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LOW DIMENSIONAL FIELD THEORY AND STATISTICAL MODEL

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T. D. Lee

* The CCAST international symposium/workshop series is published separately by Gordon and Breach Scientific Publishers.

Contents

Elliptic Zamolodchikov-Faddeev Algebra and Q-Deformed Affine Algebra

Heng Fan[a,b], Bo-yu Hou[b], Kang-jie Shi[a,b]
Wen-li Yang[a,b], Rui-hong Yue[a]
[a]CCAST(World Laboratory)
P.O.Box 8730, Beijing 100080, China
[b]Institute of Modern Physics, Northwest University
P.O.Box 105, Xian, 710069, China

Abstract

We propose a method to construct the factorizable L-operator of elliptic Affine algebra in term of the vertex-face intertwiners. In the level one representation, we give explicitly $L^{\pm}$ operator and vertex operators. The L operator can be used to set up the comodule of elliptic Zamolodchikov- Faddeev algebra.

1 introduction

Recently. the studies on the systems with the degrees of infinite freedom have been successed. Of the important ones [1] is that the one-dimensional infinite size lattice six vertex model (H_{xxz}) at the principle regime favours a q-deformed affine algebra symmetry (with no-zero central extension) [2,3]. It is different from the q-deformed lie algebra $su_q(2)$ symmetry in same model with finite size lattice and fixed boundary condition [4]. By using the crystal bases. Davies. Foda. Jimbo. Miwa and Nakayashiki give an approach to set up the infinite dimensional representation spaces of an affine algebra $U_q(\widehat{sl_2})$ in physical system [1.5]. They also study systemly the trigonometric statistical models and related Sine-Gordon and Thirring models in term of vertex operator (dynamical operator) in admissible configuration space. With these mathematical tools they derive the difference equation satisfied by correlation functions and get some physical quantities such as stragged spontaneous polarization. Besides. they generalize the idea into elliptic eight vertex model (H_{xyz} chain) [7]. We find that the elliptic generalization of affine algebra has a deep mathematical background. The symmetry of physical systems related to elliptic R matrices is the elliptic q-deformation of affine algebra (Sklyanin algebra with centre extension) [8.9,10].

2 Vertex-Face Intertwiner and ZF Algebra

Belavin [12] $Z_n \otimes Z_n$ model can be defined as following. For a vertex. an intersection of a vertical line and a horizontal line. we associate a Boltzmann weight $R_{ij,kl}(z)$ (see Figure 1). The subscripts i, j, k, l go around the vertex from down edge in the clockwise order and take value in $\{0, n-1\}$.

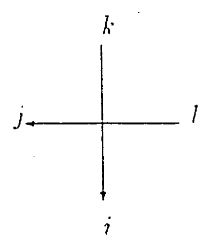


Figure 1: Vertex Boltzmann weight R

$$R_{ij,kl}(z) = \begin{cases} \frac{h(z)\theta^{(i-j)}(z+w)}{\theta^{(i-k)}(w)\theta^{(k-j)}(z)} & \text{if } i+j = k+l \mod n \\ 0 & \text{if } i+j \neq k+l \mod n \end{cases} \quad (1)$$

where

$$h(z) = \frac{\prod_{j=0}^{n-1}\theta^{(j)}(z)}{\prod_{j=1}^{n-1}\theta^{(j)}(0)}$$

$$\theta^{(j)}(z) = \theta\left[\begin{array}{c} 1/2 - j/n \\ 1/2 \end{array}\right](z, n\tau)$$

$$\theta\left[\begin{array}{c} r \\ s \end{array}\right](z,\tau) = \sum_{m\in Z}\exp\{i\pi\tau(m+r)^2 + i2\pi(m+r)(z+s)\} \quad (2)$$

where z and w are spectral and crossing parameters respectively. We point out here that the R matrix defined by equation (1) is same as one given by Jimbo et al [13]. It is the transposition of one in reference [14]. This R matrix satisfies Yang-Baxter equation

$$R_{12}(z_1 - z_2)R_{13}(z_1 - z_3)R_{23}(z_2 - z_3) = R_{23}(z_2 - z_3)R_{13}(z_1 - z_3)R_{12}(z_1 - z_2) \quad (3)$$

and some useful properties [14,15,16,19].

1.Initial condition $\quad R_{ij,kl}(0) = \delta_{il}\delta_{jk} \quad (4)$

2.Unitary condition $\quad R_{ik,jl}(z)R_{lj,k'i'}(-z) = \frac{h(z+w)h(z-w)}{h^2(w)}\delta_{ii'}\delta_{kk'} \quad (5)$

3.Crossing condition $\quad R_{il,kj}(z)R_{k'j,i'l}(-z-nw) = \frac{h(z)h(-z-nw)}{h^2(w)}\delta_{jj'}\delta_{kk'} \quad (6)$

In the dual representation the R matrix is [2,3]

$$\begin{aligned} R^{V,V^*}(z) &= R^{-1,t_2}(z) \qquad R^{V,^*V}(z) = R^{t_2,-1}(z) \\ R^{^*V,V}(z) &= R^{t_1,-1}(z) \qquad R^{V^*,V}(z) = R^{-1,t_1}(z) \\ R^{V^*,V^*}(z) &= R^{t_1,t_2}(z) \end{aligned} \quad (7)$$

