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MINIMAL REPRESENTATIONS OF U-DUALITY GROUPS

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Abstract

Equations of motion of eleven dimensional supergravity compactified to d dimensions on $(11-d)$-torus have a hidden non-compact global symmetry group $\mathrm{E}_{11-d(11-d)}$, being examples of U-duality groups. U-duality groups are conjectured to be broken to their discrete subgroups $\mathrm{E}_{11-d(11-d)}(\mathbb{Z})$ in the non-perturbatve M-theory.

In this thesis we study algebraic structures and unitary realizations of U-duality groups, in particular those arising from Maxwell-Einstein supergravity (MESGT) theories in $d = 5$, 4, 3 dimensions. These algebraic structures and their related symmetry groups arise also in the study of generalized space-times defined by Jordan algebras and related algebraic constructs. After reviewing the previous work on the Lorentz groups and conformal groups of spacetimes defined by Jordan algebras we study their quasi-conformal groups. In particular we give the geometric realization of these quasi-conformal groups in a basis covariant with respect to their generalized Lorentz groups.

Minimal realization of U-duality groups associated with $d = 3$ supergravities is constructed as an extension of conformal quantum mechanics using ideas and methods pioneered by Günaydin, Koepsell and Nicolai. Minimal unitary representations of such U-duality groups are presented ellucidating connection with alternative construction of minimal representations by Kazhdan, Pioline and Waldron.

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To my parents

Chapter 1

Introduction

In the 1970's Veneziano [110] proposed the dual amplitude

$$\mathcal{A}(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))} \qquad \alpha(s) = \alpha_0 + \alpha' s \tag{1.1}$$

in order to fit experimentally observed data from scattering of hadrons. Nambu and Goto [81] realized that the Veneziano amplitude could be derived from the theory of one-dimensional closed strings that propagate in $\mathbb{R}^{1,25}$ so as to minimise the area of 2 dimensional surface swept out. The model, however, was soon abandoned since it predicted a massless spin 2 particle but no massless hadrons of spin 2 were observed. Moreover, fixed angle large energy limit of the scattering amplitude did not agree with refined experimental data.

This deficiency of the dual model has been turned into its virtue by Scherk and Schwarz [97] and, independently by Yoneya [122] who proposed to identify the massless spin 2 particle of closed string theory with the graviton and suggested that string theory be interpreted as a quantum theory of gravity and that the scale α' should be identified with the Planck scale (see [96] for review).

Bosonic string theory, however, faced problems since its spectrum contained a tachyon, a particle of negative mass squared rendering a theory unstable (see [102] for recent reinterpretations of tachyon). To overcome this and to introduce fermions into play, a supersymmetric extension of bosonic string theory has been put forward which is tachyon free [94, 84].

One way to study perturbative super-string theory is to define a 2-dimensional superconformal field theory "living" on a stringy world-sheet, i.e. the time-history of 1-dimensional extended object, a string. Fields of 2D conformal field theory (CFT) are taken to be coordinates of string, being a map from world-sheet into target space where the string is said to propagate, i.e. $X^\mu(\sigma,\tau)$ and $\psi(\sigma,\tau)$. Strings interact "geometrically", by splitting and joining with each other. Dynamics of strings is governed by action functional that measures the "super-area" of the stringy world-sheet [33]. To account for interactions one should sum over world-sheets of different topology, which for two dimensional oriented surfaces is characterised by genus and number of punctures. Vibrational modes of a string are then interpreted as particles, and for each string coordinate there is a tower of vibrational modes, carrying different energies and spins, with the natural mass scale being that of Plank mass

$m_p \propto 1/\alpha'$. Modes with lowest energies are, after appropriate projection is applied, massless and are easiest to excite. Thus in a low energy limit $\mathcal{E} \lll m_p c^2$ of string theory only those massless modes should be relevant.

Five consistent perturbative super-string theories were constructed during the 70-s and the 80-s, referred to as type I with gauge group $\mathrm{SO}(32)$, type IIA, type IIB, and two heterotic string theories, with gauge groups $\mathrm{SO}(32)$ and $\mathrm{E}_8 \times \mathrm{E}_8$ all living in critical space-time dimension $d = 10$ ensuring that 2D CFT is invariant under the world-sheet reparameterisation [33, 91]. All the five string theories, among their low-energy modes, have target space-time metric $g_{\mu\nu}$, anti-symmetric field $B_{\mu\nu}$, dilaton ϕ and gravitino ψ_μ as well as possess target space-time supersymmetry. Thus all of them in the low-energy limit describe some 10-dimensional supergravity theory. Superstring theories are perturbatively finite, meaning that contribution of each term in genus expansion is ultra-violet divergencies free and therefore are consistent candidates for quantum theory of gravity.

Our current experiments indicate only four observable dimensions around us, thus one hopes that vacuum of string theory corresponds to space-time of geometry $\mathcal{M}^{1,3} \times \mathcal{T}$ where $\mathcal{T}$ is some manifold with its characteristic size sufficiently small not to be detectable at energies within reach of current technology. Such ideas were put forward by Kaluza and, independently by Klein at the beginning of 20-th century in their attempt to explain electromagnetism starting with pure general relativity in 5 dimensions. Having one of the dimensions compact implies that string center of mass momentum in the compact direction is quantised. Besides, being one dimensional object, the string can wind around compact dimension arbitrary number w of times:

$$X^9 = 2\sqrt{\alpha'}\frac{n}{R}\tau + 2\sqrt{\alpha'}wR\sigma + \text{oscillators} \tag{1.2}$$

where τ parametrises time-like direction and σ – space-like direction on the world sheet. We observe that interchanging momentum and winding numbers $n \leftrightarrow w$ supplemented by change of radius $R \to R' = \sqrt{\alpha'}R^{-1}$ leaves the string coordinate X^9 invariant. String theory, compactified on a torus of radius R, indeed turns out to be invariant with respect to this T-duality transformation. T-duality invariance is perturbative and holds for every order of genus expansion. It means that string compactified on torus of large radius R produces the same theory as if it was compactified on smaller radius R'.

Narain [82] showed that T-duality group of string theory compactified on torus T^n is $\mathrm{SO}(n, n; \mathbb{Z})$ – a discrete subgroup of maximally non-compact real form of $\mathrm{SO}(2n)$.

It was known [33, 92] that two heterotic superstring theories toroidally compactified down to $d = 9$ yield the same theory. Similarly type IIA and type IIB theories toroidally compactified to $d = 9$ also result in the same theory, see figure 1.1.

Supersymmetry provides a powerful tool to gain glimpses into non-perturbative aspects

of string theories. It is made possible by studying Bogomolnyi-Prasad-Sommerfield (BPS) states, i.e. states that transform in non-generic (short) representations of supersymmetry algebra, and by studying supersymmetry protected quantities (see [86, 53, 120, 100]). Perturbative corrections cannot change non-generic (short) representation into generic (long) one, and thus the property of state being BPS will be preserved for all values of coupling constant. The goal then is to identify physical quantities in the theory protected, thanks to supersymmetry, by non-renormalisation theorems and to understand their behaviour in the non-perturbative regime.

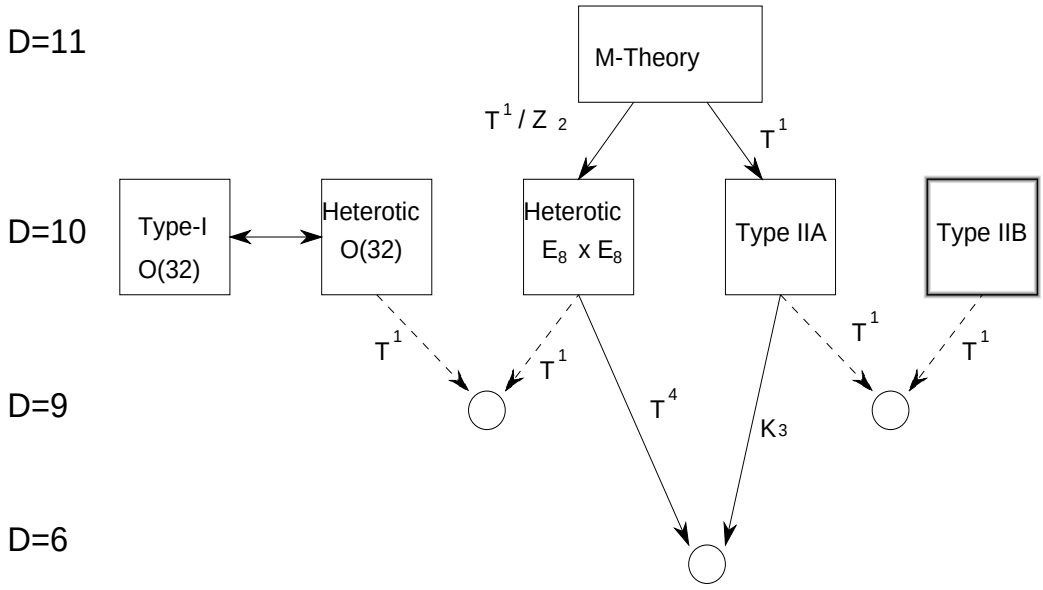

Figure 1.1: The web of dualities between string theories.
Broken lines correspond to perturbative duality connections. Type IIB in $d = 10$ is self-dual under $\mathrm{SL}(2, \mathbb{Z})$. Figure adopted from [66].

A breakthrough in understanding of non-perturbative string dynamics, known in folklore as second string revolution, came with discovery of non-perturbative dualities, first among field theories [99] (see also [72] for review and references) and soon after also among string theories (review and further references can be found for instance in [98, 109, 101]).

Web of perturbative and non-perturbative dualities (see Fig. 1.1) connect different regimes of perturbative string theories. Moreover in strong-coupling regime of type IIA an extra 11th dimension unfolds and theory becomes effectively described by eleven dimensional supergravity [119]. An existence of some unique eleven dimensional quantum theory has been conjectured [119, 108], dubbed M-theory while awaiting a better name. All the known consistent superstring theories and 11D supergravity are believed to correspond to different limits of M-theory. M-theory has received much of string theory community's attention

Dimension	Hidden sugra symmetry	T-duality	U-duality
10A	$\mathrm{SO}(1,1;\mathbb{R})/\mathbb{Z}_2$	1	1
10B	$\mathrm{SL}(2,\mathbb{R})$	1	$\mathrm{SL}(2,\mathbb{Z})$
9	$\mathrm{SL}(2,\mathbb{R})\times\mathrm{O}(1,1;\mathbb{R})$	$\mathbb{Z}_2$	$\mathrm{SL}(2,\mathbb{Z})\times\mathbb{Z}_2$
8	$\mathrm{SL}(3,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$	$\mathrm{O}(2,2,\mathbb{Z})$	$\mathrm{SL}(3,\mathbb{Z})\times\mathrm{SL}(2,\mathbb{Z})$
7	$\mathrm{SL}(5,\mathbb{R})$	$\mathrm{O}(3,3,\mathbb{Z})$	$\mathrm{SL}(5,\mathbb{Z})$
6	$\mathrm{O}(5,5,\mathbb{R})$	$\mathrm{O}(4,4,\mathbb{Z})$	$\mathrm{O}(5,5,\mathbb{Z})$
5	$\mathrm{E}_{6(6)}$	$\mathrm{O}(5,5,\mathbb{Z})$	$\mathrm{E}_{6(6)}(\mathbb{Z})$
4	$\mathrm{E}_{7(7)}$	$\mathrm{O}(6,6,\mathbb{Z})$	$\mathrm{E}_{7(7)}(\mathbb{Z})$
3	$\mathrm{E}_{8(8)}$	$\mathrm{O}(7,7,\mathbb{Z})$	$\mathrm{E}_{8(8)}(\mathbb{Z})$

Table 1.1: Duality symmetries of toroidally compactified type II string.

(see [85] for review and references) but despite the effort even its fundamental degrees of freedom are not known as yet.

Both perturbative T-dualities and non-perturbative S-dualities[1] act on the moduli space of M-theory. These two types of duality transformations commute and can be embedded into larger discrete group as it has been demonstrated for type IIB theory [54]. This wider group is termed U-duality group, where symbol U stands allegedly for unity. U-duality group, conjectured to be exact symmetry of non-perturbative M-theory compactified down to correspondent dimension, was shown to be a discrete subgroup of continuous non-compact symmetries of maximal effective supergravity theories (see table 1.1).

With every additional dimension compactified, theory acquires more and more scalar degrees of freedom, and its U-duality group which acts on those scalars non-linearly becomes larger. Bosonic field content of eleven dimension supergravity on shell is beautifully simple, and comprises a graviton g_{MN} and a rank 3 anti-symmetric field A_{MNK}. After toroidal compactification down to $d = 3$ the theory acquires 128 scalars degrees of freedom which transform non-linearly under U-duality symmetry. Since some of those scalars appear as a result of dualisation of higher rank gauge fields, we conclude that U-duality symmetry of $d = 3$ theory should mix elementary excitations with solitonic collective modes.

M-theory and 11 dimensional supergravity allow for compactification on some more elaborate backgrounds – Calabi-Yau manifolds, which are 6 real dimensional manifolds with special properties ensuring that resulting theory in $d = 5$ is supersymmetric with 8 supercharges [11, 25, 6]. Resulting supergravity theories depend on moduli of Calabi-Yau manifold and are known as $N = 2$ Maxwell-Einstein supergravity theories (MESGT) [39] coupled to hypermultiplets [13, 7].

U-duality groups similar to the one compiled in table 1.1 starting with $d = 5$ for toroidal compactifications also arise in five and lower dimensional supergravity theories coupled

[1] S-duality is the identification of strong coupling regime of one theory with a weak coupling regime of another (possibly the same) theory. It is also often referred to as strong/weak duality. See [72] for review.

d	$\mathcal{M}_{sc}$	U-duality
5	$\frac{\mathrm{SO}(n-1,1)}{\mathrm{SO}(n-1)} \times \mathrm{SO}(1,1)$	$\mathrm{SO}(n-1,1) \times \mathrm{SO}(1,1)$
	$\frac{\mathrm{SL}(3,\mathbb{R})}{\mathrm{SO}(3)}$	$\mathrm{SL}(3,\mathbb{R})$
	$\frac{\mathrm{SL}(3,\mathbb{C})}{\mathrm{SU}(3)}$	$\mathrm{SL}(3,\mathbb{C})$
	$\frac{\mathrm{SU}^*(6)}{\mathrm{Usp}(6)}$	$\mathrm{SU}^*(6)$
	$\frac{\mathrm{E}_{6(-26)}}{\mathrm{F}_4}$	$\mathrm{E}_{6(-26)}$
4	$\frac{\mathrm{SO}(2,1)\times\mathrm{SO}(n,2)}{\mathrm{SU}(2)\times\mathrm{SO}(n)\times\mathrm{SO}(2)}$	$\mathrm{SO}(2,1) \times \mathrm{SO}(n,2)$
	$\frac{\mathrm{Sp}(6,\mathbb{R})}{\mathrm{U}(3)}$	$\mathrm{Sp}(6,\mathbb{R})$
	$\frac{\mathrm{SU}(3,3)}{\mathrm{S}(\mathrm{U}(3)\times\mathrm{U}(3))}$	$\mathrm{SU}(3,3)$
	$\frac{\mathrm{SO}^*(12)}{\mathrm{U}(6)}$	$\mathrm{SO}^*(12)$
	$\frac{\mathrm{E}_{7(-25)}}{\mathrm{E}_6\times\mathrm{U}(1)}$	
3	$\frac{\mathrm{SO}(n+2,4)}{\mathrm{SO}(n+2)\times\mathrm{SO}(4)}$	$\mathrm{SO}(n+2,4)$
	$\frac{\mathrm{F}_{4(4)}}{\mathrm{USp}(6)\times\mathrm{SU}(2)}$	$\mathrm{F}_{4(4)}$
	$\frac{\mathrm{E}_{6(2)}}{\mathrm{SU}(6)\times\mathrm{SU}(2)}$	$\mathrm{E}_{6(2)}$
	$\frac{\mathrm{E}_{7(-5)}}{\mathrm{SO}(12)\times SU(2)}$	$\mathrm{E}_{7(-5)}$
	$\frac{\mathrm{E}_{8(-24)}}{\mathrm{E}_7\times\mathrm{SU}(2)}$	$\mathrm{E}_{8(-24)}$

Table 1.2: Duality symmetries of supergravities obtained by compactifications of $N = 2$ $d = 5$ MESGTs with scalar manifolds being symmetric spaces.

to matter [39]. They are summarised in table 1.2 for MESGTs with symmetric scalar manifolds.

In the present thesis we discuss the algebraic structure of U-duality groups arising both from MESGTs and from maximal supergravities. Algebraic structures of U-duality groups of maximal supergravity theories were previously studied in [45, 46]. We build a geometric realization of U-duality Lie algebras relevant to supergravity. For real forms of U-duality groups relevant to MESGTs it corresponds to Lorentz, conformal and quasi-conformal actions on generalised space-times associated with formally real Jordan algebras defining Maxwell Einstein supergravity theories [39, 43, 50]. This is achieved in the manner emphasising the algebraic structure of underlying Jordan algebras.

We also construct minimal [59] realizations of MESGT U-duality groups [48] as an extension of conformal quantum mechanics [1] in parallel to minimal realization of $\mathfrak{e}_{8(8)}$ [46]. Minimal representation is constructed in simple case of $\mathfrak{so}(4,4)$, and a connection with alternative studies [64] (see [88] for overview) of minimal representations is pointed out.

The structure of the thesis is as follows. We start with a brief review of relevant supergravity theories introducing the notion of scalar manifold, and that of hidden symmetries, by considering toroidal compactifications of eleven dimensional supergravity. Appearance of exceptional Lie groups is discussed for compactifications to five, four and three dimensions.

In chapter 3 we focus on the the general structure of U-duality groups as well as their geometric interpretation as Lorentz, conformal and quasi-conformal groups of Jordan algebras following [45] and [50]. The construction of geometric action of Lie algebras of U-duality groups is built using Jordan algebras. The inter-relation and connection of U-duality groups in different dimensions is discussed in details. Geometric realization of exceptional U-duality Lie algebras is given as a spinorial extension of quasi-conformal algebra $\mathfrak{so}\left(d+2,4\right)$ associated with d-dimensional Minkowski space-time $\mathbb{R}^{1,d-1}$.

Minimal realizations of U-duality Lie algebras is discussed in chapter 4. Minimal realizations of U-duality algebras related to $N=2$ Maxwell-Einstein supergravity theories is given following [48]. Minimal realization is viewed as extension of conformal quantum mechanics studied by de Alfaro, Fubini and Furlan [1]. Minimally realized U-duality algebras is then viewed as spectrum generating symmetry of this quantum mechanics. Extension by bosonic as well as fermionic fields are discussed.

Minimal unitary representation of U-duality algebras is analysed from the point of view of spherical vectors in the last chapter which is based on unpublished work.

Chapter 2

A glimpse of supergravity theories

Supergravity theories [19, 28] (or SUGRAs) were constructed as a supersymmetric extensions of Einstein theory of gravity soon after the discovery of space-time supersymmetry [115], in hope that extending symmetry group of Einstein gravity theory would render it quantizable. This turned out to be a false hope and SUGRAs were later determined not to be UV-finite [20]. They are understood, according to common wisdom, as a low-energy effective field theories [114] of M-/string theory [91, 92] and remain a topic of active research in this context.

If supersymmetric theory in d dimensions is to have no degrees of freedom with spin greater than $s = 2$ upon dimensional reduction to 4 dimensions, then supersymmetry restricts such theory to reside in space-time of dimension d no greater than 11 [79].

This 11 dimensional supergravity [14] can be dimensionally reduced to $d = 4$ by toroidal compactification. Julia and Cremmer showed [15] that, surprisingly this theory possesses a much richer global non compact hidden symmetry $E_{7(7)}$, local SU (8) symmetry and scalar manifold isomorphic to $\mathrm{E}_{7(7)}/\mathrm{SU}\,(8)$.

2.1 Short review of supersymmetry

Supersymmetry in four dimensional Minkowski space-time can be best illustrated by investigating symmetries of non-interacting Wess-Zumino [115] model:

$$\mathcal{S}_{WZ} = \int d^4x \left(\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi + i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi\right) \tag{2.1}$$

where ψ is Majorana spinor. Besides the well-known space time symmetries, Lorentz rotation and translation symmetries, this action is also invariant under the following supersymmetry transformation [118]:

$$\begin{aligned} &\partial_\epsilon \phi = \bar{\epsilon}\,(1-\gamma_5)\,\psi \qquad\qquad \partial_\epsilon \phi^* = \bar{\epsilon}\,(1+\gamma_5)\,\psi \\ &\partial_\epsilon \psi = -\left(i\gamma^\mu \partial_\mu \phi + m\phi\right)\frac{1+\gamma_5}{2}\epsilon - \left(i\gamma^\mu \partial_\mu \phi^* + m\phi^*\right)\frac{1-\gamma_5}{2}\epsilon \end{aligned} \tag{2.2}$$

provided that fields ϕ and ψ satisfy equations of motion of (2.1). Transformation parameter ϵ is Grassmann spinor. Transformations (2.2) intermix bosonic and fermionic degrees of freedom. Commutator of two supersymmetry transformations

$$\begin{aligned}[\delta_\epsilon, \delta_\eta]\,\phi &= -2i\,(\bar{\epsilon}\gamma^\mu\eta)\,\partial_\mu\phi = 2\,(\bar{\epsilon}\gamma^\mu\eta)\,P_\mu\phi \\ [\delta_\epsilon, \delta_\eta]\,\psi &= -2i\,(\bar{\epsilon}\gamma^\mu\eta)\,\partial_\mu\psi = 2\,(\bar{\epsilon}\gamma^\mu\eta)\,P_\mu\psi\end{aligned} \tag{2.3}$$

amounts to translation transformation provided that fields satisfy equation of motion. Referring to the fact that equations of motion are needed to show closure of supersymmetry algebra, one says that supersymmetry closes "on-shell". Thus Pòincare algebra gets extended by the supersymmetry transformations. Noether charges of supersymmetry transformation Q^a, translation P_μ and rotation $M_{\mu\nu}$ form supersymmetry algebra. Generators M and P form standard Pòincare algebra

$$\begin{aligned}[M_{\mu\nu}, M_{\lambda\rho}] &= \eta_{\nu\lambda}M_{\mu\rho} - \eta_{\mu\lambda}M_{\nu\rho} + \eta_{\mu\rho}M_{\nu\lambda} - \eta_{\nu\rho}M_{\mu\lambda} \\ [M_{\mu\nu}, P_\rho] &= \eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu \qquad\qquad [P_\mu, P_\nu] = 0\end{aligned} \tag{2.4a}$$

Supercharges Q are Majorana spinors of Lorentz algebra, and commute with all translations:

$$[M_{\mu\nu}, Q^a] = -(J_{\mu\nu})^a{}_b Q^b \qquad [P_\mu, Q^a] = 0 \tag{2.4b}$$

Supercharges Q anti-commute into translations

$$\left\{Q^a, Q^b\right\} = 2\left(\Gamma_\mu C^{-1}\right)^{ab} P^\mu \tag{2.4c}$$

which, from a group theoretic viewpoint is made possible because

$$((\mathbf{0},\mathbf{2}) + (\mathbf{2},\mathbf{0})) \otimes_S ((\mathbf{0},\mathbf{2}) + (\mathbf{2},\mathbf{0})) = \mathbf{4}_{\text{vector}}.$$

In here $(\mathbf{0},\mathbf{2})$ corresponds to chiral Weyl spinors $\Gamma_5\psi = +\psi$ and $(\mathbf{2},\mathbf{0})$ corresponds to anti-chiral Weyl spinors $\Gamma_5\psi = -\psi$.

Fields that irreducibly transform into each other under Poincarè supersymmetry transformation are called supersymmetry multiplet. All fields in the multiplet will have the same mass since $P_\mu P^\mu$ commutes with Q. Fields within a multiplet have spins which form arithmetic progression with step 1/2, in virtue of Q being a spinor of $\mathfrak{so}(3)$.

Possible multiplets of supersymmetry are classified by observing that (2.4c) evaluated on momentum eigenstates is isomorphic to a Clifford algebra, and thus its irreducible representations are finite dimensional [79]. An important implication of this fact is that one can not have more that 32 independent supercharges if we disallow fields with spin greater than 2 [79], that of a graviton.

Algebra (2.4) can be generalised by allowing independent supercharges Q_i^a, where $i = 1..N$. Then (2.4c) gets modified

$$\left\{Q_i^a, Q_j^b\right\} = 2\delta_{ij}\left(\Gamma_\mu C^{-1}\right)^{ab} P^\mu \tag{2.4c'}$$

and (2.4b) modified trivially. This algebra admits a central extension [52] by a set of generators Z_{ij} and Y_{ij} - central charges - which commute with every other generator in supersymmetry algebra and between themselves. Then (2.4c) becomes

$$\left\{Q_i^a, Q_j^b\right\} = 2\delta_{ij}\left(\Gamma_\mu C^{-1}\right)^{ab} P^\mu + Z_{ij}\left(C^{-1}\right)^{ab} + Y_{ij}\left(\Gamma_5 C^{-1}\right)^{ab} \tag{2.4c''}$$

In case of extended supersymmetry, the algebra gets also supplemented by R-symmetry algebra acting on i indices of supercharges:

$$[R_I, Q_k^a] = (\tau_I)^k{}_i Q_i^a \tag{2.4d}$$

Note the above restriction on number of supercharges implies that N can be at most 8. Supersymmetric theories with 32 supercharges are called *maximally supersymmetric* [105].

Witten and Olive [121] showed that non-trivial central charges appear in theories that possess topologically non-trivial solutions, solitons. Then central charges are related to topological charges of these solutions.

Supersymmetry algebra can be introduced for any flat space time $\mathbb{R}^{t,s}$ with t time-like and s space-like coordinates [106, 79]. Possible reality conditions that can be imposed on spinors are summarised in appendix A.1. Possible supersymmetry algebras are found requiring that $\{Q, Q\}$ closes into P_μ and some central charges. Classification of all such algebras was given by W. Nahm [79]. He found in particular, that if a d-dimensional theory is to contain fields of spin no greater than 2 upon toroidal compactification to four dimensions then $d \leqslant 11$. Also if a d-dimensional theory, when compactified to four dimensions should have particles of spin at most one, then $d \leqslant 10$.

Supersymmetry discussed so far was a global symmetry, in that super-symmetry transformation acted the same at all points in space-time. By making supersymmetry transformation local, i.e. depending on space-time coordinates x, we also make parameters of translation transformation local (c.f. (2.3)). We thus promote translations to infinitesimal general coordinate transformations. A theory invariant under the local supersymmetry is thus necessarily invariant under general coordinate transformations, and must be a supersymmetric extension of gravity theory.

2.2 11-dimensional supergravity theory

Eleven-dimensional supergravity [14] theory stands out, because 11 is maximum possible dimension for an admissible theory of supergravity [79]. Moreover, all other maximal supergravity theories in lower dimensions can be obtained from 11 dimensional one by dimensional reduction.

In $\mathbb{R}^{1,10}$ supercharges are 32 component Majorana-Weyl spinors of $\mathfrak{so}(1,10)$, thus $N=1$. Anti-commutator of supercharges hence takes on the following form

$$\left\{Q^a, Q^b\right\} = 2\left(\Gamma_\mu C^{-1}\right)^{ab} P^\mu \tag{2.5}$$

where $a, b = 1, \ldots, 32$. When applied to momentum operator eigenstate with finite momenta, the right-hand side of (2.5) is a constant symmetric matrix which can be diagonalised. It is easy to see that

$$\det\left(\Gamma_\mu C^{-1} P^\mu\right) = (P\cdot P)\ \text{Polynomial}\,(P\cdot P) \tag{2.6}$$

It therefore follows that massless representations are special. Indeed, choosing a rest frame with $P^\mu = (E, E, 0, \ldots, 0)$, we see that r.h.s. of (2.5) is $2E(\Gamma_0 - \Gamma_1)C^{-1}$ and thus 16 supercharges for which $\left((\Gamma_0-\Gamma_1)C^{-1}\right)^{\alpha\alpha} = 0$ would act trivially because for such supercharges

$$\left\langle \Omega \left| (Q^\alpha)^\dagger Q^\alpha + Q^\alpha (Q^\alpha)^\dagger \right| \Omega \right\rangle = \left\langle \Omega \left| Q^\alpha (Q^\alpha)^\dagger \right| \Omega \right\rangle = \left\| (Q^\alpha)^\dagger |\Omega\rangle \right\|^2 \tag{2.7}$$

and therefore $(Q^\alpha)^\dagger |\Omega\rangle = 0$. Size of the massless representation is therefore reduced, and massless multiplets are shorter [105][1]. Thus, total number of states in massless representation in 11D is $2^{(16/2)} = 256$. There will be 128 fermionic and 128 bosonic states, depending whether odd or even number of supercharges were involved in exciting a particular state. Because vacuum $|\Omega\rangle$ is invariant w.r.t. to rotations $\mathrm{SO}(9)$ of transverse directions, these states should fall into representations of this group. Indeed, a decomposition relevant for supergravity is

$$\begin{aligned} \mathbf{128}_b &= \mathbf{44} + \mathbf{84} = [g_{\mu\nu}] + [A_{\mu\nu\rho}] \\ \mathbf{128}_f &= \mathbf{128} = \left[\psi^\alpha_\mu \,|\, \Gamma^\mu \psi_\mu = 0\right] \end{aligned} \tag{2.8}$$

where $g_{\mu\nu}$ correspond to a metric, $A_{\mu\nu\rho}$ to a 3-form, and fermion ψ^μ to gravitino.

A supergravity theory with the above content was constructed by Cremmer and Julia

[1] Phenomenon of shortening occurs whenever right hand-side of supercharges anti-commutator acquires null space. This can also occur for massive states in the presence of central charges, whenever mass saturates so called Bogomolnyi's bound [105].

and Scherk [14] in 1978. Its action reads as follows

$$S_{11} = \frac{1}{2}\int d^{11}x\sqrt{-g}\left(R - \frac{1}{2}F\wedge *F\right) - \frac{1}{6}\int A\wedge F\wedge F + \text{fermionic terms} \tag{2.9}$$

where $F = dA$ is a 4-form field strength of 3-form field A. In order to write a supersymmetry transformations it is necessary to introduce a elfbein [14, 21] which is square-root of metric g:

$$E^a{}_\mu E_b{}^\mu = \delta^a{}_b \qquad E_a{}^\nu E^a{}_\mu = \delta^\nu{}_\mu \qquad E^a{}_\mu \eta_{ab} E^b{}_\nu = g_{\mu\nu} \tag{2.10}$$

Index a is a local Lorentz index labelling coordinate of tangent space TM_x to space-time M at a point x. This tangent space has a flat metric η_{ab} with the same signature as that of space-time metric g.

The supersymmetry transformations can be written as follows:

$$\delta E^a{}_\mu = \frac{1}{2}\bar{\epsilon}\Gamma^a\psi_\mu \tag{2.11a}$$

$$\delta A_{\mu\nu\rho} = -\frac{\sqrt{2}}{8}\bar{\epsilon}\Gamma_{[\mu\nu}\psi_{\rho]} \tag{2.11b}$$

$$\delta\psi_\mu = D_\mu\left(\hat{\omega}\right)\epsilon + \frac{\sqrt{2}}{288}\left(\eta_{\mu\nu}E_f{}^\nu\Gamma^{abcdf} - 8E^a{}_\mu\Gamma^{bcd}\right)\epsilon\hat{F}_{abcd} \tag{2.11c}$$

where the following "supercovariant" combinations appear:

$$\begin{aligned}(\hat{\omega}_\mu)^a{}_b &= (\omega_\mu)^a{}_b + \frac{1}{8}\bar{\psi}^\nu\Gamma_{\nu\mu\lambda\rho\tau}\psi^\tau E^\rho{}_b E^\lambda{}_c\eta^{ac}\\ \hat{F}_{abcd} &= F_{abcd} - 3\bar{\psi}_\mu E_{[a}{}^\mu\Gamma_{bc}E_{d]}{}^\nu\psi_\nu\,.\end{aligned}$$

These particular combinations are called supercovariant because their supersymmetry transformation does not involve derivatives of ϵ. Here ω denotes so called spin-connection [21, 80] which determines parallel transport of "elfbein" and, for torsion-free manifolds becomes an analog of Christoffel symbol:

$$(\omega_\mu)^a{}_b = \frac{1}{2}E^c{}_\mu\eta^{ad}\left(\Omega_{dbc} - \Omega_{cdb} + \Omega_{bcd}\right) \quad \text{where} \quad \Omega_{abc} = 2\eta_{cd}E_{[a}{}^\mu E_{b]}{}^\nu\partial_\mu E^d{}_\nu \tag{2.12}$$

These supersymmetry transformations close on-shell [14]. Action (2.9) of 11D supegravity is also invariant under general coordinate transformations and local $\mathrm{SO}(1,10)$ transformations that act on flat indices.

Furthermore, action (2.9) is invariant under Abelian tensor gauge transformation $A \to A + \delta A = A + d\Lambda$. Indeed, the field strength is manifestly invariant $F = dA \to dA + d^2\Lambda =$

$dA = F$ and we only have to investigate invariance of Chern-Simons term

$$\int \delta\left(F \wedge F \wedge A\right) = \int F \wedge F \wedge d\Lambda = \int d\left(F \wedge F \wedge \Lambda\right) = 0 \tag{2.13}$$

where we used

$$d\left(F \wedge F \wedge \Lambda\right) = dF \wedge F \wedge \Lambda + F \wedge dF \wedge \Lambda + F \wedge F \wedge d\Lambda = F \wedge F \wedge d\Lambda \tag{2.14}$$

as a consequence of Bianchi identity $dF = d^2A = 0$, as well as assumptions that fields fall-off fast enough at infinity.

And in conclusion of this section let us note the following scaling transformation

$$\delta g_{\mu\nu} = \lambda^2 g_{\mu\nu} \qquad \delta A_{\mu\nu\rho} = \lambda^3 A_{\mu\nu\rho} \tag{2.15}$$

which amounts to rescaling of action (2.9) $S_{11} \to \lambda^9 S_{11}$ and thus is a symmetry of equations of motion.

2.3 Extension of supersymmetry algebra in $d = 11$

As it was mentioned before, supersymmetry algebras allow for cental extensions [105], i.e. we allow central charges to appear in the anti-commutator of supercharges $\{Q, Q\}$. These central charges were shown to arise when a theory allows for topologically non-trivial solutions [121], like instantons, monopoles or black holes and are related to topological charges of these solutions.

Gauge field $A_{\mu\nu\lambda}$ can either carry "electric" or "magnetic" charge, coupling to 2- or 5-dimensional extended objects respectively. One therefore would expect charges which are 2 and 5 forms in space-time coordinates. Indeed, anti-commutator of two supercharges belongs to a symmetrised tensor product of **32** representations of $\mathfrak{so}\left(1, 10\right)$, which decomposes into irreducible components as follows:

$$\mathbf{32} \otimes_S \mathbf{32} = \mathbf{11} \oplus \mathbf{55} \oplus \mathbf{462}. \tag{2.16}$$

Representation **55** corresponds to anti-symmetric rank 2 tensors in 11D, and **462** corresponds to rank 5 antisymmetric tensors. Thus such extension appears to be the most general:

$$\{Q_A, Q_B\} = 2\left(C\Gamma_\mu\right)_{AB} P^\mu + \left(C\Gamma_{\mu\nu}\right)_{AB} Z^{\mu\nu} + \left(C\Gamma_{\mu\nu\lambda\rho\tau}\right)_{AB} Z^{\mu\nu\lambda\rho\tau} \tag{2.17}$$

We note that the extension in question is different from central extensions discussed earlier in that central charges are not singlets of Lorentz group, and are not strictly speaking central

charges of supersymmetry algebras, rather they appear on the same footings as momentum P and do commute with supercharges:

$$[Q_A, P^\mu] = 0 \qquad [Q_A, Z^{\mu\nu}] = 0 \qquad \left[Q_A, Z^{\mu\nu\lambda\rho\tau}\right] = 0 \tag{2.18}$$

2.4 Dimensional reduction

Let us start by discussing ideas behind symmetry reduction, and then proceed to details of dimensional reduction by toroidal compactification. Say Lagrangian of a field theory, as well as its vacuum, possesses an invariance group G, and let H be its proper subgroup. Excitations of the theory then fall into irreducible representations of group G, leading to decomposition of Hilbert space:

$$\mathcal{H} = \bigoplus_r \mathcal{H}_r \tag{2.19}$$

In the same manner a ring of quantum operators becomes graded with respect to representations of G. Symmetry would impose super-selection rules

$$\langle \psi_{\rho_1} | \mathcal{O}_{\rho_2} | \phi_{\rho_3} \rangle = 0 \tag{2.20}$$

unless representations ρ_1, ρ_2 and ρ_3 are such that decomposition of tensor representation $\rho_1^* \otimes \rho_2 \otimes \rho_3$ into irreducible components contains a singlet.

In the case of toroidal compactification H is a translation symmetry along some direction, which we denote z. Accordingly all our fields are independent on this coordinate. Because translation acts trivially on the tensor structure of fields, the dimensional reduction amounts to restricting excitations of supergravity to a co-dimension one hyperplane.

Dimensional reduction of bosonic part of 11D supergravity was worked out in [15]. We shall briefly review their result necessary for explanation of ideas presented in this thesis, following [74] closely. Let use hatted indices for coordinates of $\mathbb{R}^{1,d}$, and un-hatted indices for coordinates of $\mathbb{R}^{1,d-1}$: $x^{\hat{\mu}} = (x^\mu, z)$. Further in this section all fields are assumed independent on z.

Using local Lorenz symmetry $\mathrm{SO}(1,d)$ we fix $\hat{E}^a{}_z = 0$:

$$\hat{E}^{\hat{a}}{}_{\hat{\mu}} = \begin{pmatrix} e^{\alpha\phi} E^a{}_\mu & e^{(2-d)\alpha\phi} B_\mu \\ 0 & e^{(2-d)\alpha\phi} \end{pmatrix}. \tag{2.21}$$

Coefficient α is taken to be $\alpha^{-2} = 2(d-1)(d-2)$ so as to simplify resulting Lagrangian in d dimensions. Adopting ansätz (2.21) amounts to the following space-time metric

$$ds^2_{d+1} = e^{2\alpha\phi} ds^2_d + e^{2(2-d)\alpha\phi} (dz + B_\mu dx^\mu)^2 \tag{2.22}$$

With a little algebra it follows that

$$\hat{E}R^{(d+1)} = ER^{(d)} - \frac{1}{2}E\left(\partial\phi\right)^2 - \frac{1}{4}Ee^{(2-d)\alpha\phi}\left(dB\right)^2 \tag{2.23}$$

Appearance of dB is easy to understand as it is an invariant under "induced" gauge transformation resulting from the following transformation:

$$z \to z + \xi\left(x^\mu\right) \qquad \Longrightarrow \qquad B_\mu \to B_\mu - \partial_\mu \xi \tag{2.24}$$

Let us now discuss how to reduce n-form gauge potential:

$$\begin{aligned} \hat{A}_{(n)} &\to A_{(n)} + A_{(n-1)} \wedge dz \\ \hat{F}_{(n+1)} = d\hat{A}_{(n)} &\to dA_{(n)} + dA_{(n-1)} \wedge dz = F_{(n+1)} + F_{(n)} \wedge \left(dz + B_{(1)}\right) \end{aligned} \tag{2.25}$$

From which we deduce that field strength acquire so-called Kaluza-Klein correction

$$F_{(n)} = dA_{(n-1)} - dA_{(n-2)} \wedge B_{(1)} \tag{2.26}$$

Analogously we define for a later use a twisted field strength associated with forms B_n:

$$\mathcal{F}_{(n)} = dB_{(n-1)} - dB_{(n-2)} \wedge B_{(1)}. \tag{2.27}$$

A kinetic terms of $d+1$ dimensional field strength $\hat{F}_{(n)}$ reduces to kinetic term of d-dimensional field strengths of $F_{(n)}$ and $F_{(n-1)}$ [74]:

$$\frac{1}{2}\hat{E}\hat{F}_{(n)} \wedge *\hat{F}_{(n)} \to \frac{1}{2}Ee^{2(1-n)\alpha\phi}F_{(n)} \wedge *F_{(n)} - \frac{1}{2}Ee^{2(d-n)\alpha\phi}F_{(n-1)} \wedge *F_{(n-1)}. \tag{2.28}$$

The procedure outlined above can be successively applied to reduce 11D supergravity to lower dimension d. Bosonic content of the compactified theory would be as follows [74]:

$$\begin{aligned} &E^{\hat{a}}{}_{\hat{\mu}} \to E^a{}_\mu, \quad \vec{\phi}, \quad B^m_{(1)}, \quad B^m_{(0)n} \\ &\hat{A}_{(3)} \to A_{(3)}, \quad A_{(2)m}, \quad A_{(1)mn}, \quad A_{(0)mnp} \end{aligned} \tag{2.29}$$

where indices are split as $\mu, \nu, a = 0, \ldots, d-1$ and $m, n, p = d, \ldots, 10$. Naturally forms $A_{(2)}$, $A_{(1)}$ and $A_{(0)}$ are antisymmetric in their compactified indices. Kaluza-Klein potential $B^m_{(0)n}$ is only defined for $n < m$. Each reduction of dimension gave rise to one dilaton, resulting in $11-d$ dilatonic scalars organised in a vector ϕ. Dilatons characterise the size or rather volume, of compact manifold as seen from (2.22). All other scalar fields will be referred to as axions [74]. The Lagrangian [74] of bosonic sector of d-dimensional supergravity resulting

d	11	10	9	8	7	6	5	4	3
no. of scalars	-	1	3	7	14	25	41	63	92
no. of $F_{(4)}$	1	1	1	1	1	1	1	1	-
no. of $F_{(3)}$	-	1	2	3	4	5	6	7	8
no. of $F_{(2)}$	-	1	3	6	10	15	21	28	36

Table 2.1: Number of scalars and k-form field strengths in theories obtained from 11D supergravity by toroidal compactification

upon toroidal compactification of $11D$ supergravity has the following form:

$$\begin{aligned}\mathcal{L} = ER &- \frac{1}{2}E\left(\partial\vec{\phi}\right)^2 - \frac{1}{2\cdot 4!}Ee^{\vec{a}\cdot\vec{\phi}}F_{(4)}^2 - \frac{1}{2\cdot 3!}E\sum_i e^{\vec{a}_i\cdot\vec{\phi}}\left(F_{(3)}^i\right)^2 \\ &- \frac{1}{2\cdot 2!}E\sum_{i<j} e^{\vec{a}_{ij}\cdot\vec{\phi}}\left(F_{(2)}^{ij}\right)^2 - \frac{1}{2\cdot 2!}E\sum_i e^{\vec{b}_i\cdot\vec{\phi}}\left(\mathcal{F}_{(2)}^i\right)^2 - \frac{1}{2}E\sum_{i<j<k} e^{\vec{a}_{ijk}\cdot\vec{\phi}}\left(F_{(1)}^{ijk}\right)^2 \\ &- \frac{1}{2\cdot 2!}E\sum_{i<j} e^{\vec{a}_{ij}\cdot\vec{\phi}}\left(\mathcal{F}_{(1)}^{ij}\right)^2 + \mathcal{L}_{FFA}\end{aligned} \tag{2.30}$$

where coefficients a and b are as follows:

$$\begin{array}{lll} & F_{\hat{\mu}\hat{\nu}\hat{\lambda}\hat{\rho}} & \text{vielbein} \\ 4\text{-form}: & \vec{a} = -\vec{g} & \\ 3\text{-form}: & \vec{a}_i = \vec{f}_i - \vec{g} & \\ 2\text{-form}: & \vec{a}_{ij} = \vec{f}_i + \vec{f}_j - \vec{g} & \vec{b}_i = -\vec{f}_i \\ 1\text{-form}: & \vec{a}_{ijk} = \vec{f}_i + \vec{f}_j + \vec{f}_k - \vec{g} & \vec{b}_{ij} = -\vec{f}_i + \vec{f}_j \end{array} \tag{2.31}$$

where

$$\begin{aligned}&\vec{g} = 3\,(s_1, s_2, \ldots, s_{11-d}) \\ &\vec{f}_i = \left(\underbrace{0, 0, \ldots, 0}_{i-1}, (10-i)\,s_i, s_{i+1}, \ldots, s_{11-d}\right), \qquad s_k = \sqrt{\frac{2}{(10-k)(9-k)}}.\end{aligned} \tag{2.32}$$

Notice that $F_{(1)}^2$ term is nothing but a Kinetic term for axions. Numbers of scalars, and of rank 4 to 2 forms for compactified theories down to 3 dimensions are summarised in table 2.1. Few observations, to be used later, are in order. Rank n field strength in a d dimensional theory can be expressed via dual $d-n$ rank field strength via Hodge duality. This paves a way to define a "dual" effective theory where instead of $n-1$ dimensional gauge potential as a degree of freedom one has $d-n-1$ dimensional gauge potential.

This duality is called electric-magnetic. Indeed, recalling expressions for electric and

magnetic charges with respect to rank $n-1$ gauge field

$$\int *F_{(n)} = q_e \qquad \int F_{(n)} = q_m \tag{2.33}$$

enclosed in $d-n$ and n dimensional volumes of $\mathbb{R}^d$. Hodge duality thus would map electrically charged degrees of freedom into magnetically charges and vice versa. But coming back to matters of compactified theories we observe that gauge potentials of rank $d-2$ can be dualised to scalars.

