bar{\partial}}^{0,1}(K,V[\bar{\underline{s}}_j])| = \frac{1}{2}\left|\int_K c_3(V[s_j])\right| \quad \text{(III.93)}$$

Thus, the net number of generations of four dimensional particles in the representation of $\underline{t}_j$ of T is given by one-half of the third Chern number of the vector bundle $V[\underline{s}_j]$. A topological solution is obtained for the family replication problem.

To illustrate these ideas, we apply them in the familiar context of T = E_6 and S = SU(3). The $\underline{248}$ of E_8 decomposes as in (III.89). We deal with each term in turn.

The $\underline{78}$ in ($\underline{78}$,$\underline{1}$) is the adjoint of E_6. The $\underline{1}$, of course, is a trivial SU(3) bundle, and hence, from our general analysis the number of spin-one fields arising from this component of the $\underline{248}$ is $H_{\bar{\partial}}^{0,0}(K,\mathbb{C})$. As noted previously $h^{0,0}$ = dim $H_{\bar{\partial}}^{0,0}(K,\mathbb{C})$ is equal to one. Thus, as expected since the unbroken gauge group is E_6, we have one E_6 adjoint representation of spin-one gauge bosons. The number of scalar fields is given by $H_{\bar{\partial}}^{0,1}(K,\mathbb{C})$ which vanishes as $h^{0,1}$ equals zero for a Calabi-Yau manifold. The spin $\frac{1}{2}$ gaugino partners for the gauge bosons arise from $H_{\bar{\partial}}^{0,3}(K,\mathbb{C})$ as discussed.

In the ($\underline{27}$,$\underline{3}$), the $\underline{27}$ is the fundamental representation of E_6 and the $\underline{3}$ is the fundamental representation associated with the S = SU(3) principle Yang-Mills sub-bundle. This component of the $\underline{248}$ thus gives rise to dim $H_{\bar{\partial}}^{0,0}(K,V)$ four dimensional spin-one $\underline{27}$-multiplets, and dim $H_{\bar{\partial}}^{0,1}(K,V)$ four dimensional spin-zero $\underline{27}$-multiplets, where V is the vector bundle for the $\underline{3}$ of SU(3). In this scenario, though, the identification of the SU(3) spin connection with the S = SU(3) Yang-Mills connection allows us to identify the vector bundle V with the holomorphic tangent bundle $T'(K)$ of K. From precisely the same reasoning used to derive (III.75), we have

$$\dim H_{\bar{\partial}}^{0,0}(K,T'(K)) = \dim H_{\bar{\partial}}^{2,0}(K,\mathbb{C}) \quad . \tag{III.94}$$

Since K is a Calabi-Yau manifold, the analysis at the end of the last section implies that this dimension vanishes, and hence the $(\underline{27},\underline{3})$ gives rise to no spin–one fields. For the spin–zero fields we have from (III.75)

$$\dim H_{\bar{\partial}}^{0,1}(K,T'(K)) = \dim H_{\bar{\partial}}^{2,1}(K,\mathbb{C}) \quad . \tag{III.95}$$

The $(\underline{27},\underline{3})$ component of $A_M{}^i$ thus gives rise to $h^{1,2}$ four dimensional $\underline{27}$–multiplets of scalar fields.

The analysis of the $(\overline{\underline{27}},\overline{\underline{3}})$ component is similar, except now the vector bundle associated with the Yang–Mills $\overline{\underline{3}}$ representation must be identified with the holomorphic contangent bundle, $T^{*\prime}(K)$. The resulting number of spin–1 fields is again zero since

$$\dim H_{\bar{\partial}}^{0,0}(K,T^{*\prime}(K)) = \dim H_{\bar{\partial}}^{1,0}(K,\mathbb{C}) = 0 \tag{III.96}$$

as shown. The number of spin–0 fields is given by

$$\dim H_{\bar{\partial}}^{0,1}(K,T^{*\prime}(K)) = \dim H_{\bar{\partial}}^{1,1}(K,\mathbb{C}) \tag{III.96b}$$

and therefore gives $h^{1,1}$ $\overline{\underline{27}}$–multiplets of scalar fields. The fermionic partners of the scalar fields arising from the $(\underline{27},\underline{3})$ and $(\overline{\underline{27}},\overline{\underline{3}})$ sectors may be identified with elements from the same cohomology groups, and we therefore find $h^{1,2}\cdot(\underline{27}$ of $E_6)$ and $h^{1,1}(\overline{\underline{27}}$ of $E_6)$ chiral superfields. (Of course, we remark that the assignment of $\underline{27}$ and $\overline{\underline{27}}$ is a matter of convention. If $h^{1,1} > h^{1,2}$ we will reverse the assignment made above). We may write this as

$$|h^{1,1} - h^{1,2}|\cdot(\underline{27} \text{ of } E_6) + \min\,(h^{1,1},h^{1,2})\cdot(\underline{27} \oplus \overline{\underline{27}} \text{ of } E_6) \ . \tag{III.97}$$

The net generation number $|h^{1,1} - h^{1,2}|$ was calculated in (III.76). We have $|h^{1,1} - h^{1,2}| = \frac{1}{2}|\chi(K)|$ and therefore find that the net number of generations is determined topologically, being given by one-half of the absolute value of the Euler characteristic of the compactified component of the universe.

The $\underline{8}$ representation of SU(3) arising from the $(\underline{1},\underline{8})$ component of the $\underline{248}$ is built from the antisymmetric product of the $\underline{3}$ and the $\underline{\bar{3}}$ representations. That is

$$3 \otimes \bar{3} = [3 \otimes \bar{3}]_{AS} \oplus [3 \otimes \bar{3}]_S = \underline{8} \oplus \underline{1} \tag{III.98}$$

where S denotes symmetric part and AS denotes antisymmetric part. By making the identification of $\underline{\bar{3}}$ with the holomorphic cotangent space we see that the $\underline{8}$ is associated with the bundle $[T'(K) \otimes T^{*\prime}(K)]_{AS}$. Since $T^{*\prime}(K)$ is the dual space of $T'(K)$, the latter is just the space of linear maps acting on $T'(K)$. The anti-symmetrization ensures that these maps have determinant one (when viewed as matrices), and hence this bundle is End T : the bundle whose fibre at a given point is the set of unit determinant endomorphisms of the holomorphic tangent space residing at the same point.

Following the field content analysis elucidated above, the number of four dimensional massless spin-one fields arising from the $(\underline{1},\underline{8})$ is given by dim $H_{\bar{\partial}}^{0,0}(K,\text{End } T)$, and the number of spin-zero E_6-singlet fields is given by dim $H_{\bar{\partial}}^{0,1}(K,\text{End } T)$.

The dimensions of the former cohomology group is readily computed from

$$\dim H_{\bar{\partial}}^{0,0}(K,\text{End } T) = \dim H_{\bar{\partial}}^{0,0}(K,T\otimes T^*) - \dim H_{\bar{\partial}}^{0,0}(K,\mathbb{C}) \, . \tag{III.99}$$

The right hand side, using familiar manipulations, is equal to

$$\dim H_{\bar\partial}^{1,0}(K,T) - 1 \tag{III.100}$$

$$= \dim H_{\bar\partial}^{0,2}(K,T) - 1$$

$$= \dim H_{\bar\partial}^{2,2}(K,\mathbb{C}) - 1$$

$$= h^{1,1} - 1 \tag{III.101}$$

Unlike in the previous examples in which we could easily relate the relevant bundle valued Dolbeault cohomology groups to ordinary complex valued Dolbeault cohomology, we do not have an immediate prescription for determining the dimensions of the $H_{\bar\partial}^{0,1}(K,\text{End } T)$ valued cohomology group. We therefore follow the procedure presented in [14].

The vector space of differential forms of a given degree (at an arbitrary fixed point on the base manifold K) forms a representation of the tangent space symmetry group. If we ignore the complex structure of K, the tangent space symmetry group is SO(6); differential forms fall into representations of this group. When we impose a complex structure on K, and in particular demand a connection of SU(3) holonomy, these SO(6) representations will decompose into irreducible SU(3) representations. This gives an additional tool for studying the harmonic structure of K.

In particular, the vector spaces of differential forms decompose in the following manner:

$$\begin{aligned}
\Omega_{0;1} &= \Omega_{0,0;1} \\
\Omega_{1;6} &= \Omega_{1,0;3} \oplus \Omega_{0,1;\bar{3}} \\
\Omega_{2;15} &= \Omega_{2,0;\bar{3}} \oplus \Omega_{1,1;(1\oplus 8)} \oplus \Omega_{0,2;3} \\
\Omega_{3;(10\oplus\overline{10})} &= \Omega_{3,0;1} \oplus \Omega_{2,1;(3\oplus 6)} \oplus \Omega_{1,2;(\bar{3}\oplus\bar{6})} \oplus \Omega_{0,3;1}
\end{aligned} \tag{III.102}$$

where indices before a semi-colon denote the real or complex degree of the form, and indices after a semi-colon denote SO(6) or SU(3) representations. This group theoretic decomposition induces the same decomposition on the subspaces of harmonic forms, and hence we can refine the Hodge numbers $h^{p,q}$ by labelling them with the SU(3) representation which they carry, $h^{p,q;r}$. The important point shown in [14] is that on a Calabi-Yau manifold $h^{p,q;r}$ changes only with $\underline{r}$. In fact, it is shown that the only non-zero refined Hodge numbers are

$$h^{0,0;1} = 1,\ h^{1,1;1} = 1,\ h^{1,1;8} = h^{1,1} \text{and } h^{1,2;6} = h^{1,2}\ . \qquad \text{(III.103)}$$

Thus we may exploit this group theoretic property when calculating dimensions of cohomology groups. We now apply this to $H_{\bar{\partial}}{}^{0,1}(K,\text{End } T)$.

From a group theoretic point of view, (0,1) forms with values in End T transform in the $\underline{3} \otimes \underline{8}$ of SU(3). This representation reduces according to

$$\underline{3} \otimes \underline{8} = \underline{3} \oplus \bar{\underline{6}} \oplus \underline{15} \qquad \text{(III.104)}$$

Referring to (III.102) we note that the $\underline{15}$ does not appear, and is hence not representable by a differential form on K. This is not surprising as a tensor product, as opposed to a wedge product, appears in (III.104). We shall not be able to compute the dimension of this component and shall simply refer to it as d^{15}. The other two terms are representable by forms, and hence from (III.103) we have

$$\dim H_{\bar{\partial}}{}^{0,1}(K,\text{End } T) = h^{1,2} + d^{15}. \qquad \text{(III.105)}$$

We may thus summarize our results for this example of S = SU(3) and T = E_6:

d = 10	d = 4	Spin	Yang-Mills Representation	No of Multiplets
$A_M{}^i$	A_μ	1	$\underline{78}$	1
	φ	0	$\underline{27}$	$2 \cdot h^{1,2}$
	φ'	0	$\underline{\overline{27}}$	$2 \cdot h^{1,1}$
	ζ	1	$\underline{1}$	$h^{1,2} - 1$
	ξ	0	$\underline{1}$	$2 \cdot (h^{1,2}+d^{15})$
χ^i	χ	$\frac{1}{2}$	$\underline{78}$	1
	ψ	$\frac{1}{2}$	$\underline{27}$	$h^{1,2}$
	ψ'	$\frac{1}{2}$	$\underline{\overline{27}}$	$h^{1,1}$
	$\tilde{\zeta}$	$\frac{1}{2}$	$\underline{1}$	$h^{1,2} - 1$
	$\tilde{\xi}$	$\frac{1}{2}$	$\underline{1}$	$h^{1,2} + d^{15}$

TABLE III.2

Massless spacetime fields arising from the ten-dimensional Yang-Mills multiplet.

Similar reasoning easily shows that the hitherto ignored $E_8{}'$ Yang-Mills sector of the ten dimensional theory descends to give a $\underline{248}$ adjoint vector superfield in four dimensions. These fields shall not be discussed further.

At first sight, the entry for ζ is rather disturbing since Calabi-Yau manifolds have no isometries [10] and thus should not give rise to gauge bosons through the compactification process. However, as noted in [14], a mechanism described in

[15] causes the $h^{1,2} - 1$ E_6–singlet vector superfields to combine with $h^{1,2} - 1$ chiral superfields from the supergravity multiplet, to form a massive vector superfield.

We see, therefore, that in this special case of embedding the spin–connection in the gauge group, the dimensions of the relevant bundle valued cohomology groups can be related to those of the ordinary complex valued Dolbeault cohomology groups of K. This allows us to use our knowledge about the Hodge numbers of Calabi–Yau manifolds to determine the massless four–dimensional field content. For arbitrary structure groups, this is not the case, and the general results derived prior to this example can not be further refined.

In summary, we have employed harmonic theory in the guise of holomorphic bundle valued Dolbeault cohomology over Calabi–Yau manifolds to deduce the massless four dimensional fields which descend from ten–dimensional N=1 supergravity coupled to $E_8' \otimes E_8$ super Yang–Mills. We have seen that this field content depends upon the topology of certain principle sub–bundles of the Yang–Mills bundle, and their associated vector bundles. The observable gauge and matter fields in the uncompactified four dimensions of common experience are thus determined by the topological structure of certain holomorphic fibre bundles constructed on the compactified component of the universe.

III.3 THE FLUX MECHANISM

The goal of addressing the hierarchy problem has led us to consider compactifications on Calabi–Yau manifolds, as this ensures the survival of four dimensional N=1 supersymmetry. Although Calabi–Yau manifolds do not admit any continuous symmetries, they may have discrete symmetries. As we shall discuss in Chapter V, our attention focuses on Calabi–Yau manifolds realized as the zero locus of algebraic varieties (zeroes of polynomial functions) in complex projective space. Such Calabi–Yau manifolds often have an abundance of discrete symmetries. Besides restricting the allowed interactions of the theory (discussed in the next section), these discrete symmetries are instrumental in resolving four additional issues.

First, recall that in the previous section we described how the requirement of equality between the second Chern class of the tangent bundle of K and the Yang–Mills bundle built on K provides a mechanism for breaking the observable E_8 gauge group to smaller and phenomenologically attractive unifying groups. Typically, these groups (e.g. E_6, SO(10), SU(5)) must be further reduced towards the SU(3) $\otimes$ SU(2) $\otimes$ U(1) gauge group of the standard model. As the massless four dimensional field content is topologically determined, one cannot, as in ordinary grand unified theories, over–extend the benevolence of nature and blithely postulate the existence of an all too accommodating Higgs representation. Fortunately, there is an alternative. In this section we shall describe how the discrete symmetries of the Calabi–Yau base manifold may be used to construct a novel and extremely useful "Higgsless" method for breaking gauge symmetry, known as the flux mechanism [16,17].

Second, multiplets of the unifying group often contain particles which can potentially mediate proton decay at a rate exceeding experimental limits. We are therefore faced with the task of causing these particles to acquire large masses, thereby rendering them innocuous, while leaving their multiplet partners (containing standard model matter and Higgs fields) light. This problem is familiar from ordinary grand unified theories in which, for example, the SU(5) coloured triplet partners of the standard model weak Higgs doublets must be made heavy. This is accomplished in the context of grand unified theories by unattractive *ad hoc* fine tuning. We shall demonstrate in this section that the flux mechanism offers an elegant alternative.

Third, in ordinary grand unified theories, the Yukawa couplings for particles in a given representation of the grand unifying group are subject to simple group theoretic relations which are generally in conflict with experiment. We shall see that as far as *counting* states is concerned, the massless fields fill out T representations. However, in actuality the massless fields are not related by simple T transformations, and therefore the phenomenologically distressful relations amongst Yukawa couplings do not hold.

Finally, in the fairly well studied $T = E_6$, $S = SU(3)$ scenarios, the number of generations, as given by one-half of the modulus of the Euler characteristic of the base manifold, is typically quite large [12,14]. The same discrete symmetries used in addressing the last three issues also provide a means for reducing the predicted number of generations and hence making compactifications more realistic. We now describe the flux mechanism.

Let K_0 be a simply connected Calabi-Yau manifold, i.e. $\pi_1(K_0) = 0$. Let D denote the group of discrete symmetries of K_0. By this we mean the group of all holomorphic bijective maps f with $f : K_0 \to K_0$. Finally, let G be a subgroup of D. We will consider compactifications on $K = K_0/G$, as this also can be shown to be a Calabi-Yau manifold [12]. Now, how do we specify a vacuum configuration based upon this compactification? From our previous discussion, we need a hermitian Yang–Mills connection satisfying (II.48) which fills out a structure group $S \subset E_8' \otimes E_8$. The curvature components residing in the unbroken group T vanish, and we therefore have a flat T bundle over K. This, however, does not uniquely determine the Yang–Mills configuration. Parallel transport in a flat bundle depends upon the homotopy of the projected path on the base manifold. For simply connected manifolds K_0, in which all paths are homotopic, the flatness condition nails down the connection. However, on non–simply connected manifolds, the flatness condition does not uniquely determine the parallel transport law for paths which are not homotopic to the identity. The only constraint for such paths, is that the parallel transport law respect the path multiplication structure of the homotopy group. With every element in $\pi_1(K) = G$, therefore we may associate an element $t \in T$ as long as this provides a homomorphic embedding, Φ. We therefore have complete choice in defining Φ as long as

1. $\Phi : \pi_1(K) \longrightarrow T$
2. $\Phi(g_1 g_2) = \Phi(g_1)\Phi(g_2)$.

If we denote the image of $\pi_1(K)$ under Φ by $\bar{G}$, which is a discrete subgroup of T, it is clear that $\bar{G}$ defines a non–trivial global Yang–Mills "holonomy" [18], which in conjunction with the Yang–Mills structure group S determines the vacuum configuration. The unbroken gauge group H is computed according to

$$
\begin{array}{lll}
E_8' \otimes E_8 & & \\
\quad \llcorner\!\xrightarrow{\quad (1) \quad} & T & \\
 & \quad \llcorner\!\xrightarrow{\quad (2) \quad} & H
\end{array}
\qquad \text{(III.106)}
$$

where

(1) depends upon the structure subgroup S of $E_8' \otimes E_8$ filled out by the Yang–Mills connection, with T being the maximal subgroup of $E_8' \otimes E_8$ which commutes with S

and

(2) depends upon the choice of the homomorphic embedding of $G = \pi_1(K)$ into T, with H being the maximal subgroup of T which commutes with $\bar{G}$.

We thus see that choosing a non–trival homomorphism Φ results in a further reduction of the gauge group from T to $H \subset T$. This gauge symmetry breaking has been accomplished completly within the context of ingredients naturally present in the original theory. There has been no need to overextend our faith into the realm of Higgsianity.

The image $\bar{G}$ of $\pi_1(K)$ in T may be given a more physical basis by the use of Wilson lines.

Let the contour γ_g be a representative of the abstract element $g \in \pi_1(K)$. The image $\bar{g}$ in $\bar{G}$ of the element g may be explicitly realized by the Wilson loop

$$\bar{g} = U_g = P \exp\left[-i \int_{\gamma_g} T^a A^a{}_m dy^m\right] \tag{III.107}$$

where $A^a{}_m$ are the vacuum T gauge fields, and T^a are the group generators. For non–trivial contours γ_g, this integral can be different from the identity since, as mentioned, there are non–trivial flat T bundles over non–simply connected K. One may picture the loop γ_g surrounding a hole in K through which non–zero vector potential "flux" passes. This is the origin of the term "flux breaking".

There are two commonly employed means of determining the maximal subgroup H contained in T which commutes with $\bar{G}$. The first involves choosing a convenient maximal subgroup of T (e.g. one in which the standard model is manifest) and considering all possible embeddings of G which leave $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ of the standard model unbroken. It is then straighforward to discern the unbroken generators which constitute H [17]. An example should make this clear. Consider the case of $T = E_6$ (we suppress the E_8' factor) which has a maximal subgroup (with suggestive notation) $SU(3)_C \otimes SU(3)_L \otimes SU(3)_R$. Identifying the first SU(3) with the colour group of the standard model and noting that the subgroup of E_6 that commutes with this $SU(3)_C$ (ignoring the centre of E_6) is in fact $SU(3) \otimes SU(3)$, any standard model preserving U_g must take the form

$$(I) \otimes (U_L) \otimes (U_R) \quad . \tag{III.108}$$

In this expression we identify $SU(2)_L$ as the obvious subgroup of the second SU(3) factor. For simplicity we assume $\pi_1(K)$ is abelian and take $\bar{G} = Z_n$. Then U_g may be written [17]

$$U_g = (I) \otimes \begin{bmatrix} \alpha_1 & & \\ & \alpha_1 & \\ & & \alpha_2 \end{bmatrix} \otimes \begin{bmatrix} \beta_1 & & \\ & \beta_2 & \\ & & \beta_3 \end{bmatrix} \tag{III.109}$$

where the requirement of unit determinant forces $\alpha_2 = \alpha_1^{-2}$, $\beta_1\beta_2\beta_3 = 1$, and $\bar{G} = Z_n$ constrains all α_i and β_j to be n^{th} roots of unity.

The unbroken subgroup H is then determined in two steps:

(1) The unbroken subgroup contained within $SU(3) \otimes SU(3) \otimes SU(3)$ is manifest from the form of (III.109).

(2) The unbroken E_6 generators not contained within this maximal subgroup are determined by noting that they lie in the first two summands in the

decomposition of the adjoint of E_6 under $SU(3) \otimes SU(3) \otimes SU(3)$

$$\underline{78} = (\underline{3},\underline{3},\underline{3}) \oplus (\bar{\underline{3}},\bar{\underline{3}},\bar{\underline{3}}) \oplus (\underline{8},\underline{1},\underline{1}) \oplus (\underline{1},\underline{8},\underline{1}) \oplus (\underline{1},\underline{1},\underline{8}) \quad . \qquad \text{(III.110)}$$

A generator in thee subpaces remains unbroken if it is invariant under the U_g. That is, if the appropriate product $\alpha_i\beta_j$ is unity.

For example, if $\alpha_1 \neq \alpha_2$, $\beta_1 = \beta_2 \neq \beta_3$ and $\alpha_i\beta_j \neq 1$ for all i,j, then $H = SU(3)_C \otimes SU(2)_L \otimes SU(2)_R \otimes U(1) \otimes U(1)$. If, however, $\alpha_2\beta_3 = 1$ with the same choice of β_i, then $H = SU(4) \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)$. For use in Chapter V, note that if $G = Z_3$ and $\alpha_1 = \beta_1 = \beta_2$, then $H = SU(3) \otimes SU(3) \otimes SU(3)$ is the unbroken gauge group.

Although adequate, this method of determining H is rather cumbersome. For the special case of abelian $\pi_1(K)$ (the above method still applies if $\pi_1(K)$ is non-abelian) there is a far more efficient procedure based on the Dynkin formalism [19] as adopted by Breit, Ovrut and Segré [20]. This method will be illustrated and extensively applied in Chapter IV.

Having elucidated this "Higgsless" method of gauge symmetry breaking, we now examine its consequences on the massless matter content of the four dimensional theory. From the last section we know that every massless four dimensional field (aside from fields arising from the metric, which shall not be of concern) is associated with a section of the bundle-valued harmonic forms based on the manifold of compactifiction. These harmonic sections on $K = K_o/G$ are generally only a subset of those on K_o. This is because on passing from K_o to K, all points on K_o which lie in a single G-orbit are identified with a single point on K. Consistency then demands that only those sections which are constant on G-orbits survive as sections on K_o/G. (Obviously no new harmonic bundle valued forms are created by passing to K_0/G since all such forms are harmonic and consistently defined on K_0. Thus, the bundle valued harmonic forms on K_0/G are in fact a subset of those on K_0). To phrase this consistency requirement mathematically, we need to know how to compare differential forms at two different points in K_0 (i.e. compare the values of a section at two different base points) which lie in a G–orbit. Naively one might require

$$\Psi(gy_0) = \Psi(y_0) \qquad \text{(III.110)}$$

for all $y_0 \in K_0$ and $g \in G$, where typically $\Psi \in H_{\bar{\partial}}^{0,q}(K_0,E)$. However, one must bear in mind that, in the cases of interest, Ψ carries a representation $\underline{t}_j$ of T as well. Thus, in parallel transporting Ψ from y_0 to gy_0, the Wilson loop–generated T element U_g (in the representation appropriate to Ψ) acts according to

$$\Psi(y_0) \longrightarrow U_g^{\underline{t}_j}\, \Psi(gy_0) \qquad \text{(III.111)}$$

where the superscript indicates the T representaiton (and will henceforth be suppressed as it is clear from context).

Thus, consistency requires

$$U_g\Psi(gy_0) = \Psi(y_0), \qquad \text{(III.112)}$$

which is the statement

$$(G \oplus \bar{G})\ \Psi = \Psi \qquad \text{(III.113)}$$

where $\oplus$ in this context denotes diagonal sum. (To avoid confusion we emphasize that $\bar{G}$ acts on the T indices of Ψ while G acts on $H_{\bar{\partial}}^{0,q}(K,E)$.) Only harmonic sections which satisfy this constraint survive the passage from K_0 to K_0/G. In the literature [10], fields which do not satisfy (III.113) are sometimes said to have acquired superheavy masses. It is more accurate to view such fields as simply not existing in the theory formulated on K_0/G as they are not well defined on this quotient manifold.

The consistency requirement (III.113) has two important implications. First, the T components of a fixed Ψ generally have different $\bar{G}$ transformation properties. However, since G acts on the vector space $H_{\bar{\partial}}^{0,q}(K,E)$, all components in a fixed Ψ transform identically under G. Therefore, only *some* of the T components of Ψ satisfy (III.113) and hence survive. There is thus the possibility of being able to choose a particularly felicitous embedding Φ for which certain dangerous components in a particular T multiplet vanish while their welcome multiplet partners survive. This multiplet splitting mechanism, however, must be considered in conjunction with the index theorem described in section (III.1) and applied in section (III.2). Specifically, notice that (III.93) is a topological statement pertaining to the bundle $V[\underline{s}_j]$. It is therefore independent of the holomorphic connection provided by the Uhlenbeck and Yau theorem and is, in particular, independent of the Wilson loops which generate $\bar{G}$.

For every representation $\underline{t}_j$ of T, which is associated with the vector bundle $V[\underline{s}_j]$, we are therefore guaranteed to have $|\frac{1}{2}\int_K C_3(V[\underline{s}_j])|$ net generations of complete T multiplets. The incomplete T multiplets discussed above are in addition to these complete multiplets, and of course constitute complete H multiplets. The index theorem also dictates that for every additional T component from the representation $\underline{t}_j$ which remains in theory, the corresponding component from the representation $\bar{\underline{t}}_j$ must remain in the theory as well. This is the only way of preserving the net number of generations in the representation $\underline{t}_j$.

The second implication of (III.113) is that the components of a particular complete T multiplet, in general, do not descend from the same harmonic section on K_0. As noted, the $\bar{G}$ transformation properties of these components are different; to satisfy (III.113) they necessarily must have different G transformation properties. However, the components of a fixed Ψ on K_0 have the same G transformation properties and we therefore conclude that the surviving components in a T multiplet must have originated from different harmonic sections Ψ. As far as *counting* states we have a well established number of net complete T multiplets; as far as group theory goes, these components are not related by simple T transformations. This implies that the ordinary group theoretic relations amongst Yukawa couplings do not apply.

These two obervations indicate the solutions which the flux mechanism offers to the second and third problems raised at the outset of this section.

The final issue is readily addressed by noting the following useful relation [12,14]:

$$\chi(K_0/G) = \chi(K_0)/|G| \qquad \text{(III.114)}$$

where $|G|$ denotes the number of elements in G. Passing from K_0 to K_0/G therefore has the additional virtue, in the $T = E_6$ case, of reducing the net number of generations.

Compactifications on non-simply connected manifolds are thus seen to provide elegant solutions to the issues raised at the outset of this section.

III.4 DISCRETE SYMMETRIES AND TOPOLOGICAL YUKAWA COUPLINGS

A recurrent theme in this thesis is the topological and geometric determination of physical observables which in the past were simply hypothesized in order to explain experimental results. We have seen that the topology and geometry of the compactified component of the universe determines whether spacetime supersymmetry survives and implies a particular massless spacetime field content, (thus providing a topological reason for family replication) and four–space unifying gauge group. To this point, however, we have not addressed interactions. Remarkably, even this aspect of the theory is susceptible to topological and geometrical analysis [21,22]. Effective four dimensional Yukawa couplings arising from compactification, as shall be shown below, may be expressed as harmonic mode overlap integrals on the compact space. A similar statement applies to the normalization matrix for kinetic terms. Surprisingly, these integrals depend only on the cohomology ring of the coefficient vector. bundles. Therefore, in theory, the entire set of tree level Yukawa couplings can be computed topologically. In practice, however, there are calculational obstacles which lead one to develop simpler means of achieving the less ambitious goal of determining whether a particular coupling vanishes. The study of naturally occurring discrete symmetries is a fruitful means of achieving the latter aim. In this section we shall briefly discuss the use of discrete symmetries in restricting couplings, and indicate the computational algorithm for their topological evaluation. As our focus in the original work of Chapter V centres on the discrete symmetry approach with, as yet, no dedicated attempt to extend the results to the full–fledged topological computation, our description of the latter procedure aims at conveying the central ideas without regard for complete detail. We begin with discrete symmetries.

As discussed in the last section, discrete symmetries often abound in

Calabi–Yau compactification. For a holomorphic discrete symmetry f of the base manifold K (which are the only type of symmetries we shall consider) we have from (III.14´) that a map f^* is induced on the holomorphic bundle valued cohomology based on K. We know that four dimensional massless fields arise from elements in these cohomology groups, and hence the discrete symmetries of K induce non–trivial symmetry transformations on the relevant fields in the theory. In particular, four dimensional massless fields can only couple if the corresponding coupling of their six dimensional harmonic modes forms a discrete symmetry invariant. This gives a procedure for determining whether a particular coupling vanishes, without providing its magnitude should it be non–zero. In a viable model, one hopes that certain dangerous couplings, leading for example to fast proton decay, are not present. Although a very small coupling may be a sufficient cure, the imposition of these inevitable discrete symmetries provides a powerful mechanism for their elimination. Additionally, as we shall discuss in Chapter V, the coupling restrictions implied by the discrete symmetries have important ramifications for flat directions in the superpotential (which are important for breaking gauge groups at a high energy scale), as well as other phenomenological implications.

Our task, then, is to determine the transformation properties on the relevant cohomology groups, induced by the discrete symmetries possessed by K. If one were to explicitly compute the harmonic forms on K in terms of local coordinates, the tranformation laws would be manifest. In practice, at least for the T = E_6, S = SU(3) case on algebraic variety Calabi–Yau compactifications, there are easier ways which invoke some powerful theorems from algebraic geometry. The Kodaira–Spencer theory relating to deformations of the complex structure of complex manifolds [4], and a number of the celebrated Lefschetz theorems [1] play a prominent role. Since these notions will be fully explained in Chapter V and applied to a rather pertinent example, we will not go into any further detail here.

