PTEP

Prog. Theor. Exp. Phys. **2018**, 033E02 (20 pages)
DOI: 10.1093/ptep/pty024

The essence of the Blandford–Znajek process

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Received November 14, 2017; Revised January 22, 2018; Accepted February 6, 2018; Published March 27, 2018

From a spacetime perspective, the dynamics of magnetic field lines of force-free electromagnetic fields can be rewritten into a quite similar form for the dynamics of strings, i.e., dynamics of "field sheets". Using this formalism, we explicitly show that the field sheets of stationary and axisymmetric force-free electromagnetic fields have identical intrinsic properties to the world sheets of rigidly rotating Nambu–Goto strings. Thus, we conclude that the Blandford–Znajek process is kinematically identical to an energy-extraction mechanism by the Nambu–Goto string with an effective magnetic tension.

Subject Index E01, E14, E15, E31, E35

1. Introduction and summary

The rotational energy of rotating black holes is a promising energy source for the formation of relativistic jets, which are ubiquitous in astrophysics. The Blandford–Znajek process [1], which is an energy-extraction mechanism by force-free electromagnetic fields, can efficiently achieve powerful energy fluxes and thus has been widely believed to be a viable mechanism to extract the rotational energy of a black hole. There have been a number of analytical and numerical investigations from various aspects during the four decades since this process was proposed. A detailed analysis of the extraction mechanism was in terms of the membrane paradigm [2]. In recent years interpretations and explanations of this mechanism have been discussed (see, e.g., Refs. [3–5]). Moreover, various numerical simulations have been developed and demonstrated [6–8]. In this paper we will analytically reveal the essence of the energy-extraction mechanism in the Blandford–Znajek process from an alternative perspective.

Recently, it has been elucidated that rigidly rotating Nambu–Goto strings [9] twining around a rotating black hole can highly efficiently extract the rotational energy from the black hole [10]. The order of the possible energy flux, namely, the energy-extraction rate, is given by the so-called Dyson luminosity[1] $\sim c^5/G_{\mathrm{N}} \simeq 10^{59}$erg/s [12], consisting of only fundamental constants: the speed of light c and Newton's constant G_{N}, and a dimensionless string tension $G_{\mathrm{N}}\mu/c^2$ as a coefficient. Such energy flux and angular-momentum flux are locally determined by the locus where the string intersects the light surface associated with its angular velocity, at which the velocity of the corotating frame coincides with the speed of light. Moreover, a necessary condition for the energy extraction to occur is that the light surface enters into the ergoregion and the angular velocity of the string is less than that of the black hole.

[1] For historical details, see Ref. [11].

In this mechanism, if we replace the tension of the Nambu–Goto string with a typical magnetic tension of electromagnetic fields surrounding a rotating black hole $\sim B^2 r_{\rm h}^2$, we can reproduce the energy-extraction rate of the Blandford–Znajek process while assuming conventional values of the magnetic field and the black hole mass. This fact suggests that both the energy-extraction mechanisms are closely related and the magnetic field lines with magnetic tension play an essential role in the Blandford–Znajek process. The purpose of this paper is to exhibit that such observations are exactly true in a quantitative and theoretical sense as well as a qualitative and intuitive sense. We show that the energy-extraction mechanism by stationary, axisymmetric force-free electromagnetic fields is essentially identical to that by rigidly rotating Nambu–Goto strings. It follows that *the Blandford–Znajek process, i.e., the energy-extraction mechanism from rotating black holes via force-free electromagnetic fields, is the Penrose process for magnetic field lines with angular-momentum and energy transport mediated by their magnetic tension.*

For this purpose, we will first reformulate the dynamics of force-free electromagnetic fields in accordance with a "field-sheet" formalism [13–16]. In this formalism, magnetic field lines are fundamental objects to describe dynamics rather than electric and magnetic fields. A time evolution of a magnetic field line can be regarded as a 2D extended object in a spacetime. We will call such objects "field sheets", a name that was adopted in Ref. [16], with a similar connotation to the term "world sheet" of strings in a spacetime. In the case of magnetically dominated force-free electromagnetic fields ($F_{\mu\nu}F^{\mu\nu} > 0$) in particular, the field sheets become 2D timelike surfaces characterized by the electromagnetic field strength $F_{\mu\nu}$. We can recast the equations of motion for force-free electromagnetic fields, equivalent to the Maxwell equations with the force-free condition, in the equations of motion in terms of the field sheets. Thus, it turns out that the dynamics of field sheets for force-free electromagnetic fields is similar to the dynamics of a world sheet for a string, and they both belong to the same category of dynamics of a 2D surface in a spacetime (a similar approach to magnetohydrodynamics and its phenomenological applications were discussed in, e.g., Refs. [17–19]). This provides an insight into a correspondence between extraction mechanisms by force-free electromagnetic fields and rigidly rotating strings.

Needless to say, electric and magnetic fields are not components of vector fields, but some of the components of a tensor field, i.e., the electromagnetic field strength $F_{\mu\nu}$. Therefore, depending on coordinate systems or reference frames, their physical interpretations can change as well as their values. One may, for instance, move to a frame at which only magnetic fields can be observed even if there exists a net energy flow. Thus, the Poynting flux is only an interpretation of the energy flow in another frame. Furthermore, in a strong gravitational field, namely, in a curved spacetime, the spacetime metric affects conventions or interpretations of the electric and magnetic field without physical significance. This means that explanations based on electric and magnetic fields will vary depending on the choice of frames. This seems to interfere with the concise understanding of the mechanism. What we should emphasize is that the field sheet is a geometrical object irrelevant to any coordinate system as well as a physically intelligible object, thought of as the time evolution of a magnetic field line. It is expected that we can grasp the essence without suffering from coordinate systems.

With the above perspective in mind, in Sect. 3, we focus on stationary and axisymmetric force-free electromagnetic fields in a stationary and axisymmetric spacetime to examine the energy-extraction process from a rotating black hole. Since the basic properties of such electromagnetic fields have been widely examined and are well known in the literature (see, e.g., Ref. [20] and references therein), we translate those into expressions based on the field sheet. We demonstrate that the field sheet of the electromagnetic field and the world sheet of the Nambu–Goto string have identical intrinsic properties such as their induced geometries. In particular, the specific angular-momentum and energy fluxes per unit tension, flowing on each field sheet, are determined by local configurations of the magnetic field lines in the same manner as the string configurations. Most importantly, on the light surface these specific quantities depend only on its locus without global configurations determined by the equations of motion and must satisfy there identical relations regardless of whether magnetic field lines or strings; if the angular velocity of a magnetic field line is less than that of a black hole, the angular momentum can be extracted, and in addition if the locus where the magnetic field line crosses the light surface enters into the ergoregion, the energy can be extracted. (The importance of the light surface and other characteristic surfaces such as the Alfvén surface has been discussed in the literature [21–23].) However, because the dynamics of magnetic field lines and strings are quite similar but different, both configurations cannot be identical in general. This fact indicates that global configurations of the magnetic fields have little significance for this energy-extraction mechanism and a local, kinematical process in the ergoregion should govern the energy-extraction mechanism. Hence, we can conclude that the essence of both the energy-extraction mechanism by the force-free electromagnetic fields and the rigidly rotating strings is identical.

In contrast with the tension of Nambu–Goto strings, the magnetic tension can vary and should be determined by solving the equations of motion. This means that global configurations of the magnetic fields can affect the value of the magnetic tension. However, a role of the magnetic tension proportionally provides efficiency of extraction rate. Roughly speaking, the larger the magnetic tension is, the more efficient the extraction rate becomes. In this process, the magnetic tension transports the angular-momentum and energy fluxes on the magnetic lines.

The fact that the essence of the extraction mechanism is local kinematics in the ergoregion irrelevant to global configurations offers some instructive insights to clarify the whole picture of the Blandford–Znajek process. It is not so significant whether magnetic field lines can penetrate the event horizon or reach the outer-light cylinder. These cannot be necessary conditions for the Blandford–Znajek process to work. Moreover, the event horizon rather than the ergoregion does not play an essential role in the energy extraction (this issue was addressed in, e.g., Refs. [24,25] on the basis of numerical simulations).

