Quasinormal Modes of Black Holes Localized on the Randall-Sundrum 2-brane

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Abstract

We explore the quasinormal modes of a black hole localized on the Randall-Sundrum 2-brane against the test field perturbations [1]. The background metric admits a conformal Killing tensor, which allows us to obtain tractable–indeed separable–field equations for the conformally coupled scalar, the massless Dirac and the Maxwell fields. We find that each radial equation obeys the same equation as the corresponding equation in the Schwarzschild background with the same horizon radius. The angular equation reflects the impact of brane tension, resulting in a distinct quasinormal mode prediction.

1 Exact description of a braneworld black hole

The Randall-Sundrum scenario has attracted much attention for intriguing phenomenological prediction in particle physics and cosmological scenario [2]. In order to further validate this model, the strong gravity test is imperative. Black holes are the best experimental fields for this purpose. Despite that a lot of effort has been devoted toward finding an exact solution describing a black hole localized on the Randall-Sundrum 3-brane, none of them are successful yet. Some authors discussed based on the AdS/CFT correspondence that the static (and large) back holes fail to exist [3, 4]. Under the present circumstances, it is very illustrative to look in a bird's-eye view. We consider a simple toy model and extract useful information. Emparan, Horowitz and Myers succeeded in constructing the desired black hole solution in a 3+1-dimensional Randall-Sundrum model [5], on which we will focus hereafter.

The Randall-Sundrum braneworld has a negative cosmological constant in the bulk, so as to reproduce a Newtonian gravity on the brane. An important point to notice is the fact that the brane undergoes a uniform acceleration in the background AdS. So the repelling acceleration is required to obtain a static black hole. In 4-spacetime dimensions, the C-metric in AdS is the desired solution. For the Z_2-symmetry across the brane, the metric describing a localized black hole is given by

$$g_{ab}\mathrm{d}x^a\mathrm{d}x^b = \frac{\ell^2}{(|x|-y)^2}\left[F(y)\mathrm{d}t^2 - \frac{\mathrm{d}y^2}{F(y)} + \frac{\mathrm{d}x^2}{G(|x|)} + G(|x|)\mathrm{d}\varphi^2\right], \tag{1}$$

where

$$F(y) = -y^2 - 2\mu y^3, \qquad G(x) = 1 - x^2 - 2\mu x^3. \tag{2}$$

The metric (1) solves Einstein's equations with a negative cosmological constant, $R_{ab} = -(3/\ell^2)g_{ab}$, for $|x| > 0$. The brane locus is $x = 0$, across which the bulk has a mirror symmetry. Israel's junction condition shows that the brane tension is proportional to $1/\ell$. The x and y-coordinates are the direction cosine $x = \cos\theta$ and the radial coordinate $y = -\ell/r$. x ranges from the brane $x = 0$ to the axis $x = x_2$–the largest root of $G(x_2)$–and y from the horizon $y = -1/2\mu$ to infinity $y = x$. Observe that in order to avoid conical singularities on the axis $x = x_2$, the period of φ is fixed by $\Delta\varphi = 2\pi\beta$, where $\beta^{-1} = -G'(x_2)/2$.

The parameter μ corresponds to the mass of the black hole, hence we take $\mu \geq 0$. The black hole with small μ mimics the isolated black hole. Indeed, one can confirm easily that the $\mu \to 0$ limit with $r_h = 2\mu\ell$ fixed recovers the Schwarzschild black hole with the horizon radius $r_h(\ll \ell)$. On the other hand, the horizon will become flattened in the $\mu \gg 1$ limit (see Fig. 1). In the latter case, the brane tension has a distinguished effect on the properties of black holes from the isolated ones.

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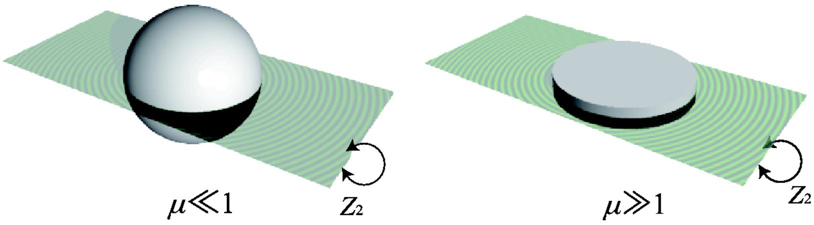

Figure 1: Schematic picture of a black hole localized on the brane. For $\mu \ll 1$, the horizon is round (left). As μ increases, the horizon tends to deform into a flattened pancake (right).