The central ideas in computing effective four dimensional Yukawa couplings are best conveyed through an example. Our treatment closely follows [21,22]. Consider a ten dimensional cubic interaction of the form

$$\int d^4x\, d^6y \sqrt{-g_4}\sqrt{g_6}\, \bar{\Psi}_A(x,y)\Gamma^M A_{MB}(x,y)\Psi_C(x,y)F^{ABC} \qquad \text{(III.115)}$$

where g_4 and g_6 are the metric determinants in M_4 and K, and A,B,C are $E_8' \otimes E_8$ gauge indices with F^{ABC} providing the gauge invariant coupling. Upon compactification, our interest shifts to the harmonic modes on K and their corresponding four space Fourier coefficients. Using (III.92) in (III.115) we find that the latter may be written as a sum of terms of the form

$$\int d^4x\, d^6y \sqrt{-g_4}\sqrt{g_6}\{\bar{\Psi}_{Di}(x)\varphi_{Ej}(x)\Psi_{Fk}(x)\Psi^{\dagger}_{\bar{d}i}(y)\gamma^{\bar{m}}A_{\bar{m}e\,j}(y)$$
$$\Psi_{fk}(y)\, f^{DEF}h^{def}\} \qquad \text{(III.116)}$$

where the y−dependent fields are the K−space harmonic modes discussed in section III.2, the indices i,j,k label different supermultiplets (i.e. zero modes) descending from the uncompactified theory, D,E,F are group indices for the surviving gauge group T while d,e,f are group indices for the broken gauge component, S. The f and h coefficients provide the appropriate T and S invariant couplings (and, of course, vanish if no such invariant coupling is possible). Splitting this integral into a product of two factors, it may be expressed as (no sum on i,j,k)

$$g_{ijk}\int d^4x \sqrt{-g_4}\,\bar{\Psi}_{Di}(x)\varphi_{Ej}(x)\Psi_{Fk}(x)\, f^{DEF} \qquad \text{(III.117)}$$

with

$$g_{ijk} = \int_K d^6y\sqrt{g_6}\,\Psi^{\dagger}_{\bar{d}i}(y)\gamma^{\bar{m}}A_{\bar{m}ej}(y)\Psi_{fk}(y)h^{def} \qquad \text{(III.118)}$$

Using the relation given after (II.46), we see that this expression becomes

$$g_{ijk} = \int d^6y\sqrt{g_6}\,\omega^{\overline{mnp}}A_{\bar{m}di}(y)A_{\bar{n}ej}(y)A_{\bar{p}fk}(y)h^{def} \quad . \qquad \text{(III.119)}$$

Expressing this in terms of $A_{di} = A_{\bar{m}di}dz^{\bar{m}}$ and $\omega = \omega_{mnp}dz^m \wedge dz^n \wedge dz^p$ (note ω only has one-independent component) we have

$$g_{ijk} = \int \omega \wedge A_{di} \wedge A_{ej} \wedge A_{fk}h^{def}. \qquad \text{(III.120)}$$

In this form it is clear that g_{ijk} depends only on the cohomology classes of the one-forms A (by Stoke's theorem and integration by parts) and the complex structure of K through the ω factor. However, as shown in [21] this expression is actually independent of the complex structure and hence depends only on the cohomology ring of the relevant vector bundles over K. The interpretation of g_{ijk} as a Yukawa coupling is contingent upon the four dimensional kinetic terms being diagonal and having the standard normalization. These coefficients arise from the zero mode inner product [21]

$$\int F_{im\bar{n}}F_j^{\ m\bar{n}} \qquad \text{(III.121)}$$

and therefore apparently depend upon the metric. However, as shown in [22], (III.121) is only sensitive to the cohomology class of the Kähler form and thus only depends upon the $h^{1,1}(K)$ real parameters which constitute part of the moduli space. Hence, only topological information regarding fibre bundles on K as well as the cohomology class of the Kähler form are required to properly normalize (III.120) and determine the effective Yukawa couplings

At the present time, this program although theoretically transparent has only been carried out for relatively simple examples [23]. We shall concentrate on the discrete symmetry approach in the work presented in Chapter V.

We now take a closer look at superstring models with SU(5) and SO(10) unifying groups.

REFERENCES

CHAPTER III

1. P. Griffiths and J. Harris, Principles of Algebraic Geometry (Wiley Interscience, 1978).
2. K. Uhlenbeck and S.T. Yau, On the Existence of Hermitian–Yang–Mills Connections in Stable Vector Bundles, to appear.
3. S.K. Donaldson, private communication.
4. K. Kodaira, Complex Manifolds and Deformations of Complex Structures (Springer–Verlag 1985).
5. W.V.D. Hodge, Theory and Application of Harmonic Integrals (Cambridge University Press, 1952).
6. M.F. Atiyah and I.M. Singer, Ann. Math., 87 (1968) 485–545.
7. G. Horowitz in Proceedings of Santa Barbara Symposium on Unified String Theories (1985).
8. A. Strominger and E. Witten, Princeton University Preprint (to appear in Comm. Math. Phys.) 1985.
9. N. Hitchin, D. Phil. Thesis, Oxford University 1972.
10. M.J. Duff, B.E.W. Nilsson and C.N. Pope, Phys. Rep., 130 (1986) 1.
11. V.S. Kaplunosvky, Phys. Lett., 155B (1985) 151.
12. P. Candelas, G.T. Horowitz, A. Strominger and E. Witten, Nucl. Phys., B258 (1985) 75.
13. E. Witten, Nucl. Phys., B268 (1986) 79.
14. T. Hübsch, University of Maryland Preprint (1986).
15. E. Witten, Phys. Lett., 149B
16. Y. Hosotani, Phys. Lett., 126B (1983) 309; Phys. Lett., 129B (1983) 193.

17. E. Witten, Nucl. Phys., B258 (1985) 85.

18. X-G. Wen and E. Witten, Nucl. Phys., B261 (1985) 651.

19. E.B. Dynkin, Am. Soc. Math. Trans. Ser 2 (1957) 111.

20. J.D. Breit, B. Ovrut and G. Segré, Phys. Lett., 158B (1985) 33.

21. A. Stromger and E. Witten, Princeton University Preprint (to appear in Comm. Math. Phys.) 1985.

22. A. Strominger, ITP Preprint NSF-ITP-105 (1985).

23. A. Strominger, ITP Preprint NSF-ITP-109 (1985).

CHAPTER IV

SUPERSTRING MODELS WITH SU(5) AND SO(10) UNIFYING GROUPS

IV.1 INTRODUCTION

Superstring theories appear to provide the first mathematically consistent (and possibly perturbatively finite) unification of the four fundamental forces of nature. In the context of these theories, one is naturally led to consider phenomenology based on an E_6 unifying group. Although potentially viable, E_6 is less attractive than the economical SU(5) unifying group where known fermions fit like a glove into two representations, or SO(10) where they are accommodated (with a right handed neutrino) in a single representation. The recent plausibility argument [1] for the natural emergence of the latter two unifying groups in the context of superstring theories is therefore of great interest.

In this chapter we shall enumerate all possible flux breaking scenarios with SU(5) and SO(10) unifying groups, including the possibility of intermediate mass scales (IMS) for the case of SO(10). In each case we determine constraints which sufficiently inhibit 'coloured Higgs' induced proton decay, and ensure the emergence of naturally light weak Higgs doublets. Using standard techniques, we analyse the phenomenology ($\sin^2\theta_W$ and unification scale M_X) predicted by the more tenable of these schemes, and are thereby led to a highly restricted class of acceptable models.

In section 2 we provide necessary background material, including a description of flux breaking and the topological origins of the matter spectra in these new models. An explanation of the group theoretical methodology employed in the determination of breaking patterns is also given. The constraints required for viable models are discussed in section 3. We carry out the analysis of SU(5) and SO(10) models in section 4. In section 5 we summarize our results and give some concluding remarks.

IV.2 BASIC CONCEPTS

The $E_8' \otimes E_8$ heterotic superstring [2] can lead to effective ten dimensional supergravity theories with many desirable phenomenological characterstics. These theories have the vacuum state $M \otimes K_0$, where M is four dimensional Minkowski space, and K_0 is a compact six dimensional manifold [3]. Models constructed on Ricci–flat, Kähler manifolds K_0 with SU(3) holonomy [4] (Calabi–Yau manifolds) give an $E_8' \otimes E_6$ internal symmetry group when the spin connection of K_0 is embedded in the gauge group $E_8' \otimes E_8$ [3,5]. This leads to a four dimensional theory with N=1 supersymmetry, $N_g \times \underline{27}$ families of chiral superfields (where N_g is half the absolute value of the Euler characteristic of K_0, $\chi(K_0)$) and possibly extra light fields in $\delta(\underline{27} + \overline{27})$ representations, where δ = minimum $(h_{1,1}, h_{2,1})$ ($h_{p,q}$ is the number of independent harmonic (p,q) forms on K_0) [3]. If K_0 is simply connected and admits a freely acting discrete group of transformations G, it is worthwhile to consider compactification on $K = K_0/G$ [3,6]. K is also a Calabi–Yau manifold, and the resulting theories have the attractive properties of those built on K_0(1) [3,6]. This serves the dual purpose of reducing the number of generations by a factor of $|G|$ (order of G) and allowing for the possibility of symmetry breaking by Wilson loops [6,8]

$$U_g = P \exp\left[-i\int_{\gamma_g} T^a A^a{}_m \, dy^m \right] \qquad \text{(IV.1)}$$

where $A^a{}_m$ are the vacuum state E_6 gauge fields, T^a the group generators, the contour γ_g a representative of the abstract element g of K's fundamental group G $= \pi_1(K)$, and P denotes path ordering. If the contour γ_g is not homotopic to a point (i.e. $g \neq 1$) then we can have $U_g \neq 1$, even though the vacuum field strength vanishes everywhere. In this way, we induce a homomorphism of G into

E_6 [6] taking g to U_g. The image $\bar{G}$ of G can be regarded as an effective adjoint Higgs vacuum expectation value (VEV) at the compactification scale. This 'flux–breaking' mechanism can lead to acceptable low–energy gauge groups [6,9,10,11,12].

Recently, Witten [1] has pointed out that it may be possible to construct stable, irreducible holomorphic SU(4) or SU(5) vector bundles over Calabi–Yau manifolds, that result in unbroken $E_8 \otimes SU(5)$ or $E_8 \otimes SO(10)$ (respectively) gauge groups and unbroken, four dimensional, N=1 supersymmetry. This is achieved by embedding the particular SU(5) or SU(4) gauge connection (whose existence is guaranteed by the theorems of Donaldson, Uhlenbeck and Yau [13]) of the vector bundle in question in one of the factors of $E_8' \otimes E_8$. We assume that known physics lies within this single E_8, and from now on, we ingore the E_8' factor.

The field content of any such scenario is determined by decomposing the $\underline{248}$ of E_8 under $T \otimes S$, where S is the structure group of the bundle, and T is the maximal subgroup of E_8 commuting with S. When the tangent bundle is chosen (S = SU(3), T = E_6), chiral superfields arise in $\underline{27}$'s ($\overline{\underline{27}}$'s) of E_6 which contain many presently unobserved particles. For S = SU(5) or SU(4) (T = SU(5) or SO(10)), however, we have the phenomenologically pleasing field content:

$$T = SU(5) \qquad \underline{248} = (\underline{1},\underline{24}) + (\underline{24},\underline{1}) + (\underline{5},\overline{\underline{10}}) + (\overline{\underline{5}},\underline{10}) + (\underline{10},\underline{5}) + (\overline{\underline{10}},\overline{\underline{5}})$$

$$T = SO(10) \qquad \underline{248} = (\underline{1},\underline{15}) + (\underline{16},\underline{4}) + (\underline{10},\underline{6}) + (\overline{\underline{16}},\overline{\underline{4}}) + (\underline{45},\underline{1})$$

in which a family of observed fermions fills a $\overline{\underline{5}} + \underline{10}$ of SU(5), or a $\underline{16}$ of SO(10) (with the addition of a right handed neutrino).

A model is specified over a given K by the construction of a stable, irreducible, holomorphic SU(5) or SU(4) bundle (meeting the topological requirement that its second Chern class equals that of the tangent bundle of K).

The massless fields arising in each representation can be related to elements of appropriate cohomology groups [1]. For SU(5), chiral superfields in the $(\underline{10},\underline{5})$ and $(\overline{\underline{10}},\overline{\underline{5}})$ transform in the fundamental representations of $S = SU(5)$ and arise as elements of $H^1(B)$ and $H^1(B^*)$ (B denotes the SU(5) bundle, B^* its dual, and $H^1(E)$ is the first Dolbeault cohomology group [14] with values in E for any holomorphic vector bundle E). The $\underline{10}$ of $S = SU(5)$ is built from $(\underline{5} \otimes \underline{5})_{AS}$ (AS denotes the antisymmetric part), and hence the $(\overline{\underline{5}},\underline{10})$ arise as elements of $H^1((B \otimes B)_{AS})$ ($\otimes$ here indicates the fibre-wise tensor product). Similarly, the $(\underline{5},\overline{\underline{10}})$ are associated with elements of $H^1((B^* \otimes B^*)_{AS})$. The $\underline{24}$ is the adjoint of SU(5) and hence the relevant bundle for $(\underline{1},\underline{24})$ is EndB (fibre-wise endomorphisms of B) [1]; fermions in this representation arise from $H^1(\mathrm{EndB})$. The number of zero modes arising in each representation is given by the complex dimension of the associated cohomology group.(2) The $(\underline{24},\underline{1})$ are the gauge bosons of the low energy (i.e. less than the Planck mass) SU(5) theory. Similarly, massless fields in each SO(10) representation arise from the following cohomology groups: the $(\underline{16},\underline{4})$ and $(\overline{\underline{16}},\overline{\underline{4}})$ from elements of $H^1(B')$ and $H^1(B'^*)$, the $(\underline{10},\underline{6})$ from $H^1((B' \otimes B')_{AS})$, and the $(\underline{1},\underline{15})$ from $H^1(\mathrm{EndB}')$ (B' is the SU(4) bundle, B'^* its dual). The $(\underline{45},\underline{1})$ are the gauge bosons of the low energy SO(10) theory. As we have no specific examples of these bundles over Calabi-Yau manifolds, we treat their respective dimensions as parameters of our models. For both cases we neglect the singlet fields arising from $H^1(\mathrm{EndB})$ ($H^1(\mathrm{EndB}')$).

Following Breit, Ovrut and Segrè [10] we use the method of Weyl weights to determine the unbroken gauge group and extra matter fields which remain light after employing the flux-breaking mechanism. As we concentrate on compactifying with Calabi-Yau manifolds having abelian fundamental groups, general Wilson loops take the form $U_g = \exp(\Sigma\lambda^i H^i)$ where the H^i are generators of the Cartan subalgebra of the unification group T (SU(5) or SO(10) in our case) and the λ^i are real parameters corresponding to a breaking direction in root space. The subgroup of T generated by the $\{U_g\}$ is denoted $\bar{G}$ (as above). The unbroken gauge group is then the subgroup H of T spanned by the generators of T whose root vectors α satisfy $\lambda^i\alpha^i = 0$ — see Tables I and V — (as the corresponding gauge bosons remain massless). The extra matter fields remaining light are those which are invariant under the diagonal sum $G \oplus \bar{G}$, where this action on a covering space state Ψ is given by $(G \oplus \bar{G})\Psi = U_g\Psi(g(x))$, $x \epsilon K_0$, $g \epsilon G$, $U_g \epsilon \bar{G}$, and Ψ transforms in some representation R of T [6]. Using the Weyl formalism, we determine the $\bar{G}$ transformation properties of Ψ by computing $(U_g\Psi)_j = (\exp(\Sigma\lambda^i H^i)\Psi)_j = \exp(\Sigma\lambda^i\beta_j^{\,i})\psi_j$ (no sum on j) where $\Psi = (\psi_1,\ldots,\psi_r)$ $(r = \dim(R))$ and $\beta_j = (\beta_j^{\,1},\ldots,\beta_j^{\,n})$ $(j = 1,\ldots,r$, and $n = \mathrm{rank}(T))$ is the weight of ψ_j in the Dynkin basis. The G transformation properties of a given Ψ will, in general, depend upon the specific model constructed. We tabulate all possibilities.

IV.3 PHENOMENOLOGICAL CONSTRAINTS

Our analysis utilises the following procedure:

i) For either choice of T (SU(5) or SO(10)) we consult Tables I and V to constrain λ to ensure $SU(3)_c \otimes SU(2)_L$ is unbroken.

ii) For any choice of $\bar{G}$ we consult Tables I and V to compute the possible flux-breakings corresponding to embeddings of G in T allowed by (i). $\bar{G}$ will generally be of the form $\mathbb{Z}_m \otimes \mathbb{Z}_n$ $m,n \epsilon (1,2,3,\ldots)$, and hence admits a number of two-element genrating sets. An embedding in T consists of choosing the values of the free parameters in λ' from one of these sets. This will become clearer when we work an example below.

iii) For each flux-breaking induced symmetry breaking pattern, we determine the resulting light matter fields. As discussed, such fields are $G \oplus \bar{G}$ singlets, where the $\bar{G}$ transformation properties are given in Tables II and VI, and the G transformation properties are generally dependent on K_0, K and B(B′). For generality, we analyse all possible G transformations. Note that after flux breaking from T to *H*, the multiplets of *H* which arise from the same T multiplet have identical G transformation properties.

iv) For a model to be considered viable, it must be possible to choose the G transformation properties of the zero modes to ensure:

a) A sufficient number of Higgs doublets (from the appropriate representation of the unifying group) remain light, in order to perform supersymmetric weak gauge symmetry breaking and give masses to ordinary fermions. (We assume that the low energy N=1 supersymmetry survives down to the weak scale where the remaining unobserved fields obtain masses along with the supersymmetric partners of observed particles.)

b) Extra coloured fields ("coloured Higgs") which could result in fast proton decay must not remain light. (One can of course envisage a situation in which topological and discrete symmetry restrictions eliminate all the offensive couplings of extra light coloured fields with quarks. We view this possibility as somewhat unnatural and seek a more general solution).

c) If the unbroken gauge group is larger than that of the standard model then a sufficient number of $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ singlets remain light in order to trigger an intermediate scale gauge symmetry breaking to the standard model.

IV.4 ANALYSIS FOR SU(5) AND SO(10)

A) SU(5):

Without loss of generality, we assume that the cohomology groups discussed above lead to a T = SU(5) field content of the form $m(\underline{10}) + n(\overline{\underline{5}}) + \delta(\underline{10} + \overline{\underline{10}}) + \epsilon(\underline{5} + \overline{\underline{5}})$ where n, m, δ, ϵ are non-negative integers. n is given by $|C_3(B)|/2$ (where $C_3(B)$ is the third Chern number of the bundle B; i.e. the integral of the bundle's third Chern class over the base manifold K) and m by $|C_3((B \otimes B)_{AS})|/2$ [1]; $\delta = (\dim H^1(B))$, and $\epsilon = \dim(H^1((B^* \otimes B^*)_{AS})$. The index theeorem which yields $n = |C_3(B)|1/2$ implies that if $Q \epsilon \underline{5}$ (or $\underline{10}$) from the $(\underline{5} + \overline{\underline{5}})$ sector (or $(\underline{10} + \overline{\underline{10}})$ sector) remains light, the corresponding state $\overline{Q} \epsilon \overline{\underline{5}}$ (or $\overline{\underline{10}}$) to which it is paired will also remain light. Note that condition (iv.a) requires $\epsilon \geqslant 1$. In accordance with standard low energy anomaly cancellation arguments, we have n= m, as the symmetric trace $|\mathrm{Tr}\{T^{[a}T^{b}T^{c]}\}|$ is equal for the $\overline{\underline{5}}$ and $\underline{10}$ representations.

Since SU(5) has rank four, a general λ is written [a,b,c,d]. Requirement (i) imposes $a = 2c$, $d = -2c$, so we write $\lambda' = [2c, -c, c, -2c]$. Since λ' contains only one free paramete, $\overline{G}$ is singly generated (i.e. a finite cyclic group) and has a unique embedding in SU(5) consistent with (i). (Note that this exhausts all possible $\overline{G}$ as the standard model has rank equal to that of SU(5), thereby excluding the use of non-abelian $\overline{G}$). We systematically exhaust the possibilities to determine Table III. It is instructive to work a few examples in detail:

(1) $G \supseteq \overline{G} = Z_3$

Let Z_3 be generated by p, i.e. $Z_3 = \{0, p, 2p\}$ with $3p = 0$. To embed Z_3 in SU(5) we identify c with p, and hence set $3c = 0$. Define $\eta = e^{ic}$.

From Table I, we see that $SU(5) \rightarrow SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$. In order to meet requirements (iv.a) and (iv.b) we must be able to choose the G transformation properties of the massless modes so that $A \epsilon \underline{5}$ stays light and B, D, and E acquire large masses. (See Table II for definitions of the multiplets A,B,C,D,E). Table III shows the appropriate G transformation properties of the H $= SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ multiplets satisfying this constraint. Note that none of our requirements constrain C; C (and $\bar{C} \epsilon \overline{\underline{10}}$) will remain light or acquire a large mass depending on whether or not it is G invariant, since from Table II we see that it is $\bar{G}$ invariant. Thus, if one constructs a holomorphic, stable SU(5) bundle over a Calabi–Yau manifold with fundamental group $\mathbb{Z}_3$, and the $\mathbb{Z}_3$ transformation properties of the massless modes do not satisfy the constraints shown in Table III, either of requirements (iv.a) or (iv.b) will necessarily be violated, making model building fruitless. Of course, the constraint $A \rightarrow A$ need only be satisfied by one of our extra(3) multiplets Ψ (since we only need one pair of Higgs doublets), while the remaining constraints must strictly be satisfied by all Ψ so as not to violate (iv.b).

(2) $G \supseteq G = \mathbb{Z}_5$

As with $\mathbb{Z}_3$ we identify c with a $\mathbb{Z}_5$ generator, so $5c = 0$.

From Table I we see that SU(5) is left unbroken. As no flux breaking occurs, and we do not pursue this case any further.

(3) $G \supseteq \bar{G} = \mathbb{Z}_{n \geqslant 7}$

Identify c with a $\mathbb{Z}_n$ generator, so $nc = 0$. Set $\eta = e^{ic}$.

From Table I, $SU(5) \rightarrow SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$. Constraint (iv.b) requires $B \not\rightarrow \eta^2 B$, $D \not\rightarrow \eta D$, $E \not\rightarrow \eta^{-4} E$, for all extra multiplets. Constraint (iv.a) requires $A \rightarrow \eta^{-3} A$ for at least one of the extra multiplets. None of our requirements constrain C and hence it will remain light if $C \to \eta^6 C$.

The analysis for other choices of $\bar{G}$ is similar, and the results are given in Table III. It is interesting to note that if the extra multiplets from the ($\underline{5}$ + $\bar{\underline{5}}$) and ($\underline{10}$ + $\overline{\underline{10}}$) sectors are all G invariant for some model, then the fundamental group of the manifold of compactification must be mapped onto $\mathbb{Z}_3$ by the Wilson loop homomorphism.

We emphasize that, unlike in ordinary SU(5) GUTs where dangerous coloured Higgs triplets are rendered inoccuous by *ad hoc* remedies, the flux mechanism gives a natural solution to this problem. Furthermore, this mechanism simultaneously gives the SU(5) gauge bosons which lead to baryon number violating processes compactification scale masses.

Note that the resulting models are indexed by two free parameters: the number of light Higgs doublets and the number of extra light charged singlets. For any specific construction of a stable, holomorphic SU(5) bundle over a Calabi–Yau manifold, these parameters will be determined. We constrain the phenomenologically allowed values of these parameters by considering the resulting one–loop values of $\sin^2\theta_W$ and M_X [15]. The results are compiled in Table IV. If we demand values of $\sin^2\theta_W$ and $M_X^{(4)}$ in the range 0.22 ± .02 and $10^{17\pm3}$ respectively, acceptable models have two Higgs doublets (the minimal number for supersymmetric weak gauge symmetry breaking) and no more than four extra charged singlets. We also note that the unification coupling constant is perturbative for less than five generations.

B) SO(10):

We assume the topological analysis of a specific T = SO(10) model leads to the spectrum $n(\underline{16})$ + $\delta(\underline{16}$ + $\overline{\underline{16}})$ + $\epsilon(\underline{10})$ where n, δ, ϵ are non–negative integers given by $n = |C_3(B')|/2$; $\delta = \dim H^1(B')$, $\epsilon = \dim H^1((B' \otimes B')_{AS})$.

Constraint (iv.a) implies $\epsilon \geqslant 1$. Note that the index theorem which gives $n = |C_3(B')|/2$ demands that if $Q \epsilon \underline{16}$ from the $(\underline{16} + \overline{\underline{16}})$ sector remains light, the corresponding piece $\bar{Q}$ of the $\overline{\underline{16}}$ to which it is paired remains light as well.

A general λ for SO(10) may be expressed as $\lambda = [a,b,c,d,e]$. Requirement (i) imposes $a = -2e$, $b = 0$, $d = -e$ and hence we write $\lambda' = [-2e,0,c,-e,e]$. Since λ' has two free parameters, an abelian $\bar{G}$ is of the form $Z_m \otimes Z_n$ $m,n \epsilon \{1,2,3,...\}$, and may have allowed embeddings in SO(10) which are inequivalent. Moreover, flux breaking with an abelian $\bar{G}$ will result in a rank five unbroken gauge group, thereby requiring additional gauge symmetry breaking. The scale of this symmetry breaking for any rank five extension of the standard model except $J = SU(3)_c \otimes SU(2)_L \otimes U(1) \otimes U(1)$ must be at least 10^9 GeV to ensure acceptable values of $\sin^2\theta_W$. Following Dine *et al* [9], we see that VEVs of this magnitude are only possible for fields in the $(\underline{16} + \overline{\underline{16}})$ sector, for which we can have D-flat directions in the superpotential. It has also been shown that such VEVs may be as large as 10^{15} GeV [12] .in models with appropriate discrete symmetries. We therefore require light $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ singlets from the $(\underline{16} + \overline{\underline{16}})$ to remain in the spectrum if the unbroken group is different from J, and assume they can acquire VEVs $\approx 10^{10}$ to 10^{15} GeV. If the unbroken group is J, it is phenomenologically acceptable to break to the standard model at ≈ 1 TeV. The above constraint is thus discarded since a VEV of this magnitude may be attained by the 'right handed neutrino' component from one of the n $\underline{16}$'s. (We do not consider possible mixings between the two U(1)'s which could be significant in the latter case).

The possible scenarios consistent with these constraints are shown in Table VII. There are five possible flux breaking patterns (SO(10) $\rightarrow$ SU(5) $\otimes$ U(1); SU(4) $\otimes$ SU(2) $\otimes$ SU(2); SU(4) $\otimes$ SU(2) $\otimes$ U(1); SU(3) $\otimes$ SU(2) $\otimes$ SU(2) $\otimes$ U(1); SU(3) $\otimes$ SU(2) $\otimes$ U(1) $\otimes$ U(1)). We discard the first three breaking patterns as we cannot have light $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ singlets from the ($\underline{16}$ + $\overline{\underline{16}}$) (for intermediate scale breaking) without also violating requirement (iv.b). This may be seen by noting that in $SU(4) \otimes SU(2) \otimes SU(2)$, for example, this singlet lies in the same $\underline{4}$ of SU(4) as a coloured triplet. Therefore all four transform identically under $G \oplus \bar{G}$; if the singlet is light, so is the colour triplet. Thus two viable patterns remain. For each choice of $\bar{G}$ and its embedding in SO(10) which leads to either of these patterns (see Table VII), there corresponds a specific constraint on the G transformarion properties of the resulting H multiplets. For example, if $\bar{G} = Z_3$ and we choose the embedding $c = 0$, $|e| = 3^{(4)}$, the unbroken group is $H = SU(3)_c \otimes SU(2) \otimes SU(2) \otimes U(1)$. Now, under H the $\underline{16}$ of SO(10) decomposes into the four multiplets A, (B,C), D, (E,F) and the $\underline{10}$ into three (W,X), Y, Z; see Table VI for clarification. (The parentheses indicate, for example, that W and X lie in the same H multiplet). To meet (iv.a) we must keep at least one (W,X) light. As each (W,X) is $\bar{G}$ invariant (from Table VI), we therefore require that at least one is G invariant as well. To meet constraint (iv.b) we want D, (E,F), Y and Z to acquire large masses. Thus we require $D \nrightarrow \eta D$, $(E,F) \nrightarrow \eta^{-1}(E,F)$, $Y \nrightarrow \eta^2 Y$, $Z \nrightarrow \eta Z$ under G (where, as usual, $\eta = e^{ie}$ with $\eta^3 = 1$) therefore ensuring that none is a $G \oplus \bar{G}$ singlet.

For ease of presentation, in Table VII we only indicate whether there exists a choice of G transformations consistent with our requirements, for a particular breaking pattern. One may find the explicit transformation law by using the reasoning above. As also indicated in Table VII, there are a number of unconstrained fields which may be light without violating any of our requirements. For a particular choice of K, construction of B', and embedding of $\bar{G}$, the number of each such fields remaining light will be determined. We place phenomenological constraints on these parameters by considering the one-loop $\sin^2\theta_W$ and M_X for various numbers of light fields, for each breaking pattern. Our results are tabulated in Tables VIII and IX. We find that favourable models have no more than four Higgs doublets from the 10 sector ($2 \times (G,H)$) by the requirement that $M_X \approx 10^{17\pm3}$ and $\sin^2\theta_W = 0.22 \pm .02$. We find that the unification coupling constant is perturbative for less than five generations. The effects of the additional charged singlets and weak doublets are readily seen. As with SU(5), acceptable models give compactification scale masses to gauge bosons and additional coloured particles which can mediate baryon number violating processes.

It is an intriguing coincidence that if we demand G invariance in all extra multiplets (from the ($\underline{16} + \overline{\underline{16}}$) and $\underline{10}$ sectors) we are led to the same conclusions as in the SU(5) case, namely that the image of the Wilson loop homomorphism is Z_3 (i.e. the fundamental group of K must contain a Z_3 subgroup).

Since SO(10) has rank five, it is conceivable that flux-breaking via a non-abelian $\bar{G}$ may be relevant. We analyse this possibility using the method of maximal subgroups [6]. Consider the SU(4) $\otimes$ SU(2) $\otimes$ SU(2) maximal subgroup of SO(10). Without loss of generality we embed $SU(3)_C \otimes SU(2)_L$ in the following way:

$$\left[\begin{array}{c|c} SU(3)_C & \\ \hline & - \end{array}\right] \otimes \left[SU(2)_L \right] \otimes \left[SU(2) \right] \qquad (IV.2)$$

Now, let $g \to V_g$ be a two dimensional irreducible representation of $\bar{G}$. In order to preserve the standard model, V_g may only be embedded in the right-most factor:

$$\begin{bmatrix} \mu_g & & & \\ & \mu_g & & \\ & & \mu_g & \\ & & & \mu_g^{-3} \end{bmatrix} \otimes \begin{bmatrix} \nu_g & \\ & \nu_g \end{bmatrix} \otimes \left[V_g \right] \qquad (IV.3)$$

where $g \to \mu_g$ and $g \to \nu_g$ are one dimensional representations with $\nu_g{}^2 = 1$ and

$\det(V_g) = 1$. Such an embedding can break SO(10) to $SU(3)_c \otimes SU(2)_L \otimes U(1)$, but one can easily check that the U(1) factor cannot be consistently identified with that of the standard model. We thus find no viable cases with non–abelian flux breaking.

IV.5 SUMMARY AND CONCLUSIONS

We have analysed all possible symmetry breaking patterns and basic phenomenology in the context of Witten's proposal for constructing superstring models with an SU(5) or SO(10) unifying group. For compactification on a Calabi–Yau manifold with fundamental group G, we have determined all flux breaking patterns and allowed transformation properties of the massless modes to avoid fast proton decay and to allow naturally light Higgs doublets for supersymmetric weak gauge symmetry breaking. In the case of SO(10) we have also examined the possibility of intermediate scale symmetry breaking. In all such scenarios, an analysis of the predicted values of the unification scale and the weak mixing angle have placed constraints on parameters, which in any specific example will be determined topologically.

A favoured class of models is based on an SU(5) unifying group. These models avoid the problem of the right handed neutrino arising from SO(10) (or the host of unobserved particles arising from E_6) and do not require the introduction of intermediate mass scales. Unlike ordinary supersymmetric SU(5) models which surmount the proton decay problem only by relying on *ad hoc* remedies, we have seen that the flux breaking mechanism offers the possibility of a simple and natural solution.

The feasibility of these scenarios cannot be further analysed until the apropriate conjectured bundle constructions are performed. Even so, it is quite inspiring that low energy phenomenology, the physics of the present day universe, can be used to place such stringent constraints on physics at the Planck scale.

FOOTNOTES

(1) For the remainder of this chapter, we denote by K the Calabi–Yau manifold used in compactification, and by K_0 its simply connected covering space. If $\chi(K) \neq 0$ (i.e. $N_g > 0$), it can be shown that $\pi_1(K)$ is finite, and that a simply connected covering space exists.

(2) By 'extra' we mean those multiplets associated with the $\delta(\underline{10} + \overline{\underline{10}})$ and $\epsilon(\underline{5} + \overline{\underline{5}})$ sectors of the spectrum of SU(5). For SO(10), 'extra' refers to multiplets in the $\delta(\underline{16} + \overline{\underline{16}})$ and $\epsilon(\underline{10})$ sectors.

(3) Witten has shown that the standard Georgi–Quinn–Weinberg calculation of $\sin^2\theta_W$ and M_X is valid in this context [6].

(4) This notation is used to indicate that e is an element of order 3 in $\mathbb{Z}_3$.

TABLE CAPTIONS

Table I: λ denotes a general root vector, whilst λ' is the most general root vector leaving $SU(3)_c \otimes SU(2)_L$ unbroken.

Table II: Capital letters denote fields that transform in the same $SU(3)_c \otimes SU(2)_L$ multiplet. The elements for the $\bar{\underline{5}}$ and $\underline{10}$ are shown.