From the known results of reference [17,18,19], we can show

$$
\begin{aligned}
R^{V,V^*}(z) &= -\frac{\sigma_0(w)}{\sigma_0(z+w)}\prod_{p=1}^{n-1}\frac{\sigma_0(w)}{\sigma_0(z-pw)}(1\otimes P^-_{n-1}) \\
&\quad \times R^{1,2}(z-w)\cdots R^{1,n}(z-(n-1)w)(1\otimes P^-_{n-1}) \\
R^{V,^*V}(z) &= -\frac{\sigma_0(w)}{\sigma_0(z+(n+1)w)}\prod_{p=1}^{n-1}\frac{\sigma_0(w)}{\sigma_0(z+pw)}(1\otimes P^-_{n-1}) \\
&\quad \times(1\otimes P^-_{n-1})R^{1,2}(z+(n-1)w)\cdots R^{1,n}(z+w)(1\otimes P^-_{n-1}) \\
R^{V^*,V}(z) &= -\frac{\sigma_0(w)}{\sigma_0(z+w)}\prod_{p=1}^{n-1}\frac{\sigma_0(w)}{\sigma_0(z-pw)}(P^-_{n-1}\otimes 1) \\
&\quad \times R^{1,n}(z-w)\cdots R^{n-1,n}(z-(n-1)w)(P^-_{n-1}\otimes 1) \\
R^{^*V,V}(z) &= -\frac{\sigma_0(w)}{\sigma_0(z+(n+1)w)}\prod_{p=1}^{n-1}\frac{\sigma_0(w)}{\sigma_0(z+pw)}(P^-_{n-1}\otimes 1) \\
&\quad \times R^{1,n}(z-(n-1)w)\cdots R^{n-1,n}(z+w)(P^-_{n-1}\otimes 1)
\end{aligned}
\tag{8}
$$

where P^-_{n-1} is an antisymmetric projecting operator in tensor space $V^{\otimes n-1}$

Before introducing the IRF model we recall several notations. Let ϵ_μ be set of orthogonal bases and Λ_μ be the fundamental weight $A^{(1)}_{n-1}$ satisfying

$$
\begin{aligned}
\langle\epsilon_\mu,\epsilon_\nu\rangle &= \delta_{\mu\nu} & \hat{\mu} &= \epsilon_\mu-\frac{1}{n}\sum_{\nu=0}^{n-1}\epsilon_\nu \\
\langle\Lambda_0,\epsilon_\mu\rangle &= 0 & \Lambda_{\mu+1} &= \Lambda_\mu+\hat{\mu}
\end{aligned}
\tag{9}
$$

or

$$
\begin{aligned}
\hat{\mu} &= (\underbrace{0,\cdots,0}_{\mu},1,\cdots,0) \\
\Lambda_\mu &= (\underbrace{1,\cdots,1}_{\mu},0,\cdots,0)
\end{aligned}
\tag{10}
$$

Denote by $P_+(n,l)$ the set of dominate integral weights of level l

$$
P_+(n,l)=\left\{a=l\Lambda_0+\sum_{\mu=0}^{n-1}|l_\mu\in Z,l_0\geq l_1\cdots\geq l_{n-1}\right\} \tag{11}
$$

For an element of $P_+(n,l)$, there exists a one-to-one correspondence with a young diagram of $(n-1)\times l$ rectangle. l_μ represents the length of μ-th row. An ordered pair (a,b) is said to be admissible if

$$
a-b = \hat{\mu} \quad \text{if} \quad \mu\in(0,n-1) \tag{12}
$$

We further define

$$
\begin{aligned}
a_{\mu} &= l_{\mu} - \frac{1}{n}\sum_{\nu=0}^{n-1} l_{\nu} + w_{\mu} \\
a_{\mu\nu} &= a_{\mu} - a_{\nu} \qquad (13)
\end{aligned}
$$

where w_μ is an arbitrary complex parameter. Now, we introduce the IRF model [13]. For a vertex lattice, we can get a dual lattice by putting four indexes a, b, c and d on the corners in the clockwise order counting from northwest one instead of on lines (Figure 2).

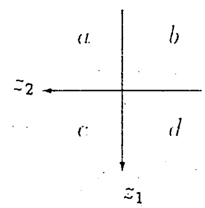


Figure 2: IRF Boltzmann weight W.

The boltzmann weight represented by Figure 2 takes the form

$$
W\left(\begin{array}{cc} a & b \\ d & c \end{array}\middle| \frac{z_1 - z_2}{w}\right)
$$

which is not zero unless $(b, a).(c, b).(d, a)$ and $(c. d)$ are all admissible

$$
\begin{aligned}
W\left(\begin{array}{cc} a & a-\hat{\mu} \\ a-\hat{\mu} & a-2\hat{\mu} \end{array}\middle| u\right) &= \frac{[1+u]}{[1]} \\
W\left(\begin{array}{cc} a & a-\hat{\mu} \\ a-\hat{\mu} & a-\hat{\mu}-\hat{\nu} \end{array}\middle| u\right) &= \frac{[a_{\mu\nu}+u]}{[a_{\mu\nu}]} \qquad \mu \neq \nu \qquad (14) \\
W\left(\begin{array}{cc} a & a-\hat{\mu} \\ a-\hat{\nu} & a-\hat{\mu}-\hat{\nu} \end{array}\middle| u\right) &= \frac{[u]}{[1]}\frac{[a_{\mu\nu}-1]}{[a_{\mu\nu}]} \qquad \mu \neq \nu
\end{aligned}
$$

where

$$
[u] = \theta\left[\begin{array}{c} 1/2 \\ 1/2 \end{array}\right](uw, \tau)
$$

The notation of our paper is different from one appearing in reference [13]. The W defined by equation (14) satisfies Yang-Baxter equation

$$\sum_g W\left(\begin{matrix} b & g \\ c & d \end{matrix}\middle| \frac{z_1-z_2}{w}\right) W\left(\begin{matrix} a & f \\ b & g \end{matrix}\middle| \frac{z_1-z_3}{w}\right) W\left(\begin{matrix} f & e \\ g & d \end{matrix}\middle| \frac{z_2-z_3}{w}\right)$$
$$= \sum_g W\left(\begin{matrix} a & g \\ b & c \end{matrix}\middle| \frac{z_2-z_3}{w}\right) W\left(\begin{matrix} g & e \\ c & d \end{matrix}\middle| \frac{z_1-z_3}{w}\right) W\left(\begin{matrix} a & f \\ g & e \end{matrix}\middle| \frac{z_1-z_2}{w}\right) \quad (15)$$