As a concluding note, let us make few remarks about reduction of fermionic degrees of freedom. Because translation acts trivially on spinorial indices, theories in d dimensions will have 32 supercharges just as their parental supergravity, which is why toroidal compactification of 11D proved such a handy tool to construct maximal supergravities in various dimensions. Thus every resulting supergravity would have exactly 128 fermionic degrees of freedom per point. Because upon dimensional reduction the rotation group of the resulting theory is a subgroup of the rotation group of the original theory, Rarita-Schwinger field $\hat{\psi}_{\hat{\mu}}$ would give rise to Rarita-Schwinger field in lower dimensions ψ_μ as well as additional spinors.

2.5 Scalar manifold and hidden symmetry

It is natural to expect the dimensionally reduced theory (2.30) to have symmetries induced by symmetries of the parental 11D supergravity.

General coordinate transformations group $\mathrm{GL}(11,\mathbb{R})_{\mathrm{local}}$ in 11D theory induces the following symmetry on the compactified theory

$$\mathrm{GL}(11,\mathbb{R})_{\mathrm{local}} \supset \mathrm{GL}(d,\mathbb{R})_{\mathrm{local}} \otimes \mathrm{SL}(11-d,\mathbb{R})_{\mathrm{global}}$$

that preserves a volume of internal manifold. Combined with trombone symmetry (2.15) this can be extended to $\mathrm{GL}_{\mathrm{global}}(11-d,\mathbb{R})$.

Original local Abelian gauge symmetry reduces to global shift symmetry for axions:

$$\mathbb{R}^q: \quad \delta A_{(0)mnk} = \lambda_{mnk} \qquad \text{where} \qquad q = \binom{11-d}{3}. \tag{2.34}$$

It is clear that these two symmetries do not commute, and result into

$$G_{\mathrm{manifest}} = \mathrm{GL}(11-d,\mathbb{R}) \ltimes \mathbb{R}^q \tag{2.35}$$

It was however shown that [15, 16] these symmetries can be extended to what is collectively called $E_{11-d(11-d)}$ - non-compact groups of rank $11-d$ that have $11-d$ more non-compact

dimension d	10	9	8	7	6
$E_{11-d(11-d)}$	$\mathbb{R}$	$\mathrm{GL}(2,\mathbb{R})$	$\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$	$\mathrm{SL}(5,\mathbb{R})$	$\mathrm{O}(5,5)$
$\dim E_{11-d(11-d)}$	1	4	11	24	45
$\dim G_{\text{manifest}}$	1	4	10	20	35

Table 2.2: Hidden symmetries of dimensionally reduced supergravity theories $d > 5$

generators than compact, which are symmetry groups of dimensionally reduced to d dimensions supergravities. These non-compact hidden symmetry groups are examples of *U-duality* groups. The notation of $\mathrm{E}_{r(r)}$ will be justified in few moments.

In dimensions $d = 10, \ldots, 6$ there is no need to dualise degrees of freedom to make this symmetry enhancement manifest, that is extended symmetries act on supergravities' local degrees of freedom only. They are listed in Table 2.2. From it one sees that G_{manifest} coincides with full symmetry group for $d = 10$ and $d = 9$ where there are no axions coming from 11D gauge field A.

When compactifying to $d = 5$ one notices that field strength $F_{(4)}$ has only 5 independent degrees of freedom, as becomes manifest after applying the Hodge-$*$ operation to it

$$\left(*F_{(4)}\right)^{\mu} = \frac{1}{24}\epsilon^{\mu\nu\rho\tau\lambda}F_{\nu\rho\tau\lambda} \tag{2.36}$$

and thus the underlying gauge potential is just a scalar. This scalar could not however be expressed through $A_{\mu\nu\lambda}$ locally. Similarly rank 2 gauge field $A_{(2)}$ with field strength $F_{(3)}$ would dualise to vector gauge potential $\tilde{A}_{\mu}$. Consulting table 2.1 one concludes that supergravity dimensionally reduced to $d = 5$ should have 42 scalars, and 27 gauge fields. Supplemented with the graviton, this is exactly the bosonic content of $d = 5$, $N = 8$ supergravity [17].

These additional scalars appearing after dualisation allow for extension of supergravity's symmetry group. Resulting groups are collected in table 2.3

dimension d	5	4	3
$E_{11-d(11-d)}$	$\mathrm{E}_{6(6)}$	$\mathrm{E}_{7(7)}$	$\mathrm{E}_{8(8)}$
$\dim E_{11-d(11-d)}$	78	133	248
$\dim G_{\text{manifest}}$	56	84	120

Table 2.3: Hidden symmetries of dimensionally reduced supergravity theories $d \leqslant 5$

Scalars of maximal supergravity theories form a homogeneous manifold G/H where G is the supergavity's U-duality group, and H is its maximal compact subgroup. The number of scalars in d-dimensional maximal supergravity naturally equals to the dimension of the coset $E_{11-d(11-d)}/H$. Maximal compact subgroups H of U-duality groups G are listed in table 2.4. This manifold is referred to as scalar manifold. In fact supersymmetry and scalar manifold determine supergravity theory uniquely [103, 117]. It is worth noting that scalar

d	G	H
10	$\mathbb{R}$	$\emptyset$
9	$\mathrm{GL}(2,\mathbb{R})$	$\mathrm{SO}(2)$
8	$\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(3) \times \mathrm{SO}(2)$
7	$\mathrm{SL}(5,\mathbb{R})$	$\mathrm{SO}(5)$
6	$\mathrm{O}(5,5)$	$\mathrm{S}(\mathrm{O}(5) \times \mathrm{O}(5))$
5	$\mathrm{E}_{6(6)}$	$\mathrm{USp}(8)$
4	$\mathrm{E}_{7(7)}$	$\mathrm{SU}(8)$
3	$\mathrm{E}_{8(8)}$	$\mathrm{SO}(16)$

Table 2.4: Maximal compact subgroups of maximally split maximal supergravity symmetry groups $G = E_{d(d)}$

manifold need not be homogeneous or symmetric (see for example [47]) in supergravities with lower amount of supersymmetry.

In order to get a glimpse into a structure of U-duality group we observe that coefficients $\vec{a}_{ijk}$ and $\vec{b}_{ij}$ associated with scalar kinetic terms of dimensionally reduced supergravity (2.31) obey the following relations [74]

$$\vec{b}_{ij} + \vec{b}_{jk} = \vec{b}_{ik} \qquad \vec{a}_{ijk} + \vec{b}_{kn} = \vec{a}_{ijn} \tag{2.37}$$

and thus only $\vec{a}_{123}$ and $\vec{b}_{k(k+1)}$ for $k = 1, \ldots, 10-d$ are independent, since any other $\vec{b}_{ij}$ and $\vec{a}_{ijk}$ can be obtained as a linear combination of these. One thus can regard vectors $\vec{b}_{ij}$ and $\vec{a}_{ijk}$ as an integer lattice with $\vec{a}_{123}$ and $\left\{\vec{b}_{k(k+1)}\right\}$ being simple positive roots. Further noticing that scalar products of generators between themselves is always an even integer:

$$\left(\vec{b}_{i,i+1}, \vec{b}_{j,j+1}\right) = 4\delta_{ij} - 2\delta_{i,j+1} - 2\delta_{j,i+1} \quad (\vec{a}_{123}, \vec{a}_{123}) = 4 \quad \left(\vec{a}_{123}, \vec{b}_{i,i+1}\right) = -2\delta_{i,3} \tag{2.38}$$

we conclude that the lattice must be a restriction of root lattice of a Lie algebra. Dynkin diagram[2] corresponding to generators of the resulting root lattice for $d = 3$ is depicted on Fig. 2.1. This is a Dynkin diagram of U-duality group E_8 of $d = 3$ $N = 16$ maximal

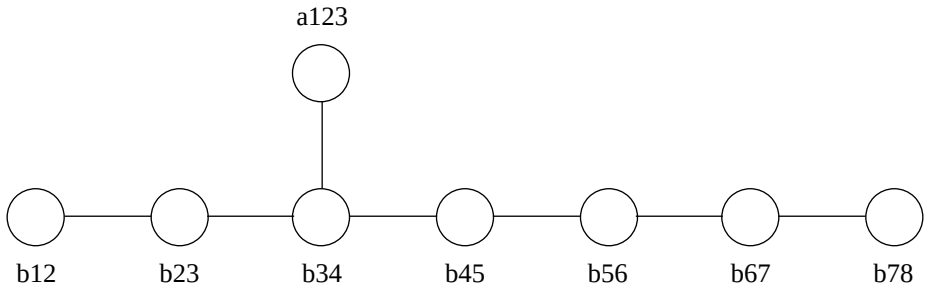

Figure 2.1: Dynkin diagram corresponding to the Lie algebra E_8. Labels denote generators of the root lattice formed by $\vec{a}_{ijk}$ and $\vec{b}_{ij}$.

supergravity. Notice the positive root lattice of E_8 is bigger that that of $\vec{a}_{ijk}$ and $\vec{b}_{ij}$, because positive roots of E_8 allow for vectors which are sums of $\vec{a}$. For instance $\vec{a}_{456} + \vec{a}_{123}$ is a

[2] See appendix A.2 for a short review of Lie algebra terminology.

positive root of E_8, but is not a root of $\vec{a}$, $\vec{b}$ lattice.

Appearance of exceptional hidden symmetry groups in supergravity theories is an astounding fact first discovered by Julia and Cremmer [15].

2.6 $N = 2$ supergravity in $d = 5$

Exceptional U-duality groups appear also in matter coupled supergravity theories [39, 41], in particular in $N = 2$ supergravity in $d = 5$ coupled to certain number of Abelian vector fields, so called Maxwell-Einstein supergravity theories, and in their dimensionally reduced to $d = 4$ and $d = 3$ theories.

We start by considering a pure supergravity in $d = 5$. It has been constructed in [12, 17, 18]. The field content of pure supergravity theory is as follows – graviton ${e_\mu}^m$, gravitini ψ^i_μ which form doublet of supersymmetry algebra's R-group $\mathrm{SU}(2)_R$, and the Abelian gauge field A_μ. All spinors in $d = 5$ are assumed symplectic-Majorana, i.e.

$$\bar{\psi}_{\mu,i} = \epsilon_{ij} \left(\psi^j_\mu\right)^t C$$

Five dimensions is remarkable because it is next odd dimension after $d = 11$ admitting Chern-Simons $F \wedge F \wedge A$ terms. Pure $5D$ supergravity, in fact, resembles $11D$ supergravity in many ways and has been studied in the literature with $11D$ theory in mind [78, 93]. Bosonic part of pure $5D$ sugra Lagrangian reads (c.f. (2.9)):

$$\mathcal{L} = -\frac{1}{2}E\left(R + \frac{1}{2}F_{\mu\nu}F^{\mu\nu}\right) + \frac{1}{6\sqrt{6}}\epsilon^{\mu\nu\lambda\rho\sigma}F_{\mu\nu}F_{\lambda\rho}A_\sigma \tag{2.39}$$

In particular the toroidal compactification of (2.39) to three dimensions possesses scalar manifold isomorphic to $\mathrm{G}_{2(2)}/\mathrm{SO}(4)$, while toroidal compactification of (2.9) to three dimensions has scalar manifold isomorphic to $\mathrm{E}_{8(8)}/\mathrm{SO}(16)$.

Vector multiplet in $d = 5$ contains one scalar ϕ, one $\mathrm{SU}(2)_R$ doublet symplectic Majorana spinor λ^i and a gauge field A_μ. Let us consider supergravity multiplet coupled to n vector multiplets. There are $n + 1$ gauge fields organised into A^I_μ, one coming from supergravity multiplet and others from vector multiplets; n scalars denoted as ϕ_x. The bosonic part of the Lagrangian [39] reads

$$\begin{aligned} E^{-1}\mathcal{L}_{\text{bosonic}} = &-\frac{1}{2}R - \frac{1}{4}\mathring{a}_{IJ}F^I_{\mu\nu}F^{J\mu\nu} - \frac{1}{2}g_{xy}(\partial_\mu\varphi^x)(\partial^\mu\varphi^y) \\ &+ \frac{E^{-1}}{6\sqrt{6}}C_{IJK}\varepsilon^{\mu\nu\rho\sigma\lambda}F^I_{\mu\nu}F^J_{\rho\sigma}A^K_\lambda, \end{aligned} \tag{2.40}$$

where E and R denote the fünfbein determinant and the scalar curvature in $d = 5$, respectively. $F^I_{\mu\nu}$ are the field strengths of the Abelian vector fields A^I_μ, $(I = 0, 1, 2 \ldots, n)$

with A^0_μ denoting the "bare" graviphoton. The metric, g_{xy}, of the scalar manifold $\mathcal{M}$ and the "metric" $\overset{\circ}{a}_{IJ}$ of the kinetic energy term of the vector fields both depend on the scalar fields φ^x $(x, y, \ldots = 1, 2, \ldots, n)$. For the Chern-Simons term in the Lagrangian (2.40) to be invariant under the Abelian gauge transformations of the vector fields, the completely symmetric tensor C_{IJK} has to be constant. Moreover, the entire $N = 2$, $d = 5$ MESGT is uniquely determined by the constant tensor C_{IJK} [39]. In particular, the metrics of the kinetic energy terms of the vector and scalar fields are determined by C_{IJK}. More specifically, consider the cubic polynomial, $\mathcal{V}(h)$, in $(n+1)$ real variables h^I, with $I = 0, 1, \ldots, n$, defined by the C_{IJK}

$$\mathcal{V}(h) = C_{IJK} h^I h^J h^K \,. \tag{2.41}$$

Using this polynomial as a real "Kähler potential" for a metric, a_{IJ}, in an $n+1$ dimensional ambient space with the coordinates h^I:

$$a_{IJ}(h) = -\frac{1}{3} \frac{\partial}{\partial h^I} \frac{\partial}{\partial h^J} \ln \mathcal{V}(h) \tag{2.42}$$

one finds that the n-dimensional target space, $\mathcal{M}$, of the scalar fields φ^x can be identified with the hypersurface [39]

$$\mathcal{V}(h) = C_{IJK} h^I h^J h^K = 1 \tag{2.43}$$

in this space. The metric g_{xy} of the scalar manifold is simply the pull-back of (2.42) to $\mathcal{M}$

$$g_{xy} = h^I_x h^J_y \overset{\circ}{a}_{IJ} \tag{2.44}$$

where

$$h^I_x = -\sqrt{\frac{3}{2}} \frac{\partial}{\partial \phi^x} h^i \tag{2.45}$$

and one finds that the Riemann curvature of the scalar manifold has the simple form

$$K_{xyzu} = \frac{4}{3} \left(g_{x[u} g_{z]y} + T_{x[i}{}^w T_{z]yw} \right) \tag{2.46}$$

where T_{xyz} is the symmetric tensor

$$T_{xyz} = h^I_x h^J_y h^K_z C_{IJK} \tag{2.47}$$

The "metric" $\overset{\circ}{a}_{IJ}(\varphi)$ of the kinetic energy term of the vector fields appearing in (2.40) is given by the component-wise restriction of a_{IJ} to $\mathcal{M}$:

$$\overset{\circ}{a}_{IJ}(\varphi) = a_{IJ}|_{\mathcal{V}=1} \,. \tag{2.48}$$

The physical requirement of positivity of kinetic energy requires that g_{xy} and $\overset{\circ}{a}_{IJ}$ be positive

definite metrics. This requirement induces constraints on the possible C_{IJK}, and in [39] it was shown that any C_{IJK} that satisfy these constraints can be brought to the following form

$$C_{000} = 1, \quad C_{0ij} = -\frac{1}{2}\delta_{ij}, \quad C_{00i} = 0, \tag{2.49}$$

with the remaining coefficients C_{ijk} $(i, j, k = 1, 2, \ldots, n)$ being completely arbitrary. This basis is referred to as the canonical basis for C_{IJK}.

Denoting the symmetry group of the tensor C_{IJK} as G one finds that the full symmetry group of $N = 2$ MESGT in $d = 5$ is of the form $G \times \mathrm{SU}(2)_R$ where $\mathrm{SU}(2)_R$ denotes the local R-symmetry group of $N = 2$ supersymmetry algebra.

From the form of the Riemann curvature tensor K_{xyzu} it is clear that the covariant constancy of T_{xyz} implies the covariant constancy of K_{xyzu}:

$$T_{xyz;w} = 0 \implies K_{xyzu;w} = 0 \tag{2.50}$$

Therefore the scalar manifolds $\mathcal{M}_5$ with covariant constantly constant T tensor are locally symmetric spaces.

If $\mathcal{M}_5$ is a homogeneous space the covariant constancy of T_{xyz} was shown to be equivalent to the following identity [39]:

$$C^{IJK} C_{J(MN} C_{PQ)K} = \delta^{I}{}_{(M} C_{NPQ)} \tag{2.51}$$

where the indices are raised by $\overset{\circ}{a}{}^{IJ}$.[3]

Remarkably the cubic forms defined by C_{IJK} of the $N = 2$ MESGT's with $n \geqslant 2$ and with a symmetric target space $\mathcal{M}_5$ and a covariantly constant T tensor are in one-to-one correspondence with the norm forms of Euclidean (formally real) Jordan algebras of degree three.

The precise connection between Jordan algebras of degree 3 and the geometries of MESGT's with symmetric target spaces in $d = 5$ was established [39] through a novel formulation of the corresponding Jordan algebras. This formulation is due to McCrimmon [76], who generalised and unified previous constructions by Freudenthal, Springer and Tits [30], which we outline here following [48].

Let V be a vector space over the field of reals $\mathbb{R}$, and let $\mathcal{V}$: $V \times V \times V \to \mathbb{R}$ be a cubic norm on V. Furthermore, assume that there exists a quadratic map $\sharp : x \to x^{\sharp}$ of V into

[3] For proof of this equivalence an expression for constants C_{IJK} in terms of scalar field dependent quantities was used

$$C_{IJK} = \frac{5}{2} h_I h_K h_K - \frac{3}{2} \overset{\circ}{a}{}_{(IJ} h_{K)} + T_{xyz} h^x_I h^y_J h^z_K$$

as well as algebraic constraints $h_I h^I = 1$ and $h^I_x h_I = 0$ that follows from susy.

itself and a "base point" $c \in V$ such that

$$\mathcal{V}(c) = 1 \qquad \text{and} \qquad c^\sharp = c \tag{i), (ii)}$$

$$T\left(x^\sharp, y\right) = y^I \partial_I \mathcal{V}\big|_x \tag{iii}$$

$$c \times y = T(y, c)\, c - y \tag{iv}$$

$$\left(x^\sharp\right)^\sharp = \mathcal{V}(x)\, x \tag{v}$$

The last equation is referred to as the adjoint identity. The map $T : V \times V \to \mathbb{R}$ is defined as

$$T(x, y) = -\, x^I y^J \partial_I \partial_J \ln \mathcal{V}\big|_c \tag{2.52a}$$

and the Freudenthal product $\times$ of two elements x and y is defined as

$$x \times y = (x + y)^\sharp - x^\sharp - y^\sharp \tag{2.52b}$$

McCrimmon showed that a vector space with the above properties defines a unital Jordan algebra with Jordan product $\circ$ given by

$$x \circ y = \frac{1}{2}\Big(T(c, x)\, y + T(c, y)\, x - T(c, x \times y)\, c + x \times y\Big) \tag{2.53}$$

and a quadratic operator U_x given by

$$U_x y = T(x, y)\, x - x^\sharp \times y \tag{2.54}$$

In [39] it was shown that the properties (i) and (iv) are satisfied by the cubic norm form defined by the tensor C_{IJK} of $N = 2$ MESGT's in $d = 5$. The condition of adjoint identity is equivalent to the requirement that the scalar manifold be symmetric space with a covariantly constant T-tensor [39]. The corresponding symmetric spaces are of the form

$$\mathcal{M} = \frac{\mathrm{Str}_0(J)}{\mathrm{Aut}(J)} \tag{2.55}$$

where $\mathrm{Str}_0(J)$ and $\mathrm{Aut}(J)$ are the reduced structure group and automorphism group of the Jordan algebra J respectively.

From the foregoing we see that the classification of locally symmetric spaces $\mathcal{M}$ for which the tensor T_{xyz} is covariantly constant reduces to the classification of Jordan algebras with cubic norm forms. Following Schafers [95] the possibilities were listed in [39]:

1. $J = \mathbb{R}$, $\mathcal{V}(x) = x^3$. The base point may be chosen as $c = 1$. This case supplies $n = 0$, i.e. pure $d = 5$ supergravity.

2. $J = \mathbb{R} \oplus \Gamma$, where Γ is a simple algebra with identity $\mathbf{e}_2$ and quadratic norm $Q(\mathbf{x})$, for $\mathbf{x} \in \Gamma$, such that $Q(\mathbf{e}_2) = 1$. The norm is given as $\mathcal{V}(x) = aQ(\mathbf{x})$, with $x = (a, \mathbf{x})$. The base point may be chosen as $c = (1, \mathbf{e}_2)$. This includes two special cases

   (a) $\Gamma = \mathbb{R}$ and $Q = b^2$, with $\mathcal{V} = ab^2$. This is applicable to $n = 1$.

   (b) $\Gamma = \mathbb{R} \oplus \mathbb{R}$ and $Q = bc$, and $\mathcal{V} = abc$ and is applicable to $n = 2$.

   Notice that for these special cases the norm is completely factorised, so that the space $\mathcal{C}$ and therefore $\mathcal{M}$, is flat. For $n > 2$, $\mathcal{V}$ is still factorised into a linear and quadratic parts, so that $\mathcal{M}$ is still reducible. The positive definiteness of the metric a_{IJ} of $\mathcal{C}$, which is required on the physical grounds, requires that Q have Minkowski signature $(+, -, -, \ldots, -)$. The point $\mathbf{e}_2$ can be chosen as $(1, 0, \ldots, 0)$. It is then obvious that the invariance group of the norm is

$$\mathrm{Str}_0(J) = \mathrm{SO}(n-1, 1) \times \mathrm{SO}(1, 1) \tag{2.56}$$

   where the $\mathrm{SO}(1,1)$ factor arises from the invariance of $\mathcal{V}$ under the dilatation $(a, \mathbf{x}) \to \left(e^{-2\lambda}a, e^{\lambda}\mathbf{x}\right)$ for $\lambda \in \mathbb{R}$, and that $\mathrm{SO}(n-1)$ is $\mathrm{Aut}(J)$. Hence

$$\mathcal{M} = \frac{\mathrm{SO}(n-1, 1)}{\mathrm{SO}(n-1)} \times \mathrm{SO}(1, 1) \tag{2.57}$$

3. Simple Euclidean Jordan algebras $J = J_3^{\mathbb{A}}$ generated by 3×3 Hermitian matrices over the four division algebras $\mathbb{A} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$. In these four cases an element $x \in J$ can be written as

$$x = \begin{pmatrix} \alpha_1 & a_3 & a_2^* \\ a_3^* & \alpha_2 & a_1 \\ a_2 & a_1^* & \alpha_3 \end{pmatrix} \tag{2.58}$$

   where $\alpha_k \in \mathbb{R}$ and $a_k \in \mathbb{A}$ with $*$ indicating the conjugation in the underlying division algebra. The cubic norm $\mathcal{V}$, following Freudenthal [30], is given by

$$\mathcal{V}(x) = \alpha_1\alpha_2\alpha_3 - \alpha_1 |a_1|^2 - \alpha_2 |a_2|^2 - \alpha_3 |a_3|^2 + a_1a_2a_3 + (a_1a_2a_3)^* \tag{2.59}$$

   For $\mathbb{A} = \mathbb{R}$ or $\mathbb{C}$ it coincides with the usual definition of determinant $Det(x)$. The corresponding spaces $\mathcal{M}$ are irreducible of dimension $3(1 + \dim \mathbb{A}) - 1$, which we list below:

$$\begin{aligned} \mathcal{M}(J_3^{\mathbb{R}}) &= \frac{\mathrm{SL}(3, \mathbb{R})}{\mathrm{SO}(3)} & \mathcal{M}(J_3^{\mathbb{H}}) &= \frac{\mathrm{SU}^*(6)}{\mathrm{USp}(6)} \\ \mathcal{M}(J_3^{\mathbb{C}}) &= \frac{\mathrm{SL}(3, \mathbb{C})}{\mathrm{SU}(3)} & \mathcal{M}(J_3^{\mathbb{O}}) &= \frac{\mathrm{E}_{6(-26)}}{\mathrm{F}_4} \end{aligned} \tag{2.60}$$

The magical supergravity theories described by simple Jordan algebras $J_A^{\mathbb{A}}$ (A = R, C, H or O) can be truncated to theories belonging to the generic families. This is achieved by restricting the elements of $J_A^{\mathbb{A}}$ [40]

$$\begin{pmatrix} \alpha_1 & a_3 & \overline{a}_2 \\ \overline{a}_3 & \alpha_2 & a_1 \\ a_2 & \overline{a}_1 & \alpha_3 \end{pmatrix} \tag{2.61}$$

to lie in their subalgebra $J = \mathbb{R} \oplus J_2^{\mathbb{A}}$ be setting $a_1 = a_2 = 0$. Their symmetry groups are as follows:

$$\begin{aligned} J &= \mathbb{R} \oplus J_2^{\mathbb{R}} : \mathrm{SO}(1,1) \times \mathrm{SO}(2,1) \subset \mathrm{SL}(3,\mathbb{R}) \\ J &= \mathbb{R} \oplus J_2^{\mathbb{C}} : \mathrm{SO}(1,1) \times \mathrm{SO}(3,1) \subset \mathrm{SL}(3,\mathbb{C}) \\ J &= \mathbb{R} \oplus J_2^{\mathbb{H}} : \mathrm{SO}(1,1) \times \mathrm{SO}(5,1) \subset \mathrm{SU}^*(6) \\ J &= \mathbb{R} \oplus J_2^{\mathbb{O}} : \mathrm{SO}(1,1) \times \mathrm{SO}(9,1) \subset \mathrm{E}_{6(-26)} \end{aligned} \tag{2.62}$$

2.7 Dimensional reduction to $d = 4$

Five dimensional MESGT theory with n vector multiplets toroidally compactified to $d = 4$ theory will have $2n + 2$ scalars, $n + 5$ vector fields and a graviton.

Under dimensional reduction to the four dimensions the kinetic energy of the scalar fields of the five dimensional $N = 2$ MESGTs can be written as [39]

$$E^{-1}\mathcal{L}_{\text{scalars}} = -g_{IJ}\partial_\mu Z^I \partial^\mu \overline{Z}^J \tag{2.63}$$

where

$$g_{IJ} = \hat{a}_{IJ}\left(Z - \overline{Z}\right) = -\frac{1}{2}\frac{\partial}{\partial Z^I}\frac{\partial}{\partial \overline{Z}^J}\ln \mathcal{V}\left(Z - \overline{Z}\right) \tag{2.64}$$

and Z^I are complex scalar fields

$$Z^I = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{2}{3}}A^I + i\hat{h}^I\right) \tag{2.65}$$

where the real parts A^I are scalars coming from the vectors in 5 dimensions and $\hat{h}^I$ are

$$\hat{h}^I = e^{\sigma} h^I\left(\phi^x\right) \tag{2.66}$$

where σ is the scalar coming from the graviton in the five dimensions. Since $\mathcal{V}\left(\hat{h}\right) = e^{3\sigma} > 0$ the scalar manifold in $4D$ theories corresponds to the "upper half-plane" with respect to the cubic norm. For Euclidean Jordan algebras of degree three these are the Koecher upper

half-spaces [69] of the corresponding Jordan algebras

$$\mathcal{M}_4 = \mathcal{D}(J) = J + i\mathcal{C}(J) \tag{2.67}$$

where $\mathcal{C}(J)$ denotes elements of the Jordan algebra with positive cubic norm. The Koecher half-spaces are bi-holomorphically equivalent to bounded symmetric domains (see [22]) whose Bergman kernel is simply $\mathcal{V}\left(Z-\overline{Z}\right)$. As was first shown in [116] the scalar manifold of the $4D$ MESGTs must be special Kähler. For the theories coming from $5D$ the Kähler potential reads

$$F\left(Z,\overline{Z}\right) = -\frac{1}{2}\ln \mathcal{V}\left(Z-\overline{Z}\right) \tag{2.68}$$

and are called very special Kähler geometries.

The bounded symmetric domains associated with the upper half-spaces of Jordan algebras are isomorphic to certain hermitian symmetric spaces. For the Euclidean Jordan algebras of degree 3 these spaces are as follows:

$$\begin{aligned}
\mathcal{M}_4\left(\mathbb{R}+\Gamma(Q)\right) &= \frac{\mathrm{SO}(2,1)\times \mathrm{SO}(n,2)}{\mathrm{SO}(2)\times \mathrm{SO}(n)\times \mathrm{SO}(2)} \\
\mathcal{M}_4\left(J_3^{\mathbb{R}}\right) &= \frac{\mathrm{Sp}(6,\mathbb{R})}{\mathrm{U}(3)} \\
\mathcal{M}_4\left(J_3^{\mathbb{C}}\right) &= \frac{\mathrm{SU}(3,3)}{\mathrm{S}\left(\mathrm{U}(3)\times \mathrm{U}(3)\right)} \\
\mathcal{M}_4\left(J_3^{\mathbb{H}}\right) &= \frac{\mathrm{SO}^*(12)}{\mathrm{U}(6)} \\
\mathcal{M}_4\left(J_3^{\mathbb{O}}\right) &= \frac{\mathrm{E}_{7(-25)}}{\mathrm{E}_6\times \mathrm{U}(1)}
\end{aligned} \tag{2.69}$$

These symmetric spaces are simply the quotients of the conformal groups of the corresponding Jordan algebras by their maximal compact subgroups:

$$\mathcal{M}_4 = \frac{\mathrm{Conf}(J)}{K(J)}$$

The correspondence between the vector fields and the elements of the underlying Jordan algebras in five dimensions gets extended to a correspondence between the vector field strengths $F^A_{\mu\nu}$ plus their magnetic duals $G^A_{\mu\nu}$ with the elements of the Freudenthal triple system defined by the Jordan algebra of degree three

$$F^A_{\mu\nu} \oplus G^A_{\mu\nu} \Leftrightarrow \mathcal{FTS}(J) \tag{2.70}$$

The automorphism group of this FTS is isomorphic to the four dimensional U-duality group and it acts as the spectrum generating conformal group on the charge space of the

BPS black hole solutions of five dimensional MESGT's [45, 49].

2.8 Geometries of the three dimensional MESGTs defined by Jordan algebras of degree 3

Upon further dimensional reduction to 3 space-time dimensions, the MESGTs defined by Euclidean Jordan algebras of degree three have target spaces that are quaternionic symmetric spaces [41]. The corresponding symmetric spaces are:

$$\begin{aligned} \mathcal{M}_3\left(J=\mathbb{R}+\Gamma\left(Q\right)\right) &= \frac{\mathrm{SO}(n+2,4)}{\mathrm{SO}(n+2)\times\mathrm{SO}(4)} \\ \mathcal{M}_3\left(J_3^{\mathbb{R}}\right) = \frac{\mathrm{F}_{4(4)}}{\mathrm{USp}(6)\times\mathrm{SU}(2)} & \qquad \mathcal{M}_3\left(J_3^{\mathbb{H}}\right) = \frac{\mathrm{E}_{7(-5)}}{\mathrm{SO}(12)\times SU(2)} \\ \mathcal{M}_3\left(J_3^{\mathbb{C}}\right) = \frac{\mathrm{E}_{6(2)}}{\mathrm{SU}(6)\times\mathrm{SU}(2)} & \qquad \mathcal{M}_3\left(J_3^{\mathbb{O}}\right) = \frac{\mathrm{E}_{8(-24)}}{\mathrm{E}_7\times\mathrm{SU}(2)} \end{aligned} \tag{2.71}$$

The pure $5d$, $N=2$ supergravity under dimensional reduction to three dimensions leads to the target space

$$\frac{\mathrm{G}_{2(2)}}{\mathrm{SU}(2)\times\mathrm{SU}(2)} \tag{2.72}$$

which can be embedded in the coset space

$$\frac{\mathrm{SO}(3,4)}{\mathrm{SO}(3)\times\mathrm{SO}(4)} \tag{2.73}$$

We should note that the above target spaces are obtained after dualising all the bosonic propagating fields to scalar fields which is special to three dimensions. The Lie algebras of the three dimensional U-duality groups have a 5-graded decomposition with respect to the four dimensional U-duality groups. They are isomorphic to the quasiconformal groups constructed over the corresponding FTS's, which act as spectrum generating symmetry group on the charge-entropy space of BPS black hole solutions in four dimensional MESGT's [45, 49].

2.9 U-duality groups and entropy of BPS black holes in supergravity theories

Both in maximally extended supergravity, and in supergravities coupled to matter the entropy of BPS black hole solutions is invariant under the corresponding U-duality groups. Indeed, according to Bekenstein's formula (the leading order of) entropy is proportional to area of event horizon. Since U-duality group acts trivially on the graviton, entropy must

be a singlet of U-duality group. For instance in five-dimensional $N = 8$ supergravity the entropy S of a BPS black hole can be cast into the form [26]:

$$S \propto \sqrt{I_3} = \sqrt{d_{IJK} q^I q^J q^k} \tag{2.74}$$

where I_3 is the cubic form invariant under $\mathrm{E}_{6(6)}$ with q^I being 27 charges, coupling to 27 vector fields of the theory. The entropy of BPS black hole solutions of five dimensional $N = 2$ MESGT's is given by cubic form defined by the constant tensor C_{IJK} [26]:

$$S \propto \sqrt{\mathcal{V}} = \sqrt{C_{IJK} q^I q^J q^K} \tag{2.75}$$

For $N = 2$ MESGT theories defined by Jordan algebras of degree 3 this cubic form is the norm form and the global symmetry group G is the norm invariance group.

Because of supersymmetry and BPS property of black hole solutions in question, they are in one-to-one correspondence with charges q^I [26]. This fact was used by authors of [27] to classify orbits of the BPS black hole solutions of $N = 2$ five dimensional MESGTs defined by Euclidean Jordan algebras under the action of their U-duality groups. It was instrumental to associate to each BPS solution with charges q^I an element

$$J = \sum_{I=0}^{n} e_I q^I \tag{2.76}$$

of Jordan algebra of degree 3, where set $\{e_I\}$ stands for a basis of the Jordan algebra. Similarly, classification of the orbits of BPS black hole solutions of $N = 8$ sugra in $d = 5$ as given in [27] associates a BPS solution an element of the split exceptional Jordan algebra $J_3^{\mathbb{O}_s}$. The cubic invariant $I_3(q)$ is then given by the norm form $\mathcal{N}$ of the split exceptional Jordan algebra. Invariance of the norm (i.e. reduced structure group of the Jordan algebra) is $E_{6(6)}$ which coincides with U-duality group of the maximal $N = 8$ supergravity in $d = 5$.

In $d = 4$ magical $N = 2$ MESGTs obtained by toroidal dimensional reduction from $d = 5$, as well as in maximal 4d supergravity the entropies of BPS black holes are given by quartic invariants of their U-duality groups [60]

$$S \propto \sqrt{I_4} = \sqrt{d_{IJKL} q^I q^J q^K q^L} \tag{2.77}$$

were d_{IJKL} are the completely symmetric tensors defined by the Freudenthal-Kantor triple systems associated with the corresponding simple Jordan algebras of degree three [27] and q now denote both electric and magnetic charges.

Chapter 3

Structure of U-duality algebras and Jordan algebras

Appearance of non-compact exceptional groups as hidden symmetry groups of supergravity theories in $d = 5$, $d = 4$ and $d = 3$ dimensions is a fascinating fact. Classical groups $\mathrm{SU}(n)$, $\mathrm{SO}(n)$ and $\mathrm{Sp}(2n)$ have been known through their geometric definitions as invariance groups of Hermitian, Euclidean and symplectic scalar products. Exceptional groups were discovered by Elie Cartan in his thesis on Lie algebra classification (see appendix A.2 for short review). Geometric interpretation of exceptional groups has been associated with division algebras of quaternions and octonions [56] and Jordan algebras [29, 30] (see [3] for recent account and references).

3.1 Jordan algebras

Jordan algebras have been introduced by P. Jordan [57] in an attempt to generalise quantum mechanics. Jordan algebra J is equipped with commutative Jordan product operation

$$\forall x, y \in J \qquad x \circ y = y \circ x \in J\,. \tag{3.1a}$$

Jordan product should satisfy alternating associativity requirement:

$$(x \circ x) \circ (x \circ y) = x \circ ((x \circ x) \circ y) \tag{3.1b}$$

which assures that subalgebra formed by any two elements x and y of Jordan algebra J is associative. In general, however, Jordan algebra is not associative, and one introduces an associator

$$\{a, b, c\} = a \circ (b \circ c) - (a \circ b) \circ c \tag{3.2}$$

to measure the degree of non-associativity in the same way as commutator measures degree of non-commutativity of the algebra.

An example of Jordan algebra to keep in mind is the algebra of $n \times n$ Hermitian matrices over an associative division algebra $\mathbb{A}$ (i.e. $\mathbb{A}$ can be $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$) with Jordan product given

by

$$M_1 \circ M_2 = \frac{1}{2}\left(M_1 M_2 + M_2 M_1\right) \tag{3.3}$$

Such algebra will be denoted $J_n^{\mathbb{A}}$. In this case associator becomes

$$\{M_1, M_2, M_3\} = \frac{1}{4}\left[\left[M_1, M_2\right], M_3\right] \tag{3.4}$$

Finite dimensional Jordan algebras have been classified by Jordan, von Neumann and Wigner [58]. They showed that all finite dimensional Jordan algebras, but one exceptional case, have realizations in terms of associative matrices with Jordan product defined as in (3.3). The exception is the algebra of 3×3 Hermitian matrices over octonions (which is not associative), again with Jordan product defined by (3.3).

Another, so called quadratic formulation of Jordan algebras [55, 56] will be important in what follows. The reader is referred to the book of McCrimmon [76] for a review and references. We start by noticing that Jordan algebra is its own module, and introduce and operator L_x such that

$$\forall y \in J \qquad L_x y = x \circ y \tag{3.5}$$

We then define, for all $x \in J$, an operator U_x as

$$U_x = 2L_x L_x - L_{x^2} \qquad U_x y = (x, y, x) \tag{3.6}$$

where

$$(x, y, z) = x \circ (y \circ z) + (x \circ y) \circ z - y \circ (x \circ z) \tag{3.7}$$

is Jordan triple product. In this formulation the axioms of *unital* Jordan algebra, with U_e being denoting $U_e = \mathrm{id}$, can be rewritten as

$$U_{U_x y} = U_x U_y U_x \qquad\qquad U_x\left(y, x, z\right) = \left(U_x y, z, x\right) \tag{3.8}$$

In terms of the quadratic operator U the Jordan triple product reads

$$(x, y, z) = \left[U_{x+z} - U_x - U_z\right] y \tag{3.9}$$

One of the remarkable properties of this formulation, making it relevant for supergravity [39], is that for every Jordan algebra J one can define a norm $\mathcal{N} : J \to \mathbb{R}$ satisfying the composition property

$$\mathcal{N}\left(U_x y\right) = \left(\mathcal{N}\left(x\right)\right)^2 \mathcal{N}\left(y\right) \tag{3.10}$$

Jordan algebras relevant to supergravity are those where such norm is cubic.

3.2 Linear fractional group of Jordan algebras as generalised conformal group

Following Kantor [62] consider an n-dimensional vector space V endowed with a non-degenerate form $\mathcal{N} : \underbrace{V \otimes V \otimes \cdots \otimes V}_{p \text{ times}} \to \mathbb{R}$ of degree p:

$$\mathcal{N}(x) \stackrel{def}{=} \mathcal{N}(x, x, \ldots, x) \qquad \mathcal{N}(\lambda x) = \lambda^p \mathcal{N}(x)$$

For every set of distinct four vectors $x, y, z, u \in V$ define a cross-ratio

$$\frac{\mathcal{N}(x-z)}{\mathcal{N}(y-z)} \frac{\mathcal{N}(y-w)}{\mathcal{N}(x-w)} \tag{3.11}$$

and let $G_{\text{c.r.}}$ be its invariance group. For each set of distinct non-vanishing $x_i \in V$, $i = 1, \ldots, p$ define

$$\frac{\mathcal{N}(x_1, \ldots, x_p)^p}{\mathcal{N}(x_1)\mathcal{N}(x_2)\cdots\mathcal{N}(x_p)} \tag{3.12}$$

and let $G_{p-\text{angle}}$ be its invariance group. Kantor [62] proved that if $G_{p-\text{angle}}$ is finite dimensional then it is isomorphic to $G_{\text{c.r.}}$. When $\mathcal{N}$ is usual scalar product bilinear form in $\mathbb{R}^n$ the invariance group of (3.11) and (3.12) is conformal group $\mathrm{SO}(1, n+1)$.

If vector space V is taken to be semi-simple Jordan algebra J with a generic form $\mathcal{N}$ invariance group of (3.11) and (3.12) defines generalised conformal transformation groups [43], provided J is sufficiently nice. The action of group G on J can be written as a "linear fractional transformation" J, and generated by inversions, translations and Lorentz rotations [69, 22].

Further we describe a construction of the Lie algebra of the above "linear fractional transformation group" due to Tits, Kantor and Koecher [61, 70, 107], closely following [43, 50].

The reduced structure group H of a Jordan algebra J is defined as the invariance group of its norm $\mathcal{N}$. By adjoining to it the constant scale transformation we obtain the full structure group of J. The Lie algebra $\mathfrak{g}$ of the conformal group of J can be given a three-graded structure with the respect to the Lie algebra $\mathfrak{g}^0$ of its structure group:

$$\mathfrak{g} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^{+1} = \mathfrak{g}^{-1} \oplus (\mathfrak{h} \oplus \Delta) \oplus \mathfrak{g}^{+1} \tag{3.13}$$

with $\mathfrak{h}$ denoting the Lie algebra of H and Δ being the generator of constant scale transformations. The negative grade generators correspond to translations, and can thus be labelled

by elements of Jordan algebra:

$$\forall a \in J \qquad U_a \in \mathfrak{g}^{-1} \qquad U_a(x) = a \tag{3.14}$$

Every such Lie algebra admits an involutive automorphism (conjugation) which maps $\tau : \mathfrak{g}^{-1} \mapsto \mathfrak{g}^{+1}$. Hence elements of $\mathfrak{g}^{+1}$ are also labelled by elements of Jordan algebra

$$V_a = \tau(U_a) \in \mathfrak{g}^{+1} \qquad V_a(x) = -\frac{1}{2}(x, a, x) \tag{3.15}$$

The Lie algebra is then given as follows [43]:

$$\begin{aligned} [U_a, V_b] &= S_{ab} & [S_{ab}, U_c] &= U_{(abc)} & [U_a, U_b] &= 0 \\ [S_{ab}, S_{cd}] &= S_{(abc)d} - S_{c(bad)} & [S_{ab}, V_c] &= -V_{(bac)} & [V_a, V_b] &= 0 \end{aligned} \tag{3.16}$$

where (abc) is Jordan triple product, and

$$S_{ab} \in \mathfrak{g}^0 \qquad S_{ab}(x) = (abx). \tag{3.17}$$

The Jacobi identities of (3.16) require Jordan triple product to satisfy the following identities

$$(abc) = (cba) \qquad (ab(cdx)) - (cd(abx)) = ((abc)dx) - (c(bad)x) \tag{3.18}$$

which follow from defining identities (3.1) of Jordan algebra. However, because Lie algebra is defined entirely in terms of Jordan triple product, and (3.18) are defining identities of Jordan triple system (JTS) [43] this construction extends to JTS.

It is gratifying to examine relationship between JTS and Lie algebras in the opposite direction [83]. Let $\mathfrak{g}$ be a graded Lie algebra with a graded involution $\iota : \mathfrak{g}^n \mapsto \mathfrak{g}^{-n}$. Then any Lie algebra admitting 3-graded decomposition is defining JTS via

$$(x, y, z) = [[x, \iota(y)], z] \tag{3.19}$$

where $x, y, z \in \mathfrak{g}^{-1}$.

3.3 Generalised rotation, Lorentz and conformal groups

The first proposal to use Jordan algebras to define generalised space-times was made in the early days of supersymmetry in attempts to find the super-analogs of the exceptional Lie algebras [36]. This proposal is very natural, since, using twistor formalism [87], coordinates

of Minkowski space-time can be organised into 2×2 Hermitian matrix over $\mathbb{C}$:

$$x = x^{\mu}\sigma_{\mu} = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \tag{3.20}$$

which is an element of $J_2^{\mathbb{C}}$. Rotation group $\mathrm{SO}(3)$ is exactly an automorphism group of Jordan algebra since its action

$$x \mapsto UxU^{-1} \qquad U = \exp\left(\sum_{i,j,k=1}^{3} \omega_i \, \epsilon^{ijk} \, \sigma_{jk}\right) \qquad \sigma_{jk} = \frac{1}{4}\left[\sigma_j, \sigma_k\right] \qquad U^{\dagger}U = 1 \tag{3.21}$$

preserves hermiticity of x as well as Jordan product. Lorentz group action

$$x \mapsto \Lambda x \Lambda^{\dagger} \qquad \Lambda = \exp\left(\sum_{j=1}^{3} \beta^j \sigma_0 \sigma_j + \sum_{i,j,k=1}^{3} \omega_i \, \epsilon^{ijk} \, \sigma_{jk}\right) \tag{3.22}$$

also preserves hermiticity of x. It also preserves the norm of the Jordan algebra

$$\mathcal{N}(x) = \det x = \left(x^0\right)^2 - \sum_{i=1}^{3} \left(x^i\right)^2 \qquad \mathcal{N}(x) = \mathcal{N}\left(\Lambda x \Lambda^{\dagger}\right) = \left|\det \Lambda\right|^2 \mathcal{N}(x)\,. \tag{3.23}$$

if $\det \Lambda = 1$. Dilatation transformation which rescales the norm by a constant factor is achieved with $\Lambda = \delta \cdot \mathrm{id}$. Thus the Lorentz group coincides with reduced structure group[1] of $J_2^{\mathbb{C}}$. Conformal group of Minkowski space-time is then generated from structure group, comprising Lorentz group and dilatation $x \mapsto e^t x$, translations and inversion

$$x \mapsto (x)^{-1} \qquad \Longrightarrow \qquad x^{\mu} \mapsto \eta_{\mu\nu} \frac{x^{\nu}}{x_{\lambda} x^{\lambda}} \tag{3.24}$$

One can extend the notion of rotation, Lorentz and conformal group to any Jordan algebra, thus establishing coordinatization of generalised space-times by Jordan algebras.

