

Thermodynamic phase structure of charged anti-de Sitter scalar-tensor black holes

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Abstract. When electromagnetic field with nonlinear lagrangian acts as a source of gravity the no-scalar-hair theorems can be eluded and black holes with non-trivial scalar field can be found in scalar tensor theories [1]. Black holes with secondary scalar hair exist also when a cosmological constant is added in the theory. The thermodynamics of black holes in anti-de Sitter (AdS) space-time has attracted considerable interest due to the AdS/CFT conjecture. A natural question that arises is whether the non-trivial scalar field would alter the black-hole thermodynamical phase structure. In the current work we present the phase structure of charged hairy black holes coupled to nonlinear Born-Infeld electrodynamics in canonical ensemble which is naturally related to AdS space-time. In certain regions of the parameter space we find the existence of a first-order phase transition between small and very large black holes. An unexpected result is that for a small subinterval of charge values two phase transitions are observed – one of zeroth and one of first order.

1. Introduction

In the current paper we present the thermodynamical phase structure in canonical ensemble of charged anti-de Sitter black holes coupled to nonlinear Born-Infeld electrodynamics within a certain class of scalar-tensor theories. The thermodynamics of black holes in AdS space-time has attracted considerable research interest in the last decade especially in the context of the AdS/CFT conjecture according to which the thermodynamics of the AdS black holes is related to the thermodynamics of the dual conformal field theory (CFT) residing on the boundary of the AdS space [3, 4]. In their pioneering work Hawking and Page [5] showed the existence of a phase transition between the AdS black hole and thermal AdS space. The AdS/CFT duality provides us with a new tool to study the phase transitions in the dual CFT theories on the base of studying the phase transitions of the AdS black holes.

On the other side, scalar tensor theories (STT) attract significant research interest since they are among the most natural generalizations of General Relativity (GR). Fundamental scalar fields are a possible remnant of a fundamental unified theory like string theory or higher dimensional gravity theories [6]. STT and GR can hardly be distinguished in the weak field limit. Significant



differences between these theories are more likely to occur in strong fields, hence the interest in black holes in STT. According to the no scalar-hair conjecture, the black-hole solutions in the STT coincide with the solutions from GR. No-scalar-hair theorems treating the cases of static, spherically symmetric, asymptotically flat, electrically neutral black holes and charged black holes in the Maxwell electrodynamics have been proved for a large class of scalar-tensor theories (see [7, 8] and papers referring to them for more details). The no-scalar-hair conjecture can be circumvented when nonlinear matter models, e.g. non-linear electrodynamics (NLED), are considered. In [1, 9] black holes with non-trivial scalar field which are significantly different from the corresponding Einstein-Born-Infeld solutions in GR have been constructed numerically. The solutions obtained in [1] have been generalized in [2] where black holes with AdS asymptotic have been numerically studied. Here we present in details the thermodynamical structure in canonical ensemble of the latter.

2. Black holes in STT coupled to Born-Infeld electrodynamics

The general action of STT in Jordan frame has the following form

$$S = \frac{1}{16\pi G_*} \int d^4x \sqrt{-\tilde{g}} \left(F(\Phi) \tilde{R} - Z(\Phi) \tilde{g}^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - 2U(\Phi) \right) + S_m.$$

The action of matter which acts as a gravitational source is denoted by S_m and it is constructed as in General relativity. STT are often studied in Einstein frame in which the action takes the form

$$S = \frac{1}{16\pi G_*} \int d^4x \sqrt{-g} \left(R - 2g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - 4V(\varphi) \right) + S_m[\Psi_m; \mathcal{A}^2(\varphi) g_{\mu\nu}].$$

In this frame the scalar field is not directly coupled to the Ricci scalar curvature but appears explicitly in the action of the sources. In the particular class of theories that we have considered the potential of the scalar field and the coupling function are given by

$$V(\varphi) = \frac{1}{2}\Lambda \quad \text{and} \quad \mathcal{A}(\varphi) = e^{\alpha\varphi},$$

where Λ is a negative constant that ensures the AdS asymptotic and α is the coupling parameter between the electromagnetic field and the scalar field. This choice of potential for the scalar field allows solutions which are asymptotically AdS to be found. The coupling function is the same as in Brans-Dicke theory. The action of the nonlinear electrodynamics is

$$S_m = \frac{1}{4\pi G_*} \int d^4x \sqrt{-g} \mathcal{A}^4(\varphi) L(X, Y),$$

where $L(X, Y)$ denotes the Lagrangian of the nonlinear electrodynamics. The equations defining the functions X and Y are

$$X = \frac{\mathcal{A}^{-4}(\varphi)}{4} F_{\mu\nu} F^{\mu\nu}, \tag{1}$$

$$Y = \frac{\mathcal{A}^{-4}(\varphi)}{4} F_{\mu\nu} (\star F)^{\mu\nu}, \tag{2}$$

where $F_{\mu\nu}$ is the electromagnetic field strength tensor and \star stands for the Hodge dual with respect to the metric $g_{\mu\nu}$.