Generally speaking, the most important issue for extracting the rotational energy from black holes is angular-momentum transport to gain the energy in the ergoregion. The outward energy flux is just a by-product of the angular-momentum transport. In the Einstein gravity the spacetime metric can couple to energy–momentum tensors for any matters or fields. If one wishes to extract the energy and angular momentum from the (black hole) spacetime, any other method does not exist except for the method via the energy–momentum tensor (for gravitational waves, the energy–momentum pseudotensor). To elucidate an extraction mechanism of the rotational energy, we should examine what contents of the energy–momentum tensor mainly contribute to the angular-momentum transport (the extraction mechanism for a general energy–momentum tensor is discussed in Ref. [26]). Such angular-momentum and energy transfer, i.e., local conservation of the energy–momentum tensor, should be governed by local physics causally connected. Thus, the extraction process can be roughly dissected into three parts: generating an energy by angular-momentum transfer in the ergoregion, transporting the gained energy to a region far away from the black hole, and disposing of garbage that has lost its angular momentum and energy to the black hole. The essence of the energy extraction mentioned previously is nothing but this generating process. On the other hand, global configurations of the magnetic fields at the event horizon and at a far region are related to the disposing process and the transporting one, respectively.

After all, "boundary conditions" at the event horizon such as whether the magnetic field lines can penetrate the horizon cannot be a necessary condition for the energy extraction. In fact, even if the magnetic field lines failed to penetrate the horizon and could never get drawn into the black hole, the energy extraction by the Blandford–Znajek process can succeed. One may say that the energy is not extracted from any black hole in that case. However, the fact remains that the gravitational energy measured in an asymptotic region is extracted from the total system including the spacetime. Moreover, even though the magnetic field lines cannot reach the outer-light cylinder, we can say that the Blandford–Znajek process is at work if the magnetic field lines extend to a region sufficiently far away from the black hole and the gained energy in the ergoregion can be transferred there.

In this paper, we show that the essential mechanism of the Blandford–Znajek process is determined by local physics in the neighborhood of the ergoregion. Of course, global configurations of the magnetic field and the electric current are important to make the Blandford–Znajek process efficiently and successfully sustaining. However, this is just a stage rather than a principal role. We stress that, in general, how to extract rotational energy from the spacetime and how to arrange appropriate configurations of the magnetic field or electric current for extracting the energy are different questions. Furthermore, we should separately consider the kinematical properties without globally solving the equations of motion and dynamical properties.

The rest of the paper is organized as follows. We first review the field-sheet formalism for force-free electromagnetic fields. Then, we explicitly show a correspondence between the energy-extraction mechanisms by stationary, axisymmetric force-free electromagnetic fields and rigidly rotating strings. In Appendix A we briefly summarize some results for the rigidly rotating strings shown in Ref. [10].

2. Field-sheet formalism for force-free electromagnetic fields

In this section, in order to elucidate the similarity between Nambu–Goto strings and magnetic field lines of force-free fields, we will rewrite the equations of motion for force-free electromagnetic fields according to a "field-sheet" formalism [13–16]. Unless otherwise specified, Newton's constant G_{N} and the speed of light c are set to unity hereafter.

Let $F_{\mu\nu}$ be the electromagnetic field strength and let j_μ be the current density four-vector of electric charge. In terms of $F_{\mu\nu}$, the Maxwell equations are given by

$$\nabla_\alpha F^{\mu\alpha} = 4\pi j^\mu, \quad \nabla_{[\mu} F_{\nu\lambda]} = 0. \tag{1}$$

In general, when an electromagnetic field interacts with charged matter such as plasma, the energy–momentum tensor for the electromagnetic field,

$$T_{\mu\nu} \equiv F_{\mu\alpha} F_\nu{}^\alpha - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} g_{\mu\nu}, \tag{2}$$

satisfies

$$\nabla_\mu T^\mu{}_\nu = -4\pi F_{\nu\alpha} j^\alpha, \tag{3}$$

where the right-hand side of the above equation means the Lorentz force. In the situation where the force density four-vector $F_{\nu\alpha} j^\alpha$ can be neglected, i.e.,

$$F_{\mu\alpha} j^\alpha = 0, \tag{4}$$

which is known as the *force-free condition*, the energy–momentum tensor of the electromagnetic field is individually conserved, $\nabla_\mu T^\mu{}_\nu = 0$, so that neither angular momentum nor energy is exchanged between the electromagnetic field and the other matter. The Maxwell equations (1) together with the force-free condition (4) yield the equations

$$F_{\mu\alpha}\nabla_\beta F^{\alpha\beta} = 0, \quad \nabla_{[\mu}F_{\nu\lambda]} = 0. \tag{5}$$

The dynamics described by these equations is *force-free electrodynamics* (FFE).

An important property for the force-free electromagnetic fields satisfies

$$F_{\alpha\beta}{}^*F^{\alpha\beta} = 0, \tag{6}$$

where ${}^*F_{\mu\nu}$ is the dual of $F_{\mu\nu}$, defined by ${}^*F_{\mu\nu} \equiv F^{\alpha\beta}\epsilon_{\alpha\beta\mu\nu}/2$. These fields are called *degenerate*. Note that the force-free condition for nonzero j^μ implies that $F_{\mu\nu}$ is degenerate,[2] but degeneracy does not always lead to force-freeness. To clarify the physical meanings, let t^μ be a four-velocity of a timelike observer. Then, the electric and magnetic fields measured by this observer are given by $\mathsf{E}^\mu = F^{\mu\nu}t_\nu$ and $\mathsf{B}^\mu = -{}^*F^{\mu\nu}t_\nu$, respectively. The degenerate condition physically means that $\mathsf{E}^\alpha\mathsf{B}_\alpha = 0$, where this relation holds even for an arbitrary observer t^μ because $F_{\alpha\beta}{}^*F^{\alpha\beta}$ is scalar. Now, the fact that $F_{\mu\nu}$ is closed together with the degeneracy implies that ${}^*F_{\mu\nu}$ is tangent to a 2D submanifold, $\mathscr{S}$. Furthermore, we assume that $F_{\mu\nu}$ is *magnetically dominated*, i.e.,

$$F_{\alpha\beta}F^{\alpha\beta} = 2(\mathsf{B}^\alpha\mathsf{B}_\alpha - \mathsf{E}^\alpha\mathsf{E}_\alpha) > 0. \tag{7}$$

Naively, this condition implies that the magnetic field should be stronger than the electric field. Because this condition is also described by a scalar, there is a notion independent of observers. The degeneracy and magnetically dominated condition for $F_{\mu\nu}$ guarantee the existence of a pure magnetic frame in which $\mathsf{E}^\mu = 0$. Therefore, $F_{\alpha\beta}F^{\alpha\beta} > 0$ states that the electromagnetic field $F_{\mu\nu}$ is purely magnetic with its magnitude defined by

$$B \equiv \sqrt{\frac{F_{\mu\nu}F^{\mu\nu}}{2}}. \tag{8}$$

Namely, we can take ${}^*F_{\mu\nu}$ in the form

$${}^*F_{\mu\nu} = B\sigma_{\mu\nu}, \tag{9}$$

where $\sigma_{\mu\nu}$ denotes the 2D volume element on $\mathscr{S}$. It turns out that the magnetically dominated condition for $F_{\mu\nu}$ is equivalent to $\sigma_{\mu\nu}\sigma^{\mu\nu} = -2$, and then $\mathscr{S}$ becomes timelike. We call such $\mathscr{S}$ a *field sheet*. The scalar function $B(> 0)$ irrelevant to coordinate systems means the proper magnitude of the magnetic field, which can be observed at the rest frame of the magnetic field lines. Moreover, the magnetic tension and pressure are given by this quantity, so that they are also proper quantities independent of coordinate systems.

We describe the dynamics of field sheets as string world sheets. In terms of ${}^*F_{\mu\nu}$, the equations of FFE are rewritten as

$${}^*F^{\alpha\beta}\nabla_{[\mu}{}^*F_{\alpha\beta]} = 0, \quad \nabla_\alpha{}^*F^{\alpha\mu} = 0. \tag{10}$$

[2] In four dimensions, the relation $F_{[\alpha\beta}F_{\mu\nu]} = -\epsilon_{\alpha\beta\mu\nu}F_{\lambda\rho}{}^*F^{\lambda\rho}/12$ is satisfied. While $F_{\mu\nu}j^\mu = 0$, we have $F_{\lambda\rho}{}^*F^{\lambda\rho} = 0$ if j^μ is nonzero.