The induced metric on the brane reads

$$\mathrm{d}s^2 = -\left(1 - \frac{r_h}{r}\right)\mathrm{d}\tilde{t}^2 + \left(1 - \frac{r_h}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\varphi^2, \tag{3}$$

where $\tilde{t} = \ell t$. Since $\Delta\varphi$ is less than 2π, the 2+1 dimensional gravity is recovered on the brane, as we desired.

2 Quasinormal frequencies

Since the black hole metric (1) is so involved, little is known about the quantitative properties of the solution. Quasinormal modes of black holes are the highly suggestive quantities to identify the intrinsic properties black holes.

At first glance, the background metric would not allow a separation of variables. Indeed, the massless scalar field equation are not separable, due to the overall factor $(x - y)^{-2}$ in front of the metric. We will circumvent this difficulty by restricting ourselves to conformally invariant field equations–the conformally coupled scalar field, the massless Dirac field and the Maxwell fields. For these fields, we can discard the annoying factor $(x - y)^{-2}$ from the equations of motion via the simple conformal transformation

$$\hat{g}_{ab} = \Omega^2 g_{ab}, \qquad \Omega = x - y. \tag{4}$$

In this article, only the conformally coupled scalar field is discussed. But we can proceed analogously for the Dirac and Maxwell fields, the behavior of which are qualitatively the same. A more detailed analysis can be found in Ref. [1].

A conformally coupled scalar field evolves according to

$$\left(\nabla_a\nabla^a - \frac{1}{6}R\right)\Phi = 0, \tag{5}$$

which is form-invariant under the conformal transformation (4) with $\Phi \to \hat{\Phi} = \Omega^{-1}\Phi$. Let us assume the separable form $\hat{\Phi} = e^{-i\omega t + im\varphi/\beta}R_0(y)S_0(x)$. Noticing that the Ricci scalar of the conformally related metric $\hat{g}_{ab}$ is $\hat{R} = 12\mu(x - y)$ and working in the $x > 0$ region, we can reduce the field equation to "angular" and "radial" equations as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[G(x)\frac{\mathrm{d}}{\mathrm{d}x}S_0\right] + \left[\nu(\nu+1) - 2\mu x - \frac{m^2/\beta^2}{G(x)}\right]S_0 = 0, \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[F(y)\frac{\mathrm{d}}{\mathrm{d}y}R_0\right] + \left[\nu(\nu+1) - 2\mu y + \frac{\omega^2}{F(y)}\right]R_0 = 0, \tag{7}$$

where ν is the separation constant. Taking into account the Z_2-symmetry across the brane, equation (6) gives rise to the Neumann boundary condition at the brane, $\mathrm{d}S_\mathrm{s}/\mathrm{d}x|_{x=0} = 0$. The boundary condition at

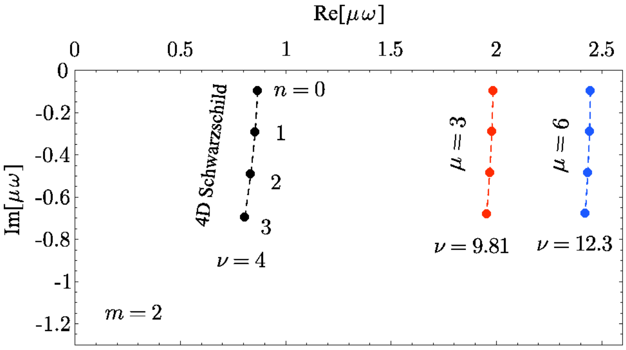

Figure 2: Quasinormal modes of a conformally coupled scalar for $m = 2$. The plots are for the second lowest ν modes.

$x = x_2$, on the other hand, is read off by defining $\tilde{S}_0 := (x_2 - x)^{-|m|/2} S_0$ and requiring the regularity for $\tilde{S}_0$ at $x = x_2$. The angular eigenvalue is obtained numerically [1]. The numerical result shows that as μ increases, the angular eigenvalue ν will become large. This tendency is also found for the codimension-two brane (see, e.g., [6]). For $\mu \ll 1$, $\nu - m$ is approximated by an even integer, recovering the (even mode) spectrum in the Schwarzschild background.