In the particular case of Born-Infeld electrodynamics the nonlinear lagrangian has the following form

$$L = 2b \left[1 - \sqrt{1 + \frac{X}{b} - \frac{Y^2}{4b^2}} \right], \tag{3}$$

where b is the Born-Infeld parameter. The black holes that we have constructed numerically in that class of theories [2] are

- asymptotically anti-de Sitter
- have a single, non-degenerate event horizons (no extremal black holes exist)
- purely magnetically charged (the metric, the scalar field and the thermodynamical variables are preserved under electric-magnetic duality rotations)
- have three free parameters: α, b, P (Due to invariance of the solution with respect to rigid rescaling the parameter Λ can be excluded from the list of free parameters.)

The major difference between them and the pure Einstein-Born-Infeld AdS black holes is that for the latter as the mass M and, respectively, the radius of the event horizon r_H are decreased an extremal black hole is reached, while for the former – a naked singularity.

3. Thermodynamics of scalar-tensor black holes coupled to Born-Infeld electrodynamics

For the temperature of the event horizon we apply a standard definition

$$T = \left. \frac{f'(r) e^{-\delta(r)}}{4\pi} \right|_{r=r_H}.$$

The formula for the entropy in Jordan frame needs to be generalized in the following way

$$S_J = \frac{1}{4G_*} \int d^2x \sqrt{-^{(2)}\tilde{g}} F(\Phi) = \frac{1}{4G_*} \int d^2x \sqrt{-^{(2)}g} = S_E = S.$$

This definition of entropy is invariant with respect to conformal transformations of the metric.

The thermodynamic potential that is naturally related to the canonical ensemble is the free energy (see Appendix B in [2] for more details concerning the derivation of the thermodynamical quantities and the first law of thermodynamics for the system):

$$F(T, P) = M - TS.$$

The phase structure is usually presented on conjugate diagrams. The two conjugate variables are

$$-S(T) = \left(\frac{\partial F}{\partial T} \right)_P \quad \text{and} \quad \Psi(P) = \left(\frac{\partial F}{\partial P} \right)_T,$$

where S is the entropy and Ψ is the magnetic potential. We will consider only the case with fixed magnetic charge P . Since the entropy is a monotonous function of the radius of the event horizon r_H the thermodynamic phase structure of the black holes is usually presented on the $T(r_H)$ diagram instead of the $T(S)$. Our aim is to study the effect that the different free parameters have on the thermodynamical phase structure. First, we will briefly discuss the role of the parameters b and α . When b is lower than some critical value $b < b_{\text{crit}}$ the phase structure is trivial – there is just one stable phase so no phase transitions can occur. For $b > b_{\text{crit}}$ the number of phases is either two, one stable and one unstable, or four – two stable and two unstable, depending on the value of the magnetic charge. Phase transitions occur between the two stable phases. The parameter coming from the coupling function of the STT α has a negligible effect on the value of b_{crit} .

Of all three parameters the magnetic charge has the most significant effect on the thermodynamical phase structure of the solutions. The function $T = T(r_H, P)$ has two inflection points $P_{\text{crit}}^{(1)}$ and $P_{\text{crit}}^{(2)}$ defined by