Substituting Eq. (9) into the above, we obtain the equations of FFE in terms of B and $\sigma_{\mu\nu}$ given by

$$\sigma^{\alpha\beta}\nabla_\alpha\sigma_{\beta\mu} = N_\mu{}^\alpha\nabla_\alpha \ln B, \tag{11}$$

$$\nabla_\alpha(B\sigma^{\alpha\mu}) = 0, \tag{12}$$

where $N_\mu{}^\nu$ is defined by

$$N_\mu{}^\nu \equiv g_\mu{}^\nu - h_\mu{}^\nu, \tag{13}$$

$$h_\mu{}^\nu \equiv \sigma_{\mu\alpha}\sigma^{\alpha\nu}. \tag{14}$$

Note that $h_{\mu\nu}$ is the induced metric on the field sheet and $N_\mu{}^\nu$ is the projection tensor onto directions normal to the field sheet. The former equation (11) describes dynamics of field sheets acted on by a force associated with B, i.e., magnetic pressure; the latter equation (12) does conservation of the magnetic flux. This expression tells us that the dynamics of field sheets in FFE is similar to the dynamics of world sheets for Nambu–Goto strings. In fact, it is known that the equations of motion for a Nambu–Goto string can be written as

$$\sigma^{\alpha\beta}\nabla_\alpha\sigma_{\beta\mu} = 0, \tag{15}$$

where $\sigma_{\mu\nu}$ denotes the volume element of the world sheet in this case [28]. Comparing the above equation with Eqs. (11) and (12), we can easily notice that the only difference between the dynamics of field sheets and world sheets is the addition of the extra scalar quantity B, which describes the magnetic tension and pressure. (In the flat spacetime such a correspondence was discussed in Refs. [29–31].)

From a geometrical point of view, their meanings are so clear. In general, the extrinsic curvature of a submanifold is defined by

$$K^\lambda{}_{\mu\nu} \equiv -h_\mu{}^\alpha h_\nu{}^\beta\nabla_\beta h_\alpha{}^\lambda, \tag{16}$$

where $h_{\mu\nu}$ is the induced metric on the submanifold. If the submanifold is 2D, we have

$$K^\lambda{}_{\mu\nu}h^{\mu\nu} = -\sigma^{\alpha\beta}\nabla_\alpha\sigma_\beta{}^\lambda. \tag{17}$$

Thus, the dynamics of field sheets and world sheets belong to the same class of dynamics of a 2D surface (see Ref. [32] and references therein). Because every degenerate, closed two-form defines a foliation of spacetime,[3] the field sheets represented by $F_{\mu\nu}$ can define a 2D timelike foliation with a coordinate transformation $x^\mu = X^\mu(\tau, \sigma, \alpha, \beta)$, where (τ, σ) denote local coordinates on field sheets $\mathscr{S}$ defined by $\alpha =$ const. and $\beta =$ const. Note that, once α and β are fixed, X^μ give embedding functions of $\mathscr{S}$. Let ∂_a be coordinate derivatives with respect to 2D local coordinates on the field sheet. The Latin indices denote intrinsic components on the field sheet. The volume element $\sigma_{\mu\nu}$ is related to X^μ as

$$\sigma^{\mu\nu} = \sigma^{ab}h_a{}^\mu h_b{}^\nu, \tag{18}$$

[3] See Appendix A in Ref. [27].

where $h_a{}^\mu \equiv \partial_a X^\mu$, and σ_{ab} is the intrinsic volume element on $\mathscr{S}$. Moreover, the induced metric can be intrinsically written as

$$h_{ab} = g_{\mu\nu}(X) h_a{}^\mu h_b{}^\nu = h_{\mu\nu} h_a{}^\mu h_b{}^\nu. \tag{19}$$

By using these intrinsic quantities, we can rewrite Eq. (11) in terms of X^μ as

$$D^2 X^\mu + \Gamma^\mu{}_{\alpha\beta} D_a X^\alpha D^a X^\beta = N^{\mu\alpha} \partial_\alpha \ln B, \tag{20}$$

where D_a denotes the covariant derivative with respect to h_{ab} and $\Gamma^\mu{}_{\alpha\beta}$ is the Christoffel symbol associated with $g_{\mu\nu}$. Note that, intriguingly, this equation is derived from the following action: $S[X^\mu] = -\int d^2\sigma B(X)\sqrt{-h}$, where h denotes the determinant of the induced metric h_{ab}.

It is instructive to compare a perfect fluid in terms of conservation of the energy–momentum tensor. As is well known, the energy–momentum tensor of a perfect fluid is given by

$$T^{\mu\nu} = \rho u^\mu u^\nu + p(g^{\mu\nu} + u^\mu u^\nu), \tag{21}$$

where u^μ is a four-velocity of the fluid, and ρ and p are the energy density and pressure in its rest frame, respectively. Decomposing the conservation law $\nabla_\mu T^{\mu\nu} = 0$ into tangential components and normal ones with respect to u^μ yields

$$\begin{aligned} (\rho + p) u^\mu \nabla_\mu u^\nu &= -(g^{\mu\nu} + u^\mu u^\nu)\nabla_\mu p, \\ u^\mu \nabla_\mu \rho + (\rho + p)\nabla_\mu u^\mu &= 0. \end{aligned} \tag{22}$$

If the fluid is pressureless (namely, a dust fluid), the four-velocity of the fluid obeys geodesic equations. In a parallel manner, we consider an energy–momentum tensor given by

$$\begin{aligned} T^{\mu\nu} &= -\mu h^{\mu\nu} + \tilde{p}(g^{\mu\nu} - h^{\mu\nu}) \\ &= (\mu + \tilde{p})\sigma^{\mu\alpha}\sigma^\nu{}_\alpha + \tilde{p} g^{\mu\nu}, \end{aligned} \tag{23}$$

where μ is a tension equal to its energy density and $\tilde{p}$ is a normal pressure. Note that Nambu–Goto strings have a constant μ and $\tilde{p} = 0$, and magnetically dominated electromagnetic fields have $\mu = \tilde{p} = B^2/2$. Decomposing the energy–momentum conservation into components tangential and normal to the field sheet yields

$$\begin{aligned} (\mu + \tilde{p})\sigma^{\alpha\beta}\nabla_\alpha \sigma_{\beta\mu} &= (g_\mu{}^\alpha - h_\mu{}^\alpha)\nabla_\alpha \tilde{p}, \\ \sigma^{\alpha\mu}\nabla_\alpha \mu + (\mu + \tilde{p}) h^\mu{}_\nu \nabla_\alpha \sigma^{\alpha\nu} &= 0. \end{aligned} \tag{24}$$

Thus, it turns out that the world line and its volume element u_μ for a fluid element of perfect fluids correspond to the world sheet (field sheet) and its volume element $\sigma_{\mu\nu}$ for a Nambu–Goto string or a magnetic field line. Moreover, in each pressureless case the world lines of dusts become geodesics and the world sheets of Nambu–Goto strings are extremal surfaces.

If the system admits some symmetries and such a symmetry is characterized by a Killing vector field ξ^μ, the conservation law of the energy–momentum tensor in terms of ξ^μ yields

$$\begin{aligned} 0 &= \nabla_\mu (T^{\mu\nu}\xi_\nu) \\ &= -\nabla_\mu[(\mu + \tilde{p})h^{\mu\nu}\xi_\nu] + \xi^\mu \nabla_\mu \tilde{p} \\ &= -\nabla_\mu[(\mu + \tilde{p})h^{\mu\nu}\xi_\nu], \end{aligned} \tag{25}$$

where we have used $\xi^\mu \nabla_\mu \tilde{p} = 0$ because of the symmetry. This conservation law is equivalent to that of a string with an effective tension $\mu + \tilde{p}$. Thus, even though the pressure does not vanish, $\tilde{p} \neq 0$, kinematic properties such as the conservation law can be identical thanks to the symmetry.

3. Energy extraction by force-free electromagnetic fields

3.1. Stationary and axisymmetric force-free electromagnetic fields

In this section, we consider stationary and axisymmetric force-free electromagnetic fields in a rotating black hole, and reinterpret energy-extraction processes via the force-free electromagnetic fields from a perspective of the field-sheet formalism. Such stationary and axisymmetric force-free electromagnetic fields in a stationary and axisymmetric spacetime have been comprehensively investigated in the literature, and their basic properties are well known [20]. Note that the above assumptions are conventional ones when the Blandford–Znajek process has been discussed. In the field-sheet approach, we will follow Ref. [16].