The rotation $Rot(J)$, Lorentz $Lor(J)$ and conformal $Conf(J)$ groups of these generalised space-times are then identified with the automorphism $Aut(J)$, reduced structure $Str_0(J)$ and Möbius $\mathrm{Mö}(J)$ groups of the corresponding Jordan algebras [36, 42, 37, 43]. Let $J_n^{\mathbb{A}}$ be the Jordan algebra of $n \times n$ Hermitian matrices over the *division* algebra $\mathbb{A}$ and let the Jordan algebra of Dirac gamma matrices in $\mathbb{R}^d$ be $\Gamma(d)$. The symmetry groups of generalised space-times defined by simple Euclidean Jordan algebras are then collected in table 3.1 [43]. The symbols $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ represent the four division algebras. For the Jordan algebras $J_n^{\mathbb{A}}$ the norm form is the determinantal form (or its generalisation to the quaternionic and

[1] Structure group of a Jordan algebra with a norm, is defined as set of transformations that rescale the norm by a constant positive factor.

J	$Rot(J)$	$Lor(J)$	$Conf(J)$
$J_n^{\mathbb{R}}$	$SO(n)$	$SL(n,\mathbb{R})$	$Sp(2n,\mathbb{R})$
$J_n^{\mathbb{C}}$	$SU(n)$	$SL(n,\mathbb{C})$	$SU(n,n)$
$J_n^{\mathbb{H}}$	$USp(2n)$	$SU^*(2n)$	$SO^*(4n)$
$J_3^{\mathbb{O}}$	F_4	$E_{6(-26)}$	$E_{7(-25)}$
$\Gamma(d)$	$SO(d)$	$SO(d,1)$	$SO(d,2)$

Table 3.1: Rotation, Lorentz and conformal groups for generalised space-times defined by simple Euclidean Jordan algebras

octonionic matrices). For the Jordan algebra $\Gamma(d)$ generated by Dirac gamma matrices Γ_i $(i = 1, 2, ...d)$

$$\{\Gamma_i, \Gamma_j\} = \delta_{ij}\mathbf{1}; \qquad i, j, \ldots = 1, 2, \ldots, d \tag{3.25}$$

the norm of a general element $x = x_0\mathbf{1} + x_i\Gamma_i$ of $\Gamma(d)$ is quadratic and given by

$$\mathcal{N}(x) = x\bar{x} = x_0^2 - x_i x_i \tag{3.26}$$

where $\bar{x} = x_0\mathbf{1} - x_i\Gamma_i$. Its automorphism, reduced structure and Möbius groups are simply the rotation, Lorentz and conformal groups of $(d+1)$-dimensional Minkowski spacetime. One finds the following special isomorphisms between the Jordan algebras of 2×2 Hermitian matrices over the four division algebras and the Jordan algebras of gamma matrices:

$$J_2^{\mathbb{R}} \simeq \Gamma(2) \quad ; \quad J_2^{\mathbb{C}} \simeq \Gamma(3) \quad ; \quad J_2^{\mathbb{H}} \simeq \Gamma(5) \quad ; \quad J_2^{\mathbb{O}} \simeq \Gamma(9) \tag{3.27}$$

The Minkowski space-times they correspond to are precisely the critical dimensions for the existence of super Yang-Mills theories as well as of the classical Green-Schwarz superstrings. These Jordan algebras are all quadratic and their norm forms are precisely the quadratic invariants constructed using the Minkowski metric.

We should note two remarkable facts about the above table. First is the fact that the maximal compact subgroups of the generalised conformal groups of formally real Jordan algebras are simply the compact forms of their structure groups (which are the products of their generalised Lorentz groups with dilatations). Second, the conformal groups of generalised space-times defined by Euclidean (formally real) Jordan algebras all admit positive energy unitary representations[2]. Hence they can be given a causal structure with a unitary time evolution as in four dimensional Minkowski space-time [44].

[2]Similarly, the generalised conformal groups defined by Hermitian Jordan triple systems all admit positive energy unitary representations [43]. In fact the conformal groups of simple Hermitian Jordan triple systems exhaust the list of simple non-compact groups that admit positive energy unitary representations. They include the conformal groups of simple Euclidean Jordan algebra since the latter form an Hermitian Jordan triple system under the Jordan triple product [43].

3.4 Positive energy unitary representations of generalised conformal groups

A Lie algebra $\mathfrak{g}$ of a non-compact group G that admits unitary lowest weight representation (ULWR), also known as positive energy representations, admits a 3-graded decomposition with respect to the Lie algebra $\mathfrak{h}$ of its maximal compact subgroup H: $\mathfrak{g} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{+1}$, where $\mathfrak{g}^{0} = \mathfrak{h}$.

In [38] the general oscillator construction of unitary lowest weight representations of non-compact groups was given. One starts realizing G in terms of bilinears of oscillators transforming in certain (possibly reducible) representation of H. Then, in the corresponding Fock space $\mathcal{F}$ of all oscillators one chooses a set of states $|\Omega\rangle$, referred to as the "lowest weight vector", which transforms irreducibly under H and which are annihilated by generators of $\mathfrak{g}^{-1}$. Then, ULWR is built by repeatedly applying $\mathfrak{g}^{+1}$ to this set of lowest weight states:

$$|\Omega\rangle\,, \quad \mathfrak{g}^{+1}\,|\Omega\rangle\,, \quad \mathfrak{g}^{+1}\mathfrak{g}^{+1}\,|\Omega\rangle\,, \ldots \tag{3.28}$$

The irreducibility of the representation of $\mathfrak{g}$ follows from the irreducibility of LWV $|\Omega\rangle$ under H.

As an illustration consider ULWR for $\mathrm{Sp}\,(2n, \mathbb{R})$, conformal group of generalised space-time defined by $J_n^{\mathbb{R}}$. In the compact basis, the 3-graded decomposition reads

$$\begin{aligned} \mathfrak{sp}\,(2n, \mathbb{R}) &= \frac{n\,(n+1)}{2} \oplus (\mathfrak{su}\,(n) \oplus \mathfrak{u}\,(1)) \oplus \frac{n\,(n+1)}{2} \\ &= \{U_{ij} = a_i a_j\} \oplus \left\{ S^i{}_j = \frac{1}{2}\left(a^i a_j + a_j a^i\right) \right\} \oplus \left\{U^{ij} = a^i a^j\right\} \end{aligned} \tag{3.29}$$

where we have chosen oscillators a^i, a_j

$$\left[a_i, a^j\right] = \delta_i{}^j \qquad a_j = \left(a^j\right)^{\dagger} \tag{3.30}$$

transforming in $\mathbf{n} \oplus \mathbf{n}^*$ of $\mathfrak{su}\,(n)$. There are only two non-equivalent irreducible LWVs:

$$\begin{aligned} |\Omega_1\rangle &= |0\rangle \\ |\Omega_2\rangle &= c^i\,|0\rangle \end{aligned} \tag{3.31}$$

vacuum and a "one-particle state". The representations above are known in the literature

as singleton representations [38]. Commutation relation of $\mathfrak{sp}(2n,\mathbb{R})$ read

$$\begin{aligned}
\left[S^i{}_j, S^k{}_m\right] &= \delta_j{}^k S^i{}_m - \delta_m{}^i S^k{}_j \\
\left[S^i{}_j, U_{km}\right] &= -\delta_k{}^i U_{jm} - \delta_m{}^i U_{kj} \\
\left[S^i{}_j, U^{km}\right] &= \delta_j{}^k U^{im} + \delta_j{}^m U^{ki} \\
\left[U_{ij}, U^{km}\right] &= \delta_j{}^k S^m{}_i + \delta_i{}^k S^m{}_j + \delta_j{}^m S^k{}_i + \delta_i{}^m S^k{}_j
\end{aligned} \tag{3.32}$$

resulting in quadratic Casimir of $\mathfrak{sp}(2n,\mathbb{R})$

$$\begin{aligned}
\mathcal{C}_2 &= S^i{}_j S^j{}_i - \frac{1}{2}\left(U_{ij}U^{ij} + U^{ij}U_{ij}\right) \\
&= S^i{}_j S^j{}_i - (n+1)\, S^i{}_i - U^{ij}U_{ij} = -\frac{1}{4} n\,(2n+1)\,.
\end{aligned} \tag{3.33}$$

We thus see that quadratic Casimir of $\mathfrak{sp}(2n,\mathbb{R})$ takes on the same values on both lwv $|\Omega_{1,2}\rangle$.

Conformal group has a natural 3-graded decomposition with respect to non-compact Lorentz algebra in $\mathfrak{g}^0$, so called covariant picture, as opposed to compact picture considered above. These two pictures can be connected [44] observing existence of intertwining operator W such that

$$\forall a \in J \qquad U_a W\,|\Omega\rangle = 0 \tag{3.34}$$

where U_a is negative grade generator of $Conf(J)$ (c.f. (3.16)) in covariant picture.

The starting point is the observation that $W\,|\Omega\rangle$ transforms under the structure group just like $|\Omega\rangle$ transforms under the maximal compact subgroup H: the conformal dimension of the vector $W\,|\Omega\rangle$ equals the negative of the conformal energy of $|\Omega\rangle$. Let e_μ be a basis for the Jordan algebra J. Let V_μ be the generators of generalised translations in the positive grade space of covariant picture that corresponds to e_μ. The covariant basis of unitary lowest weight representation of the generalised conformal group $Conf(J)$ is given by non-compact coherent states

$$|\Phi(x_\mu)\rangle := e^{ix^\mu V_\mu} W\,|\Omega\rangle \tag{3.35}$$

Conformal fields, eigenstates of dilatation operator, and covariant under Lorentz group are in one-to-one correspondence with coherent states $|\Phi(x_\mu)\rangle$. Irreducible ULWR's are equivalent to representations induced by finite dimensional irreps of the Lorentz group with a definite conformal dimension and trivial special conformal transformation properties, because the state $W\,|\Omega\rangle$ is annihilated by the generators of special conformal transformations U_μ that belong to the negative grade in covariant picture.

The outlined above procedure generalises the construction of the positive energy representations of the 4-dimensional conformal group $SU(2,2)$ [75] to all generalised confor-

mal groups of formally real Jordan algebras as well as Hermitian Jordan triple systems. They are simply induced representations with respect to the maximal parabolic subgroup $Str(J) \ltimes S_J$, where S_J is the Abelian subgroup generated by generalised special conformal transformations [44].

The generalised Poincaré groups associated with the space-times defined by Jordan algebras have the following form

$$\mathcal{PG}(J) = Lor(J) \ltimes T_J \tag{3.36}$$

where T_J is a group formed by generalised translations V_μ, which commute with each other. For quadratic Jordan algebras, $\Gamma(d)$, $\mathcal{PG}\left(\Gamma(d)\right)$ equals the Poincaré group in d dimensional Minkowski space. A quadratic Casimir $M^2 = P_\mu P^\mu$ of the group $\mathcal{PG}\left(\Gamma(d)\right)$ is the familiar mass squared operator.

For Jordan algebras J of degree n the corresponding Casimir invariant will be constructed as n-norm of translation generators. For instance for the real exceptional Jordan algebra $J_3^{\mathbb{O}}$ the corresponding Casimir invariant is cubic and equals [44]

$$M^3 = C_{\mu\nu\rho} V^\mu V^\nu V^\rho \tag{3.37}$$

where $C_{\mu\nu\rho}$ is the symmetric invariant tensor of the generalised Lorentz group $\mathrm{E}_{6(-26)}$ of $J_3^{\mathbb{O}}$ $(\mu, \nu, \rho, \ldots = 0, 1, \ldots, 26)$.

3.5 Quasiconformal groups and Freudenthal triple systems

Not every Lie algebra admits a 3-graded decomposition, examples being exceptional Lie algebras $\mathfrak{e}_8$, $\mathfrak{f}_4$ and $\mathfrak{g}_2$, which are among Lie algebras of U-duality groups. One can prove, however, that every simple Lie algebra, except three dimensional $\mathfrak{sl}(2)$, admits a five graded decomposition (c.f. A.2):

$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^{+1} \oplus \mathfrak{g}^{+2} \tag{3.38}$$

with $\mathfrak{g}^{\pm 2}$ being one-dimensional spaces. In this case the grade -1 space $\mathfrak{g}^{-1}$ is naturally endowed with (Gel'fand-Kirillov) symplectic structure

$$\forall E_x, E_y \in \mathfrak{g}^{-1} \qquad [E_x, E_y] = 2\,\langle x, y \rangle\, E \tag{3.39}$$

where $E \in \mathfrak{g}^{-2}$, turning $\mathfrak{g}^{-1}$ into Heisenberg-Weyl algebra. Subspace $\mathfrak{g}^{-1}$ also has a natural triple system associated to it by means of (3.19) studied by Freudenthal [30]. Having motivated an appearance of FTS we now give more axiomatic definitions.

A Freudenthal triple system (FTS) is a vector space $\mathfrak{M}$ with a trilinear product (X, Y, Z)

and a skew symmetric bilinear form $\langle X, Y\rangle$ such that[3]:

$$\begin{aligned}
(X,Y,Z) &= (Y,X,Z) + 2\,\langle X,Y\rangle\, Z \\
(X,Y,Z) &= (Z,Y,X) - 2\,\langle X,Z\rangle\, Y \\
\langle (X,Y,Z),W\rangle &= \langle (X,W,Z),Y\rangle - 2\,\langle X,Z\rangle\langle Y,W\rangle \\
(X,Y,(V,W,Z)) &= (V,W,(X,Y,Z)) + ((X,Y,V),W,Z) \\
&\quad + (V,(Y,X,W),Z)
\end{aligned} \tag{3.40}$$

A quartic invariant $\mathcal{I}_4$ can be constructed over the FTS by means of the triple product and the bilinear form as

$$\mathcal{I}_4(X) := \langle (X,X,X),X\rangle \tag{3.41}$$

Every FTS defines a Lie algebra with 5-graded decomposition (3.38) and one-dimensional $\mathfrak{g}^{\pm 2}$, establishing a one-to-one correspondence between simple Lie algebras (except $\mathfrak{sl}(2)$) and Freudenthal triple systems. Following [45] we shall label the Lie algebra generators belonging to grade $+1$ and grade -1 subspaces as U_A and $\tilde{U}_A$, where $A \in \mathfrak{M}$. The generators S_{AB} belonging to grade zero subspace are labelled by a pair of elements $A, B \in \mathfrak{M}$. For the grade ± 2 subspaces one would in general need another set of generators K_{AB} and $\tilde{K}_{AB}$ labelled by two elements, but since these subspaces are one-dimensional we can write them as

$$K_{AB} = \langle A,B\rangle\, K_a \qquad \tilde{K}_{AB} = \langle A,B\rangle\, \tilde{K}_a \tag{3.42}$$

where a is a real parameter.

One can realize the Lie algebra $\mathfrak{g}$ as a quasiconformal Lie algebra over a vector space whose coordinates $\mathcal{X}$ are labelled by a pair (X,x), where $X \in \mathfrak{M}$ and x is an extra single

[3] We should note that the triple product (3.40) could be modified by terms involving the symplectic invariant, such as $\langle X,Y\rangle\, Z$. The choice given above was made in [45] in order to obtain agreement with the formulas of [23].

variable as follows [45]:

$$\begin{array}{lll} K_a(X)=0 & U_A(X)=A & S_{AB}(X)=(A,B,X) \\ K_a(x)=2\,a & U_A(x)=\langle A,X\rangle & S_{AB}(x)=2\,\langle A,B\rangle\,x \\ & \tilde{U}_A(X)=\dfrac{1}{2}(X,A,X)-Ax & \\ & \tilde{U}_A(x)=-\dfrac{1}{6}\langle (X,X,X),A\rangle+\langle X,A\rangle\,x & \\ & \tilde{K}_a(X)=-\dfrac{1}{6}\,a\,(X,X,X)+aXx & \\ & \tilde{K}_a(x)=\dfrac{1}{6}\,a\,\langle (X,X,X),X\rangle+2\,a\,x^2 & \end{array} \tag{3.43}$$

From these formulas it is straightforward to determine the commutation relations of the transformations [45]:

$$\begin{array}{lll} \left[U_A,\tilde{U}_B\right]=S_{AB} & [U_A,U_B]=-K_{AB} & [S_{AB},U_C]=-U_{(A,B,C)} \\ & \left[\tilde{U}_A,\tilde{U}_B\right]=-\tilde{K}_{AB} & \left[S_{AB},\tilde{U}_C\right]=-\tilde{U}_{(B,A,C)} \\ \left[K_{AB},\tilde{U}_C\right]=U_{(A,C,B)}-U_{(B,C,A)} & [S_{AB},S_{CD}]=-S_{(A,B,C)D}-S_{C(B,A,D)} & \\ \left[\tilde{K}_{AB},U_C\right]=\tilde{U}_{(B,C,A)}-\tilde{U}_{(A,C,B)} & [S_{AB},K_{CD}]=K_{A(C,B,D)}-K_{A(D,B,C)} & \\ \left[S_{AB},\tilde{K}_{CD}\right]=\tilde{K}_{(D,A,C)B}-\tilde{K}_{(C,A,D)B} & & \\ \left[K_{AB},\tilde{K}_{CD}\right]=S_{(B,C,A)D}-S_{(A,C,B)D}-S_{(B,D,A)C}+S_{(A,D,B)C} & & \end{array} \tag{3.44}$$

where $K_{AB}=K_{\langle A,B\rangle}$, and $\tilde{K}_{AB}=\tilde{K}_{\langle A,B\rangle}$. The quasi-conformal groups leave invariant a suitably defined light-cone

$$\mathcal{I}_4(X-Y)=-2\,(x-y+2\,\langle X,Y\rangle)^2 \tag{3.45}$$

with respect to a quartic norm involving the quartic invariant of $\mathfrak{M}$ [45].

Freudenthal introduced the triple systems associated with his name in his study of the metasymplectic geometries associated with exceptional groups [29]. The geometries associated with FTSs were further studied in [2, 23, 77, 63]. A classification of FTS's may be found in [63], where it is also shown that there is a one-to-one correspondence between simple Lie algebras and simple FTS's with a non-degenerate skew symmetric bilinear form. Hence there is a quasiconformal realization of every Lie group acting on a generalised light-cone.

The Freudenthal triple systems associated with exceptional groups can be represented

by formal 2×2 "matrices" of the form

$$A = \begin{pmatrix} \alpha_1 & x_1 \\ x_2 & \alpha_2 \end{pmatrix}, \tag{3.46}$$

where α_1, α_2 are real numbers and x_1, x_2 are elements of a simple Jordan algebra $J_3^{\mathbb{A}}$ of degree three. One can define a triple product over the space of such formal matrices such that they close under it. There are only four simple Euclidean Jordan algebras J of this type, namely the 3×3 Hermitian matrices over the four division algebras $\mathbb{A} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. We shall denote the corresponding FTS's as $\mathfrak{M}(J)$.

One may ask which Freudenthal triple systems can be realized in the above form in terms of an underlying Jordan algebra. This question was investigated by Ferrar [24] who proved that such a realization is possible only if the underlying Jordan algebra is of degree three. Remarkably, if one further requires that the underlying Jordan algebra be formally real then the list of Jordan algebras over which FTS's can be defined as above coincides with the list of Jordan algebras that occur in five dimensional $N = 2$ MESGT's whose target spaces are symmetric spaces of the form G/H such that G is a symmetry of the Lagrangian [39].

Here we will focus only on the quasiconformal groups defined over formally real Jordan algebras. The Freudenthal triple product of the elements of $\mathfrak{M}(J)$ is defined as [23]

$$\langle X_1, X_2, X_3 \rangle = \begin{pmatrix} \gamma & c \\ d & \delta \end{pmatrix} \quad \text{with} \quad X_i = \begin{pmatrix} \alpha_i & a_i \\ b_i & \beta_i \end{pmatrix} \tag{3.47}$$

where

$$\begin{aligned}
\gamma &= \alpha_1\beta_2\alpha_3 + 2\alpha_1\alpha_2\beta_3 - \alpha_3 T(a_1, b_2) - \alpha_2 T(a_1, b_3) \\
&\quad - \alpha_1 T(a_2, b_3) + T(a_1, a_2 \times a_3) \\
c &= (\alpha_2\beta_3 + T(b_2, a_3))\, a_1 + (\alpha_1\beta_3 + T(b_1, a_3))\, a_2 + (\alpha_1\beta_2 + T(b_1, a_2))\, a_3 \\
&\quad - \alpha_1\, b_2 \times b_3 - \alpha_2\, b_1 \times b_3 - \alpha_3 b_1 \times b_2 \\
&\quad - \{a_1, b_2, a_3\} - \{a_1, b_3, a_2\} - \{a_2, b_1, a_3\} \\
\delta &= -\gamma^\sigma \qquad d = -c^\sigma \qquad \text{where} \qquad \sigma = (\alpha \leftrightarrow \beta)\,(a \leftrightarrow b)\,.
\end{aligned}$$

Here σ denotes a permutation of α with β and a with b, and

$$\{a, b, c\} = U_{a+c}b - U_a b - U_c b \tag{3.48}$$

where $U_a b$ is defined as in (2.54). Quartic invariant of $\mathfrak{M}(J)$ is given by [23]

$$\mathcal{I}_4\left(\begin{pmatrix}\alpha & a\\ b & \beta\end{pmatrix}\right) = (\alpha\beta - T(a,b))^2 + 6\left(\alpha\mathcal{V}(b) + \beta\mathcal{V}(a) - T\left(a^\sharp, b^\sharp\right)\right)^2 \tag{3.49}$$

3.6 Space-times over $J_3^{\mathbb{A}}$ as dilatonic and spinorial extensions of Minkowskian space-times

As stated above we will restrict our studies of generalised space-times to those defined by formally real Jordan algebras of degree 3. A unified geometric realization of the conformal and quasiconformal groups of generalised space-times defined by Jordan algebras of degree three and the FTS's defined over them was given in [50]. It is reproduced in sections 3.6, 3.7 and 3.8.

The Jordan algebras of degree three that arose in the study of MESGT's were later studied by Sierra who showed that there exists a correspondence between them and classical relativistic point particle actions [104]. In the same work Sierra showed that this could be extended to a correspondence between classical relativistic bosonic strings and the Freudenthal triple systems defined over them.

Consider now the space-times coordinatized by the generic Jordan family

$$J = \mathbb{R} \oplus \Gamma(Q) \tag{3.50}$$

we shall interpret the extra coordinate corresponding to $\mathbb{R}$ as a dilatonic coordinate ρ and label the coordinates defined by J as $(\rho, x_m, m = 0, 1, 2, ...(d-1))$. The automorphism group $SO(d-1)$ will then be the rotation group of this space-time under which both the time coordinate x_0 and the dilatonic coordinate ρ will be singlets. The Lorentz group of this spacetime is the reduced structure group which is simply

$$\mathrm{SO}(d-1,1) \times \mathrm{SO}(1,1) \tag{3.51}$$

It leaves invariant the cubic norm which, following [104], we normalise as

$$\mathcal{V}(\rho, x_m) = \sqrt{2}\rho\, x_m x_n \eta^{mn} \tag{3.52}$$

Under the action of $SO(d-1,1)$, the dilaton ρ is a singlet and under $SO(1,1)$ we have

$$SO(1,1): \begin{array}{c}\rho \Rightarrow e^{2\lambda}\rho\\ x_m \Rightarrow e^{\lambda} x_m\end{array} \tag{3.53}$$

Freudenthal product of two elements of $J = \mathbb{R} \oplus \Gamma(Q)$ is simply

$$(\rho, x) \times (\sigma, y) = \left(\sqrt{2} x_m y^m, \sqrt{2}\left(\rho y_m + \sigma x_m\right)\right) \tag{3.54}$$

The conformal group of the spacetime is the Möbius group of J which is

$$\mathrm{SO}(d, 2) \times \mathrm{SO}(2, 1) \tag{3.55}$$

The Freudenthal triple systems defined over the generic Jordan family can be represented by 2×2 matrices

$$\mathfrak{M}\left(J = \mathbb{R} \oplus \Gamma\left(Q\right)\right) = \begin{pmatrix} x_d^1 & J^1 \\ J^2 & x_d^2 \end{pmatrix} = X \tag{3.56}$$

where $J^1, J^2 \in J$ and x_d^1 and x_d^2 are real coordinates. The automorphism group of $\mathfrak{M}$ is $\mathrm{SO}(d, 2) \otimes \mathrm{Sp}(2, \mathbb{R})$ under which an element of $\mathfrak{M}$ transforms in the representation $(\mathbf{d} + \mathbf{2}, \mathbf{2})$. We shall label the "coordinates" of $\mathfrak{M}$ as

$$x_\mu^a = (x_m^a, x_d^a, \rho^a) \quad \text{where} \quad a = 1, 2$$

and interpret it as coordinates of a conformally covariant phase space (so that $a = 1$ labels the coordinates and $a = 2$ labels the momenta).

Skew-symmetric invariant form over $\mathfrak{M}$ is given by

$$\langle X, Y \rangle = \epsilon_{ab} \eta^{\mu\nu} X_\mu^a Y_\nu^b \tag{3.57}$$

We should stress the important fact that the conformal group of the spacetime defined by J is isomorphic to the automorphism group of the Freudenthal triple system $\mathfrak{M}(J)$!

To define the quasi-conformal group over the conformal phase space represented by $\mathfrak{M}(J)$ we need to extend it by an extra coordinate corresponding to the cocycle (symplectic form) over $\mathfrak{M}(J)$. We shall denote the elements of $\mathfrak{M}(J)$ as X and the extra coordinate as x. The quasi-conformal group of $\mathfrak{M}(J) \oplus \mathbb{R}$ is the group $\mathrm{SO}(d + 2, 4)$.

The space-times defined by simple Jordan algebras of degree 3 $J_3^{\mathbb{A}}$ correspond to extensions of Minkowski space-times in the critical dimensions $d = 3, 4, 6, 10$ by a dilatonic (ρ) *and* commuting spinorial coordinates (ξ^a).

$$\begin{aligned}
J_3^{\mathbb{R}} &\Longleftrightarrow (\rho, x_m, \xi^\alpha) \quad m = 0, 1, 2 \quad \alpha = 1, 2 \\
J_3^{\mathbb{C}} &\Longleftrightarrow (\rho, x_m, \xi^\alpha) \quad m = 0, 1, 2, 3 \quad \alpha = 1, 2, 3, 4 \\
J_3^{\mathbb{H}} &\Longleftrightarrow (\rho, x_m, \xi^\alpha) \quad m = 0, \ldots, 5 \quad \alpha = 1, \ldots, 8 \\
J_3^{\mathbb{C}} &\Longleftrightarrow (\rho, x_m, \xi^\alpha) \quad m = 0, \ldots, 9 \quad \alpha = 1, \ldots, 16
\end{aligned} \tag{3.58}$$

The commuting spinors ξ are represented by a 2×1 matrix over $\mathbb{A}=\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}$. The cubic norm of a "vector" with coordinates (ρ,x_m,ξ^α) is given by [104]

$$\mathcal{V}\left(\rho,x_m,\xi^\alpha\right)=\sqrt{2}\rho x_m x_n\eta^{mn}+x^m\bar{\xi}\gamma_m\xi \tag{3.59}$$

The Lorentz groups of the space-times over $J_3^{\mathbb{A}}$ are

$$\mathrm{SL}\left(3,\mathbb{R}\right),\quad \mathrm{SL}\left(3,\mathbb{C}\right),\quad \mathrm{SU}^*\left(6\right),\quad \text{and}\quad \mathrm{E}_{6(-26)} \tag{3.60}$$

respectively, corresponding to the invariance groups of their cubic norm. The Freudenthal product of two vectors in the corresponding space-time is given by

$$\begin{aligned}&\left(\rho,x_m,\xi^\alpha\right)\times\left(\sigma,y_m,\zeta^\alpha\right)=\\&\qquad\left(\sqrt{2}\,x_m y^m,\ \frac{1}{2}\left(\bar{\xi}\gamma_m\zeta+\bar{\zeta}\gamma_m\xi\right)+\sqrt{2}\left(\rho\,y_m+\sigma\,x_m\right),\ x^m\,\bar{\zeta}\gamma_m+y^m\,\bar{\xi}\gamma_m\right)\end{aligned} \tag{3.61}$$

The conformal groups of these space-times are

$$\mathrm{Sp}\left(6,\mathbb{R}\right),\quad \mathrm{SU}\left(3,3\right),\quad \mathrm{SO}^*\left(12\right),\quad \text{and}\quad \mathrm{E}_{7(-25)} \tag{3.62}$$

respectively. The automorphism groups of the FTS $\mathfrak{M}\left(J_3^{\mathbb{A}}\right)$ are isomorphic to their conformal groups.

The quasi-conformal groups acting on $\mathfrak{M}\left(J_3^{\mathbb{A}}\oplus\mathbb{R}\right)$, where $\mathbb{R}$ represents the extra "cocyle" coordinate, are

$$\mathrm{F}_{4(4)},\quad \mathrm{E}_{6(2)},\quad \mathrm{E}_{7(-5)},\quad \text{and}\quad \mathrm{E}_{8(-24)} \tag{3.63}$$

whose minimal unitary irreducible representations were constructed in [48].

3.7 Geometric realizations of $\mathrm{SO}\left(d+2,4\right)$ as quasiconformal groups

Lie algebra of $\mathrm{SO}\left(d+2,4\right)$ admits the following 5-graded decomposition

$$\mathfrak{so}\left(d+2,4\right)=\mathbf{1}\oplus\left(\mathbf{d+2},\mathbf{2}\right)\oplus\left(\Delta\oplus\mathfrak{sp}\left(2,\mathbb{R}\right)\oplus\mathfrak{so}\left(d,2\right)\right)\oplus\left(\mathbf{d+2},\mathbf{2}\right)\oplus\mathbf{1} \tag{3.64}$$

Generators are realized as differential operators in $2d+5$ coordinates corresponding to $\mathfrak{g}^{-2}\oplus\mathfrak{g}^{-1}$ subspace which we shall denote as x and $X^{\mu,a}$ where $a=1,2$ is an index of representation $\mathbf{2}$ of $\mathfrak{sp}\left(2,\mathbb{R}\right)$ and we shall let the indices μ run from 1 to $d+2$ with the indices $d+1$ and $d+2$ labelling the two time-like coordinates, i.e x_μ transforms like a vector of $SO(d,2)$.

Let ϵ_{ab} be symplectic real-valued matrix, and $\eta_{\mu\nu}$ denote signature $(d,2)$ metric preserved by $SO(d,2)$. Then

$$\mathcal{I}_4 = \eta_{\mu\nu}\eta_{\rho\tau}\epsilon_{ac}\epsilon_{bd}X^{\mu,a}X^{\nu,b}X^{\rho,c}X^{\tau,d} \tag{3.65}$$

is a 4th-order polynomial invariant under the semisimple part of $\mathfrak{g}^0$. Define

$$\begin{aligned}
K_+ &= \frac{1}{2}\left(2x^2 - \mathcal{I}_4\right)\frac{\partial}{\partial x} - \frac{1}{4}\frac{\partial \mathcal{I}_4}{\partial X^{\mu,a}}\eta^{\mu\nu}\epsilon^{ab}\frac{\partial}{\partial X^{\nu,b}} + x\, X^{\mu,a}\frac{\partial}{\partial X^{\mu,a}} \\
U_{\mu,a} &= \frac{\partial}{\partial X^{\mu,a}} - \eta_{\mu,\nu}\epsilon_{ab}X^{\nu,b}\frac{\partial}{\partial x} \\
M_{\mu\nu} &= \eta_{\mu\rho}X^{\rho,a}\frac{\partial}{\partial X^{\nu,a}} - \eta_{\nu\rho}X^{\rho,a}\frac{\partial}{\partial X^{\mu,a}} \\
J_{ab} &= \epsilon_{ac}X^{\mu,c}\frac{\partial}{\partial X^{\mu,b}} + \epsilon_{bc}X^{\mu,c}\frac{\partial}{\partial X^{\mu,a}} \\
K_- &= \frac{\partial}{\partial x} \qquad \Delta = 2x\frac{\partial}{\partial x} + X^{\mu,a}\frac{\partial}{\partial X^{\mu,a}} \qquad \tilde{U}_{\mu,a} = [U_{\mu,a}, K_+]
\end{aligned} \tag{3.66}$$

where ϵ^{ab} denotes an inverse symplectic metric: $\epsilon^{ab}\epsilon_{bc} = \delta^a{}_c$ and $\tilde{U}_{\mu,a}$ evaluates to

$$\begin{aligned}
\tilde{U}_{\mu,a} = \eta_{\mu\nu}\epsilon_{ad}&\left(\eta_{\lambda\rho}\epsilon_{bc}X^{\nu,b}X^{\lambda,c}X^{\rho,d} - xX^{\nu,d}\right)\frac{\partial}{\partial x} + x\frac{\partial}{\partial X^{\mu,a}} \\
&- \eta_{\mu\nu}\epsilon_{ab}X^{\nu,b}X^{\rho,c}\frac{\partial}{\partial X^{\rho,c}} - \epsilon_{ad}\eta_{\lambda\rho}X^{\rho,d}X^{\lambda,c}\frac{\partial}{\partial X^{\mu,c}} \\
&+ \epsilon_{ad}\eta_{\mu\nu}X^{\rho,d}X^{\nu,b}\frac{\partial}{\partial X^{\rho,b}} + \eta_{\mu\nu}\epsilon_{bc}X^{\nu,b}X^{\rho,c}\frac{\partial}{\partial X^{\rho,a}}
\end{aligned} \tag{3.67}$$

we have

$$\frac{\partial \mathcal{I}_4}{\partial X^{\mu,a}} = -4\,\eta_{\mu\nu}\,\eta_{\lambda\rho}\,X^{\nu,b}X^{\lambda,c}X^{\rho,d}\,\epsilon_{bc}\epsilon_{ad}$$

These generators satisfy the following commutation relation:

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\tau}] &= \eta_{\nu\rho}M_{\mu\tau} - \eta_{\mu\rho}M_{\nu\tau} + \eta_{\mu\tau}M_{\nu\rho} - \eta_{\nu\tau}M_{\mu\rho} \\
[J_{ab}, J_{cd}] &= \epsilon_{cb}J_{ad} + \epsilon_{ca}J_{bd} + \epsilon_{db}J_{ac} + \epsilon_{da}J_{bc} \\
[\Delta, K_\pm] &= \pm 2K_\pm \qquad [K_-, K_+] = \Delta \\
[\Delta, U_{\mu,a}] &= -U_{\mu,a} \qquad \left[\Delta, \tilde{U}_{\mu,a}\right] = \tilde{U}_{\mu,a} \\
[U_{\mu,a}, K_+] &= \tilde{U}_{\mu,a} \qquad \left[\tilde{U}_{\mu,a}, K_-\right] = -U_{\mu,a} \\
[U_{\mu,a}, U_{\nu,b}] &= 2\eta_{\mu\nu}\epsilon_{ab}K_- \qquad \left[\tilde{U}_{\mu,a}, \tilde{U}_{\nu,b}\right] = 2\eta_{\mu\nu}\epsilon_{ab}K_+
\end{aligned} \tag{3.68a}$$

$$\begin{aligned}
[M_{\mu\nu}, U_{\rho,a}] &= \eta_{\nu\rho}U_{\mu,a} - \eta_{\mu\rho}U_{\nu,a} \qquad \left[M_{\mu\nu}, \tilde{U}_{\rho,a}\right] = \eta_{\nu\rho}\tilde{U}_{\mu,a} - \eta_{\mu\rho}\tilde{U}_{\nu,a} \\
[J_{ab}, U_{\mu,c}] &= \epsilon_{cb}U_{\mu,a} + \epsilon_{ca}U_{\mu,b} \qquad \left[J_{ab}, \tilde{U}_{\mu,c}\right] = \epsilon_{cb}\tilde{U}_{\mu,a} + \epsilon_{ca}\tilde{U}_{\mu,b}
\end{aligned} \tag{3.68b}$$

$$\left[U_{\mu,a}, \tilde{U}_{\nu,b}\right] = \eta_{\mu\nu}\epsilon_{ab}\,\Delta - 2\,\epsilon_{ab}M_{\mu\nu} - \eta_{\mu\nu}J_{ab} \tag{3.68c}$$

The distance invariant under $\mathrm{SO}(d+2,4)$ can be constructed following [45]. Let us first introduce a difference between two vectors $\mathcal{X}$ and $\mathcal{Y}$ on $\mathfrak{g}^{-1} \oplus \mathfrak{g}^{-2}$:

$$\delta(\mathcal{X},\mathcal{Y}) = \left(X^{\mu,a} - Y^{\mu,a}, x - y - \eta_{\mu\nu}\epsilon_{ab}X^{\mu,a}Y^{\nu,b}\right) \tag{3.69}$$

and define the "length" of a vector $\mathcal{X}$ as

$$\ell(\mathcal{X}) = \mathcal{I}_4(X) + 2x^2 \tag{3.70}$$

Then the cone defined by $\ell(\delta(\mathcal{X},\mathcal{Y})) = 0$ is invariant w.r.t. the full group $SO(d+2,4)$, because of the following identities:

$$\begin{aligned}
\Delta \circ \ell(\delta(\mathcal{X},\mathcal{Y})) &= 4\,\ell(\delta(\mathcal{X},\mathcal{Y})) \\
\tilde{U}_{\mu,a} \circ \ell(\delta(\mathcal{X},\mathcal{Y})) &= -2\eta_{\mu\nu}\epsilon_{ab}\left(X^{\nu,b} + Y^{\nu,b}\right)\ell(\delta(\mathcal{X},\mathcal{Y})) \\
K_+ \circ \ell(\delta(\mathcal{X},\mathcal{Y})) &= 2(x+y)\,\ell(\delta(\mathcal{X},\mathcal{Y})) \\
\text{any other generator} \circ \ell(\delta(\mathcal{X},\mathcal{Y})) &= 0
\end{aligned} \tag{3.71}$$

This result provides an example of the cone referred to in (3.45).

3.8 Geometric realizations of $\mathrm{E}_{8(-24)}, \mathrm{E}_{7(-5)}, \mathrm{E}_{6(2)}$ and $\mathrm{F}_{4(4)}$ as quasiconformal groups

The minimal unitary representations of the quasiconformal groups of the space-times defined by simple formally real Jordan algebras of degree three were given in [48]. In this section we will give their geometric realizations as quasiconformal groups in an $SO(d,2) \times Sp(2,\mathbb{R})$ covariant basis where d is equal to one of the critical dimensions $3, 4, 6, 10$.

3.8.1 Geometric realization of the quasiconformal group $\mathrm{E}_{8(-24)}$

For realizing the geometric action of the quasiconformal group $\mathrm{E}_{8(-24)}$ in an $\mathrm{SO}(10,2) \times \mathrm{SO}(2,1)$ covariant basis we shall use the following 5-graded decomposition of its Lie algebra

$$\mathfrak{e}_{8(-24)} = \tilde{\mathbf{1}} \oplus \tilde{\mathbf{56}} \oplus \left[\mathfrak{so}(1,1) \oplus \mathfrak{e}_{7(-25)}\right] \oplus \mathbf{56} \oplus \mathbf{1} \tag{3.72}$$

$$\mathfrak{e}_{8(-24)} = \mathbf{1} \oplus \begin{pmatrix} (\mathbf{2},\mathbf{12}) \\ (\mathbf{1},\mathbf{32}_c) \end{pmatrix} \oplus \left[\Delta \oplus \begin{pmatrix} (\mathbf{2},\mathbf{32}_s) \\ \mathfrak{sp}(2,\mathbb{R}) \oplus \mathfrak{so}(10,2) \end{pmatrix}\right] \oplus \begin{pmatrix} (\mathbf{2},\mathbf{12}) \\ (\mathbf{1},\mathbf{32}_c) \end{pmatrix} \oplus \mathbf{1}.$$

The generators of the simple subalgebra $\mathfrak{e}_{7(-25)}$ in $\mathfrak{g}^0$ satisfy the following SO(10, 2) covariant commutation relations.

$$\begin{aligned}
[M_{\mu\nu}, Q_{a\dot{\alpha}}] &= Q_{a\dot{\beta}} \left(\Gamma_{\mu\nu}\right)^{\dot{\beta}}{}_{\dot{\alpha}} \\
[J_{ab}, Q_{c\dot{\alpha}}] &= \epsilon_{cb} Q_{a\dot{\alpha}} + \epsilon_{ca} Q_{b\dot{\alpha}} \\
\left[Q_{a\dot{\alpha}}, Q_{b\dot{\beta}}\right] &= \epsilon_{ab} \left(C\Gamma_{\mu\nu}\right)_{\dot{\alpha}\dot{\beta}} M^{\mu\nu} + C_{\dot{\alpha}\dot{\beta}} J_{ab}
\end{aligned} \tag{3.73}$$

where $M_{\mu\nu}$ and J_{ab} are the generators of SO(10, 2) and Sp$(2, \mathbb{R})$, respectively and $Q_{a\dot{\alpha}}$ are the remaining generators transforming in the $(\mathbf{32}_s, \mathbf{2})$) of SO$(10, 2) \times Sp(2, \mathbb{R})$. C is the charge conjugation matrix in $(10, 2)$ dimensions and is antisymmetric

$$C^t = -C \tag{3.74}$$

The generators of $\mathrm{E}_{7(-25)}$ are realized in terms of the "coordinates" $X^{\mu,a}$ and ψ^α transforming in the $(\mathbf{12}, \mathbf{2})$ and $(\mathbf{32}_c, \mathbf{1})$ representation of SO$(10, 2) \times Sp(2, \mathbb{R})$ as follows

$$\begin{aligned}
M_{\mu\nu} &= \eta_{\mu\rho} X^{\rho,a} \frac{\partial}{\partial X^{\nu,a}} - \eta_{\nu\rho} X^{\rho,a} \frac{\partial}{\partial X^{\mu,a}} - \psi^\alpha (\Gamma_{\mu\nu})^\beta{}_\alpha \frac{\partial}{\partial \psi^\beta} \\
J_{ab} &= \epsilon_{ac} X^{\mu,c} \frac{\partial}{\partial X^{\mu,b}} + \epsilon_{bc} X^{\mu,c} \frac{\partial}{\partial X^{\mu,a}} \\
Q_{a\dot{\alpha}} &= \epsilon_{ab} X^{\mu,b} (\Gamma_\mu)^\beta{}_{\dot{\alpha}} \frac{\partial}{\partial \psi^\beta} - \psi^\beta \left(C\Gamma_\mu\right)_{\beta\dot{\alpha}} \eta^{\mu\nu} \frac{\partial}{\partial X^{\nu,a}}
\end{aligned} \tag{3.75}$$

where Γ_μ are the gamma matrices and $\Gamma_{\mu\nu} = \frac{1}{4}\left(\Gamma_\mu \Gamma_\nu - \Gamma_\nu \Gamma_\mu\right)$. $\alpha, \beta, ..$ and $\dot{\alpha}, \dot{\beta}, ...$ are chiral and anti-chiral spinor indices that run from 1 to 32, respectively. Γ matrices are taken to be in a chiral basis (with Γ_{13} being diagonal) . The spinorial "coordinates" ψ^α transform as a Majorana-Weyl spinor of SO(10, 2). One convenient choice for gamma matrices is

$$\begin{aligned}
&\Gamma_i = \sigma_1 \otimes \sigma_1 \otimes \Gamma_i^{(8)} \quad \Gamma_9 = \sigma_1 \otimes \sigma_1 \otimes \Gamma_9^{(8)} \quad \Gamma_{10} = \sigma_1 \otimes \sigma_3 \otimes 1_{16} \\
&\Gamma_{11} = \sigma_1 \otimes i\sigma_2 \otimes 1_{16} \quad \Gamma_{12} = i\sigma_2 \otimes 1_{32} \quad C = 1_2 \otimes i\sigma_2 \otimes 1_{16}
\end{aligned} \tag{3.76}$$

where $i = 1, \ldots, 8$ and $\Gamma_9^{(8)} = \Gamma_1^{(8)} \ldots \Gamma_8^{(8)}$. Matrices $\Gamma_i^{(8)}$ are those of Clifford algebra of $\mathbb{R}^8$. The chiral realization (3.76) assumes mostly plus signature convention:

$$\eta_{\mu\nu} = \mathrm{diag}\left((+)^{10}, (-)^2\right) \qquad \mu, \nu = 1, \ldots, 12. \tag{3.77}$$

The fourth order invariant of $\mathfrak{e}_{7(-25)}$ in the above basis reads as

$$\begin{aligned}
\mathcal{I}_4 = &\eta_{\mu\nu}\eta_{\rho\tau}\epsilon_{ac}\epsilon_{bd} X^{\mu,a} X^{\nu,b} X^{\rho,c} X^{\tau,d} + 2\epsilon_{ab} X^{\mu,a} X^{\nu,b} \psi^\alpha \left(C\Gamma_{\mu\nu}\right)_{\alpha\beta} \psi^\beta \\
&+ \frac{1}{6} \psi^\alpha \left(C\Gamma_{\mu\nu}\right)_{\alpha\beta} \psi^\beta \psi^\gamma \left(C\Gamma^{\mu\nu}\right)_{\gamma\delta} \psi^\delta
\end{aligned} \tag{3.78}$$

Given the above data, it is straightforward to realize the generators of $\mathfrak{e}_{8(-24)}$ on a $56{+}1 = 57$ dimensional space following [45, 46, 48]. We start with negative grade generators

$$\begin{aligned} K_- &= \frac{\partial}{\partial x} \qquad U_\alpha = \frac{\partial}{\partial \psi^\alpha} - C_{\alpha\beta}\psi^\beta \frac{\partial}{\partial x} \\ U_{\mu,a} &= \frac{\partial}{\partial X^{\mu,a}} - \eta_{\mu,\nu}\epsilon_{ab} X^{\nu,b} \frac{\partial}{\partial x} \end{aligned} \tag{3.79}$$

where x is the singlet "cocycle" coordinate. Grade $+2$ generator is

$$\begin{aligned} K_+ = &\frac{1}{2}\left(2x^2 - \mathcal{I}_4\right)\frac{\partial}{\partial x} - \frac{1}{4}\frac{\partial \mathcal{I}_4}{\partial X^{\mu,a}}\eta^{\mu\nu}\epsilon^{ab}\frac{\partial}{\partial X^{\nu,b}} \\ &+ \frac{1}{4}\frac{\partial \mathcal{I}_4}{\partial \psi^\alpha}\left(C^{-1}\right)^{\alpha\beta}\frac{\partial}{\partial \psi^\beta} + x\, X^{\mu,a}\frac{\partial}{\partial X^{\mu,a}} + x\,\psi^\alpha\frac{\partial}{\partial \psi^\alpha} \end{aligned} \tag{3.80}$$

Generators of grade $+1$ space are obtained by commuting K_+ with corresponding generators of $\mathfrak{g}^{-1}$:

$$\tilde{U}_{\mu,a} = [U_{\mu,a}, K_+] \qquad \tilde{U}_\alpha = [U_\alpha, K_+]\,. \tag{3.81}$$

The generator that determines the five grading is simply

$$\Delta = 2x\frac{\partial}{\partial x} + X^{\mu,a}\frac{\partial}{\partial X^{\mu,a}} + \psi^\alpha\frac{\partial}{\partial \psi^\alpha} \tag{3.82}$$

The commutation relations of these generators are those of (3.68) for $d = 10$ supplemented with (3.73) and the following:

$$\begin{aligned} &[U_\alpha, U_\beta] = 2\,C_{\alpha\beta}K_- && [U_\alpha, K_+] = \tilde{U}_\alpha \\ &\left[\tilde{U}_\alpha, \tilde{U}_\beta\right] = 2\,C_{\alpha\beta}K_+ && \left[\tilde{U}_\alpha, K_-\right] = -U_\alpha \\ &[Q_{a\dot\alpha}, U_{\mu,b}] = -\epsilon_{ab}(\Gamma_\mu)^\alpha{}_{\dot\alpha}U_\alpha && [Q_{a\dot\alpha}, U_\beta] = (C\Gamma_\mu)_{\beta\dot\alpha}\,\eta^{\mu\nu}U_{\nu,a} \\ &\left[Q_{a\dot\alpha}, \tilde{U}_{\mu,b}\right] = -\epsilon_{ab}(\Gamma_\mu)^\alpha{}_{\dot\alpha}\tilde{U}_\alpha && \left[Q_{a\dot\alpha}, \tilde{U}_\beta\right] = (C\Gamma_\mu)_{\beta\dot\alpha}\,\eta^{\mu\nu}\tilde{U}_{\nu,a} \\ &\left[\tilde{U}_\alpha, U_{\mu,a}\right] = -\left(C\Gamma_\mu C^{-1}\right)_\alpha{}^{\dot\alpha}Q_{a\dot\alpha} && \\ &\left[U_\alpha, \tilde{U}_{\mu,a}\right] = \left(C\Gamma_\mu C^{-1}\right)_\alpha{}^{\dot\alpha}Q_{a\dot\alpha} && \left[U_\alpha, \tilde{U}_\beta\right] = C_{\alpha\beta}\Delta - (C\Gamma_{\mu\nu})_{\alpha\beta}\,M^{\mu\nu} \end{aligned} \tag{3.83}$$

with all the remaining commutators vanishing. The explicit expressions for the grade $+1$ generators are

$$\begin{aligned} \tilde{U}_{\mu,a} = &-\frac{1}{4}\frac{\partial \mathcal{I}_4}{\partial X^{\mu,a}}\frac{\partial}{\partial x} - x\,\eta_{\mu\nu}\epsilon_{ab}X^{\nu,b}\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial X^{\mu,a}} \\ &- \frac{1}{4}\frac{\partial^2 \mathcal{I}_4}{\partial X^{\mu,a}\partial X^{\nu,b}}\eta^{\nu\lambda}\epsilon^{bc}\frac{\partial}{\partial X^{\lambda,c}} - \frac{1}{4}\frac{\partial^2 \mathcal{I}_4}{\partial X^{\mu,a}\partial \psi^\alpha}\left(C^{-1}\right)^{\alpha\beta}\frac{\partial}{\partial \psi^\beta} \\ &- \eta_{\mu\nu}\epsilon_{ab}X^{\nu,b}\left(X^{\lambda,c}\frac{\partial}{\partial X^{\lambda,c}} + \psi^\gamma\frac{\partial}{\partial \psi^\gamma}\right) \end{aligned} \tag{3.84}$$

$$\begin{aligned}\tilde{U}_\alpha = &-\frac{1}{4}\frac{\partial \mathcal{I}_4}{\partial \psi^\alpha}\frac{\partial}{\partial x} - x\left(C_{\alpha\beta}\psi^\beta\right)\frac{\partial}{\partial x} - C_{\alpha\beta}\psi^\beta\left(X^{\mu,a}\frac{\partial}{\partial X^{\mu,a}} + \psi^\gamma\frac{\partial}{\partial \psi^\gamma}\right)\\ &-\frac{1}{4}\frac{\partial^2 \mathcal{I}_4}{\partial X^{\mu,a}\partial\psi^\alpha}\eta^{\mu\nu}\epsilon^{ab}\frac{\partial}{\partial X^{\nu,b}} - \frac{1}{4}\frac{\partial^2 \mathcal{I}_4}{\partial\psi^\alpha\partial\psi^\beta}\left(C^{-1}\right)^{\beta\gamma}\frac{\partial}{\partial\psi^\gamma} + x\frac{\partial}{\partial\psi^\alpha}\end{aligned} \tag{3.85}$$

The above geometric realization of the quasiconformal action of the Lie algebra of $\mathrm{E}_{8(-24)}$ can be consistently truncated to the quasiconformal realizations of $\mathrm{E}_{7(-5)}$, $\mathrm{E}_{6(2)}$ and $\mathrm{F}_{4(4)}$, which we discuss in the following subsections. We should stress that for all these groups one can define a quartic norm such that they leave the generalised light-cone defined with respect to this quartic norm invariant as was shown for the maximally split exceptional groups in [45] and for $SO(d+2,4)$ above.