Table III: Capital letters refer to multiplets in Table II. The symbol § means that only one pair of Higgs doublets from a $(\underline{5} + \bar{\underline{5}})$ need transform in this way. In addition to fields listed in the 'light fields' column, the corresponding barred fields also remain light.

Table IV: Weinberg angle and compactification scale for SU(5) scenarios.

Table V: λ denotes a general root vector, whilst λ' is the most general root vector leaving $SU(3)_c \otimes SU(2)_L$ unbroken.

Table VI: Capital letters denote fields that transform in the same $SU(3)_c \otimes SU(2)_L$ multiplet. The elements for the $\underline{16}$ and $\underline{10}$ are shown.

Table VII: Capital letters refer to multiplets in Table VI. States enclosed in parentheses transform as a single multiplet under the corresponding gauge group. In addition to fields listed in the 'possible light fields' column, the corresponding barred fields also remain light. By $|c| = n$ ($|e| = n$) we mean that c (e) is an element of order n (a generating element for G).

Table VIII: The quantity, n, in column one denotes the number of (G,H) multiplets, and hence the total number of Higgs doublets is 2n.

ROOT α	(λ,α)	(λ',α)	ROOT α	(λ,α)	(λ',α)
(1,0,0,1)	a+d	0	(-1,2,-1,0)	-a+2b-c	-5c
(1,1,-1,0)	a+b-c	0	(-2,1,0,0)	-2a+b	-5c
(0,-1,1,1)	-b+c+d	0	(0,0,-1,2)	-c+2d	-5c
(-1,1,1,-1)	-a+b+c-d	0	(0,1,-2,1)	b-2c+d	-5c
(-1,1,0,1)	-a+b+d	-5c	(-1,0,-1,1)	-a-c+d	-5c

TABLE I: NON-ZERO SU(5) ROOTS

U_g^{diag}	$SU(2)_L \otimes SU(3)_C$	SU(5)
exp[3ic]	$(2,1) \equiv A$	$\underline{\bar{5}}$
exp[-2ic]	$(1,\bar{3}) \equiv B$	
exp[-6ic]	$(1,1) \equiv C$	$\underline{10}$
exp[-ic]	$(2,3) \equiv D$	
exp[4ic]	$(1,\bar{3}) \equiv E$	

TABLE II: SU(5) WILSON LOOP DIAGONAL ELEMENTS

IMAGE OF $\pi_1(K)$ (G)	UNBROKEN GROUP AFTER FLUX BREAKING (H)	CONSTRAINED G TRANSFORMATIONS	CHOICES OF G TRANSFORMATIONS	LIGHT FIELDS
Z_2	SU(3)⊗SU(2)⊗ ⊗U(1)	$A\to\eta A$ § $B\not\to B$ $D\not\to\eta D$ $E\not\to E$	——	A
Z_3	SU(3)⊗SU(2)⊗ ⊗U(1)	$A\to A$ $B\not\to\eta^2 B$ $D\not\to\eta D$ $E\not\to\eta^2 E$	$C\to C$ $C\not\to C$	A,C A
Z_4	SU(3)⊗SU(2)⊗ ⊗U(1)	$A\to\eta A$ § $B\not\to\eta^2 B$ $D\not\to\eta D$ $E\not\to E$	$C\to\eta^2 C$ $C\not\to\eta^2 C$	A,C A
Z_5	SU(5)	——	——	——
Z_6	SU(3)⊗SU(2)⊗ ⊗U(1)	$A\to\eta^3 A$ § $B\not\to\eta^2 B$ $D\not\to\eta D$ $E\not\to\eta^2 E$	$C\to C$ $C\not\to C$	A,C A
$Z_{n\geqslant 7}$	SU(3)⊗SU(2)⊗ ⊗U(1)	$A\to\eta^{-3} A$ § $B\not\to\eta^2 B$ $D\not\to\eta D$ $E\not\to\eta^{-4} E$	$C\to\eta^6 C$ $C\not\to\eta^6 C$	A,C A

<u>TABLE III: BREAKING PATTERNS FOR SU(5)</u>

NO. HIGGS DOUBLETS FROM THE $5+\bar{5}$	NO. EXTRA CHARGED SINGLETS FROM THE $10+\overline{10}$	$\sin^2\theta$	$\log(M_X)$ (GeV)
2	0	.230	16.3
2	1	.215	15.0
2	2	.203	13.9
4	0	.253	15.0
4	1	.237	13.9
4	2	.224	13.0
4	3	.213	12.2
6	1	.256	13.0
6	2	.242	12.2
6	3	.230	11.5

TABLE IV: $\sin^2\theta, M_X$ FOR $SU(5) \to SU(3) \otimes SU(2) \otimes U(1)$

ROOT α	(λ,α)	(λ',α)	ROOT α	(λ,α)	(λ',α)
(0,1,0,0,0)	b	0	(-1,0,1,-1,-1)	-a+c-d-e	c+2e
(1,0,0,-1,1)	a-d+e	0	(1,-1,1,0,0)	a-b+c	c-2e
(-1,1,0,1,-1)	-a+b+d-e	0	(0,-1,1,1,-1)	-b+c+d-e	c-2e
(0,-1,0,1,1)	-b+d+e	0	(1,-2,1,0,0)	a-2b+c	c-2e
(0,-1,2,-1,-1)	-b+2c-d-e	2c	(1,0,1,-1,-1)	a+c-d-e	c-2e
(-1,0,1,0,0)	-a+c	c+2e	(0,0,1,0,-2)	c-2e	c-2e
(0,-1,1,-1,1)	-b+c-d+e	c+2e	(1,-1,1,-1,-1)	a-b+c-d-e	c-2e
(-1,-1,1,0,0)	-a-b+c	c+2e	(1,0,0,1,-1)	a+d-e	-4e
(-1,1,1,-1,-1)	-a+b+c-d-e	c+2e	(2,-1,0,0,0)	2a-b	
(0,0,1,-2,0)	c-2d	c+2e	(1,-1,0,1,-1)	a-b+d-e	-4e

TABLE V : NON-ZERO SO(10) ROOTS

U_g^{diag}	$SU(2)_L \otimes SU(3)_C$	SO(10)
exp[-3ie]	$(2,1) \equiv A$	<u>16</u>
exp[i(c+3e)]	$(1,1) \equiv B$	
exp[i(3e-c)]	$(1,1) \equiv C$	
exp[ie]	$(2,3) \equiv D$	
exp[i(c-e)]	$(1,\bar{3}) \equiv E$	
exp[i(-c-e)]	$(1,\bar{3}) \equiv F$	
exp[ic]	$(2,1) \equiv W$	<u>10</u>
exp[-ic]	$(2,1) \equiv X$	
exp[-2ie]	$(1,3) \equiv Y$	
exp[2ie]	$(1,\bar{3}) \equiv Z$	

<u>TABLE VI: SO(10) WILSON LOOP DIAGONAL ELEMENTS</u>

IMAGE OF $\pi_1(K)$ (G)	EMBEDDING	UNBROKEN GROUP AFTER FLUX BREAKING (H)	$\exists$ G TRANSFORMATION CONSISTENT WITH RESTRICTIONS	POSSIBLE LIGHT FIELDS
$\mathbb{Z}_2$	e=0	$SU(4)\otimes SU(2)\otimes SU(2)$	NO	———
	c=0	$SO(10)$	NO	———
	c=e	$SU(4)\otimes SU(2)\otimes SU(2)$	NO	———
$\mathbb{Z}_3$	e=0	$SU(4)\otimes SU(2)\otimes U(1)$	NO	———
	c=0	$SU(3)\otimes SU(2)^2\otimes U(1)$	YES	A, (B,C), (W,X)
	c=e	$SU(5)\otimes U(1)$	NO	———
	c=2e	$SU(5)\otimes U(1)$	NO	———
$\mathbb{Z}_4$	e=0	$SU(4)\otimes SU(2)\otimes U(1)$	NO	———
	c=0	$SU(4)\otimes SU(2)\otimes SU(2)$	NO	———
	c=e	$SU(4)\otimes SU(2)\otimes U(1)$	NO	———
	c=2e	$SO(10)$	NO	———
	c=3e	$SU(4)\otimes SU(2)\otimes U(1)$	NO	———
$\mathbb{Z}_{n\geq 5}$ n odd	e=0	$SU(4)\otimes SU(2)\otimes U(1)$	NO	———
	c=0	$SU(3)\otimes SU(2)^2\otimes U(1)$	YES	A, (B,C), (W,X)
	e=(n±1)c/2	$SU(5)\otimes U(1)$	NO	———
	OTHERWISE	$SU(3)\otimes SU(2)\otimes U(1)^2$	YES	A,B,C,W,X
$\mathbb{Z}_{n\geq 5}$ n even	e=0	$SU(4)\otimes SU(2)\otimes U(1)$	NO	———
	c=0	$SU(3)\otimes SU(2)^2\otimes U(1)$	YES	A, (B,C), (W,X)
	e=(n/2)c	$SU(4)\otimes SU(2)\otimes U(1)$	NO	———
	OTHERWISE	$SU(3)\otimes SU(2)\otimes U(1)^2$	YES	A,B,C,W,X
$\mathbb{Z}_2\otimes\mathbb{Z}_2$	$c\in\mathbb{Z}_2, e\in\mathbb{Z}_2$	$SU(4)\otimes SU(2)\otimes SU(2)$	NO	———
$\mathbb{Z}_2\otimes\mathbb{Z}_3$	$c\in\mathbb{Z}_2, e\in\mathbb{Z}_3$	$SU(3)\otimes SU(2)^2\otimes U(1)$	YES	A, (B,C), (W,X)
$\mathbb{Z}_2\otimes\mathbb{Z}_4$	$c\in\mathbb{Z}_2, e\in\mathbb{Z}_4$	$SU(4)\otimes SU(2)\otimes SU(2)$	NO	———
$\mathbb{Z}_2\otimes\mathbb{Z}_{m\geq 5}$	$c\in\mathbb{Z}_2, e\in\mathbb{Z}_m$	$SU(3)\otimes SU(2)^2\otimes U(1)$	YES	A, (B,C), (W,X)
$\mathbb{Z}_{n\geq 3}\otimes\mathbb{Z}_2$	$c\in\mathbb{Z}_n, e\in\mathbb{Z}_2$	$SU(4)\otimes SU(2)\otimes U(1)$	NO	———
$\mathbb{Z}_{n\geq 3}\otimes\mathbb{Z}_3$	$c\in\mathbb{Z}_n, e\in\mathbb{Z}_3$	$SU(3)\otimes SU(2)\otimes U(1)^2$	YES	A,B,C,W,X
$\mathbb{Z}_{n\geq 3}\otimes\mathbb{Z}_4$	$c\in\mathbb{Z}_n, e\in\mathbb{Z}_4$	$SU(4)\otimes SU(2)\otimes U(1)$	NO	———
$\mathbb{Z}_{n\geq 3}\otimes\mathbb{Z}_{m\geq 5}$	$c\in\mathbb{Z}_n, e\in\mathbb{Z}_m$	$SU(3)\otimes SU(2)\otimes U(1)^2$	YES	A,B,C,W,X

TABLE VII: SO(10) BREAKING PATTERNS

NO. HIGGS DOUBLETS FROM 10 (W,X)	NO. EXTRA (CHARGED SINGLETS,RHν) FROM (16+16) (B,C)	NO.EXTRA WEAK DOUBLETS FROM (16+ 16) (A)	log(IMS) (GeV)	$\sin^2\theta$	log(M_X) (GeV)
			10	.237	16.9
2	2	0	12	.228	16.2
			14	.220	15.5
2	2	2	10	.256	15.2
			12	.247	14.6
2	4	0	10	.218	15.3
			12	.210	14.6
2	4	2	10	.237	13.9
			12	.228	13.3
4	2	0	10	.256	15.3
			12	.247	14.6
4	2	2	10	.271	13.9
			12	.261	13.3
4	4	0	10	.237	13.9
			12	.228	13.3
4	4	2	10	.253	12.8
			12	.243	12.3

TABLE VIII: $\sin^2\theta$,M_X FOR SO(10) VIA SU(3)$\otimes$SU(2)$\otimes$SU(2)$\otimes$U(1)

NO. HIGGS DOUBLETS FROM 10 (W,X)	NO. EXTRA CHARGED SINGLETS FROM 16+16 (B)	NO. EXTRA WEAK DOUBLETS FROM 16+16 (A)	$\sin^2\theta$	$\log(M_X)$ (GeV)
2	0	0	.230	16.3
2	0	2	.253	15.0
2	2	0	.215	15.0
2	2	2	.237	13.9
4	0	0	.253	15.0
4	0	2	.272	13.9
4	2	0	.237	13.9
4	2	2	.256	13.0

TABLE IX: $\sin^2\theta, M_X$ FOR SO(10) VIA $SU(3)\otimes SU(2)\otimes U(1)\otimes U(1)$

REFERENCES

CHAPTER IV

1. E. Witten, Nucl. Phys., B268 (1986) 79.
2. M.B. Green and J.H. Schwarz, Phys.Lett., 151B (1985) 21;
   D.J.Gross, J.A. Harvey, E. Martinec and R. Rohm Phys. Rev. Lett. 54 (1985) 502 and Nucl.Phys. B256 (1985) 253.
3. P. Candelas, G.T. Horowitz, A. Strominger and E. Witten, Nucl. Phys. B258 (1985) 46 and "Superstring Phenomenology" in Proceedings of the Argonne symposium on Anomalies, Geometry and Topology, Eds. W.A. Bardeen and A.R. White, World Scientific (1985).
4. S.T. Yau, Proc. Natl. Acad. Sci. 74 (1977) 1798; E. Calabi in "Algebraic Geometry and Topology: A symposium in honour of S.Lefschetz, Princeton University Press (1957)
5. E. Witten. Phys. Lett., 155B (1985) 151.
6. E. Witten, Nucl. Phys. B258 (1985) 75.
7. D. Gromoll and W. Meyer, J. Diff. Geometry, 6 (1971)
8. Y. Hosotani, Phys.Lett. 126B (1983) 309; 129B (1983) 193;
   E. Witten, Phys.Lett. 126B (1984) 351.
9. M. Dine, V. Kaplunovsky, M. Mangano, C. Nappi and N. Seiberg, Nucl.Phys. B259 (1985) 549.
10. J.D. Breit, B. Ovrut and G. Segre, Phys. Lett. 158B (1985) 33. See for a complete review, R. Slansky, Phys. Rep. 79 (1981) 1.
11. S. Cecotti, J.P. Derendinger, S. Ferrara, L. Girardello and M. Roncadelli Phys. Lett. 156B (1985) 318; J.L.Rosner, University of Chicago preprint EFI-85-34(1985); E. Cohen, J. Ellis, K. Enqvist and D.V. Nanopolous, CERN preprints TH-4195, TH-4222(1985); J.P. Derendinger, L.E. Ibañez

and H.P. Nilles, CERN preprint TH-4228(1985)

12. F.del Aguila, G. Blair, M. Daniel and G.G. Ross, CERN preprint (1985).

13. S.K. Donaldson, to appear in J. Diff. Geometry; K. Uhlenbeck and S.T.Yau to appear

14. see, for example, P. Griffiths and J. Harris "Principles of Algebraic Geometry" Wiley Interscience (1978)

15. S.Dimopoulos, S. Raby and F. Wilczek, Phys. Lett., 112B (1982) 133; M.B. Einhorn and D.R.T. Jones, Nucl. Phys., B196 (1982) 475; H. Georgi, H. Quinn and S. Weinberg, Phys. Rev. Lett., 33 (1974) 451; L. Ibañez and G.G.Ross, Phys. Lett., 105B (1981) 439; D.R.T. Jones, Nucl. Phys., B87 (1975) 127.

CHAPTER V

A THREE GENERATION SUPERSTRING MODEL : COMPACTIFICATION AND DISCRETE SYMMETRIES

V.1 INTRODUCTION

The recent enthusiastic interest in superstring theories [1] rests on the belief that a suitable compactification of the ten dimensional theory can be found which reproduces observed phenomenology in four dimensions. Compactification on $M_4 \times K_0$ where M_4 is four dimensional Minkowski space–time and K_0 is a compact six dimensional Ricci–flat Kähler manifold with SU(3) holonomy (Calabi–Yau manifold) ensures an N=1 supersymmetry in four dimensions [2], and hence offers the possibility of addressing the hierarchy problem [3]. Of the known superstring theories, the heterotic theory with $E_8 \otimes E_8{}'$ gauge group [4] appears to be the most phenomenologically viable.

When these theories are compactified on a Calabi–Yau manifold K_0, with the spin connection of K_0 embedded in one of the E_8 factors, an E_6 unifying group emerges (ignoring the $E_8{}'$ factor) with chiral superfields appearing in copies of its fundamental ($\underline{27}$ and $\overline{\underline{27}}$) representations. The compactification manifold K_0 is crucial in determining the other low energy properties of the model. In particular, the Hodge numbers of the manifold determine the number of light chiral supermultiplets transforming as $\underline{27}$ and $\overline{\underline{27}}$ [2], and the vacuum structure of the manifold determines the discrete symmetries of the theory which restrict the allowed Yukawa couplings of the chiral supermultiplets [5]. Indeed, it is in principle possible to determine the Yukawa couplings completely if one knows enough about the cohomology of the manifold [6]. For these reasons it is important to explore the properties of manifolds which may lead to realistic models. It is thought that only finitely many Calabi–Yau manifolds giving a small number of generations exist, and only three are known giving just three generations. We will discuss the determination of the multiplet structure and symmetries following from one of these three generation manifolds.

The number of generations for such a model is one half of the Euler characteristic of K_0 (often a large number), which can naturally be reduced if K_0 admits a freely acting group of discrete symmetries, G. In such a situation we are led to compactification on $K=K_0/G$ (which is also a Calabi–Yau manifold) whose Euler characteristic (and hence number of generations) is less than that of K_0 by a factor of $|G|$ (the number of elements in G) [2].

This procedure can, in addition to giving rise to a reasonable number of generations, set the stage for a novel method of gauge symmetry breaking. The quotient manifold K is multiply connected, and in fact has fundamental group G if K_0 is simply connected (which we henceforth assume). On such non–simply connected manifolds $K=K_0/G$ there will be contours which are not contractable to a point (homotopic to the identity) due to 'holes' in the manifold. As discussed by Hosotani, Witten and others [5,7,8], the possibility exists for symmetry breaking via Wilson loops ("flux breaking")

$$U_g = P \exp \left[-i \int_{\gamma_g} T^a A_m{}^a dy^m \right] \qquad (V.1)$$

where γ_g is a contour representing the abstract element g of the fundamental group $\pi_1(K)=G$, T^a are the E_6 generators, $A_m{}^a$ are the vacuum state E_6 gauge fields, and P denotes path ordering. If γ_g is not homotopic to a point (i.e. $g\neq 1$), then we can have $U_g\neq 1$ even though the vacuum field strength vanishes everywhere. In this way we induce a homomorphism of G into E_6 with image $\bar{G}$.

If we consider a field $\psi(x)$ on K_0, we find it is equivalent to a field $\psi(x)$ on $K=K_0/G$ if $\psi(g(x))=\psi(x)$ for all $g\epsilon G$. However, for a theory with an E_6 gauge group we can generalise this statement to

$$\psi(g(x))=U_g\psi(x) \qquad (V.2)$$

where $U_g \epsilon E_6$ as described above [5]. This is a consistent definition if $g \rightarrow U_g$ is a homomorphic map of the fundamental group $G = \pi_1(K)$ into E_6. Now, if we make a gauge transformation on the fields $\psi \rightarrow V\psi$, consistency with (V.2) demands that $[V,U_g]=0$ for all $g \epsilon G$. So the subgroup of E_6 left unbroken by the choice of the image of G in E_6, $(\bar{G})$, is the subgroup which commutes with all of the U_g. Furthermore, from equation (V.2) we see that fields that remain in the theory in passing from K_0 to K must be singlets under the diagonal sum $G \oplus \bar{G}$.

One difference between superstring-based supergravity models and standard supergravity theories [3] which we shall exploit heavily in this chapter is that, whereas in supergravity it is necessary to postulate some discrete symmetries which prevent, for example, baryon number-violating couplings of quark superfields, in superstring theories choosing a manifold for compactification can *naturally* give rise to discrete symmetries. The particular choice of a manifold automatically determines discrete symmetries that restrict the form of the superpotential. However, to implement these restrictions we require knowledge of the transformation properties of those fields appearing in the superpotential. As emphasised in [2,5,9], this is most readily accomplished by invoking the relation between massless four dimensional matter fields and harmonic forms on K. In particular, chiral superfields arise from the $H_{\bar{\partial}}^{1,1}(K)$ and $H_{\bar{\partial}}^{2,1}(K)$ cohomology groups. A discrete symmetry of K naturally induces transformations on these vector spaces and hence on the massless modes in the theory. As we shall describe in Section 3, these transformation laws are easily derived for fields arising from $H_{\bar{\partial}}^{2,1}(K)$ as these may be identified with deformations of the complex structure of K [10], and hence in certain cases with polynomials in the coordinates on K. Calculation of the transformation properties for fields arising from $H_{\bar{\partial}}^{1,1}(K)$ is straightforward although a bit more involved, making extensive use of the Lefschetz fixed point theorem from algebraic geometry [11].

In this chapter we shall give a detailed description of these techniques illustrating their use in the context of a Calabi–Yau manifold which gives rise to three generations. Three generation Calabi–Yau manifolds are obviously exceedingly attractive from the phenomenological viewpoint and apparently rather difficult to find [12]. Of the three known such manifolds [13], one is simply connected and hence not very useful since it does not admit flux breaking and cannot break at a high scale (via $\underline{27}$ and $\overline{\underline{27}}$ fields acquiring vacuum expectation values) beyond a group containing SU(5) [14]. We concentrate on one of the remaining two. This manifold was initially described by Yau as the first example in the appendix of ref. [13], with a crucial error in the calculation of certain Hodge numbers.

The organisation of this chapter is as follows. Section 2 gives some general background on compactification and makes the association between fields and forms more apparent. Section 3 presents some of the machinery of algebraic geometry useful in analysing R and similar manifolds to derive the values of $h^{1,1}(R)$ and $h^{2,1}(R)$, correcting the result of reference 13, and insofar as possible, to find explicit representatives for superfields in the low energy theory. Section 4 describes some promising patterns of symmetry breaking for models built on this manifold R, and discusses the discrete symmetries arising for a particular vacuum form of R. Some notions from elementary group theory are presented and applied to catalogue the transformation properties of the low energy superfields under these symmetries. Section 5 summarizes our results and briefly indicates the phenomenological prospects of the model.

V.2 COMPACTIFICATION AND CALABI–YAU MANIFOLDS

Effective four dimensional theories arising in the field theory limit of the compactified $E_8 \otimes E_8'$ heterotic superstring are largely characterised by the topological structure of the compact six–dimensional manifold K of the vacuum state. Demanding various phenomenologically desirable properties in four dimensions (such as unbroken N=1 supersymmetry and three or possibly four generations) places stringent restrictions on the structure of the manifold K and thereby limits the more detailed characteristics of viable models such as Yukawa couplings.

Low–energy superfields in these theories arise as zero modes of particular differential operators acting on sections of holomorphic Yang–Mills bundles built over K. These bundles are associated vector bundles of a principle bundle B specified in the compactification. In particular, if K is a Kähler manifold and its tangent bundle has vanishing first Chern class, low energy superfields can be identified with harmonic bundle–valued forms. Thus, we can elucidate many characteristics of the low energy theories using the powerful techniques of algebraic geometry. For more general K we can still manipulate the low energy superfields topologically, although the formalism is less straightforward [15]. For the remainder of this chapter we shall be concerned with the case in which K is a Calabi–Yau manifold (with SU(3) holonomy, rather than a subgroup of SU(3)).

The principle bundle B with Lie structure group $\Gamma \subset E_8 \otimes E_8'$ is a sub–bundle of the $E_8 \otimes E_8'$ principle Yang–Mills bundle over K along which the vacuum field strength is taken to be non zero. B must be chosen so that the condition for anomaly cancellation and unbroken supersymmetry is satisfied [2,4]. With vanishing torsion, this requirement is

$$(1/30)\ \mathrm{Tr}\ F\wedge F = \mathrm{tr}\ R\wedge R \qquad (V.3)$$

where F is the vacuum $E_8 \otimes E_8'$ Yang–Mills field strength and R is the curvature two form of K. Tr denotes a trace in the adjoint representation of $E_8 \otimes E_8'$, and tr a trace in the fundamental of O(6) (i.e. the $\underline{3} \oplus \bar{\underline{3}}$ of SU(3)). This condition can be written as

$$c_2(B) = c_2(T(K)) \tag{V.4}$$

where B is the associated vector bundle of B built from the fundamental representation of Γ (i.e. the second Chern class of B equals that of the tangent bundle).

All known particle interactions are then assumed to be unified in the maximal subgroup H of $E_8 \otimes E_8'$ which commutes with Γ. It is the usual, although not unique, approach to embed Γ entirely in the unprimed E_8 and to assume the maximal subgroup of this E_8 commuting with Γ is the 'unifying group'. In this scenario, the primed E_8' is the "hidden sector" communicating only gravitationally with the world of everyday experience and may be the source of supersymmetry breaking via gaugino condensation [16]. In either case, the coefficient vector bundles for the holomorphic forms representing superfields are the associated vector bundles of B built on the representations of Γ, arising when the adjoint of $E_8 \otimes E_8'$ is broken down under $\Gamma \otimes H$.

A particularly simple example of this apparatus at work is the case in which B is taken to be the natural SU(3) principle bundle over the Calabi–Yau manifold K (the principle bundle T associated with the holonomy of K). In this case, B $\equiv$ T(K) and equation (V.4) is certainly satisfied. This is another way of saying that we equate the spin connection of K with the vacuum Yang–Mills connection of an SU(3) subgroup of the unprimed E_8.(1) This solution has an E_6 unifying group and an E_8' hidden sector. Under $SU(3) \otimes E_6 \subset E_8$,

$$\underline{248} = (\underline{8},\underline{1}) \oplus (\underline{3},\underline{27}) \oplus (\overline{\underline{3}},\overline{\underline{27}}) \oplus (\underline{1},\underline{78}).$$

(we adopt the usual convention in assigning the labels $\underline{27}$ and $\overline{\underline{27}}$ to the fundamental representations of E_6). Thus E_6 singlet superfields arise as harmonic holomorphic forms with coefficients in the associated vector bundle of T built from the adjoint of SU(3), EndT (the bundle of fibrewise endomorphisms of the holomorphic tangent bundle of K), $\underline{27}$'s from harmonic forms with coefficients in the holomorphic tangent bundle T of K (built from the $\underline{3}$ of SU(3) and T); and $\overline{\underline{27}}$'s as harmonic forms with coefficients in the holomorphic cotangent bundle $T^* = \Omega^1$ of K (built from the $\overline{\underline{3}}$ of SU(3) and T). The $\underline{78}$ of low energy gauge superfields does not 'see' the bundle T and descends unaffected into the low energy regime. (A similar analysis, along with a study of all possible flux breaking scenarios, has been done for the cases where G = SU(4) or SU(5) and the unifying group *H* is SO(10) or SU(5) respectively [17]). We shall henceforth restrict ourselves to the case where **B** = T; explicit examples of other solutions have not yet appeared in the literature.

There are additional simplifications for **B** = T over the Calabi–Yau manifold K. That $c_1(T(K)) = 0$ implies the existence of a non–vanishing, covariantly constant (3,0) form (a (p,q) form is the exterior product of p holomorphic one forms, and q antiholomorphic one forms). This form gives a bundle isomorphism $T \cong \Omega^2$ (Ω^p denotes the bundle of holomorphic p–forms). Using the close relationship between the cohomology of bundles over K and the space of harmonic forms taking coefficients in those bundles (given by the appropriate generalisations of the Hodge decomposition theorem [11]) we have the following facts. The chiral superfields transforming as $\underline{27}$'s of E_6 can be represented as linearly independent elements of the cohomology groups $H_{\bar\partial}^{0,p}(K,\Omega^2) \cong H_{\bar\partial}^{2,p}(K)$. For p even these are right handed chiral superfields, for p odd they are left handed. Similarly, the chiral $\overline{\underline{27}}$'s arise as linearly independent elements of $H_{\bar\partial}^{0,p}(K;\Omega^1) \cong H_{\bar\partial}^{1,p}(K)$. Again, even p gives rise to right handed superfields and odd p, left handed superfields. For all Calabi–Yau manifolds K (with holonomy precisely SU(3)),

$$H_{\bar\partial}^{1,0}(K) = H_{\bar\partial}^{2,0}(K) = H_{\bar\partial}^{1,3}(K) = H_{\bar\partial}^{2,3}(K) = 0.$$

Therefore if we denote $\dim_{\mathbb{C}}(H_{\bar\partial}^{p,q}(K)) = h^{p,q}$ (Hodge numbers of K), we have in the 'low' energy theory ($h^{2,2}$.right–handed + $h^{2,1}$.left–handed) $\underline{27}$'s and ($h^{1,2}$.right–handed + $h^{1,1}$.left–handed) $\overline{\underline{27}}$'s. The $h^{2,1}$.left–handed $\underline{27}$'s and $h^{1,2}$.right–handed $\overline{\underline{27}}$'s are related by complex conjugation as are the $h^{1,1}$.left–handed $\overline{\underline{27}}$'s and $h^{2,2}$.right–handed $\underline{27}$'s ($h^{1,1} = h^{2,2}$ by duality). We can thus speak of $h^{2,1}$ $\underline{27}$'s and $h^{1,1}$ $\overline{\underline{27}}$'s in the four dimensional theory.

The other Hodge numbers of a Calabi–Yau manifold K are $h^{0,0}=h^{0,3}=h^{3,3}=h^{3,0}=1$ ($h^{p,q} = h^{q,p}$) which lead to the relation $\chi(K)/2 = h^{1,1}-h^{2,1}$ where $\chi(K)$ is the Euler characteristic of K (since $\chi(K) = \Sigma(-1)^{p+q} h^{p,q}$). The $\underline{27}$'s and $\overline{\underline{27}}$'s contain all observed fermions, and the $\underline{78}$ all observed gauge bosons; there are, however, possibly additional E_6 singlet superfields arising as linearly independent elements of $H_{\bar\partial}^{0,p}(K,\mathrm{End}T)$ [9]. These fields have phenomenological interest – however they shall not be discussed here.

The analysis of the low energy theory using topological techniques can be pushed much further. In fact, Strominger and Witten [6] have shown that, in principle, all the tree level dimension four contributions to the effective action of the 'low' energy theory are calculable as various topological products of the superfield–representing harmonic forms. Their results are surprisingly free from explicit dependence on the unknown Ricci flat metrics of the Calabi–Yau manifolds

K. They do, however, depend on the moduli of the metric and complex structure of K in the vacuum state (i.e. the vacuum expectation values of $h^{1,1}$ real and $h^{2,1}$ complex scalar fields roughly equivalent to the 'size and shape' of the vacuum K) which must presumably be determined by some sort of minimisation in the exact model. A less ambitious, but perhaps more tractable method of gaining information about the couplings of four dimensional superfields is the natural imposition of the discrete symmetries of K (and of its covering space, K_0) on the possible couplings. The problem of the undetermined moduli of K is still present in this approach as we must choose a vacuum 'size and shape' for K to determine which discrete symmetries to impose. This method, due to Witten [5] shall be further discussed and applied to a model of particular interest in Section 4.

V.3 MULTIPLET STRUCTURE AND DISCRETE SYMMETRIES

Calabi–Yau manifolds have been constructed in several general ways: as the zeros of certain homogeneous polynomials in complex projective spaces (or in products of complex projective spaces), as the quotient of certain manifolds by non–freely acting discrete groups with singularities repaired in a well prescribed fashion, and by constructions combining these techniques. We analyse in some detail a three generation manifold R ($\chi(R) = -6$) belonging to the first category. This manifold was originally constructed by Yau [13]. We calculate the Hodge numbers of this manifold and present methods that may be applicable in the analysis of other Calabi–Yau manifolds.