Let us move on to the analysis of equation (7). This equation can be written in a familiar form using $r = -\ell/y$ and $\mathrm{d}r_* = (1 - 2\mu\ell/r)^{-1}\mathrm{d}r$. The black hole horizon is mapped to $r_* = -\infty$ $(r = 2\mu\ell)$, while the acceleration horizon $y = 0$ corresponds to $r_* = +\infty$. In terms of r_*, equation (7) is translated into the Schrödinger-type equation

$$\frac{\mathrm{d}^2}{\mathrm{d}r_*^2} R_0 + [\tilde{\omega}^2 - V_0(r)] R_0 = 0, \qquad V_0(r) = \left(1 - \frac{2\mu\ell}{r}\right)\left(\frac{\nu(\nu+1)}{r^2} + \frac{2\mu\ell}{r^3}\right), \tag{8}$$

where $\tilde{\omega} := \omega/\ell$. It is important to note that equation (8) is apparently identical to the radial equation for the (conformally coupled) scalar field perturbation in the four dimensional Schwarzschild background with the horizon radius $r_h = 2\mu\ell$. *The only distinction arises from the different angular eigenvalues.*

We impose the following quasinormal boundary conditions for the radial equation

$$R_0 \to e^{+i\tilde{\omega} r_*} \quad (r_* \to \infty), \qquad R_0 \to e^{-i\tilde{\omega} r_*} \quad (r_* \to -\infty). \tag{9}$$

having only an incoming wave at the black hole horizon and an outgoing wave at the acceleration horizon. These boundary conditions give rise to a discrete frequency, corresponding to the quasinormal modes of black holes. We employ the third order WKB method developed by Iyer and Will [7] in order to evaluate the quasinormal modes. We show the result in Fig. 2. We have normalized the frequencies by (half of) the horizon radius: $\mu\ell \cdot \tilde{\omega} = \mu\omega$. As μ increases, the real part of quasinormal modes are likely to enlarge, as expected from the results in Ref. [6].

The WKB approximation will fail for higher overtones with $n > \nu$. The asymptotic quasinormal modes $(n \gg 1)$ are obtained using the monodromy method [8] as

$$\mu\omega_n \approx \frac{\ln 3}{8\pi} - \frac{\mathrm{i}}{4}\left(n + \frac{1}{2}\right), \tag{10}$$

where one takes $n \to \infty$. This is independent of the angular eigenvalue and therefore the leading order behavior will be the same for the brane-localized and braneless black holes.

3 Summary

In this article we explored the quasinormal modes of black holes localized on the Randall-Sundrum 2-brane. The background is the exact black hole solution found in Ref. [5]. Taking advantage the fact that the background metric allows a conformal Killing tensor, which is a direct consequence of Petrov type D, we have investigated the behavior of conformally invariant field perturbations around the brane-localized black hole.

For all types of fields we considered, we found that each radial equation is identical to the corresponding field equation in the four Schwarzschild background. However, the angular equations differ from their Schwarzschild counterparts. We have determined the angular eigenfunctions and eigenvalues numerically. In the case of the conformal scalar field, the angular eigenvalues ν are given by $\nu \simeq l = 0, 1, 2, ...$ for a small black hole ($\mu \ll 1$), recovering the Schwarzschild result. As the size of the black hole increases, ν becomes larger ($\nu > l$) and hence become sensitive to the magnetic quantum number m. Accordingly, each quasinormal modes of the large localized black hole (with the horizon radius $2\mu\ell$ on the brane) behaves like the mode having a larger angular mode number in the four dimensional Schwarzschild background with the same horizon radius $r_h = 2\mu\ell$. The situation is basically the same for electromagnetic and massless Dirac field perturbations. In particular, we have found no unstable modes for any types of fields we investigated.

It has been widely known that the Randall-Sundrum braneworlds have a rich structure concerning the AdS/CFT correspondence [3, 4]. It would be also interesting to discuss the implications of our result from the viewpoint of the AdS/CFT correspondence.

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