The IRF model and vertex model have the following correspondence relation

$$\sum_{\gamma,\sigma} R_{\alpha\beta,\gamma\sigma}(z_1-z_2)\phi^{\gamma}_{a,a-\hat{\mu}}(z_1)\otimes\phi^{\sigma}_{a-\hat{\mu},a-\hat{\mu}-\hat{\nu}}(z_2)$$
$$= \sum_k \phi^{\alpha}_{a-\hat{k},a-\hat{\mu}-\hat{\nu}}(z_1)\otimes\phi^{\beta}_{a,a-\hat{k}}(z_2) W\left(\begin{matrix} a & a-\hat{\mu} \\ a-\hat{k} & a-\hat{\mu}-\hat{\nu} \end{matrix}\middle| \frac{z_1-z_2}{w}\right) \quad (16)$$

where $\phi^{\alpha}_{a,a-\hat{\mu}}(z) = \theta^{(\alpha)}(z+nwa_\mu)$. In the right hand side of the above equation, there exists one term if $\mu=\nu$, and if $\mu\neq\nu$ two terms $k=\mu,\nu$ contribute.

For the further consideration, it is convenient to extend the definition of $\hat{\mu}, \mu \in (0, n-1)$ by

$$\widehat{\mu+n} = \hat{\mu} - \delta \quad (17)$$

where δ is the image root of A^1_{n-1}. In this case, the definition of a_μ changes into

$$a_\mu = l_\mu - \frac{1}{n}\sum_{\nu=0}^{\infty} l_\nu + w_\mu \quad (18)$$

From equation (16), we know that there exist n^2 intertwinning vectors $\phi^i_{a,a-\hat{\mu}}$, $i, \mu \in (0, n-1)$. After the extension, we have infinite number of intertwinning vectors. However, the extension of a_μ only contributes a spectral shift. The number of independent intertwinning vectors is still n^2. On other language, for a given k, we can define a new weight a' such that $a'_\mu = a_{nk+\mu}$, $\mu \in (0, n-1)$. In this way, one can also get n^2 intertwinning vectors, but translates the initial weight a into a'. Therefore, the infinite number of extended $\phi^i_{a,a-\hat{\mu}}$ can be written as a linear combination of n^2 independent intertwinning vectors. In the following we will take $w_\mu = \mu$ in equation (17). Let $a = \Lambda_\mu$, $\hat{\mu} \to \widehat{\mu-1}$ and $\hat{\nu} \to \widehat{\mu-2}$. The left hand side of equation (16) contains originally two terms. Due to $(\Lambda_\mu)_{\mu,\mu-1} = 1$, one can find

$$W\left(\begin{matrix} \Lambda_\mu & \Lambda_{\mu-1} \\ \Lambda_\mu - \widehat{\mu-2} & \Lambda_{\mu-2} \end{matrix}\middle| \frac{z_1-z_2}{w}\right) = 0$$

So, we have derived the following relation from equation (16)

$$\begin{aligned}\sum_{k,l} R_{ij,kl}(z_1 - z_2)\phi^k_{\Lambda_\mu,\Lambda_{\mu-1}}(z_1) \otimes \phi^l_{\Lambda_{\mu-1},\Lambda_{\mu-2}}(z_2) \\ = \frac{h(z_1 - z_2 + w)}{h(w)}\phi^i_{\Lambda_{\mu-1},\Lambda_{\mu-2}}(z_1) \otimes \phi^j_{\Lambda_\mu,\Lambda_{\mu-1}}(z_2)\end{aligned} \tag{19}$$

Rescaling the R matrix by a factor as $\tilde{R} = R \times \frac{h(w)}{h(z_1 - z_2 + w)}$, one finally obtain

$$\begin{aligned}\sum_{k,l} \tilde{R}_{ij,kl}(z_1 - z_2)\phi^k_{\Lambda_\mu,\Lambda_{\mu-1}}(z_1) \otimes \phi^l_{\Lambda_{\mu-1},\Lambda_{\mu-2}}(z_2) \\ = \phi^i_{\Lambda_{\mu-1},\Lambda_{\mu-2}}(z_1) \otimes \phi^j_{\Lambda_\mu,\Lambda_{\mu-1}}(z_2)\end{aligned} \tag{20}$$

This is nothing but the definition of Zamolodchikov-Faddeev algebra. Precisely speaking, we get a realization of ZF algebra in term of intertwinning vectors. It is worthy to note that the rescaled R satisfies only unitarity but crossing symmetry, i.e.

$$\tilde{R}_{ik,jl}(z)\tilde{R}_{lj,k'i'}(-z) = \delta_{ii'}\delta_{kk'} \tag{21}$$

$$\tilde{R}_{il,kj}(z)\tilde{R}_{k'j,i'l}(-z - nw) = \frac{h(z)h(-z - nw)}{h(z + w)h(w - z - nw)}\delta_{ii'}\delta_{kk'} \tag{22}$$

Arranging the $\phi^i_{a,a-\hat{\mu}}$ as the element of a matrix ϕ located in i-th row and μ-column. Similar to the definition of R^{V,V^*} we can define the conjugate of ϕ as

$$\begin{aligned}\phi^*_{\Lambda_\mu\Lambda_{\mu+1-n}} &= P^-_{n-1} \overset{n-1}{\phi}_{\Lambda_\mu,\Lambda_{\mu-1}}(z + (n-1)w) \overset{n-2}{\phi}_{\Lambda_{\mu-1},\Lambda_{\mu-2}}(z + (n-2)w) \\ &\times \cdots \overset{1}{\phi}_{\Lambda_{\mu-n+2},\Lambda_{\mu-n+1}}(z + w)P^-_{n-1}\end{aligned} \tag{23}$$

$$\begin{aligned}{}^*\phi_{\Lambda_\mu\Lambda_{\mu+1-n}} &= \times P^-_{n-1} \overset{n-1}{\phi}_{\Lambda_\mu,\Lambda_{\mu-1}}(z - w) \overset{n-2}{\phi}_{\Lambda_{\mu-1},\Lambda_{\mu-2}}(z - 2w) \\ &\cdots \overset{1}{\phi}_{\Lambda_{\mu-n+2},\Lambda_{\mu-n+1}}(z - (n-1)w)P^-_{n-1} \,.\end{aligned} \tag{24}$$

$$\overset{j}{\phi} = \overbrace{\underbrace{1 \otimes 1 \otimes \cdots \phi}_{j} \otimes \cdots \otimes 1}^{n-1} \tag{25}$$