3.8.2 Geometric realization of the quasiconformal group $\mathrm{E}_{7(-5)}$

Truncation of the geometric realization of the quasiconformal group $\mathrm{E}_{8(-24)}$ to $\mathrm{E}_{7(-5)}$ is achieved by "dimensional reduction" from 10 to 6 dimensions. Reduction of 32-component Majorana-Weyl spinor of $\mathfrak{so}(10,2)$ is done by using the projection operators:

$$\mathcal{P}^\alpha{}_\beta = \frac{1}{2}(1+\Gamma_1\Gamma_2\Gamma_3\Gamma_4)^\alpha{}_\beta \qquad \mathcal{P}^{\dot\alpha}{}_{\dot\beta} = \frac{1}{2}(1+\Gamma_1\Gamma_2\Gamma_3\Gamma_4)^{\dot\alpha}{}_{\dot\beta} \tag{3.86}$$

where we assumed we compactify first 4 compact directions. This projection will reduce number of spinor components down to 16. It is clear that projected spinors will have the same chirality as their ancestors:

$$\mathcal{P}\Gamma_5\ldots\Gamma_{12}\mathcal{P} = \mathcal{P}\Gamma_{13}\mathcal{P} \tag{3.87}$$

This 16-component spinor would thus comprise 2 same chirality 8-components spinors of $\mathfrak{so}(6,2)$ satisfying symplectic Majorana-Weyl reality condition. Their R-group is $\mathfrak{su}(2)$ - part of the $\mathfrak{so}(4)$ of the transverse directions that leaves the projection operator invariant. Thus the relevant 5-graded decomposition of $\mathfrak{e}_{7(-5)}$

$$\mathfrak{e}_{7(-5)} = \tilde{\mathbf{1}} \oplus \tilde{\mathbf{32}} \oplus [\mathfrak{so}^*(12)\oplus\mathfrak{so}(1,1)] \oplus \mathbf{32} \oplus \mathbf{1}$$

reads

$$\begin{aligned}\mathfrak{e}_{7(-5)} = \mathbf{1} \oplus &\begin{pmatrix}(\mathbf{2},\mathbf{1},\mathbf{8}_v)\\(\mathbf{1},\mathbf{2},\mathbf{8}_c)\end{pmatrix} \oplus\\ &\left[\Delta \oplus \begin{bmatrix}(\mathbf{2},\mathbf{2},\mathbf{8}_s)\\ \mathfrak{sp}(2,\mathbb{R})\oplus\mathfrak{su}(2)\oplus\mathfrak{so}(6,2)\end{bmatrix}\right] \oplus \begin{pmatrix}(\mathbf{2},\mathbf{1},\mathbf{8}_v)\\(\mathbf{1},\mathbf{2},\mathbf{8}_c)\end{pmatrix} \oplus \mathbf{1}\end{aligned} \tag{3.88}$$

Let $\xi^{i,\alpha}$ be an $\mathfrak{su}(2)$ doublet of $\mathfrak{so}(6,2)$ chiral spinors (symplectic Majorana-Weyl spinor) with $a,b,..=1,2$ and $\alpha,\beta,...=1,2,..,8$. Then one can realize the Lie algebra of $\mathfrak{so}^*(12)$ of grade zero subspace as

$$\begin{aligned} M_{\mu\nu} &= \eta_{\mu\rho}X^{\rho,a}\frac{\partial}{\partial X^{\nu,a}} - \eta_{\nu\rho}X^{\rho,a}\frac{\partial}{\partial X^{\mu,a}} - \xi^{i,\alpha}(\Gamma_{\mu\nu})^{\beta}{}_{\alpha}\frac{\partial}{\partial \xi^{i,\beta}} \\ J_{ab} &= \epsilon_{ac}X^{\mu,c}\frac{\partial}{\partial X^{\mu,b}} + \epsilon_{bc}X^{\mu,c}\frac{\partial}{\partial X^{\mu,a}} \\ L_{ij} &= \epsilon_{ik}\xi^{k,\alpha}\frac{\partial}{\partial \xi^{j,\alpha}} + \epsilon_{jk}\xi^{k,\alpha}\frac{\partial}{\partial \xi^{i,\alpha}} \\ Q_{ia\dot{\alpha}} &= \epsilon_{ab}X^{\mu,b}(\Gamma_{\mu})^{\beta}{}_{\dot{\alpha}}\frac{\partial}{\partial \xi^{i,\beta}} - \epsilon_{ij}\psi^{j,\beta}\left(C\Gamma_{\mu}\right)_{\beta\dot{\alpha}}\eta^{\mu\nu}\frac{\partial}{\partial X^{\nu,a}} \end{aligned} \tag{3.89}$$

where $C^t = C$ and $\mu,\nu,..=1,2,..,8$. Generators M,J,L,Q form $\mathfrak{so}^*(12)$ algebra:

$$\begin{aligned} [M_{\mu\nu},Q_{ia\dot{\alpha}}] &= Q_{ia\dot{\beta}}\left(\Gamma_{\mu\nu}\right)^{\dot{\beta}}{}_{\dot{\alpha}} \\ [L_{ij},L_{km}] &= \epsilon_{kj}L_{im} + \epsilon_{ki}L_{jm} + \epsilon_{mj}L_{ik} + \epsilon_{mi}L_{jk} \\ [J_{ab},Q_{ic\dot{\alpha}}] &= \epsilon_{cb}Q_{ia\dot{\alpha}} + \epsilon_{ca}Q_{ib\dot{\alpha}} \\ \left[Q_{ia\dot{\alpha}},Q_{jb\dot{\beta}}\right] &= \epsilon_{ij}\epsilon_{ab}\left(C\Gamma_{\mu\nu}\right)_{\dot{\alpha}\dot{\beta}}M^{\mu\nu} + \epsilon_{ij}C_{\dot{\alpha}\dot{\beta}}J_{ab} + \epsilon_{ab}C_{\dot{\alpha}\dot{\beta}}L_{ij} \end{aligned} \tag{3.90}$$

corresponding to the decomposition

$$\mathfrak{so}^*(12) \supset \mathfrak{so}^*(8) \oplus \mathfrak{so}^*(4) \equiv \mathfrak{so}(6,2) \oplus \mathfrak{su}(2) \oplus \mathfrak{sp}(2,\mathbb{R}) \tag{3.91}$$

The fourth order invariant of $\mathfrak{so}^*(12)$ in the above basis is given by

$$\begin{aligned} \mathcal{I}_4 &= \eta_{\mu\nu}\eta_{\rho\tau}\epsilon_{ac}\epsilon_{bd}X^{\mu,a}X^{\nu,b}X^{\rho,c}X^{\tau,d} - 2\epsilon_{ij}\epsilon_{ab}X^{\mu,a}X^{\nu,b}\xi^{i,\alpha}\left(C\Gamma_{\mu\nu}\right)_{\alpha\beta}\xi^{j,\beta} \\ &+ \frac{1}{4}\xi^{i,\alpha}\left(C\Gamma_{\mu\nu}\right)_{\alpha\beta}\xi^{j,\beta}\xi^{k,\gamma}\left(C\Gamma^{\mu\nu}\right)_{\gamma\delta}\xi^{l,\delta}\epsilon_{ij}\epsilon_{kl} \end{aligned} \tag{3.92}$$

We can now write generators of $\mathfrak{e}_{7(-5)}$, starting with negative grade generators

$$\begin{aligned} K_- &= \frac{\partial}{\partial x} \qquad U_{i,\alpha} = \frac{\partial}{\partial \xi^{i,\alpha}} + \epsilon_{ij}C_{\alpha\beta}\xi^{j,\beta}\frac{\partial}{\partial x} \\ U_{\mu,a} &= \frac{\partial}{\partial X^{\mu,a}} - \eta_{\mu,\nu}\epsilon_{ab}X^{\nu,b}\frac{\partial}{\partial x} \end{aligned} \tag{3.93}$$

Positive grade $+2$ generator K_+ is

$$\begin{aligned} K_+ &= \frac{1}{2}\left(2x^2 - \mathcal{I}_4\right)\frac{\partial}{\partial x} - \frac{1}{4}\frac{\partial \mathcal{I}_4}{\partial X^{\mu,a}}\eta^{\mu\nu}\epsilon^{ab}\frac{\partial}{\partial X^{\nu,b}} \\ &+ \frac{1}{4}\epsilon^{ij}\frac{\partial \mathcal{I}_4}{\partial \xi^{i,\alpha}}\left(C^{-1}\right)^{\alpha\beta}\frac{\partial}{\partial \xi^{j,\beta}} + x\,X^{\mu,a}\frac{\partial}{\partial X^{\mu,a}} + x\,\xi^{i,\alpha}\frac{\partial}{\partial \xi^{i,\alpha}} \end{aligned} \tag{3.94}$$

Commutation relations of $\mathfrak{g}^{-1}$ and $\mathfrak{g}^{+1}$ specific to $6+2=8$ dimensions are :

$$\begin{aligned}\left[U_{i,\alpha},\tilde{U}_{j,\beta}\right] &= \epsilon_{ij}\left(C\Gamma_{\mu\nu}\right)_{\alpha\beta}M^{\mu\nu}+C_{\alpha\beta}L_{ij}-\epsilon_{ij}C_{\alpha\beta}\Delta\\ \left[U_{i,\alpha},\tilde{U}_{\mu,a}\right] &= -\left(C\Gamma_{\mu}C^{-1}\right)_{\alpha}^{\ \dot{\alpha}}Q_{ia\dot{\alpha}}.\end{aligned} \tag{3.95}$$

Grade +1 generators have the following form:

$$\begin{aligned}\tilde{U}_{\mu,a} = &-\frac{1}{4}\frac{\partial\mathcal{I}_4}{\partial X^{\mu,a}}\frac{\partial}{\partial y}-\eta_{\mu\nu}\epsilon_{ab}X^{\nu,b}\,y\frac{\partial}{\partial y}+y\frac{\partial}{\partial X^{\mu,a}}\\ &-\frac{1}{4}\frac{\partial^2\mathcal{I}_4}{\partial X^{\mu,a}\partial X^{\nu,b}}\eta^{\nu\rho}\epsilon^{bc}\frac{\partial}{\partial X^{\rho,c}}+\frac{1}{4}\frac{\partial^2\mathcal{I}_4}{\partial X^{\mu,a}\xi^{i,\alpha}}\epsilon^{ij}\left(C^{-1}\right)^{\alpha\beta}\frac{\partial}{\partial\xi^{j,\beta}}\\ &-\eta_{\mu\nu}\epsilon_{ab}X^{\nu,b}\left(X^{\lambda,c}\frac{\partial}{\partial X^{\lambda,c}}+\xi^{i,\alpha}\frac{\partial}{\partial\xi^{i,\alpha}}\right)\end{aligned} \tag{3.96}$$

$$\begin{aligned}\tilde{U}_{i,\alpha} = &-\frac{1}{4}\frac{\partial\mathcal{I}_4}{\partial\xi^{i,\alpha}}\frac{\partial}{\partial y}+C_{\alpha\beta}\epsilon_{ij}\xi^{j,\beta}\,y\frac{\partial}{\partial y}+y\frac{\partial}{\partial\xi^{i,\alpha}}\\ &+\frac{1}{4}\frac{\partial^2\mathcal{I}_4}{\partial\xi^{i,\alpha}\partial\xi^{j,\beta}}\left(C^{-1}\right)^{\beta\gamma}\epsilon^{jk}\frac{\partial}{\partial\xi^{k,\gamma}}-\frac{1}{4}\frac{\partial^2\mathcal{I}_4}{\partial\xi^{i,\alpha}X^{\mu,a}}\eta^{\mu\nu}\epsilon^{ab}\frac{\partial}{\partial X^{\nu,b}}\\ &+C_{\alpha\beta}\epsilon_{ij}\xi^{j,\beta}\left(X^{\lambda,c}\frac{\partial}{\partial X^{\lambda,c}}+\xi^{i,\alpha}\frac{\partial}{\partial\xi^{i,\alpha}}\right)\end{aligned} \tag{3.97}$$

3.8.3 Geometric realization of the quasiconformal group $\mathrm{E}_{6(2)}$

Truncation to $\mathfrak{e}_{6(2)}$ is done by further dimensional reduction from $d=6$ to $d=4$. Projecting spinors is done in a similar way and results in breaking R-symmetry algebra to $\mathfrak{u}(1)$. The resulting 5-graded decomposition of $\mathfrak{e}_{6(2)}$ is:

$$\mathfrak{e}_{6(2)} = \tilde{\mathbf{1}}\oplus\tilde{\mathbf{20}}\oplus\left[\mathfrak{su}(3,3)\oplus\mathfrak{so}(1,1)\right]\oplus\mathbf{20}\oplus\mathbf{1} \tag{3.98}$$

$$\begin{aligned}\mathfrak{e}_{6(2)} = \mathbf{1}\oplus&\begin{pmatrix}(\mathbf{2},\mathbf{6}_v)\\(\mathbf{1},\mathbf{4}_c)^{+}\\(\mathbf{1},\mathbf{4}_s)^{-}\end{pmatrix}\oplus\\ &\left[\Delta\oplus\begin{bmatrix}(\mathbf{2},\mathbf{4}_s)^{+}\oplus(\mathbf{2},\mathbf{4}_c)^{-}\\ \mathfrak{sp}(2,\mathbb{R})\oplus\mathfrak{u}(1)\oplus\mathfrak{so}(4,2)\end{bmatrix}\right]\oplus\begin{pmatrix}(\mathbf{2},\mathbf{6}_v)\\(\mathbf{1},\mathbf{4}_c)^{+}\\(\mathbf{1},\mathbf{4}_s)^{-}\end{pmatrix}\oplus\mathbf{1}\end{aligned} \tag{3.99}$$

where + and − refer to ±1 charges of $\mathfrak{u}(1)$. Let ζ^{α} be a chiral spinor of $\mathfrak{so}(4,2)$ and $\zeta^{\dot{\alpha}}$ the corresponding anti-chiral spinor. We shall combine these two chiral spinors into one Majorana spinor ψ^{A} of $\mathfrak{so}(4,2)$. The decomposition of the The generators of $\mathfrak{su}(3,3)$ in $\mathfrak{g}^{0}$

subspace read as follows

$$
\begin{aligned}
M_{\mu\nu} &= \eta_{\mu\rho} X^{\rho,a} \frac{\partial}{\partial X^{\nu,a}} - \eta_{\nu\rho} X^{\rho,a} \frac{\partial}{\partial X^{\mu,a}} - \psi^A (\Gamma_{\mu\nu})^A{}_B \frac{\partial}{\partial \psi^B} \\
H &= \zeta^\alpha \frac{\partial}{\partial \zeta^\alpha} - \zeta^{\dot\alpha} \frac{\partial}{\partial \zeta^{\dot\alpha}} = \psi^A (\Gamma_7)^B{}_A \frac{\partial}{\partial \psi^B} \\
Q_{a,A} &= \epsilon_{ab} X^{\mu,a} (\Gamma_\mu)^B{}_A \frac{\partial}{\partial \psi^B} + \eta^{\mu\nu} \psi^B \left(C\Gamma_\mu \Gamma_7\right)_{BA} \frac{\partial}{\partial X^{\nu,a}}
\end{aligned}
$$

while J_{ab} is defined as before and the charge conjugation matrix is now symmetric $C^t = C$. These generators of $\mathfrak{su}(3,3)$ satisfy the commutation relations

$$
\begin{aligned}
[Q_{a,A}, Q_{b,B}] &= \frac{3}{2} \epsilon_{ab} C_{AB} H - \epsilon_{ab} \left(C\Gamma_{\mu\nu}\right)_{AB} M^{\mu\nu} - \left(C\Gamma_7\right)_{AB} J_{ab} \\
[H, Q_{a,A}] &= (\Gamma_7)^B{}_A Q_{a,B}
\end{aligned}
\tag{3.100}
$$

The chiral components of the generators of $Q_{a,A}$ are given by

$$
\begin{aligned}
Q_{a,\alpha} &= \epsilon_{ab} X^{\mu,b} (\Gamma_\mu)^{\dot\alpha}{}_\alpha \frac{\partial}{\partial \zeta^{\dot\alpha}} + \eta^{\mu\nu} \zeta^\beta \left(C\Gamma_\mu\right)_{\beta\alpha} \frac{\partial}{\partial X^{\nu,a}} \\
Q_{a,\dot\alpha} &= \epsilon_{ab} X^{\mu,b} (\Gamma_\mu)^\alpha{}_{\dot\alpha} \frac{\partial}{\partial \zeta^\alpha} - \eta^{\mu\nu} \zeta^{\dot\beta} \left(C\Gamma_\mu\right)_{\dot\beta\dot\alpha} \frac{\partial}{\partial X^{\nu,a}}
\end{aligned}
$$

The 4-th order invariant of $\mathfrak{su}(3,3)$ written in terms of X and ψ reads as follows:

$$
\begin{aligned}
\mathcal{I}_4 = {} & \eta_{\mu\nu}\eta_{\rho\tau}\epsilon_{ac}\epsilon_{bd} X^{\mu,a} X^{\nu,b} X^{\rho,c} X^{\tau,d} - 2 X^{\mu,a} \epsilon_{ab} X^{\nu,b} \psi^A \psi^B \left(C\Gamma_{\mu\nu}\right)_{AB} \\
& + \frac{1}{3} \eta^{\mu\nu}\eta^{\rho\tau} \psi^A \psi^B \left(C\Gamma_{\mu\rho}\right)_{AB} \psi^E \psi^F \left(C\Gamma_{\nu\tau}\right)_{EF}
\end{aligned}
\tag{3.101}
$$

The spinorial generators of $\mathrm{E}_{6(2)}$ belonging to $\mathfrak{g}^{-1}$ subspace are realized as

$$
U_A = \frac{\partial}{\partial \psi^A} + \left(C\Gamma_7\right)_{AB} \psi^B \frac{\partial}{\partial y} \tag{3.102}
$$

which commute into the grade -2 generator K_-

$$
[U_A, U_B] = 2 \left(C\Gamma_7\right)_{AB} K_- \tag{3.103}
$$

The commutators of the generators $Q_{a,A}$ belonging to grade zero subspace with those in grade -1 subspace read as

$$
\begin{aligned}
[Q_{a,A}, U_B] &= - \left(C\Gamma_\mu\Gamma_7\right)_{AB} \eta^{\mu\nu} U_{\nu,a} \\
[Q_{a,A}, U_{\mu,b}] &= -\epsilon_{ab} U_B (\Gamma_\mu)^B{}_A
\end{aligned}
\tag{3.104}
$$

Commutation relations of the form $\left[\mathfrak{g}^{-1},\mathfrak{g}^{+1}\right]\subset\mathfrak{g}^{0}$ are

$$\begin{aligned}\left[U_A,\tilde{U}_B\right]&=-\frac{3}{2}\left(C\Gamma_7\right)_{AB}H-C_{AB}\Delta+\left(C\Gamma_{\mu\nu}\right)_{AB}M^{\mu\nu}\\ \left[U_A,\tilde{U}_{\mu,a}\right]&=\left(C\Gamma_\mu C^{-1}\right)_A{}^B Q_{a,B}\\ \left[\tilde{U}_A,U_{\mu,a}\right]&=-\left(C\Gamma_\mu C^{-1}\right)_A{}^B Q_{a,B}\end{aligned}\tag{3.105}$$

Explicit expressions for positive grade generators are as follows:

$$\begin{aligned}\tilde{U}_{\mu,a}=&-\frac{1}{4}\frac{\partial\mathcal{I}_4}{\partial X^{\mu,a}}\frac{\partial}{\partial y}-\eta_{\mu\nu}\epsilon_{ab}X^{\nu,b}y\frac{\partial}{\partial y}+y\frac{\partial}{\partial X^{\mu,a}}\\ &-\frac{1}{4}\frac{\partial^2\mathcal{I}_4}{\partial X^{\mu,a}X^{\nu,b}}\eta^{\nu\rho}\epsilon^{bc}\frac{\partial}{\partial X^{\rho,c}}-\frac{1}{4}\frac{\partial^2\mathcal{I}_4}{\partial X^{\mu,a}\psi^A}\left(\Gamma_7C^{-1}\right)^{AB}\frac{\partial}{\partial\psi^B}\\ &-\eta_{\mu\nu}\epsilon_{ab}X^{\nu,b}\left(X^{\lambda,c}\frac{\partial}{\partial X^{\lambda,c}}+\psi^A\frac{\partial}{\partial\psi^A}\right)\end{aligned}\tag{3.106}$$

$$\begin{aligned}\tilde{U}_A=&-\frac{1}{4}\frac{\partial\mathcal{I}_4}{\partial\psi^A}\frac{\partial}{\partial y}-\left(C\Gamma_7\right)_{AB}\psi^B y\frac{\partial}{\partial y}+y\frac{\partial}{\partial\psi^A}\\ &-\frac{1}{4}\frac{\partial^2\mathcal{I}_4}{\partial\psi^A\psi^B}\left(\Gamma_7C^{-1}\right)^{BC}\frac{\partial}{\partial\psi^C}-\frac{1}{4}\frac{\partial^2\mathcal{I}_4}{\partial X^{\mu,a}\psi^A}\eta^{\mu\nu}\epsilon^{ab}\frac{\partial}{\partial X^{\nu,b}}\\ &-\left(C\Gamma_7\right)_{AB}\psi^B\left(X^{\lambda,c}\frac{\partial}{\partial X^{\lambda,c}}+\psi^C\frac{\partial}{\partial\psi^C}\right)\end{aligned}\tag{3.107}$$

$$\begin{aligned}K_+=&\frac{1}{2}\left(2y^2-\mathcal{I}_4\right)\frac{\partial}{\partial y}+y\left(X^{\mu,a}\frac{\partial}{\partial X^{\mu,a}}+\psi^A\frac{\partial}{\partial\psi^A}\right)\\ &-\frac{1}{4}\frac{\partial\mathcal{I}_4}{\partial X^{\mu,a}}\epsilon^{ab}\eta^{\mu\nu}\frac{\partial}{\partial X^{\nu,b}}-\frac{1}{4}\frac{\partial\mathcal{I}_4}{\partial\psi^A}\left(\Gamma_7C^{-1}\right)^{AB}\frac{\partial}{\partial\psi^B}\end{aligned}\tag{3.108}$$

3.8.4 Geometric realization of the quasiconformal group $F_{4(4)}$

Further truncation to $\mathfrak{f}_{4(4)}$ is performed by reducing from $d=4$ to $d=3$. The 5-grading in this case is

$$\mathfrak{f}_{4(4)}=\tilde{\mathbf{1}}\oplus\tilde{\mathbf{14}}\oplus\left[\mathfrak{sp}(6,\mathbb{R})\oplus\mathfrak{so}(1,1)\right]\oplus\mathbf{14}\oplus\mathbf{1}\tag{3.109}$$

$$\begin{aligned}\mathfrak{f}_{4(4)}=&\,\mathbf{1}\oplus\begin{pmatrix}(\mathbf{2},\mathbf{5})\\(\mathbf{1},\mathbf{4})\end{pmatrix}\oplus\\ &\left[\Delta\oplus\begin{bmatrix}(\mathbf{2},\mathbf{4})\\ \mathfrak{sp}\,(2,\mathbb{R})\oplus\mathfrak{so}\,(3,2)\end{bmatrix}\right]\oplus\begin{pmatrix}(\mathbf{2},\mathbf{5})\\(\mathbf{1},\mathbf{4})\end{pmatrix}\oplus\mathbf{1}\end{aligned}\tag{3.110}$$

We use the same notations for spinors as above, assuming that now $A = 1, \ldots, 4$.

$$M_{\mu\nu} = \eta_{\mu\rho} X^{\rho,a} \frac{\partial}{\partial X^{\nu,a}} - \eta_{\nu\rho} X^{\rho,a} \frac{\partial}{\partial X^{\mu,a}} - \psi^A (\Gamma_{\mu\nu})^A{}_B \frac{\partial}{\partial \psi^B}$$
$$Q_{a,A} = \epsilon_{ab} X^{\mu,a} (\Gamma_\mu)^B{}_A \frac{\partial}{\partial \psi^B} + \eta^{\mu\nu} \psi^B \left(C\Gamma_\mu\right)_{BA} \frac{\partial}{\partial X^{\nu,a}}$$

where $C^t = -C$. The generators $Q_{a,A}$ close into $\mathfrak{sp}\left(2,\mathbb{R}\right) \oplus \mathfrak{so}\left(3,2\right)$ as follows:

$$[Q_{a,A}, Q_{b,B}] = \epsilon_{ab} \left(C\Gamma_{\mu\nu}\right)_{AB} M^{\mu\nu} + C_{AB} J_{ab} \tag{3.111}$$

and transform under $\mathfrak{sp}\left(2,\mathbb{R}\right) \oplus \mathfrak{so}\left(3,2\right)$ in the $(\mathbf{2},\mathbf{4})$ representation. The generators $Q_{a,A}$, $M^{\mu\nu}$ and J_{ab} form the $\mathfrak{sp}(6,\mathbb{R})$ subalgebra.

Generators $K_- \in \mathfrak{g}^{-2}$ and $U_{\mu,a} \in \mathfrak{g}^{-1}$ are as above and spinorial generators of $\mathfrak{g}^{-1}$ are given by

$$U_A = \frac{\partial}{\partial \psi^A} + \left(C\right)_{AB} \psi^B \frac{\partial}{\partial y} \tag{3.112}$$

Spinorial generators form an Heisenberg subalgebra with charge conjugation matrix C serving as symplectic metric:

$$[U_A, U_B] = -2 C_{AB} K_- \tag{3.113}$$

The generators Q act on $\mathfrak{g}^{-1}$ subspace as follows

$$\begin{aligned} [Q_{a,A}, U_B] &= \left(C\Gamma_\mu\right)_{AB} \eta^{\mu\nu} U_{\nu,a} \\ [Q_{a,A}, U_{\mu,b}] &= -\epsilon_{ab} (\Gamma_\mu)^B{}_A U_B \end{aligned} \tag{3.114}$$

Quartic invariant $\mathcal{I}_4$ of $\mathfrak{sp}(6,\mathbb{R})$ in $\mathfrak{sp}\left(2,\mathbb{R}\right) \oplus \mathfrak{so}\left(3,2\right)$ basis is given by

$$\mathcal{I}_4 = \eta_{\mu\nu}\eta_{\rho\tau}\epsilon_{ac}\epsilon_{bd} X^{\mu,a} X^{\nu,b} X^{\rho,c} X^{\tau,d} - 2 X^{\mu,a} \epsilon_{ab} X^{\nu,b} \psi^A \psi^B \left(C\Gamma_{\mu\nu}\right)_{AB} \tag{3.115}$$

Notice that the quartic term involving purely spinorial coordinates ,present in previous cases, now vanishes, since there is no symmetric rank 4 invariant tensor of $\mathfrak{so}\left(3,2\right) \simeq \mathfrak{sp}\left(4,\mathbb{R}\right)$ over its spinorial representation space. Then the positive grade generators are

$$\begin{aligned} K_+ = &\frac{1}{2}\left(2y^2 - \mathcal{I}_4\right) \frac{\partial}{\partial y} + y \left(X^{\mu,a} \frac{\partial}{\partial X^{\mu,a}} + \psi^A \frac{\partial}{\partial \psi^A} \right) \\ &- \frac{1}{4} \frac{\partial \mathcal{I}_4}{\partial X^{\mu,a}} \epsilon^{ab} \eta^{\mu\nu} \frac{\partial}{\partial X^{\nu,b}} + \frac{1}{4} \frac{\partial \mathcal{I}_4}{\partial \psi^A} \left(C^{-1}\right)^{AB} \frac{\partial}{\partial \psi^B} \end{aligned} \tag{3.116}$$

$$
\begin{aligned}
\tilde{U}_{\mu,a} = & -\frac{1}{4}\frac{\partial \mathcal{I}_4}{\partial X^{\mu,a}}\frac{\partial}{\partial y} - \eta_{\mu\nu}\epsilon_{ab}X^{\nu,b}y\frac{\partial}{\partial y} + y\frac{\partial}{\partial X^{\mu,a}} \\
& - \frac{1}{4}\frac{\partial^2 \mathcal{I}_4}{\partial X^{\mu,a}X^{\nu,b}}\eta^{\nu\rho}\epsilon^{bc}\frac{\partial}{\partial X^{\rho,c}} + \frac{1}{4}\frac{\partial^2 \mathcal{I}_4}{\partial X^{\mu,a}\psi^A}\left(C^{-1}\right)^{AB}\frac{\partial}{\partial \psi^B} \\
& - \eta_{\mu\nu}\epsilon_{ab}X^{\nu,b}\left(X^{\lambda,c}\frac{\partial}{\partial X^{\lambda,c}} + \psi^A\frac{\partial}{\partial \psi^A}\right)
\end{aligned}
\tag{3.117}
$$

$$
\begin{aligned}
\tilde{U}_A = & -\frac{1}{4}\frac{\partial \mathcal{I}_4}{\partial \psi^A}\frac{\partial}{\partial y} + C_{AB}\psi^B y\frac{\partial}{\partial y} + y\frac{\partial}{\partial \psi^A} \\
& + \frac{1}{4}\frac{\partial^2 \mathcal{I}_4}{\partial \psi^A\psi^B}\left(C^{-1}\right)^{BC}\frac{\partial}{\partial \psi^C} - \frac{1}{4}\frac{\partial^2 \mathcal{I}_4}{\partial X^{\mu,a}\psi^A}\eta^{\mu\nu}\epsilon^{ab}\frac{\partial}{\partial X^{\nu,b}} \\
& + (C)_{AB}\,\psi^B\left(X^{\lambda,c}\frac{\partial}{\partial X^{\lambda,c}} + \psi^C\frac{\partial}{\partial \psi^C}\right)
\end{aligned}
\tag{3.118}
$$

Commutation relations of generators belonging to $\mathfrak{g}^{-1}$ and to $\mathfrak{g}^{+1}$ are

$$
\begin{aligned}
\left[U_A, \tilde{U}_B\right] &= (C\Gamma_{\mu\nu})_{AB}\, M^{\mu\nu} - C_{AB}\Delta \\
\left[U_A, \tilde{U}_{\mu,a}\right] &= -\left(C\Gamma_\mu C^{-1}\right)_A{}^B Q_{a,B} \\
\left[\tilde{U}_A, U_{\mu,a}\right] &= \left(C\Gamma_\mu C^{-1}\right)_A{}^B Q_{a,B}
\end{aligned}
\tag{3.119}
$$

3.9 Summary

In conclusion, we find that U-dualities of MESGT supergravity theories in $d = 5$ whose scalar manifolds are symmetry spaces with covariantly constant T-tensor are Lorentz group of generalised space-times, associated with Jordan algebras $J_3^{\mathbb{A}}$ and generic Jordan family. U-dualities of theories obtained by compactification of $d = 5$ theories to $d = 4$ are conformal groups of these space-times. Further compactification to $d = 3$, where all degrees of freedom are dualised to scalars, U-duality group is isomorphic to quasi-conformal group.

U-duality groups of MESGTs, summarised in the table 3.2, form what is known as Magic Square [41] obtained by Freudenthal [29] in his study of relation between division algebras and exceptional groups.

	$d = 5$ compact	$d = 5$	$d = 4$	$d = 3$
$J_3^{\mathbb{R}}$	$\mathrm{SO}(3)$	$\mathrm{SL}(3,\mathbb{R})$	$\mathrm{Sp}(6,\mathbb{R})$	$\mathrm{F}_{4(4)}$
$J_3^{\mathbb{C}}$	$\mathrm{SU}(3)$	$\mathrm{SL}(3,\mathbb{C})$	$\mathrm{SU}(3,3)$	$\mathrm{E}_{6(2)}$
$J_3^{\mathbb{H}}$	$\mathrm{Usp}(6)$	$\mathrm{SU}^*(6)$	$\mathrm{SO}^*(12)$	$\mathrm{E}_{7(-5)}$
$J_3^{\mathbb{O}}$	F_4	$\mathrm{E}_{6(-26)}$	$\mathrm{E}_{7(-25)}$	$\mathrm{E}_{8(-24)}$

Table 3.2: U-duality groups for exceptional MESGTs in $d = 3, 4, 5$.

It is important to stress that using Jordan algebras to define generalised space-times with conformal groups admitting positive energy unitary irreducible representations requires

that the underlying Jordan algebra be Euclidean. These algebras are relevant for $\mathcal{N} = 2$ MESGT geometries. The conformal groups of simple Jordan algebras of degree 3 over the split composition algebras do not admit positive energy unitary irreducible representations. Correspondingly, U-duality groups of maximal supergravities do not admit lowest weight unitary representations. One can however carry out algebraic constructions of Lie algebras based on Jordan triple systems and Freudenthal triple systems associated with these Jordan algebras. Resulting Lie algebras will be U-duality algebras corresponding to maximal supergravity theories in $d = 5$, $d = 4$ and $d = 3$ dimensions [46].

Chapter 4

Minimal representations of U-duality groups

In 1974 Joseph solved [59] the following problem:

> Determine the least number of degrees of freedom for which a quantum mechanical system admits a given semi-simple Lie algebra and construct the corresponding class of realizations.

The realizations he found are termed minimal realizations. Joseph gave minimal realizations for classical groups as well as for the exceptional Lie algebra $\mathfrak{g}_2$, and showed that minimal realizations are determined by a unique completely prime two-sided ideal (known in literature as Josephs ideal) J_0 in enveloping algebra $U(\mathfrak{g})$ of Lie algebra $\mathfrak{g}$ in question. Minimal number of degrees of freedom necessary to realize simple Lie algebras are summarised in table 4.1. It is computed as $\dim\left(\mathfrak{g}^{-1}\right)/2+1$, where $\mathfrak{g}^{-1}$ refers to -1 grade in the five-graded decomposition (3.38). Indeed since $\mathfrak{g}^{-1}$ is endowed with Kirillov-Konstant symplectic structure half of $\mathfrak{g}^{-1}$ can be identified as coordinates and complementary half as momentum. Extra coordinate comes from associating a coordinate to $\mathfrak{g}^{-2}$.

Studying minimal realizations is very relevant in the context of spectrum generating algebras [9]. Indeed, because the latter are required to possess an irreducible representation which exhaust the spectrum of a given system, their construction involves the realization of a Lie algebra within a minimal numbers of degrees of freedom.

Minimal unitary representations were proved to exist by Vogan [111] who identified them within the framework of Langlands classification. Minimal representations for all simply-laced algebras, were constructed by Kazhdan and Savin [65], and by Brylinski and Konstant [10] using rather different methods. Gross and Wallach [35] gave yet another construction of minimal unitary representation of exceptional Lie algebras of real rank 4

Cartan label:	A_n	B_n	C_n	D_n	G_2	F_4	E_6	E_7	E_8
# d.o.f. :	n	$2n-2$	n	$2n-3$	3	8	11	17	29

Table 4.1: Minimal number of degrees of freedom necessary to realize given simple Lie algebra as a symmetry of quantum mechanical system.
Number of degrees of freedom (d.o.f.) is also minimal "dimension" of representation.

which includes $\mathfrak{e}_{8(-24)}$. An explicit realization of Chevalley generators in terms of pseudo-differential operators was recently given for all simply-laced algebras by Kazhdan, Pioline and Waldron [64] who used minimal representation of continuous U-duality groups G to construct automorphic forms of discrete subgroups $G(\mathbb{Z})$ of U-duality groups G.

Construction of minimal realization using Freudenthal triple systems was done in [46] by Günaydin, Koepsell and Nicolai for maximally split U-duality groups and by Günaydin and Pavlyk in [48, 50] for U-duality groups of MESGTs. These constructions arise naturally from geometric construction of quasi-conformal algebras [46] since there is a naturally defined (3.39) symplectic structure on $\mathfrak{g}^{-1}$ and thus one can introduce a notion of coordinates and momenta.

4.1 Conformal quantum mechanics

Consider a quantum mechanical model with one degree freedom, an associated canonical coordinate y and momentum p[1]

$$[y, p] = i \tag{4.1}$$

De Alfaro, Fubini and Furlan [1] considered a motion of such a particle in inverse square potential and explored conformal symmetry $\mathfrak{sl}(2, \mathbb{R})$ that it admits. Let

$$E = \frac{1}{2}y^2 \qquad F = \frac{1}{2}p^2 + \frac{g(g+1)}{2y^2} \qquad \Delta = \frac{1}{2}(yp + py) \tag{4.2}$$

with commutation relations

$$\begin{aligned} [\Delta, E] &= -2iE \\ [\Delta, F] &= +2iF \end{aligned} \qquad\qquad [E, F] = i\Delta \tag{4.3}$$

of $\mathfrak{sp}(2, \mathbb{R})$. Since each generator is Hermitian and the Casimir of the above algebra reads

$$\mathcal{C}_{\mathfrak{sp}(2,\mathbb{R})} = \Delta^2 + (F - E)^2 - (F + E)^2 \tag{4.4}$$

we conclude that $F - E$ and Δ are non-compact generators and $F + E$ is compact. The implication of this is that only $F + E$ has a spectrum bounded from below and hence is suitable for the role of Hamiltonian of quantum mechanical model. Evaluating quadratic Casimir on the realization given by (4.2) we find

$$\mathcal{C}_{\mathfrak{sp}(2,\mathbb{R})} = \frac{3}{4} - g(g+1) \tag{4.5}$$

[1] Throughout the thesis we set $\hbar = 1$ as well as $m = 1$ for convenience.

it to reduce to c-number. In coordinate representation $E+F$ reads as follows:

$$-\frac{1}{2}\frac{\partial^2}{\partial x^2}f\left(x\right)+\frac{1}{2}\left(x^2+\frac{g\left(g+1\right)}{x^2}\right)f\left(x\right)=\mathcal{E}f\left(x\right). \tag{4.6}$$

Seeking for solutions regular at the origin we find

$$f_{\mathcal{E}}\left(x\right)=e^{-\frac{x^2}{2}}x^{g+1}L^{g+\frac{1}{2}}_{\frac{1}{4}(-2g+2\mathcal{E}-3)}\left(x^2\right) \tag{4.7}$$

where function L stands for Laguerre function which can be expressed in terms of confluent hypergeometric function

$$L^{\mu}_{\nu}\left(z\right)=\frac{\Gamma\left(\mu+\nu+1\right)}{\Gamma\left(\mu+1\right)\Gamma\left(\nu+1\right)}\,{}_1F_1\left(\begin{array}{c|c}-\nu & \\ \mu+1 & z\end{array}\right) \tag{4.8}$$

Regularity at infinity requires L to be polynomial of some degree n. This determines the spectrum

$$\mathcal{E}=2n+g+\frac{3}{2} \tag{4.9}$$

The resemblance of the spectrum to that of harmonic oscillator is easily seen by noting that (4.6) is in fact the radial part of Shrödinger equation for three-dimensional oscillator with coupling constant g being angular momentum quantum number.

Because all U-duality algebras feature $\mathfrak{sl}\left(2,\mathbb{R}\right)\simeq\mathfrak{sp}\left(2,\mathbb{R}\right)\simeq\mathfrak{su}\left(1,1\right)$ as subalgebra generated by grade ±2 subspaces, we shall encounter the above construction as a common theme for minimal unitary realizations of U-duality algebras.