First we make a few remarks about manifolds defined as the zeros of homogeneous polynomials in $\mathbb{C}P^n$ spaces; such manifolds are known as algebraic varieties and have been the subject of intense mathematical research for many decades. $\mathbb{C}P^n$ is the complex Kähler manifold obtained from $\mathbb{C}^{n+1}-\{0\}$ by associating $(z_0,\ldots,z_n)\sim\lambda(z_0,\ldots,z_n)$ (the z_i are coordinates on $\mathbb{C}^{n+1}$) for all $\lambda\in\mathbb{C}-\{0\}$ (i.e. $\mathbb{C}P^n$ is the space of complex 'lines' in $\mathbb{C}^{n+1}$). The coordinates $(z_0,\ldots,z_n)$ are called the homogeneous coordinates of $\mathbb{C}P^n$. Obviously, some of the information in these coordinates is redundant. In fact, we can think of $\mathbb{C}P^n$ as being covered by $n + 1$ coordinate patches (one where $z_0 = 1$, one where $z_1 = 1$, etc.). A simple algebraic variety $M \subset \mathbb{C}P^n$ is the subspace of $\mathbb{C}P^n$ given by the vanishing of $k<n$ homogeneous polynomials $\{F_i(z_0,\ldots,z_n)\,|\,i=1,\ldots,k\}$ [11] where (z_j) are the homogeneous coordinates on $\mathbb{C}P^n$ and the order of the polynomial F_i is d_i. M is a nonsingular $n-k$ complex dimensional manifold if the $\mathbb{C}P^n$ differential form $\omega = dF_1\wedge\ldots\wedge dF_k$ does not vanish anywhere on M. Provided this condition is satisfied, the gross topology(2) of the manifold M is completely determined by the integers $n,d_1,\ldots,d_k$. It can be shown that M always inherits the Kähler condition from

$\mathbb{C}P^n$. Similar algebraic varieties, which we shall call product varieties, can be defined in the product of two projective spaces $\mathbb{C}P^n \times \mathbb{C}P^m$, say, by the vanishing locus of k polynomials in the homogeneous coordinates on $\mathbb{C}P^n$ and $\mathbb{C}P^m$. If $(x_0,\ldots,x_n)$ are $\mathbb{C}P^n$ homogeneous coordinates and $(y_0,\ldots,y_m)$ are $\mathbb{C}P^m$ homogeneous coordinates, these polynomials $\{G_j(x_0,\ldots,x_n,y_0,\ldots,y_m) \mid j=1,\ldots,k\}$ must have the property that for fixed (x_i) G_j is homogeneous of degree q_j in the (y_i) and for fixed (y_i) G_j is homogeneous of degree p_j in the (x_i); such polynomials are said to be of bidegree (p_j,q_j). A condition analogous to that for the simple varieties is imposed to define a nonsingular Kähler manifold in this fashion and the numbers $n,m,p_1,q_1,\ldots,p_k,q_k$ determine the gross topological invariants such as the Hodge numbers of this manifold.

The Chern classes of nonsingular varieties Y defined in a single $\mathbb{C}P^n$ space are easily calculated using the formula for the total Chern class [2,11]

$$c(Y) = \frac{(1+J)^{n+1}}{(1+d_1 J)\ldots(1+d_k J)} \tag{V.5}$$

where J is the Kahler form of $\mathbb{C}P^n$ normalised so that

$$\int_{\mathbb{C}P^1} J = 1 \tag{V.6}$$

over any $\mathbb{C}P^1$ subspace of $\mathbb{C}P^n$. This normalisation also implies that

$$\int_Y J^{n-k} = d_1\ldots d_k \tag{V.7}$$

As the highest Chern class is the Euler class, this formula also determines the Euler characteristic of Y. For product varieties the situation is a bit more complex. We state here only the facts we shall need in dealing with the manifold *R* below. Consider the product $\mathbb{C}P^3 \times \mathbb{C}P^3$. Nonsingular three complex dimensional manifolds V defined by polynomials of bidegree (n,0),(0,m), and (1,1) in this product have vanishing first Chern class and Euler characteristic given by $\chi(V) = -2nm(4-n)(4-m)$ [13].

In computing the Hodge numbers of both simple and product algebraic varieties a very powerful theorem due to Lefschetz comes in handy [11]. In its simplest form this theorem relates certain of the Hodge numbers of an algebraic variety to those of the space (e.g. $\mathbb{CP}^n$) in which it is defined. The two applications important to our analysis are:

(i) Suppose M is the variety defined by a single polynomial in $\mathbb{CP}^n$. Then

$$h^{p,q}(M) \begin{cases} =h^{p,q}(\mathbb{CP}^n) \text{ for } p+q<n-1 \\ \geqslant h^{p,q}(\mathbb{CP}^n) \text{ for } p+q=n-1. \end{cases} \qquad (V.8)$$

(The Hodge numbers of $\mathbb{CP}^n$ are:

$$h^{p,q}(\mathbb{CP}^n) = \begin{cases} \delta^{p,q} \text{ for } p\leqslant n \\ 0 \text{ for } p>n. \end{cases} \qquad (V.9))$$

This theorem can be applied sequentially to varieties defined by more than one polynomial in $\mathbb{CP}^n$. Suppose, for instance, V is defined by two polynomials in $\mathbb{CP}^n$, F and G. Call the variety defined by F only U. Then:

$$h^{p,q}(V) \begin{cases} =h^{p,q}(U) \text{ for } p+q<n-2 \\ \geqslant h^{p,q}(U) \text{ for } p+q=n-2 \end{cases} \qquad (V.10)$$

and

$$h^{p,q}(U) \begin{cases} =h^{p,q}(\mathbb{CP}^n) \text{ for } p+q<n-1 \\ \geqslant h^{p,q}(\mathbb{CP}^n) \text{ for } p+q=n-1 \end{cases} \qquad (V.11)$$

implies

$$h^{p,q}(V) \begin{cases} =h^{p,q}(\mathbb{CP}^n) \text{ for } p+q<n-2 \\ \geqslant h^{p,q}(\mathbb{CP}^n) \text{ for } p+q=n-2. \end{cases} \qquad (V.12)$$

(ii) Suppose X is the product variety defined by polynomials of bidegree (0,a) and (b,0) in $\mathbb{CP}^n \times \mathbb{CP}^n$ (i.e. the product of two varieties in $\mathbb{CP}^n$). Then if the variety Y is defined by an additional polynomial of bidegree (1,1) in X:

$$h^{p,q}(Y) \quad \begin{cases} = h^{p,q}(X) & \text{for } p+q<2n-3 \\ \leq h^{p,q}(X) & \text{for } p+q=2n-3. \end{cases} \tag{V.13}$$

It is often helpful to write the Hodge numbers of a complex manifold in the form of the Hodge diamond:

$$\begin{array}{ccccccccc}
 & & & & |B & & & & \\
 & & & & | & & & & \\
 & & & & h^{0,0} & & & & \\
 & & & h^{1,0} & & h^{0,1} & & & \\
 & & h^{2,0} & & h^{1,1} & & h^{0,2} & & \\
 & & \vdots & & \vdots & & \vdots & & \\
---- & h^{n,0} & h^{n-1,1} & \ldots & \ldots & \ldots & h^{1,n-1} & h^{0,n} & --- A \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & h^{n-2,n} & & h^{n-1,n-1} & & h^{n,n-2} & & \\
 & & & h^{n-1,n} & & h^{n,n-1} & & & \\
 & & & & h^{n,n} & & & & \\
 & & & & | & & & & \\
 & & & & | & & & &
\end{array}$$

Poincare duality asserts $h^{p,q} = h^{n-p,n-q}$ and this implies that the Hodge Diamond is symmetric about the horizontal axis A. If the manifold is Kähler (as are all manifolds in our discussion) $h^{p,q} = h^{q,p}$ and the Hodge diamond is symmetric about the vertical axis B as well.

A final useful fact is the Kunneth formula relating the Hodge numbers of a product of Kähler manifolds to those of the 'factors':

$$h^{p,q}(M \times N) = \sum_{\substack{p=r+m \\ q=s+n}} h^{r,s}(M).h^{m,n}(N) \tag{V.14}$$

We now define the Calabi–Yau manifold R [13]. Consider the product variety R_0 in $\mathbb{CP}^3 \times \mathbb{CP}^3$ defined by homogeneous polynomials C,C',H of bidegree (3,0), (0,3) and (1,1), respectively; we assume that the $\mathbb{CP}^3 \times \mathbb{CP}^3$ differential form $dC \wedge dC' \wedge dH$ does not vanish anywhere on R_0 (i.e. R_0 is a nonsingular three complex dimensional Kähler manifold). From the considerations discussed above $c_1(R_0) = 0$ and $\chi(R_0) = -18$. R_0 is the simply connected covering space of R and we define R by 'modding out' a freely acting Z_3 symmetry G on R_0. If $(x_0,\ldots,x_3),(y_0,\ldots,y_3)$ are the homogeneous coordinates on the two defining $\mathbb{CP}^3$'s, G is generated by the restriction of g: $(x_0,x_1,x_2,x_3) \times (y_0,y_1,y_2,y_3) \rightarrow (x_0,\alpha^2 x_1,\alpha x_2,\alpha x_3) \times (y_0,\alpha y_1,\alpha^2 y_2,\alpha^2 y_3)$, $\alpha = \exp(2\pi i/3)$, to $R_0{}^{(3)}$. Obviously for this G to be a symmetry of R_0, the polynomials C,C' and H must be chosen to be invariant under g. It follows that $c_1(R) = 0$ and $\chi(R) = -18/3 = -6$. R_0 and R are Calabi–Yau manifolds with (number of generations) $= |\chi|/2 = 9$ and 3 respectively.

We now apply the theorems discussed above to calculate the Hodge numbers of the covering space R_0. First consider a cubic U in $\mathbb{CP}^3$. Equations (V.5)–(V.7) imply $\chi(U) = 9$ and the Lefschetz theorem in form (i) above implies $h^{0,0}(U) = 1$; $h^{0,1}(U) = h^{1,0}(U) = 0$. Since $\chi(U) = \Sigma(-1)^{p+q}h^{p,q}(U)$ this implies $h^{0,2}(U) + h^{1,1}(U) + h^{2,0}(U) = 7$. However, the fact that $c_1(U) = J \neq 0$ implies $h^{0,2}(U) = h^{2,0}(U) = 0$. Thus we have a Hodge diamond for the cubic:

$$\text{Hodge diamond}(U) = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 7 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Now we use the Kunneth formula described above to find the Hodge diamond of the product of U and a second $\mathbb{CP}^3$ cubic U':

$$\text{Hodge diamond } (U\times U') = \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 14 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 0 & & 0 & & 51 & & 0 & & 0 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 0 & & 14 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

Now we apply the Lefshetz theorem in form (ii) above to find some of the Hodge numbers of R_0: $h^{0,0}(R_0) = 1$, $h^{1,0}(R_0) = h^{0,1}(R_0) = h^{2,0}(R_0) = h^{0,2}(R_0) = 0$, $h^{1,1}(R_0) = 14$. Since R_0 is a Calabi–Yau manifold $h^{0,3}(R_0) = 1$ and $\chi(R_0)/2 = -18/2 = h^{1,1}(R_0) - h^{2,1}(R_0)$ implies $h^{2,1}(R_0) = 23$. We therefore have all the Hodge numbers for R_0:

$$\text{Hodge diamond } (R_0) = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 14 & & 0 & \\ 1 & & 23 & & 23 & & 1 \\ & 0 & & 14 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Therefore, R_0 is a simply connected 9 generation Calabi–Yau manifold. Compactification on R_0 would lead to models with 23 $\underline{27}$'s and 14 $\overline{\underline{27}}$'s. Before considering a similar analysis for $R = R_0/G$, we need more machinery.

In the late 1950's, Kodaira and Spencer [10] developed an extremely powerful formalism for manipulating the space of complex structure deformations of a compact complex manifold M. This space is the cohomology group $H_{\bar{\partial}}^{0,1}(M,T)$ of holomorphic one forms on M taking coefficients in the holomorphic tangent bundle T of M. For M an algebraic variety, they found that (with certain exceptions) the linearly independent deformations of M can be represented by the linearly independent homogeneous monomials that can be used to define M. An example may be in order here. Suppose V is the variety in $\mathbb{CP}^3$ defined by a homogeneous cubic polynomial. The most general homogeneous cubic polynomial is of the form $F = a_{ijk}x_ix_jx_k$ where the (x_i) are the homogeneous coordinates on $\mathbb{CP}^3$. The a_{ijk} are 19 complex parameters. However 15 of them (the dimension of the projective general linear group PGL on $\mathbb{CP}^3$) may be removed by a suitable linear transformation amongst the (x_i). Thus the space of all cubics in $\mathbb{CP}^3$ is indexed by four complex parameters. This is the dimension of $H_{\bar{\partial}}^{0,1}(V,T)$. The linear transformation amongst the (x_i) may in fact be chosen to put the cubic into the form:

$$F = \sum_{i=0}^{3} x_i^{\,3} + ax_0x_1x_2 + bx_0x_1x_3 + cx_0x_2x_3 + dx_1x_2x_3 \qquad \text{(V.15)}$$

(provided the variety is nonsingular) and the linearly independent elements of $H_{\bar{\partial}}^{0,1}(V,T)$ may be represented by the monomials $x_0x_1x_2$, $x_0x_1x_3$, $x_0x_2x_3$, and $x_1x_2x_3$. Infinitesimal variations of the parameters a,b,c,d represent complex structure deformations. This approach may be extended to product varieties in a straightforward way.

As discussed in Section 2 above, we have for Calabi–Yau manifolds M, a bundle isomorphism $T \cong \Omega^2$ and therefore an isomorphism $H_{\bar{\partial}}^{0,1}(M,T) \cong H_{\bar{\partial}}^{2,1}(M)$. Thus, for Calabi–Yau manifolds defined as algebraic varieties, the holomorphic harmonic forms representing superfields in the $\underline{27}$ of E_6 can be manipulated as the independent monomials in the variety's defining polynomials. We apply these concepts on R_0 in the following fashion. Suppose $(x_0,\dots,x_3)$ are homogeneous coordinates on one $\mathbb{CP}^3$ and $(y_0,\dots,y_3)$ homogeneous coordinates on the other. We choose to use the separate linear (PGL) freedom in the (x_i) and (y_i) to put the cubic (bidegree (0,3) and (3,0)) defining polynomials, C and C', in the standard form (V.15) above. This gives 4 + 4 = 8 independent parameters for R_0 from the cubic sector. Then no additional linear freedom exists with which to eliminate terms in the hyperplane H (bidegree (1,1)) defining polynomial and all 16 − 1 = 15 terms (one is removed by scaling a nonvanishing term to 1). Thus we have 4 + 4 + 16 − 1 = 23 independent deformations of the complex structure of R_0 and thus $h^{2,1}(R_0) = 23$. This confirms our calculation using the Lefschetz theorem. In this way, the most general vacuum state polynomials defining R_0 can be written

$$\begin{aligned}
&\Sigma x_i^3 + a_1x_0x_1x_2 + a_2x_0x_1x_3 + a_3x_0x_2x_3 + a_4x_1x_2x_3 = 0 \\
&\Sigma y_i^3 + b_1y_0y_1y_2 + b_2y_0y_1y_3 + b_3y_0y_2y_3 + b_4y_1y_2y_3 = 0 \qquad \text{(V.16)} \\
&\Sigma c_{ij}x_iy_j = 0 \quad (c_{00} = 1)
\end{aligned}$$

Also, the 23 independent harmonic (2,1) forms on R_0 can be represented (in a similar fashion to the example above) by the 23 monomials

$$\begin{array}{cccc}
x_ix_jx_k & y_iy_jy_k & x_iy_j & \\
i,j,k\epsilon\{0,1,2,3\} & i,j,k\epsilon\{0,1,2,3\} & \text{except } i=j=0 & \text{(V.17)} \\
i,j,k \textit{ all } \text{distinct} & i,j,k \textit{ all } \text{distinct} & &
\end{array}$$

(We have assumed here that the term x_0y_0 is present in the defining polynomial H as this shall be necessary for nonsingularity when we move to $R = R_0/G$). Note that this analysis implies both that the complex structure of a vacuum state R_0 manifold depends on 23 independent parameters and that there are 23 independent infinitesimal deformations of the complex structure of any such vacuum state.

We are more interested in analysing the three generation quotient manifold R

$= R_0/G$. How can we determine the Hodge diamond of R and represent its harmonic forms? The independent (1,1) and (2,1) forms on R_0 transform in certain generally reducible representations of the freely acting group G. It is the independent harmonic forms on R_0 transforming *trivially* under G which descend to harmonic forms on R. The (1,1) and (2,1) forms transforming in other representations of G will become important when flux breaking is implemented, in which case we shall be primarily interested, not in G singlet, but in diagonal $G \oplus \bar{G}$ singlet superfields (where $\bar{G}$ is the image of G in E_6 given by the Wilson loop homomorphism). If, for instance, superfields transforming in a certain representation of the low energy gauge group $H \subset E_6$ transform nontrivially under $\bar{G}$, they will be topologically represented by forms on the covering space R_0 transforming nontrivially under G.

For algebraic variety Calabi–Yau manifolds, we have explicit representations for the harmonic (2,1) forms by their polynomial counterparts; we can thus categorise the (2,1) forms directly according to their G transformation properties. We apply this procedure to R_0 using the monomials (V.17) found above, and present the result in Table 1. In this table, G(1), G(α), and G(α^2) denote the three representations of $G \cong Z_3$ in which the generator g of G is represented by multiplication by $1,\alpha,\alpha^2$ respectively. Thus the 23 harmonic (2,1) forms transform as follows under G:

9 are G invariant; G(1)

7 go to α.themselves under G; G(α)

7 go to α^2.themselves under G; G(α^2).

This implies that $h^{2,1}(R) = 9$, and consequently, since $\chi(R)/2 = -3 = h^{1,1}(R) - h^{2,1}(R)$ (as R is a Calabi–Yau manifold) we have $h^{1,1}(R) = 6$ and

$$
\text{Hodge diamond(R)} = \begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 6 & & 0 & \\
1 & & 9 & & 9 & & 1 \\
 & 0 & & 6 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Note that the most general vacuum state defining polynomials for R are given by the equations (V.16) with $a_3 = a_4 = b_3 = b_4 = 0$ and $c_{ij} = 0$ except for $c_{00} = 1$, c_{11}, c_{22}, c_{33}, c_{23} and c_{32} (i.e. the complex structure of the vacuum state of R is determined by 9 complex numbers).

We can reach the same conclusion by considering indirectly the action of G on the harmonic (1,1) forms of R_0. A result that is very useful to this end is another theorem due to Lefschetz, the fixed point theorem [11]. This theorem shall also be used extensively in the consideration of discrete symmetries in Section 4. Suppose f is an isometry of the n−complex dimensional Kähler manifold M, we define the Lefschetz number of the map f as follows:

$$L(f) = \sum_{\substack{p=0 \\ q=0}}^{n} (-1)^{p+q} \mathrm{Tr}_{H^{p,q}(M)}(f) \qquad \text{(V.18)}$$

where $\mathrm{Tr}_{H^{p,q}(M)}(f)$ denotes the trace in the vector space $H_{\bar\partial}^{p,q}(M)$ of the action of f on (p,q) forms. Note that

$$L(\text{identity}) = \chi(M). \qquad \text{(V.19)}$$

The Lefschetz fixed point theorem asserts that

$$L(f) = \sum_{\{\mu_j\}} \chi\,(\mu_j) \qquad \text{(V.20)}$$

where the μ_j are the submanifolds of M left fixed by f and $\chi(\mu_j)$ is the Euler characteristic of μ_j. To apply the result to the (1,1) forms of R_0, we note in the derivation of $h^{1,1}(R_0)=14$ above, that 7 of these forms originate totally in one cubic, and 7 in the other. Therefore, if we denote the two $\mathbb{CP}^3$ cubics used in defining R_0 by $\sum_1$ and $\sum_2$, $H_{\bar\partial}{}^{1,1}(R_0) \cong H_{\bar\partial}{}^{1,1}(\Sigma_1) \oplus H_{\bar\partial}{}^{1,1}(\sum_2)$. It is the assertion of the Lefschetz hyperplane theorem that these forms are 'independent' of the hyperplane (bidegree (1,1)) defining polynomial H. Thus we can consider the action of extensions $\tilde{g}$ of the generator of G on the (1,1) forms of the $\mathbb{CP}^3$ cubic simple varieties seperately, and it is to these extensions that we apply the Lefschetz fixed point theorem. Suppose we denote by $\tilde{g}_1$ and $\tilde{g}_2$ the extensions of g to $\sum_1$ and $\sum_2$ respectively, given by

$$\tilde{g}_1 : (x_0, x_1, x_2, x_3) \to (x_0, \alpha^2 x_1, \alpha x_2, \alpha x_3)$$

$$\tilde{g}_2 : (y_0, y_1, y_2, y_3) \to (y_0, \alpha y_1, \alpha^2 y_2, \alpha^2 y_3) . \tag{V.21}$$

$$\left[\alpha = \exp\left\{\frac{2\pi i}{3}\right\}\right]$$

Using the Hodge diamond for such cubics derived above, the fixed point theorem asserts

$$\begin{aligned} L(\tilde{g}_1) &= \chi(\text{fixed pts. of } \tilde{g}_1) = 2 + \mathrm{Tr}_{H^{1,1}}(\textstyle\sum_1)(\tilde{g}_1) \\ L(\tilde{g}_2) &= \chi(\text{fixed pts. of } \tilde{g}_2) = 2 + \mathrm{Tr}_{H^{1,1}}(\textstyle\sum_2)(\tilde{g}_2) \end{aligned} \tag{V.22}$$

(the (0,0) and (2,2) forms of $\sum_1$ and $\sum_2$ must be invariant under $\tilde{g}_{1,2}$ since a constant function certainly <u>is</u> invariant).

From the argument above,

$$\mathrm{Tr}_{H^{1,1}(R_0)}(g) = \mathrm{Tr}_{H^{1,1}(\Sigma_1)}(\tilde{g}_1) + \mathrm{Tr}_{H^{1,1}(\Sigma_2)}(\tilde{g}_2) =$$

$$(\chi(\text{fixed points of } \tilde{g}_1 \text{ on } \Sigma_1) +$$

$$\chi(\text{fixed points of } \tilde{g}_2 \text{ on } \Sigma_2) - 4). \qquad \text{(V.23)}$$

It is easy to show that $\tilde{g}_1, \tilde{g}_2$ have three fixed points on the most general g invariant cubics Σ_1, Σ_2, respectively. Then (V.20) asserts that $2 = \mathrm{Tr}_{H^{1,1}(R_0)}(g)$. Because we know the action of g on $H^{1,1}(R_0)$ can be taken to be a 14×14 diagonal matrix with all entries $1, \alpha$ or α^2 (g generates a $\mathbb{Z}_3$ symmetry of R_0), we know that this implies that the 14 (1,1) forms break down into:

6 invariant under g; $G(1)$

4 go to α.themselves under g; $G(\alpha)$

4 go to α^2.themselves under g; $G(\alpha^2)$.

This confirms $h^{1,1}(R)=6$ as found via deformation theory above.

V.4 SYMMETRY BREAKING PATTERNS AND DISCRETE SYMMETRIES

In this section we will lay the groundwork for analysing the phenomenology of compactification of heterotic superstring theories on the manifold R.

On a multiply connected manifold M with fundamental group G≠identity, the flux trapping mechanism offers a natural means for breaking the E_6 gauge group to a phenomenologically viable low energy gauge group [2,5]. As outlined in Section 1, this is accomplished by homomorphically embedding the fundamental group G onto $\bar{G} \subset E_6$; the resulting unbroken gauge group is then the largest subgroup of E_6 which commutes with $\bar{G}$ [5]. In our case, the fundamental group of R is $G \cong \mathbb{Z}_3$ which has two homomorphic embeddings in E_6 resulting in an $(SU(3))^3$ unbroken gauge group and an extended colour group.

Further analysis shows the possibility of using an intermediate scale Higgs mechanism (see below) to break the former gauge group to the standard model, so we shall concentrate on this route [14].

The embedding of $\mathbb{Z}_3$ in E_6 resulting in $(SU(3))^3$ unbroken gauge symmetry is given by

$$g \epsilon \mathbb{Z}_3 \to U_g = \quad (I) \otimes \begin{bmatrix} \alpha & & \\ & \alpha & \\ & & \alpha \end{bmatrix} \otimes \begin{bmatrix} \alpha & & \\ & \alpha & \\ & & \alpha \end{bmatrix} \qquad (V.24)$$

where α is a non-trivial cube root of unity and we are embedding in the $SU(3)_C \otimes SU(3)_L \otimes SU(3)_R$ maximal subgroup of E_6. Now, the $\underline{27}$ of E_6 decomposes under $SU(3)_C \otimes SU(3)_L \otimes SU(3)_R$ as

$$\underline{27} = \quad (1,3,\bar{3}) + (\bar{3},1,\bar{3}) + (3,3,1).$$

Thus, for example, leptons lie in $(1,3,\bar{3})$ and hence transform under $\bar{G}$ as $\alpha \cdot \bar{\alpha} =$

1, and are therefore invariant; similarly, left handed anti-quarks Q transform under $\bar{G}$ as $Q \to \bar{\alpha}Q = \alpha^2 Q$ and left handed quarks q as $q \to \alpha q$. Using this information we can determine the light fields which remain in the theory after flux breaking. In addition to the three full generations of $\underline{27}$'s of E_6 which are guaranteed by the index theorem ($|\chi|/2 = |n_L - n_R| = 3$), any fields belonging to the $\overline{\underline{27}}$ sector which are $G \oplus \bar{G}$ singlets remain light as well. This implies that an additional corresponding component of a $\underline{27}$ remains light, in order to preserve the relation dictated by the index theorem.

In the previous section, we catalogued the G transformation properties of the $\underline{27}$ and $\overline{\underline{27}}$ sectors of the theory, when formulated on the covering space R_0 of R. In passing from R_0 to R, we thus see that the $G \oplus \bar{G}$ invariance requirement yields an additional six pairs of leptons (six from each of the $\underline{27}$ and $\overline{\underline{27}}$ sectors), four pairs of left handed anti-quarks, and four pairs of left handed quarks (beyond three full generations).

Extra light leptons are quite useful, and in fact necessary, in further reducing the $(SU(3))^3$ gauge group. Identifying elements in the $\underline{27}$ by their transformation properties under SO(10) and SU(5), we note that if the (1,1) and (16,1) scalar lepton components from each of the two $\underline{27}$'s acquires vacuum expectation values (VEVs) then $(SU(3))^3$ breaks according to

$$SU(3) \otimes SU(3) \otimes SU(3) \xrightarrow{\langle(1,1)\rangle} SU(3) \otimes SU(2) \otimes SU(2) \otimes U(1)$$

$$\xrightarrow{\langle 16,1\rangle} SU(3) \otimes SU(2) \otimes U(1) \ .$$

In order to yield an acceptable value of $\sin^2\theta_W$, the scale for these VEVs must be $\gtrsim O(10^{14}$ GeV) [14]. We shall shortly address whether this is possible.

The extra quarks, however, are unwanted, as they may lead to rapid proton decay and unrealistic values for $\sin^2\theta_W$. We thus seek to make them very massive, hopefully utilizing the intermediate scale VEVs required for phenomenologically reasonable gauge symmetry breaking.

In approaching these tasks, we are thus led to a number of questions:

1) Are there sufficiently flat directions in the superpotential to yield VEVs of order 10^{14} GeV?

2) Are the Yukawa couplings allowed by discrete symmetries of R (and its covering space) sufficient to give large masses to unwanted fields?

and, more generally

3) Do the allowed Yukawa couplings give rise to realistic mass matrices for the low lying fermions?

4) Are the allowed Yukawa couplings sufficiently restricted to yield a reasonable proton lifetime?

The answers to these questions depend crucially on the transformation properties of the fields in the theory under discrete symmetries possessed by the vacuum state of the manifold R. We shall now derive these transformation laws for a particularly symmetric vacuum.

A simple way to find the symmetries of R is to ascertain which symmetries on the simply connected covering space R_0 survive the passage to R. As discussed in [5], the requirement for a discrete symmetry d of R_0 to be a symmetry of R is that d belong to the group of automorphisms of G. That is, for any $g \epsilon G$, there is a $g' \epsilon G$ such that

$$dgd^{-1}=g' . \qquad (V.25)$$

To implement this requirement it is helpful to express the discrete symmetries as matrices acting on the two $\mathbb{CP}^3$ spaces. The generator $g \epsilon G$ takes the form

$$\left[\begin{array}{cccc|cccc} 1 & & & & & & & \\ & \alpha^2 & & & & & & \\ & & \alpha & & & & & \\ & & & \alpha & & & & \\ \hline & & & & 1 & & & \\ & & & & & \alpha & & \\ & & & & & & \alpha^2 & \\ & & & & & & & \alpha^2 \end{array}\right] \tag{V.26}$$

Now assume that an 8×8 matrix acting on the homogeneous coordinates of the two defining $\mathbb{CP}^3$ spaces

$$U = \begin{bmatrix} U_1 & U_3 \\ U_4 & U_2 \end{bmatrix} \tag{V.27}$$

(with each U_i being a 4×4 matrix) satisfies (V.25). A little algebra shows that the most general form which any such non-singular U may take has

$$U_1 = \begin{bmatrix} a & & & \\ & b & & \\ & & c & e \\ & & f & d \end{bmatrix}$$

$$U_2 = \begin{bmatrix} a' & & & \\ & b' & & \\ & & c' & e' \\ & & f' & d' \end{bmatrix} \tag{V.28}$$

and $U_3 = U_4 = 0$.

In fact, these matrices commute with $g \epsilon G$.

To be of interest, the matrix U must belong to the discrete symmetry group of R_0, which depends upon a choice of the independent moduli that determine the complex structure of the manifold. As we at present have no dynamical means to determine this vacuum configuration, we simply posit a particular form. Other choices amount to picking different vacuum states for the theory. We assume the R_0 defining polynomials take the form

$$\begin{aligned} \Sigma x_i{}^3 &= 0 \\ \Sigma y_i{}^3 &= 0 \qquad\qquad (\text{V.29}) \\ \Sigma x_i y_i &= 0. \end{aligned}$$

This highly symmetric vacuum state leads to a rich set of discrete symmetries. It is a group of order 1296, generated by

i) diagonal permutations of the coordinates of each $\mathbb{CP}^3$ space (i.e. general coordinate permutations acting identically on each space),

ii) Diagonal scaling of the coordinates by cube roots of unity (i.e. $x_i \to \alpha^{n_i} x_i$, $y_i \to \alpha^{-n_i} y_i$ with $\alpha^3 = 1$ and $\Sigma n_i = 0$ (mod3)),

iii) Swapping of x and y coordinates (i.e. $x_i \to y_i$, $y_i \to x_i$).

Which elements in this group are of the form given in equation (V.28)? Clearly, matrix elements in U_1 may be any of the three cube roots of unity with e = f = 0 or c = d = 0. The corresponding primed elements in U_2 given by their inverses. This yields a fifty-four element non-abelian discrete symmetry group D which we take to be generated by:

$$A = \left[\begin{array}{cccc|cccc} \alpha &&&&&&& \\ & \alpha &&&&&& \\ && 1 &&&&& \\ &&& 1 &&&& \\ \hline &&&& \alpha^2 &&& \\ &&&&& \alpha^2 && \\ &&&&&& 1 & \\ &&&&&&& 1 \end{array}\right]$$

$$B = \left[\begin{array}{cccc|cccc} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 0 & 1 & & & & \\ & & 1 & 0 & & & & \\ \hline & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 0 & 1 \\ & & & & & & 1 & 0 \end{array}\right] \qquad \text{(V.30)}$$

$$C = \left[\begin{array}{cccc|cccc} 1 & & & & & & & \\ & 1 & & & & & & \\ & & \alpha & & & & & \\ & & & 1 & & & & \\ \hline & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & \alpha^2 & \\ & & & & & & & 1 \end{array}\right] .$$

The symmetries of R_0 which do not survive (often referred to as 'pseudosymmetries' [5]) consist of diagonal cyclic permutations of the coordinates and the x,y swap operation, multiplied by any symmetry which does survive. These form a set of order 1134 which, with G and D, reconstitutes the original group of order 1296.

As discussed in [5], flux breaking may further reduce the group of compatible discrete symmetries. In general, an element d of a discrete symmetry group of a manifold M with fundamental group $\pi_1(M)$ survives flux breaking if there exists a homomorphic map of d to an E_6 element V_d such that

$$V_d^{-1} U_h V_d = U_h' . \qquad \text{(V.31)}$$

In this expression U_h is the E_6 image of $h \epsilon \pi_1(M)$ under flux breaking, and $d^{-1}hd = h'$. In our case, as we have shown, any d satisfying (V.25) commutes with the fundamental group G of R. Hence (V.31) is satisfied for all $d \epsilon D$ by the trivial homomorphism. We find, therefore, that the full group D survives flux breaking.