It is clear that there exists a homomorphism $\phi^*(z) \rightarrow {}^*\phi(z) = \phi^*(z - nw)$, which is a spectral shift operator. It is not difficult to show

$$\begin{aligned}\tilde{R}^{V,V^*}(z_1 - z_2)\phi_{\Lambda_\mu,\Lambda_{\mu-1}}(z_1) \otimes \phi^*_{\Lambda_{\mu-1},\Lambda_{\mu-n}}(z_2) \\ = \phi_{\Lambda_{\mu-1-n},\Lambda_{\mu-n}}(z_1) \otimes \phi^*_{\Lambda_\mu,\Lambda_{\mu+1-n}}(z_2)\end{aligned} \tag{26}$$

$$\begin{aligned}\phi^{*t}_{\Lambda_\mu,\Lambda_{\mu+1-n}}(z_1) \otimes \phi^{*t}_{\Lambda_{\mu-1},\Lambda_{\mu-n}}(z_2)R(z_1 - z_2) \\ = \phi^{*t}_{\Lambda_{\mu-1},\Lambda_{\mu-n}}(z_1)\phi^{*t}_{\Lambda_\mu,\Lambda_{\mu+1-n}}(z_2)\end{aligned} \tag{27}$$

Notice that the parameter w_μ is not generic and the crossing parameter $w \neq 1/N$. Thus, our model is not cyclic one. From equations (20) and (27) we can get a comodule of ZF algebra.

3 Vertex Operator

In this section we will introduce a vertex operator acting on the path. For the convenience we pay our attention on level 1 case. The high level case will be considered in future. Considering a half-infinite chain in north-south direction, we can introduce a face-like path F(k) (see Figure 3).

F(0) F(1) F(2)

Figure 3: Face path

where k is the coordinate of chain counting from south. F(k) takes value in the weight lattice P(n.l) and satisfying that the neighbour pair are admissible. Physically, the configurations of face model in principle regime must be ground states at the low temperature. On the other hand, W contributes only one term in low temperature limit,

$$W\begin{pmatrix} \Lambda_{\mu+2} & \Lambda_{\mu+1} \\ \Lambda_{\mu+1} & \Lambda_{\mu} \end{pmatrix}$$

Thus, the ground state pathes are

$$F_\mu(k) = \Lambda_{\mu+k}\ , \quad \mu = 0, \cdots, n-1\ , \quad k = 0, 1, \cdots$$

and the configuration space decomposes into n subspaces (μ-like sectors) according to the ground state properties. In a μ-like sector, the pathes there exists many pathes which have the same structure as the ground state in infinite, i.e

$$F(k) = \Lambda_{\mu+k} = F_\mu(k) \quad \text{if} \quad k \gg 0.$$

These pathes stand for the excited states. By putting some vectors $\phi_{F(k),F(k+1)}(\zeta) \in C^n$ on (i,i+1) sides of a face path step by step, we can get a vertex-like path (Figure 4).

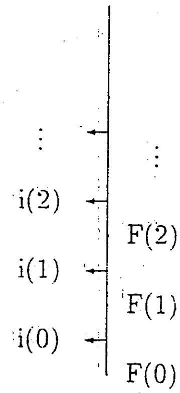


Figure 4: Vertex path

We denote by P(k) the set of vertex-like pathes. In this graphic representation, the ϕ and $\overline{\phi}$ in above section can be depicted as

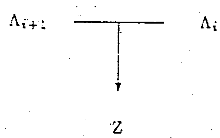
$\phi_{\Lambda_{i+1},\Lambda_i}(z) =$ 

Figure 5: vertex ϕ

Now, we introduce an operator $\psi_{P(\Lambda_{\mu+1}),P(\Lambda_\mu)}$ acting on $P(\Lambda)$. The graphic representation is

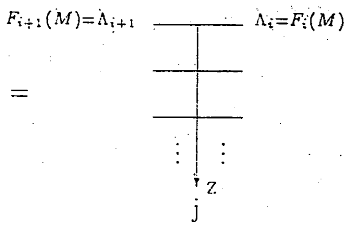
$\psi^j_{P(\Lambda_{i+1}),P(\Lambda_i)}(z) =$ 

Figure 6: ψ

For example, we explicitly write out ψ in a finite N-lattice as

$$\left(\psi^k_{\Lambda_{\mu+1}\Lambda_\mu}(z)\right)_{IJ} = \sum_{\{l_i\}} R^{j_1 l_1}_{i_1 k}(z) R^{j_2 l_2}_{i_2 l_1} \cdots$$

$$\cdots R^{j_N l_N}_{i_N l_{N-1}}(z) \phi^{l_N}_{\Lambda_{\mu+1}\Lambda_\mu}(z) \qquad (28)$$

It is clear that $\psi_{P(\Lambda_{\mu+1}),P(\Lambda_\mu)}$ acting on a μ-like path gives a $\mu+1$-like path. i.e.

$$P(\Lambda_\mu) \overset{\psi_{P(\Lambda_{\mu+1}),P(\Lambda_\mu)}}{\longrightarrow} P(\Lambda_{\mu+1}) \otimes V$$