$$\frac{\partial T}{\partial r_H} = \frac{\partial^2 T}{\partial r_H^2} = 0.$$

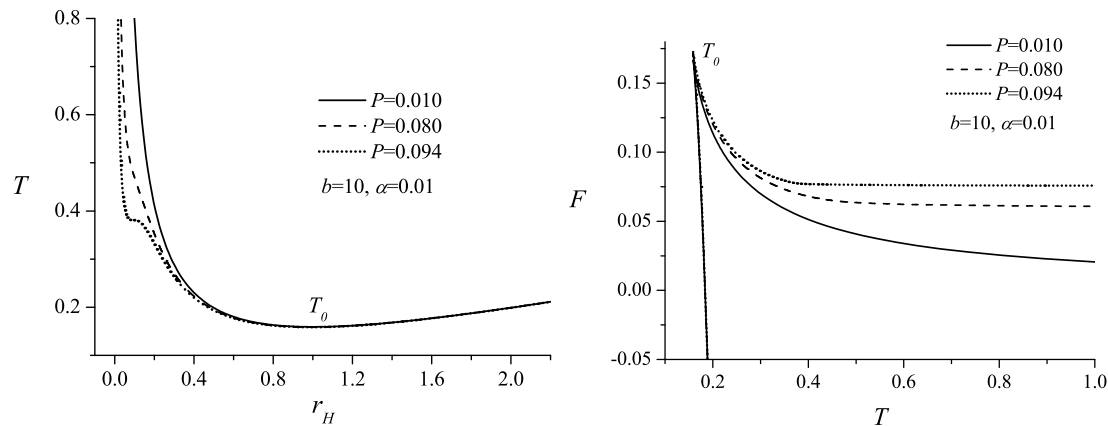


Figure 1. The $T(r_H)$ and the $F(T)$ diagrams for low magnetic charge interval.

They determine three major intervals of values of P : low charge $P < P_{\text{crit}}^{(1)}$, middle charge $P_{\text{crit}}^{(1)} < P < P_{\text{crit}}^{(2)}$ (this one has three subintervals) and high charge $P_{\text{crit}}^{(2)} < P$. Each of them will be discussed separately below.

3.1. Low magnetic charge interval

For the case of low magnetic charge the conjugate diagram $T(r_H)$ and the free energy as function of the temperature $F(T)$ are presented, respectively, on the left and on the right panel of Fig. (1). In that interval of the magnetic charge the thermodynamical system has two phases. The phase with negative slope of the temperature and higher free energy is unstable. The presence of only one stable phase means that no phase transitions occur in that interval of values of the magnetic charge.

3.2. Middle magnetic charge interval

For all values of the magnetic charge between $P_{\text{crit}}^{(1)}$ and $P_{\text{crit}}^{(2)}$ the thermodynamical system has four phases which we term: very small black holes (VSBH), small black holes (SBH), large black holes (LBH) and very large black holes (VLBH). The temperature T has three extrema – two local minima at T_{min} and T_0 and one local maximum at T_{max} . The branch of VSBH spans from T_{min} to infinite temperatures, the branch of SBH spans between T_{min} and T_{max} , the branch of LBH spans between T_{max} and T_0 and the last branch corresponding to VLBH – from T_0 to infinity.

For two of the phases, the SBH and the VLBH, the slope of the temperature is positive which means that these phases are stable. The other two phases are unstable. The thermodynamical system is dominated by the stable phase which has the lowest value of the free energy. As T is varied while P is kept fixed the system passes from one stable phase to another stable phase, i.e. a phase transition is observed.

The middle magnetic charge interval is divided in the following three subintervals. The number and the order of the observed phase transition is different in each of them.

I) The first subinterval is presented on Fig. (2). For this subinterval $T_0 < T_{\text{min}}$ so when $T_0 < T < T_{\text{min}}$ the only stable phase is the VLBH. For $T_{\text{min}} < T < T_{\text{max}}$ there is a second stable phase – the SBH – but since for all values of the temperature $F_{\text{VLBH}} < F_{\text{SBH}}$ no phase transitions occur.

II) The second subinterval is presented on Fig. (3). This is the subinterval with the largest number of phase transitions. Here still $T_0 < T_{\text{min}}$. Again for $T_0 < T < T_{\text{min}}$ the system is in the

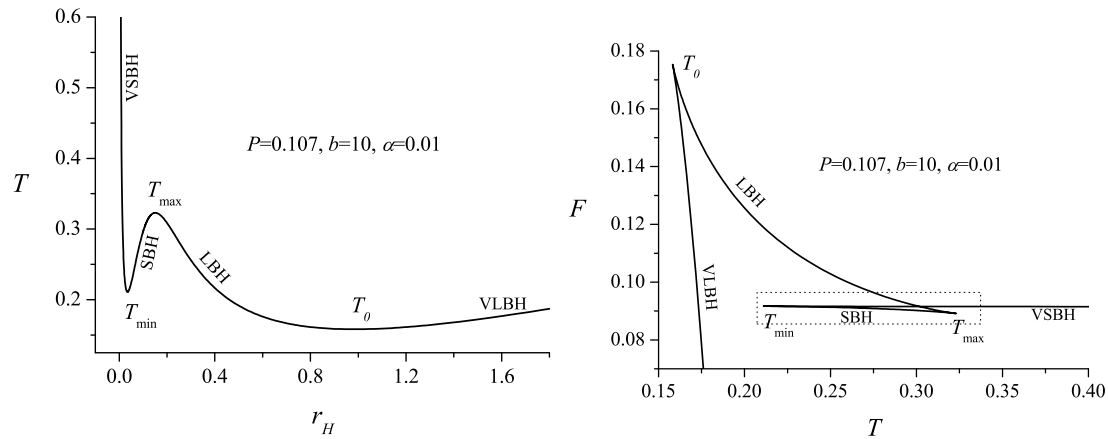


Figure 2. Middle magnetic charge interval (first subinterval).