We now turn to construction of minimal realization of $\mathfrak{e}_{8(-24)}$ and construct minimal realizations of supergravity pertaining subalgebras by truncation. Results of sections 4.2, 4.3 and 4.4 were published in [48].

4.2 Minimal realization of $\mathfrak{e}_{8(-24)}$

The Lie algebra $\mathfrak{e}_{8(-24)}$ of $\mathrm{E}_{8(-24)}$ admits a 5-grading with respect its subalgebra $\mathfrak{e}_{7(-25)}\oplus\mathfrak{so}(1,1)$ determined by the generator Δ of a dilatation subgroup $SO(1,1)$

$$\begin{array}{ccccccccccc}\mathfrak{e}_{8(-24)} & = & \mathfrak{g}^{-2} & \oplus & \mathfrak{g}^{-1} & \oplus & \mathfrak{g}^{0} & \oplus & \mathfrak{g}^{+1} & \oplus & \mathfrak{g}^{+2}\\ & & \mathbf{1} & \oplus & \mathbf{56} & \oplus & (\mathbf{133}\oplus\mathbf{1}) & \oplus & \mathbf{56} & \oplus & \mathbf{1}\end{array} \tag{4.10}$$

such that $\mathfrak{g}^{\pm2}$ generators together with Δ form an $\mathfrak{sl}(2,\mathbb{R})$ subalgebra.

To construct the minimal unitary representation of $\mathfrak{e}_{8(-24)}$ we find it convenient to work in a basis covariant with respect to $\mathfrak{su}^*(8)$ subalgebra of $\mathfrak{e}_{7(-25)}$[2]. In the $\mathfrak{su}^*(8)$ basis the

[2] The $\mathfrak{su}^*(8)$-covariant basis of $\mathfrak{e}_{7(-25)}$ is the analog of $\mathfrak{sl}(8,\mathbb{R})$ basis of $\mathfrak{e}_{7(7)}$ [45]

generators of $\mathfrak{e}_{7(-25)}$ can be labelled as follows

$$\mathbf{133} = \mathbf{63} \oplus \mathbf{70} = J^A{}_B \oplus J^{ABCD} \tag{4.11}$$

where $J^A{}_B$ denote the generators of $\mathfrak{su}^*(8)$ and J^{ABCD} is completely antisymmetric in its indices $A, B, \ldots = 1, 2, \ldots, 8$. They satisfy the commutation relations

$$\begin{aligned} \left[J^A{}_B, J^C{}_D\right] &= \delta^C{}_B J^A{}_D - \delta^A{}_D J^C{}_B \\ \left[J^A{}_B, J^{CDEF}\right] &= -4\delta^{[C}{}_B J^{DEF]A} - \frac{1}{2}\delta^A{}_B J^{CDEF} \\ \left[J^{ABCD}, J^{EFGH}\right] &= -\frac{1}{36}\epsilon^{ABCDK[EFG} J^{H]}{}_K \end{aligned} \tag{4.12}$$

and the following reality conditions

$$\begin{aligned} \left(J^A{}_B\right)^\dagger &= J_A{}^B = \Omega_{AC}\Omega^{BD} J^C{}_D \\ \left(J^{ABCD}\right)^\dagger &= -J_{ABCD} = -\Omega_{AE}\Omega_{BF}\Omega_{CG}\Omega_{DH} J^{EFGH} \end{aligned} \tag{4.13}$$

where Ω is a symplectic matrix such that $\Omega_{AB} = -\Omega_{BA} = \left(\Omega^{BA}\right)^*$, $\Omega_{AB}\Omega^{BC} = \delta^C{}_A$. The quadratic Casimir operator of $\mathrm{E}_{7(-25)}$ in the basis (4.12) is given by

$$\begin{aligned} \mathcal{C}_2 &= \frac{1}{6} J^A{}_B J^B{}_A - \frac{1}{24}\epsilon_{ABCDEFGH} J^{ABCD} J^{EFGH} \\ &= \frac{1}{6} J^A{}_B J^B{}_A - J^{ABCD}(\epsilon J)_{ABCD} \end{aligned} \tag{4.14}$$

where $(\epsilon J)_{ABCD} = \frac{1}{4!}\epsilon_{ABCDEFGH} J^{EFGH}$.

The fundamental representation **56** of $\mathfrak{e}_{7(-25)}$ decomposes as $\mathbf{28} \oplus \tilde{\mathbf{28}}$ under its $\mathfrak{su}^*(8)$ subalgebra, where **28** (X^{AB}) and $\tilde{\mathbf{28}}$ ($\tilde{X}_{AB}$) are anti-symmetric tensors satisfying the following reality condition

$$\left(X^{AB}\right)^\dagger = X_{AB} = \Omega_{AC}\Omega_{BD} X^{CD}\,, \quad \left(\tilde{X}_{AB}\right)^\dagger = \tilde{X}^{AB} = \Omega^{AC}\Omega^{BD}\tilde{X}_{CD}\,. \tag{4.15}$$

Under the action of $\mathfrak{e}_{7(-25)}$ they transform as

$$\begin{aligned} \delta X^{AB} &= \Sigma^A{}_C X^{CB} + \Sigma^B{}_C X^{AC} - \Sigma^{ABCD}\tilde{X}_{CD} \\ \delta\tilde{X}_{CD} &= -\Sigma^A{}_C \tilde{X}_{AD} - \Sigma^A{}_D \tilde{X}_{CA} + \Sigma_{CDAB} X^{AB} \end{aligned} \tag{4.16}$$

where $\Sigma^A{}_C$ and $\Sigma^{ABCD} = -\left(\Sigma_{ABCD}\right)^\dagger$ denote parameters of $\mathrm{SU}^*(8)$ transformation and those of the coset generators $\mathrm{E}_{7(-25)}/\mathrm{SU}^*(8)$, respectively.

4.2.1 Exceptional Lie Algebra $\mathfrak{e}_{8(-24)}$

Note that **56** is a real representation of $\mathfrak{e}_{7(-25)}$ just as **28** and $\tilde{\mathbf{28}}$ are real representations of $\mathfrak{su}^*(8)$. Thus in $\mathfrak{su}^*(8)$ covariant basis we can label generators belonging to grade -1 space as E^{AB} and $\tilde{E}_{AB}$ and grade +1 space as F^{AB} and $\tilde{F}_{AB}$. The 5-graded decomposition of $\mathfrak{e}_{8(-24)}$ in $\mathfrak{su}^*(8)$ basis takes the form

$$\mathfrak{e}_{8(-24)} = E \oplus \left\{E^{AB}\,, \tilde{E}_{CD}\right\} \oplus \left\{J^A{}_B\,, J^{ABCD}\,; \Delta\right\} \oplus \left\{F^{AB}\,, \tilde{F}_{CD}\right\} \oplus F \tag{4.10}$$

The grading is defined by the generator Δ of $SO(1,1)$

$$\begin{aligned} &[\Delta, E] = -2E\,, && [\Delta, F] = +2F \\ &\left[\Delta, E^{AB}\right] = -E^{AB}\,, && \left[\Delta, F^{AB}\right] = +F^{AB} \\ &\left[\Delta, \tilde{E}_{CD}\right] = -\tilde{E}_{CD}\,, && \left[\Delta, \tilde{F}_{CD}\right] = +\tilde{F}_{CD} \end{aligned} \tag{4.17}$$

Positive and negative generators form two separate maximal Heisenberg subalgebras with commutation relations

$$\left[E^{AB},\, \tilde{E}_{CD}\right] = 2\,\delta^{AB}_{CD}\,E \qquad \left[E,\, E^{AB}\right] = 0 \qquad \left[E,\, \tilde{E}_{AB}\right] = 0 \tag{4.18}$$

and

$$\left[F^{AB}\,, \tilde{F}_{CD}\right] = 2\,\delta^{AB}_{CD}\,F \qquad \left[F,\, F^{AB}\right] = 0 \qquad \left[F,\, \tilde{F}_{AB}\right] = 0\,. \tag{4.19}$$

However these two Heisenberg subalgebras do not commute with each other (see eqs. (4.22) below). Generators of $\mathfrak{g}^{\pm 2}$ are invariant under $\mathfrak{e}_{7(-25)}$

$$\left[J^A{}_B,\, F\right] = 0 \quad \left[J^{ABCD},\, F\right] = 0 \quad \left[J^A{}_B,\, E\right] = 0 \quad \left[J^{ABCD},\, E\right] = 0 \tag{4.20}$$

while generators of $\mathfrak{g}^{\pm 1}$ transform under $\mathfrak{su}^*(8)$ as follows

$$\begin{aligned} \left[J^A{}_B\,, E^{CD}\right] &= \delta^C{}_B E^{AD} + \delta^D{}_B E^{CA} - \frac{1}{4}\delta^A{}_B E^{CD} \\ \left[J^A{}_B\,, F^{CD}\right] &= \delta^C{}_B F^{AD} + \delta^D{}_B F^{CA} - \frac{1}{4}\delta^A{}_B F^{CD} \\ \left[J^A{}_B\,, \tilde{E}_{CD}\right] &= -\delta^A{}_C \tilde{E}_{BD} - \delta^A{}_D \tilde{E}_{CB} + \frac{1}{4}\delta^A{}_B \tilde{E}_{CD} \\ \left[J^A{}_B\,, \tilde{F}_{CD}\right] &= -\delta^A{}_C \tilde{F}_{BD} - \delta^A{}_D \tilde{F}_{CB} + \frac{1}{4}\delta^A{}_B \tilde{F}_{CD} \end{aligned} \tag{4.21}$$

The remaining commutation relations read as follows

$$\begin{aligned}
\left[J^{ABCD}\,,\tilde{E}_{EF}\right] &= \delta^{[AB}_{EF}E^{CD]}\,, \quad \left[J^{ABCD}\,,E^{EF}\right] = -\frac{1}{24}\epsilon^{ABCDEFGH}\tilde{E}_{GH}\\
\left[J^{ABCD}\,,\tilde{F}_{EF}\right] &= \delta^{[AB}_{EF}F^{CD]}\,, \quad \left[J^{ABCD}\,,F^{EF}\right] = -\frac{1}{24}\epsilon^{ABCDEFGH}\tilde{F}_{GH}
\end{aligned}$$

$$\begin{aligned}
\left[E^{AB}\,,F^{CD}\right] &= -12\,J^{ABCD}\,, \qquad \left[\tilde{E}_{AB}\,,F^{CD}\right] = 4\,\delta^{[C}{}_{[A}J^{D]}{}_{B]} + \delta^{CD}_{AB}\Delta\\
\left[\tilde{E}_{AB}\,,\tilde{F}_{CD}\right] &= -12\,(\epsilon J)_{ABCD}\,, \quad \left[E^{AB}\,,\tilde{F}_{CD}\right] = 4\,\delta^{[A}{}_{[C}J^{B]}{}_{D]} - \delta^{AB}_{CD}\Delta
\end{aligned} \tag{4.22}$$

$$\begin{aligned}
\left[E,F^{AB}\right] &= -E^{AB}\,, \quad \left[E,\tilde{F}_{AB}\right] = -\tilde{E}_{AB}\\
\left[F,E^{AB}\right] &= +F^{AB}\,, \quad \left[F,\tilde{E}_{AB}\right] = +\tilde{F}_{AB}
\end{aligned} \qquad [E,F] = \Delta$$

Reality properties for generators belonging to grade ±1 and ±2 are as follows

$$\begin{aligned}
\left(F^{AB}\right)^\dagger &= -\Omega_{AC}\Omega_{BD}F^{CD}\,, \qquad \left(\tilde{F}_{AB}\right)^\dagger = -\Omega^{AC}\Omega^{BD}\tilde{F}_{CD}\,,\\
\left(E^{AB}\right)^\dagger &= -\Omega_{AC}\Omega_{BD}E^{CD}\,, \qquad \left(\tilde{E}_{AB}\right)^\dagger = -\Omega^{AC}\Omega^{BD}\tilde{E}_{CD}\,,\\
E^\dagger &= -E\,, \qquad\qquad F^\dagger = -F\,,
\end{aligned} \tag{4.23}$$

The quadratic Casimir operator of the above Lie algebra is given by

$$\begin{aligned}
\mathcal{C}_2\left(\mathfrak{e}_{8(-24)}\right) = &\frac{1}{6}J^A{}_BJ^B{}_A - J^{ABCD}(\epsilon J)_{ABCD}\\
&+\frac{1}{12}\Delta^2 - \frac{1}{6}\left(FE+EF\right)\\
&-\frac{1}{12}\left(\tilde{E}_{AB}F^{AB} + F^{AB}\tilde{E}_{AB} - \tilde{F}_{AB}E^{AB} - E^{AB}\tilde{F}_{AB}\right)
\end{aligned} \tag{4.24}$$

In order to make manifest the fact that the above Lie algebra is of the real form $\mathfrak{e}_{8(-24)}$ with the maximal compact subalgebra $\mathfrak{e}_7\oplus\mathfrak{su}(2)$ let us write down the compact and non-compact generators explicitly. Under the maximal compact subalgebra $\mathfrak{usp}(8)$ of $\mathfrak{su}^*(8)$ we have the following decompositions of the adjoint and fundamental representations of $\mathfrak{e}_{8(-24)}$

$$\begin{aligned}
\mathbf{133} &= \mathbf{63}\oplus\mathbf{70} = (\mathbf{36}\oplus\mathbf{27})\oplus(\mathbf{1}\oplus\mathbf{27}\oplus\mathbf{42})\\
\mathbf{56} &= \mathbf{28}\oplus\tilde{\mathbf{28}} = (\mathbf{1}\oplus\mathbf{27})\oplus(\mathbf{1}\oplus\mathbf{27})
\end{aligned}$$

where **27** and **42** correspond to symplectic traceless antisymmetric 2-tensor and 4-tensor of $\mathfrak{usp}(8)$ respectively.[3] Note that the generators in the representations $\mathbf{1}\oplus\mathbf{36}\oplus\mathbf{42}$ of $\mathfrak{usp}(8)$ in the decomposition of the adjoint representation of $\mathfrak{e}_{7(-25)}$ form the maximal compact subalgebra $\mathfrak{e}_6\oplus\mathfrak{u}(1)$ of $\mathfrak{e}_{7(-25)}$.

[3] The group $\mathrm{SU}^*(8)$ is defined as a subgroup of $\mathrm{SL}(8,\mathbb{C})$ generated by elements $U\in\mathrm{SL}(8,\mathbb{C})$ such that $U^\dagger U = 1$ and $U\Omega = U^*\Omega$. U^* is obtained from U by component-wise complex conjugation.

Denoting the generators (T) transforming covariantly under the $\mathfrak{usp}(8)$ subalgebra of $\mathfrak{su}^*(8)$ with a check ($\check{T}$) we find that the generators in $\mathbf{36} \oplus \mathbf{27}$ are given by $\check{G}^{(\pm)}_{AB} = \Omega_{AC}J^C{}_B \pm \Omega_{BC}J^C{}_A$, while generators coming from the decomposition of $\mathbf{70}$ with respect to $\mathfrak{usp}(8)$ are given by

$$\check{J}^{AB} = J^{ABCD}\Omega_{CD} + \frac{1}{8}\Omega^{AB}\check{J}\,.$$

$$\check{J}^{ABCD} := J^{ABCD} + \frac{3}{2}\Omega^{[AB}\Omega_{EF}J^{CD]EF} + \frac{1}{8}\Omega^{[AB}\Omega^{CD]}\check{J}$$

$$\check{J} := \Omega_{EF}\Omega_{GH}J^{EFGH}$$

Thus we find that

$$J^{ABCD}\,(\epsilon J)_{ABCD} = \check{J}^{ABCD}\check{J}_{ABCD} - \frac{3}{2}\,\check{J}^{AB}\check{J}_{AB} + \frac{1}{16}\check{J}^2 \tag{4.25}$$

The decomposition of $\mathbf{56}$ of $\mathfrak{e}_{7(-25)}$ into $\mathfrak{usp}(8)$ irreducible components leads to the following generators that transform in the $\mathbf{27}$ of $\mathfrak{usp}(8)$:

$$\begin{aligned}
\check{C}^{\pm}_{AB} &= \tilde{E}_{AB} + F_{AB} \pm \left(\tilde{F}_{AB} - E_{AB}\right) \\
&\quad + \frac{1}{8}\Omega_{AB}\Omega^{CD}\left[\tilde{E}_{CD} + F_{CD} \pm \left(\tilde{F}_{CD} - E_{CD}\right)\right] \\
\check{N}^{\pm}_{AB} &= \tilde{F}_{AB} + E_{AB} \pm \left(\tilde{E}_{AB} - F_{AB}\right) \\
&\quad + \frac{1}{8}\Omega_{AB}\Omega^{CD}\left[\tilde{F}_{CD} + E_{CD} \pm \left(\tilde{E}_{CD} - F_{CD}\right)\right]
\end{aligned} \tag{4.26}$$

and to the following singlets of $\mathfrak{usp}(8)$:

$$\begin{aligned}
\check{C}^{\pm} &= \Omega^{CD}[\tilde{E}_{CD} + F_{CD} \pm \left(\tilde{F}_{CD} - E_{CD}\right)] \\
\check{N}^{\pm} &= \Omega^{CD}[\tilde{F}_{CD} + E_{CD} \pm \left(\tilde{E}_{CD} - F_{CD}\right)]
\end{aligned}$$

Then the following 133 operators

$$\check{G}^{(+)}_{AB}\,, \quad \check{J}^{ABCD}\,, \quad \check{J} + 2\,(E+F)\,, \quad \check{C}^{\pm}_{AB} \tag{4.27}$$

generate the compact E_7 subgroup and the operators $\check{C}^{\pm}$ and $2(E+F) - 3\check{J}$ generate the compact $SU(2)$ subgroup. The remaining 112 generators are non-compact:

$$G^{(-)}_{AB}\,, \quad \check{J}^{AB}\,, \quad \Delta\,, \quad F - E\,, \quad \check{N}^{\pm}_{AB}\,, \quad \check{N}^{\pm}. \tag{4.28}$$

4.2.2 The Minimal Unitary Realization of $\mathfrak{e}_{8(-24)}$ in $\mathfrak{su}^*(8)$ Basis

It was noted earlier that elements of the subspace $\mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \subset \mathfrak{e}_{8(-24)}$ form an Heisenberg algebra with 28 "coordinates" and 28 "momenta" with the generator of $\mathfrak{g}^{-2}$ acting as its central charge. As it was done for $\mathfrak{e}_{8(8)}$ [45] we shall realize these Heisenberg algebra generators using canonically conjugate position (X^{AB}) and momentum (P_{AB}) operators:

$$\left[X^{AB}\,, P_{CD}\right] = i\,\delta^{AB}_{CD}\,. \tag{4.29}$$

satisfying the following reality properties

$$\left(X^{AB}\right)^\dagger = X_{AB} = \Omega_{AC}\Omega_{BD}X^{CD}\,, \quad (P_{AB})^\dagger = P^{AB} = \Omega^{AC}\Omega^{BD}P_{CD} \tag{4.30}$$

The commutation relations (4.29) can also be rewritten in more $\mathfrak{usp}(8)$ covariant fashion

$$[X_{AB}\,, P_{CD}] = \frac{i}{2}\left(\Omega_{AC}\Omega_{BD} - \Omega_{BC}\Omega_{AD}\right). \tag{4.29$'$}$$

The generators of $\mathfrak{g}^{-1} \oplus \mathfrak{g}^{-2}$ subalgebra are then realized as

$$E^{AB} = -iy\,X^{AB} \qquad \tilde{E}_{AB} = -iy\,P_{AB} \qquad E = -\frac{i}{2}\,y^2 \tag{4.31}$$

where y is an extra coordinate related to central charge. In order to be able to realize $\mathfrak{g}^{+1} \oplus \mathfrak{g}^{+2}$ generators we need to introduce a momentum operator p conjugate to y:

$$[y\,, p] = i\,. \tag{4.32}$$

The grade zero $\mathfrak{g}^0$ generators, realized linearly on operators X^{AB} and P_{AB}, take on the form

$$\begin{aligned} J^A{}_B &= -2iX^{AC}P_{CB} - \frac{i}{4}\delta^A{}_B X^{CD}P_{CD} \\ J^{ABCD} &= -\frac{i}{2}X^{[AB}X^{CD]} - \frac{i}{48}\epsilon^{ABCDEFGH}P_{EF}P_{GH} \end{aligned} \tag{4.33}$$

The dilatation generator Δ that defines the grading is simply

$$\Delta = -\frac{i}{2}\left(py + yp\right). \tag{4.34}$$

Since $\mathfrak{g}^{-1}$ generators are linear and $\mathfrak{g}^0$ generators are quadratic polynomials in X and P we expect $\mathfrak{g}^{+1}$ generators to be cubic. Furthermore, $\mathfrak{g}^{+1} = \left[\mathfrak{g}^{+2}, \mathfrak{g}^{-1}\right]$ suggests that F must be a quartic polynomial in X and P. Since it is an $\mathfrak{e}_{7(-25)}$ singlet, this quartic must be the

quartic invariant of $\mathfrak{e}_{7(-25)}$. Indeed we find

$$\begin{aligned} F &= \frac{1}{2i}p^2 + \frac{2}{iy^2} I_4\left(X\,,P\right) \\ F^{AB} &= ipX^{AB} + \frac{2}{y}\left[X^{AB}\,, I_4\left(X,P\right)\right] \\ \tilde{F}_{AB} &= ipP_{AB} + \frac{2}{y}\left[P_{AB}\,, I_4\left(X,P\right)\right]. \end{aligned} \tag{4.35}$$

The quartic invariant I_4 coincides with quadratic Casimir of $\mathfrak{e}_{7(-25)}$ modulo an additive constant:

$$\begin{aligned} I_4\left(X\,,P\right) = \mathcal{C}_2\left(\mathfrak{e}_{7(-25)}\right) + \frac{323}{16} = \frac{547}{16} + \\ - \frac{1}{2}\left(X^{AB}P_{BC}X^{CD}P_{DA} + P_{AB}X^{BC}P_{CD}X^{DA}\right) \\ + \frac{1}{8}\left(X^{AB}P_{AB}X^{CD}P_{CD} + P_{AB}X^{AB}P_{CD}X^{CD}\right) \\ + \frac{1}{96}\epsilon^{ABCDMNKL}P_{AB}P_{CD}P_{MN}P_{KL} \\ + \frac{1}{96}\epsilon_{ABCDMNKL}X^{AB}X^{CD}X^{MN}X^{KL} \end{aligned} \tag{4.36}$$

The quadratic Casimir of $\mathfrak{e}_{8(-24)}$ (4.24) evaluated in the above realization reduces to a c-number as required by the irreducibility. In order to demonstrate that we decompose the quadratic Casimir (4.24) into three $\mathfrak{e}_{7(-25)}$-invariant pieces

$$\mathcal{C}_2\left(\mathfrak{e}_8\right) = \mathcal{C}_2\left(\mathfrak{e}_7\right) + \mathcal{C}_2\left(\mathfrak{sl}\left(2,\mathbb{R}\right)\right) + \mathcal{C}'$$

according to the first, second and the third lines of (4.24) respectively. From (4.36) we find that

$$\mathcal{C}_2\left(\mathfrak{e}_7\right) = I_4 - \frac{323}{16}\,.$$

Using definitions of Δ, E, F we obtain

$$\mathcal{C}_2\left(\mathfrak{sl}\left(2,\mathbb{R}\right)\right) = \frac{1}{3}I_4 - \frac{1}{16}$$

Using definitions for $\mathfrak{g}^{-1} \oplus \mathfrak{g}^{+1}$ generators we find

$$\begin{aligned} 3\mathcal{C}' &= 7 - 28I_4 - iX^{AB}I_4P_{AB} + iP_{AB}I_4X^{AB} \\ &= 7 - 28I_4 + \left(32I_4 - \frac{265}{4}\right) = 4I_4 - \frac{237}{4} \end{aligned}$$

and therefore

$$\mathcal{C}_2\left(\mathfrak{e}_8\right) = -40 \tag{4.37}$$

Since E_8 does not have any invariant tensors in 58 dimensions (corresponding to 29 position and 29 momentum operators) all higher Casimir operators of $\mathfrak{e}_{8(-24)}$ in the above realization must also reduce to c-numbers as was argued for the case of $\mathfrak{e}_{8(8)}$ in [46]. By integrating the above Lie algebra one obtains the minimal unitary representation of the group $E_{8(-24)}$ over the Hilbert space of square integrable complex functions in 29 variables.

4.2.3 The Minimal Unitary Realization of $\mathfrak{e}_{8(-24)}$ in $\mathfrak{su}(6,2)$ Basis

Analysis above was done in $\mathfrak{su}^*(8)$ covariant basis (see footnote on the page 57). Since covariant operators X^{AB} and P_{AB} are position and momenta we refer to this basis as the Schrödinger picture. One can consider an oscillator basis where the natural operators are 28 creation and 28 annihilation operators constructed out of X and P's. Being complex, we expect them to transform as $\mathbf{28} \oplus \overline{\mathbf{28}}$ of some non-compact version of $\mathfrak{su}(8)$ within $\mathfrak{e}_{7(-25)} \subset \mathfrak{g}^0$. This algebra turns out to be $\mathfrak{su}(6,2)$ and the creation and annihilation operators are given as follows

$$\begin{aligned} Z^{ab} &= \frac{1}{4}\,\Gamma^{ab}{}_{CD}\left(X^{CD} - iP_{CD}\right) \\ \tilde{Z}^{ab} &= \frac{1}{4}\,\Gamma^{ab}{}_{CD}\left(X^{CD} + iP_{CD}\right) \end{aligned} \tag{4.38}$$

where transformation coefficient $\Gamma^{ab}{}_{CD}$ are related to gamma-matrices of

$$\mathfrak{so}(6,2) \simeq \mathfrak{so}^*(8) \simeq \mathfrak{su}^*(8) \cap \mathfrak{su}(6,2)$$

as spelled out in appendix A. Operators Z and $\tilde{Z}$ satisfy

$$\left[\tilde{Z}^{ab}\,, Z^{cd}\right] = \frac{1}{2}\left(\eta^{ca}\eta^{db} - \eta^{cb}\eta^{da}\right)\,. \tag{4.39}$$

with the following reality conditions

$$\left(Z^{ab}\right)^{\dagger} = \tilde{Z}^{ab} = \eta^{ac}\eta^{bd}\tilde{Z}_{cd} \tag{4.40}$$

where $\eta = \mathrm{Diag}\,(+,+,+,+,+,+,-,-)$ is used to raise and lower indexes. Generators of $\mathfrak{e}_{7(-25)}$ in this basis take the following form

$$\begin{aligned} J^a{}_b &= 2Z^{ac}\tilde{Z}_{bc} - \frac{1}{4}\delta^a{}_b Z^{cd}\tilde{Z}_{cd} \\ J^{abcd} &= \frac{1}{2}Z^{[ab}Z^{cd]} - \frac{1}{48}\epsilon^{abcdefgh}\tilde{Z}_{ef}\tilde{Z}_{gh} \end{aligned} \tag{4.41}$$

with Hermiticity conditions

$$(J^a{}_b)^\dagger = \eta^{ad}\eta_{bc}J^c{}_d \qquad \left(J^{abcd}\right)^\dagger = -\frac{1}{24}\epsilon^{abcd}{}_{efgh}J^{efgh} \tag{4.42}$$

Their commutation relations are

$$\begin{aligned}
\left[J^a{}_b\,,J^c{}_d\right] &= \delta^c{}_b J^a{}_d - \delta^d{}_a J^c{}_b \\
\left[J^a{}_b\,,J^{cdef}\right] &= -4\delta^{[c}{}_b J^{def]a} - \frac{1}{2}\delta^a{}_b J^{cdef} \\
\left[J^{abcd}\,,J^{efgh}\right] &= \frac{1}{36}\epsilon^{abcdp[efg}J^{h]}{}_p
\end{aligned} \tag{4.43}$$

which have the same form as $\mathfrak{su}^*(8)$ covariant eqs. (4.12). Quadratic Casimir in this basis reads as

$$\begin{aligned}
\mathcal{C}_2\left(\mathfrak{e}_7\right) &= \frac{1}{6}J^a{}_b J^b{}_a + J^{abcd}\left(\epsilon J\right)_{abcd} = I_4\left(Z\,,\tilde{Z}\right) - \frac{323}{16} = \\
&= \frac{1}{2}\left(\tilde{Z}_{ab}Z^{bc}\tilde{Z}_{cd}Z^{da} + Z^{ab}\tilde{Z}_{bc}Z^{cd}\tilde{Z}_{da}\right) \\
&- \frac{1}{8}\left(\tilde{Z}_{ab}Z^{ab}\tilde{Z}_{cd}Z^{cd} + Z^{ab}\tilde{Z}_{ab}Z^{cd}\tilde{Z}_{cd}\right) + 14 \\
&+ \frac{1}{96}\epsilon_{abcdefgh}Z^{ab}Z^{cd}Z^{ef}Z^{gh} + \frac{1}{96}\epsilon^{abcdefgh}\tilde{Z}_{ab}\tilde{Z}_{cd}\tilde{Z}_{ef}\tilde{Z}_{gh}
\end{aligned} \tag{4.44}$$

Negative grade generators of $\mathfrak{e}_{8(-24)}$ are then simply

$$E = \frac{1}{2}y^2 \qquad E^{ab} = yZ^{ab} \qquad \tilde{E}_{ab} = y\tilde{Z}_{ab} \tag{4.45}$$

Generators in $\mathfrak{g}^{+1}$ can be inferred commuting $\mathfrak{g}^{+2}$ generator

$$F = \frac{1}{2}p^2 + 2y^{-2}I_4 \tag{4.46}$$

with generators in $\mathfrak{g}^{-1}$

$$\begin{aligned}
F^{ab} &= i\left[E^{ab}\,,F\right] = -pZ^{ab} + 2iy^{-1}\left[Z^{ab}\,,I_4\right] \\
\tilde{F}_{ab} &= i\left[\tilde{E}_{ab}\,,F\right] = -p\tilde{Z}_{ab} + 2iy^{-1}\left[\tilde{Z}_{ab}\,,I_4\right]
\end{aligned} \tag{4.47}$$

or more explicitly

$$\begin{aligned} F^{ab} =& -pZ^{ab} - \frac{i}{12} y^{-1} \epsilon^{abcdefgh} \tilde{Z}_{cd} \tilde{Z}_{ef} \tilde{Z}_{gh} \\ &+ 4iy^{-1} Z^{c[a} \tilde{Z}_{cd} Z^{b]d} + \frac{i}{2} y^{-1} \left(Z^{ab} \tilde{Z}_{cd} Z^{cd} + Z^{cd} \tilde{Z}_{cd} Z^{ab} \right) \\ F_{ab} =& -p\tilde{Z}_{ab} + \frac{i}{12} y^{-1} \epsilon_{abcdefgh} Z^{cd} Z^{ef} Z^{gh} \\ &- 4iy^{-1} \tilde{Z}_{c[a} Z^{cd} Z_{b]d} - \frac{i}{2} y^{-1} \left(\tilde{Z}_{ab} Z^{cd} \tilde{Z}_{cd} + \tilde{Z}_{cd} Z^{cd} \tilde{Z}_{ab} \right) \end{aligned} \tag{4.48}$$

We see that commutation relations in this basis closely follow those in $\mathfrak{su}^*(8)$ basis, with modified reality conditions (cf. (4.45) and (4.31) as well as (4.47) with (4.35)). The $SU(6,2)$ covariant commutation relations follow closely those given in section 4.2.1

$$\begin{gathered} [E, F] = -\Delta \qquad \left[E\,, F^{ab}\right] = -iE^{ab} \\ [\Delta,\, F] = 2F \qquad [\Delta,\, E] = -2E \qquad \left[E\,, \tilde{F}_{ab}\right] = -i\tilde{E}_{ab} \\ \left[\Delta,\, F^{ab}\right] = F^{ab} \qquad \left[\Delta,\, E^{ab}\right] = -E^{ab} \qquad \left[F\,, E^{ab}\right] = iF^{ab} \\ \left[\Delta,\, \tilde{F}_{ab}\right] = \tilde{F}_{ab} \qquad \left[\Delta,\, \tilde{E}_{ab}\right] = -\tilde{E}_{ab} \qquad \left[F\,, \tilde{E}_{ab}\right] = i\tilde{F}_{ab} \end{gathered} \tag{4.49}$$

$$\begin{gathered} \left[E,\, E^{ab}\right] = 0 \qquad \left[E,\, \tilde{E}_{ab}\right] = 0 \qquad \left[\tilde{E}_{ab}\,, E^{cd}\right] = 2\,\delta^{cd}_{ab}\,E \\ \left[F,\, F^{ab}\right] = 0 \qquad \left[F,\, \tilde{F}_{ab}\right] = 0 \qquad \left[\tilde{F}_{ab}\,, F^{cd}\right] = 2\,\delta^{cd}_{ab}\,F \end{gathered} \tag{4.50}$$

$$\begin{gathered} \left[E^{ab}\,, F^{cd}\right] = -12iJ^{abcd} \qquad \left[\tilde{E}_{ab},\, F^{cd}\right] = -4i\delta^{[c}{}_{[a}J^{d]}{}_{b]} - i\delta^{cd}_{ab}\Delta \\ \left[\tilde{E}_{ab},\, \tilde{F}_{cd}\right] = 12i\,(\epsilon J)_{abcd} \qquad \left[E^{ab},\, \tilde{F}_{cd}\right] = -4i\delta^{[a}{}_{[c}J^{b]}{}_{d]} + i\delta^{ab}_{cd}\Delta \end{gathered} \tag{4.51}$$

$$\begin{gathered} \left[J^a{}_b\,, E^{cd}\right] = \delta^c{}_b E^{ad} + \delta^d{}_b E^{ca} - \frac{1}{4}\delta^a{}_b E^{cd} \qquad \left[J^{abcd}\,, \tilde{E}_{ef}\right] = \delta^{[ab}_{ef} E^{cd]} \\ \left[\tilde{E}_{cd}\,, J^a{}_b\right] = \delta^a{}_c \tilde{E}_{bd} + \delta^a{}_d \tilde{E}_{cb} - \frac{1}{4}\delta^a{}_b \tilde{E}_{cd} \qquad \left[J^{abcd}\,, E^{ef}\right] = \frac{-1}{24}\epsilon^{abcdefgh}\tilde{E}_{gh} \end{gathered}$$

The quadratic Casimir of $\mathfrak{e}_{8(-24)}$ in this basis reads as follows

$$\begin{aligned} \mathcal{C}_2 =& \frac{1}{6}\, J^a{}_b J^b{}_a + J^{abcd}\,(\epsilon J)_{abcd} + \frac{1}{6}\left(EF + FE + \frac{1}{2}\,\Delta^2 \right) \\ &- \frac{i}{12}\left(\tilde{F}_{ab} E^{ab} + E^{ab}\tilde{F}_{ab} - \tilde{E}_{ab} F^{ab} - F^{ab}\tilde{E}_{ab} \right) \end{aligned} \tag{4.52}$$

and reduces to the same c-number as (4.37).

4.3 Truncations of the minimal unitary realization of $\mathfrak{e}_{8(-24)}$

Since our realization of $\mathfrak{e}_{8(-24)}$ is non-linear, not every subalgebra of $\mathfrak{e}_{8(-24)}$ can be obtained by a consistent truncation. We consider consistent truncations to subalgebras that are quasi-conformal. Since quasi-conformal algebras admit a 5-grading

$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{+1} \oplus \mathfrak{g}^{+2}$$

with $\mathfrak{g}^{\pm 2}$ being one-dimensional, they have an $\mathfrak{sl}(2,\mathbb{R})$ subalgebra generated by elements of $\mathfrak{g}^{\pm 2}$ and the generator Δ that determines 5-grading. However, the quartic invariant $\mathcal{I}_4$ will now be that of a subalgebra $\mathfrak{g}^0$ of the linearly realized $\mathfrak{e}_{7(-25)}$ within $\mathfrak{e}_{8(-24)}$. Furthermore, this subalgebra must act on the grade ± 1 subspaces via a *symplectic representation*.

Hence, the problem is reduced to enumeration of subalgebras of linearly realized $\mathfrak{e}_{7(-25)}$ admitting a non-degenerate quartic invariant on the symplectic representation. Before giving the explicit truncations below we shall first indicate a partial web of consistent truncations as quasiconformal subalgebras.

Firstly, we can truncate $\mathfrak{e}_{8(-24)}$ down to either $\mathfrak{e}_{7(5)}$ or $\mathfrak{e}_{7(-25)}$, by keeping singlets of either $\mathfrak{su}(2)$ or $\mathfrak{su}(1,1)$ within $\mathfrak{su}(6,2) \subset \mathfrak{e}_{7(-25)}$ correspondingly. Further truncations of $\mathfrak{e}_{7(-25)}$ to rank 6 quasi-conformal algebras can lead to either $\mathfrak{so}(10,2)$ or $\mathfrak{e}_{6(-14)}$, while truncations of $\mathfrak{e}_{7(5)}$ lead to either $\mathfrak{e}_{6(-14)}$ or $\mathfrak{e}_{6(2)}$:

$$\begin{array}{ccccccccc}
 & & & & \mathfrak{so}(10,2) & \to & \mathfrak{so}(6,2) & \to & \mathfrak{so}(4,2) \\
 & & & \nearrow & & & & & \\
 & & \mathfrak{e}_{7(-25)} & & & & & & \\
 & \nearrow & & \searrow & & & & & \\
\mathfrak{e}_{8(-24)} & & & & \mathfrak{e}_{6(-14)} & \to & \mathfrak{so}(8,2) & \to & \mathfrak{su}(4,1) \to \mathfrak{su}(2,1) \\
 & \searrow & & \nearrow & & & & & \\
 & & \mathfrak{e}_{7(5)} & & & & & & \\
 & & & \searrow & & & & & \\
 & & & & \mathfrak{e}_{6(2)} & \to & \mathfrak{f}_{4(4)} & \to & \mathfrak{so}(4,4) \to \mathfrak{g}_{2(2)} \to \mathfrak{sl}(3,\mathbb{R})
\end{array} \tag{4.53}$$

The minimal unitary realization of $\mathrm{SU}(2,1)$ was given in [46].

4.3.1 Truncation to the minimal unitary realization of $\mathfrak{e}_{7(-5)}$ as a quasi-conformal subalgebra

In order to truncate the above minimal unitary realization of $\mathfrak{e}_{8(-24)}$ down to its subalgebra $\mathfrak{e}_{7(-5)}$ whose maximal compact subalgebra is $\mathfrak{so}^*(12) \oplus \mathfrak{su}(2)$ we first observe that $\mathfrak{e}_{7(-5)}$ has

the 5-grading

$$\begin{array}{rcccccccccc} \mathfrak{e}_{7(-5)} & = & \mathfrak{g}^{-2} & \oplus & \mathfrak{g}^{-1} & \oplus & \mathfrak{g}^{0} & \oplus & \mathfrak{g}^{+1} & \oplus & \mathfrak{g}^{+2} \\ & & \mathbf{1} & \oplus & \mathbf{32} & \oplus & (\mathfrak{so}^*(12)\oplus\mathbf{1}) & \oplus & \mathbf{32} & \oplus & \mathbf{1} \end{array} \tag{4.54}$$

Furthermore, we note that $\mathfrak{e}_{7(-25)}$ has a subalgebra $\mathfrak{so}^*(12)\oplus\mathfrak{su}(2)$. Hence $\mathfrak{e}_{7(-5)}$ is centralised by an $\mathfrak{su}(2)$ subalgebra, which can be identified with the one in $\mathfrak{su}(6)\oplus\mathfrak{su}(2)\oplus\mathfrak{u}(1)\subset\mathfrak{su}(6,2)\subset\mathfrak{e}_{7(-25)}$. Under the subalgebra $\mathfrak{su}(6)$ the adjoint **66** and the spinor representation **32** of $\mathfrak{so}^*(12)$ decompose as follows:

$$\mathbf{32}=\mathbf{15}\oplus\mathbf{1}\oplus\overline{\mathbf{15}}\oplus\mathbf{1}\qquad\text{and}\qquad\mathbf{66}=\mathbf{35}\oplus\mathbf{15}\oplus\overline{\mathbf{15}}\oplus\mathbf{1}\,.$$

This truncation is thus implemented by setting

$$\begin{aligned}\tilde{Z}_{7b}=0\text{ and }Z^{7b}=0\text{ where }b\neq 8\,,\\ \tilde{Z}_{7b}=0\text{ and }Z^{8b}=0\text{ where }b\neq 7\,,\end{aligned}$$

i.e. by restricting to the $\mathfrak{su}(2)$ singlet sector.