With the discrete symmetry group of R in hand, we now proceed to determine the transformation properties of the matter fields in the theory.

Determining the transformation properties of the $\underline{27}$ sector is straightforward [5]; as noted earlier, elements in $H_{\bar{\partial}}^{2,1}(R)$ may be identified with deformations of the complex structure of R and hence with polynomials in the coordinates of $\mathbb{CP}^3 \times \mathbb{CP}^3$. Under A,B and C, elements in $H_{\bar{\partial}}^{2,1}(R)$ transform like their corresponding monomial representations. We thus immediately write down the $\underline{27}$ transformation properties (Table 2).

Fields arising in the $\overline{27}$ sector (arising from $H_{\bar{\partial}}^{1,1}(R)$) are not readily identified with simple coordinate representations; to compute their transformation properties we therefore use more indirect means.

For completeness, we first briefly describe some notions from finite group theory [19] which will be necessary in our calculation.

By a matrix representation R(H) of a finite group H, we mean a homomorphic mapping of H into some matrix group. The character of $h \in H$ in the representation R(h) is the trace of the image of h, TrR(h). We shall often speak of the character of a representation R(H), by which we mean the collection of all of the characters TrR(h) for all $h \in H$. Notice that characters are constant on conjugacy classes.

$$\mathrm{TrR}(h'^{-1} h h') = \mathrm{TrR}(h) \text{ for all } h, h' \in H.$$

It is a well known result from finite group theory that the number of irreducible representations of H is equal to the number of conjugacy classes of H. Furthermore, the characters of these representations form an orthonormal and complete set of functions for the space of characters of arbitrary representations of H. By this we mean that if χ^{λ} denotes the character for an irreducible representation of a group H, and χ_k^{λ} is the value of the character on the conjugacy class H_k of H [19]:

$$\sum \left[\frac{|H_k|}{|H|}\right]^{\frac{1}{2}} \left[\chi_k^{\lambda}\right]^* \left[\frac{|H_k|}{|H|}\right]^{\frac{1}{2}} \left[\chi_k^{\mu}\right] = \delta^{\lambda\mu} \qquad \text{(V.32)}$$

where $|H|$ is the number of elements in H, and $|H_k|$ is the number of elements in the conjugacy class H_k. Thus, in order to express an arbitrary representation R(H) in terms of a direct sum of irreducible representations, we must simply express its characters as a sum of characters of H's irreducible representations. It is therefore necessary to have the characters of all of the irreducible representations of H; these are usually presented in a 'character table'.

In practice, for groups with a small number of elements, one may construct the character table by finding a few obvious representations and then filling in the remainder guided by equation (V.32) (orthonormality between 'rows') together with

$$\sum_{\lambda} \left[\frac{|H_k|}{|H|}\right]^{\frac{1}{2}} \left[\chi_k^{(\lambda)}\right]^* \left[\frac{|H_k|}{|H|}\right]^{\frac{1}{2}} \left[\chi_{\ell}^{\lambda}\right] = \delta_{k\ell} \qquad \text{(V.33)}$$

(orthonormality between 'columns') and

$$\sum_{\lambda} |n_{\lambda}|^2 = |H| \qquad \text{(V.34)}$$

where n_{λ} is the dimension of the λth irreducible representation. This formalism shall now be used extensively in dealing with transformation properties of the $\overline{\underline{27}}$ sector of the theory.

Let us take a closer look at the group generated by A,B and C. B and C generate an 18 element non-abelian group $\mathcal{V}$ (consisting of all 2 × 2 diagonal or antidiagonal matrices with entries taken from the three cube roots of unity)[4] while A generates a Z_3 group which commutes with $\mho$. Hence A,B and C generate $D = Z_3 \otimes \mho$. We will denote this Z_3 by W.

Following the methodology above, we can determine how the $\overline{\underline{27}}$ sector transforms under the discrete group if we

1) compute the character table of D, and
2) compute the character of the action of D on the vector space $H_{\bar{\partial}}^{1,1}(R)$.

To accomplish 1) we recall that the irreducible representations of a product group may be obtained from those for each factor. We thus treat W and $\mho$ separately.

Since W is abelian, its irreducible representations are all one dimensional. It has three conjugacy classes and its character table is then trivially computed to be

	I	A	A^2
S1	1	1	1
S2	1	α	α^2
S3	1	α^2	α

(V.35)

where $\alpha = \exp(2\pi i/3)$. The required orthogonality properties are manifest.

To analyse $\mho$, we first note that it admits nine conjugacy classes

(1) $\{1\}$

(2) $\{CBCB\}$

(3) $\{CB^2CB^2\}$

(4) $\{B\ ,\ CBC\}$

(5) $\{B^2\ ,\ CB^2C\}$

(6) $\{CB^2CB\ ,\ CBCB^2\}$

(7) $\{C\ ,\ BCB^2\ ,\ B^2CB\}$

(8) $\{BCB\ ,\ B^2C\ ,\ CB^2\}$

(9) $\{BC\ ,\ CB\ ,\ B^2CB^2\}$

Applying equation (V.34), we see that

$$\sum_{i=1}^{9} n_i{}^2=18 \tag{V.36}$$

only has the solution $n_1 = n_2 = \ldots = n_6 = 1$, $n_7 = n_8 = n_9 = 2$. The six one dimensional representations are seen to be the product of the three one dimensional representations associated with the $\mathbb{Z}_3$ subgroup generated by B and the two one dimensional representations from the subgroup generated by C. One of the three two dimensional representations is the defining representation generated by the 2 × 2 matrices $B = \begin{pmatrix}\alpha & \\ & 1\end{pmatrix}$ and $C = \begin{pmatrix} & 1\\ 1 & \end{pmatrix}$. The other two are easily checked to be generated by

$$B = \left\{ \begin{bmatrix}\alpha^2 & \\ & 1\end{bmatrix},\ C = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\right\} \text{ and } \left\{B = \begin{bmatrix}\alpha & \\ & \alpha^2\end{bmatrix},\ C = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\right\} . \tag{V.37}$$

These representations give rise to the character table (Table 3).

One may readily check that the required orthogonality relations (V.32) and (V.33) hold for this table. We have thus completed 1). In order to use this information, we need to determine the character of D acting on $H_{\mathfrak{J}}{}^{1,1}(R)$. To accomplish this, we again make use of the theorems of Lefschetz described earlier.

The Lefshetz Fixed Point Theorem, as discussed in Section 3, asserts that on a complex manifold of dimension n

$$\sum_{s,t=0}^{n} (-1)^{s+t}\ \chi_{s,t}(f) = \sum\ \chi(\mu_j) \tag{V.38}$$

where f is any map of M to itself, $\chi_{s,t}(f) = Tr_{H^{s,t}(M)}(f)$ (i.e. the character of f acting on $H_{\bar\partial}^{s,t}(M)$, μ_j are the fixed point sets of f, and $\chi(\mu_j)$ is the Euler characteristic of μ_j. The Lefschetz Hyperplane Theorem allows us to effectively ignore the hyperplane connecting the two $\mathbb{CP}^3$ spaces, and hence to compute the action of D on $H_{\bar\partial}^{1,1}(\Sigma_1) \oplus H_{\bar\partial}^{1,1}(\Sigma_2)$.

By recalling the Hodge diamond for Σ_1 and Σ_2, we see that (V.38) on Σ_i becomes

$$2 + \chi_{1,1}(f)_{\Sigma_i} = \sum_{\{\mu_j\} \text{ in } \Sigma_i} \chi(\mu_j) \qquad \text{(V.39)}$$

since $H^{0,0}$ and $H^{2,2}$ are both invariant. Thus, when acting on $H_{\bar\partial}^{1,1}(\Sigma_1) \oplus H_{\bar\partial}^{1,1}(\Sigma_2)$ we have

$$\chi_{1,1}(f) = \chi_{1,1}(f)_{\Sigma_1} + \chi_{1,1}(f)_{\Sigma_2} = \sum_{\substack{\{\mu_j\} \\ \text{in } \Sigma_1}} \chi(\mu_j) + \sum_{\substack{\{\mu_j\} \\ \text{in } \Sigma_1}} \chi(\mu_j) - 4 \quad . \qquad \text{(V.40)}$$

By choosing $f \epsilon D$, this gives us an explicit algorithm for computing the character of D acting on the $\overline{\underline{27}}$ sector of our theory.

For our purposes, it is more useful to compte the character of D on each of the subspaces of the $\overline{\underline{27}}$ sector which we identify with leptons, left handed quarks and anti-quarks, i.e. the harmonic (1,1) forms which are invariant under g, those which scale by α under g, and those which scale like $\bar\alpha = \alpha^2$ under g, where g is the generator of $G = \mathbb{Z}_3$. This is readily accomplished in the following manner [18].

Consider the operator

$$P_k = (1/3) \sum_{m=0}^{2} \alpha^{-km} g^m \qquad (V.41)$$

which projects the full $\overline{\underline{27}}$ sector onto the leptons, left handed quarks, and left handed anti-quarks for k = 0,1,2 respectively. The action of $f \epsilon \underset{\cdot}{D}$ on the subspace labelled by k is then

$$\chi^k{}_{1,1}(f) = (1/3) \sum_{m=0} \alpha^{-km} \chi_{1,1}(g^m f) \qquad (V.42)$$

Thus by calculating the relevant $\chi^{1,1}(g^j f$ and using (V.42) we can compute the character of D acting on leptons and quarks in the $\overline{\underline{27}}$ sector [18]. An example should help clarify this procedure:

Consider $T = (\alpha_1) \epsilon D$. What are the fixed points of T? (Again, we are restricting ourselves to the first $\mathbb{CP}^3$ factor)

$$T: [x_0, x_1, x_2, x_3] \to [x_0, x_1, \alpha x_2, x_3]$$

The fixed points are thus $\{x_2 = 0;\ x_0{}^3 + x_1{}^3 + x_3{}^3 = 0\}$. The Euler characteristic of this set is easily calculated to be zero, so $\chi_{1,1}(T)_{\Sigma_1} = -2$ (this can be seen from equations (V.5) and (V.7) after noting that the fixed point set is a cubic in $\mathbb{CP}^2$). For gT the fixed point set is $\{x_0 = x_3 = 0;\ x_1{}^3 + x_2{}^3 = 0\}$. The Euler characteristic of this fixed point set (which consists of three discrete points) is $+3$. A similar calculation for g^2T yields the same result, and hence $\chi_{1,1}(gT)_{\Sigma_1} = \chi_{1,1}(g^2T)_{\Sigma_1} = 1$. Using these results in (V.42) gives

$$\begin{aligned}
\chi^0{}_{1,1}(T)_{\Sigma_1} &= (-2+1+1)/3 = 0 \\
\chi^1{}_{1,1}(T)_{\Sigma_1} &= (-2+\alpha^2+\alpha)/3 = -1 \qquad (V.43) \\
\chi^2{}_{1,1}(T)_{\Sigma_1} &= (-2+\alpha^2+\alpha)/3 = -1.
\end{aligned}$$

Incorporating the second $\mathbb{C}P^3$ space as well leads to $\chi^0{}_{1,1}(T) = 0$, $\chi^1{}_{1,1}(T) = -2$, $\chi^2{}_{1,1}(T) = -2$ over the entire manifold R_0. We use this procedure to calculate the characters for the nine conjugacy classes of $\mathbb{V}$, and three conjugacy classes of W. The results are given below (see Table 3). For the W subgroup, this analysis reveals

$$\begin{array}{c|ccc} & I & p & p^2 \\ \hline \chi_{1,1}{}^0 & 6 & 0 & 0 \\ \chi_{1,1}{}^1 & 4 & 4 & 4 \\ \chi_{1,1}{}^2 & 4 & 4 & 4 \end{array} \qquad (V.44)$$

Now that we have all of the characters of D acting on the lepton, left handed quark, and right handed quark subspaces arising from the $\overline{\underline{27}}$ sectors, we use the representation reduction methodology outlined above to express these as the sum of irreducible representations.

For $\mathbb{V}$ we find:

$\overline{\underline{27}}$ leptons transform as	2.R1 + 2.R5 + 2.R6
$\overline{\underline{27}}$ quarks transform as	2.R9
$\overline{\underline{27}}$ antiquarks transform as	2.R9.

For W we find:

$\overline{\underline{27}}$ leptons transform as	2.S1 + 2.S2 + 2.S3
$\overline{\underline{27}}$ quarks transform as	4.S1
$\overline{\underline{27}}$ antiquarks transform as	4.S1.

Finally, inspection of the characters of AB and AB^2 (for the leptons, in particular) allows us to correlate W and $\mathbb{V}$ transformation properties and hence arrive at Table 4 of transformation properties for the full $\overline{\underline{27}}$ sector.

These tables provide the necessary information for detailed examination of phenomenology based upon this compactification (once again, we point out that such models may be further restricted by the topological considerations of Strominger and Witten [6]). Preliminary investigations [20] indicate that the discrete symmetries of the vacuum configuration analysed above ensure flat directions in the superpotential which allow vacuum expectation values of greater than 10^{14} GeV. These VEVs, in addition to breaking the gauge group to the standard model, have the potential to give large masses to all unwanted fields, essentially yielding the supersymmetric standard model. Furthermore, the form of the discrete symmetries also restricts the Yukawa couplings giving rise to light quark and fermion masses. Although more work needs to be done, the general form of these mass matrices appears to be consistent with experiment.

Pseudosymmetries, as shown in [5], place further restrictions on Yukawa couplings. As the full group of symmetries (honest symmetries and pseudosymmetries) is quite large, the calculation of transformation properties is a bit unwieldy. One approach is to choose a different form for R_0 which preserves the honest symmetries of R and reduces the number of pseudosymmetries (e.g. in (V.16) take $c_{22} = c_{33} \neq c_{11} \neq c_{0,0}$). Although making the problem tractable, this scheme is rather *ad hoc*. Further investigation is certainly necessary.

V SUMMARY AND CONCLUSIONS

In this chapter we have presented the first detailed analysis of a three generation superstring model. It is based on a Calabi–Yau manifold R which was initially presented in ref. [13]. We have presented some results from algebraic geometry and finite group theory, and have illustrated their utility in calculating the Hodge numbers of R and discrete symmetry transformation properties of differentiable forms on this manifold. For a particular choice of vacuum moduli and flux breaking, we have catalogued the discrete symmetry transformation properties of the low energy superfields. Preliminary study indicates that the resulting model may be phenomenologically viable, and certainly warrants more detailed investigation. In any event, the techniques illustrated in our analysis of this (potentially pertinent) compactification, should carry over to solutions ultimately generated by dynamical means. Hopefully, then, this work will be a useful prototype for extracting low energy phenomenology from the superstring.

FOOTNOTES

(1) This scenario is precisely that found to be necessary by Candelas et. al. [2] for the preservation of N = 1 supersymmetry with the one somewhat controversial assumption that the torsion field strength H_{pqr} vanishes in the vacuum.

(2) By 'gross' topology here we mean the topological invariants of M such as the Hodge numbers $h^{p,q} = \dim H_{\bar\partial}^{p,q}(M)$. The moduli of M (its Kähler metric and complex structure) which can be thought of as the 'size and shape' of M are not uniquely determined.

(3) We choose a different (but equivalent) freely acting $\mathbb{Z}_3$ from that used in refernce 13. This choice makes the calculations of this section and of Section 4 more straightforward.

(4) We omit the 2 × 2 identity matrix which would be in the upper left hand corner if we wrote these as 4 × 4 matrices, and do not write the corresponding action on the second $\mathbb{CP}^3$ space.

TABLE CAPTIONS

TABLE 1

Explicit monomial representatives for the 23 independent $\underline{27}$ superfields on R_0 given by deformation theory. The representatives are catalogued according to their G transformation properties. G(1), G(α) and G(α^2) are the representatives of G $\cong$ $\mathbb{Z}_3$ where the generator g of G corresponds to multiplication by 1,α and α^2 respectively.

TABLE 2

Transformation properties of the $\underline{27}$ superfields under the discrete symmetry group D $\cong$ W $\times$ $\mho$ of the manifold R with vacuum moduli described in the text. The representations S1–S3 and R1–R9 are defined in (V.35) of the text and Table 3, respectively.

TABLE 3

Character table of the 18 element nonabelian group $\mho$ along with the characters of $\mho$ acting on the $\overline{\underline{27}}$ lepton, left handed quark, and left handed antiquark subspaces of $H_{\bar{\partial}}^{1,1}(R_0)$.

TABLE 4

Transformation properties of $\overline{\underline{27}}$ superfields under the discrete symmetry group D $\cong$ W $\times$ $\mho$ of the manifold R with vacuum moduli described in the text. The representations S1–S3 and R1–R9 are described in (V.35) of the text and Table 3 respectively.

$G(1)$	$G(\alpha)$	$G(\alpha^2)$
$\lambda_1 \equiv x_0 x_1 x_2$	$q_1 \equiv x_1 x_2 x_3$	$Q_1 \equiv x_0 x_2 x_3$
$\lambda_2 \equiv x_0 x_1 x_3$	$q_2 \equiv y_0 y_2 y_3$	$Q_2 \equiv y_1 y_2 y_3$
$\lambda_3 \equiv y_0 y_1 y_2$	$q_3 \equiv x_0 y_1$	$Q_3 \equiv x_0 y_2$
$\lambda_4 \equiv y_0 y_1 y_3$	$q_4 \equiv x_1 y_2$	$Q_4 \equiv x_0 y_3$
$\lambda_5 \equiv x_1 y_1$	$q_5 \equiv x_1 y_3$	$Q_5 \equiv x_1 y_0$
$\lambda_6 \equiv x_2 y_2$	$q_6 \equiv x_2 y_0$	$Q_6 \equiv x_2 y_1$
$\lambda_7 \equiv x_3 y_3$	$q_7 \equiv x_3 y_0$	$Q_7 \equiv x_3 y_1$
$\lambda_8 \equiv x_2 y_3$		
$\lambda_9 \equiv x_3 y_2$		

TABLE 1

Field		W		U		
		A	Rep.	B	C	Rep.
Leptons	λ_1	α^2	S3	α	$\to\lambda_2$	R7
	λ_2	α^2	S3	1	$\to\lambda_1$	
	λ_3	α	S2	α^2	$\to\lambda_4$	R8
	λ_4	α	S2	1	$\to\lambda_3$	
	λ_5	1	S1	1	+1	R1
	$\lambda_6+\lambda_7$	1	S1	1	+1	R1
	$\lambda_6-\lambda_7$	1	S1	1	-1	R4
	λ_8	1	S1	α	$\to\lambda_9$	R9
	λ_9	1	S1	α^2	$\to\lambda_8$	
Left handed quarks	q_1	α	S2	α	+1	R2
	q_2	α^2	S3	α^2	+1	R3
	q_3	1	S1	1	+1	R1
	q_4	α	S2	α^2	$\to q_5$	R8
	q_5	α	S2	1	$\to q_4$	
	q_6	α^2	S3	α	$\to q_7$	R7
	q_7	α^2	S3	1	$\to q_6$	
Left handed anti quarks	Q_1	α	S2	α	+1	R2
	Q_2	α^2	S3	α^2	+1	R3
	Q_3	α	S2	α^2	$\to Q_4$	R8
	Q_4	α	S2	1	$\to Q_3$	
	Q_5	1	S1	1	+1	R1
	Q_6	α^2	S3	α	$\to Q_7$	R7
	Q_7	α^2	S3	1	$\to Q_6$	

TABLE 2

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
R1	1	1	1	1	1	1	1	1	1
R2	1	α^2	α	α	α^2	1	1	α^2	α
R3	1	α	α^2	α^2	α	1	1	α	α^2
R4	1	1	1	1	1	1	-1	-1	-1
R5	1	α^2	α	α	α^2	1	-1	$-\alpha^2$	$-\alpha$
R6	1	α	α^2	α^2	α	1	-1	$-\alpha$	$-\alpha^2$
R7	2	2α	$2\alpha^2$	$1+\alpha$	$1+\alpha^2$	-1	0	0	0
R8	2	$2\alpha^2$	2α	$1+\alpha^2$	$1+\alpha$	-1	0	0	0
R9	2	2	2	-1	-1	-1	0	0	0
$\overline{27}$ Leptons	6	0	0	0	0	6	-2	4	4
$\overline{27}$ LH Quarks	4	4	-2	-2	-2	0	0	0	0
$\overline{27}$ LH Anti-Quarks	4	4	-2	-2	-2	0	0	0	0

TABLE 3

Field		W		U		
		A	Rep.	B	C	Rep.
Leptons	$\overline{\lambda_1}$	1	S1	1	+1	R1
	$\overline{\lambda_2}$	1	S1	1	+1	R1
	$\overline{\lambda_3}$	α	S2	α	-1	R5
	$\overline{\lambda_4}$	α	S2	α	-1	R5
	$\overline{\lambda_5}$	α^2	S3	α^2	-1	R6
	$\overline{\lambda_6}$	α^2	S3	α^2	-1	R6
Left Handed Quarks	$\overline{Q_1}$	1	S1	α	$\to Q_2$	R9
	$\overline{Q_2}$	1	S1	α^2	$\to Q_1$	
	$\overline{Q_3}$	1	S1	α	$\to Q_4$	R9
	$\overline{Q_4}$	1	S1	α^2	$\to Q_3$	
Left Handed Anti-Quarks	$\overline{q_1}$	1	S1	α	$\to q_2$	R9
	$\overline{q_2}$	1	S1	α^2	$\to q_1$	
	$\overline{q_3}$	1	S1	α	$\to q_4$	R9
	$\overline{q_4}$	1	S1	α^2	$\to q_3$	

TABLE 4

REFERENCES
CHAPTER V

1. For reviews, see:

   J.H. Schwarz, Phys.Rep., 89 (1982) 223.

   M.B.Green, Surv. High Energy Phys., 3, (1982) 127.

2. P.Candelas, G.T.Horowitz, A.Strominger and E.Witten, Nucl. Phys., B258, (1985) 46.

3. See, for example

   H.P. Nilles, Phys. Rep., 110, (1984) 1.

4. D.J. Gross, J.A. Harvey, E. Martinec and R. Rohm, Phys. Rev. Lett. 54, (1985) 502; Nucl. Phys. B256, (1985) 253; Princeton University Preprint (1985); M.B. Green and J.H. Schwarz, Phys.Lett. 151B, (1985) 21.

5. E. Witten, Nucl.Phys., B258, (1985) 75.

6. A. Strominger and E. Witten, Princeton University Preprint (to appear in Comm. Math. Phys.) 1985; A.Strominger ITP Preprint (1985)

7. Y. Hosotani, Phys.Lett. 126B, (1983) 309; E.Witten, Phys.Lett.126B, (1984) 351.

8. J.D. Breit, B. Ovrut and G. Segre, Phys.Lett.,158B, (1985) 33.

9. E.Witten, "New Issues in Manifolds of SU(3) Holonomy" Princeton University Preprint (1985)

10. For a rigorous presentation and further references, see

    K.Kodaira "Complex Manifolds and Deformations of Complex Structures" (Springer- Verlag 1985).

11. P. Griffiths and J. Harris "Principles of Algebraic Geometry" (Wiley Interscience, 1978); T. Eguchi, P.B. Gilkey and A.J. Hanson, Phys.Rep., 66, (1980) 213.

12. S-T. Yau, Private Communication

13. S–T. Yau, in Proceedings of the Argonne Symposium on Anomalies, Geometry and Topology (World Scientific, 1985)

14. M. Dine, V. Kaplunovsky, M. Mangano, C. Nappi and N. Seiberg Nucl.Phys., B259, (1985) 519; F. del Aguila, G. Blair, M. Daniel and G.G. Ross, CERN preprint 4336 (1985)

15. N.J. Hitchin, D.Phil. Thesis (University of Oxford, 1972)

16. M. Dine, R. Rohm, N. Seiberg and E. Witten "Gluino Condensation in Superstring Models", Princeton University Preprint (1985); J–P. Derendinger, L.E. Ibanez and P. Nilles, Phys. Letts., 155B, (1985) 65.

17. B.R. Greene, K.H. Kirklin and P.J. Miron, Nucl. Phys. 274B (1986) 574.

18. M.W. Goodman and E. Witten, "Global Symmetries in Four and Higher Dimensions" Princeton University Preprint (1985)

19. J–P Serre, "A course in Arithmetic" (Springer Verlag)

20. B.R. Greene, K.H. Kirklin, P.J. Miron and G.G. Ross,in preparation.

SUPERSTRINGS :
TOPOLOGY, GEOMETRY and PHENOMENOLOGY
&
ASTROPHYSICAL IMPLICATIONS OF SUPERSYMMETRIC MODELS

VOLUME II

Brian Randolph Greene
Magdalen College

Department of Theoretical Physics
University of Oxford

Thesis submitted for the degree of Doctor of Philosophy
in the University of Oxford

July 1986

ASTROPHYSICAL IMPLICATIONS OF SUPERSYMMETRIC MODELS

Brian Randolph Greene

Magdalen College

ABSTRACT

We derive cosmological constraints on a supersymmetric extension of the standard model in which weak gauge symmery breaking is triggered at the tree-level by a Higgs singlet superfield. The fermionic component of this gauge singlet (the "nino") is shown to be the lightest supersymmetric particle with a relic abundance near the critical closure density for a remarkably wide range of the unconstrained parameters.

The previously favoured photino dark matter scenario has been eliminated by the non-observation of high energy solar neutrinos. After briefly reviewing this argument, we extend the analysis to eliminate Higgsino dark matter scenarios with $<H_1{}^0> \neq <H_2{}^0>$. The nino produces an *acceptably low level* of solar neutrinos and may also account for the anomalously high level of cosmic ray antiproton flux.

Thesis submitted for the degree of Doctor of Philosophy
in the University of Oxford

July 1986

VOLUME I
CONTENTS

VOLUME II

CONTENTS

CHAPTER VI

SUPERSYMMETRIC MODEL BUILDING

VI.1 ASPECTS OF THE SUPERSYMMETRIC STANDARD MODEL

Symmetries, and their use in deepening our understanding of the laws of nature through unification, have been a major guiding principle in physics during the last century. In the 1960's, it became clear that in addition to the space–time symmetries elucidated by Einstein, internal symmetries (e.g. $SU(2)_L \times U(1)_Y$ of the electroweak interaction) have a profound position in the underlying structure of the universe. In the quest for unification, attempts were made to combine, in a non–trivial manner, the space–time symmetry of the Poincaré group with internal symmetry groups. These efforts, as shown by Coleman and Mandula [1], were in vain. In 1967, they proved under rather general assumptions that any Lie group containing the Poincaré group P and an internal Lie symmetry group G, which leaves a non–trivial S–matrix invariant, is necessarily a direct product of P and G.

This 'no–go' theorem, however, depends critically on the assumption that the symmetries considered form Lie groups. By slackening the reins of this assumption to allow graded Lie groups, Gol'fand and Likhtman [2] showed that a non–trivial union of spacetime and internal symmetries could be achieved. In particular, a $\mathbb{Z}_2$ grading leads to a graded Lie group in which the odd generators transform non–trivially under the Lorentz group. Choosing these generators to be in the $(0,\frac{1}{2})$ and $(\frac{1}{2},0)$ representations of the Lorentz group leads to the graded Lie algebra presented in (A.2). (See the Appendix for a more detailed discussion). We see from (A.2) the intriguing property that the anticommutator of two odd generators (which act internally) is proportional to a space–time translation. This hints at a possible connection between local supersymmetry (in which these space–time translations would also be localised) and general relatively. As discussed in the Appendix, this is in fact the case, and hence local supersymmetry is generally referred to as supergravity.

The physical import of Q_α and $\bar{Q}_{\dot{\alpha}}$ (the odd generators; see the Appendix for notation) is readily apparent. The result of acting on a quantum field with these generators is to change the spin by $\frac{1}{2}$. Thus, fermions are mapped to bosons and bosons to fermions! Supersymmetric particle multiplets therefore contain both fermions and bosons. Furthermore, as we shall now sketch, one can prove that in any representation of the supersymmetry algebra (in which P_μ is a one–to–one onto operator) there are equal numbers of fermionic and bosonic degrees of freedom [3]. Consider one particular Q_α, say Q_1. Notice that $\{Q_1,\bar{Q}_{\dot{1}}\}$ maps bosons to bosons and fermions to fermions. Let us concentrate on the bosonic set. The above anticommutator may be expressed as a linear combination of the operators P_μ and hence is a one–to–one onto operator. Thus we may conclude the same property holds for each $Q_\alpha,\bar{Q}_{\dot{\alpha}}$, and hence our assertion is true.

When one attempts to put these ideas into phenomenological practice by constructing a supersymmetric version of the standard model, one encounters an ineluctable aesthetic conundrum: the particle content of the standard model must be doubled. As shown, supersymmetry requires any model to contain equal numbers of fermionic and bosonic degrees of freedom. Component fields in the same supermultiplet, though, have the same gauge transformation properties. Inspection of the quantum numbers of particles in the standard model reveals that the only possible bosonic–fermionic pairing is that of the scalar Higgs doublet with a left handed lepton doublet. If this pairing is made, however, the spontaneous breakdown of $SU(2)_L \times U(1)_Y$ to $U(1)_{EM}$, through a Higgs vacuum expectation value, would result in the breakdown of global lepton number symmetry. Thus, there are no phenomenologically viable bosonic–fermionic pairings into supermultiplets with the fields contained in the standard model. For each boson (fermion) in the standard model, therefore, a new presently unobserved fermionic (bosonic) partner must be added to fill out the structure imposed by superymmetry. Even this enlargement is not sufficient. To provide masses for all of the quarks and leptons in the theory as well as to avoid anomalies, a second Higgs superfield must be included. When coupled to N=1 supergravity, the model also contains the spin 2 graviton and the spin 3/2 gravitino. We thus see that less than half of the particle content of the minimal locally supersymmetric version of the standard model has been observed. We maintain our belief in supersymmetry by blithely interpreting this as a breakdown in supergravity resulting in masses for the unobserved superpartners above that which can be attained by present day particle accelerators.

From a cosmological viewpoint, these (presently) unobserved superparticles can have profound implications. In many supersymmetric models one may naturally introduce a new conserved quantum number known as R−parity (this will be elaborated upon later). The standard model fields are R−parity even while their superpartners are R−parity odd; the lightest supersymmetric particle (LSP) is therefore absolutely stable. The LSP, however, can pair annihilate, the rate of which determines its present cosmological relic abundance. In any proposed supersymmetric model, the cosmological constraint of not overclosing the universe must be imposed: the LSP relic abundance cannot be in excess of the critical closure density. (For completeness we note that this constraint may be relaxed by a factor of three to five and still be in agreement with experimental evidence [4]. We will maintain the requirement asserted above).

An analysis of this form was originally carried out by Ellis *et al.* [5] in which it was determined that the most likely candidates for the LSP are photinos and neutral Higgsinos. Cosmological density considerations require the former to have minimum mass around $\frac{1}{2}$ to 5 GeV (depending on the Hubble constant and squark masses) and the latter to have mass greater than that of the top quark (for $<H_1{}^0> = <H_2{}^0>$, where the brackets denote vacuum expectation value) or the bottom quark (for $<H_1{}^0>/<H_2{}^0> \gtrsim 2$).

It is natural to pursue this track one highly restrictive step further. Galactic rotation curves are incompatible with the observed (luminous) mass content and, in fact, imply that we are apparently ignorant as to the composition of ninety percent of the mass of our galaxy [6]. The latter experimental conclusion jibes with the theoretical prediction of inflationary cosmological scenarios which predict $\Omega = \rho_{universe}/\rho_{critical} = 1$. Is it possible that this "dark matter" is the LSP from supersymmetric particle physics?