In order that we will later focus on the Kerr spacetime as an explicit example, we suppose that the spacetime is stationary and axisymmetric and admits two Killing vector fields $(\partial_t)^\mu$ and $(\partial_\phi)^\mu$, which respectively represent time translation symmetry and axisymmetry of the spacetime as $\mathcal{L}_{\partial_t} g_{\mu\nu} = \mathcal{L}_{\partial_\phi} g_{\mu\nu} = 0$. Here, $\mathcal{L}$ denotes the Lie derivative. Its metric can be written as

$$g_{\mu\nu} dx^\mu dx^\nu = g_{tt} dt^2 + 2 g_{t\phi} dt d\phi + g_{\phi\phi} d\phi^2 + g_{rr} dr^2 + g_{\theta\theta} d\theta^2, \tag{26}$$

where we assume that $g_{tt} g_{\phi\phi} - g_{t\phi}^2 = 0$ characterizes the event horizon and $g_{tt} g_{\phi\phi} - g_{t\phi}^2 < 0$ implies the outside of the black hole. Here, we have taken a coordinate system in which the Killing vector fields $(\partial_t)^\mu$ and $(\partial_\phi)^\mu$ are manifestly orthogonal to 2D surfaces spanned by (r, θ) such as the Boyer–Lindquist coordinates for the Kerr spacetime. Moreover, we suppose that the electromagnetic fields share the same Killing symmetries such that $\mathcal{L}_{\partial_t} F_{\mu\nu} = \mathcal{L}_{\partial_\phi} F_{\mu\nu} = 0$.

It is known that stationary and axisymmetric force-free electromagnetic fields are given by the following field strength:

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = d\psi \wedge (\eta - d\varphi), \tag{27}$$

where $\eta \equiv d\phi - \omega(\psi) dt$, and both ψ and φ are scalar functions independent of t and ϕ. This yields a common component representation of the field strength:

$$F_{\mu t} = F_{\mu\nu} (\partial_t)^\nu = -\omega(\psi) \partial_\mu \psi, \quad F_{\mu\phi} = F_{\mu\nu} (\partial_\phi)^\nu = \partial_\mu \psi, \quad F_{r\theta} = \frac{\partial \varphi}{\partial r} \frac{\partial \psi}{\partial \theta} - \frac{\partial \varphi}{\partial \theta} \frac{\partial \psi}{\partial r}, \tag{28}$$

which are related to an electric field, and poloidal and toroidal components of the magnetic field, respectively. Note that we can easily confirm that $dF = 0$ is satisfied. The form of this field strength (27) can be expressed as $F = d\psi \wedge d\hat{\phi}$ in terms of the so-called Euler potentials given by the following scalar functions: ψ and $\hat{\phi} \equiv \phi - \omega(\psi) t - \varphi$ (see, e.g., Ref. [16]). This means that the field sheets are represented by the intersections of the hypersurfaces of constant ψ and $\hat{\phi}$. Each magnetic field line given by the field sheets is rotating with angular velocity $\omega(\psi)$. Obviously, the field sheets described by the same ψ have the same angular velocity because of the axisymmetry. Such a ψ-constant surface is a so-called magnetic surface, and $2\pi\psi = \int F$ gives the magnetic flux across the region enclosed by the magnetic surface.

The "proper" magnetic field B is given by

$$B^2 \equiv \frac{1}{2} F_{\mu\nu} F^{\mu\nu} = \eta^2 |\nabla\psi|^2 + |\nabla\psi|^2 |\nabla\varphi|^2 - (\nabla\psi \cdot \nabla\varphi)^2. \tag{29}$$

Here, we have written the contractions for vector fields u and v as $u \cdot v = u_\mu v^\mu$ and $u^2 = u_\mu u^\mu$ in the abbreviated notation. In addition, we have

$$F_{r\theta}F^{r\theta} = g^{rr}g^{\theta\theta}F_{r\theta}^2 = |\nabla\psi|^2|\nabla\varphi|^2 - (\nabla\psi \cdot \nabla\varphi)^2. \tag{30}$$

As we mentioned in the previous section, $B > 0$ is a scalar function irrelevant to coordinate systems, which gives the magnetic tension and pressure.

Now, χ is a corotating vector defined by

$$\chi^\mu = \left(\frac{\partial}{\partial t}\right)^\mu + \omega(\psi)\left(\frac{\partial}{\partial \phi}\right)^\mu, \tag{31}$$

which satisfies $\chi^\mu\eta_\mu = 0$ and $\chi^2 = g_{tt} + 2g_{t\phi}\omega + g_{\phi\phi}\omega^2 = -(g_{t\phi}^2 - g_{tt}g_{\phi\phi})\eta^2$. Since the corotating vector χ satisfies $F_{\mu\nu}\chi^\nu = 0$, χ is tangential to the field sheet. It turns out that χ represents a corotating frame with the angular velocity $\omega(\psi)$ for each field sheet labeled by constant ψ. If χ (also, η) becomes null, the velocity of such a corotating frame has reached the speed of light. The locus characterized by $\chi^2 = 0$ is called the light surface.[4] Meanwhile, because a field sheet is a 2D surface, there is another tangential vector linearly independent of χ. We will introduce the other tangential vector as

$$\begin{aligned}\lambda^\mu &\equiv {}^*F^{\mu\nu}\nabla_\nu t \\ &= \frac{1}{\sqrt{-g}}\left[F_{\theta\phi}\left(\frac{\partial}{\partial r}\right)^\mu + F_{\phi r}\left(\frac{\partial}{\partial \theta}\right)^\mu + F_{r\theta}\left(\frac{\partial}{\partial \phi}\right)^\mu\right].\end{aligned} \tag{32}$$

By construction, this vector is tangential to the field sheet and normal to $\nabla_\mu t$, so that λ becomes a generator of the intersection of the field sheet and a constant-t surface. This means that the integral curves of λ are magnetic field lines at a time t. The direction of λ is radially outward for $F_{\theta\phi} = \partial_\theta\psi > 0$, while its direction is radially inward for $\partial_\theta\psi < 0$. It turns out that the direction of λ corresponds to that of the magnetic field. Note that, since $[\lambda, \chi] = \mathcal{L}_\lambda\chi = 0$ are satisfied, χ and λ can be coordinate bases on the field sheet as $\chi = \partial/\partial\tau$ and $\lambda = \partial/\partial\sigma$, where τ and σ denote local coordinates on the field sheet. The dual of $F_{\mu\nu}$ is expressed as

$${}^*F^{\mu\nu} = \lambda^\mu\chi^\nu - \chi^\mu\lambda^\nu. \tag{33}$$

This expression manifestly shows that it is a two-form tangential to the field sheet, i.e., proportional to the volume form of the field sheet as ${}^*F_{\mu\nu} = B\sigma_{\mu\nu}$.

3.2. *Induced geometry on the field sheet*

As we mentioned, χ and λ can constitute the tangential coordinate bases on the field sheet, so that components of the induced metric on the field sheet in terms of the coordinate system (τ, σ) are

[4] In general, two light surfaces can exist in a black hole spacetime: one is the inner-light surface near the black hole, and the other is the outer-light surface (light cylinder) in the far region. Since the inner-light surface is important for extracting the rotational energy from the black hole [10], we will focus only on the inner-light surface hereafter.

given by

$$\begin{aligned} h_{\tau\tau} &\equiv \chi^\mu \chi^\nu g_{\mu\nu} = g_{tt} + 2g_{t\phi}\omega + g_{\phi\phi}\omega^2, \\ h_{\tau\sigma} &\equiv \chi^\mu \lambda^\nu g_{\mu\nu} = \frac{F_{r\theta}}{\sqrt{-g}}(g_{t\phi} + g_{\phi\phi}\omega), \\ h_{\sigma\sigma} &\equiv \lambda^\mu \lambda^\nu g_{\mu\nu} = \frac{1}{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}|\nabla\psi|^2 + g_{\phi\phi}\left(\frac{F_{r\theta}}{\sqrt{-g}}\right)^2. \end{aligned} \tag{34}$$

Since χ is the corotating vector, the light surface is characterized by $\chi^2 = h_{\tau\tau} = 0$. In addition, χ is the Killing vector with respect to the induced metric on the field sheet,[5] so that the light surface corresponds to the Killing horizon on the field sheet. This Killing horizon works as a causal boundary for various phenomena governed by the induced metric [16]. We find the following correspondence between the induced metric of the field sheet and that of the rigidly rotating string (A3):

$$\frac{F_{r\theta}}{\sqrt{-g}} \leftrightarrow \varphi', \quad \frac{1}{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}|\nabla\psi|^2 \leftrightarrow g_{rr}r'^2 + g_{\theta\theta}\theta'^2. \tag{35}$$

Indeed, we can explicitly show

$$\mathcal{L}_\lambda \varphi = \lambda^\mu \partial_\mu \varphi = \frac{1}{\sqrt{-g}}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial\varphi}{\partial r} - \frac{\partial\psi}{\partial r}\frac{\partial\varphi}{\partial\theta}\right) = \frac{F_{r\theta}}{\sqrt{-g}}, \tag{36}$$

and

$$\mathcal{L}_\lambda r = \lambda^\mu \partial_\mu r = \frac{\partial_\theta\psi}{\sqrt{-g}}, \quad \mathcal{L}_\lambda \theta = \lambda^\mu \partial_\mu \theta = -\frac{\partial_r\psi}{\sqrt{-g}}. \tag{37}$$

If we introduce the local coordinates as $\chi = \partial/\partial\tau$ and $\lambda = \partial/\partial\sigma$ restricted on the field sheet, the correspondence becomes more apparent. Thus, the intrinsic geometry on the field sheet of the stationary, axisymmetric force-free electromagnetic field and that on the world sheet of the rigidly rotating string have entirely identical properties.