For the sake of notational convenience, we would retain symbols Z^{ab} and $\tilde{Z}_{ab}$ to denote creation and annihilation operators transforming as **15** and $\overline{\mathbf{15}}$ of $\mathfrak{su}(6)\subset\mathfrak{so}^*(12)$, where a and b now run from 1 to 6. Then, generators in $\mathfrak{g}^{-1}\oplus\mathfrak{g}^{-2}$ of $\mathfrak{e}_{7(-5)}$ are given as follows

$$E=\frac{1}{2}y^2\quad E^{ab}=yZ^{ab}\quad E^{+}=yZ^{78}\quad \tilde{E}_{ab}=y\tilde{Z}_{ab}\quad E_{-}=y\tilde{Z}_{78} \tag{4.55}$$

The grade zero generators are Δ and

$$\begin{aligned} J^a{}_b &= 2Z^{ac}\tilde{Z}_{bc}-\frac{1}{3}\delta^a{}_bZ^{cd}\tilde{Z}_{cd}\\ J^{ab} &= \frac{1}{6}Z^{ab}Z^{78}-\frac{1}{48}\epsilon^{abefgh}\tilde{Z}_{ef}\tilde{Z}_{gh}\\ \tilde{J}_{ab} &= -\frac{1}{6}\tilde{Z}_{ab}\tilde{Z}_{78}+\frac{1}{48}\epsilon_{abefgh}Z^{ef}Z^{gh}\\ H &= -\frac{1}{4}\left(Z^{78}\tilde{Z}_{78}+\tilde{Z}_{78}Z^{78}\right)+\frac{1}{24}\left(Z^{ab}\tilde{Z}_{ab}+\tilde{Z}_{ab}Z^{ab}\right)\end{aligned} \tag{4.56}$$

which form the $\mathfrak{so}^*(12)$ subalgebra. They satisfy the following commutation relations

$$\begin{array}{ll} [J^a{}_b\,,J^c{}_d]=\delta^c{}_bJ^a{}_d-\delta^d{}_aJ^c{}_b & \left[J^{ab}\,,\tilde{J}_{cd}\right]=\frac{1}{18}\left(2\delta^{[a}{}_{[c}J^{b]}{}_{d]}-\delta^{ab}_{cd}H\right)\\ \left[J^a{}_b\,,J^{cd}\right]=-2\delta^{[c}{}_bJ^{d]a}-\frac{1}{3}\delta^a{}_bJ^{cd} & \left[H\,,\tilde{J}_{ab}\right]=-\frac{1}{6}\tilde{J}_{ab}\\ \left[J^a{}_b\,,\tilde{J}_{cd}\right]=2\delta^a{}_{[c}\tilde{J}_{d]b}+\frac{1}{3}\delta^a{}_b\tilde{J}_{cd} & \left[H\,,J^{ab}\right]=\frac{1}{6}J^{ab}\end{array} \tag{4.57}$$

In order to construct positive grade generator we need quadratic Casimir of $\mathfrak{so}^*(12)$:

$$\begin{aligned}\mathcal{C}_2\left(\mathfrak{so}^*(12)\right) &= \frac{1}{6}J^a{}_b J^b{}_a + 4H^2 + 24\left(J^{ab}\tilde{J}_{ab} + \tilde{J}_{ab}J^{ab}\right) = I_4 - \frac{99}{16} = \\ &= \frac{1}{2}\left(\tilde{Z}_{ab}Z^{bc}\tilde{Z}_{cd}Z^{da} + Z^{ab}\tilde{Z}_{bc}Z^{cd}\tilde{Z}_{da}\right) \\ &+ \frac{1}{2}\left(Z^{78}\tilde{Z}_{78}Z^{78}\tilde{Z}_{78} + \tilde{Z}_{78}Z^{78}\tilde{Z}_{78}Z^{78}\right) + \\ &- \frac{1}{8}\left(\tilde{Z}_{ab}Z^{ab}\tilde{Z}_{cd}Z^{cd} + Z^{ab}\tilde{Z}_{ab}Z^{cd}\tilde{Z}_{cd}\right) + 4 \\ &- \frac{1}{4}\left(Z^{ab}\tilde{Z}_{ab}Z^{78}\tilde{Z}_{78} + Z^{78}\tilde{Z}_{78}Z^{ab}\tilde{Z}_{ab}\right) \\ &- \frac{1}{4}\left(\tilde{Z}_{ab}Z^{ab}\tilde{Z}_{78}Z^{78} + \tilde{Z}_{78}Z^{78}\tilde{Z}_{ab}Z^{ab}\right) \\ &+ \frac{1}{12}\epsilon_{abcdef}Z^{ab}Z^{cd}Z^{ef}Z^{78} + \frac{1}{12}\epsilon^{abcdef}\tilde{Z}_{ab}\tilde{Z}_{cd}\tilde{Z}_{ef}\tilde{Z}_{78}\end{aligned} \tag{4.58}$$

where the quartic invariant is built out of the spinor representation **32** of $\mathfrak{so}^*(12)$. Then generators of $\mathfrak{g}^{+1}$ are defined via (4.47). Commutation relations of $\mathfrak{g}^0$ with $\mathfrak{g}^{-1}$ read

$$\begin{aligned}&\left[J^a{}_b\,,E^{cd}\right] = -2\delta^{[c}{}_b E^{d]a} - \frac{1}{3}\delta^a{}_b E^{cd} \qquad \left[J^a{}_b\,,\tilde{E}_{cd}\right] = 2\delta^a{}_{[c}\tilde{E}_{d]a} + \frac{1}{3}\delta^a{}_b\tilde{E}_{cd} \\ &\left[J^{ab}\,,E^{cd}\right] = -\frac{1}{24}\epsilon^{abcdef}\tilde{E}_{ef} \qquad \left[\tilde{J}_{ab}\,,E^{cd}\right] = -\frac{1}{6}\delta^{cd}_{ab}\tilde{E}_{78} \\ &\left[\tilde{J}_{ab}\,,\tilde{E}_{cd}\right] = -\frac{1}{24}\epsilon_{abcdef}E^{ef} \qquad \left[J^{ab}\,,\tilde{E}_{cd}\right] = -\frac{1}{6}\delta^{ab}_{cd}E^{78}\end{aligned}$$

$$\begin{aligned}&\left[H\,,E^{ab}\right] = \frac{1}{12}E^{ab} \qquad \left[H\,,\tilde{E}_{ab}\right] = -\frac{1}{12}\tilde{E}_{ab} \\ &\left[J^{ab}\,,E^{78}\right] = 0 \qquad \left[\tilde{J}_{ab}\,,E^{78}\right] = -\frac{1}{12}\tilde{E}_{ab} \qquad \left[H\,,E^{78}\right] = -\frac{1}{4}E^{78} \\ &\left[\tilde{J}_{ab}\,,\tilde{E}_{78}\right] = 0 \qquad \left[J^{ab}\,,\tilde{E}_{78}\right] = -\frac{1}{12}E^{ab} \qquad \left[H\,,\tilde{E}_{78}\right] = +\frac{1}{4}\tilde{E}_{78}\end{aligned}$$

Commutators of $\mathfrak{so}^*(12)$ generators and the generators belonging to $\mathfrak{g}^{+1}$ subspace are obtained by substituting E^{ab} with F^{ab} and $\tilde{E}_{ab}$ with $\tilde{F}_{ab}$ in equations above. Spaces $\mathfrak{g}^{\pm 2}$ are $\mathfrak{so}^*(12)$ singlets each. Elements of $\mathfrak{g}^{\pm 2}$ together with Δ generate an $\mathfrak{sl}(2,\mathbb{R}) \subset \mathfrak{e}_{7(-5)}$ subalgebra

$$[E,F] = -\Delta \qquad [\Delta\,,E] = -2E \qquad [\Delta\,,F] = +2F\,. \tag{4.59}$$

Generators in $\mathfrak{g}^{-1}$ and $\mathfrak{g}^{+1}$ close into $\mathfrak{g}^0$ as follows

$$\begin{aligned}\left[E^{ab}\,,F^{cd}\right] &= -6i\epsilon^{abcdef}\tilde{J}_{ef} \\ \left[E^{ab}\,,\tilde{F}_{cd}\right] &= -i\delta^{ab}_{cd}\left(4H - \Delta\right) - 4i\delta^{[a}{}_{[c}J^{b]}{}_{d]} \\ \left[E^{ab}\,,F^{78}\right] &= -12iJ^{ab} \qquad \left[E^{ab}\,,\tilde{F}_{78}\right] = 0\end{aligned} \tag{4.60}$$

$$\begin{aligned}
\left[\tilde{E}_{ab}\,,F^{cd}\right] &= -i\delta^{cd}_{ab}\,(4H+\Delta) - 4i\delta^{[c}{}_{[a}J^{d]}{}_{b]} \\
\left[\tilde{E}_{ab}\,,\tilde{F}_{cd}\right] &= +6i\epsilon_{abcdef}J^{ef} \\
\left[\tilde{E}_{ab}\,,\tilde{F}_{78}\right] &= +12i\tilde{J}_{ab} \qquad \left[\tilde{E}_{ab}\,,F^{78}\right] = 0
\end{aligned} \tag{4.61}$$

$$\begin{aligned}
&\left[E^{78}\,,F^{ab}\right] = -12iJ^{ab} \quad \left[E^{78}\,,\tilde{F}_{78}\right] = i\left(\frac{1}{2}\Delta + 6H\right) \\
&\left[E^{78}\,,\tilde{F}_{ab}\right] = 0 \qquad \left[E^{78}\,,F^{78}\right] = 0 \\
&\left[\tilde{E}_{78}\,,F^{ab}\right] = 0 \qquad \left[\tilde{E}_{78}\,,\tilde{F}_{78}\right] = 0 \\
&\left[\tilde{E}_{78}\,,F^{78}\right] = i\left(-\frac{1}{2}\Delta + 6H\right) \quad \left[\tilde{E}_{78}\,,\tilde{F}_{ab}\right] = +12i\tilde{J}_{ab}
\end{aligned} \tag{4.62}$$

The resulting realization of $\mathfrak{e}_{7(-5)}$ is that of the minimal unitary representation and the quadratic Casimir of $\mathfrak{e}_{7(-5)}$ reduces to a c-number as required by irreducibility of the minimal unitary representation

$$\begin{aligned}
\mathcal{C}_2\left(\mathfrak{e}_{7(-5)}\right) &= \mathcal{C}_2\left(\mathfrak{so}^*(12)\right) + \frac{1}{12}\Delta^2 + \frac{1}{6}\left(FE+EF\right) \\
&- \frac{1}{12}\left(\tilde{E}_{ab}F^{ab} + F^{ab}\tilde{E}_{ab} - \tilde{F}_{ab}E^{ab} - E^{ab}\tilde{F}_{ab}\right) \\
&- \frac{1}{6}\left(\tilde{E}_{78}F^{78} + F^{78}\tilde{E}_{78} - \tilde{F}_{78}E^{78} - E^{78}\tilde{F}_{78}\right) \\
&= \left(I_4 - \frac{99}{16}\right) + \left(\frac{1}{3}I_4 - \frac{1}{16}\right) + \left(-\frac{4}{3}I_4 - \frac{31}{4}\right) = -14
\end{aligned} \tag{4.63}$$

4.3.2 Truncation to the minimal unitary realization of $\mathfrak{e}_{6(2)}$ as a quasiconformal subalgebra

Quasi-conformal algebra $\mathfrak{e}_{6(2)}$ with the maximal compact subalgebra $\mathfrak{su}(6)\oplus\mathfrak{su}(2)$ has the following 5-graded decomposition

$$\mathbf{78} = \mathbf{1}\oplus\mathbf{20}\oplus\left(\mathfrak{su}(3,3)\oplus\Delta\right)\oplus\mathbf{20}\oplus\mathbf{1}$$

and since $\mathfrak{su}(3,3)\subset\mathfrak{so}^*(12)$ it can be obtained by the further truncation of $\mathfrak{e}_{7(-5)}$. The maximal compact subalgebra $\mathfrak{su}(3)\oplus\mathfrak{su}(3)\oplus\mathfrak{u}(1)$ of $\mathfrak{su}(3,3)$ is also a subalgebra of $\mathfrak{su}(6)\subset\mathfrak{so}^*(12)$. This suggests that we split $\mathfrak{su}(6)$ indices $a=1,\ldots,6$ into two subsets, $\check{a}=(1,2,3)$ and $\hat{a}=(4,5,6)$, and keep only oscillators which have both types of indices in addition to singlets $\tilde{Z}_{78}$ and Z^{78}, i.e. set

$$Z^{\check{a}\check{c}} = 0 \qquad \tilde{Z}_{\check{a}\check{c}} = 0 \qquad Z^{\hat{a}\hat{c}} = 0 \qquad \tilde{Z}_{\hat{a}\hat{c}} = 0 \tag{4.64}$$

Indeed corresponding $\mathfrak{su}(3) \oplus \mathfrak{su}(3) \subset \mathfrak{su}(3,3)$ branching reads

$$\mathbf{20} = (\mathbf{1},\, \mathbf{1}) \oplus (\mathbf{3},\, \mathbf{3}) \oplus \left(\overline{\mathbf{3}},\, \overline{\mathbf{3}}\right) \oplus (\mathbf{1},\, \mathbf{1})$$

This reduction is quite straightforward, and we shall not give here complete commutation relations. All of the formulae of $\mathfrak{e}_{7(-5)}$ carry over to this case provided we set to zero appropriate operators. The quadratic Casimir of $\mathfrak{su}(3,3)$ that is needed to construct generators of $\mathfrak{g}^{+1} \oplus \mathfrak{g}^{+2}$ reads as follows

$$\begin{aligned}
&\mathcal{C}_2\left(\mathfrak{su}\left(3,\,3\right)\right) = \frac{1}{6} J^{\check{a}}{}_{\check{c}} J^{\check{c}}{}_{\check{a}} + \frac{1}{6} J^{\hat{a}}{}_{\hat{c}} J^{\hat{c}}{}_{\hat{a}} + 4H^2 + 24\left(J^{\check{a}\check{c}} \tilde{J}_{\check{a}\check{c}} + \tilde{J}_{\check{a}\check{c}} J^{\check{a}\check{c}}\right) \\
&+ 24\left(J^{\hat{a}\hat{c}} \tilde{J}_{\hat{a}\hat{c}} + \tilde{J}_{\hat{a}\hat{c}} J^{\hat{a}\hat{c}}\right) = I_4 - \frac{35}{16} = -\frac{1}{8}\left(\tilde{Z}_{ab} Z^{ab} \tilde{Z}_{cd} Z^{cd} + Z^{ab} \tilde{Z}_{ab} Z^{cd} \tilde{Z}_{cd}\right) \\
&+ \frac{1}{2}\left(\tilde{Z}_{ab} Z^{bc} \tilde{Z}_{cd} Z^{da} + Z^{ab} \tilde{Z}_{bc} Z^{cd} \tilde{Z}_{da}\right) + \frac{1}{2}\left(Z^{78} \tilde{Z}_{78} Z^{78} \tilde{Z}_{78} + \tilde{Z}_{78} Z^{78} \tilde{Z}_{78} Z^{78}\right) \\
&- \frac{1}{4}\left(Z^{ab} \tilde{Z}_{ab} Z^{78} \tilde{Z}_{78} + Z^{78} \tilde{Z}_{78} Z^{ab} \tilde{Z}_{ab}\right) - \frac{1}{4}\left(\tilde{Z}_{ab} Z^{ab} \tilde{Z}_{78} Z^{78} + \tilde{Z}_{78} Z^{78} \tilde{Z}_{ab} Z^{ab}\right) \\
&+ \frac{1}{12} \epsilon_{abcdef} Z^{ab} Z^{cd} Z^{ef} Z^{78} + \frac{1}{12} \epsilon^{abcdef} \tilde{Z}_{ab} \tilde{Z}_{cd} \tilde{Z}_{ef} \tilde{Z}_{78} + \frac{5}{4}
\end{aligned} \tag{4.65}$$

where Z^{ab} and $\tilde{Z}_{ab}$ are as described above, and hence I_4 is the quadratic invariant of $\mathfrak{su}(3,3)$ in the representation **20**. The resulting realization of $\mathfrak{e}_{6(2)}$ is again that of the minimal unitary representation. Because some of the oscillators were set equal to zero in the truncation, they do not contribute to the value of the quadratic Casimir of the algebra, the c-number to which it reduces is now different

$$\mathcal{C}_2\left(\mathfrak{e}_{6(2)}\right) = \left(I_4 - \frac{35}{16}\right) + \left(\frac{1}{3} I_4 - \frac{1}{16}\right) + \left(-\frac{4}{3} I_4 - \frac{15}{4}\right) = -6 \tag{4.66}$$

4.3.3 Truncation to the minimal unitary realization of $\mathfrak{e}_{6(-14)}$ as a quasiconformal subalgebra

Quasiconformal realization of another real form of $\mathfrak{e}_6$, namely $\mathfrak{e}_{6(-14)}$ with the maximal compact subalgebra $\mathfrak{so}(10) \oplus \mathfrak{so}(2)$, can also be obtained by truncation of $\mathfrak{e}_{7(-5)}$. Its five-graded decomposition reads as follows

$$\mathfrak{e}_{6(-14)} = \mathbf{1} \oplus \mathbf{20} \oplus (\mathfrak{su}(5,1) \oplus \Delta) \oplus \mathbf{20} \oplus \mathbf{1}$$

In order to implement this truncation we observe the following chain of inclusions

$$\mathfrak{su}(5,1) \subset \mathfrak{so}(10,2) \subset \mathfrak{e}_{7(-25)}$$

Subalgebra $\mathfrak{so}(10,2)$ is centralised by $\mathfrak{su}(1,1)$ while $\mathfrak{su}(5,1)$ is centralised by $\mathfrak{su}(2,1)$ within $\mathfrak{e}_{7(-25)}$, suggesting that we only keep oscillators Z^{ab} and $\tilde{Z}_{ab}$ with indexes now running from 1 to 5 as follows from $\mathfrak{u}(5) \subset \mathfrak{su}(5,1)$ branching of $\mathbf{20} = \mathbf{10} \oplus \overline{\mathbf{10}}$. Then generators of $\mathfrak{su}(5,1)$ are given as follows

$$\begin{aligned} J^a{}_b &= 2Z^{ac}\tilde{Z}_{bc} - \frac{2}{5}\delta^a{}_b Z^{cd}\tilde{Z}_{cd} \qquad H = \frac{1}{24}\left(Z^{ab}\tilde{Z}_{ab} + \tilde{Z}_{ab}Z^{ab}\right) \\ J^a &= -\frac{1}{48}\epsilon^{abcde}\tilde{Z}_{bc}\tilde{Z}_{de} \qquad \tilde{J}_a = +\frac{1}{48}\epsilon_{abcde}Z^{bc}Z^{de} \end{aligned} \tag{4.67}$$

with commutation relations

$$\begin{aligned} &[J^a{}_b\,, J^c{}_d] = \delta^c{}_b J^a{}_d - \delta^d{}_a J^c{}_b \qquad [J^a{}_b\,, J^c] = \delta^c{}_b J^a - \frac{1}{5}\delta^a{}_b J^c \\ &\left[J^a{}_b\,, \tilde{J}_c\right] = -\delta^a{}_c\tilde{J}_b + \frac{1}{5}\delta^a{}_b\tilde{J}_c \quad [H, J^a] = -\frac{1}{6}J^a \quad \left[H, \tilde{J}_a\right] = +\frac{1}{6}\tilde{J}_a \\ &\left[J^a\,, \tilde{J}_b\right] = \frac{1}{144}J^a{}_b - \frac{1}{20}\delta^a{}_b H \qquad \left[J^a\,, J^b\right] = 0 \qquad \left[\tilde{J}_a\,, \tilde{J}_b\right] = 0 \end{aligned} \tag{4.68}$$

resulting in the following Casimir

$$\begin{aligned} \mathcal{C}_2\left(\mathfrak{su}(5,1)\right) &= \frac{1}{6}J^a{}_b J^b{}_a + \frac{36}{5}H^2 + 24\left(J^a\tilde{J}_a + \tilde{J}_a J^a\right) = I_4 - \frac{35}{16} \\ &= \frac{1}{2}\left(\tilde{Z}_{ab}Z^{bc}\tilde{Z}_{cd}Z^{da} + Z^{ab}\tilde{Z}_{bc}Z^{cd}\tilde{Z}_{da}\right) \\ &- \frac{1}{8}\left(\tilde{Z}_{ab}Z^{ab}\tilde{Z}_{cd}Z^{cd} + Z^{ab}\tilde{Z}_{ab}Z^{cd}\tilde{Z}_{cd}\right) + \frac{5}{4} \end{aligned} \tag{4.69}$$

Remaining generators of $\mathfrak{e}_{6(-14)}$ and their commutators straightforwardly follow from those of $\mathfrak{e}_{7(-5)}$. We shall only present the c-number to which the quadratic Casimir of $\mathfrak{e}_{6(-14)}$ reduces upon evaluation on the resulting minimal unitary realization

$$\mathcal{C}_2\left(\mathfrak{e}_{6(-14)}\right) = \left(I_4 - \frac{35}{16}\right) + \left(\frac{1}{3}I_4 - \frac{1}{16}\right) + \left(-\frac{4}{3}I_4 - \frac{15}{4}\right) = -6 \tag{4.70}$$

4.3.4 Truncation to the minimal unitary realization of $\mathfrak{f}_{4(4)}$ as a quasiconformal subalgebra

The realization of the Lie algebra $\mathfrak{e}_{6(2)}$ given above can be further truncated to obtain the minimal unitary realization of the Lie algebra $\mathfrak{f}_{4(4)}$ with the maximal compact subalgebra $\mathfrak{usp}(6) \oplus \mathfrak{usp}(2)$. The five graded structure of $\mathfrak{f}_{4(4)}$ as a quasiconformal algebra reads as follows

$$\mathbf{52} = \mathfrak{f}_{4(4)} = \mathbf{1} \oplus \mathbf{14} \oplus \left(\mathfrak{sp}\left(6\,, \mathbb{R}\right) \oplus \Delta\right) \oplus \mathbf{14} \oplus \mathbf{1} \tag{4.71}$$

One way to obtain the truncation of $\mathfrak{e}_{6(2)}$ to $\mathfrak{f}_{4(4)}$ is suggested by $\mathfrak{u}(3) \subset \mathfrak{sp}\left(6,\, \mathbb{R}\right)$ branching of $\mathbf{14} = \mathbf{1} \oplus \mathbf{6} \oplus \overline{\mathbf{6}} \oplus \mathbf{1}$. It amounts to identifying the two $\mathfrak{su}(3)$ subalgebra of $\mathfrak{su}(3,3) \subset \mathfrak{e}_{6(2)}$

and discarding the antisymmetric components $Z^{[ab]}$ of Z^{ab} .

Let us define the symmetric tensor oscillators $S^{ac} = Z^{(ac)}$ and $\tilde{S}_{ac} = \tilde{Z}_{(ac)}$, $a, b, ... = 1, 2, 3$, which correspond to independent oscillators left after the identification. They satisfy the following commutation relations

$$\left[\tilde{S}_{cd}\,, S^{ab}\right] = \frac{1}{4}\left(\delta^a{}_c\delta^b{}_d + \delta^b{}_c\delta^a{}_d\right) \tag{4.72}$$

With these oscillators we build generators of $\mathfrak{sp}\,(6\,,\mathbb{R})$:

$$\begin{aligned}
J^a{}_b &= 2S^{ac}\tilde{S}_{bc} - \frac{2}{3}\delta^a{}_b S^{cd}\tilde{S}_{cd} \\
H &= -\frac{1}{4}\left(Z^{78}\tilde{Z}_{78} + \tilde{Z}_{78}Z^{78}\right) + \frac{1}{12}\left(S^{ab}\tilde{S}_{ab} + \tilde{S}_{ab}S^{ab}\right) \\
J^{ab} &= \frac{1}{6}S^{ab}Z^{78} + \frac{1}{12}\epsilon^{acd}\epsilon^{bef}\tilde{S}_{ce}\tilde{S}_{df} \\
\tilde{J}_{ab} &= -\frac{1}{6}\tilde{S}_{ab}\tilde{Z}_{78} - \frac{1}{12}\epsilon_{acd}\epsilon_{bef}S^{ce}S^{df}
\end{aligned} \tag{4.73}$$

satisfying the following commutation relations

$$\begin{aligned}
&\left[J^a{}_b\,, J^{cd}\right] = \delta^{(c}{}_b J^{d)a} - \frac{1}{3}\delta^a{}_b J^{cd} && [J^a{}_b\,, J^c{}_d] = \delta^c{}_b J^a{}_d - \delta^d{}_a J^c{}_b \\
&\left[J^a{}_b\,, \tilde{J}_{cd}\right] = -\delta^a{}_{(c}\tilde{J}_{d)b} + \frac{1}{3}\delta^a{}_b\tilde{J}_{cd} && \left[H\,, J^{ab}\right] = -\frac{1}{6}J^{ab} \\
&\left[J^{ab}\,, \tilde{J}_{cd}\right] = \frac{1}{72}\left(\delta^{(a}{}_{(c}J^{b)}{}_{d)} - 2\delta^{(a}{}_{(c}\delta^{b)}{}_{d)}H\right) && \left[H\,, \tilde{J}_{ab}\right] = \frac{1}{6}\tilde{J}_{ab}
\end{aligned} \tag{4.74}$$

The quadratic Casimir of $\mathfrak{sp}(6, \mathbb{R})$ is then given by

$$\begin{aligned}
\mathcal{C}_2\left(\mathfrak{sp}(6,\mathbb{R})\right) &= \frac{1}{3}\,J^a{}_b J^b{}_a + 4\,H^2 + 24\left(J^{ab}\tilde{J}_{ab} + \tilde{J}_{ab}J^{ab}\right) = I_4 - \frac{15}{16} \\
&= \left(\tilde{S}_{ab}S^{bc}\tilde{S}_{cd}S^{da} + S^{ab}\tilde{S}_{bc}S^{cd}\tilde{S}_{da}\right) \\
&+ \frac{1}{2}\left(Z^{78}\tilde{Z}_{78}Z^{78}\tilde{Z}_{78} + \tilde{Z}_{78}Z^{78}\tilde{Z}_{78}Z^{78}\right) + \\
&- \frac{1}{2}\left(\tilde{S}_{ab}S^{ab}\tilde{S}_{cd}S^{cd} + S^{ab}\tilde{S}_{ab}S^{cd}\tilde{S}_{cd}\right) + \frac{7}{16} \\
&- \frac{1}{2}\left(S^{ab}\tilde{S}_{ab}Z^{78}\tilde{Z}_{78} + Z^{78}\tilde{Z}_{78}S^{ab}\tilde{S}_{ab}\right) \\
&- \frac{1}{2}\left(\tilde{S}_{ab}S^{ab}\tilde{Z}_{78}Z^{78} + \tilde{Z}_{78}Z^{78}\tilde{S}_{ab}S^{ab}\right) \\
&- \frac{2}{3}\epsilon_{abc}\epsilon_{def}S^{ad}S^{be}S^{cf}Z^{78} - \frac{2}{3}\epsilon^{abc}\epsilon^{def}\tilde{S}_{ad}\tilde{S}_{be}\tilde{S}_{cf}\tilde{Z}_{78}
\end{aligned} \tag{4.75}$$

Negative grade generators are defined as

$$E = \frac{1}{2}y \quad E^{ab} = yS^{ab} \quad E^{+} = yZ^{78} \quad \tilde{E}_{ab} = y\tilde{S}_{ab} \quad E_{-} = y\tilde{Z}_{78} \tag{4.76}$$

They satisfy commutation relations, different from those of negative grade generators of $\mathfrak{e}_{7(-5)}$

$$\left[\tilde{E}_{ab}\,,E^{cd}\right]=\delta^{(c}{}_{(b}\delta^{b)}{}_{d)}E \tag{4.77}$$

reflecting that S^{ab} and $\tilde{S}_{ab}$ are now symmetric tensor oscillators. Positive grade generators, and their commutator, are given by the following equations

$$\begin{aligned} F&=\frac{1}{2}p^2+2iy^{-2}I_4\\ F^{ab}&=-pS^{ab}+2iy^{-1}\left[S^{ab}\,,I_4\right] \qquad \left[\tilde{F}_{ab}\,,F^{cd}\right]=\delta^{(c}{}_{(b}\delta^{b)}{}_{d)}F\\ \tilde{F}_{ab}&=-p\tilde{S}_{ab}+2iy^{-1}\left[\tilde{S}_{ab}\,,I_4\right] \end{aligned} \tag{4.78}$$

Quadratic Casimir of the resulting minimal unitary realization of $\mathfrak{f}_{4(4)}$

$$\begin{aligned} \mathcal{C}_2\left(\mathfrak{f}_{4(4)}\right)=&\,\mathcal{C}_2\left(\mathfrak{sp}(6,\mathbb{R})\right)+\frac{1}{12}\Delta^2+\frac{1}{6}\left(FE+EF\right)\\ &+i\left(\tilde{E}_{ab}F^{ab}+F^{ab}\tilde{E}_{ab}-\tilde{F}_{ab}E^{ab}-E^{ab}\tilde{F}_{ab}\right)\\ &-\frac{i}{6}\left(\tilde{E}_{78}F^{78}+F^{78}\tilde{E}_{78}-\tilde{F}_{78}E^{78}-E^{78}\tilde{F}_{78}\right) \end{aligned} \tag{4.79}$$

reduces to a c-number

$$\mathcal{C}_2\left(\mathfrak{f}_{4(4)}\right)=\left(I_4-\frac{15}{16}\right)+\left(\frac{1}{3}I_4-\frac{1}{16}\right)+\left(-\frac{4}{3}I_4-\frac{9}{4}\right)=-\frac{13}{4} \tag{4.80}$$

in agreement with parental algebras and as required by irreducibility.

4.3.5 Truncation to the minimal unitary realization of $\mathfrak{so}\,(4,4)$ as a quasiconformal subalgebra

We further truncate $\mathfrak{f}_{4(4)}$ to obtain the minimal unitary realization of $\mathfrak{so}\,(4,4)$ which has the following 5-graded decomposition

$$\mathbf{28}=\mathbf{1}\oplus(\mathbf{2},\mathbf{2},\mathbf{2})\oplus(\mathfrak{sp}\,(2,\,\mathbb{R})\oplus\mathfrak{sp}\,(2,\,\mathbb{R})\oplus\mathfrak{sp}\,(2,\,\mathbb{R})\oplus\Delta)\oplus(\mathbf{2},\mathbf{2},\mathbf{2})\oplus\mathbf{1} \tag{4.81}$$

This truncation is achieved by restricting S^{ab} and $\tilde{S}_{ab}$ operators to their diagonal components

$$S^{ab}=\delta^{ab}S^a \qquad \tilde{S}_{ab}=\delta_{ab}\tilde{S}_a \qquad \left[\tilde{S}_a,\,S^b\right]=\frac{1}{2}\delta^b{}_a \tag{4.82}$$

where $a, b, .. = 1, 2, 3$, and discarding the off-diagonal oscillators. Three copies of $\mathfrak{sp}\,(2,\,\mathbb{R})$ are generated by

$$\begin{aligned} J^a_- &= -\frac{1}{6}\tilde{S}_a\tilde{Z}_{78} - \frac{1}{12}\epsilon_{abc}S^bS^c \\ J^a_+ &= \frac{1}{6}S^aZ^{78} + \frac{1}{12}\epsilon^{abc}\tilde{S}_b\tilde{S}_c \end{aligned} \qquad J^a_0 = Z^{78}\tilde{Z}_{78} + \sum_{b=1}^{3}\left(2\delta^{ab}-1\right)S^b\tilde{S}_b \tag{4.83}$$

The quadratic Casimir of $\mathfrak{sp}\,(2,\,\mathbb{R}) \oplus \mathfrak{sp}\,(2,\,\mathbb{R}) \oplus \mathfrak{sp}\,(2,\,\mathbb{R})$ then reads

$$\begin{aligned} &\mathcal{C}_2\left(\mathfrak{sp}\,(2,\,\mathbb{R}) \oplus \mathfrak{sp}\,(2,\,\mathbb{R}) \oplus \mathfrak{sp}\,(2,\,\mathbb{R})\right) = \sum_{a=1}^{3}\left[\frac{1}{3}J^a_0J^a_0 + 24\left(J^a_-J^a_+ + J^a_+J^a_-\right)\right] \\ &= \sum_{a=1}^{3}\left(\left(\tilde{S}_aS^a\right)^2 + \left(S^a\tilde{S}_a\right)^2\right) + \left(Z^{78}\tilde{Z}_{78}\right)^2 + \left(\tilde{Z}_{78}Z^{78}\right)^2 \\ &- \frac{1}{2}\left(\sum_{a=1}^{3}S^a\tilde{S}_a + Z^{78}\tilde{Z}_{78}\right)^2 - \frac{1}{2}\left(\sum_{a=1}^{3}\tilde{S}_aS^a + \tilde{Z}_{78}Z^{78}\right)^2 \\ &- 4\,S^1\,S^2\,S^3\,Z^{78} - 4\,\tilde{S}_1\,\tilde{S}_2\,\tilde{S}_3\,\tilde{Z}_{78} = I_4 - \frac{3}{16} \end{aligned} \tag{4.84}$$

The commutation relations of the generators in $\mathfrak{g}^0$ are

$$\left[J^a_0,\,J^b_\pm\right] = \pm\delta^{ab}J^a_\pm \qquad \left[J^a_+,\,J^b_-\right] = \frac{1}{72}\,\delta^{ab}J^a_0 \tag{4.85}$$

Negative grade generators are

$$E = \frac{1}{2}y^2 \qquad E^a = yS^a \qquad \tilde{E}_a = y\tilde{S}_a \qquad E^{78} = yZ^{78} \qquad \tilde{E}_{78} = y\tilde{Z}_{78} \tag{4.86}$$

and positive grade generators are

$$F = \frac{1}{2}p^2 + 2iy^{-2}I_4 \qquad \begin{aligned} F^a &= -pS^a + 2iy^{-1}\left[S^a,\,I_4\right] \\ F^{78} &= -pZ^{78} + 2iy^{-1}\left[Z^{78},\,I_4\right] \\ \tilde{F}_a &= -p\tilde{S}_a + 2iy^{-1}\left[\tilde{S}_a,\,I_4\right] \\ \tilde{F}_{78} &= -p\tilde{Z}_{78} + 2iy^{-1}\left[\tilde{Z}_{78},\,I_4\right] \end{aligned} \tag{4.87}$$

The quadratic Casimir of $\mathfrak{so}\,(4,4)$

$$\begin{aligned} \mathcal{C}_2\left(\mathfrak{so}\,(4,4)\right) = \mathcal{C}_2\left(\mathfrak{g}^0\right) &+ \frac{1}{12}\Delta^2 + \frac{1}{6}\left(FE + EF\right) \\ &+ \frac{i}{6}\left(E^a\tilde{F}_a + \tilde{F}_aE^a - \tilde{E}_aF^a - F^a\tilde{E}_a\right) \\ &+ \frac{i}{6}\left(E^{78}\tilde{F}_{78} + \tilde{F}_{78}E^{78} - F^{78}\tilde{E}_{78} - \tilde{E}_{78}F^{78}\right) \end{aligned} \tag{4.88}$$

reduces to c-number as before

$$\mathcal{C}_2\left(\mathfrak{so}\left(4,4\right)\right)=\left(I_4-\frac{3}{16}\right)+\left(\frac{1}{3}\,I_4-\frac{1}{16}\right)+\left(-\frac{4}{3}I_4-\frac{13}{12}\right)=-\frac{4}{3} \tag{4.89}$$

4.4 Truncation to the minimal unitary realization of $\mathfrak{e}_{7(-25)}$ as a quasiconformal subalgebra

The group $\mathrm{E}_{7(-25)}$ has the maximal compact subgroup $E_6\times U(1)$ and arises as the U-duality group of exceptional $\mathcal{N}=2$ Maxwell-Einstein supergravity in $d=4$ whose scalar manifold is $\mathrm{E}_{7(-25)}/\left(E_6\times U(1)\right)$. Its action on the 27 complex scalar fields can be represented as a generalised conformal group [40, 43, 45]. As a generalised conformal group its Lie algebra has a natural 3-graded structure

$$\mathfrak{e}_{7(-25)}=\overline{27}\oplus\left(\mathfrak{e}_{6(-26)}\oplus\mathfrak{so}(1,1)\right)\oplus 27$$

The quasiconformal realization of $E_{8(-24)}$ can be truncated to the conformal realization of $E_{7(-25)}$ in essentially two different ways.

In this section we will however consider a different truncation of $E_{8(-24)}$ such that the resulting realization of $E_{7(-25)}$ is quasiconformal corresponding to its minimal unitary representation.

Just as the subalgebra $\mathfrak{e}_{7(-5)}$ is normalised by $\mathfrak{su}(2)\subset\mathfrak{su}(6,2)\subset\mathfrak{g}^0=\mathfrak{e}_{7(-25)}$, the subalgebra $\mathfrak{e}_{7(-25)}$ is normalised by $\mathfrak{su}(1,1)\subset\mathfrak{su}(6,2)\subset\mathfrak{g}^0\mathfrak{e}_{7(-25)}$ within $\mathfrak{e}_{8(-24)}$. Similarly to $\mathfrak{e}_{7(-5)}$ we obtain

$$\mathfrak{e}_{7(-25)}=\mathbf{133}=\mathbf{1}\oplus\mathbf{32}\oplus\left(\mathfrak{so}(10,2)\oplus\Delta\right)\oplus\mathbf{32}\oplus\mathbf{1} \tag{4.90}$$

We identify the $\mathfrak{su}(1,1)$ in question with the one generated by $J^6{}_7$, $J^7{}_6$ and $J^6{}_6-J^7{}_7$ generators of $\mathfrak{su}(6,2)\subset\mathfrak{e}_{7(-25)}\subset\mathfrak{e}_{8(-24)}$. The truncation will then amount to setting $Z^{a6}=Z^{6a}=0$ where $a\neq 7$, as well as $Z^{a7}=Z^{7a}=0$ for $a\neq 6$. Let us relabel coefficients and introduce $\dot{a}=1,\ldots,5,8$. Then $\mathfrak{su}(5,1)$ is generated by

$$J^{\dot{a}}{}_{\dot{b}}=2Z^{\dot{a}\dot{c}}\tilde{Z}_{\dot{b}\dot{c}}-\frac{1}{3}\delta^{\dot{a}}{}_{\dot{b}}Z^{\dot{d}\dot{c}}\tilde{Z}_{\dot{d}\dot{c}} \tag{4.91}$$

The other generators of $\mathfrak{so}(10,2)$ are then given as follows

$$\begin{aligned} U &= \frac{3}{2}\left(Z^{67}\tilde{Z}_{67}+\tilde{Z}_{67}Z^{67}\right)-\frac{1}{4}\left(Z^{\dot{a}\dot{b}}\tilde{Z}_{\dot{a}\dot{b}}+\tilde{Z}_{\dot{a}\dot{b}}Z^{\dot{a}\dot{b}}\right) \\ J^{-}_{\dot{a}\dot{b}} &= -\frac{1}{6}\tilde{Z}_{\dot{a}\dot{b}}\tilde{Z}_{67}+\frac{1}{48}\epsilon_{\dot{a}\dot{b}\dot{c}\dot{d}\dot{e}\dot{f}}Z^{\dot{c}\dot{d}}Z^{\dot{e}\dot{f}} \\ J^{\dot{a}\dot{b}}_{+} &= \frac{1}{6}Z^{\dot{a}\dot{b}}Z^{67}-\frac{1}{48}\epsilon^{\dot{a}\dot{b}\dot{c}\dot{d}\dot{e}\dot{f}}\tilde{Z}_{\dot{c}\dot{d}}\tilde{Z}_{\dot{e}\dot{f}} \end{aligned} \tag{4.92}$$

satisfying the following Hermiticity condition

$$\left(J^{\dot{a}}{}_{\dot{b}}\right)^{\dagger}=\eta^{\dot{a}\dot{c}}\eta_{\dot{b}\dot{d}}J^{\dot{d}}{}_{\dot{c}} \qquad U^{\dagger}=U \qquad \left(J^{-}_{\dot{a}\dot{b}}\right)^{\dagger}=J^{\dot{c}\dot{d}}_{+}\eta_{\dot{a}\dot{c}}\eta_{\dot{b}\dot{d}} \tag{4.93}$$

where $\eta_{\dot{a}\dot{b}}=\mathrm{diag}\,(+1,+1,+1,+1,+1,-1)$; and the commutation relations read as follows

$$\begin{aligned} \left[J^{\dot{a}}{}_{\dot{b}},\,J^{\dot{c}}{}_{\dot{d}}\right] &= \delta^{\dot{c}}{}_{\dot{b}}J^{\dot{a}}{}_{\dot{d}}-\delta^{\dot{a}}{}_{\dot{d}}J^{\dot{c}}{}_{\dot{b}} \\ \left[J^{\dot{a}}{}_{\dot{b}},\,J^{\dot{c}\dot{d}}_{+}\right] &= \delta^{\dot{c}}{}_{\dot{b}}J^{\dot{a}\dot{d}}_{+}+\delta^{\dot{d}}{}_{\dot{b}}J^{\dot{c}\dot{a}}_{+}-\frac{1}{3}\delta^{\dot{a}}{}_{\dot{b}}J^{\dot{c}\dot{d}}_{+} \\ \left[J^{\dot{a}}{}_{\dot{b}},\,J^{-}_{\dot{c}\dot{d}}\right] &= -\delta^{\dot{a}}{}_{\dot{c}}J^{-}_{\dot{b}\dot{d}}-\delta^{\dot{a}}{}_{\dot{d}}J^{-}_{\dot{c}\dot{b}}+\frac{1}{3}\delta^{\dot{a}}{}_{\dot{b}}J^{-}_{\dot{c}\dot{d}} \\ \left[U,\,J^{-}_{\dot{c}\dot{d}}\right] &= -J^{-}_{\dot{c}\dot{d}} \qquad \left[U,\,J^{\dot{c}\dot{d}}_{+}\right]=+J^{\dot{c}\dot{d}}_{+} \qquad \left[U,\,J^{\dot{c}}{}_{\dot{d}}\right]=0 \end{aligned} \tag{4.94}$$

The quadratic Casimir of the algebra reads

$$\mathcal{C}_2\left(\mathfrak{so}\,(10,2)\right)=\frac{1}{6}J^{\dot{a}}{}_{\dot{b}}J^{\dot{b}}{}_{\dot{a}}+\frac{1}{9}U^2+12\left(J^{\dot{a}\dot{b}}_{+}J^{-}_{\dot{a}\dot{b}}+J^{-}_{\dot{a}\dot{b}}J^{\dot{a}\dot{b}}_{+}\right)=I_4-\frac{99}{16} \tag{4.95}$$

Definition of the grade ±1 generators goes along the same lines as for $\mathfrak{e}_{7(-5)}$ so we omit them here. Let us only note that the quadratic Casimir of the minimal unitary realization of $\mathfrak{e}_{7(-25)}$ takes on the same value as that of $\mathfrak{e}_{7(-5)}$ and equals to -14.

4.5 Minimal unitary realizations of the quasiconformal groups $\mathrm{SO}(d+2,4)$

The minimal unitary realization of the groups $SO\,(d+2,4)$, that were given in [50] corresponding to the quantisation of their geometric realizations as quasiconformal groups given in chapter 3 following methods of [46, 48]. Let X^{μ} and P_{μ} be canonical coordinates and momenta in $\mathbb{R}^{(2,d)}$:

$$[X^{\mu},P_{\nu}]=i\delta^{\mu}_{\nu} \tag{4.96}$$

In the earlier notation we identify $X^{\mu,a=1}$ to be coordinates X^{μ}, and $P_{\mu}=\eta_{\mu\nu}X^{\nu,a=2}$ to be conjugate momenta. Also let x be an additional "cocycle" coordinate and p be its conjugate

momentum:

$$[x, p] = i \tag{4.97}$$

The grade zero generators $(M_{\mu\nu}, J_{\pm}, J_0)$, grade -1 generators (U_μ, V^μ), grade -2 generator K_- and the 4-th order invariant $\mathcal{I}_4$ of the semi-simple part of the grade zero subalgebra are realized as follows:

$$\begin{aligned}
M_{\mu\nu} &= i\eta_{\mu\rho}X^\rho P_\nu - i\eta_{\nu\rho}X^\rho P_\mu & J_0 &= \frac{1}{2}\left(X^\mu P_\mu + P_\mu X^\mu\right) \\
U_\mu &= xP_\mu \qquad V^\mu = xX^\mu & J_- &= X^\mu X^\nu \eta_{\mu\nu} \\
K_- &= \frac{1}{2}x^2 & J_+ &= P_\mu P_\nu \eta^{\mu\nu} \\
\mathcal{I}_4 &= \left(X^\mu X^\nu \eta_{\mu\nu}\right)\left(P_\mu P_\nu \eta^{\mu\nu}\right) + \left(P_\mu P_\nu \eta^{\mu\nu}\right)\left(X^\mu X^\nu \eta_{\mu\nu}\right) \\
&\quad - \left(X^\mu P_\mu\right)\left(P_\nu X^\nu\right) - \left(P_\mu X^\mu\right)\left(X^\nu P_\nu\right)
\end{aligned} \tag{4.98}$$

Using the quartic invariant we define the grade $+2$ generator as

$$K_+ = \frac{1}{2}p^2 + \frac{1}{4\,y^2}\left(\mathcal{I}_4 + \frac{d^2+3}{2}\right) \tag{4.99}$$

It is easy to verify that the generators $M_{\mu\nu}$ and $J_{0,\pm}$ satisfy the commutation relations of $\mathfrak{so}\left(d,2\right) \oplus \mathfrak{sp}\left(2,\mathbb{R}\right)$

$$\begin{aligned}
\left[M_{\mu\nu}, M_{\rho\tau}\right] &= \eta_{\nu\rho}M_{\mu\tau} - \eta_{\mu\rho}M_{\nu\tau} + \eta_{\mu\tau}M_{\nu\rho} - \eta_{\nu\tau}M_{\mu\rho} \\
\left[J_0, J_\pm\right] &= \pm 2iJ_\pm \qquad \left[J_-, J_+\right] = 4iJ_0
\end{aligned} \tag{4.100}$$

under which coordinates X^μ and momenta P^μ transform as Lorentz vectors and form doublets of the symplectic group $Sp(2,\mathbb{R})$:

$$\begin{aligned}
\left[J_0, V^\mu\right] &= -iV^\mu \quad \left[J_-, V^\mu\right] = 0 \quad\;\; \left[J_+, V^\mu\right] = -2i\eta^{\mu\nu}U_\nu \\
\left[J_0, U_\mu\right] &= +iU_\mu \quad \left[J_-, U_\mu\right] = 2i\eta_{\mu\nu}V^\nu \quad \left[J_+, U_\mu\right] = 0
\end{aligned} \tag{4.101}$$

The generators in the subspace $\mathfrak{g}^{-1} \oplus \mathfrak{g}^{-2}$ form a Heisenberg algebra

$$\left[V^\mu, U_\nu\right] = 2i\delta^\mu{}_\nu\, K_- \,. \tag{4.102}$$

Define the grade $+1$ generators as

$$\tilde{V}^\mu = -i\left[V^\mu, K_+\right] \qquad \tilde{U}_\mu = -i\left[U_\mu, K_+\right] \tag{4.103}$$

which explicitly read as follows

$$\begin{aligned}\tilde{V}^{\mu} &= pX^{\mu} + \frac{1}{2}x^{-1}\left(P_{\nu}X^{\lambda}X^{\rho} + X^{\lambda}X^{\rho}P_{\nu}\right)\eta^{\mu\nu}\eta_{\lambda\rho} \\ &\quad - \frac{1}{4}x^{-1}\left(X^{\mu}\left(X^{\nu}P_{\nu} + P_{\nu}X^{\nu}\right) + \left(X^{\nu}P_{\nu} + P_{\nu}X^{\nu}\right)X^{\mu}\right) \\ \tilde{U}_{\mu} &= pP_{\mu} - \frac{1}{2}x^{-1}\left(X^{\nu}P_{\lambda}P_{\rho} + P_{\lambda}P_{\rho}X^{\nu}\right)\eta_{\mu\nu}\eta^{\lambda\rho} \\ &\quad + \frac{1}{4}x^{-1}\left(P_{\mu}\left(X^{\nu}P_{\nu} + P_{\nu}X^{\nu}\right) + \left(X^{\nu}P_{\nu} + P_{\nu}X^{\nu}\right)P_{\mu}\right).\end{aligned} \tag{4.104}$$

Then one finds that the generators in $\mathfrak{g}^{+1} \oplus \mathfrak{g}^{+2}$ subspace form an isomorphic Heisenberg algebra

$$\left[\tilde{V}^{\mu}, \tilde{U}_{\nu}\right] = 2i\delta^{\mu}{}_{\nu}K_{+} \qquad V^{\mu} = i\left[\tilde{V}^{\mu}, K_{-}\right] \qquad U_{\mu} = i\left[\tilde{U}_{\mu}, K_{-}\right]. \tag{4.105}$$

Commutators $\left[\mathfrak{g}^{-1}, \mathfrak{g}^{+1}\right]$ close into $\mathfrak{g}^{0}$ as follows

$$\begin{aligned}\left[U_{\mu}, \tilde{U}_{\nu}\right] &= i\eta_{\mu\nu}J_{-} \qquad \left[V^{\mu}, \tilde{V}^{\nu}\right] = i\eta^{\mu\nu}J_{+} \\ \left[V^{\mu}, \tilde{U}_{\nu}\right] &= 2\eta^{\mu\rho}M_{\rho\nu} + i\delta^{\mu}{}_{\nu}\left(J_{0} + \Delta\right) \\ \left[U_{\mu}, \tilde{V}^{\nu}\right] &= -2\eta^{\nu\rho}M_{\mu\rho} + i\delta^{\nu}{}_{\mu}\left(J_{0} - \Delta\right)\end{aligned} \tag{4.106}$$

where Δ is the generator that determines the 5-grading

$$\Delta = \frac{1}{2}\left(xp + px\right) \tag{4.107}$$

such that

$$[K_{-}, K_{+}] = i\Delta \qquad [\Delta, K_{\pm}] = \pm 2iK_{\pm} \tag{4.108}$$

$$[\Delta, U_{\mu}] = -iU_{\mu} \quad [\Delta, V^{\mu}] = -iV^{\mu} \quad \left[\Delta, \tilde{U}_{\mu}\right] = i\tilde{U}_{\mu} \quad \left[\Delta, \tilde{V}^{\mu}\right] = i\tilde{V}^{\mu} \tag{4.109}$$

The quadratic Casimir operators of subalgebras $\mathfrak{so}\left(d,2\right)$, $\mathfrak{sp}\left(2,\mathbb{R}\right)_{J}$ of grade zero subspace and $\mathfrak{sp}\left(2,\mathbb{R}\right)_{K}$ generated by $K_{\pm}$ and Δ are

$$\begin{aligned}M_{\mu\nu}M^{\mu\nu} &= -\mathcal{I}_{4} - 2\left(d+2\right) \\ J_{-}J_{+} + J_{+}J_{-} - 2\left(J_{0}\right)^{2} &= \mathcal{I}_{4} + \frac{1}{2}\left(d+2\right)^{2} \\ K_{-}K_{+} + K_{+}K_{-} - \frac{1}{2}\Delta^{2} &= \frac{1}{4}\mathcal{I}_{4} + \frac{1}{8}\left(d+2\right)^{2}\end{aligned} \tag{4.110}$$

Note that they all reduce to $\mathcal{I}_{4}$ modulo some additive constants. Noting also that

$$\left(U_{\mu}\tilde{V}^{\mu} + \tilde{V}^{\mu}U_{\mu} - V^{\mu}\tilde{U}_{\mu} - \tilde{U}_{\mu}V^{\mu}\right) = 2\mathcal{I}_{4} + \left(d+2\right)\left(d+6\right) \tag{4.111}$$

we conclude that there exists a family of degree 2 polynomials in the enveloping algebra of $\mathfrak{so}\left(d+2,4\right)$ that degenerate to a c-number for the minimal unitary realization, in accordance with Joseph's theorem [59]:

$$\begin{aligned} M_{\mu\nu}M^{\mu\nu} + \kappa_1\left(J_-J_+ + J_+J_- - 2\left(J_0\right)^2\right) + 4\kappa_2\left(K_-K_+ + K_+K_- - \frac{1}{2}\Delta^2\right) \\ - \frac{1}{2}\left(\kappa_1+\kappa_2-1\right)\left(U_\mu\tilde{V}^\mu + \tilde{V}^\mu U_\mu - V^\mu\tilde{U}_\mu - \tilde{U}_\mu V^\mu\right) \\ = \frac{1}{2}\left(d+2\right)\left(d+2-4\left(\kappa_1+\kappa_2\right)\right) \end{aligned} \tag{4.112}$$

The quadratic Casimir of $\mathfrak{so}\left(d+2,4\right)$ corresponds to the choice $2\kappa_1 = 2\kappa_2 = -1$ in (4.112). Hence the eigenvalue of the quadratic Casimir for the minimal unitary representation is equal to $\frac{1}{2}\left(d+2\right)\left(d+6\right)$.