Beyond simply restricting the analysis of ref. [1], which demands $\Omega \lesssim 1$, to the more restrictive constraint of requiring $\Omega = 1$, several authors [7] have explored the observable effects which the LSP's annihilation can have in the galaxy and in the solar system if it is the dark matter. Silk and Srednicki [7], followed by Hagelin and Kane [7], showed that in addition to solving the dark matter problem, galactic photinos and Higgsinos may also account for the anomalously high flux of observed antiprotons. Recently, however, Steigman, Gaisser and Tilav [8] have shown that, for a wide range of parameters, photinos (and scalar neutrinos) comprising the galactic halo would be captured in the Sun and through subsequent annihilation produce solar neutrinos in an abundance beyond that observed (in the proton decay experiment at IMB). In Chapter VIII, we will show that this analysis extends to Higgsinos if $<H_1{}^0>/<H_2{}^0> \neq 1$, making all but a finely tuned subset of the supersymmetric dark matter scenarios proposed in ref. [7] *untenable*. Beyond the solution of Higgsinos with $<H_1{}^0>/<H_2{}^0> = 1$ (which as we shall see also requires a degree of fine tuning) there is another possibility. All of the supersymmetric models studied in refs. [5,7] assume that weak gauge symmetry breaking occurs via radiative corrections driving Higgs mass terms negative. If we demand tree level weak gauge symmetry breaking, as we shall see in the next section, then we necessarily have to enlarge the field content of the theory. The simplest possibility is the introduction of a gauge singlet Higgs superfield N [9]. (In fact, it appears rather difficult to construct phenomenologically viable models of this sort where N is anything *but* a gauge singlet [10]). Although there has been theoretical bias towards using radiative corrections to break gauge symmetry ("But radiative corrections are beautiful...."[11]), tree level breaking is equally viable. There has been some discussion of possibly destabilizing the hierarchy with the inclusion of a gauge singlet if one ultimately tries to grand unify the model [9]. However, this question has not been settled, and, additionally, we are not compelled to invoke grand unification.

Inclusion of the N field dramatically alters the cosmological evolution of the model [12], as the fermioinic component of this singlet (the "nino") is often the LSP. This cosmological analysis, which was done in collaboration with P.J. Miron, is the content of Chapter VII. In Chapter VIII we examine the astrophysical implications of demanding that the nino comprises the dark matter [13]. (This work was done in collaboration with D.N. Spergel). We show that the relic abundance of the nino is equal to the critical closure density (with $H_o = 50$ Km/sec/Mpc) for a surprisingly wide range of the free parameters. Furthermore, we demonstrate that the nino sidesteps the quagmire of overproduction of solar neutrinos due to its relatively small scattering cross section off of protons and hence low solar capture rate.

Besides finely tuned photino and Higgsino scenarios, we thereby present the only consistent supersymmetric dark matter scenario.

VI.2 SUPERSYMMETRIC WEAK GAUGE SYMMETRY BREAKING AT THE TREE LEVEL

In the Appendix we examine the scalar potential arising from a spontaneously broken supergravity Lagrangian. From equation (A.30) we have

$$V = |\hat{g}_i|^2 + m_{3/2}^2 \, |z_i|^2 + m_{3/2}[z_i\hat{g}_i + (A-3)\hat{g} + h.c.] \qquad \text{(VI.1)}$$

where $\hat{g}$ depends on the sum of superpotentials h and g from a hidden sector and an observable sector, respectively. The superpotential g, describing the observable sector must at least contain the supersymmetric generalizations of the Yakawa couplings present in the standard model:

$$g_1 = \lambda_{kj}U_k^a\bar{H}_a\bar{U}_j + \lambda'_{kj}U_k^a H^b \varepsilon_{ab}\bar{D}_j + \lambda''_{kj}L_k^a H^b \varepsilon_{ab}\bar{E}_j$$

where U_k^a, $\bar{U}_k$, $\bar{D}_k$, L_k^a, $\bar{E}_k$ are three families (k = 1,2,3) of left handed chiral superfields (see appendix A), a is an SU(2) index (colour indices are supressed), and H^a, $\bar{H}_a$ are two Higgs left handed chiral superfields. This superpotential is not the most general gauge invariant holomorphic function of the chiral superfields as, for example, it does not contain $U^a L^b \varepsilon_{ab}\bar{D}$ amongst others. Such terms are not included as they generally lead to phenomenological catastrophes like lepton or baryon number violation. To eliminate these unwanted terms in a more natural manner, R parity is introduced. R parity has a multiplicative quantum number; standard model fields are R−parity even and their superpartners are R−parity odd. R−parity invariance then restricts g to be of the form

$$g = \mu H\bar{H} + g_1 \qquad \text{(VI.3)}$$

The scalar Higgs bosons occur in the scalar potential V as $\mu^2(hh^* + \overline{hh}^*)$ plus other terms determined by (VI.2), none of which can drive spontaneous symmetry breakdown. To break weak gauge symmetry at the tree level we are therefore obliged to introduce a new superfield, N. The most economical choice is for N to be a gauge singlet with scalar component R−parity even and fermionic component R−parity odd. This allows us to choose

$$g = \lambda N(H\overline{H} - \mu^2) + g_1 \qquad \text{(VI.4)}$$

which, as shall be shown in detail in the next chapter, can break weak gauge symmetry. Notice that since the fermionic component of N (the 'nino') is R−parity odd, it is a candidate for the cosmological role of lightest supersymmetric particle.

REFERENCES

CHAPTER VI

1. S. Coleman and J. Mandula, Phys. Rev., 159 (1967) 1251.
2. Y.A. Gol'fand and E.S. Likhtman, JETP Letters, 13 (1971) 323.
3. P.C. West in Superstrings and Supergravity, Proceedings of the Twenty–eighth Scottish Universities Summer School in Physics (1985) 126.
4. D.N. Schramm, International School on Gauge Theory and the Early Universe (1986), private communication.
5. J. Ellis, J.S. Hagelin, D.V. Nonoplous, M. Srednicki, Phys. Lett., 127B (1983) 233; Nucl. Phys. B238 (1984) 453.
6. J.R. Primack, SLAC preprint (July 1984) for a review.
7. J. Silk and M. Srednicki, Phys. Rev. Lett., 53 (1984) 624; F.W. Stecker, S. Rudaz and T.F. Walsh, Phys. Rev. Lett., 55 (1985) 2622.
8. G. Steigman, T. Gaisser and S. Tilav, "Closing the Door on Cold Dark Matter", Moriond Workshop on Neutrinos in Physics and Astrophysics, ed. T. Tranh, in press.
9. J–M. Frere, D.R.T. Jones and S. Raby, Nucl. Phys., B222 (1983) 11; J.P. Derendinger and C.A. Savoy, Phys. Lett., 119B (1982) 343.
10. H.P. Nilles, Phys. Rep. 110C (1984) 1, for a review.
11. G.G. Ross, private communication (1986).
12. B.R. Greene and P.J. Miron, Phys. Lett., 168B (1986) 226.
13. B.R. Greene and D.N. Spergel, Harvard–Smithsonian Preprint and Oxford Preprint (1986).

CHAPTER VII

SUPERSYMMETRIC COSMOLOGY WITH A GAUGE SINGLET

VII.1 INTRODUCTION

We derive cosmological constraints on a supersymmetric version of the standard model coupled to N=1 supergravity in which weak gauge symmetry breaking is triggered by a Higgs singlet superfield. In section 1 we examine the scalar potential and discuss conditions that ensure that the global minimum breaks $SU(2)_L \times U(1)_Y$ to $U(1)_{EM}$. In section 2 we discuss the cosmological evolution of the lightest supersymmetric particle and in section 3 we present our results. This work was done in collaboration with P.J. Miron

VII.2 WEAK GAUGE SYMMETRY BREAKING

As shown in Chapter VI, weak gauge symmetry breaking at the tree level in the minimal supersymmetric version of the standard model can only be accomplished if the field content is enlarged to include, in the simplest case, a gauge singlet Higgs superfield. We assume, as described in the Appendix, that supergravity is spontaneously broken by the hidden sector, so from equations (VI.1), (VI.4) and (A.30) we may write the potential and supersymmetry breaking terms in the lagrangian as

$$
\begin{aligned}
L_{pot+SSB} = & -\sum_A\left|\frac{\partial f}{\partial z^A}\right|^2 - \frac{1}{2}\sum_J (D\cdot D)_J - m_{3/2}\sum_A z_A{}^* z_A \\
& - (m_{3/2}(A-3)f(z) + \sum_A \frac{\partial f}{\partial z_A} z_A + \text{h.C.}) \\
& - \lambda v' \epsilon_{ij}\tilde{H}_1{}^i\tilde{H}_2{}^j - \lambda v_1\tilde{N}\tilde{H}_2^o - \lambda v_2\tilde{N}\tilde{H}_1^o \\
& - \sum m_\chi \tilde{\lambda}\tilde{\lambda} \qquad\qquad (\text{VII.1})
\end{aligned}
$$

where $v' = \langle N \rangle$ (we write N for the scalar component of the superfield N and tildes generally denote fermionic components), $v_1 = \langle H_1{}^o \rangle$ ($H_1 = \begin{bmatrix} H_1{}^o \\ H_1{}^- \end{bmatrix}$), $v_2 = \langle H_2{}^o \rangle$ ($H_2 = \begin{bmatrix} H_2{}^+ \\ H_2{}^o \end{bmatrix}$), $D_j = g_j\, z^{*A} T_A{}^B z_B$ where g_j are coupling constants and T are gauge group generators. The most general renormalizable superpotential with this new field contains, in addition to the terms in (VI.4), $m_N{}^2 N^2$ and γN^3. If γ is non-zero, the model is severely ill. As shown by Frere *et al* [1], the global minimum responsible for the spontaneous breakdown of weak gauge symmetry also spontaneously breaks electric charge symmetry. Derendinger and Savoy [2] have argued that radiative corrections may offer a possible way of avoiding this catastrophe, while Barrosa and Romao [3] have explicitly shown that the problem is soluble if $m_N{}^2$ and γ are set equal to zero. We shall adopt the latter solutions and hence take our superpotential to be

$$f = \lambda(N\epsilon_{\alpha\beta}H_1{}^{\alpha}H_2{}^{\beta}) + rN + \lambda_{ij}U_i{}^{a}H_{2a}U_j + \lambda'_{ij}U^{a}{}_{i}H_1{}^{b}\epsilon_{ab}\bar{D}_j$$
$$+ \lambda''_{ij}L^{a}{}_{i}{}^{a}H_1{}^{b}\epsilon_{ab}\bar{E}_j \ . \qquad \text{(VII.2)}$$

In this case, it has been numerically verified [3] that the global minimum of the scalar potential breaks $SU(2)_L \times U(1)_Y$ to $U(1)_{\epsilon m}$ as long as $1 < A < \surd 8$. With A in this range we may, for the purposes of minimization, concentrate on the terms in the scalar potential which only contain the scalar components of H, $\bar{H}$ and N, i.e. $(h,\bar{h},N)$. With this proviso, V may be written

$$V = \lambda^2|\epsilon^{\alpha\beta}h_{1\alpha}h_{2\beta} - (r/\lambda)N|^2 + \lambda^2|N\epsilon^{\alpha\beta}h_{1\beta}|^2$$
$$+ \lambda^2|N\epsilon^{\alpha\beta}h_{2\beta}|^2 + m^2_{3/2}(h_1^{*\alpha}h_{1\alpha} + h_2^{*\alpha}h_{2\alpha} + N^*N)$$
$$+ \lambda m_{3/2}[AN\epsilon^{\alpha\beta}h_{1\alpha}h_{2\beta} - (A-2)(r/\lambda)N + c.c.]$$
$$+ \tfrac{1}{2}(D^aD^a + D'D') \qquad \text{(VII.3)}$$

where

$$D^a = g_2(h_1{}^{\dagger}\tfrac{1}{2}\sigma^a h^1 + h_2{}^{\dagger}\tfrac{1}{2}\sigma^a h^2)$$

and

$$D' = g_1(-\tfrac{1}{2}h_1{}^{\dagger}h_1 + \tfrac{1}{2}h_2{}^{\dagger}h_2) \qquad \text{(VII.4)}$$

The minima of V have $v_1 = \langle h_1{}^0\rangle$ and $v_2 = \langle h_2{}^0\rangle$ equal, as is easily seen since they appear symmetrically. (We neglect radiative scalar mass corrections). We then see that $v \equiv v_1 = v_2$ and v' must satisfy [3]

$$\hat{v}[(\hat{v}^2-x) + \hat{v}'^2 + A\hat{v}' + 1] = 0 \tag{VII.5a}$$

$$(2\hat{v}'+A)\hat{v}^2 + \hat{v}' - (A-2)x = 0 \tag{VII.5b}$$

where

$$\hat{v} = (\lambda/m_{3/2})v, \quad \hat{v}' = (\lambda/m_{3/2})v', \quad x = \frac{r\lambda}{m^2_{3/2}} \tag{VII.6}$$

If $\hat{v} = 0$, the minimum of V will not induce the spontaneous breakdown of $SU(2)_L$. Therefore, assuming $\hat{v} \neq 0$, we find from (VII.5) that

$$\hat{v} = (x - 1 - \hat{v}'^2 - Av')^{\frac{1}{2}} \tag{VI1.7}$$

with v' a solution to

$$2\hat{v}'^3 + 3A\hat{v}'^2 + (1-2x + A^2)\hat{v}' + (A-2x) = 0. \tag{VII.8}$$

Of the six solutions to these equations, we choose the one which corresponds to the deepest minimum of the potential, which for $1 < A < \sqrt{8}$ is always deeper than the $SU(2)_L$ preserving minimum.

Our free parameters are thus $m_{3/2}$, M_1, M_2 (the $\tilde{B}$ and $\tilde{W}$ gaugino mass terms) x and A (modulo the above constraint). For simplicity we assume that M_1 and M_2 are related as they would be in a grand unified theory based on a miminal supersymmetric model : $M_1 = (5\alpha_1/3\alpha_2)M_2$. (It would probably be more consistent to choose $M_1 = M_2$ as we avoid grand unifying the theory; the choice does not affect our results.) When these parameters are specified we use equations (VII.7) and (VII.8) to find the values of $\hat{v}$ and $\hat{v}'$ at the global minimum of V. The W$-$boson mass requires $v = 250$ GeV, and hence from (VII.6) we solve for λ, r and v', thereby determining the remaining parameters [1].

VII.3 COSMOLOGICAL ANALYSIS

Our cosmological analysis [1] follows that of Lee and Weinberg [6] as adapted by Ellis *et al* [5]. For each choice of the free parameters we find the lightest neutralino mass eigenstate, which is absolutely stable by conservation of R-parity, from the neutralino mixing matrix

$$[\tilde{W}^3\ \tilde{B}^0\ \tilde{H}_1^0\ \tilde{H}_2^0\ \tilde{N}]\begin{bmatrix} M_2 & 0 & -\frac{g_2 v}{\sqrt{2}} & \frac{g_2 v}{\sqrt{2}} & 0 \\ 0 & \frac{5}{3}\frac{\alpha_1}{\alpha_2}M_2 & \frac{g_1 v}{\sqrt{2}} & \frac{-g_1 v}{\sqrt{2}} & 0 \\ \frac{-g_2 v}{\sqrt{2}} & \frac{g_1 v}{\sqrt{2}} & 0 & \lambda\langle N\rangle & \lambda v \\ \frac{g_2 v}{\sqrt{2}} & \frac{-g_1 v}{\sqrt{2}} & \lambda\langle N\rangle & 0 & \lambda v \\ 0 & 0 & \lambda v & \lambda v & 0 \end{bmatrix}\begin{bmatrix} \tilde{W}^3 \\ \tilde{B}^0 \\ \tilde{H}_1^0 \\ \tilde{H}_2^0 \\ \tilde{N} \end{bmatrix} \qquad \text{(VII.9)}$$

where $\alpha_1 = g_i{}^2/4\pi$.

After diagonalization this gives the lightest neutralino state

$$\chi = \alpha\tilde{W}^3 + \beta\tilde{B}^0 + \gamma\tilde{H}_1^0 + \delta\tilde{H}_2^0 + \xi\tilde{N} \quad . \qquad \text{(VII.10)}$$

We compute its annihilation cross-section into fermions via:

i) s-channel Z-exchange (2)

ii) t-channel sfermion exchange

iii) s-channel Higgs exchange.

This third channel has not been previously considered, as its effect is negligible in models without the N field. In our case, we often have states with significant $\tilde{N}$ and $\tilde{H}$ components for which this channel is dominant.

Calculation of the effects of process (iii) requires the diagonalization of the Higgs boson mass matrix, which is determined by computing $\partial^2 V/\partial\varphi_i\partial\varphi_j$. In this expression, φ_i and φ_j represent real scalar fields which are the components of the complex scalar Higgs fields. For the neutral Higgs bosons we write

$$\begin{aligned} H_1^{o} &= \varphi_1 - i\varphi_2 \\ H_2^{o} &= \varphi_3 + i\varphi_4 \\ N &= \varphi_5 + i\varphi_6 \end{aligned} \tag{VII.11}$$

and find the mass matrix

$$[\varphi_1\varphi_2\varphi_3]\begin{bmatrix} \lambda^2 w+gv^2+m_{3/2} & 2\lambda^2v^2+\lambda m_{3/2}Av'-gv^2-\epsilon\lambda^2 & 2\lambda_2 vv'+\lambda m_{3/2}Av \\ 2\lambda^2v^2+\lambda m_{3/2}Av'-gv^2-\epsilon\lambda^2 & \lambda^2w+gv^2+m_{3/2} & 2\lambda^2vv'+\lambda m_{3/2}Av \\ 2\lambda^2vv'+\lambda m_{3/2}Av & 2\lambda^2vv'+\lambda m_{3/2}Av & 2\lambda^2v^2+m^2_{3/2} \end{bmatrix}\begin{bmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{bmatrix} \tag{VII.12a}$$

$$[\varphi_4\varphi_5\varphi_6]\begin{bmatrix} m^2_{3/2}+\lambda^2w & m_{3/2}\lambda Av'-\epsilon\lambda^2 & m_{3/2}\lambda Av \\ m_{3/2}\lambda Av'-\epsilon\lambda^2 & m^2_{3/2}+\lambda^2w & -m_{3/2}\lambda Av \\ m_{3/2}\lambda Av & -m_{3/2}\lambda Av & m^2_{3/2}+2\lambda^2v^2 \end{bmatrix}\begin{bmatrix}\varphi_4\\ \varphi_5\\ \varphi_6\end{bmatrix} \tag{VII.12b}$$

where $w = v^2 + v'^2$, $g = \frac{1}{2}(g_1{}^2+g_2{}^2)$, and $\epsilon = r/\lambda$.

We express the interaction eigenstates $\varphi_1,\ldots,\varphi_6$ in terms of the mass eigenstates $H_1,\ldots,H_5$ (there is an unphysical Goldstone boson) by diagonalizing the mass matrix in (VII.12).

We write

$$\begin{aligned} H_1^{o} &= K_1H_1 + K_2H_2 + K_3H_3 - i(K_4H_4 + K_5H_5) \\ H_2^{o} &= L_1H_1 + L_2H_2 + L_3H_3 + i(L_4H_4 + L_5H_5) \end{aligned} \tag{VII.13}$$

where the K_i and L_i are determined by the diagonalization.

To determine the annihilation cross-section of the neutralino into fermions we use the low energy effective form of our Lagrangian

$$L = \sum_f \bar{\chi}\gamma^{\mu}\gamma_5\chi\bar{f}\gamma_{\mu}(A_f P_L + B_f P_R)f + C_f\bar{\chi}\chi\bar{f}f + D_f\bar{\chi}\gamma_5\chi\bar{f}\gamma_5 f \tag{VII.14}$$

where

$$A_f^Z = (\gamma^2 - \delta^2)\frac{g_1\sin\theta_W + g_2\cos\theta_W}{4M_Z^2}\left[\tfrac{1}{2}Y_{f_L}g_1\sin\theta_W - T^3_{f_L}g_2\cos\theta_W\right] \tag{VII.15a}$$

$$B_f^Z = (\gamma^2 - \delta^2)\frac{g_1\sin\theta_W + g_2\cos\theta_W}{8M_Z^2}Y_{f_R}g_1\sin\theta_W \tag{VII.15b}$$

$$A_f^{\tilde{f}} = \frac{(T^3_{f_L}\alpha g_2 + \tfrac{1}{2}Y_{f_L}\beta g_1)^2}{2m^2_{\tilde{f}_L}} + \begin{cases}\gamma^2 m_f{}^2/4v^2m^2_{\tilde{f}_R} & \text{u-type}\\ \delta^2{}_m f^2/4v^2m^2_{\tilde{f}_R} & \text{d-type}\end{cases} \tag{VII.15c}$$

$$B_f^{\tilde{f}} = -\frac{(\tfrac{1}{2}Y_{f_R}\beta g_1)^2}{2m^2_{\tilde{f}_L}} - \begin{cases}\gamma^2 m_f{}^2/4v^2m^2_{\tilde{f}_L} & \text{u-type}\\ \delta^2 m_f{}^2/4v^2m^2_{\tilde{f}_L} & \text{d-type}\end{cases} \tag{VII.15d}$$

$$C_f^i = \frac{g_2 m_f}{\surd 2\ M_W m_i{}^2} \times \begin{cases}L_i(SL_i - QK_i) & \text{u-type}\\ K_i(SL_i - QK_i) & \text{d-type}\end{cases} \tag{VII.15e}$$

$$D_f^j = \frac{-g_2 m_f}{\surd 2\ M_W m_j{}^2} \times \begin{cases}L_j(SL_j+QK_j) & \text{u-type}\\ K_j(SL_j+QK_J) & \text{d-type}\end{cases} \tag{VII.15f}$$

and we use the convention $T_3 = Q - Y/2$.

We define the coefficients S and Q as

$$Q = [\gamma(g_2\beta - g_1\alpha) + \sqrt{2}\lambda\delta\zeta] \qquad \text{(VII.16a)}$$

$$S = [\delta(g_2\beta - g_1\alpha) - \sqrt{2}\lambda\gamma\zeta] \qquad \text{(VII.16b)}$$

$i = 1,2,3$, $j = 4,5$ label the Higgs channels, u–type represents u,c,t quarks, and d–type represents e, μ, τ, d, s, b.

Here, m_f and m_x are final–state fermion masses and neutralino masses respectively, $x = v_{rel}/6$, and q is the final state three momentum which, for simplicity, we take to be(4)

$$|q| = \text{Max}\ (\sqrt{m_\chi^2 - m_f^2},\ m_\chi v). \qquad \text{(VII.17)}$$

The annihilation cross-section can now be written as

$$\langle\sigma v_{rel}\rangle = \tilde{a} + \tilde{b}x \qquad \text{(VII.18)}$$

where

$$\tilde{a} = \sum_f \theta(m_x - m_f)\ \frac{|q|}{2\pi m_\chi}\ [m_f^2(A_f - B_f)^2 + 4D_f^2 m_\chi^2 + 4(A_f - B_f)D_f m_f m_\chi] \qquad \text{(VII.19a)}$$

$$\tilde{b} = \sum_f \theta(m_x - m_f)\ \frac{|q|}{2\pi m_\chi}\ [(A_f^2 + B_f^2)(4m_\chi^2 - \frac{13}{4}m_f^2) + \frac{21}{2}A_f B_f m_f^2 \qquad \text{(VII.19b)}$$

$$+ 6C_f^2(m_\chi^2 - m_f^2) + 3D_f^2 m_\chi^2 - 3D_f(A_f - B_f)m_f m_\chi]$$

From equation (VII.18) one can now calculate the relic neutralino abundance using the method of Lee and Weinberg [6]. In this method, relic abunances are computed using the evolution equation

$$dn/dt = -3(\dot{R}/R)n - \langle\sigma v_{rel}\rangle(n^2 - n_0^2), \tag{VII.20}$$

where n is the number density at time t, n_0 is the number density of heavy fermions in thermal equilibrium, and R is the cosmic scale factor. Using standard methods, this may be cast into the more convenient form

$$df/dx = (m_\chi/k^3)((45/8\pi^3)N_F G)^{\frac{1}{2}}\langle\sigma v_{rel}\rangle(f^2 - f_0^2) \tag{VII.21}$$

where x is replaced by its thermal average kT/m_χ, $f(x) = n/T^3$, $f_0(x) = n_0/T^3$, k is Boltzmann's constant, $G = (1/m_{p\ell})^2$ is Newton's constant, and N_F counts the effective number of degrees of freedom at a given temperature. Assuming an initial thermal equilibrium, and integrating this rate equation forward in time using the approximation scheme of ref. [11] yields the relic mass density [6]

$$\rho_\chi = 5.0 \times 10^{-40}(T_\chi/T_\gamma)^3(T_\gamma/2.8K)^3\sqrt{N_F}[GeV^{-2}/(\tilde{a}x_f + \tfrac{1}{2}\tilde{b}x_f^2)]\,g/cm^3 \tag{VII.22}$$

where T_χ/T_γ is a reheat factor, explained in ref [12]. If this abundance is less than the present critical closure density of the universe, ρ_c, then the chosen point in parameter space is cosmologically allowed. In our analysis, we take the present value of the Hubble constant H_0 to be 100 km/sec/Mpc which implies $\rho_c = 2.1 \times 10^{-29}$ gm/cc. Smaller choices of H_0 reduce the value of ρ_c and hence shrink the allowed portions of parameter space.

We characterize the parameter space of our Lagrangian with respect to five categories of increasing relative priority:

i) lightest neutralino is the lightest supersymmetric particle (LSP) and is cosmologically allowed

ii) lightest neutralino is the LSP, is cosmologically allowed, and has abundance within 10% of critical closure density

iii) lightest neutralino is not the LSP

iv) lightest neutralino is not cosmologically allowed

v) lightest chargino or sfermion mass $\lesssim 20$ GeV, in conflict with experimental lower bounds.

In figures one through three we display our numerical results. We categorise our lightest neutralino states into five basic types:

$\tilde{N}$: (niño) $|\zeta| \gtrsim .9$

$\tilde{\gamma}$: (photino) $|\alpha| \simeq .4$, $|\beta| \simeq .8$, $|\zeta| \simeq 0$

$\tilde{H}$: (higgsino) $|\gamma| = |\delta| \simeq .7$, $|\zeta| \simeq 0$, $|\alpha| \simeq |\beta| \simeq 0$

$\tilde{C}$: (higgsino–nino combination) $|\gamma| = |\delta| \simeq .3-.5$, $|\zeta| \simeq .7-.9$

$\tilde{M}$: (gaugino–higgsino mixture) $|\zeta| \simeq 0$, not fitting any of the above.

From the neutralino mass matrix (VII.9), it is clear that the nino is the LSP for small v/v' (for which there is an approximate eigenstate $v(\tilde{H}_1{}^0 + \tilde{H}_2{}^0) - v'\tilde{N}$), while the photino is the LSP for small M_2. We partition our parameter space with dotted lines into areas in which each state dominates. We also divide the parameter space with solid lines according to the cosmological categories given above. Following ref. [4](5) we restrict λ to be less than one to allow meaningful perturbative analysis and greater than the minimum required to yield positive scalar Higgs masses. This minimum is an increasing function of A as can be seen by comparing figure 1a with 1c. We also restrict $m_{3/2}$ to be greater than the minimum required to yield positive sfermion masses (and at least 20 GeV).

In figure 1a the gravitino mass is taken to be 100 GeV with A=1 (we actually take A slightly greater than 1 to ensure that we are sitting in the global minimum of the potential). The lightest cosmologically allowed neutralino is a 4.5 GeV nino (at $M_2 \simeq 8$ GeV, $\lambda \simeq .1$). The scalar Higgs masses are sufficiently small in this region of parameter space to allow efficient s–channel annihilation between the $\tilde{N}$ and $\tilde{H}$ components. These nino states yield a relic density near the critical closure density as shown by the cross–hatched region. The width of this region indicates the rate of change of the relic density with λ and M_2. For $\lambda < .1$ at this value of M_2, the nino states are too light and λ is too small for efficient annihilation and hence these states are not cosmologically allowed. The lightest allowed photino is 6 GeV which also yields a relic density near the critical closure density. However, we see that as λ and M_2 increase, the relic photino density quickly drops well below the critical value. We also note that in the upper right hand corner of figure 1a pure higgsino states are the LSP. At this point their mass is 85 GeV, greater than the top quark mass which we have taken to be 40 GeV. These states can therefore annihilate efficiently through t–channel scalar top exchange, explaining the absence of a plateau, which is present in the results of [5].

Increasing the gravitino mass to 200 GeV, as in figure 1b, further restricts the cosmologically allowed regions as this increases the sfermion and Higgs masses. We find that a 4.5 Gev nino state is still cosmologically allowed, but the region where the minimum chargino mass is less than 20 GeV rules out photino states of slightly higher mass.

In figure 1c, A is chosen to be 2.5. (We only take $m_{3/2}$ = 100 GeV as the requirement $\lambda < 1$ forbids $m_{3/2} > 120$ GeV[5] [4]). This increases the minimum value of λ and we see that nino states, which are generally the LSP for small λ values (for which v'/v is large), do not occur. We find the lightest cosmologically allowed neutralino to be a 6 GeV photino (at $M_2 \simeq 7.5$ GeV). As M_2 increases, the photinos evolve into mixture states which do not annihilate as efficiently as the photinos which are gauge coupled to fermions. Since the mixture state masses are increasing functions of λ and M_2, they become cosmologically allowed in the region shown at $M_2 \simeq 200$ GeV. As M_2 increases further, the mixture states evolve into pure higgsino states whose mass exceeds 40 GeV, which is the lightest sfermion mass for the chosen parameters.

In figures 2a and 2b, the gaugino mass is held constant at 100 and 200 GeV respectively. (We do not analyse the case of A = 2.5 as the parameter space is severely restricted by requiring that the minimum chargino mass is greater than 20 GeV). The minimum value of λ is an increasing function of $m_{3/2}$ which explains the diagonal lines across the graphs. The lightest cosmologically allowed neutralino in figure 2a is a 5.5 GeV nino at $m_{3/2} \simeq 20$ GeV, as shown. At the more realistic value of $m_{3/2} = 105$ GeV we find cosmologically allowed nino states of 4.5 GeV when λ is sufficiently large ($\simeq .1$) which yield the critical closure density. As λ increases for this fixed value of $m_{3/2}$, the nino evolves into a mixture state which annihilates less efficiently and is not cosmologically allowed. The minimum value of λ for cosmologically allowed ninos is an increasing function of $m_{3/2}$, as evidenced by the upward slope of the allowed region, because the nino mass is an increasing function of λ and a decreasing function of $m_{3/2}$. For larger values of $m_{3/2}$, therefore, we need larger values of λ to yield a sufficiently large nino mass for efficient annihilation to take place.

The photino states in the upper right hand corner of the graph, (which are approximately 35 GeV) are not cosmologically allowed as the large gravitino mass has forced the sfermion masses to increase, thereby inhibiting t-channel exchange.

The analysis for figure 2b is similar to that of figure 2a. We note that the change in the upper right hand corner to cosmologically allowed photinos is due to their increased mass ($\simeq$85 GeV) arising from the change of M_2 from 100 to 200 GeV.

In figure 3 we have set $M_2 = m_{3/2}$. Clearly, for $m_{3/2} \simeq 100$ and 200 GeV the analysis is the same as in figures 2a and 2b respectively. The main difference, then, is for small $m_{3/2}$ where we find that the small gaugino mass term results in photino states as opposed to mixture states being the LSP. The cosmological analysis is not significantly affected.

In summary, we have studied cosmological constraints on the parameters of a supersymmetric version of the standard model coupled to $N = 1$ supergravity, with an additional gauge singlet superfield. These constraints arise from demanding that the relic abundance of the LSP be consistent with $\Omega \leqslant 1$. Previous analysis [4], without the N−field, has shown tht LSP states of less than the top quark mass are restricted by this cosmological consideration to be photinos. We find, however, that by including the N−field and allowing for annihilation via s−channel Higgs exchange, nino states can be the lightest cosmologically allowed LSP. In particular, we find that a 4.5 GeV nino is a potential dark matter candidate and is fairly insensitive to changes in the gravitino and gaugino mass terms. However, as the minimum value of λ is sensitive to changes in the value of A, so are these nino states. We have also seen that photinos are possible dark matter candidates, but their relic density is more sensitive to these parameters and hence requires a degree of fine tuning. In the next chapter we shall concentrate more closely on the restrictions which arise from demanding that the LSP gives rise to $\Omega = 1$, that is, requiring the LSP to compose the dark matter.

FOOTNOTES

(1) We are concerned with absolute minima and do not consider the possibility of long-lived local minima, which could be consistent with present day observations.

(2) In our case, because the vevs of the two Higgs fields are equal, this channel decouples.

(3) We correct a minor sign error of Ref [4] in equations (VII.15 c,d).

(4) The approximation $q = \surd(m_\chi{}^2 - m_f{}^2)$ used in Ref. [4] breaks down near threshold ($m_\chi \approx m_f$). This is unimportant in our case, but effects the maximum relic higgsino densities computed in Ref. [4].

(5) We correct a minor error in Ref. [6] which affects their numerical results. In particular, the upper bound on the gravitino mass (for a value of A near 1) is substantially increased.

FIGURE CAPTIONS

FIGURE 1: Solid lines partition parameter space according to the cosmological categories cited in the text. Hatched areas denote regions within 10% of the critical closure density (category II), other regions are labelled numerically, following the notation of the text. Dotted lines partition parameter space according to the dominant neutralino state present in each. Each such region is labelled by one of the five basic neutralino types (capital lettes) described in the text. Note that the axes represent λ and M_2.