3.3. *Angular-momentum flux and energy flux*

Since the spacetime has time-translational symmetry and axisymmetry, we have conservation laws for the energy–momentum tensor that are associated with each symmetry. In particular, the energy–momentum tensor of the force-free electromagnetic field is conserved independently of other matter. The angular-momentum conservation $\nabla_\mu(T^\mu{}_\nu(\partial_\phi)^\nu) = 0$ yields

$$\begin{aligned} 0 &= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}T^\mu{}_\nu(\partial_\phi)^\nu) \\ &= -\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}F^{\mu\nu}\partial_\nu\psi) \\ &= -\frac{1}{\sqrt{-g}}\left(\partial_\theta\psi\frac{\partial}{\partial r} - \partial_r\psi\frac{\partial}{\partial\theta}\right)\sqrt{-g}F^{r\theta}. \end{aligned} \tag{38}$$

[5] Each component of the induced metric is a function of only r and θ. Since $\mathcal{L}_\chi r = \mathcal{L}_\chi \theta = 0$, we have $\mathcal{L}_\chi h_{ab} = 0$ on the field sheet.

This indicates that $\sqrt{-g}F^{r\theta}$ should depend only on ψ as

$$I(\psi) \equiv \sqrt{-g}F^{r\theta}. \tag{39}$$

Note that this scalar function[6] is constant on the field sheets, and it is one of the characteristic quantities characterizing the stationary and axisymmetric force-free electromagnetic field as well as ψ and $\omega(\psi)$. By using λ, this angular-momentum conservation can be rewritten as

$$\frac{1}{\sqrt{-g}}\partial_t(\sqrt{-g}T^t{}_\nu(\partial_\phi)^\nu) = \lambda^\mu\partial_\mu I(\psi). \tag{40}$$

Therefore, $I(\psi)$ is identified with the angular-momentum flux (per unit magnetic flux $d\psi$) flowing on the field sheet, which should be conserved for each field sheet. Note that $-I(\psi)$ gives the outward angular-momentum flux for $\partial_\theta\psi > 0$, while $I(\psi)$ is the outward flux for $\partial_\theta\psi < 0$. This is because the direction of λ changes depending on the sign of $\partial_\theta\psi$ (for instance, recall that $\mathcal{L}_\lambda r = \partial_\theta\psi/\sqrt{-g}$ in Eq. (37)). It is well known that $I(\psi)$ is connected with the electric current according to Ampère's law. However, we stress that this is *electromagnetic* angular-momentum flux without involving matter such as charged particles, because the relevant energy–momentum tensor consists only of the electromagnetic field and it is individually conserved thanks to the force-free condition. Similarly, the energy conservation $\nabla_\mu(T^\mu{}_\nu(\partial_t)^\nu) = 0$ leads to

$$\frac{1}{\sqrt{-g}}\partial_t(-\sqrt{-g}T^t{}_\nu(\partial_t)^\nu) = \lambda^\mu\partial_\mu[\omega(\psi)I(\psi)], \tag{41}$$

where $\omega(\psi)I(\psi)$ is identified with the conserved energy flux flowing on the field sheet. As in the case of the angular-momentum flux, $-\omega(\psi)I(\psi)$ gives the outward energy flux for $\partial_\theta\psi > 0$, while $\omega(\psi)I(\psi)$ is the outward flux for $\partial_\theta\psi < 0$. Integrating Eqs. (40) and (41) over a 3D volume on a t-constant hypersurface surrounding the black hole, we have total fluxes of angular momentum $\mathcal{J}$ and energy $\mathcal{M}$, i.e., each extraction rate from the black hole,[7] as

$$\frac{d\mathcal{J}}{dt} = -2\pi\int I(\psi)d\psi, \quad \frac{d\mathcal{M}}{dt} = -2\pi\int \omega(\psi)I(\psi)d\psi. \tag{42}$$

Now, we find from Eq. (36) that

$$I(\psi) = \sqrt{-g}F^{r\theta} = (-g)g^{rr}g^{\theta\theta}\frac{F_{r\theta}}{\sqrt{-g}} = (g_{t\phi}^2 - g_{tt}g_{\phi\phi})\mathcal{L}_\lambda\varphi, \tag{43}$$

and also we find from Eqs. (29) and (30) that

$$\begin{aligned} I(\psi)^2 &= (g_{t\phi}^2 - g_{tt}g_{\phi\phi})(B^2 - \eta^2|\nabla\psi|^2) \\ &= (g_{t\phi}^2 - g_{tt}g_{\phi\phi})B^2 + \chi^2|\nabla\psi|^2. \end{aligned} \tag{44}$$

Combining the above two results, we can obtain

$$\hat{q} \equiv \mp\frac{I(\psi)}{B} = \mp\frac{(g_{t\phi}^2 - g_{tt}g_{\phi\phi})\mathcal{L}_\lambda\varphi}{\sqrt{(g_{t\phi}^2 - g_{tt}g_{\phi\phi})(\mathcal{L}_\lambda\varphi)^2 - \chi^2[g_{rr}(\mathcal{L}_\lambda r)^2 + g_{\theta\theta}(\mathcal{L}_\lambda\theta)^2]}}, \tag{45}$$

[6] Since $I(\psi) = {}^*F_{\mu\nu}(\partial_t)^\mu(\partial_\phi)^\nu$, we can see that $I(\psi)$ is scalar.

[7] We denote the mass and angular momentum of the black hole as $\mathcal{M}_{\rm BH}$ and $\mathcal{J}_{\rm BH}$. Each conservation law yields $\frac{d\mathcal{M}}{dt} + \frac{d\mathcal{M}_{\rm BH}}{dt} = 0$ and $\frac{d\mathcal{J}}{dt} + \frac{d\mathcal{J}_{\rm BH}}{dt} = 0$.

where we take the upper minus sign for $\partial_\theta \psi > 0$ and the lower plus sign for $\partial_\theta \psi < 0$. In a unified manner, alternatively, it can be rewritten as

$$\hat{q} = -\frac{(g_{t\phi}^2 - g_{tt} g_{\phi\phi}) d\varphi/dr}{\sqrt{(g_{t\phi}^2 - g_{tt} g_{\phi\phi})(d\varphi/dr)^2 - \chi^2 [g_{rr} + g_{\theta\theta}(d\theta/dr)^2]}}, \tag{46}$$

where $d\varphi/dr \equiv \mathcal{L}_\lambda \varphi / \mathcal{L}_\lambda r$ and $d\theta/dr \equiv \mathcal{L}_\lambda \theta / \mathcal{L}_\lambda r$. This quantity $\hat{q}$ means the specific angular-momentum flux per unit tension, and its expression is identical to that of the specific angular-momentum flux for the rigidly rotating strings [10] (see also Eq. (A7)). Similarly, $\omega(\psi)\hat{q}$ is the specific energy flux per unit tension. Since the sign of $\hat{q}$ has been defined such that $\hat{q}$ should be positive if a radially outward flux, $\hat{q} > 0$ and $\omega \hat{q} > 0$ mean angular momentum extracting and energy extracting, respectively. It is worth noting that the sign of $\hat{q}$ is irrelevant to the direction of λ as is clear from Eq. (46). In other words, whether it is an extracting process or an injecting one does not depend on the direction of the magnetic field but the configuration of the magnetic field line. One thing to keep in mind as a difference from the cases of the rigidly rotating strings is that the specific angular-momentum flux $\hat{q}$ is not conserved while $I(\psi)$ is conserved on the field sheet. The tension of Nambu–Goto strings is constant, while the magnetic tension associated with B can vary even on the field sheet. Therefore, the specific angular-momentum flux per tension should be not necessarily conserved, or more accurately the scalar function B can work as an effective tension for the field sheet.[8]