4.6 Minimal realizations as non-compact groups and conformal quantum mechanics

In this and the next sections we outline the ongoing work on the unified construction of minimal unitary realizations of non-compact groups and non-compact supergroups [51].

In this section we explore possibilities of extensions of conformal quantum mechanics in the way inspired by minimal realizations of U-duality algebras, so that role of $g\left(g+1\right)$ in (4.2) is played by a quartic polynomial built out of Weyl algebra elements (i.e. coordinates and momentums or oscillators).

We shall restrict ourselves to such algebras where grade zero is a direct sum of a simple algebra $\mathfrak{g}^0$ and $\mathfrak{so}\left(1,1\right)$ generator Δ:

$$\mathfrak{g} = E \oplus E^\alpha \oplus \left(\mathfrak{g}^0 \oplus \Delta\right) \oplus F^\alpha \oplus F \tag{4.113}$$

Let J^a denote generators of $\mathfrak{g}^0$ with commutation relations

$$\left[J^a\,,J^b\right] = f^{ab}{}_c J^c \tag{4.114a}$$

and let ρ denote the symplectic representation by which $\mathfrak{g}^0$ acts on $\mathfrak{g}^{\pm1}$

$$\left[J^a\,,E^\alpha\right] = \left(\lambda^a\right)^\alpha{}_\beta E^\beta \qquad \left[J^a\,,F^\alpha\right] = \left(\lambda^a\right)^\alpha{}_\beta F^\beta \tag{4.114b}$$

where E^α, $\alpha,\beta,.. = 1,..,N = \dim(\rho)$ are generators that span $\mathfrak{g}^{-1}$

$$\left[E^\alpha\,,E^\beta\right] = 2\Omega^{\alpha\beta}E \tag{4.114c}$$

and F^α are generators that span $\mathfrak{g}^{+1}$

$$\left[F^\alpha\,,F^\beta\right] = 2\Omega^{\alpha\beta}F \tag{4.114d}$$

and $\Omega^{\alpha\beta}$ is the symplectic invariant "metric" of the representation ρ. The negative grade generators form a Heisenberg subalgebra since

$$[E^\alpha, E] = 0 \tag{4.114e}$$

with the grade -2 generator E acting as its central charge. Similarly the positive grade generators form a Heisenberg algebra with the grade +2 generator F acting as its central charge. The remaining non-vanishing commutation relations of g are

$$\begin{aligned} F^\alpha &= [E^\alpha\,,F] & [\Delta, E^\alpha] &= -E^\alpha \\ E^\alpha &= [E\,,F^\alpha] & [\Delta, F^\alpha] &= F^\alpha \\ \left[E^\alpha, F^\beta\right] &= -\Omega^{\alpha\beta}\Delta + \epsilon\lambda_a^{\alpha\beta}J^a & [\Delta, E] &= -2E \\ & & [\Delta, F] &= 2F \end{aligned} \tag{4.114f}$$

where Δ is the generator that determines the five grading and ϵ is a parameter to be determined.

Let us now realize the generators using bosonic oscillators ξ^α satisfying the canonical commutation relations

$$\left[\xi^\alpha\,,\xi^\beta\right] = \Omega^{\alpha\beta} \tag{4.115}$$

The grade -1, -2 generators and those of $\mathfrak{g}^0$ can be realized easily as

$$E = \frac{1}{2}y^2 \qquad E^\alpha = y\,\xi^\alpha \qquad J^a = -\frac{1}{2}\lambda^a{}_{\alpha\beta}\xi^\alpha\xi^\beta \tag{4.116}$$

where y, at this point, is an extra "coordinate" such that $\frac{1}{2}y^2$ acts as the central charge of the Heisenberg algebra formed by the negative grade generators. We should note that we are following the conventions of [8]. The quadratic Casimir operator of the Lie algebra $\mathfrak{g}^0$ is

$$\mathcal{C}_2\left(\mathfrak{g}^0\right) = \eta_{ab}J^aJ^b \tag{4.117}$$

where η_{ab} is the Killing metric of $\mathfrak{g}^0$. We make an Ansätz for the grade +2 generator F of the form

$$F = \frac{1}{2}p^2 + \kappa y^{-2}\left(\mathcal{C}_2 + \mathfrak{C}\right) \tag{4.118}$$

where p is the momentum conjugate to the coordinate y (c.f. (4.1)) and κ and $\mathfrak{C}$ are some

constants to be determined later. This implies then

$$F^{\alpha} = [E^{\alpha}, F] = ip\,\xi^{\alpha} + \kappa y^{-1}\,[\xi^{\alpha}\,, \mathcal{C}_2]$$
$$F^{\alpha} = ip\,\xi^{\alpha} - \kappa y^{-1}\left[2\,(\lambda^a)^{\alpha}{}_{\beta}\xi^{\beta}J_a + \,C_\rho\,\xi^{\alpha}\right] \tag{4.119}$$

where C_ρ is the eigenvalue of the second order Casimir of $\mathfrak{g}^0$ in the representation ρ.[4]

$$\lambda^a_{\alpha\beta}\lambda_a^{\beta\gamma} = -C_\rho\delta^{\gamma}_{\alpha} \tag{4.120}$$

Next, we determine unknown constants requiring that generators close into the algebra (4.114). We first consider commutators of elements of $\mathfrak{g}^1$ and $\mathfrak{g}^{-1}$. To calculate the commutators

$$\left[E^{\alpha}\,, F^{\beta}\right] = i\,(y\,p)\,\Omega^{\alpha\beta} - \xi^{\beta}\,\xi^{\alpha} + \kappa\left[\xi^{\alpha}\,, \left[\xi^{\beta}\,, \mathcal{C}_2\right]\right] \tag{4.121}$$

we use the following almost trivial result

$$[\xi^{\alpha}\,, \mathcal{C}_2] = -2\,(\lambda^a)^{\alpha}{}_{\beta}\xi^{\beta}J_a - \,C_\rho\,\xi^{\alpha} \tag{4.122}$$

which upon substitution into (4.121) yields

$$\left[E^{\alpha}\,, F^{\beta}\right] = -\Delta\Omega^{\alpha\beta} + \left\{\frac{3\kappa C_\rho}{1+N} - \frac{1}{2}\right\}\left(\xi^{\alpha}\xi^{\beta} + \xi^{\beta}\xi^{\alpha}\right) - 6\kappa\,(\lambda^a)^{\alpha\beta}\,J_a \tag{4.123}$$

where $\Delta = -\frac{i}{2}\,(yp + py)$. Hence closure requires

$$\frac{3\kappa C_\rho}{1+N} - \frac{1}{2} = 0 \tag{4.124}$$

Next, we require that $\left[\mathfrak{g}^1,\,\mathfrak{g}^1\right] = \mathfrak{g}^2$. Computation of this commutator

$$\left[F^{\alpha}\,, F^{\beta}\right] = -p^2\Omega^{\alpha\beta} + \frac{\kappa}{y^2}\left(-\xi^{\alpha}\left[\xi^{\beta}\,, \mathcal{C}_2\right] + \xi^{\beta}\,[\xi^{\alpha}\,, \mathcal{C}_2] + \kappa\left[[\xi^{\alpha}\,, \mathcal{C}_2]\,, \left[\xi^{\beta}\,, \mathcal{C}_2\right]\right]\right) \tag{4.125}$$

is slightly more involved. Doing some algebra reveals

$$-\,\xi^{\alpha}\left[\xi^{\beta}\,, \mathcal{C}_2\right] + \xi^{\beta}\,[\xi^{\alpha}\,, \mathcal{C}_2] = C_\rho\Omega^{\alpha\beta} + 2\left(\xi^{\alpha}(\lambda^a)^{\beta}{}_{\gamma} - \xi^{\beta}(\lambda^a)^{\alpha}{}_{\gamma}\right)\xi^{\gamma}J_a \tag{4.126}$$

[4] Note that the indices $\alpha, \beta, ..$ are raised and lowered with the antisymmetric symplectic metric $\Omega^{\alpha\beta} = -\Omega^{\beta\alpha}$ that satisfies $\Omega^{\alpha\beta}\Omega_{\gamma\beta} = \delta^{\alpha}_{\beta}$ and $V^{\alpha} = \Omega^{\alpha\beta}V_{\beta}$, $\quad V_{\alpha} = V^{\beta}\Omega_{\beta\alpha}$. In particular, we have $V^{\alpha}W_{\alpha} = -V_{\alpha}W^{\alpha}$.

and

$$\kappa\left[\left[\xi^{\alpha},\mathcal{C}_2\right],\left[\xi^{\beta},\mathcal{C}_2\right]\right]=\frac{12\kappa C_\rho}{N+1}\left(\xi^{\alpha}(\lambda^a)^{\beta}{}_{\gamma}-\xi^{\beta}(\lambda^a)^{\alpha}{}_{\gamma}\right)\xi^{\gamma}J_a-\kappa C_\rho^2\Omega^{\alpha\beta}$$
$$+4\kappa\left(3\left(\lambda^b\lambda^a\right)^{\alpha\beta}J_aJ_b-2\left(\lambda^b\lambda^a\right)^{\beta\alpha}J_aJ_b+f_{ab}{}^{c}(\lambda^a)^{\alpha}{}_{\mu}\left(\lambda^b\right)^{\beta}{}_{\nu}\xi^{\mu}\xi^{\nu}J_c\right) \tag{4.127}$$

Hence combining the two expressions above we get for (4.125)

$$\begin{aligned}\left[F^{\alpha},F^{\beta}\right]&=-2\left(\frac{1}{2}p^2+\frac{1}{y^2}\left(\frac{\kappa^2}{2}C_\rho^2-\frac{\kappa}{2}C_\rho\right)\right)\Omega^{\alpha\beta}\\&+\frac{4\kappa}{y^2}\left(\xi^{\alpha}(\lambda^a)^{\beta}{}_{\gamma}-\xi^{\beta}(\lambda^a)^{\alpha}{}_{\gamma}\right)\xi^{\gamma}J_a\\&+\frac{4\kappa^2}{y^2}\left(3\left(\lambda^b\lambda^a\right)^{\alpha\beta}J_aJ_b-2\left(\lambda^b\lambda^a\right)^{\beta\alpha}J_aJ_b+f_{ab}{}^{c}(\lambda^a)^{\alpha}{}_{\mu}\left(\lambda^b\right)^{\beta}{}_{\nu}\xi^{\mu}\xi^{\nu}J_c\right)\end{aligned} \tag{4.128}$$

We need to prove that the right hand side is equal to $2\Omega^{\alpha\beta}F$ with $F=\frac{1}{2}p^2+\kappa y^{-2}\left(\mathcal{C}_2+\mathfrak{C}\right)$. Therefore contracting the right hand side with $\Omega_{\beta\alpha}$ we find

$$-N\left(p^2+\frac{1}{y^2}\left(\kappa^2C_\rho^2-\kappa C_\rho\right)\right)-\frac{1}{y^2}\kappa\left(-16+20\kappa i_\rho\ell^2-4\kappa C_{\text{adj}}\right)\mathcal{C}_2 \tag{4.129}$$

where i_ρ is the Dynkin index of the representation ρ of $\mathfrak{g}^0$ and C_{adj} is the eigenvalue of the second order Casimir in the adjoint of $\mathfrak{g}^0$. To obtain this result we used the fact that

$$\lambda^a_{\alpha\beta}\lambda^{b,\alpha\beta}=-i_\rho\ell^2\eta^{\alpha\beta}$$

where ℓ is the length of the longest root of $\mathfrak{g}^0$.[5] Using the fact that

$$C_{adj}=-\ell^2h^{\vee} \tag{4.130}$$

the closure then requires[6]

$$\left(-8+10\kappa i_\rho\ell^2+2\kappa h^{\vee}\ell^2\right)=N \tag{4.131}$$

Consistency of (4.124) and (4.131), combined with

$$i_\rho\ell^2=\frac{N}{D}C_\rho \tag{4.132}$$

[5]The length squared ℓ^2 of the longest root is normalised such that it is 2 for the simply-laced algebras, and 4 for B_n, C_n and F_4 and 6 for G_2. The i_ρ, C_ρ and ℓ are related by $i_\rho=\frac{NC_\rho}{D\ell^2}$ where $D=\dim(g^0)$.

[6]$h^{\vee}$ is the dual Coxeter number of $\mathfrak{g}^0$ subalgebra of $\mathfrak{g}$.

g	D	$\check{g}$	ρ	i_ρ
$sp(2n)$	$n(2n+1)$	$n+1$	$2n$	$\frac{1}{2}$
$sl(6)$	35	6	20	3
$so(12)$	66	10	32	4
E_7	133	18	56	6
$sp(6)$	21	4	14	$\frac{5}{2}$
$sl(2)$	3	2	4	5

Table 4.2: Quasi-conformal algebras based on simple (complex) g and irreducible ρ

implies

$$\frac{h^\vee}{i_\rho} = \frac{3D}{N(N+1)}(N+8) - 5 \tag{4.133}$$

This can be checked explicitly to work by checking number against the table 1 of [8] relevant part of which is collected in table 4.2 for convenience. In [8] it was shown that all the groups and the corresponding symplectic representations listed in the above table satisfy the equation

$$h^\vee = 2i_\rho \left(\frac{D}{N} + \frac{3D}{N(1+N)} - 1 \right) \tag{4.134}$$

which was obtained as a consistency condition for the existence of certain class of infinite dimensional nonlinear quasi-superconformal algebras. Comparing this equation with the equation (4.133) we see that they agree if

$$D = \frac{3N\,(N+1)}{N+16} \tag{4.135}$$

This magically holds true for all the groups G listed in Table 4.2 except for the generic family of non-compact symplectic groups $\mathrm{Sp}(2n,\mathbb{R})$ with $\dim \rho = 2n$. The minimal unitary realization of the generic $\mathrm{Sp}(2n,\mathbb{R})$ family is obtained as a degenerate limit of our Ansatz [51].

Now, let us make sure that $[F\,, F^\alpha] = 0$. This is true provided

$$\xi^\alpha\,(\mathcal{C}_2 + \mathfrak{C}) + (\mathcal{C}_2 + \mathfrak{C})\,\xi^\alpha + \kappa\,[\mathcal{C}_2\,, [\xi^\alpha\,, \mathcal{C}_2]] = 0 \tag{4.136}$$

Using (4.122) and $[\mathcal{C}_2\,, J^a] = 0$ we arrive at

$$\begin{aligned} 2\,\xi^\alpha\,(\mathcal{C}_2 + \mathfrak{C}) + 2\,(1 - \kappa C_\rho)\,C_\rho \xi^\alpha + 2\,(1 - \kappa C_\rho)\,(\lambda^a)^\alpha{}_\beta\,\xi^\beta J_a \\ -4\,\kappa \left(\lambda^a \lambda^b\right)^\alpha{}_\beta\,\xi^\beta J_b J_a = 0 \end{aligned} \tag{4.137}$$

In order to extract restrictions on $\mathfrak{g}$ implied by the above equation we contract it with $\xi^\gamma \Omega_{\gamma\alpha}$

$\mathfrak{g}$	$\dim(\mathfrak{g})$	$g^\vee$	Does (4.139) hold ?
$\mathfrak{a}_n$	n^2+2n	$n+1$	for $\mathfrak{sl}(3)$ only
$\mathfrak{b}_n$	$2n^2+n$	$2n-1$	no
$\mathfrak{c}_n$	$2n^2+n$	$n+1$	no
$\mathfrak{d}_n$	$2n^2-n$	$2n-2$	for $\mathfrak{so}(8)$ only
$\mathfrak{e}_6$	78	12	yes
$\mathfrak{e}_7$	133	18	yes
$\mathfrak{e}_8$	248	30	yes
$\mathfrak{f}_4$	52	9	yes
$\mathfrak{g}_2$	14	4	yes

Table 4.3: Dimensions and dual Coxeter numbers of simple complex Lie algebras. In order for the algebra to admit 5-graded decomposition its dimension must be greater than 6. This rules out $\mathfrak{sl}(2)$ for which (4.139) also holds.

and obtain

$$\frac{h^\vee}{i_\rho}=\frac{D}{N(N+1)}(N-8)+1\,. \tag{4.138}$$

It agrees with (4.133) provided (4.135) holds true.

It is of interest to investigate implications of (4.135) for the 5-graded Lie algebra $\mathfrak{g}$. It is well known [43, 113] that $N=2(g^\vee-2)$ where $g^\vee$ is the dual Coxeter element of algebra $\mathfrak{g}$ in question. Since

$$\dim(\mathfrak{g})=2+2N+1+\dim\left(\mathfrak{g}^0\right)=1+2\,(N+1)+D=2\frac{(g^\vee+1)\;(5g^\vee-6)}{g^\vee+6} \tag{4.139}$$

Algebras in this triangle enjoy some really exceptional properties. In addition The quadratic Casimir of the algebra constructed above reads as follows

$$\mathcal{C}_2(\mathfrak{g})=J^aJ_a+\frac{2\,C_\rho}{N+1}\left(\frac{1}{2}\,\Delta^2+EF+FE\right)-\frac{C_\rho}{N+1}\Omega_{\alpha\beta}\left(E^\alpha F^\beta+F^\beta E^\alpha\right) \tag{4.140}$$

which upon using (4.124) and the following results

$$\begin{aligned}\frac{1}{2}\,\Delta^2+EF+FE&=\kappa\,(J^aJ_a+\mathfrak{C})-\frac{3}{8}\\ \Omega_{\alpha\beta}\left(E^\alpha F^\beta+F^\beta E^\alpha\right)&=8\,\kappa J^aJ_a+\frac{N}{2}+\kappa C_\rho N\end{aligned} \tag{4.141}$$

is seen to reduce to c-number

$$\begin{aligned}\mathcal{C}_2(\mathfrak{g})&=\mathfrak{C}\left(\frac{8\kappa C_\rho}{N+1}-1\right)-\frac{3}{4}\,\frac{C_\rho}{N+1}-\frac{N}{2}\,\frac{C_\rho}{N+1}-\frac{\kappa C_\rho^2 N}{N+1}\\ &=_{(\text{using eq.}(4.124))}-\frac{C_\rho}{36}\,\frac{(N+4)\,(5\,N+8)}{N+1}\end{aligned} \tag{4.142}$$

in agreement with irreducibility. We note that this result agrees with explicit calculations for magic square algebras in ref. [48]. In the normalisation chosen there $\kappa = 2$ and hence $12C_\rho = N + 1$. Then, using $N = 2g^\vee - 4$ we get

$$\mathcal{C}_2\left(\mathfrak{g}\right) = -\frac{1}{108}\left(5g^\vee - 6\right)g^\vee. \tag{4.143}$$

Few remarks are in order. Construction presented in section 4.5 sheds some light on what happens if requirement of simplicity of $\mathfrak{g}^0$ is relaxed. Assuming $\mathfrak{g}^0$ semi-simple, quadratic Casimir of each simple component would be some linear combination of quartic invariant of $\mathfrak{g}^0$. This, of course, assumes uniqueness of quartic invariant of symplectic representation of $\mathfrak{g}^0$.

4.7 Representations of non-compact supergroups and conformal quantum mechanics

The same steps undertaken in the previous chapter for bosonic oscillators ξ^α can be repeated to fermionic oscillators, leading to fermionic extension of conformal quantum mechanics.

We start with the same 5-graded decomposition, assuming $\mathfrak{g}^0$ to be simple. Subspace $\mathfrak{g}^{-1}$ now can not be assumed symplectic. Indeed, assumptions of 5-graded decomposition imply that $\mathfrak{g}^{-1}$ is isomorphic to some Clifford algebra, by the same token as it was isomorphic to Heisenberg-Weyl algebra in bosonic case.

We adopt structure of (4.114), with Ω now denoting non-degenerate bilinear form associated with aforementioned Clifford algebra.
Now we realize them using anti-commuting oscillators ξ^α such that

$$\left\{\xi^\alpha, \xi^\beta\right\} = \Omega^{\alpha\beta} \tag{4.144}$$

this way

$$E = \frac{1}{2}y^2 \qquad E^\alpha = y\xi^\alpha \qquad J^a = -\frac{1}{2}\lambda^a{}_{\alpha\beta}\xi^\alpha\xi^\beta \tag{4.145}$$

We are following conventions of [8]. The quadratic Casimir of $\mathfrak{g}^0$ is taken to be

$$\mathcal{C}_2\left(\mathfrak{g}^0\right) = \eta_{ab}J^aJ^b \tag{4.146}$$

The grade $+2$ generator F is taken to be

$$F = \frac{1}{2}p^2 + \kappa y^{-2}\left(\mathcal{C}_2 + \mathfrak{C}\right) \tag{4.147}$$

for some constants κ and $\mathfrak{C}$ to be determined later. This makes

$$F^{\alpha} = ip\,\xi^{\alpha} + \kappa y^{-1}\left[\xi^{\alpha}, \mathcal{C}_2\right] \tag{4.148}$$

The following will be useful later

$$\left\{E^{\alpha}, F^{\beta}\right\} = i\,(y\,p)\,\Omega^{\alpha\beta} + \xi^{\beta}\,\xi^{\alpha} + \kappa\left\{\xi^{\alpha}, \left[\xi^{\beta}, \mathcal{C}_2\right]\right\} \tag{4.149}$$

and

$$\left\{F^{\alpha}, F^{\beta}\right\} = -p^2\Omega^{\alpha\beta} - \frac{\kappa}{y^2}\left(\xi^{\alpha}\left[\xi^{\beta}, \mathcal{C}_2\right] + \xi^{\beta}\left[\xi^{\alpha}, \mathcal{C}_2\right] - \kappa\left\{\left[\xi^{\alpha}, \mathcal{C}_2\right], \left[\xi^{\beta}, \mathcal{C}_2\right]\right\}\right) \tag{4.150}$$

and

$$[F,\, F^{\alpha}] = \frac{\kappa}{x^3}\left(\left(\mathcal{C}_2 + \mathfrak{C}\right)\xi^{\alpha} + \xi^{\alpha}\left(\mathcal{C}_2 + \mathfrak{C}\right) + \kappa\left[\mathcal{C}_2, [\xi^{\alpha},\, \mathcal{C}_2]\right]\right) \tag{4.151}$$

We find the following identities useful

$$\begin{aligned}
[AB,\, C\} &= A\,[B,\, C\} + (-)^{BC}\,[A,\, C\}\,B \\
[A,\, BC\} &= [A,\, B\}\,C + (-)^{AB}\,B\,[A,\, C\} \\
[AB,\, CD\} &= A\,[B,\, C\}\,D + (-)^{BC+AC+BD}\,C\,[A,\, D\}\,B \\
&\quad + (-)^{BC}\,AC\,[B,\, D\} + (-)^{BC+BD}\,[A,\, C\}\,DB
\end{aligned} \tag{4.152}$$

By inspection

$$\left[\xi^{\alpha}, \mathcal{C}_2\right] = -2\,(\lambda^a)^{\alpha}{}_{\beta}\xi^{\beta}J_a + \,C_{\rho}\,\xi^{\alpha} \tag{4.153}$$

and using the following ansatz

$$(\lambda^a)^{\beta}{}_{\gamma}(\lambda_a)^{\alpha}{}_{\delta} + (\lambda^a)^{\beta\alpha}\,(\lambda_a)_{\gamma\delta} = \frac{C_{\rho}}{N-1}\left(\Omega^{\alpha\beta}\Omega_{\gamma\delta} + \delta^{\beta}{}_{\gamma}\delta^{\alpha}{}_{\delta} - 2\delta^{\alpha}{}_{\gamma}\delta^{\beta}{}_{\delta}\right) \tag{4.154}$$

we find

$$\left\{E^{\alpha},\, F^{\beta}\right\} = -\Omega^{\alpha\beta}\Delta - 6\kappa\,(\lambda^a)^{\beta\alpha}\,J_a + \left(\frac{3\kappa C_{\rho}}{N-1} - \frac{1}{2}\right)\left(\xi^{\beta}\xi^{\alpha} - \xi^{\alpha}\xi^{\beta}\right). \tag{4.155}$$

Computing $\left\{F^{\alpha},\, F^{\beta}\right\}$ we get

$$\begin{aligned}
\left\{F^{\alpha},\, F^{\beta}\right\} = -2F\Omega^{\alpha\beta} &= -2\left(\frac{p^2}{2} + \frac{k}{x^2}\left(\frac{1}{2}\kappa C_{\rho}^2 + \frac{1}{2}C_{\rho}\right)\right)\Omega^{\alpha\beta} \\
&\quad - \frac{\kappa}{x^2}\left(-4\left(\xi^{\alpha}(\lambda_a)^{\beta}{}_{\gamma} + \xi^{\beta}(\lambda_a)^{\alpha}{}_{\gamma}\right)\xi^{\gamma}J^a + 12\kappa\,(\lambda_a\lambda_b)^{\alpha\beta}\,J^bJ^a\right. \\
&\quad \left. + 8\kappa\,(\lambda_a\lambda_b)^{\beta\alpha}\,J^bJ^a - 4\kappa(\lambda_a)^{\alpha}{}_{\delta}(\lambda_b)^{\beta}{}_{\gamma}\xi^{\delta}\xi^{\gamma}f^{ab}{}_{c}J^c\right)
\end{aligned} \tag{4.156}$$

Taking the Ω trace we obtain

$$N = 8 - 10\kappa i_\rho \ell^2 + 2\kappa C_{\text{adj}} \qquad 2\mathfrak{C} = \kappa C_\rho^2 + C_\rho \tag{4.157}$$

Taking into account that

$$\frac{i_\rho \ell^2}{C_\rho} = \frac{N}{D} \qquad C_{\text{adj}} = +\ell^2 g_0^\vee \tag{4.158}$$

we obtain our first restriction for fermionic super-conformal algebra data

$$\frac{g_0^\vee}{i_\rho} = 5 + \frac{3D}{N(N-1)}\,(N-8)\,. \tag{4.159}$$

Now, we look at $[F,\, F^\alpha] = 0$. This yields the following condition

$$2\xi^\alpha\,(\mathcal{C}_2 + \mathfrak{C}) - C_\rho \xi^\alpha + (2 + 4\kappa C_\rho)\,(\lambda_a)^\alpha{}_\beta - \kappa C_\rho^2 \xi^\alpha - 4\kappa(\lambda_a\lambda_b)^\alpha{}_\beta \xi^\beta J^b J^a = 0 \tag{4.160}$$

which, upon contraction with $\xi^\gamma \Omega_{\gamma\alpha}$ results into the following condition

$$\frac{g_0^\vee}{i_\rho} = -1 + \frac{D}{N\,(N-1)}\,(N+8)\,. \tag{4.161}$$

These two conditions (4.159) and (4.161) agree provided

$$D = \frac{3N\,(N-1)}{16-N} \tag{4.162}$$

which is also the condition for them to agree with eq. (2-32) in [8]. Looking at the table 1 of [8] and verifying these conditions we find the following fermionic super-conformal algebras:

$\mathfrak{g}$	$\mathfrak{g}_0$	D	N
$\mathfrak{osp}\,(10\|2,\mathbb{R})$	$\mathfrak{so}(10)$	45	10
$F(4)$	$\mathfrak{so}(7)$	21	8_s
$G(3)$	$\mathfrak{g}_2$	14	7

(4.163)

Chapter 5

Minimal unitary representations of U-duality Lie algebras

Minimal unitary representation of a non-compact simple Lie algebra $\mathfrak{g}$ is defined by minimal realization described in previous chapter, realized on the Hilbert space of square integrable functions.

Minimal unitary representations of non-compact reductive groups have been studied by mathematicians (see [73] for review and further references).

5.1 K-types

Consider a linear connected reductive group[1] G and its representation π on a Hilbert space V. Let K be the maximal compact subgroup of G and consider representation π such that $\pi|_K$ is unitary (i.e. if π is unitary or it is an induced representation). Then $\pi|_K$ decomposes into orthogonal sum of irreducible representations of K:

$$\pi|_K \simeq \bigoplus_{\tau\in\hat{K}} n_\tau \tau \tag{5.1}$$

where $\hat{K}$ denotes the space of unitary irreducible representations of group K, referred to in mathematical literature as dual of K. Multiplicity n_τ is either a non-negative integer or it is $+\infty$. Any unitary irreducible representation $\tau \in \hat{K}$ is finite dimensional. An equivalence class τ that occur in $\pi|_K$ with positive multiplicity are called *K-types* of π. For unitary irreducible representation π multiplicities of K-types are integers and are bounded from above $n_\tau \leqslant \dim \tau$ for all $\tau \in \hat{K}$ [68]. Among irreducible representations $\tau \in \hat{K}$ occurring in π there is a finite number of *minimal* K-types minimising

$$\|\Lambda(\tau) + 2\delta_K\|^2 \tag{5.2}$$

[1] A complex analytic group is called reductive if its linear analytic representation is completely reducible. Semi-simple groups are reductive. $\mathrm{GL}(n,\mathbb{C})$ is reductive though not semi-simple because it is direct sum of simple and Abelian group.

where $\Lambda(\tau)$ is the highest weight of the representation τ and $2\delta_K$ is sum of all positive roots of Lie algebra $\mathfrak{k}$ of compact group K.

A representation π is called *admissible* if its restriction to the compact subgroup $\pi|_K$ acts unitarily on Hilbert space V and if multiplicities of its K-types are all finite. For instance lowest energy representations considered in Section 3.4 are admissible.

Consider the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. The universal enveloping algebra has a natural grading

$$\mathcal{U}(\mathfrak{g}) = 1 \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}) \oplus \cdots = \bigoplus_{k=0}^{\infty} \mathcal{T}_k(\mathfrak{g}) \tag{5.3}$$

with respect to order of monomial in $\mathfrak{g}$. Consider action of $\mathcal{U}(\mathfrak{g})$ on a vector space V of representation of $\mathfrak{g}$. It is clear that dimension of $\mathcal{T}_k(\mathfrak{g})V$ will grow polynomially with k. Define *Gelfand-Kirillov dimension* to be the rate of growth for large k. For generic representations Gelfand-Kirillov dimension is equal to the number of positive roots in Lie algebra $\mathfrak{g}$, and is lesser than this for special representations. Minimal representation corresponds to minimal Gelfand-Kirillov dimension.

Gelfand-Kirillov dimension of a representation can be less than that of a generic representation if some ideal $\mathcal{I}$ of universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ vanishes on this representation, i.e. $\pi(\mathcal{I})V = 0$. Joseph showed [59] that for minimal representations this ideal is two-sided prime ideal $\mathcal{I}_0$ generated by its members in $\mathcal{I}_0 \cap \mathcal{T}_2(\mathfrak{g})$ bearing Joseph's name (cf. (4.112)). Minimal representation can also be understood as quantisation of coadjoint orbit going through maximal root of $\mathfrak{g}$ and thus having the smallest possible functional dimension [59, 112].

Minimal representation of U-duality groups are studied in connection with automorphic forms [64, 88] of discrete U-duality groups. Construction of automorphic forms of weight zero, i.e. functions invariant under discrete U-duality groups, involves a spherical vector of continuous version of U-duality group (see [88] for physicist's oriented review). A spherical vector is a vector of one-dimensional minimal K-type, and thus it is annihilated by all the compact generators of $\mathfrak{g}$:

$$\pi(\mathfrak{k})|\phi_{\text{spherical}}\rangle = 0 \tag{5.4}$$

Clearly not every Lie algebra $\mathfrak{g}$ would admit a spherical vector in the minimal representation, in particular algebras of type $\mathfrak{g}_{2(2)}$ and $\mathfrak{f}_{4(4)}$ do not admit spherical vector [89].

5.2 Constructing spherical vector for split algebras

As it is explained in the Appendix A.2 every Lie algebras except $\mathfrak{su}(2)$ admits 5-graded decomposition (3.38) associated with highest root ω of $\mathfrak{g}$:[2]

$$\hat{E}_{-\omega} \oplus \mathfrak{g}_{-1} \oplus \left(\hat{H}_{\omega} \oplus \mathfrak{g}_0\right) \oplus \mathfrak{g}_{+1} \oplus \hat{E}_{\omega} \tag{5.5}$$

Coadjoint orbit passing through highest root vector $\hat{E}_{-\omega}$ is generated by the action of $\hat{H}_{\omega}$, $\hat{E}_{\omega}$ and $\mathfrak{g}_{+1}$. The orbit is endowed with $\mathfrak{g}$-invariant Kirillov-Konstant symplectic form, which decomposes into symplectic form on $\mathfrak{g}_{-1}$, as it was explained at the beginning of chapter 4 and in Appendix A.2, and symplectic form on $\hat{H}_{\omega}$ and E_{ω}. In order to carry out quantisation of this orbit one has to introduce polarisation, i.e. call some variables coordinates and some momenta, and then represent generators of algebra $\mathfrak{g}$ as differential operators acting on function of the chosen coordinates as it was done in chapter 4.

The choice made by authors of [64] is based on distinguished root $\hat{E}_{\beta_0} \in \mathfrak{g}_{+1}$, unique simple root which is not orthogonal to ω (which is also the root to which affine root attached on extended Dynkin diagram). Uniqueness fails for algebras of type A_n which were not considered in [64]. One declares parameters associated with $\mathfrak{g}_0$ orbit of $\hat{E}_{\beta_0}$ to be coordinates in our phase space, with x_0 corresponding to E_{β_0} (compare this to (3.46)):

$$\hat{E}_{\omega} = \hat{y} \qquad \hat{E}_{\beta_k} = -i\hat{y}\hat{p}_k \qquad \hat{E}_{\gamma_k} = i\hat{x}^k \tag{5.6}$$

with momenta realized as $\hat{p} = i\partial_{\hat{y}}$ and $\hat{p}_k = i\partial_{\hat{x}^k}$. Let $\mathfrak{h}_0$ be subalgebra of $\mathfrak{g}_0$ such that

$$\left[\hat{E}_{\beta_0}, \mathfrak{h}_0\right] = 0 \tag{5.7}$$

and let $\mathcal{N}(x_a)$ be cubic invariant of $\mathfrak{h}_0$. Then $\hat{H}_{\beta_0}$, $\hat{E}_{\beta_k}$ and $\hat{E}_{-\delta_k}$ (for some δ) and $\mathfrak{h}_0$ form conformal algebra to which one can associate Jordan triple system as explained earlier and $\mathcal{N}(x_a)$ coincides with cubic norm of underlying Jordan algebra. Giving the above data one can reconstruct the whole algebra thanks to two Weyl reflection realized as integral transforms on functions:

$$(S \circ f)(\hat{y}, \hat{x}_0, \ldots, \hat{x}_{d-1}) = \int \frac{\prod_{k=0}^{d-1} dp_k}{(2\pi\hat{y})^{d/2}} f(\hat{y}, p_0, \ldots, p_{d-1})\, e^{\frac{i}{\hat{y}}\sum_{k=0}^{d-1}\hat{x}_k p_k} \tag{5.8}$$

and

$$(A \circ f)(\hat{y}, \hat{x}_0, \ldots, \hat{x}_{d-1}) = e^{-\frac{1}{\hat{x}_0\hat{y}}\mathcal{N}(\hat{x}_a)} f(-\hat{x}_0, \hat{y}, \hat{x}_1, \ldots, \hat{x}_{d-1}) \tag{5.9}$$

In particular reflection A allows to map between negative and positive grade subspaces.

[2] We shall denote all the generators in the construction of Kazhdan, Pioline and Waldron [64] with hat to distinguish it from ours.

Resulting structure of selected generators has the form

$$\begin{aligned}\hat{H}_{\beta_0} &= -\hat{y}\partial_{\hat{y}} + \hat{x}_0\partial_{\hat{x}_0} \\ \hat{H}_{\omega} &= -\mu - 2\hat{y}\partial_{\hat{y}} - \hat{x}_k\partial_{\hat{x}_k}\end{aligned} \qquad \begin{aligned}\hat{E}_{-\beta_0} &= -\hat{x}_0\partial_{\hat{y}} + \frac{i}{\hat{y}^2}\mathcal{N}(\hat{x}_a) \\ \hat{E}_{-\omega} &= \hat{y}\hat{p}^2 + \hat{p}\hat{x}_k\hat{p}^k + \hat{x}_0\mathcal{N}(\hat{p}) \\ &\quad - \frac{\hat{p}_0}{\hat{y}^2}\mathcal{N}(\hat{x}) + \frac{1}{\hat{y}}\frac{\partial\mathcal{N}(\hat{x})}{\partial\hat{x}_k}\frac{\partial\mathcal{N}(\hat{p})}{\partial\hat{p}_k}\end{aligned} \tag{5.10}$$

with generators $\mathfrak{h}_0$ being bilinears in $\hat{x}_k$, $\hat{p}_k$ and $\hat{y}$, $\hat{p}$. The spherical vector then has the following form (for $\mathfrak{g}$ not being of D series)

$$f_K = \frac{1}{|z|^{2\nu+1}}\hat{K}_\nu(S_1)\, e^{-iS_2} \tag{5.11}$$

where $\hat{K}_\nu(z) = z^{-\nu}K_\nu(z)$ and K_ν is modified Bessel function of the second kind. Here $z = \hat{y} + i\hat{x}_0$ and

$$S_1 = \sqrt{\sum_a \hat{Z}_a^2 + \left(\partial_{\hat{Z}_a}\frac{\mathcal{N}(\hat{x})}{\sqrt{\hat{y}^2+\hat{x}_0^2}}\right)^2} \qquad S_2 = \frac{\hat{x}_0\mathcal{N}(\hat{x})}{\hat{y}\left(\hat{y}^2+\hat{x}_0^2\right)} \tag{5.12}$$

where $Z = (\hat{y}, \hat{x}_0, \hat{x}_a)$. For full details of this construction we refer the reader to the original paper [64] which is also reviewed in [89, 88]. We shall try to understand this construction for D_4 and the resulting spherical vector in more details below.

5.3 Spherical vector of $D_{4(4)}$

Maximal split real form of Lie algebra of type D_4 corresponds to SO$(4,4)$ which corresponds to 3-dimensional U-duality groups of compactification of 5-dimensional STU model [39], i.e. pure 5-d supergravity coupled to two vector multiplets.

Let α_1, α_2, α_3 be three mutually orthogonal with respect to Killing form simple roots on which triality acts by permutations (see Fig. 5.1). Let β_0 be the remaining simple root, invariant under triality homomorphism. The five-graded decomposition looks as follows

$$\hat{E}_\omega \oplus \begin{Bmatrix}\hat{E}_{\beta_k}\\ \hat{E}_{\gamma_k}\end{Bmatrix} \oplus \begin{pmatrix}\hat{H}_\omega \\ \hat{H}_{\beta_0}\end{pmatrix} \oplus \begin{matrix}\hat{E}_{\alpha_1}, \hat{H}_{\alpha_1}, \hat{E}_{-\alpha_1} \\ \oplus\ \hat{E}_{\alpha_2}, \hat{H}_{\alpha_2}, \hat{E}_{-\alpha_2} \\ \hat{E}_{\alpha_3}, \hat{H}_{\alpha_3}, \hat{E}_{-\alpha_3}\end{matrix} \oplus \begin{Bmatrix}\hat{E}_{-\beta_k}\\ \hat{E}_{-\gamma_k}\end{Bmatrix} \oplus \hat{E}_{-\omega} \tag{5.13}$$

where $k = 0, 1, 2, 3$ and $\beta_k + \gamma_k = \omega$. For every root the Hermiticity condition corresponds to that of maximal split case (see Appendix A.2):

$$\left(\hat{H}_\alpha\right)^\dagger = -\hat{H}_\alpha \quad \left(\hat{E}_\alpha\right)^\dagger = \hat{E}_{-\alpha} \tag{5.14}$$

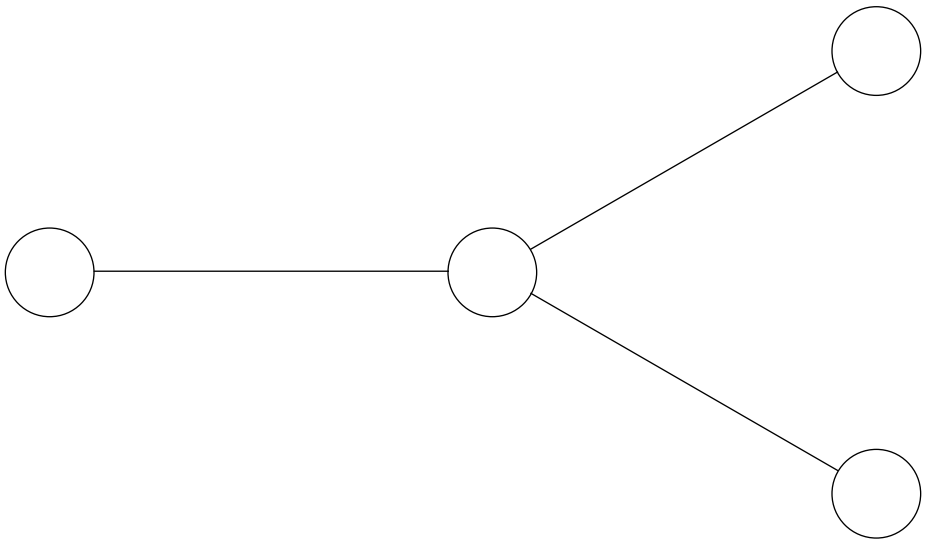

Figure 5.1: Root system of $\mathfrak{d}_4$.
The central root is denoted β_0, and other roots which connect to it are denoted α_1, α_2, α_3. Triality is an outer automorphism that permutes α_k roots.

and thus compact generators are given by $\hat{E}_\alpha + \hat{E}_{-\alpha}$ for every positive root of $\mathfrak{so}(4,4)$. Thus there will be 12 compact generators that form $\mathfrak{so}(4) \oplus \mathfrak{so}(4)$. Commutation relation of the algebra is given in standard Chevalley basis:

$$\left[\hat{E}_\alpha, \hat{E}_{-\alpha}\right] = \hat{H}_\alpha \qquad \left[\hat{H}_\alpha, \hat{E}_{\pm\alpha}\right] = \mp 2\hat{H}_{\pm\alpha} \tag{5.15}$$

for all simple roots α. And for $k = 1, 2, 3$ we have

$$\hat{E}_{\beta_k} = \left[\hat{E}_{\alpha_k}, \hat{E}_{\beta_0}\right] \qquad \hat{E}_{\gamma_k} = \left[\hat{E}_{\alpha_i}, \hat{E}_{\beta_j}\right] = \left[\hat{E}_{\alpha_j}, \hat{E}_{\beta_i}\right] \tag{5.16}$$

for i, j, k being cyclic permutation of $1, 2, 3$. Then

$$\left[\hat{E}_{\beta_I}, \hat{E}_{\gamma_J}\right] = \delta_{IJ}\hat{E}_\omega \tag{5.17}$$

where I, J run over $0, 1, 2, 3$. The remaining commutation relation follow from given and hermitian conjugation involution.