(a) $m_{3/2} = 100$ GeV, $A = 1$.

(b) $m_{3/2} = 200$ GeV, $A = 1$.

(c) $m_{3/2} = 100$ GeV, $A = 2.5$.

FIGURE 2: As in figure 1 with the axes representing λ and $m_{3/2}$.

(a) $M_2 = 100$ GeV, $A = 1$.

(b) $M_2 = 200$ GeV, $A = 1$.

FIGURE 3: As in figure 1.

$M_2 = m_{3/2}$, $A = 1$.

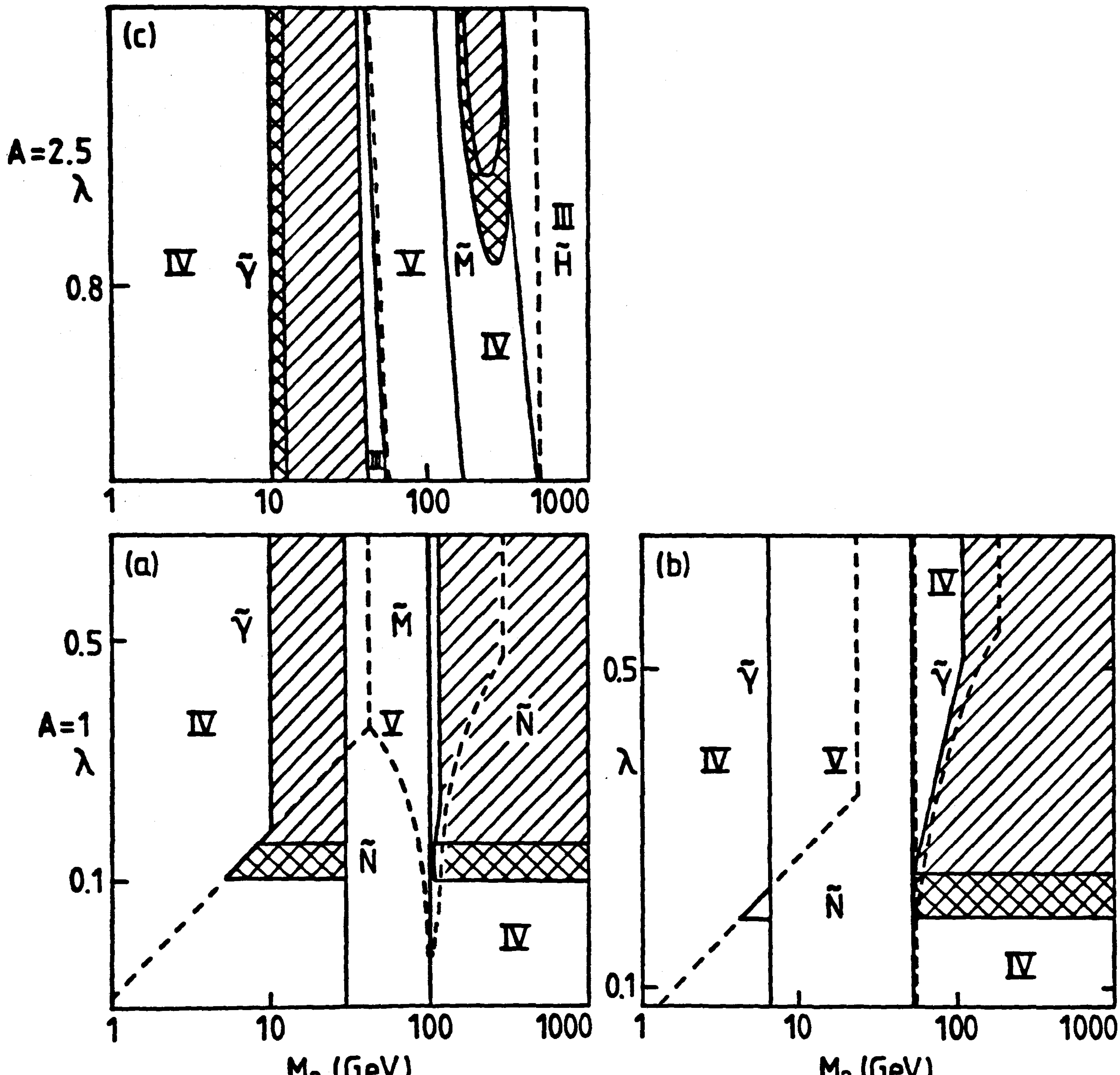
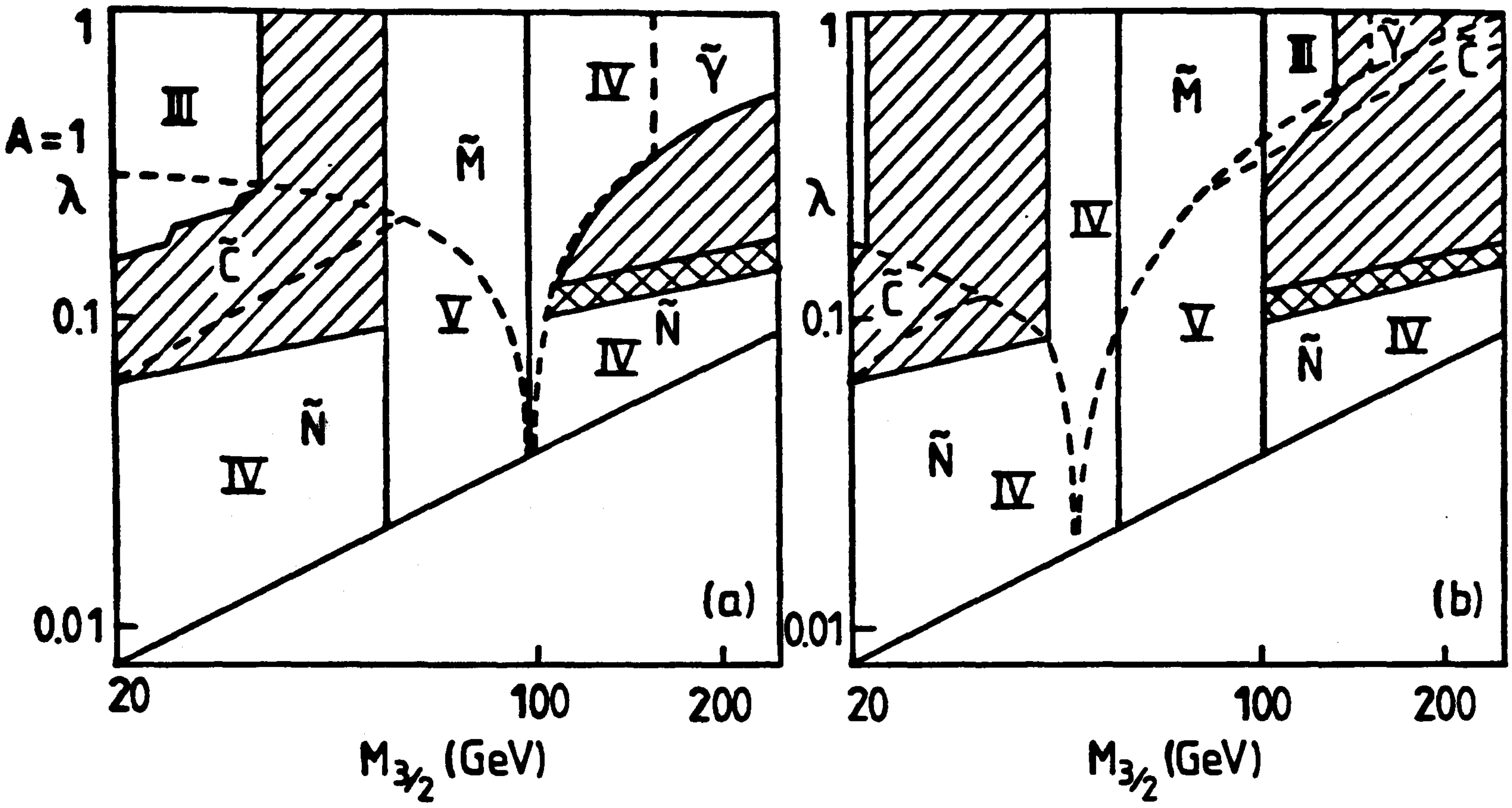
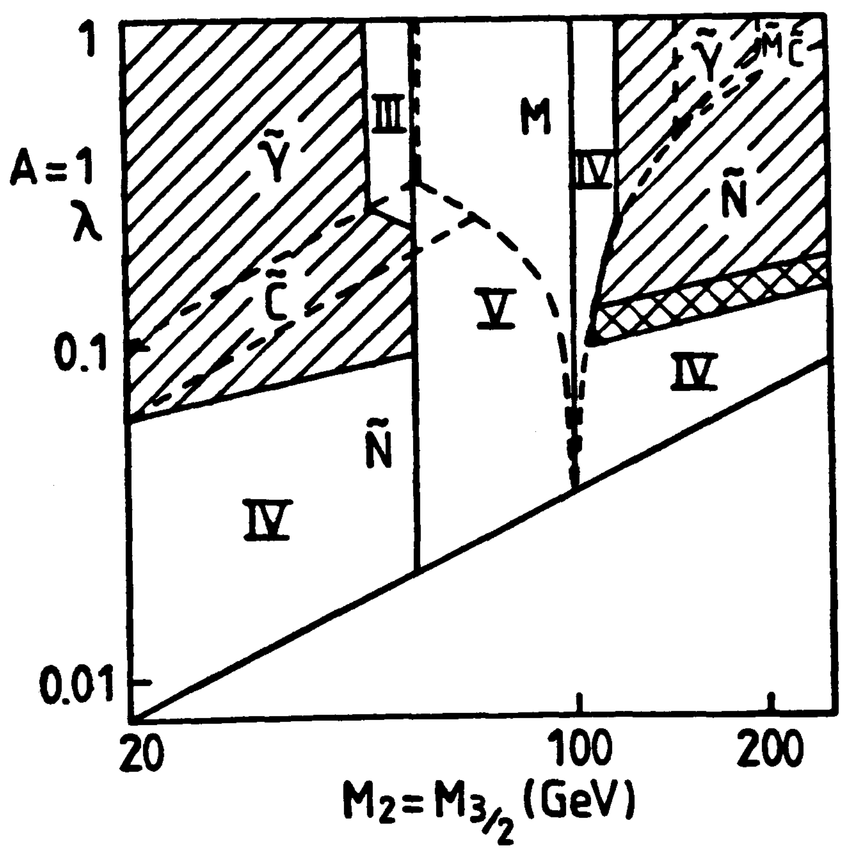

(c)
A=2.5
λ
0.8
IV
$\tilde{\gamma}$
V
$\tilde{M}$
III
$\tilde{H}$
IV
III
1
10
100
1000
(a)
$\tilde{\gamma}$
$\tilde{M}$
0.5
A=1
λ
IV
V
$\tilde{N}$
$\tilde{N}$
0.1
IV
1
10
100
1000
M_2 (GeV)
(b)
IV
0.5
$\tilde{\gamma}$
$\tilde{\gamma}$
λ
IV
V
$\tilde{N}$
0.1
IV
1
10
100
1000
M_2 (GeV)

Fig. 1



1
0.1
0.01
A = 1
λ
III
IV
V
$\tilde{M}$
$\tilde{C}$
$\tilde{N}$
$\tilde{\gamma}$
(a)
(b)
20
100
200
$M_{3/2}$ (GeV)

Fig. 2



1
0.1
0.01
A = 1
λ
III
IV
V
M
$\tilde{M}$
$\tilde{C}$
$\tilde{N}$
$\tilde{\gamma}$
20
100
200
$M_2 = M_{3/2}$ (GeV)

Fig. 3

CHAPTER VII

REFERENCES

1. B.R. Greene and P.J. Miron, Phys. Lett., 168B (1986) 226.
2. J.M. Frére, D.R.T. Jones and S. Raby, Nucl. Phys., B222 (1983) 11.
3. J.P. Derendinger and C.A. Savoy, CEN-Saclay preprint, SPht/83-98 (1983).
4. A. Barroso and J.C. Romao, Phys. Lett., 158B (1985) 51.
5. J. Ellis, J.S. Hagelin, D.V. Nanopolous, M. Srednicki, Phys. Lett., 127B (1983) 233; Nucl. Phys., B238 (1984) 453.
6. B.W. Lee and S. Weinberg, Phys. Rev. Lett., 39 (1977) 165.

CHAPTER VIII

ASTROPHYSICAL IMPLICATIONS OF SUPERSYMMETRIC MODELS

VIII.1 INTRODUCTION

We discuss the observational implications of the hypothesis that the galactic dark matter is composed of supersymmetric particles [1]. The requirement of $\Omega = 1$, as indicated in the graphs of the last chapter, is highly restrictive. We shall see that such constraints are further refined by demanding consistency with observations of high energy solar neutrinos. In section 1 we review the dynamics of solar capture and the conclusion of Steigman, Gaisser and Tilav [2] that the observed level of high energy neutrinos from the sun rules out photinos as the missing mass for a large range of parameters. In section 2 we extend this analysis to Higgsinos with $v_1/v_2 \gtrsim 2$, and also show that the nino (and Higgsinos with $v_1/v_2 \approx 1$) sidesteps this problem as its scattering cross section off of protons is comparatively small. In section 3 we discuss the possibility that the nino (or Higgsinos with $v_1 \simeq v_2$) could be responsible for the observed excess of GeV cosmic ray anti-protons. In section 4 we give our conclusions. This chapter contains the contents of ref. [1] which was researched and written in collaboration with D.N. Spergel.

VIII.2 CAPTURE AND ANNIHILATION OF THE LSP IN THE SUN

In the standard "cold dark matter" galaxy formation scenario [3], density perturbations induced during the inflationary epoch grow gravitationally, become non-linear and stop expanding with the Hubble flow. These perturbations will then collapse to form galaxies. As a result of the formation of the galaxy, the local density of the LSP is enhanced by a factor of 10^6 over the cosmological density and through "violent relaxation" the particles acquire velocities of $\simeq$ 300 km/s and a nearly thermal distribution function.

The local halo density of $\simeq$ 1 GeV/cm^3 implies high flux rate of halo particles through the Sun. Since the Sun's surface escape velocity of 617 km/s exceeds the mean velocity of particles in the halo, gravitational focusing will increase the flux. We will follow the treatment of ref. [4] and calculate the fraction of the incident flux which is captured.

Let us consider a particle with velocity at infinity, v_∞, hence, total energy, $m_\chi v_\infty^2/2$, where m_χ is the LSP's mass. Falling into the potential of the Sun, the particle will acquire additional kinetic energy; its velocity, v_χ, will greatly exceed v_∞.

$$v_\chi^2 = v_\infty^2 + \mu v_{esc}^2, \qquad \text{(VIII.1)}$$

where $\mu^{1/2} v_{esc}$ is the local escape velocity and v_{esc} is the escape velocity from the surface of the Sun (617 km/s). At the half-mass radius of the Sun, $\mu \approx 3.4$. The earth is much less centrally concentrated; at its half-mass radius, $\mu \approx 1.4$.

If in a typical elastic scatter off of a nucleus of mass m_n, the LSP energy loss,

$$\Delta E = \frac{m_\chi^2 \, m_n}{(m_\chi + m_n)^2} v_\chi^2 \qquad \text{(IVII.2)}$$

exceeds its energy at infinity, the particle will become bound. This implies a maximum capturable velocity,

$$v_{max} = \left[\frac{2\mu m_\chi m_n}{m_\chi^2 + m_n^2}\right]^{\frac{1}{2}} v_{esc} \ . \qquad \text{(VIII.3)}$$

If the LSP's velocity exceeds v_{max} at infinity, it will not be captured in a typical collision. Note that if the particle is capturable, then $v_\infty \lesssim v_{esc}$; hence, gravitational focusing is always important for capturable particles regardless of whether it is important for "typical" particles.

Since all of the capturable flux will be gravitationally focused, the incident flux as a function of velocity is:

$$\frac{dF}{d(v^2)} = \sqrt{13.5\pi} \, \frac{v_{esc}^2 R^2}{\bar{v}} \qquad \text{(VIII.4)}$$

where $\bar{v}$ is the r.m.s. velocity of the particles in the halo, and R is the radius of the Sun. The capturable flux can be estimated by integrating equation (VIII.4) up to either the scale height of the distribution, $2\bar{v}^2/3$, or the maximum capturable velocity, v_{max}^2, whichever is smaller. The fraction of this flux that will actually be captured will depend upon the scattering cross-section off of nuclei in the sun, σ. Ref. [4] estimates that the fraction of the incident capturable flux that will be trapped in the Sun scales as $0.89\sigma/\sigma_{crit}$, where σ_{crit} is the cross-section for which the radius of the Sun is one mean free path. For the Sun, $\sigma_{crit} = 4 \times 10^{-36}$; while for the Earth, $\sigma_{crit} = 1 \times 10^{-34}$.

Scattering off of Hydrogen is the primary capture mechanism for low mass particles with axial couplings to matter. Elastic collisions with Helium are the dominant means of trapping particles like scalar neutrinos which have vector couplings to matter. The more massive particles, however, are not always stopped by a single scatter off of a Hydrogen nucleus. Despite their low abundances, heavier elements play an important role in trapping more massive particles, especially those more massive than 30 GeV. This motivates us to sum over various isotopes in the Sun when estimating the particle capture rate, $\dot{N}_{capt}$,

$$\dot{N}_{capt} = \sqrt{6\pi}\,\frac{v_{esc}^2 R^2}{\bar{v}}\,\frac{\rho_\chi}{m_\chi}\,\sum_i \left[x_i\,\frac{\sigma_i}{\sigma_{crit}}\,\min\left[1.0\,\frac{3v_{max}^2}{2\bar{v}^2}\right]\right] \qquad \text{(VIII.5)}$$

where x_i is the isotopic abundance of nucleus of mass m_i and σ_i is the scattering cross-section of the WIMP (weakly interacting massive particle) off of the given isotope. Note that for WIMPs more massive than iron, the dominant capture mechanism for particles with spin-dependent axial couplings is still scatter off of ^1H with an additional 40% from N and ^{57}Fe, while elastic collisions with ^{56}Fe is the dominant mechanism of capturing particles with spin-independent vector couplings.

There are two processes that will counterbalance the capture of weakly interacting particles: evaporation and annihilation. Evaporation is the dominant loss mechanism for particles less massive than ≃4 GeV in the Sun. Annihilation is important for more massive particles. As long as enough particles are captured for the timescale for annihilation to be shorter than the age of the Sun ($n\sigma_{ann}v\tau_\odot >$ 1), we can assume that the WIMP population in the Sun is in equilibrium and that annihilation balances capture [5].

One of the end products of annihilation in the Sun are neutrinos. These high energy neutrinos could be detected by the IMB detector in Ohio, thus providing evidence for particles in the Sun and the halo [2,6,7]. If capture and anihilation are in equilibrium, $\dot{N}_{ann} = \dot{N}_{capt}/2$. The factor of 2 is due to every annihilation destroying 2 particles. The neutrino flux at the earth, F_ν, will depend upon the number of (prompt) neutrinos produced in a given annihilation $N_\nu(E_\nu)$, and the earth–Sun distance, D:

$$F_\nu(E_\nu) = \frac{\dot{N}_{capt}}{8\pi D^2} N_\nu(E_\nu)$$

$$= 4 \times 10^2 N_\nu(E_\nu) \left[\sum\left(\frac{\sigma_i}{\sigma_{crit}}\right)\min\left(1,\frac{3v_{max}^2}{2\bar{v}^2}\right)\right]\left(\frac{m_\chi}{1\mathrm{GeV}}\right)^{-1}\left(\frac{\rho}{\mathrm{GeV\ cm^{-3}}}\right)\mathrm{cm}^{-2}\mathrm{s}^{-1} \qquad \text{(VIII.6)}$$

The production of prompt neutrinos from fermion decays has been extensively studied in [2]. Adopting these results we have

$$N_\nu(E_\nu) = \sum_f B_{\chi f}(M_\chi)B_{fi}g_{fi}(y)/m_\chi \qquad \text{(VIII.7)}$$

where $B_{\chi f}$ denotes the branching ratio for $\chi\bar{\chi}$ annihilation into $f\bar{f}$ pairs. B_{fi} denotes the branching ratio for f into ν_i and $g_{fi}(y)$ is a normalized function with average value .3 (for ν_e and ν_μ), and somewhat less (due to the comparatively large τ mass) for ν_τ [2] that describe the energy distribution of the neutrinos produced in the annihilation. The parameter y denotes E_ν/m_χ

This large flux can produce a detectable signal in the IMB proton decay experiment. Using $(\sigma_\nu + \sigma_{\bar{\nu}}) \simeq 10^{-38}\ E_\nu/\mathrm{GeV\ cm^{-2}}$, predicts a count rate/kiloton/year in the IMB detector:

$$R_\nu = 40 N_\nu(E_\nu) \left[\sum\left(\frac{\sigma_i}{\sigma_{crit}}\right)\min\left(1,\frac{3v_{max}^2}{2\bar{v}^2}\right)\right]\left(\frac{m_\chi}{1\mathrm{GeV}}\right)^{-1}\left(\frac{\rho}{\mathrm{GeV\ cm^{-3}}}\right)\left(\frac{E_\nu}{1\mathrm{GeV}}\right)\mathrm{kT^{-1}yr^{-1}} . \qquad \text{(VIII.8)}$$

VIII.3 AVOIDING THE SOLAR NEUTRINO FLUX PROBLEM

The critical factor in equation (VIII.8) which determines the neutrino flux from the Sun is the scattering cross-section of the dark matter off of the protons in the Sun. Kane and Kani [8] have calculated this cross-section for both higgsinos and photinos. For photinos, this cross section is:

$$\sigma = 8 \times 10^{-39} \frac{M_{\tilde{\gamma}}^2}{(M_{\tilde{\gamma}}+M_p)^2} \left[\frac{M_W}{M_{\tilde{u}}}\right]^4 \mathrm{cm}^2 \qquad \text{(VIII.9)}$$

where $M_{\tilde{\gamma}}$ is the photino mass, M_W is the mass of the W, and $M_{\tilde{u}}$ is the mass of the up and down squarks. Our desire to have the LSP provide the closure density of the universe constrains the photino annihilation cross-section in the early universe to be 5×10^{-37} cm^2, thus effectively fixing the ratio $M_{\tilde{\gamma}}^2/M_{\tilde{u}}^4 \approx 10^{-5}$. While for higgsinos [8] this cross section is:

$$\sigma = 4 \times 10^{-38} \frac{(1-v_1^2/v_2^2)^2}{(1+v_1^2/v_2^2)^2} \frac{M_{\tilde{h}}^2}{(M_{\tilde{h}}+M_p)^2} \mathrm{cm}^2 \qquad \text{(VIII.10)}$$

where v_1 and v_2 are the vacuum expectation values of the Higgs superfields and $M_{\tilde{h}}$ is the mass of the higgsino.

Using the formalism reviewed in the preceding section, Steigman *et al* [2] have considered the annihilation of photinos in the Sun in detail. They conclude that the predicted neutrino flux from the Sun will exceed experimental limits if photinos dominate the halo and their mass is less than 27 GeV. For squark masses greater than 125 GeV, the solar capture rate and hence neutrino flux is acceptably small; the constraint of closure then requires a photino mass of greater than 27 GeV.

With $v_1/v_2 > 2$, higgsino annihilation predominantly proceeds through Z^0

exchange, and thus depends only weakly on squark and slepton masses. The closure constraint (for a choice of H_O) thus determines the higgsino mass [7]. As shown by Hagelin and Kane [7] for H_O = 50, $M_{\tilde{h}}$ = 5.9 GeV and for H_O = 100, $M_{\tilde{h}}$ = 4.5 GeV. We see from (VIII.9) and (VIII.10) that for this range of v_1/v_2 higgsino scattering off of solar nuclei has an efficiency comparable to that of photinos with $M_{\tilde{u}} \approx M_W$. As the only difference in the predicted solar neutrino flux (from VIII.8) for higgsinos, in comparison with photinos, is the branching ratios to fermions (higgsinos annihilation is not subject to charge suppression) we expect a similar rate. Computation confirms this and we find an event rate in excess of 1 event $(KT-yr)^{-1}$. Notice that this is rather insensitive to squark masses, and hence if $v_1/v_2 > 2$ the neutrino flux from a halo composed of higgsinos should have been detected at IMB.

One loophole in the neutrino constraint is to suppress higgsino–nucleon scattering by setting $v_1/v_2 \approx 1$. This however, also suppresses annihilation in the early universe. The cosmological constraint that $\Omega \leqslant 1$ then forces the higgsino to be more massive than the top quark.

Another solution is offered by the supergravity model with the tree level gauge symmetry breaking discussed in the last chapter. The requirement of closure in this model restricts the LSP to be a nino, a photino or a higgsino–gaugino mixture. The additional requirement of not overproducing neutrinos in the sun eliminates photinos, and makes the unlikely higgsino–gaugino candidate (see figures of Chapter VII) at best marginally acceptable. As this model has $v_1 = v_2$, the higgsino behaves as it has been described in the minimal model.

If Hubble's constant H_O = 100 km/s/Mpc, and if the nino is to provide the closure density, its mass must be 5 GeV [9]. This conclusion is fairly insensitive to the choice of $m_{3/2}$. On the other hand, if H_O = 50 km/s/Mpc, then the nino can satisfy the closure constraint, over a fairly wide mass range. As λ is increased, there is a cancelling competition between increasing scalar Higgs masses and increasing nino mass. For H_0 = 50 Km/s/MpC, the relic nino abundance provides the closure density for a nino mass between 9.5 GeV and the top mass. If the nino mass exceeds the top mass, annihilation through the $\tilde{H}_1^0$ amd $\tilde{H}_2^0$ components of the nino significantly reduces its relic abundance.

The nino scattering cross-section from up and down quarks through virtual scalar higgs exchange is computed by rotating the annihilation equations VII.14-VII.19 from an s-channel process to a t-channel process. Notice that this cross-section involves the relatively small (m_u^2/M_W^2) Hff Yukawa couplings, which enter through the C and D coefficients.

For the nino to close the universe, its mass must be greater than that of the bottom quark [9], into which it predominantly annihilates. The m_f^2 dependence then implies that its scattering off of u and d quarks is supressed relative to its annihilation rate by $(m_u/m_b)^2 \approx 10^{-4}$. Notice that photinos and higgsinos (with $v_1/v_2 > 2$) have gauge coupling interactions with fermions and hence do not exhibit this mass dependence. Demanding Ω = 1 implies similar early universe annihilation rates for ninos, photinos and higgsinos, but vastly different solar capture rates through scattering off of protons. The low nino scattering cross-section implies a low solar capture and hence annihilation rate. The neutrino annihilation signature of a halo composed of ninos would be far below background. The annihilation of the LSP in the halo into anti-protons may provide the only possible experimental signature of ninos and higgsinos (with $v_1 \approx v_2$).

VIII.4 ANTIPROTON FLUX FROM THE MISSING MASS

Cosmic rays have played an important role in the development of particle physics, where they have served as probes of energies beyond the scales attainable by particle accelerators. Recently, several authors [2,7] have proposed that they may reveal the composition of the galactic halo, if the galactic "missing mass" is comprised of supersymmetric relics of the early universe.

One of the end products of LSP annihilation is anti-protons. If the LSP annihilates into quarks, hadronization will produce protons and anti-protons. Current observation of the anti-proton flux suggests the existence of an anomaly [10]; more anti-protons are observed than are predicted. We will explore the possibility that this excess anti-proton flux is the result of the annihilation of the LSP that composes most of the galactic halo.

The current universe is on average not dense enough to produce a significant annihilation rate. However, in galactic halos where the local density exceeds the cosmological average density by over 5 orders of magnitude, annihilation can produce a detectable signal. As we saw in section VIII.2, capture into stars can further enhance LSP density and lead to a high annihilation rate.

The anti-proton flux rate depends upon its local density, n_χ, and the rate of annihilation into anti-protons, $<\sigma v>_{\bar{p}}$,

$$\dot{n}_{\bar{p}} = n_\chi^2 <\sigma v>_{\bar{p}} \qquad \text{(VIII.11)}$$

In the halo of the galaxy, $v \approx 300$ km/s $\approx 10^{-3}$ c. The average density of halo at 8 kpc, the earth's distance from the galactic centre, is 0.3 GeV/cm^3. This density is inferred from galactic rotation curves [11] and has an uncertainty of a factor 2. Locally, in the disc of our galaxy, the halo density is probably enhanced by the gravitational effects of the formation of the disc. Binney, May, and Ostriker [12] estimate that this enhancement increases the halo density by a factor 3. This increases the predicted anti-proton production rate by an order of magnitude.

The anti-protons will be trapped in the galactic disc by the galactic magnetic field. The interstellar density is too low for the anti-protons to annihilate, and the interstellar magnetic fields are too weak to generate significant synchotron radiation losses. Eventually, the anti-protons will leak out along open field lines and escape into the intra-galactic medium. The cosmic-ray confinement time in the galaxy, τ can be estimated from measurements of the isotopic composition of cosmic-rays, particularly Beryllium. The half-life of ^{10}Be, 1.6×10^6 years, is on the order the cosmic ray lifetime, thus, it makes an excellent tracer of cosmic ray history. Weibenback and Greiner [13] deduce $\beta\tau = 8.4(+4.0,-2.4) \times 10^6$ years from their measurements of Be isotopic abundances, where β is the cosmic ray particle's velocity in units of c.

The estimated flux of anti-protons at the earth will be proportional to the anti-proton density, $n_{\bar{p}} = \dot{n}_{\bar{p}}\tau$ and the particle velocity, βc.

$$F_{\bar{p}} \approx 2.4\times10^{-2}\left[\frac{\langle\rho^2\rangle}{1\mathrm{GeV}^2\mathrm{cm}^{-6}}\right]\left[\frac{m}{1\mathrm{GeV}}\right]^{-2}\left[\frac{\langle\sigma v\rangle_{\bar{p}}}{3\times10^{-27}\mathrm{cm}^3\mathrm{s}^{-1}}\right]\left[\frac{\tau}{8.4\times10^6\mathrm{yr}}\right]\ \mathrm{cm}^{-2}\mathrm{s}^{-1} \tag{VIII.12}$$

Note that the annihilation rate is lower for more massive particles because their *number* density is lower. Equation (VIII.12) can also be used to calculate the cosmic ray positron flux.

The number of anti-protons produced in each LSP annihilation is determined by the LSP's branching ratio to the available quark pairs. Due to the $m_q^2/m_{\tilde{H}}^2$ dependence of the nino annihilation cross-section, it annihilates primarily to the most massive quark– anti–quark pair: $b\bar{b}$. Following Silk and Srednicki [7], we use the values inferred from e^+e^- annihilations:

$$\begin{aligned} \chi\chi &\to \bar{\tau}\tau \to 1.5(e^+e^-) + 0(\bar{p}p) + 1\gamma + 5.5\nu \qquad \text{(VIII.13)} \\ &\to \bar{c}c \to 4.0(e^+e^-) + 0.2(\bar{p}p) + 7\gamma + 22\nu \\ &\to \bar{b}b \to 7.5(e^+e^-) + 0.3(\bar{p}p) + 13\gamma + 41\nu \end{aligned}$$

In the $v_1 \approx v_2$ higgsino scenario, annihilation proceeds mainly into top quarks. We conservatively estimate that .4 $\bar{p}p$ pairs are produced from an initial annihilation into a $t\bar{t}$ pair. Gamma rays produced by the LSP annihilating in the halo may also provide a detectable signal [14]. The gamma rays and neutrinos are, of course, not trapped by the galactic magnetic fields. These fluxes can be determined by Gauss' law: all of the photons produced in a sphere around the galactic centre will eventually exit through its surface:

$$F_\gamma \simeq \frac{\int_0^{R_\odot} \dot{n}_\gamma dV}{4\pi R_\odot^2} \simeq 2\text{x}10^{-5}\left[\frac{<\rho^2>}{1\text{GeV}^2\text{cm}^{-6}}\right]\left[\frac{m_\chi}{1\text{GeV}}\right]^{-2}\left[\frac{<\sigma v>_{\bar{p}}}{3\text{x}10^{-27}\text{cm}^3\text{s}^{-1}}\right]\text{cm}^{-2}\text{s}^{-1} \qquad \text{(VIII.14)}$$

For the nino, this flux will be too weak to be observable.