By using (19) and Yang-Baxter equation, one can show

$$\sum_{k,l} \tilde{R}_{ij,kl}(z_1 - z_2)\psi^k_{P(\Lambda_\mu),P(\Lambda_{\mu-1})}(z_1)\psi^j_{P(\Lambda_{\mu-1}),P(\Lambda_{\mu-2})}(z_2)$$
$$= \psi^j_{P(\Lambda_\mu),P(\Lambda_{\mu-1})}(z_2)\psi^k_{P(\Lambda_{\mu-1}),P(\Lambda_{\mu-2})}(z_1) \quad (29)$$

It is worthy to note that (29) is obtained in finite size lattice. When we consider half-infinite system (thermodynamical limit), they will be divergent. In order to cancel this divergency, we have to renormalize R, $\psi\psi$. Mathematically, there is an ambiguous to arrange the order of operators in the above equations. Thus, we take normal order as

$$\psi^i(z_1)\psi^j(z_2) = f(z_1 - z_2) : \psi^i(z_1)\psi^j(z_2) : \quad (30)$$

Taking care of the two-point function, one can find the following difference equations

$$\frac{f(z_1 - z_2)}{f(z_1 - z_2 + nw)} = \frac{h(z_1 - z_2 + nw)}{h(w)\kappa(z_1 - z_2)} \quad (31)$$

where $\kappa(z)$ appears in reference [19] . It is not difficult to find

$$f(z) = \prod_{p=0}^{\infty} \frac{\kappa(z - pnw)h(w)}{h(z + w - pnw)} \quad (32)$$

and

$$\sum_{k,l} \bar{R}_{ij,kl}(z_1 - z_2)\psi^k_{P(\Lambda_\mu),P(\Lambda_{\mu-1})}(z_1)\psi^l_{P(\Lambda_{\mu-1}),P(\Lambda_{\mu-2})}(z_2)$$
$$= \psi^j_{P(\Lambda_\mu),P(\Lambda_{\mu-1})}(z_2)\psi^i_{P(\Lambda_{\mu-1}),P(\Lambda_{\mu-2})}(z_1) \quad (33)$$

where $\bar{R}(z) = \tilde{R}(z)\kappa^{-1}(z)$ is normalized boltzmann weights which satisfies unitarity and the crossing symmetry. In a similar way we can define $\psi^* = \phi^* R \cdots$. After proper normalization it has

$$\psi^*_{P(\Lambda_\mu),P(\Lambda_{\mu-n+1})} \circ \psi_{P(\Lambda_{\mu-n+1}),P(\Lambda_{\mu-n})} = id \quad (34)$$

$$\sum_{i,j} \bar{R}_{ij,kl}(z_1 - z_2)\psi^{*i}_{P(\Lambda_\mu),P(\Lambda_{\mu+1-n})}(z_1)\psi^{*j}_{P(\Lambda_{\mu-1}),P(\Lambda_{\mu-n})}(z_2)$$
$$= \psi^{*l}_{P(\Lambda_\mu),P(\Lambda_{\mu+1-n})}(z_2)\psi^{*k}_{P(\Lambda_{\mu-1}),P(\Lambda_{\mu-n})}(z_1) \quad (35)$$

Up to now, we set up the vertex operators ψ and ψ^* in the fundamental representation in the half-infinite system. The physical space is infinite. So we need to consider another half-infinite space.

4 The dual representation and vertex operator

In this section we will construct another vertex operator in the dual representation. The dual weights of $A_n^{(1)}$ is labeled by

$$\begin{aligned} (a^*) &= (l_0^*, \cdots, l_{n-1}^*) \in Z_N^{\otimes n-1} \\ (a^*) &= (a'^*) \qquad \pmod{1, \cdots, 1} \end{aligned} \tag{36}$$

and a^* can be expressed as

$$\begin{aligned} a^* &= l^* \Lambda_0^* + \sum_{\mu=1}^{n-1} l_\mu^* \hat{\mu}^* \\ \Lambda_\mu^* &= \Lambda_0^* + \sum_{\nu=0}^{\mu-1} \hat{\nu}^* \end{aligned} \tag{37}$$

or

$$\begin{aligned} (\Lambda_\mu^*) &= (\underbrace{1, \cdots, 1}_{\mu-1}, 0, \cdots, 0) \\ (\mu^*) &= (\underbrace{0, \cdots, 0, -1}_{\mu-1}, 0, \cdots, 0) \end{aligned} \tag{38}$$

Define

$$\begin{aligned} a_\mu^* &= l_\mu^* - \frac{1}{n}\sum_{\nu=0}^{n-1} l_\nu^* + w_\mu^* \\ a_{\mu\nu}^* &= a_\mu^* - a_\nu^* \end{aligned} \tag{39}$$

Similarly, we can define the admissible path (a^*, b^*) in the dual space by setting $a^* - b^* = \hat{\mu}^*, \mu \in (0, n-1)$. The no-zero boltzmann weights are

$$\begin{aligned} W^+\left(\begin{array}{cc} a^* & a^* + \hat{\mu}^* \\ a^* + \hat{\mu}^* & a^* + 2\hat{\mu}^* \end{array}\middle| u\right) &= \frac{[1+u]}{[1]} \\ W^+\left(\begin{array}{cc} a^* & a^* + \hat{\mu}^* \\ a^* + \hat{\mu}^* & a^* + \hat{\mu}^* + \hat{\nu}^* \end{array}\middle| u\right) &= \frac{[u + a_{\mu\nu}^*]}{[a_{\mu\nu}^*]} \\ W^+\left(\begin{array}{cc} a^* & a^* + \hat{\mu}^* \\ a^* + \hat{\nu}^* & a^* + \hat{\mu}^* + \hat{\nu}^* \end{array}\middle| u\right) &= \frac{[u]}{[1]}\frac{[a_{\mu\nu}^* - 1]}{[a_{\mu\nu}^*]} \end{aligned} \tag{40}$$