Below we give the realization of Kazhdan, Pioline and Waldron [64] explicitly

$$\hat{E}_{\beta_L} = i\hat{y}\hat{p}_L \qquad \hat{E}_{\gamma_L} = i\hat{x}_L \qquad \hat{E}_\omega = i\hat{y} \tag{5.18}$$

where $L = 0, 1, 2, 3$. Grade zero generators are given as

$$\begin{aligned} \hat{E}_{+\alpha_k} &= -i\hat{x}_0\hat{p}_k - i\hat{y}^{-1}\hat{x}_i\hat{x}_j \\ \hat{E}_{-\alpha_k} &= +i\hat{x}_k\hat{p}_0 - i\hat{x}\hat{p}_i\hat{p}_j \end{aligned} \qquad \hat{H}_{\alpha_k} = -i\hat{p}_0\hat{x}_0 + 2i\hat{x}_k\hat{p}_k - i\sum_{j=1}^{3}\hat{x}_j\hat{p}_j \tag{5.19}$$

For each simple root α_k and $k = 1, 2, 3$ there correspond $\mathfrak{sp}(2, \mathbb{R})$ which is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ used in previous chapter in Section 4.3.5.

$$\hat{E}_{-\beta_0} = -i\hat{x}_0\hat{p} + \frac{i}{\hat{y}^2}\hat{x}_1\hat{x}_2\hat{x}_3 \qquad \hat{H}_{\beta_0} = -i\hat{y}\hat{p} + i\hat{x}_0\hat{p}_0 \tag{5.20}$$

from here we recognise the cubic form $\mathcal{N} = \hat{x}_1\hat{x}_2\hat{x}_3$ of generic Jordan algebra $\mathbb{R} \oplus \Gamma(2)$ discussed earlier. The algebra $\mathfrak{h}_0$ is spanned by

$$\mathfrak{h}_0 = \text{span}\left\{\hat{H}_{\gamma_3}, \hat{H}_{\gamma_2}\right\} \simeq \mathfrak{so}(1,1) \oplus \mathfrak{so}(1,1) \tag{5.21}$$

and indeed coincides with the Lorentz group of corresponding generalised space-time. Remaining generators are given as follows:

$$\begin{aligned} \hat{E}_{-\omega} &= -i\hat{y}\hat{p}^2 - 3\hat{p} + i\hat{y}^{-1} - i\hat{x}_0\hat{p}_0\hat{p} + i\hat{y}^{-2}\hat{x}_1\hat{x}_2\hat{x}_3\hat{p}_0 - i\hat{x}_0\hat{p}_1\hat{p}_2\hat{p}_3 \\ &\quad - i\hat{y}^{-1}\left(\hat{x}_1\hat{x}_2\hat{p}_1\hat{p}_2 + \hat{x}_1\hat{x}_3\hat{p}_1\hat{p}_3 + \hat{x}_2\hat{x}_3\hat{p}_2\hat{p}_3\right) - \hat{x}_k\hat{p}_k\left(i\hat{p} + \hat{y}^{-1}\right) \\ \hat{E}_{-\gamma_0} &= -3\hat{p}_0 - i\hat{y}\hat{p}\hat{p}_0 + i\hat{y}\hat{p}_1\hat{p}_2\hat{p}_3 - i\left(\hat{x}_0\hat{p}_0 + \hat{x}_k\hat{p}_k\right)\hat{p}_0 \\ \hat{E}_{-\gamma_k} &= -i\hat{y}\hat{p}\hat{p}_k - i\left(\hat{p}_0\hat{x}_0 + \hat{p}_k\hat{x}_k\right)\hat{p}_k - i\hat{y}^{-1}\hat{x}_i\hat{x}_j\hat{p}_0 \\ \hat{E}_{-\beta_k} &= i\hat{x}_k\hat{p} + i\hat{y}^{-1}\hat{x}_k\left(\hat{x}_i\hat{p}_i + \hat{p}_j\hat{x}_j\right) + i\hat{x}_0\hat{p}_i\hat{p}_j \end{aligned} \tag{5.22}$$

Here no summation over repeated indices is assumed and i, j, k is taken to be cyclic permutation of $1, 2, 3$. For convenience and further referencing we also give

$$\hat{H}_\omega = \hat{H}_{\alpha_1} + \hat{H}_{\alpha_2} + \hat{H}_{\alpha_3} + 2\hat{H}_{\beta_0} = -i\sum_{k=0}^{3}\hat{x}_k\hat{p}_k - 2i\hat{y}\hat{p} - 3 \tag{5.23}$$

It is a tedious exercise to verify that the following function

$$\hat{f}_K(\hat{y}, \hat{x}_I) = \frac{4\pi}{\sqrt{\hat{y}^2 + \hat{x}_0^2}} K_0\left(\frac{\sqrt{\left(\hat{y}^2 + \hat{x}_0^2 + \hat{x}_1^2\right)\left(\hat{y}^2 + \hat{x}_0^2 + \hat{x}_2^2\right)\left(\hat{y}^2 + \hat{x}_0^2 + \hat{x}_3^2\right)}}{\hat{y}^2 + \hat{x}_0^2}\right) e^{-\frac{i\hat{x}_0\hat{x}_1\hat{x}_2\hat{x}_3}{\hat{y}\left(\hat{y}^2 + \hat{x}_0^2\right)}} \tag{5.24}$$

satisfies 12 differential equations:

$$\left(\hat{E}_\rho + \hat{E}_{-\rho}\right) f_K = 0 \tag{5.25}$$

for ρ being either α_k or β_0 and

$$\left(\hat{E}_\rho - \hat{E}_{-\rho}\right) f_K = 0 \tag{5.26}$$

for ρ being one of β_k, γ_k, γ_0, β_0 or ω.

The spherical vector (5.24) is square integrable function, explictly invariant under triality acting as permutation on $\hat{x}_k$.

5.4 Recasting into conformal QM basis

Procedure outlined above constructs, in the language of Chevalley basis and root systems, a Freudenthal triple system from Jordan triple system (3.46) associated with Jordan algebra that has adjoint identity (2.52). Thus it could be recast into conformal quantum mechanics basis used in this thesis in hope to learn something new from this exercise.

Five graded decomposition (5.13) helps us identify generators, and comparing (4.116) with $\hat{E}_\omega$, $\hat{E}_{\beta_a}$ and $\hat{E}_{\gamma_a}$ we conclude that module generators rescaling we should make the following change of variables

$$\hat{y} = y^2 \qquad\qquad \hat{x}_k = y\, x_k \tag{5.27}$$

Such change of variables induces the following relation between vector fields:

$$\frac{\partial}{\partial \hat{x}_k} = \frac{1}{y}\frac{\partial}{\partial x_k} \qquad \frac{\partial}{\partial \hat{y}} = \frac{1}{2y}\frac{\partial}{\partial y} - \frac{1}{2y^2}\sum_{k=0}^{4} x_k \frac{\partial}{\partial x_k} \tag{5.28}$$

which recasts $\hat{H}_\omega$ into

$$H_\omega = -3 - y\frac{\partial}{\partial y} = -\frac{1}{2}\left(y\frac{\partial}{\partial y} + \frac{\partial}{\partial y}y\right) - \frac{5}{2} \tag{5.29}$$

Since $\hat{H}_\omega$ plays the role of grading operator defining 5-grading it is identified with our Δ (4.107). Again modulo generators rescaling we conclude that

$$p = -i\frac{\partial}{\partial y} + \frac{5i}{2y} = -iy^{\frac{5}{2}}\frac{\partial}{\partial y}y^{-\frac{5}{2}} \tag{5.30}$$

We hence adopt the following complement to (5.27):

$$\hat{p}_k = \frac{1}{y}p_k \qquad\qquad \hat{p} = \frac{1}{2y}\left(p + \frac{5i}{2y}\right) - \frac{1}{2y^2}\sum_{k=0}^{3} x_k p_k \tag{5.31}$$

We are now set to write out the generators

$$\begin{aligned} E_{\omega} &= iy^2 \qquad H_{\omega} = -\frac{i}{2}\left(py + yp\right) \\ E_{-\omega} &= -\frac{i}{4}p^2 - \frac{i}{2y^2}\left(\mathcal{C}_2 + \frac{3}{8}\right) \end{aligned} \tag{5.32}$$

where $\mathcal{C}_2$ stands for quadratic Casimir of each of the following $\mathfrak{sp}\left(2,\mathbb{R}\right)$:

$$\begin{aligned} H_{\alpha_k} &= ip_0x_0 + 2ix_kp_k - i\sum_{n=1}^{3} x_np_n \qquad & E_{\alpha_k} &= -i\left(x_0p_k + x_ix_j\right) \\ & & E_{-\alpha_k} &= i\left(x_kp_0 - p_ip_j\right) \end{aligned} \tag{5.33}$$

$$\mathcal{C}_2 = \frac{1}{2}H_{\alpha_k}^2 - E_{\alpha_k}E_{-\alpha_k} - E_{-\alpha_k}E_{\alpha_k} \qquad \forall k = 1,2,3 \tag{5.34}$$

Explicit expression for quadratic Casimir $\mathcal{C}_2$ reads as follows

$$\begin{aligned} \mathcal{C}_2 = \frac{1}{2} - \frac{5}{2}ix_0p_0 - \frac{1}{2}i\sum_{k=1}^{3} x_kp_k - 2x_0p_1p_2p_3 + 2x_1x_2x_3p_0 + \frac{1}{2}\sum_{k=0}^{4} x_k^2p_k^2 + \\ x_0p_0\sum_{k=1}^{3} x_kp_k - x_1x_2p_1p_2 - x_1x_3p_1p_3 - x_2x_3p_2p_3 \end{aligned} \tag{5.35}$$

Negative grade generators linear in x_k and p_k read as follows

$$E_{\beta_k} = iyp_k \qquad E_{\gamma_k} = iyx_k \qquad \forall k = 0,1,2,3 \tag{5.36}$$

and remaining generators

$$\begin{aligned} H_{\beta_0} &= -\frac{i}{4}\left(yp + py\right) + \frac{3i}{2}p_0x_0 + \frac{i}{2}\sum_{n=1}^{3} x_np_n \\ E_{-\beta_0} &= -\frac{i}{2}px_0 + \frac{i}{2y}\left(-\frac{5i}{2}x_0 + x_0\sum_{n=0}^{3} x_np_n + 2x_1x_2x_3\right) \\ E_{-\gamma_0} &= -\frac{i}{2}pp_0 + \frac{i}{2y}\left(\frac{5i}{2}x_0 - \sum_{n=0}^{3} x_np_np_0 + 2p_1p_2p_3\right) \\ E_{-\beta_k} &= \frac{i}{2}px_k + \frac{i}{2y}\left(\frac{i}{2}x_k + 2x_0p_ip_j - x_kx_0p_0 - x_kx_kp_k + x_kx_ip_i + x_kx_jp_j\right) \\ E_{-\gamma_k} &= -\frac{i}{2}pp_k + \frac{i}{2y}\left(\frac{i}{2}p_k - 2x_0p_ip_j - x_0p_0p_k - x_kp_kp_k + x_ip_ip_k + x_jp_jp_k\right) \end{aligned} \tag{5.37}$$

The spherical vector is modified traces down to (5.24) with appropriate change of variables

applied and multiplied by $y^{5/2}$ so as to untwist (5.30);

$$f_K = \frac{4\pi y^{3/2}}{\sqrt{y^2+x_0^2}} K_0 \left(\frac{y\sqrt{\left(y^2+x_0^2+x_1^2\right)\left(y^2+x_0^2+x_2^2\right)\left(y^2+x_0^2+x_3^2\right)}}{y^2+x_0^2} \right) e^{-\frac{ix_0x_1x_2x_3}{y^2+x_0^2}} \tag{5.38}$$

It is also annihilated by 12 differential equations:

$$\left(E_\rho + E_{-\rho}\right) f_K = 0 \tag{5.39}$$

for ρ being either α_k or β_0 and

$$\left(E_\rho - E_{-\rho}\right) f_K = 0 \tag{5.40}$$

for ρ being one of β_k, γ_k, γ_0, β_0 or ω.

Appendix A

Reference material

A.1 Clifford algebras and spinors

This section in part follows [71]. In a d-dimensional spacetime $\mathbb{R}^{t,s}$ with t-timelike and s-spacelike dimensions, Clifford algebra is generated by products of Γ_μ satisfying the following relation

$$\{\Gamma_\mu, \Gamma_\nu\} = \Gamma_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\mu = 2\eta_{\mu\nu} \tag{A.1}$$

where indices μ and ν run $\mu, \nu = 0, \ldots, d-1$, and $\eta_{\mu\nu} = \mathrm{diag}\left((+)^t, (-)^s\right)$. Let $\ell(\gamma)$ be a length of Clifford algebra element γ, i.e. a minimum number of generators product of which forms γ.

Thanks to diagonality of metric η a space of algebra generators of length k is isomorphic to space Λ^k of rank k anti-symmetric tensor in space-time indices. There is a natural identification between Λ^k and Λ^{d-k} provided by Hodge $*$-operation:

$$\Gamma_{\mu_1 \ldots \mu_k} \equiv \Gamma_{[\mu_1} \ldots \Gamma_{\mu_k]} \quad (*\Gamma)_{\mu_1 \ldots \mu_k} = \frac{1}{(d-k)!} \epsilon_{\mu_1 \ldots \mu_k \mu_{k+1} \ldots \mu_d} \eta^{\mu_{k+1}\nu_{k+1}} \ldots \eta^{\mu_d \nu_d} \Gamma_{\nu_{k+1} \ldots \nu_d}$$

Let Γ_{d+1} be an element of Clifford algebra of length d:

$$\Gamma_{d+1} = \Gamma_0 \Gamma_1 \ldots \Gamma_d \tag{A.2}$$

then

$$\Gamma_{\mu_1 \ldots \mu_k} = (-1)^{(d-k)(d-k-1)/2} \, \Gamma_{d+1} \, (*\Gamma)_{\mu_1 \ldots \mu_k} \tag{A.3}$$

Since there are $\binom{d}{k}$ independent elements of length k, we conclude that the total dimension of Clifford algebra $\mathcal{C}(p, q)$ is equal to

$$\sum_{k=0}^{d} \binom{d}{k} = 2^d \tag{A.4}$$

Let ρ be some faithful representation of $\mathcal{C}(t, s)$, and let V be a module of ρ, i.e. finite dimensional vector space where elements of Clifford algebra are realized as matrices. Each

ϵ	η	$(s-t)$	*Spinor*	*Reality Condition*
$+1$	$+1$	$0,1,2 \mod 8$	Majorana	$\psi^* = B\psi$
$+1$	-1	$0,6,7 \mod 8$	Pseudo-Majorana	$\psi^* = B\psi$
-1	$+1$	$4,5,6 \mod 8$	Symplectic Majorana	$\psi^{*i} = (\psi_i)^* = \Omega^{ij} B\psi_j$
-1	-1	$2,3,4 \mod 8$	Pseudo-symplectic Majorana	$\psi^{*i} = (\psi_i)^* = \Omega^{ij} B\psi_j$

Table A.1: Admissible reality conditions in various space-times.
Here Ω^{ij} is constant symplectic matrix satisfying $\Omega_{ij}\Omega^{jk} = -\delta^k{}_i$, and index i labels a pseudo-real representation of a given Lie algebra which admits such representations.

representation of Clifford algebra forms a representation of rotation algebra $\mathfrak{so}(t,s)$ with algebra generators given as

$$J_{\mu\nu} = \frac{i}{4}[\Gamma_\mu, \Gamma_\nu] = \frac{i}{4}(\Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu) \tag{A.5}$$

Indeed, it is easy to verify that Js satisfy commutation relation of generators of $\mathfrak{so}(t,s)$:

$$[J_{\mu\nu}, J_{\lambda\rho}] = i(\eta_{\nu\lambda}J_{\mu\rho} - \eta_{\mu\lambda}J_{\nu\rho} + \eta_{\mu\rho}J_{\nu\lambda} - \eta_{\nu\rho}J_{\mu\lambda}). \tag{A.6}$$

Vectors of module of this representations are called *spinors*, and the corresponding representation of $\mathfrak{so}(t,s)$ is called *spinor representation*. In even-dimensional space-time matrix Γ_{d+1} anti-commutes with all Γ_μ:

$$\Gamma_{d+1}\Gamma_\mu + \Gamma_\mu\Gamma_{d+1} = 0 \qquad (\Gamma_{d+1})^2 = (-1)^{\frac{s-t}{2}} \tag{A.7}$$

and as a consequence commutes with all $J_{\mu\nu}$. We thus find the representation defined by $J_{\mu\nu}$ is reducible for even d, since eigenspaces of Γ_{d+1} are invariant under action of $\mathfrak{so}(t,s)$. Spinors that belong to these eigenspaces are called *chiral spinors*, or *Weyl spinors*.

Let us further study the reducibility of Clifford algebra representation. We note that Hermitian conjugation maps Clifford algebra into an equivalent one:

$$(\Gamma_\mu)^\dagger = A\Gamma_\mu A^{-1}, \quad \text{with} \quad A = \Gamma_0\Gamma_1\ldots\Gamma_{t-1} \tag{A.8}$$

Matrix A is chosen in such way as to make compact generators $J_{\mu\nu}$ Hermitian. Since $\pm\Gamma^*_\mu$ also form an inequivalent representation of the algebra there exists an invertible matrix B such that

$$\Gamma^*_\mu = -\eta B\Gamma_\mu B^{-1} \qquad \eta^2 = 1 \qquad B^\dagger B = 1 \qquad B^t = \epsilon B \qquad \epsilon^2 = 1. \tag{A.9}$$

It follows that

$$\Gamma^t_\mu = -\eta C\Gamma_\mu C^{-1} \qquad C^\dagger C = 1 \qquad C = BA \qquad C^t = \epsilon\eta^t(-1)^{\frac{t(t+1)}{2}}C \tag{A.10}$$

Not any choice of ϵ and η is admissible for given space-time $\mathbb{R}^{t,s}$. The values of ϵ and η, and the allowed type of spinors, together with reality condition they satisfy are listed in Table A.1. The Weyl condition, i.e. $\Gamma_{d+1}\psi = \pm\psi$ can be consistently imposed on any type Majorana spinor provided $s - t$ is a multiple of 4.

A.2 Lie algebras

Lie algebra is a vector space $\mathfrak{g}$ equipped with a bilinear operation, called Lie bracket (also called commutator)

$$[\cdot,\cdot]:\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g} \tag{A.11}$$

that satisfies two additional requirements

$$\forall a,b\in\mathfrak{g}\qquad [a,b]=-[b,a] \tag{A.12a}$$

$$\forall a,b,c\in\mathfrak{g}\qquad [a,[b,c]]+[b,[c,a]]+[c,[a,b]]=0 \tag{A.12b}$$

known as anti-symmetry and Jacobi identity. For any element $X\in\mathfrak{g}$ one defines an *adjoint action* $\mathrm{ad}_X:\mathfrak{g}\to\mathfrak{g}$ as follows

$$\mathrm{ad}_X(Y)=[X,Y]$$

Lie algebra is said to have an ideal $\mathfrak{I}$, if

$$[\mathfrak{I},\mathfrak{g}]\subset\mathfrak{I}.$$

Lie algebra is said to be *simple* if it contains no ideals besides itself. Lie algebra is said to be *semi-simple* if it is a direct sum of simple Lie algebras.

Simple finite dimensional Lie algebras have been classified by E. Cartan. Consider a simple finite dimensional Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be its maximal Abelian subalgebra, also referred to as *Cartan subalgebra*. Because algebra $\mathfrak{g}$ is assumed simple

$$\forall H\in\mathfrak{h}\quad\text{ and }\quad\forall E\in\mathfrak{g}\ominus\mathfrak{h}\qquad\qquad [H,E]\in\mathfrak{g}\ominus\mathfrak{h}. \tag{A.13}$$

Since $\mathfrak{h}$ is Abelian there exists a basis in $\mathfrak{g}$ which diagonalises adjoint action ad_H for all $H\in\mathfrak{h}$. Cartan proved that all such eigenspaces are one-dimensional:

$$\mathfrak{g}=\mathfrak{h}\oplus\bigoplus_{\alpha\in\Phi}\mathfrak{g}_\alpha \tag{A.14}$$

i.e.

$$\mathfrak{g}_\alpha=\mathbb{R}\otimes E_\alpha\quad:\quad[H,E_\alpha]=\ell_\alpha(H)\,E_\alpha \tag{A.15}$$

for some linear form $\ell_\alpha:\mathfrak{h}\to\mathbb{R}$. He also proved that for every $\mathfrak{g}_\alpha$ there exists $\mathfrak{g}_{-\alpha}$ by constructing a Cartan involution τ such that

$$\begin{aligned}&\tau:\mathfrak{h}\to\mathfrak{h}\\&\tau:\mathfrak{g}_\alpha\to\mathfrak{g}_{-\alpha}\qquad\forall\alpha\in\mathfrak{h}^*.\end{aligned} \tag{A.16}$$

Linear forms $\alpha \in \mathfrak{h}^*$ are referred to as Lie algebra's *roots*. It is easy to see that for every two roots α and β Jacobi identity implies that

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta} \tag{A.17}$$

In particular $[E_\alpha, E_\beta] = 0$ whenever $\alpha + \beta$ is not a root. Since $\mathfrak{g}_\alpha$ are one-dimensional we conclude that $\alpha + \alpha$ is never a root. Jacobi identity commands that

$$\forall \alpha \in \mathfrak{h}^* \qquad [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}. \tag{A.18}$$

This establishes a map of $\mathfrak{h}^*$ onto $\mathfrak{h}$. Let $H_\alpha = [E_\alpha, E_{-\alpha}]$. Because ad_{H_α} is a finite dimensional matrix acting on $\mathfrak{g}$ Cartan defines Cartan bilinear form

$$(\alpha, \beta) = \mathrm{Tr}_{\mathfrak{g}} \left(\mathrm{ad}_{H_\alpha} \mathrm{ad}_{H_\beta} \right) \tag{A.19}$$

and proves that it is non-degenerate, establishing isomorphism between $\mathfrak{h}$ and $\mathfrak{h}^*$. Then

$$\ell_\alpha \left(H_\beta \right) = 2 \frac{(\alpha, \beta)}{(\alpha, \alpha)} \tag{A.20}$$

Rank of Lie algebra r is defined as dimension of its Cartan subalgebra.

Cartan involution induces an (arbitrary) decomposition of root system Φ into positive Φ^+ and negative Φ^- roots. Among positive roots, there exist a set simple roots, such that any positive root is a sum of simple roots

$$\alpha = \sum_{i=1}^{r} c_i \alpha_i \qquad c_i \in \mathbb{Z}^+ \tag{A.21}$$

with non negative coefficients c_i. Cartan's classification theorem states that for finite-dimensional simple Lie algebra $\mathfrak{g}$

$$2 \geqslant \kappa_{ij} = \ell_{\alpha_i} \left(H_{\alpha_j} \right) \in \mathbb{Z} \qquad \kappa_{ij} \kappa_{ji} = 0, 1, 2, 3, 4 \tag{A.22}$$

Thus finite-dimensional simple Lie algebras are exhausted by 4 families of classical Lie algebras A_n, B_n, C_n, D_n and 5 exceptional algebras G_2, F_4, E_6, E_7, and E_8. The subscript indicates the rank of the algebra. To each of these algebras one associates a Dynkin diagram (see Table A.2), which is a graph with simple positive roots as nodes, connected with a single line if $\kappa_{ij} = \kappa_{ji}$, with a double line if $\kappa_{ij} = 2\kappa_{ji}$ and an arrow from longer to shorter root, and with a triple line if $\kappa_{ij} = 3\kappa_{ji}$.

Among positive roots Ψ^+ of a simple finite-dimensional Lie algebra, there is a highest root ω, such that for any other root $\alpha \in \Phi^+$ sum $\alpha + \omega \notin \Phi$. Element H_ω of Cartan algebra

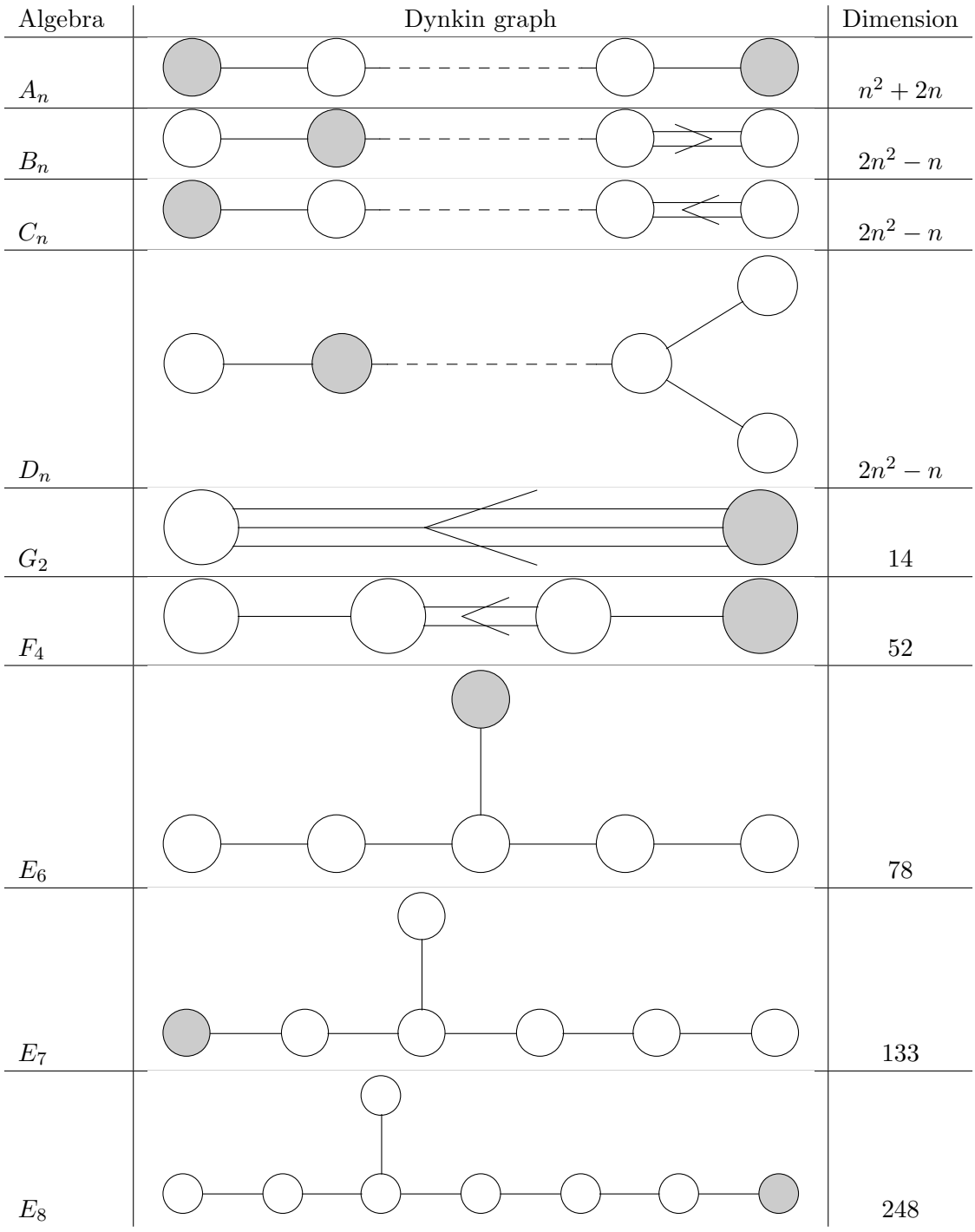

Table A.2: Dynkin diagrams of finite dimensional simple Lie algebras.
Gray nodes denote the root to which extended root $-\omega$ attaches.

induces grading of Lie algebra $\mathfrak{g}$, since

$$[H_\omega, \mathfrak{h}] = 0 \qquad [H_\omega, E_\alpha] = \ell_\omega(\alpha) E_\alpha \tag{A.23}$$

One can prove that

$$\forall \alpha \in \Phi \qquad -2 \leqslant \ell_\omega(\alpha) \leqslant 2 \tag{A.24}$$

The inequalities are saturated only if $\alpha = \pm\omega$. Thus every Lie algebra admits a 5 grading. The only exception to this rule is algebra $\mathfrak{sl}_2 \simeq A_1$ of dimension 3, where there does not exist $\alpha \in \Phi^+$ such that $\ell_\omega(\alpha) = 1$.

One can also prove that most of simple roots of a Lie algebra are orthogonal to ω, i.e. $\ell_\omega(\alpha_i) = 0$. For all Lie algebras there is unique simple root α such that $\ell_\omega(\alpha) = 1$, except for algebras of type A_n where there two such simple roots. Those simple roots that are not orthogonal to ω are drawn gray in table A.2.

A.3 Real forms of Lie algebras

Real form $\mathfrak{g}_r$ of complex Lie algebra $\mathfrak{g}_c$ is a subalgebra of $\mathfrak{g}_c$ invariant under some anti-linear involution. That is starting from complex Lie algebra classified by Cartan we make possible complex change of variables such that basis elements J^a of with structure constants $f^{ab}{}_c$

$$\left[J^a, J^b\right] = f^{ab}{}_c J^c \tag{A.25}$$

under involution τ behave

$$\forall \alpha \in \mathbb{C} \quad \tau(\alpha J)^a = \alpha^* T^a{}_b J^b\,, \qquad \left[\tau(J)^a, \tau(J)^b\right] = -f^{ab}{}_c (\tau J)^c\,. \tag{A.26}$$

Existence of compact forms of Lie algebras follow from Cartan classification, since structure constants can be chosen real. Compact real form then corresponds to involution $\tau(J)^a = -J^a$, that is all generators are anti-hermitian.

From the above discussion it is clear that classification of real forms boils down to classification of involutive automorphisms of Lie algebra. Without going into much detail, for which we refer reader to excellent textbook of Gilmore [31], we quote the list of real forms in table A.3 for further reference.

Real forms of complex Lie algebra are denoted by specifying character of real form χ equal to number of non-compact generators minus number of compact generators. For the exception of some real forms of classical Lie algebras specifying character χ suffices to identify real form uniquely.

So-called split real forms, with maximal possible character $\chi = r$ correspond to Cartan

Root space	Compact Form	Associated non-compact form	Maximal compact subgroup	$\chi =$ no. non-comp.$-$ no. comp.
A_{n-1}	$\mathfrak{su}(n)$	$\mathfrak{sl}(n,\mathbb{R})$	$\mathfrak{so}(n)$	$n-1$
	$\mathfrak{su}(2n)$	$\mathfrak{su}^*(2n)$	$\mathfrak{usp}(2n)$	$-2n-1$
	$\mathfrak{su}(p+q)$	$\mathfrak{su}(p,q)$	$\mathfrak{su}(p)\oplus\mathfrak{su}(q)\oplus\mathfrak{u}(1)$	$1-(p-q)^2$
B_n	$\mathfrak{so}(p+q)$	$\mathfrak{so}(p,q)$	$\mathfrak{so}(p)\oplus\mathfrak{so}(q)$	pq
D_n	$\mathfrak{so}(p+q)$	$\mathfrak{so}(p,q)$	$\mathfrak{so}(p)\oplus\mathfrak{so}(q)$	pq
	$\mathfrak{so}(2n)$	$\mathfrak{so}^*(2n)$	$\mathfrak{su}(n)\oplus\mathfrak{u}(1)$	$-n$
C_n	$\mathfrak{usp}(2n)$	$\mathfrak{sp}(2n,\mathbb{R})$	$\mathfrak{su}(n)\oplus\mathfrak{u}(1)$	$+n$
	$\mathfrak{usp}(2p+2q)$	$\mathfrak{usp}(2p,2q)$	$\mathfrak{usp}(2p)\oplus\mathfrak{usp}(2q)$	$4pq$
G_2	$\mathfrak{g}_{2(-14)}$	$\mathfrak{g}_{2(2)}$	$A_1\oplus A_1$	2
F_4	$\mathfrak{f}_{4(-52)}$	$\mathfrak{f}_{4(-20)}$	B_4	-20
	$\mathfrak{f}_{4(-52)}$	$\mathfrak{f}_{4(4)}$	$C_3\oplus A_1$	4
E_6	$\mathfrak{e}_{6(-78)}$	$\mathfrak{e}_{6(-26)}$	F_4	-26
	$\mathfrak{e}_{6(-78)}$	$\mathfrak{e}_{6(-14)}$	$D_5\oplus D_1$	-14
	$\mathfrak{e}_{6(-78)}$	$\mathfrak{e}_{6(2)}$	$A_5\oplus A_1$	2
	$\mathfrak{e}_{6(-78)}$	$\mathfrak{e}_{6(6)}$	C_4	6
E_7	$\mathfrak{e}_{7(-133)}$	$\mathfrak{e}_{7(-25)}$	$E_6\oplus D_1$	-25
	$\mathfrak{e}_{7(-133)}$	$\mathfrak{e}_{7(-5)}$	$D_6\oplus A_1$	-5
	$\mathfrak{e}_{7(-133)}$	$\mathfrak{e}_{7(7)}$	A_7	7
E_8	$\mathfrak{e}_{8(-248)}$	$\mathfrak{e}_{8(-24)}$	$E_7\oplus D_1$	-24
	$\mathfrak{e}_{8(-248)}$	$\mathfrak{e}_{8(8)}$	D_8	8

Table A.3: Real forms of simple Lie algebras.

$\mathrm{SL}(n,\mathbb{R})$	$A^\dagger = A^t$
$\mathrm{SU}^*(2n)$	$A^\dagger A = 1 \quad A^* J = JA$
$\mathrm{SU}\,(p,q)$	$A^\dagger \eta A = \eta$
$\mathrm{SO}^*\,(2n)$	$A^t A = 1 \quad A^* J = JA$
$\mathrm{SO}\,(p,q)$	$A^t \eta A = \eta$
$\mathrm{Sp}\,(2p,2q)$	$A^t J A = J \quad A^* \eta = \eta A$
$\mathrm{Sp}\,(2p,\mathbb{R})$	$A^t J A = J \quad A^\dagger = A^t$

Table A.4: Real forms of classical Lie algebras and reality conditions. Here J is symplectic matrix, and η is a metric of $\mathbb{R}^{p,q}$.

involution

$$\forall \alpha \in \Phi^+ \quad E_\alpha \to E_{-\alpha} \qquad E_{-\alpha} \to E_\alpha \qquad H_\alpha \to H_\alpha \tag{A.27}$$

since for each positive root we have one compact $E_\alpha - E_{-\alpha}$ and one non-compact $E_\alpha + E_{-\alpha}$ root generators, with all elements of Cartan subalgebra $\mathfrak{h}$ being non-compact.

In conclusion we list explicitly reality conditions for real forms of classical Lie groups in table A.4.

A.4 Going from $\mathfrak{su}^*(8)$ to $\mathfrak{su}(6,2)$ basis

Recall that position and momentum operators X^{AB} and P_{AB} transform as **28** and $\tilde{\mathbf{28}}$ under $\mathfrak{su}^*(8)$. To build annihilation and creation operators we need to take complex linear combinations of the form $X^{AB} \pm iP_{AB}$, which transform covariantly under $\mathfrak{so}^*(8)$ subalgebra of $\mathfrak{su}^*(8)$. We expect resulting creation and annihilation operators to transform as **28** and $\overline{\mathbf{28}}$ of some non-compact form of $\mathfrak{su}(8)$ [1]. The isomorphism $\mathfrak{so}^*(8) \simeq \mathfrak{so}(6,2)$ suggests that this non-compact form should be $\mathfrak{su}(6,2)$ as we shall establish.

In order to elucidate the role of triality of $\mathfrak{so}(8)$ we recall that adjoint representation of compact $\mathfrak{e}_7$ decomposes into four representations of $\mathfrak{so}(8)$:

$$\mathbf{133} = \mathbf{28} \oplus \mathbf{35}_v \oplus \mathbf{35}_s \oplus \mathbf{35}_c$$

where three **35** correspond to symmetric traceless tensor in $\mathbf{8}_v \otimes \mathbf{8}_v$, $\mathbf{8}_s \otimes \mathbf{8}_s$ and $\mathbf{8}_c \otimes \mathbf{8}_c$ respectively, with $\mathbf{8}_v$, $\mathbf{8}_s$ and $\mathbf{8}_c$ being three inequivalent eight dimensional representations of $\mathfrak{so}(8)$. Triality of $\mathfrak{so}(8)$ then maps **35** representations into one another. Observe also, that **28** combined with any one of three **35** generate an $\mathfrak{su}(8)$ subalgebra of $\mathfrak{e}_7$. Compact $\mathfrak{so}(8)$ becomes $\mathfrak{so}^*(8)$ if we consider $\mathfrak{e}_{7(-25)}$ instead of compact $\mathfrak{e}_7$ and $\mathfrak{su}(8)$ becomes $\mathfrak{su}^*(8)$.

Consider the Clifford algebra of $\mathbb{R}^{6,2}$

$$\left\{\Gamma^a, \Gamma^b\right\} = 2\eta^{ab} \tag{A.28}$$

and choose a basis with the following Hermiticity property

$$(\Gamma^a)^\dagger = \eta_{ab}\Gamma^b = \omega \cdot \Gamma^a \cdot \omega^{-1} \tag{A.29}$$

where $\omega = \Gamma^7 \cdot \Gamma^8$ is a 16×16 symplectic matrix. One particular choice of basis, in which chirality matrix Γ^9 is diagonal, is given as follows

$$\begin{aligned}
\Gamma^1 &= \sigma_1 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 & \Gamma^2 &= \sigma_2 \otimes \sigma_1 \otimes \mathbb{I}_2 \otimes \sigma_2 \\
\Gamma^3 &= \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 & \Gamma^4 &= \sigma_2 \otimes \sigma_2 \otimes \sigma_3 \otimes \mathbb{I}_2 \\
\Gamma^5 &= \sigma_2 \otimes \sigma_3 \otimes \mathbb{I}_2 \otimes \sigma_2 & \Gamma^6 &= \sigma_2 \otimes \sigma_2 \otimes \sigma_1 \otimes \mathbb{I}_2 \\
\Gamma^7 &= i\sigma_2 \otimes \mathbb{I}_2 \otimes \sigma_2 \otimes \sigma_3 & \Gamma^8 &= i\sigma_2 \otimes \mathbb{I}_2 \otimes \sigma_2 \otimes \sigma_1
\end{aligned} \tag{A.30}$$

Then,

$$\begin{aligned}
Z^{ab} &= \frac{1}{4}\,\Gamma^{ab}{}_{CD}\left(X^{CD} - iP_{CD}\right) \\
\tilde{Z}^{ab} &= \frac{1}{4}\,\Gamma^{ab}{}_{CD}\left(X^{CD} + iP_{CD}\right)
\end{aligned} \tag{A.31}$$

[1] Notice that compact $\mathfrak{su}(8)$ is not a subalgebra of $\mathfrak{e}_{7(-25)}$.

where transformation coefficient are given by matrix elements of chiral representation of $\mathfrak{so}(6,2)$ generators

$$\Gamma^{ab}{}_{CD} = \mathbb{P}\left(\frac{i}{4}\left[\Gamma^a, \Gamma^b\right]\right)_{CD} \tag{A.32}$$

and $\mathbb{P}$ is the chiral projection operator in spinor space. Symplectic structure (4.29) of X and P induces the symplectic structure

$$\left[\tilde{Z}^{ab}, Z^{cd}\right] = \frac{1}{8}\,\mathrm{Tr}\left[\Gamma^{ab}\,\Gamma^{cd}\right] = \frac{1}{2}\left(\eta^{ca}\eta^{db} - \eta^{cb}\eta^{da}\right). \tag{A.33}$$

on Z and $\tilde{Z}$. Gamma matrices defined above satisfy the following identities

$$\begin{gathered}
\Gamma^{ab}{}_{AB} = -\Gamma^{ab}{}_{BA} = -\Gamma^{ba}{}_{AB} \\
\Gamma^{abcd}{}_{AB} = \frac{1}{24}\epsilon^{abcd}{}_{efgh}\Gamma^{efgh}{}_{AB} = \Gamma^{abcd}{}_{BA} \\
\Gamma^{abcd}{}_{ABCD} := \Gamma^{[ab}{}_{[AB}\Gamma^{cd]}{}_{CD]} \\
\Gamma^{abcd}{}_{ABCD} = -\frac{1}{24}\epsilon^{abcd}{}_{efgh}\Gamma^{efgh}{}_{ABCD} = -\frac{1}{24}\epsilon_{ABCDEFGH}\Gamma^{abcd}{}_{EFGH} \\
\Gamma^{ac}{}_{[AB}\Gamma^{cb}{}_{CD]} = \Gamma^{bc}{}_{[AB}\Gamma^{ca}{}_{CD]} = \frac{1}{24}\epsilon_{ABCDEFGH}\Gamma^{ac}{}_{[EF}\Gamma^{cb}{}_{GH]}
\end{gathered}$$

where

$$\Gamma^{abcd}{}_{AB} = \mathbb{P}\left(\Gamma^{[a}\Gamma^{b}\Gamma^{c}\Gamma^{d]}\right)_{AB}.$$

These identities allow us to rewrite generators of $\mathfrak{e}_{7(-25)}$ in $\mathfrak{su}(6,2)$ basis:

$$\begin{aligned}
\eta^{bc}J^a{}_c - \eta^{ac}J^b{}_c &= \Gamma^{ab}{}_{AB}\left(J^A{}_B - J^B{}_A\right) \\
\eta^{bc}J^a{}_c + \eta^{ac}J^b{}_c &= \Gamma^{ab}{}_{ABCD}\left(J^{ABCD} + (\epsilon J)_{ABCD}\right) \\
J^{abcd} + \frac{1}{24}\epsilon^{abcd}{}_{efgh}J^{efgh} &= \Gamma^{abcd}{}_{AB}\left(J^A{}_B + J^B{}_A\right) \\
J^{abcd} - \frac{1}{24}\epsilon^{abcd}{}_{efgh}J^{efgh} &= \Gamma^{abcd}{}_{ABCD}\left(J^{ABCD} - (\epsilon J)_{ABCD}\right)
\end{aligned} \tag{A.34}$$

or, more succinctly,

$$\begin{aligned}
\eta^{bc}J^a{}_c &= \Gamma^{ab}{}_{AB}J^A{}_B + \Gamma^{ab}{}_{ABCD}J^{ABCD} \\
J^{abcd} &= \Gamma^{abcd}{}_{AB}J^A{}_B + \Gamma^{abcd}{}_{ABCD}J^{ABCD}
\end{aligned} \tag{A.35}$$

A.5 Minimal realization of $\mathfrak{e}_{8(8)}$ in $\mathfrak{su}^*(8)$ basis

Non-compact exceptional Lie algebra $\mathfrak{e}_{8(8)}$ also admits realization in an $\mathfrak{su}^*(8)$ basis. It is seen via the following chain of subalgebra inclusions $\mathfrak{su}^*(8) \subset \mathfrak{e}_{7(7)} \subset \mathfrak{e}_{8(8)}$.

Algebra $\mathfrak{e}_{7(7)}$ is generated as

$$\begin{aligned} J^A{}_B &= -2iX^{AC}P_{CB} - \frac{i}{4}\delta^A{}_B X^{CD}P_{CD} \\ J^{ABCD} &= -\frac{i}{2}X^{[AB}X^{CD]} + \frac{i}{48}\epsilon^{ABCDEFGH}P_{EF}P_{GH}\,. \end{aligned} \tag{A.36}$$

where $A, B, \ldots$ are $\mathfrak{su}^*(8)$ indices. Note different relative signs between XX and PP terms in (A.36) and (4.33). It amounts to change of sign in the commutator on the third line

$$\begin{aligned} \left[J^A{}_B, J^C{}_D\right] &= \delta^C{}_B J^A{}_D - \delta^A{}_D J^C{}_B \\ \left[J^A{}_B, J^{CDEF}\right] &= -4\delta^{[C}{}_B J^{DEF]A} - \frac{1}{2}\delta^A{}_B J^{CDEF} \\ \left[J^{ABCD}, J^{EFGH}\right] &= +\frac{1}{36}\epsilon^{ABCDK[EFG}J^{H]}{}_K \end{aligned} \tag{A.37}$$

as compared to that in (4.12) while does not change the Hermiticity properties (4.13) resulting in the following quadratic Casimir

$$\begin{aligned} \mathcal{C}_2 &= \frac{1}{6}J^A{}_B J^B{}_A + \frac{1}{24}\epsilon_{ABCDEFGH}J^{ABCD}J^{EFGH} \\ &= \frac{1}{6}J^A{}_B J^B{}_A + J^{ABCD}(\epsilon J)_{ABCD}\,. \end{aligned} \tag{A.38}$$

The decomposition of $\mathfrak{e}_{7(7)}$ with respect to the maximal compact subalgebra $\mathfrak{usp}(8)$ of $\mathfrak{su}^*(8)$ results now in

$$\mathbf{133} = \mathbf{63} \oplus \mathbf{70} = (\mathbf{36}_{\text{c.}} \oplus \mathbf{27}_{\text{n.c.}}) \oplus (\mathbf{42}_{\text{n.c.}} \oplus \mathbf{27}_{\text{c.}} \oplus \mathbf{1}_{\text{n.c.}})$$

and shows the the constructed $\mathfrak{e}_7$ is indeed $\mathfrak{e}_{7(7)}$. The remaining generators of algebra $\mathfrak{e}_{8(8)}$ are then given by

$$E^{AB} = -iyX^{AB} \qquad \tilde{E}_{AB} = -iyP_{AB} \qquad E = -\frac{i}{2}y^2$$

and

$$\begin{aligned} F &= \frac{1}{2i}p^2 + \frac{2}{iy^2}I_4\left(X\,,P\right) & F^{AB} &= ipX^{AB} + \frac{2}{y}\left[X^{AB}\,, I_4\left(X,P\right)\right] \\ I_4\left(X\,,P\right) &= \mathcal{C}_2 + \frac{323}{16} & \tilde{F}_{AB} &= ipP_{AB} + \frac{2}{y}\left[P_{AB}\,, I_4\left(X,P\right)\right]. \end{aligned}$$

They satisfy the same commutation relations as their counterparts of $\mathfrak{e}_{8(-24)}$ except for

$$\begin{aligned}
\left[J^{ABCD}\,,E^{EF}\right] &= +\frac{1}{24}\epsilon^{ABCDEFGH}\tilde{E}_{GH} \\
\left[J^{ABCD}\,,F^{EF}\right] &= +\frac{1}{24}\epsilon^{ABCDEFGH}\tilde{F}_{GH} \\
\left[\tilde{E}_{AB}\,,\tilde{F}_{CD}\right] &= +12\,(\epsilon J)_{ABCD}
\end{aligned} \tag{A.39}$$

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