So far we have shown that the intrinsic structures on the field sheet are identical to those on the world sheet of the rigidly rotating string. This does not mean that the global, extrinsic structures of both objects will be identical. The equations of motion for each are indeed similar, but they are not identical, as seen in the previous section. In general, the global configurations of the Nambu–Goto strings and the magnetic field lines are different, and such global configurations should be determined by each dynamics by solving the equations of motion. However, it is noteworthy that kinetic properties can be locally determined without solving the equations of motion. In what follows we will see that the specific angular-momentum flux is in fact governed by local relations on the light surface. On the light surface, which is characterized by

$$\chi^2 = g_{tt} + 2 g_{t\phi}\omega(\psi) + g_{\phi\phi}\omega(\psi)^2 = 0, \tag{47}$$

we have the specific angular-momentum flux

$$\hat{q}^2 = (g_{t\phi}^2 - g_{tt} g_{\phi\phi})|_{\chi^2=0}, \tag{48}$$

by substituting $\chi^2 = 0$ into Eq. (44). It turns out that this expression is identical to that of the specific angular-momentum flux for the rigidly rotating strings (A6). Since Eqs. (47) and (48) are equations in terms of r and θ essentially, they give relations among the angular velocity ω, the specific angular-momentum flux $\hat{q}$, and the locus of the light surface $(r_{\rm LS}, \theta_{\rm LS})$. Thus, we conclude that both the stationary, axisymmetric force-free electromagnetic fields and the rigidly rotating strings have identical relations. It is worth noting that these relations are determined by the background spacetime metric irrelevant to the dynamics of the electromagnetic field. Even though the dynamics of electromagnetic fields and rigidly rotating strings are different, i.e., their global configurations

[8] As can be seen from the energy–momentum tensor, B^2 describes the magnetic tension per unit area, which has the same dimension as pressure. Now, B describes the magnetic tension per unit magnetic flux.

are different, the same locus of the light surface provides the same angular velocity and specific angular-momentum flux. This fact implies that the specific angular-momentum flux is kinematically determined and both extracting mechanisms of energy and angular momentum via the force-free electromagnetic fields and the rigidly rotating strings are essentially identical.

3.4. Energy extraction in the Kerr spacetime

Now, to discuss the energy-extraction process from a rotating black hole explicitly, let us focus on the Kerr spacetime. Since the major properties of the force-free electromagnetic fields and the rigidly rotating strings are identical, as seen, most of the following argument and its results are the same as those shown in Ref. [10]. For more details, refer to it.

In the Boyer–Lindquist coordinates the metric of the Kerr spacetime with mass M and angular momentum aM is given by

$$g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + \frac{2Mr}{\Sigma}(dt - a\sin^2\theta d\phi)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + (r^2+a^2)\sin^2\theta d\phi^2, \tag{49}$$

where

$$\Sigma(r,\theta) = r^2 + a^2\cos^2\theta, \quad \Delta(r) = r^2 + a^2 - 2Mr. \tag{50}$$

For simplicity and without loss of generality, we assume $a > 0$. This metric gives $g_{t\phi}^2 - g_{tt}g_{\phi\phi} = \Delta\sin^2\theta$, and the event horizon lies at $r = r_{\rm h} \equiv M + \sqrt{M^2 - a^2}$ defined by $\Delta(r_{\rm h}) = 0$. Note that we have $\Delta > 0$ outside the black hole $r > r_{\rm h}$. The angular velocity of the black hole is given by $\Omega_{\rm h} \equiv a/(r_{\rm h}^2 + a^2)$. The ergosphere is characterized by the locus where the stationary Killing vector $(\partial_t)^\mu$ becomes null, i.e., $g_{tt} = 0$, and its radius is $r_{\rm ergo}(\theta) \equiv M + \sqrt{M^2 - a^2\cos^2\theta}$.

In this spacetime, the conditions (47) and (48) at the light surface $(r_{\rm LS}, \theta_{\rm LS})$ become

$$\chi^2 = -\left[1 - \frac{2Mr}{\Sigma}(1-\omega a\sin^2\theta)^2 - \omega^2(r^2+a^2)\sin^2\theta\right] = 0, \tag{51}$$

and

$$\hat{q}^2 = \Delta\sin^2\theta. \tag{52}$$

Solving these equations in terms of r and θ yields the locus of the light surface as[9]

$$r_{\rm LS}(\omega,\hat{q}) = \frac{M}{1+\hat{q}\omega} + \sqrt{\left(\frac{M}{1+\hat{q}\omega}\right)^2 - a(a-\hat{q})}, \tag{53}$$

and

$$\sin\theta_{\rm LS} = \frac{|\hat{q}|}{\sqrt{\Delta(r_{\rm LS})}}. \tag{54}$$

Here, we focus on the northern hemisphere and should take $\pi - \theta_{\rm LS}$ in the southern hemisphere. In order to satisfy $\Delta(r_{\rm LS}) > 0$, namely, the light surface being located outside the horizon, and

[9] As shown in Ref. [10], these equations have another branch of solutions, $r^-(\omega,\hat{q}) \equiv r_{\rm LS}(\omega,-\hat{q})$. This branch describes time-reversal processes rather than physically natural processes.

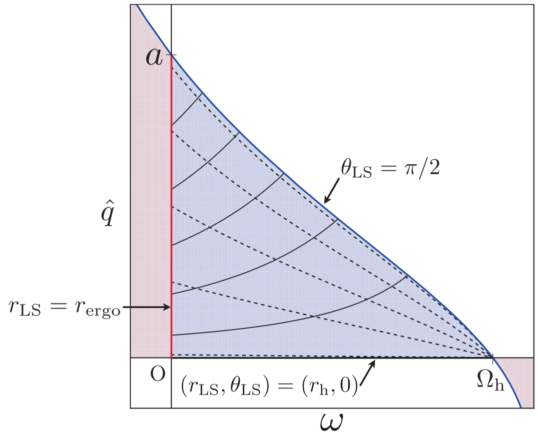

Fig. 1. Typical shape of the allowed parameter region $(\omega, \hat{q})$ focusing on the energy-extraction region (an example of $a/M = 0.9$). The shaded region is the allowed region, and in particular the triangular region $\omega\hat{q} > 0$ is where the energy extraction occurs. The solid and dotted curves denote contour lines for the locus of the light surface $r_{\rm LS}$ and $\theta_{\rm LS}$, respectively. The radius of the light surface will coincide with that of the ergosphere $r_{\rm LS} = r_{\rm ergo}$ on the vertical axis $\omega = 0$ and that of the event horizon $r_{\rm LS} = r_{\rm h}$ on the horizontal axis $\hat{q} = 0$. The curve represented by $\omega = \omega_{\rm eq}(\hat{q})$ indicates that the light surface is located on the equatorial plane $\theta_{\rm LS} = \pi/2$; the horizontal axis that it is on the rotation axis $\theta_{\rm LS} = 0$.

$0 \leq \sin^2\theta_{\rm LS} \leq 1$, we obtain an allowed parameter region in terms of $(\omega, \hat{q})$. The allowed region can be described by the intervals in which ω lies as

$$\begin{aligned} &\omega_{\rm axis}(\hat{q}) < \omega \leq \omega_{\rm eq}(\hat{q}) \quad \text{for} \quad \hat{q} > 0, \\ &\omega_{\rm eq}(\hat{q}) \leq \omega < \omega_{\rm axis}(\hat{q}) \quad \text{for} \quad \hat{q} < 0, \end{aligned} \tag{55}$$

where

$$\omega_{\rm axis}(\hat{q}) \equiv -\frac{1}{\hat{q}}, \quad \omega_{\rm eq}(\hat{q}) \equiv \frac{a - \hat{q}}{2M^2 - \hat{q}(a - \hat{q}) + 2M\sqrt{M^2 - (a^2 - \hat{q}^2)}}. \tag{56}$$

The boundaries represented by $\omega_{\rm eq}$ and $\omega_{\rm axis}$ indicate when the light surface is located at the equatorial plane ($\theta_{\rm LS} = \pi/2$) and when it approaches the rotation axis ($\theta_{\rm LS} \to 0$), respectively. Moreover, the radius of the light surface coincides with that of the ergosphere ($r_{\rm LS} = r_{\rm ergo}$) when $\omega = 0$ and that of the event horizon ($r_{\rm LS} = r_{\rm h}$) when $\hat{q} = 0$ including $\omega = \Omega_{\rm h}$. The typical shape of the allowed region is shown in Fig. 1. Since the region where the energy extraction occurs should be located in $\omega\hat{q} > 0$, a necessary condition for the energy extraction is that a magnetic field line intersects the light surface inside the ergoregion of the Kerr black hole ($r_{\rm h} < r_{\rm LS} < r_{\rm ergo}$) and the angular velocity of the magnetic field line is less than that of the black hole ($\omega < \Omega_{\rm h}$). It turns out that the curve of the upper boundary of the energy-extraction region, represented by $\omega = \omega_{\rm eq}(\hat{q})$, should pass through $(\omega, \hat{q}) = (0, a), (\Omega_{\rm h}, 0)$. Therefore, for an arbitrary a the extraction rate of the specific energy can be bounded by $\sim a\Omega_{\rm h}/4$, and such a maximum extraction rate will be achieved when the angular velocity of the magnetic field line is approximately half of the black hole angular velocity near the equatorial plane ($\omega \simeq \Omega_{\rm h}/2$ and $\theta_{\rm LS} \simeq \pi/2$). It is worth noting that these extraction rates are irrelevant to the mass scale of the central black hole.