It is easy to show that $W^+(u)$ satisfies Yang-Baxter equation. Define the intertwining vectors in the dual space as

$$\varphi_{a^*, a^* + \hat{\mu}^*}^{+k}(z) = \theta^{(k)}(z - nwa_\mu^*) , \tag{41}$$

and define $\overline{\phi}^{+}$ the cofactor matrix of ϕ^{+}. One can verify the following vertex-face correspondence relation

$$\begin{aligned}
&\sum_{k,l} R_{ij,kl}(z_1 - z_2)\phi^{+k}_{a^*,a^*+\hat{\mu}^*}(z_1)\otimes\phi^{+l}_{a^*+\hat{\mu}^*,a^*+\hat{\mu}^*+\hat{\nu}^*}(z_2)\\
&= \sum_k \phi^{+i}_{a^*+\hat{k}^*,a^*+\hat{\mu}^*+\hat{\nu}^*}(z_1)\otimes\phi^{+j}_{a^*,a^*+\hat{k}^*}(z_2)W^{+}\left(\begin{array}{cc} a^* & a^*-\hat{\mu}^* \\ a^*+\hat{k}^* & a^*+\hat{\mu}^*+\hat{\nu}^* \end{array}\middle|\, \frac{z_1-z_2)}{w}\right)
\end{aligned} \tag{42}$$

$$\begin{aligned}
&\overline{\phi}^{+}_{a^*+\hat{\mu},}(z_1)\otimes\overline{\varphi}^{+}_{a^*,a^*+\hat{\mu}^*}(z_2)R(z_1 - z_2)\\
&= \sum_k \overline{\phi}^{+}_{a^*,a^*+\hat{k}^*}(z_1)\otimes\overline{\phi}^{+}_{a^*+\hat{k}^*,a^*+\hat{\mu}^*+\hat{\nu}^*}(z_2)W^{+}\left(\begin{array}{cc} a^* & a^*+\hat{\mu}^* \\ a^*+\hat{k}^* & a^*+\hat{\mu}^*+\hat{\nu}^* \end{array}\middle|\, \frac{z_1-z_2)}{w}\right)
\end{aligned} \tag{43}$$

Taking $a^* = \Lambda_i^*, \hat{\mu} = \widehat{i-1}^*$ and $\hat{\nu}^* = \widehat{i-2}^*$, one can find from (44)

$$\begin{aligned}
&\overline{\phi}^{+}_{\Lambda^*_{i-1},\Lambda^*_{i-2}}(z_1)\otimes\overline{\phi}^{+}_{\Lambda^*_i,\Lambda^*_{i-1}}(z_2)R(z_1 - z_2)\\
&\qquad = \frac{h(z_1 - z_2 + w)}{h(w)}\overline{\phi}^{+}_{\Lambda^*_i,\Lambda^*_{i-1}}(z_1)\otimes\phi^{+}_{\Lambda^*_{i-1},\Lambda^*_{i-2}}(z_2)
\end{aligned} \tag{44}$$

In the graphic repreentation the $\overline{\phi}^{+}$ is

$\overline{\phi}^{+}_{\Lambda^*_{i+1},\Lambda^*_i}(z) =$

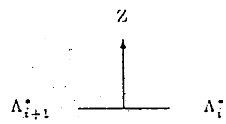


Figure 7: vertex $\overline{\phi}^{+}$

Also we define the vertex operator by

$\overline{\varphi}^{+j}_{P(\Lambda^*_{i+1}),P(\Lambda^*_i)}(z) =$

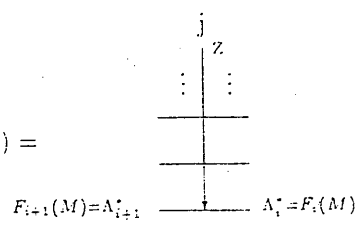


Figure 8: vertex $\overline{v}^{+}$

which maps $P(\Lambda^*_i)$ to $P(\Lambda^*_{i+1})\otimes V^*$

$$P(\Lambda^*_i) \xrightarrow{\overline{v}^{+}_{P(\Lambda^*_{i+1}),P(\Lambda^*_i)}} P(\Lambda^*_{i+1})\otimes V^*$$

In the finite size chain the $\overline{\psi}^+$ satisfies

$$\sum_{i,k}\overline{\psi}^{+i}_{P(\Lambda^*_{\mu+2}),P(\Lambda^*_{\mu+1})}(z_2)\overline{\psi}^{+k}_{P(\Lambda^*_{\mu+1}),P(\Lambda^*_{\mu})}(z_1)R_{ij,kl}(z_1-z_2)$$

$$=\frac{h(z_1-z_2+w)}{h(w)}\overline{\psi}^{+j}_{P(\Lambda^*_{\mu+2}),P(\Lambda^*_{\mu+1})}(z_1)\overline{\psi}^{+l}_{P(\Lambda^*_{\mu+1}),P(\Lambda^*_{\mu})}(z_2) \tag{45}$$

Based upon the same reason for ψ we introduce the normal order

$$\overline{\psi}^{+i}(z_1)\overline{\psi}^{+j}(z_2)=\bar{f}(z_1-z_2):\overline{\psi}^{+i}(z_1)\overline{\psi}^{+j}(z_2): \tag{46}$$

and get a difference equation of $\bar{f}$

$$\frac{\bar{f}(z_1-z_2+nw)}{\bar{f}(z_1-z_2)}=\frac{h(z_1-z_2+nw)}{h(w)\kappa(z_1-z_2)} \tag{47}$$

By using equation (48) and renormalized $\bar{R}$, one can show

$$\sum_{i,k}\overline{\psi}^{+i}_{P(\Lambda^*_{\mu+2}),P(\Lambda^*_{\mu+1})}(z_2)\overline{\psi}^{+k}_{P(\Lambda^*_{\mu+1}),P(\Lambda^*_{\mu})}(z_1)R_{ij,kl}(z_1-z_2)$$

$$=\overline{\psi}^{+j}_{P(\Lambda^*_{\mu+2}),P(\Lambda^*_{\mu+1})}(z_1)\overline{\psi}^{+l}_{P(\Lambda^*_{\mu+1}),P(\Lambda^*_{\mu})}(z_2) \tag{48}$$