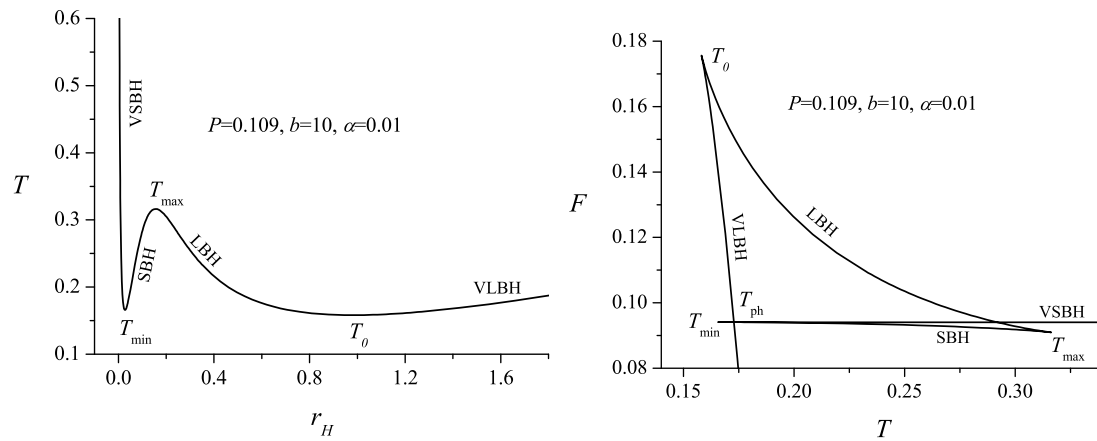


Figure 3. Middle magnetic charge interval (second subinterval).

phase of VLBH since that is the only stable phase. What happens when $T_{\min} < T < T_{\max}$ and the number of stable phases is two? As it can be seen on the right panel of (3) for $T_{\min} < T < T_{\text{ph}}$ the system is dominated by the SBH since $F_{\text{SBH}} < F_{\text{VLBH}}$. At $T = T_{\min}$ a phase transition between VLBH and SBH is observed. The free energy is discontinuous at that point so the order of the phase transition zeroth.

When the temperature rises above T_{ph} , however, $F_{\text{VLBH}} < F_{\text{SBH}}$ so a first order phase transition between SBH and VLBH occurs.

III) The last subinterval is presented on Fig. (4). Here $T_0 > T_{\min}$. When $T_{\min} < T < T_0$ the system is in the only stable state – that of the SBH. For temperatures between T_0 and T_{ph} , $F_{\text{SBH}} < F_{\text{VLBH}}$ so no phase transition is observed at T_0 where the second stable phase comes into existence. The first order phase transition at T_{ph} which was present in the previously discussed subinterval is observed also in the current subinterval of the magnetic charge.

3.3. High magnetic charge interval

The graphics for the interval of high magnetic charges are presented on Fig. (5). Again only two phases exist – one unstable and one stable, and no phase transitions are observed.