This necessary condition applies to each field sheet, i.e., each magnetic field line. It follows that each local configuration of the magnetic field line at the light surface governs whether energy and

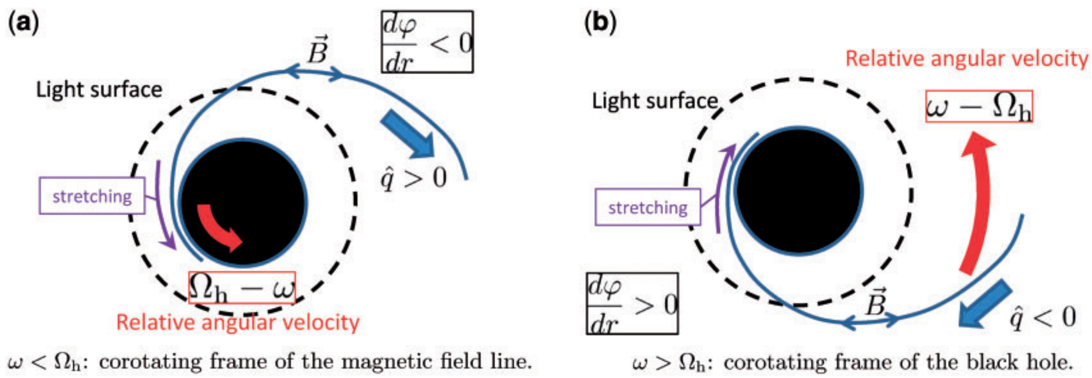

Fig. 2. Relations among local configurations of the magnetic field line and directions of the specific angular-momentum flux $\hat{q}$. The left panel shows that the black hole is rotating faster than the magnetic field line ($\omega < \Omega_{\rm h}$) in the corotating frame of the field line; the right panel shows that the magnetic field line is rotating faster than the black hole ($\omega > \Omega_{\rm h}$) in the corotating frame of the black hole. The directions of the magnetic field are irrelevant. The dashed circle and the black disk depict, respectively, a light surface and a black hole viewed from the top along the rotation axis. The magnetic field line is stretching inside the light surface.

angular-momentum extraction can occur or not, i.e., the sign of $\omega\hat{q}$ and $\hat{q}$. Figure 2 shows relations among local configurations of the magnetic field line and directions of the specific angular-momentum flux $\hat{q}$. From Eq. (46) we find that the sign of $\hat{q}$ is directly connected with the sign of $d\varphi/dr \equiv \mathcal{L}_\lambda\varphi/\mathcal{L}_\lambda r$ regardless of the direction of λ. When the black hole is rotating faster than a magnetic field line, $\omega(\psi) < \Omega_{\rm h}$, the magnetic field line is braking the rotation of the black hole, namely, extracting the angular momentum from the black hole, $\hat{q} > 0$. When a magnetic field line is rotating faster than the black hole, $\omega(\psi) > \Omega_{\rm h}$, the magnetic field line is accelerating the rotation of the black hole, namely, injecting the angular momentum into the black hole, $\hat{q} < 0$. Furthermore, on the premise that the angular-momentum extraction has occurred $\hat{q} > 0$, the magnetic field line can extract the rotational energy of the black hole $\omega(\psi)\hat{q} > 0$ if the light surface enters into the ergoregion $r_{\rm h} < r_{\rm LS} < r_{\rm ergo}$.

As in the case of the rigidly rotating strings, this energy-extraction mechanism via the force-free electromagnetic fields can be simply interpreted as an analogy of the Penrose process. Outside the light surface, each field sheet is stationary with respect to each corotating vector $\chi = \partial_t + \omega(\psi)\partial_\phi$, which is tangential to the field sheet. In other words, the configuration of the magnetic field line does not change in the corotating frame with angular velocity $\omega(\psi)$. However, inside the light surface the field sheet cannot be stationary because the corotating vector is spacelike. This means that the proper motion of each line element of the magnetic field line cannot follow the corotating angular velocity, and therefore the magnetic field line is stretching while its configuration remains unchanged. As a result, angular momentum associated with each line element is transferred toward the central black hole. When the black hole is rotating faster than the magnetic field line, this angular-momentum transfer will make the black hole spin down; when the magnetic field line is rotating faster than the black hole, it will make the black hole spin up. If angular-momentum transfer of the magnetic field line such that the black hole will be spun down occurs in the ergoregion, the magnetic field line can gain energy as a reaction to the angular-momentum transfer. The energy gain will be transferred away from the black hole by the magnetic tension of the magnetic field line. This process is quite similar to the Penrose process for particles. Roughly speaking, the stretching part of the magnetic field lines plays the role of the infalling "object" in the Penrose process.

The total amount of energy and angular-momentum flux is given by multiplying the specific quantities by an effective magnetic tension and integrating the contributions of every magnetic line. In fact, because $B\omega\hat{q}$ and $B\hat{q}$ are conserved on each field sheet, once values of B on the light surface are given, we can obtain the total energy and angular-momentum flux by integrating $B\omega\hat{q}$ and $B\hat{q}$ on the whole light surface as in Eq. (42). To know the details of B, we have to solve the equations of motion and need global information on configurations of the magnetic field. However, the order of possible fluxes together with whether they are injecting or extracting has been locally determined by the specific quantities per unit tension, so that B plays the role of a weight function of the magnetic tension.

An average magnetic tension surrounding the black hole is defined by

$$4\pi r_{\mathrm{h}}^2 \langle B^2 \rangle \equiv 2\pi \int_{\mathrm{LS}} B|d\psi|. \tag{57}$$

The energy-extraction rate can be estimated as

$$\frac{d\mathcal{M}}{dt} = \frac{\pi}{2} M^2 \langle B^2 \rangle u(\alpha) \int_{\mathrm{LS}} \frac{B|d\psi|}{2r_{\mathrm{h}}^2 \langle B^2 \rangle} \frac{\omega\hat{q}}{\Omega_{\mathrm{h}} a/4}, \tag{58}$$

where $u(\alpha) \equiv \alpha^2(1 + \sqrt{1-\alpha^2})$ and $\alpha \equiv a/M$.

The whole of the Blandford–Znajek process can be classified into what is governed by local kinematics and what is governed by dynamics by solving the equations of motion. The former is the energy-extraction mechanism characterized by the relations between the specific energy and angular-momentum fluxes, $\omega\hat{q}$ and $\hat{q}$, and the locus of the light surface, which we have shown in this paper; the latter is global configurations of the magnetic field lines and the electric current, the value of the magnetic tension B, and so on. This fact suggests the following whole picture of the Blandford–Znajek process. Depending on the environments in an outer region, the global configurations of the magnetic field lines associated with plasmas are dynamically determined according to the equations of motion. If only appropriate configurations of the magnetic field lines can be locally realized in an inner region around the ergoregion, the magnetic field lines start to extract the energy and angular momentum of the black hole by a kinematical mechanism irrelevant to situations in the outer region. For example, to obtain explicitly the directional dependence of the energy and angular-momentum fluxes in the far region, we have to solve globally the equations of motion for the electromagnetic fields and to know the entire configurations of the magnetic field lines. This implies that we should examine the details of the surrounding system, i.e., the boundary conditions in the outer region, in order to explore the directional dependence in the far region where astrophysical jets are generated. However, in order to explain and understand the energy-extraction mechanism in the Blandford–Znajek process, we should consider just local kinematics in the inner region between the light surface and the ergosphere. The directional dependence of the specific energy and angular-momentum fluxes at the light surface has been shown in the contour of, e.g., Fig. 1. This provides the potential to generate the energy and angular-momentum fluxes in the Blandford–Znajek process. We can see that the closer to the equatorial plane $\theta_{\mathrm{LS}} = \pi/2$ the locus where the magnetic field lines intersect the light surface is, the larger specific energy flux $\omega\hat{q}$ the magnetic field lines can generate in general.