With this method we can also define the vertex operator $\overline{\psi}$ satisfying $\overline{\psi}\circ\overline{\psi}^+=id.$

5 Elliptic q-deformed Affine Algebra

From the definition of vertex operator, we know a sequence mapping

$$P(\Lambda^*_i)\xrightarrow{\overline{\psi}}P(\Lambda^*_{i+1})\otimes V\xrightarrow{\overline{\psi}\otimes id}P(\Lambda^*_{i+2})\otimes V^{\otimes 2}\rightarrow\cdots\rightarrow P(\Lambda^*_i)\otimes V^{\otimes n} \tag{49}$$

This means that n $\overline{\psi}$'s acting on a path sector $P(\Lambda^*_i)$ as a identity operator. Using this property, we can define the automorphism operators $L^{\pm}$ as following: Taking anti-symmetric fusion of $n-1$ $\overline{\psi}$'s and transforming it from south to north with the corner transfer matrix in clockwise and anticlockwise. The new operators act on $P(\Lambda^*)\otimes P(\Lambda)$

$$\begin{aligned}L^+_{ij}(z) &= \psi^i(z+nw/2)\hat{\otimes}\overline{\psi}^{+j}(z-nw/2)\\ L^-_{ij}(z) &= \psi^i(z-nw/2)\hat{\otimes}\overline{\psi}^{+j}(z+nw/2)\end{aligned} \tag{50}$$

where $\hat{\otimes}$ stands for the tensor product in quantum space. They satisfy

$$\overline{R}_{12}(z_1 - z_2)L_1^{\pm}(z_1)L_2^{\pm}(z_2) = L_2^{\pm}(z_2)L_1^{\pm}(z_1)\overline{R}_{12}(z_1 - z_2) \quad (51)$$

$$\overline{R}_{12}(z_1 - z_2 + nw)L_1^{+}(z_1)L_2^{-}(z_2) = L_2^{-}(z_2)L_1^{+}(z_1)\overline{R}_{12}(z_1 - z_2 - nw) \quad (52)$$

The proof of them is very similar. Here we show (53). Taking a element of (53), we have

$$\begin{aligned}
l.h.s &= \sum_{rs} \overline{R}_{rs,kl}(z_1 - z_2 + nw)\psi^r(z_1 + nw/2)\psi^s(z_2 - nw/2) \\
&\quad \hat{\otimes}\overline{\psi}^i(z_1 - nw/2)\overline{\psi}^j(z_2 + cnw/2) \\
&= \psi^l(z_2 - cnw/2)\psi^k(z_1 - nw/2)\hat{\otimes}\overline{\psi}^i(z_1 - nw/2)\overline{\psi}^j(z_2 + nw/2) \\
r.h.s &= \sum_{rs} \psi^l(z_2 - nw/2)\psi^k(z_1 - nw/2) \\
&\quad \hat{\otimes}\overline{\psi}^s(z_2 + nw/2)\overline{\psi}^r(z_1 - nw/2)\overline{R}_{ij,rs}(z_1 - z_2 - nw) \\
&= \psi^l(z_2 - nw/2)\psi^k(z_1 - nw/2)\hat{\otimes}\overline{\psi}^i(z_1 - nw/2)\overline{\psi}^j(z_2 + nw/2) \\
&= l.h.s
\end{aligned}$$

In the rest of this section, we study the exchange relations between ψ, $\overline{\psi}$ and $L^{\pm}$. Using (29) and (46) we find

$$\begin{aligned}
&L_{ll'}^{\pm}(z_2)\left(\psi^k_{P(\Lambda_{\mu+2}),P(\Lambda_{\mu+1})}(z_1)\hat{\otimes}T_{P(\Lambda_{\mu+2}),P(\Lambda_{\mu+1})}\right) \\
&= \sum_{ij} \overline{R}_{ij,kl}(z_1 - z_2 \mp nw/2)\left(\psi^{+i}_{P(\Lambda_{\mu+2}),P(\Lambda_{\mu+1})}(z_1)\hat{\otimes}T_{P(\Lambda_{\mu+2}),P(\Lambda_{\mu+1})}\right) L_{jl'}^{\pm}(z_2) \quad (53)
\end{aligned}$$

$$\begin{aligned}
&L_{jj'}^{\pm}(z_2)\left(T_{P(\Lambda^*_{\mu-2}),P(\Lambda^*_{\mu+1})}\hat{\otimes}\psi^{+k}_{P(\Lambda_{\mu-2}),P(\Lambda_{\mu+1})}(z_1)\right) \\
&= \sum_{kl} \overline{R}_{ij,kl}(z_1 - z_2 \mp nw/2)\left(T_{P(\Lambda^*_{\mu+2}),P(\Lambda^*_{\mu+1})}\hat{\otimes}\psi^{+i}_{P(\Lambda_{\mu+2}),P(\Lambda_{\mu-1})}(z_1)\right) L_{jl}^{\pm}(z_2) \quad (54)
\end{aligned}$$

where

$$T_{P(\Lambda_{\mu+2}),P(\Lambda_{\mu+1})} = \prod_{k=1}^{\infty} \delta_{P_{\mu-2}(k+1),P_{\mu-1}(k)}$$

is a translation operator in the face path space. In the proof of equations (54), we have used the explicit expression of ψ and $\overline{\psi}$ in a fixed path. The coproduct of $L^{\pm}$ is defined as

$$\Delta(L^{\pm}(z)) = L^{\pm}(z \mp c_2 nw/2) \dot{\hat{\otimes}} L^{\pm}(z \pm c_1 nw/2) \quad (55)$$

$$(56)$$

then they satisfy

$$\overline{R}_{12}(z_1 - z_2)\Delta(L_1^{\pm}(z_1))\Delta(L_2^{\pm}(z_2)) = \Delta(L_2^{\pm}(z_2))\Delta(L_1^{\pm}(z_1))\overline{R}_{12}(z_1 - z_2) \quad (57)$$

$$(58)$$

$$\overline{R}_{12}(z_1 - z_2 + 2nw)\Delta(L_1^{+}(z_1))\Delta(L_2^{-}(z_2))$$
$$= \Delta(L_2^{-}(z_2))\delta(L_1^{+}(z_1))\overline{R}_{12}(z_1 - z_2 - 2nw) \quad (59)$$