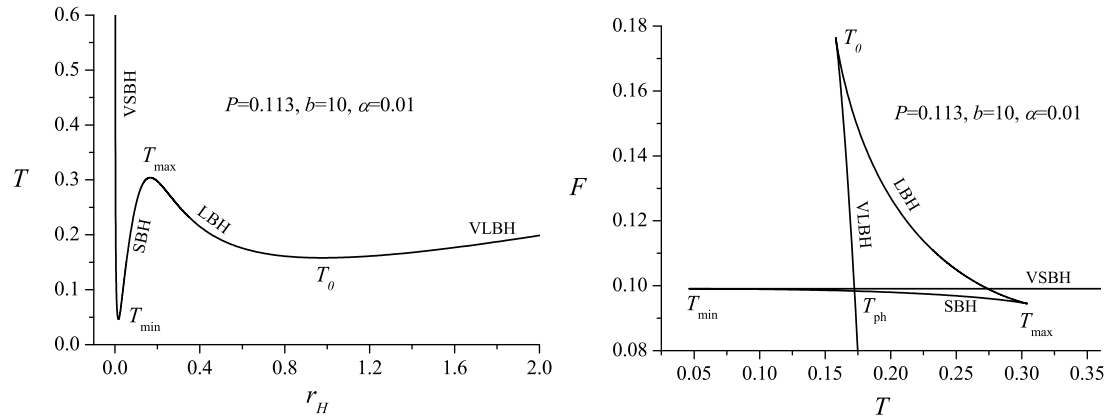


Figure 4. Middle magnetic charge interval (third subinterval).

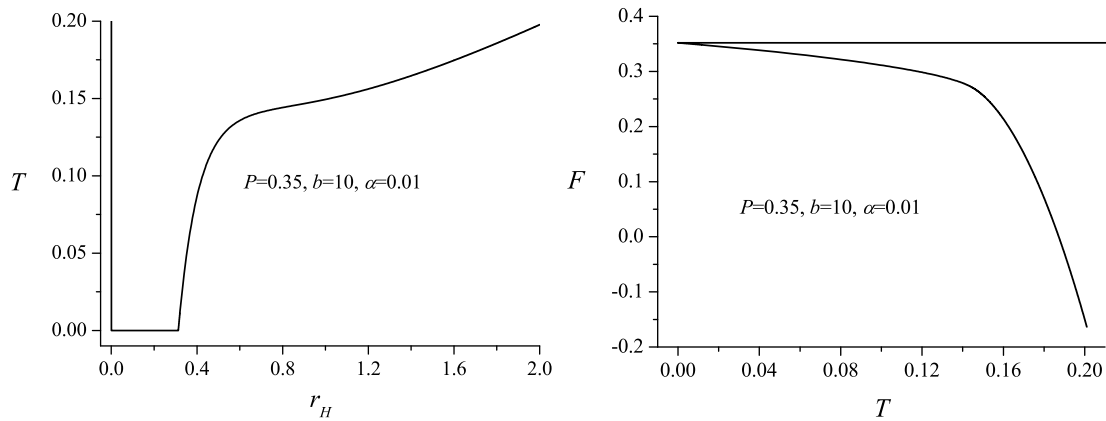


Figure 5. The $T(r_H)$ and the $F(T)$ diagrams and for high magnetic charge interval.

4. Off-shell considerations

Conclusions for the phase structure and the stability of the phases of a thermodynamical system can be made from the off-shell thermodynamic potentials. This method has been applied in [10] for the case of Einstein-Born-Infeld black holes. We will apply it here to give another perspective to the zeroth order phase transition that is observed in the middle charge interval $P_{\text{crit}}^{(1)} < P < P_{\text{crit}}^{(2)}$. In canonical ensemble the off-shell free energy is defined as

$$F_{\text{off}} = M - T_{\text{th}}S. \quad (4)$$

Here T_{th} is the temperature of the thermostat and varies independently from S unlike the temperature of the event horizon T . According to the off-shell formalism, equilibrium states occur at the extrema of the off-shell free energy, with minima corresponding to stable states and maxima – to unstable [11]. The on-shell free energy intersects the curves of the off-shell free energy in the extrema corresponding to the equilibrium phases. Equilibrium black holes have temperature equal to the that of the thermostat $T = T_{\text{th}}$.

Let us demonstrate the relation between the on-shell and the off-shell approach. As we can see on the left panel of Fig. (3) for temperatures $T \in (T_{\text{min}}, T_{\text{max}})$ there are four black holes with different radii of the event horizon and different entropies S , respectively, that have the same temperature. What is the structure of the $F_{\text{off}}(S)$ diagram? On Fig. (6) we have presented the off-shell free energy for three values of the temperature of the thermostat: $T_0 < T = 0.16 < T_{\text{min}}$,