4. Discussion

In this paper we have shown that the essence of the energy extraction by the Blandford–Znajek process is local kinematics inside the ergoregion. Therefore, global configurations of the magnetic field line, such as whether the magnetic field lines can thread the event horizon or not, should not be the essence of this process. The Znajek condition [33], which is well known as a condition to be satisfied at the event horizon, cannot be a necessary condition for the energy extraction. This condition is a consistency condition for the force-free magnetic field lines to regularly cross the event horizon (an identical condition (A11) can be derived for the rigidly rotating strings). The main process for the energy extraction occurs between the inner-light surface and the ergosphere, and, besides, the light surface is the causal boundary for this system. Therefore, the energy extraction can occur even if this condition is not necessarily satisfied. For instance, it does not matter if the force-free condition for plasma has been violated at the event horizon. For the same reason, moreover, the so-called Meissner-like effect, in which the extremal Kerr black hole tends to expel magnetic fields (see, e.g., Ref. [34] and references therein), should not have a direct connection with the Blandford–Znajek process as long as we do not require an extra assumption at the horizon for a different reason.

Throughout this paper we have discussed stationary and axisymmetric force-free electromagnetic fields for the energy extraction. At the least, we need these assumptions between the inner-light surface and the ergosphere. If such assumptions are violated, it is expected that the energy extraction by the Blandford–Znajek process will become less efficient. The reason why the Blandford–Znajek process can efficiently extract the rotational energy from the black hole and produce a highly powerful energy flux is that a relativistic tension, which is equal to its energy density, carries the angular-momentum and energy flux along the magnetic field lines in the same way as Nambu–Goto strings. If the axisymmetry, stationariness, or force-freeness are violated, the magnetic pressure or other matters begin to affect the angular-momentum and energy transfer and then alter the correspondence to the Nambu–Goto strings. This seems to be a disadvantage for the energy extraction.

What is necessary for the Blandford–Znajek process to occur is toroidal magnetic fields winding the black hole at the light surface inside the ergoregion. As we showed in Eq. (46), the angular-momentum and energy fluxes depend on toroidal configurations of the magnetic field lines, namely, the toroidal component of the magnetic field. This is because toroidal magnetic fields take the central role in the angular-momentum transfer of the magnetic field lines. The poloidal components of the magnetic fields determine the directions of the angular-momentum and energy flows.

For the Blandford–Znajek process, since magnetic field lines with magnetic tension are essential and the transferred energy and angular momentum are purely electromagnetic, it seems that the electric current or plasma has only an auxiliary role to sustain the magnetic fields. In fact, Ref. [35] recently showed that magnetic fields without plasma can extract the rotational energy of a black hole in lower spacetime dimensions.

Acknowledgements

We would like to thank Kenji Toma and Yudai Suwa for useful comments. We would like to thank the participants of the workshop "FuwakuBZ77" for helpful discussions. This work was supported by JSPS KAKENHI Grant Numbers JP16K17704 (S.K.) and the MEXT-Supported Program for the Strategic Research Foundation at Private Universities, 2014-2017, S1411024 (T.I.).

Appendix A. Rigidly rotating strings

In this appendix we summarize useful results for rigidly rotating strings around a rotating black hole, some of which were obtained in Ref. [10].

In a stationary and axisymmetric spacetime (26), a stationary rotating string with angular velocity ω is embedded as

$$t = \tau, \quad \phi = \omega\tau + \varphi(\sigma), \quad r = r(\sigma), \quad \theta = \theta(\sigma). \tag{A1}$$

The corotating (Killing) vector χ that is tangential to the world sheet becomes

$$\chi = \frac{\partial}{\partial t} + \omega\frac{\partial}{\partial\phi}. \tag{A2}$$

The induced metric of the world sheet is given by

$$h_{ab}d\sigma^a d\sigma^b = (g_{tt} + 2g_{t\phi}\omega + g_{\phi\phi}\omega^2)d\tau^2 + 2\varphi'(g_{t\phi} + g_{\phi\phi}\omega)d\tau d\sigma + (g_{rr}r'^2 + g_{\theta\theta}\theta'^2 + g_{\phi\phi}\varphi'^2)d\sigma^2, \tag{A3}$$

where the prime denotes a derivative with respect to σ. From this induced metric the Lagrangian density for the Nambu–Goto string with unit tension is given by

$$\mathscr{L} = -\sqrt{(g_{t\phi}^2 - g_{tt}g_{\phi\phi})\varphi'^2 - \chi^2(g_{rr}r'^2 + g_{\theta\theta}\theta'^2)}, \tag{A4}$$

where $\chi^2 = \chi^\mu\chi^\nu g_{\mu\nu} = h_{\tau\tau}$. The light surface is characterized by

$$\chi^2 = g_{tt} + 2g_{t\phi}\omega + g_{\phi\phi}\omega^2 = 0, \tag{A5}$$

and a consistency condition that the string can pass through the light surface regularly is given by

$$q^2 = (g_{t\phi}^2 - g_{tt}g_{\phi\phi})|_{\chi^2=0}. \tag{A6}$$

Here, q is a conserved quantity characterizing the configuration of the rigidly rotating strings, which means the specific angular-momentum flux per unit tension flowing on the string world sheet. Its expression is written as

$$q \equiv \frac{\partial\mathscr{L}}{\partial\varphi'} = -\frac{(g_{t\phi}^2 - g_{tt}g_{\phi\phi})\varphi'}{\sqrt{(g_{t\phi}^2 - g_{tt}g_{\phi\phi})\varphi'^2 - \chi^2(g_{rr}r'^2 + g_{\theta\theta}\theta'^2)}}, \tag{A7}$$

where the sign of q has been defined so that $q > 0$ describes a radially outward flux when $r' > 0$.

Using Eqs. (A4) and (A7), we have the following identity:

$$q^2\mathscr{L}^2 = (g_{t\phi}^2 - g_{tt}g_{\phi\phi})[\mathscr{L}^2 + \chi^2(g_{rr}r'^2 + g_{\theta\theta}\theta'^2)]. \tag{A8}$$

Suppose that $(g_{t\phi}^2 - g_{tt}g_{\phi\phi})|_{r=r_\mathrm{h}} = 0$ and $(g_{rr})^{-1}|_{r=r_\mathrm{h}} = 0$ should be satisfied at the event horizon $r = r_\mathrm{h}$. The determinant of the metric components g should not be degenerate, so that the equation

$$(g_{t\phi}^2 - g_{tt}g_{\phi\phi})g_{rr} = -g/g_{\theta\theta} \tag{A9}$$

is satisfied even at $r = r_\mathrm{h}$. If the string can regularly pass through the event horizon, the volume element of the string world sheet, namely, the Lagrangian density $\mathscr{L}$, should be finite and nonzero at the event horizon. Moreover, at $r = r_\mathrm{h}$ we find

$$\chi^2|_{r=r_\mathrm{h}} = g_{\phi\phi}(\omega - \Omega_\mathrm{h})^2, \tag{A10}$$

where we have used $g_{t\phi}^2 = g_{tt}g_{\phi\phi}$ and $\Omega_\mathrm{h} = -g_{t\phi}/g_{\phi\phi}$ at $r = r_\mathrm{h}$. Combining the above results (A8), (A9), and (A10), we obtain the following condition at the event horizon:

$$q^2 = (\omega - \Omega_\mathrm{h})^2 \frac{(-g)}{\mathscr{L}^2} \left(\frac{dr}{d\sigma}\right)^2 \frac{g_{\phi\phi}}{g_{\theta\theta}}\bigg|_{r=r_\mathrm{h}}. \tag{A11}$$

Note that, by definition, the combination $\mathscr{L}d\sigma$ is invariant under reparameterization of the worldsheet coordinate σ. This condition is identical to the Znajek condition for force-free electromagnetic fields:

$$I(\psi) = (\omega - \Omega_\mathrm{h})\partial_\theta\psi \sqrt{\frac{g_{\phi\phi}}{g_{\theta\theta}}}\bigg|_{r=r_\mathrm{h}}, \tag{A12}$$

which we can also obtain by evaluating Eq. (44) at the event horizon.

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