Table 1 lists the predicted cosmic densities and the expected flux of anti–protons, positrons, and gamma rays for a range of values of $m_{3/2}$ and λ. We use the same format as Hagelin and Kane [7] to facilitate comparisons. We concentrate our analysis on A near unity. The nino is not the LSP for larger values of A – the predicted LSP is a photino and gaugino–higgsino mixture state, which would produce too large a flux of neutrinos in the Sun. Larger values of λ would imply higher nino masses. Larger values of $m_{3/2}$ predict higher squark masses and lower anti–proton fluxes.

The nino does not produce an observable flux of positrons or gamma rays in the halo. Unlike the photino and the higgsino, the nino annihilation in the early universe is dominated by p–wave annihilation. This implies that the annihilation rate is strongly velocity dependent: $<\sigma v>_{pwave}$ v^2. In the early universe, $<v^2> \simeq 1/20$, while in the halo of our galaxy $<v^2> \simeq 10^{-6}$. The nino's annihilation through the scalar Higgs exchange (the C piece in equation (VII.19)) is suppressed by 5 orders of magnitude in the galactic halo. As λ increases, the nino has a larger higgsino component. The s–wave annihilation of the higgsino component is not suppressed at low velocity, thus if $m_{3/2} = 60$ GeV and if the nino mass is greater than 19 GeV, an observable flux of anti–protons will be produced in the galactic halo.

If the Hubble constant is larger ($H_o = 100$ Km/sec/Mpc) than the higher cosmological density implies a lower nino annihilation cross–section. Table 1 shows that the predicted nino annihilation rate is too slow to account for the galactic anti–proton anomaly. If $H_o = 100$ Km/sec/Mpc and the universe is closed, however, then the predicted age of the universe (7 billion years) is *shorter* than current estimates of the age of globular clusters and not much larger than the age of the solar system.

Table 2 also shows that if $v_1 = v_2$, higgsinos may also produce an observable anti–proton flux through initial annihilation into top quarks.

VIII.5 CONCLUSIONS

Supersymmetric versions of the standard model require a host of presently unobserved superparticles. These models predict that the relic abundance of the lightest such supersymmetric particle may provide the closure density of the universe. This LSP would also compose the halo of our galaxy. Through solar capture and subsequent annihilation, GeV neutrinos would be produced; if their flux is large enough, these neutrinos could be observed at the IMB detector.

We have reviewed the conclusions of Steigman *et al* that the current data rule out photinos less massive than 30 GeV and have shown that the conclusion can be extended to higgsinos.

Supersymmetric models with weak gauge symmetry breaking at the tree level offer a new LSP candidate, the nino. Because the nino scattering off of protons is suppressed by relatively small Yukawa couplings, comparatively few ninos accumulate in the Sun. Thus the neutrino signal from their solar annihilation is very weak.

If ninos do compose the halo, anti-proton flux may provide the only signal of their presence. Despite a local density of 0.3 cm^{-3}, the nino would be difficult to detect through other means. Its negligible cross-section for scatter off of nuclei makes it extremely difficult to detect the elastic recoil of a nino. The predicted count rate per kilogram per day in a low-temperature detector [14] would be below the expected neutrino background. The strong temperture dependence of the nino annihilation cross-section due to the dominance of the p-wave term results in an annihilation rate in the halo which is much smaller than in the early universe. Hence, the flux of gamma rays [14] from nino annihilation would be below background.

We have not discussed other possible LSP candidates that might avoid the neutrino constraints. The necessary condition to escape detection is either reducing the halo number density by increasing the LSP mass, or decreasing the scattering cross-section. Campbell *et al* [15] have discussed supersymmetric models that include not only a Higgs singlet but also an additional U(1) field. This superstring motivated model predicts that squark masses are much larger than the slepton masses. This suppresses scattering with baryons relative to annihilation by a factor $m_{\tilde{\gamma}}^4/m_{\tilde{q}}^4$. This small cross-section implies a lower number density in the Sun and a solar annihilation neutrino flux that is below background. The LSP in this theory is more massive (10-80 GeV) than in minimal models; hence, it produces a larger signal in recoils off of nuclei and may be detectable in terrestial experiments if the background can be sufficiently reduced.

If the annihilation of supersymmetric relic particles in the halo is responsible for the anti-proton background, then the anti-proton spectrum should have a cut-off at the LSP's mass. Extending observations of the cosmic ray background up to tens of GeV could confirm or reject this hypothesis for the origin of cosmic rays [7].

Hubble constant		50	50	50	50	50
m_χ=(GeV)		9.5	12	15.5	19	22
ρ(gm/cm^3)		5×10^{-30}	5×10^{-30}	5.2×10^{-30}	5.6×10^{-30}	5.9×10^{-30}
σv_f(cm^3/sec)		1.5×10^{-26}	1.6×10^{-26}	1.6×10^{-26}	1.5×10^{-26}	1.3×10^{-26}
σv_{cold}		1.1×10^{-29}	3.1×10^{-29}	7.0×10^{-29}	1.3×10^{-28}	2.4×10^{-28}
BR($\bar{b}b$)		.86	.86	.86	.87	.87
BR($\bar{c}c$)		8.6×10^{-2}	.08	7.8×10^{-2}	.077	.076
BR($\bar{\tau}\tau$)		5.2×10^{-2}	.049	4.8×10^{-2}	.047	.047
$\bar{p}$	$\bar{b}b$	2.8×10^{-7}	4.6×10^{-7}	6×10^{-7}	8.9×10^{-7}	1.1×10^{-6}
	$\bar{c}c$	1.9×10^{-8}	2.9×10^{-8}	4×10^{-8}	5.3×10^{-8}	6.5×10^{-8}
e_+	$\bar{b}b$	7.1×10^{-6}	1.6×10^{-5}	1.7×10^{-5}	2.2×10^{-5}	2.8×10^{-5}
	$\bar{c}c$	3.8×10^{-7}	5.8×10^{-7}	8.1×10^{-7}	1.0×10^{-6}	1.3×10^{-6}
	$\bar{\tau}\tau$	8.8×10^{-8}	1.3×10^{-7}	1.9×10^{-7}	2.4×10^{-7}	3.0×10^{-7}
γ	$\bar{b}b$	4.2×10^{-9}	6.8×10^{-9}	9.9×10^{-9}	1.3×10^{-8}	1.6×10^{-8}
	$\bar{c}c$	2.3×10^{-10}	3.4×10^{-10}	4.8×10^{-10}	6.2×10^{-10}	7.7×10^{-10}
	$\bar{\tau}\tau$	2.0×10^{-11}	3.0×10^{-11}	4.2×10^{-11}	5.5×10^{-11}	6.9×10^{-11}

TABLE I : Nino Scenario Flux Rates

BR denotes branching ratio and flux rates are in units of (cm^{-2}s^{-1}).

cont'd...

Table 1 continued.....

Hubble constant		50	50	50	50	100
m_χ=(GeV)		26.3	30.2	34.5	39	5
ρ(gm/cm^3)		6.2×10^{-30}	6×10^{-30}	6×10^{-30}	5.8×10^{-30}	1.4×10^{-29}
σv_f(cm^3/sec)		1.3×10^{-26}	1.3×10^{-26}	1.2×10^{-26}	1.2×10^{-26}	5.5×10^{-27}
σv_{cold}		3.9×10^{-28}	6×10^{-28}	9×10^{-28}	1.3×10^{-27}	7×10^{-31}
BR(bb)		0.87	·87	·87	·87	·74
BR(cc)		.076	.076	.075	.075	.16
BR($\tau\tau$)		.047	.047	.047	.047	.09
$\bar{p}$	$\bar{b}b$	1.3×10^{-6}	1.5×10^{-6}	1.8×10^{-6}	2×10^{-6}	5.6×10^{-8}
	$\bar{c}c$	7.8×10^{-8}	9×10^{-8}	1×10^{-7}	1.1×10^{-7}	7.9×10^{-9}
e^+	$\bar{b}b$	3.4×10^{-5}	3.9×10^{-5}	4.5×10^{-5}	5×10^{-5}	1.4×10^{-6}
	$\bar{c}c$	1.6×10^{-6}	1.8×10^{-6}	2.0×10^{-6}	2.3×10^{-6}	1.6×10^{-7}
	$\bar{\tau}\tau$	3.6×10^{-7}	4.2×10^{-7}	4.8×10^{-7}	5.3×10^{-7}	3.5×10^{-8}
γ	$\bar{b}b$	2.0×10^{-8}	2.3×10^{-8}	2.6×10^{-8}	2.9×10^{-8}	8.2×10^{-10}
	$\bar{c}c$	9.3×10^{-10}	1.1×10^{-9}	1.2×10^{-9}	1.3×10^{-9}	9.4×10^{-11}
	$\bar{\tau}\tau$	8.2×10^{-11}	9.5×10^{-11}	1×10^{-10}	1.2×10^{-10}	7.9×10^{-12}

Hubble const.		50	100
m_χ = (GeV)		40.1	40.1
ρ(gm/cm^3)		5×10^{-30}	2×10^{-29}
σv_f(cm^3/sec)		5.7×10^{-27}	1.4×10^{-27}
σv_{cold}		5.7×10^{-27}	1.4×10^{-27}
BR($\bar{b}b$)		3.2×10^{-4}	1.3×10^{-3}
BR($\bar{c}c$)		1.9×10^{-7}	7.6×10^{-7}
BR($\bar{\tau}\tau$)		4.2×10^{-7}	1.6×10^{-6}
BR($t\bar{t}$)		.999	.999
	$\bar{b}b$	3.2×10^{-9}	3.2×10^{-9}
$\bar{p}$	$\bar{c}c$	1.2×10^{-12}	1.2×10^{-12}
	$t\bar{t}$	1.3×10^{-5}	3.2×10^{-6}
	$\bar{b}b$	7.9×10^{-8}	7.9×10^{-8}
	$\bar{c}c$	9.2×10^{-12}	9.26×10^{-12}
e^+	$\bar{\tau}\tau$	1.0×10^{-11}	2.0×10^{-11}
	$t\bar{t}$	2.4×10^{-4}	6×10^{-5}
	$\bar{b}b$	4.6×10^{-11}	4.0×10^{-11}
	$\bar{c}c$	1.5×10^{-14}	1.5×10^{-14}
γ	$\bar{\tau}\tau$	4.6×10^{-15}	4.6×10^{-15}
	$t\bar{t}$	1.4×10^{-7}	3.5×10^{-8}

TABLE II : Higgsino Scenario ($v_1 = v_2$) Flux Rates

BR denotes branching ratio and flux rates are in units of (cm^{-2}s^{-1})

CHAPTER VIII

REFERENCES

1. B.R. Greene and D.N. Spergel, Oxford preprint.
2. G. Steigman, T. Gaisser and Tilav, "Closing the Door on Cold Dark Matter", Moriond Workshop on Neutrinos in Physics and Astrophysics, ed. T. Tranh, in press; S. Tilav, M.Sc. Thesis unpublished.
3. For Review, see G. Blumenthal, S. Faber, J. Primack and M. Rees, Nature 311 (1984).
4. W.H. Press and D.N. Spergel, Ap. J., 296 (1985) 675.
5. L. Krauss, K. Freese, W.H. Press and D.N. Spergel, Ap. J., 299, (1985) 1001.
6. J. Silk, K. Olive and M. Srednicki, Phys. Rev. Lett., 55 (1985) 257; K. Freese, "Can Scalar or Massive Neutrinos be the Missing Mass?", Phys. Lett. B, in press; L. Krauss, F. Wilczek and M. Srednicki, "Solar System Constraints and Signature for Dark Matter Candidates", ITP preprint−85−58.
7. J. Silk and M. Srednicki, Phys, Rev. Lett., 53 (1984) 624; F.W. Stecker, S. Rudaz and T.F. Walsh, Phys. Rev. Lett., 55 (1985) 2622; J. Hagelin and G. Kane, Nucl. Phys. B263 (1986) 399.
8. G.L. Kane and I. Kani, "Cross Sections for Dark Matter Particles and Implications for Allowed Masses, Interactions and Detection", Univ. of Michigan preprint UM−TH−85−20.
9. B.R. Greene and P.J. Miron, Phys. Lett., 168B (1986) 226.
10. E.A. Bogomolov, Proc. 17th Intl. CRC (Paris) 9 (1981) 146; R.L. Golden *et al.*, Apl. Lett., 24 (1984) 75.

11. For a standard model of galaxy mass distribution, see J. Bahcall and R. Soniera, Apl. J. Suppl., 55 (1984) 67.

12. J. Binney, A. May and J.P. Postriker, private communication.

13. M.E. Wiebenbeck and D.E. Griner, Apl. J. Lett., 239 (1980) L139.

14. M. Srednicki, S. Theisan and J. Silk, Phys. Rev. Lett., 56 (1986) 263.

15. B.A. Campbell, J. Ellis, K. Enqvist, D.V. Nanopoulos, J.S. Hagelin and K.A. Olive, CERN preprint 4385 (1986).

APPENDIX

APPENDIX

In this appendix we will give a brief review of N=1 supersymmetry and supersymmetric model building. It is by no means a complete treatment and is intended more for setting up notation and serving as a compendium of important basic results from supersymmetric model building which we use in the text. We closely follow some sections of [A1], [A2] and [A3].

A supersymmetry generator, Q_α, unlike an ordinary Lie group generator, is spinorial and is hence subject to anticommutation relations. Choosing $Q_\alpha \in (\frac{1}{2},0)$ representation of SO(3,1), its adjoint $\bar{Q}_{\dot\beta}$ transforms in the $(0,\frac{1}{2})$ representation. The anticommutator $\{Q_\alpha,\bar{Q}_{\dot\beta}\}$ is non-vanishing and transforms in the $(\frac{1}{2},\frac{1}{2})$ representation of the Lorentz group, i.e. like a four vector. By the Coleman-Mandula theorem [A4] this must be proportional to P_μ, the energy-momentum vector, as no other independent four-vector is conserved. Recall that the Pauli matrices allow us to map four vector indices to spinorial indices, and hence we have

$$\{Q_\alpha,\bar{Q}_{\dot\beta}\} = 2\,\sigma^\mu_{\alpha\dot\beta}\,P_\mu \ . \tag{A.1}$$

Choosing (A.1) to be the only non-trivial anticommutator amongst the Q_α (neglecting the possibility of central charges) and augmenting (A.1) with the algebra of the Poincaré group, we arrive at the simplest algebra of N=1 supersymmetry.

$$\begin{aligned}
\{Q_\alpha,\bar{Q}_{\dot\beta}\} &= 2\,\sigma^\mu_{\alpha\dot\beta}\,P_\mu \\
\{Q_\alpha,Q_\beta\} &= \{\bar{Q}_{\dot\alpha},\bar{Q}_{\dot\beta}\} = 0 \\
[P_\mu,Q_\alpha] &= [P_\mu,\bar{Q}_{\dot\alpha}] = 0
\end{aligned}$$

$$
\begin{aligned}
[Q_\alpha, M_{\mu\nu}] &= \tfrac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta \\
[P_\mu, P_\nu] &= 0 \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i\eta_{\nu\rho}M_{\mu\sigma} - i\eta_{\mu\rho}M_{\nu\sigma} - i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\mu\sigma}M_{\nu\rho} \\
[P_\mu, M_{\rho\sigma}] &= \eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho \ . \qquad \text{(A.2)}
\end{aligned}
$$

Computations in the framework of this graded Lie algebra are facilitated by the introduction of anticommuting Grassman parameters θ^α ($\alpha=1,2$) and $\bar{\theta}^\beta$ ($\beta=1,2$) which satisfy

$$
\{\theta^\alpha, \theta^\beta\} = \{\bar{\theta}^{\dot\alpha}, \bar{\theta}^\beta\} = \{\theta^\alpha, \bar{\theta}^\beta\} = 0 \ . \qquad \text{(A.3)}
$$

In terms of these parameters the graded Lie algebra of supersymmetry may be expressed as an ordinary Lie algebra over Grassman coordinates:

$$
[\theta Q, \overline{\theta Q}] = 2\theta\sigma_\mu\bar{\theta}P^\mu \qquad \text{(A.4)}
$$

$$
[\theta Q, \theta Q] = [\overline{Q\theta}, \overline{Q\theta}] = 0 \qquad \text{(A.5)}
$$

where $\theta Q \equiv \theta^\alpha Q_\alpha$, $\overline{Q\theta} = \bar{Q}_{\dot\alpha}\bar{\theta}^{\dot\alpha}$,

$$
2\theta\sigma_\mu\bar{\theta}P^\mu = 2\theta_\alpha\sigma_\mu{}^{\alpha\dot\beta}\bar{\theta}_\beta P^\mu \qquad \text{(A.6)}
$$

and spinorial indices are raised (lowered) by the antisymmetric matrix $\varepsilon^{\alpha\beta}$ ($\varepsilon_{\alpha\beta}$) with $\varepsilon^{12} = 1 = \varepsilon_{12}$.

We may now view supersymmetry transformations as acting on "superspace", points of which are specified by spacetime and Grassman coordinates: $z^A = (x^\mu, \theta^\alpha, \bar{\theta}^{\dot\alpha})$. This action is explicitly found by using superspace to parameterize supersymmetric group elements, and computing the parametric motion under supersymmetric transformations. In particular, the general group element

$$G(x,\theta,\bar{\theta}) = e^{i(-x^m P_m + \theta Q + \overline{\theta Q})} \tag{A.7}$$

satisfies (by Hansdorf's formula)

$$G(0,\xi,\bar{\xi})G(x^m,\theta,\bar{\theta}) = G(x^m + i\theta\sigma^m\bar{\xi} - i\xi\sigma^m\bar{\theta}, \theta + \xi, \bar{\theta} + \bar{\xi}) \tag{A.8}$$

and hence the motion in superspace is generated by the differential operator representations

$$\xi Q + \overline{\xi Q} = \xi^\alpha(\partial/\partial\theta^\alpha - i\sigma_{\alpha\dot{\alpha}}{}^m\bar{\theta}^{\dot{\alpha}}\partial_m) + \bar{\xi}_{\dot{\alpha}}(\partial/\partial\bar{\theta}_{\dot{\alpha}} - i\theta^\alpha\sigma_{\alpha\beta}{}^m\varepsilon^{\beta\dot{\alpha}}\partial_m). \tag{A.9}$$

Notice that although we are still in the context of global supersymmetry, the generators have a superspace dependence through the terms linearly dependent on θ. We will find it very useful in constructing invariant lagrangians to have derivatives involving the Grassman coordinates which, like ∂_μ, (anti) commute with supersymmetry transformations. It is easily checked that if

$$\begin{aligned} D_\alpha &= \partial/\partial\theta^\alpha + i\sigma_{\alpha\dot{\alpha}}{}^m\bar{\theta}^{\dot{\alpha}}\partial_m \\ \bar{D}_{\dot{\alpha}} &= -\partial/\partial\bar{\theta}^{\dot{\alpha}} - i\theta^\alpha\sigma_{\alpha\dot{\alpha}}{}^m\partial_m \end{aligned} \tag{A.10}$$

then

$$0 = \{D_\alpha, Q_\beta\} = \{D_\alpha, \bar{Q}_\beta\} = \{\bar{D}_{\dot{\alpha}}, Q_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{Q}_\beta\}. \tag{A.11}$$

On superspace we may define functions $F_{\alpha\beta\ldots\dot{\gamma}\dot{\delta}}(x,\theta,\bar{\theta})$ known as superfields

on which representations of the algebra act by the above differential operators, Q and $\bar{Q}$. These superfields are understood in terms of their power series expansion in θ and $\bar{\theta}$, which, by the Grassman nature of the coordinates, terminates at second order (in
each variable). Suppressing external Lorentz indices,

$$\begin{aligned} F(x,\theta,\bar{\theta}) = f(x) &+ \theta\varphi(x) + \overline{\theta\chi}(x) + \\ &+ \theta\theta m(x) + \overline{\theta\theta} n(x) + \theta\sigma^m\bar{\theta}\nu_m(x) \\ &+ \theta\theta\overline{\theta\lambda}(x) + \overline{\theta\theta}\theta\psi(x) + \theta\theta\overline{\theta\theta}d(x) \end{aligned} \tag{A.12}$$

We thus define the transformation law for superfields by comparing $\theta,\bar{\theta}$ expansions on the left and right hand sides of

$$\begin{aligned} \delta_\xi F(x,\theta,\bar{\theta}) &= \delta_\xi f(x) + \theta\delta_\xi\varphi(x) + \bar{\theta}\delta_\xi\bar{\chi}(x) + \dots . \\ &\equiv (\xi Q + \overline{\xi Q})F \end{aligned} \tag{A.13}$$

(This expression defines $\delta_\xi f$, $\delta_\xi\varphi$... where Q and $\bar{Q}$ are the differential generators of equation (A.9)). Supersymmetric transformations on superfields, by this formalism, may be thought of as giving translations and rotations in superspace

$$\xi Q+\overline{\xi Q} : (x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}) \to F(x^\mu + i\theta\sigma^m\bar{\xi} - i\xi\sigma^m\bar{\theta},\ \theta+\xi,\ \bar{\theta}+\bar{\xi}). \tag{A.14}$$

There is a slight subtlety in the definition of a superfield which is easily obscured by the notation. Arbitrary functions $F_{\alpha..\gamma\dot{\alpha}..\dot{\gamma}}(x,\theta,\bar{\theta})$ on superspace are *not*, in general, superfields. Recall that superfields are to be understood in terms of their Grassmanian Taylor series expansion. For (A.13) to hold under supersymmetry transformations, the components of $F_{\alpha..\gamma\dot{\alpha}..\dot{\gamma}}(x,\theta,\bar{\theta})$ in this expansions must transform in the appropriate manner. *A priori*, we can simply define the component field transformation properties so that (A.13) holds, but only if these transformation properties have not been previously determined. For example, let $F(x,\theta,\bar{\theta})$ be a superfield. The transformation properties under supersymmetry of its component fields are then determined. Now consider $\partial/\partial\theta^{\alpha}F(x,\theta,\bar{\theta}) = F'_{\alpha}(x,\theta,\bar{\theta})$. The component field transformation properties of $F'_{\alpha}(x,\theta,\bar{\theta})$ under supersymmetry transformations are determined from those imposed by our assumption that $F(x,\theta,\bar{\theta})$ is a superfield, and in fact are easily seen to be incompatible with (A.13). Thus, $F'_{\alpha}(x,\theta,\bar{\theta})$ is a function defined on superspace which is not a superfield. It is precisely for this reason that we introduce a "covariant" derivative : a differential operator involving the Grassman variables which maps superfields to superfields. Clearly, any differential operator which (anti)−commutes with the supersymmetry generators is a suitable covariant derivative; the standard choice of basis in the superspace tangent space with this property is

$$\partial/\partial x^{\mu} \equiv \partial_{\mu} \;;\; \partial/\partial\theta^{\alpha} + i\sigma^{m}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_{m} \equiv D_{\alpha} \;;$$
$$-\,\partial/\partial\bar{\theta}^{\dot{\alpha}} - i\theta^{\alpha}\sigma_{\alpha\dot{\alpha}}{}^{m}\partial_{m} \equiv \bar{D}_{\dot{\alpha}} \qquad \text{(A.15)}$$

One easily checks $\{D_{\alpha},Q_{\beta}\} = \{D_{\alpha},\bar{Q}_{\dot{\beta}}\} = \{\bar{D}_{\dot{\alpha}},Q_{\beta}\} = \{\bar{D}_{\dot{\alpha}},\bar{Q}_{\dot{\beta}}\} = 0$.

Superfields thus provide linear realizations of the supersymmetry algebra. In general, however, these representations are reducible; to arrive at irreducible representations we must place covariant constraints on the superfields. By covariant constraint we refer to constraints which are epimorphisms of the superfield ring. The two most useful constraints are

i) $\bar{D}_{\dot{\alpha}}\Phi = 0$ defining a chiral superfield

and

ii) $V = V^{+}$ defining a vector superfield.

Using the variable $y^m = X^m + i\theta\sigma^m\bar{\theta}$ in terms of which $\bar{D}_{\dot{\alpha}} = \partial/\partial\bar{\theta}^{\dot{\alpha}}$, we see that the most general solution to (i) is

$$\Phi(y,\theta,\bar{\theta}) = A(y) + \surd 2\theta\psi(y) + \theta\theta F(y) \qquad (A.16)$$

and hence chiral superfields accommodate a (complex) scalar field, a spin ½ Weyl spinor, and an auxillary field. The solutions to (ii) are most lucidly presented in the Wess–Zumin Gauge [A.1] in which many gauge artefact components vanish. In this gauge V may be written

$$V^a = -\theta\sigma^u\bar{\theta}V^a{}_\mu(x) + i\theta\theta\overline{\theta\lambda}^a(x) - i\overline{\theta\theta}\theta\lambda^a(x) + \tfrac{1}{2}\theta\theta\overline{\theta\theta}D^a(x) \qquad (A.17)$$

where $V^a{}_\mu$ are spin–1 gauge bosons, λ^a are their spin ½ fermionic partners known as gauginos, D^a are auxillary fields and a is an (adjoint) gauge group index.

Since we have shown that supersymmetry transformations are just coordinate translations in superspace (and likewise when acting on superfields) we have a ready made procedure for constructing supersymmetric invariants and hence suitable actions : superspace integrals.

$$\delta_\xi \int d^4x d^4\theta F(x,\theta,\bar{\theta}) = \int d^4x d^4\theta \delta_\xi F(x,\theta,\bar{\theta})$$

$$= \int d^4x d^4\theta \left[F(x',\theta',\bar{\theta}') - F(x,\theta,\bar{\theta}) \right] \tag{A.18}$$

$$= 0$$

as the first term reduces to the second by a change of variables with unit Jacobean.

In the above we make use of the Berezin integral of Grassman variables:

$$\int d\theta \;= 0$$

$$\int d\theta\theta = 1 \tag{A.19}$$

and

$$d^4\theta \;= d^2\theta d^2\bar{\theta} = d\theta_1 d\theta_2 d\bar{\theta}_1 d\bar{\theta}_2 \;.$$

For chiral superfields Φ, (A.18) vanishes since (as can be seen in the y representation from (A.16)) Φ has no $\bar{\theta}$ dependence. For chiral superfields we thus use the supersymmetric invariant

$$\int d^4x d^2\theta\Phi. \tag{A.20}$$

From (A.19) we see that full superspace integration projects out the highest component of the superfield integrand, and hence generates "D" terms. Similarly, the superspace integration in (A.20) projects out the highest component of Φ, commonly referred to as "F" terms. Accordingly, a supersymmetric Lagrangian may be written as

$$L = \int d^4\, x \left[\int\!\!\int d^2\theta d^2\bar{\theta} L_D + \int d^2\theta L_F + h.c. \right] . \tag{A.21}$$

Gauge interaction kinetic energy terms are generated from

$$L_D = \Phi^{+} \; e^{2gV} \; \Phi \tag{A.22}$$

where

V^a are vector superfields

T^a are the gauge group generators in the representation associated with the (chiral) superfield multiplet $\Phi = (\Phi_1, \ldots, \Phi_m)$, and

$V = V^a T^a$.

Non-gauge interactions and pure supersymmetric Yang-Mills terms are generated from

$$L_F = g(\Phi) + \frac{i}{4} W^{\alpha} W_{\alpha} + \text{h.c.} \tag{A.23}$$

where $g(\Phi)$ (known as the superpotential) is a gauge invariant holomorphic function of the chiral superfields and

$$W_{\alpha} = -\frac{1}{4} \bar{D} \bar{D} e^{-V} D_{\alpha} e^{V}$$

contains the supersymmetric Yang-Mills field strength.

The scalar potential arising in L, in terms of the scalar components, z_i, of the chiral superfields is given by

$$V = \sum_i \left|\frac{\partial g(\Phi)}{\partial \Phi_i}\right|^2_{\Phi=z} + \sum_{\alpha,i,j} |g_\alpha z^i T^\alpha{}_i{}^j z_j|^2$$

$$= |F_i|^2 + \tfrac{1}{2}|D^\alpha|^2 . \qquad (A.24)$$

In any phenomenologically viable supersymmetric model, supersymmetry must be broken since we do not observe the degenerate scalar superpartners of the known fermions. In globally supersymmetric models supersymmetry is broken if and only if the minimum of the potential is non-vanishing. From (A.24) this occurs if any of the auxillary F or D fields acquire a non-zero vacuum expectation value.

In N=1 local supersymmetry (N=1 supergravity) which is our primary interest, local invariance requires the introduction of a spin-3/2 gravitino field $\psi_{\mu\alpha}$ (along with its spin-2 graviton partner) which acts as the gauge field of local supersymmetry. Supergravity Lagrangians depend on two functions:

(i) $G = K(z_i,z_i^*) - \log|g(z_i)|^2$ which is known as the Kähler potential and

(ii) $f_{AB}(\Phi)$ which transforms under supersymmetry like a chiral superfield and transforms under the gauge group like a symmetric product of two adjoint representations. The scalar potential in such locally invariant Lagrangians takes the form:

$$V = -\exp(-G)[3 + G_K(G^{-1})_\ell{}^k G^\ell] + \tfrac{1}{2} f_{AB}{}^{-1} D^A D^B \qquad (A.25)$$

where

$G_k = \partial G/\partial z_k \ , \ G_j{}^i = \partial^2 G/\partial z_i \partial z^{*j}$

$D^A = g G^i T_i{}^{Aj} z_j .$

Supergravity is broken if $\langle G_k \rangle$ or $\langle D^A \rangle$ develop non-zero vev's. As D-breaking has not proved fruitful in model building, we will concentrate on F-breaking through non-zero $\langle G_k \rangle$.

Unlike in global supersymmetry in which the fermionic partner of the auxillary field responsible for spontaneous supersymmetry brekaing becomes a Goldstone fermion, in supergravity the gravitino eats the would be "Goldstino" and acquires a mass

$$m_{3/2} = M \exp(-G_0/2) \qquad \text{(A.26)}$$

where $M = M_p/\sqrt{8\pi}$.

Note also that the potential may be fine tuned to vanish even in the presence of supergravity breakdown.

The spontaneous breakdown of supergravity is induced in many models by invoking a hidden sector containing gauge singlet fields which communicate with our "observable" sector only through gravitational interactions. A commonly used scheme for accomplishing this is to define the superpotential by

$$\tilde{g}(y_a, z_i) = h(y_a) + g(z_i) \qquad \text{(A.27)}$$

where y_a are hidden sector fields and z_i are observable sector fields. Choosing minimal kinetic terms $G_i^j = -\delta_i^j$ with $K(w_j, w_j^*) = -w_j w^j$ for $w = y$ or z, and vanishing D-terms, the scalar potential becomes

$$V = \exp \frac{(|y_a|^2+|z_i|^2)}{M^2} \left[\left| h_a + \frac{y_a^*}{M^2} \tilde{g} \right|^2 + \left| g_i + \frac{y_i^*}{M^2} \tilde{g} \right|^2 - \frac{3}{M^2} \left| \tilde{g} \right|^2 \right] + \tfrac{1}{2} D_A D^A \quad \text{(A.28)}$$

Assuming vev's in the hidden sector take the form

$$\langle y_a \rangle = b_a M, \ \langle h_a \rangle = \langle \partial h / \partial y_a \rangle = a_a^* m M, \ \langle h \rangle = m M^2 \quad \text{(A.29)}$$

with M much larger than any vev's in the observable setor, the low energy effective potential becomes

$$V = |\hat{g}_i|^2 + m_{3/2}^2 |z_i|^2 + m_{3/2}[z_i \hat{g}_i + (A-3)\hat{g} + h.c.] \quad \text{(A.30)}$$

where

$$m_{3/2} = \exp\left(\tfrac{1}{2}|b_a|^2|m,\right.$$

$$\hat{g}(z_i) = \exp\left(\tfrac{1}{2}|b_a|^2\right) g(z_i)$$

$$z_i \hat{g}_i = \textstyle\sum_i z_i (\partial \hat{g} / \partial z_i), \text{ and}$$

$$A = b_a^*(a_a + b_a).$$

Notice that the first term is just that which appears in global supersymmetry, and that all observable scalar fields obtain a common tree level mass of $m_{3/2}$.

REFERENCES

APPENDIX

A.1 J. Wess and J. Bagger, Supersymmetry and Sueprgravity, (Princeton University Press 1983).

A.2 H.P. Nilles, Phys. Rep. 110C (1984) 1.

A.3 G.G. Ross, Grand Unified Theories (Benjamin/Cummings 1984).

A.4 S. Coleman and J. Mandula, Phys. Rev., 159 (1967) 1251.