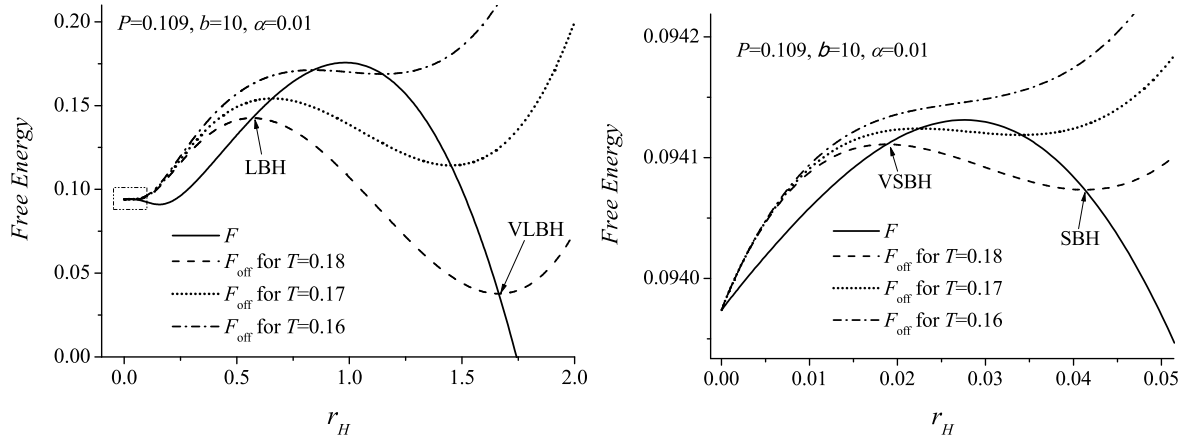


Figure 6. The off-shell free energy for three different values of the temperature and the equilibrium free energy as functions of the entropy. The right panel is a magnification of the enclosed region from the left panel.

$T_{\min} < T = 0.17 < T_{\text{ph}} < T_{\max}$ and $T_{\text{ph}} < T = 0.18 < T_{\max}$. We can clearly see the presence of the VLBH and the LBH on the left panel of Fig. (6). Due to the scale of the figure we cannot get sufficient information for the VSBH and SBH. A magnification of the enclosed region from the left panel of the figure is shown on the right panel. As it can be seen on the figure, in the latter region for the first temperature interval, $T_0 < T < T_{\min}$, the off-shell free energy is monotonous and the VSBH and SBH do not exist. Those phases, however, are present in the two other temperature intervals – $T_{\min} < T < T_{\text{ph}} < T_{\max}$ and $T_{\text{ph}} < T < T_{\max}$.

What does the $F_{\text{off}}(S)$ diagram tell us? It is usually accepted that the system is in the state which realizes not only a local but also a global minimum of the off-shell free energy.

For low enough temperatures, $T_0 < T < T_{\min}$, the thermodynamical system is in the only stable phase – the VLBH. If the temperature rises above T_{\min} but is still lower than T_{ph} the ensemble will be dominated by the SBH since the minimum corresponding to SBH is lower than the one of the VLBH. This temperature interval is presented by the dotted line on Fig. (6). In other words at T_{\min} a phase transition between VLBH and SBH is observed. It could be classified as zeroth order since the on-shell free energy is discontinuous at this point (see Fig. (3)).

What happens if the temperature is further raised? For $T_{\text{ph}} < T < T_{\max}$, presented by the dashed line, the global minimum of F_{off} is realized by the VLBH which means that a phase transition occurs at T_{ph} . The phase transition could be classified as first order since the first derivative of the on-shell free energy is discontinuous at this point (see Figs. (2)–(4)).

5. Sum-up

Let us summarize our results for the rich thermodynamical phase structure of the AdS-scalar-tensor-Born-Infeld black holes in canonical ensemble.

- At low values of the magnetic charge $P < P_{\text{crit}}^{(1)}$ the thermodynamical system has just two phases so no phase transitions are observed when the temperature T is varied while P is kept fixed.
- In the middle charge interval $P_{\text{crit}}^{(1)} < P < P_{\text{crit}}^{(2)}$ there are four thermodynamical phases – two of them stable. The middle charge interval has to be divided in three subintervals. The number and the order of the phase transition is different in each of the subintervals.
 - subinterval I: no phase transitions are observed;

- subinterval II: two phase transitions occur with the increase of the temperature.
The phase transition VLBH \rightarrow SBH is of 0-th order,
The phase transition SBH \rightarrow VLBH is of 1-st order;
- subinterval III: one phase transition is observed,
The phase transition SBH \rightarrow VLBH is classified as 1-st order phase transition;
- For high values of the magnetic charge $P_{\text{crit}}^{(2)} < P$ again no phase transitions are present.

Similar phase transitions are observed also in the case with zero scalar field, namely, the Einstein-Born-Infled black holes.

Acknowledgements

This work was partially supported by the Bulgarian National Science Fund under Grant No DMU 03/6. The authors would also like to thank the Research Group Linkage Programme of the Alexander von Humboldt Foundation for the support.

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