Define

$$S^t(L^-(z))^{ij} = \psi^{*j}(z + nw/2)\hat{\otimes}\overline{\psi}^{+i}(z - nw/2)\,. \quad (60)$$

One can show the following difference equation from equations (34). (36) and (61)

$$\Psi^i(z - nw)_{P(\Lambda_\mu),P(\Lambda_{\mu-n-1})} = L^{+ij}_{P(\Lambda_\mu),P(\Lambda_{\mu-1})}(z - 3nw/2)\Psi^k(z)_{P(\Lambda_{\mu-1}),P(\Lambda_{\mu-2})}$$
$$\times S^t(L^-(z - 3nw/2))^{jk}_{P(\Lambda_{\mu-2}),P(\Lambda_{\mu-n-1})} \quad (61)$$

6 conclusion

In this paper, we construct the vertex operator in term of the intertwinning vectors and derive out the difference equation. In fact the $L^{\pm}$ is written as tensor product in two half-infinite space. The vertex operators in up-space and down-space are all I-type. This is different from ones in reference [20]. Miki constructs the $L^{\pm}$ using I-type and II-type vertex operators. However, one can set up the relations between the I-type vertex operator in up-space and the II-type one in down-space vise vs by using the conner transfer matrix as pointed out by Miki. Precisely speaking, the construction of vertex operator in this paper is based upon the highest weight vector. One can show in principle that they are true for arbitrary degenerate vectors. But how to do it is still open.

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QUANTUM GROUPS $SU_q(N)$ AND
q-DEFORMATIONS OF CHERN CHARACTERS

Zhong-Qi Ma

Institute of High Energy Physics, P. O. Box 918(4)
Beijing 100039, P. R. of China

ABSTRACT

This talk is based on two preprints completed by Bo-Yu Hou, Bo-Yuan Hou and Zhong-Qi Ma [1,2]. For the quantum groups $SU_q(N)$, we compute some important quantities in the bicovariant differential calculus, generalize the q-deformed BRST algebra [3], define the q-deformed Chern class P_m, and find the general calculation method for the q-deformed Chern-Simons Q_{2m-1}, satisfying $dQ_{2m-1} = P_m$.

1. Introduction

Recently, quantum groups have attracted increasing attention. Since the quantum group is provided by a noncommutative algebra, the noncommutative geometry presented by Connes [4] plays a basic role like the differential geometry in the usual gauge theory. Following the general ideas of Connes, Woronowicz [5] developed the framework of the noncommutative differential calculus. There have been a lot of papers treating the differential calculus on quantum groups and the deformed gauge theories from various viewpoints.

CSWW [6] presented a systematic construction of bicovariant bimodules on the quantum groups $SU_q(N)$ and $SO_q(N)$ by using $\hat{R}_q$ matrix. They described the conjugate of the fundamental representation for $SU_q(N)$ as antisymmetrized product of $(N-1)$ fundamental representations, and showed the expressions appearing in the bicovariant differential calculus on $SU_q(N)$ both by formulas and by diagrams. However, the antisymmetrized product makes the calculation of explicit forms very complicated. In the later paper Watamura [3] investigated the q-deformation of BRST algebra for the quantum group $SU_q(2)$. Its generalization to $SU_q(N)$ depends on the explicit forms of the important quantities in the bicovariant differential calculus on $SU_q(N)$, such as the q-deformed structure constant $\mathbf{C}_{IJ}^{K}$ and the q-deformed transposition operator Λ. In fact, the key for solving this problem is to change the description for the conjugate representation. The conjugate of the fundamental representation in $SU(N)$ is also equivalent to a basic highest weight representation described by the last fundamental dominant weight λ_{N-1}. The monoid $\epsilon^{a\bar{b}}$ plays a very important role in the calculations for the quantities appearing in the bicovariant differential calculus on $SU_q(N)$.

The plan of this paper is as follows. In Sec. 2 we discuss the main properties of $\hat{R}_q$ matrices in the product representation spaces of $\lambda_1 \otimes \lambda_1$, $\lambda_{N-1} \otimes \lambda_{N-1}$, $\lambda_1 \otimes \lambda_{N-1}$, and

$\lambda_{N-1}\otimes\lambda_1$. The monoid $\epsilon^{a\bar{b}}$ is introduced to relate those $\hat{R}_q$ matrices. It is proved that the monoid transforms the conjugate of the fundamental representation λ_1 into the highest weight representation λ_{N-1}. Some important quantities in the bicovariant differential calculus on $SU_q(N)$ are computed, and the q-deformed BRST algebra for the quantum group $SU_q(N)$ is constructed in Sec. 3. ¿From the condition $\delta P_m = 0$ and $dP_m = 0$, we define the q-deformed generalized Killing form and the m-th q-deformed Chern class P_m in Sec. 4. In Sec.5, the q-deformed homotopy operator is introduced in $SU_q(N)$ to compute the q-deformed Chern-Simons Q_{2m-1} by the condition $dQ_{2m-1} = P_m$.

2. Algebra of Functions on the Quantum Group $SU_q(N)$

In the quantum enveloping algebra U_qA_{N-1} there are $(N-1)$ fundamental dominant weight λ_m. A highest weight representation is denoted by its highest weight, that is a positive integral combination of λ_j. The states in the representation are described by their weights that are the integral combinations of λ_j. The fundamental representation λ_1 is N-dimensional. The states in this representation are described by their weight $(\lambda_a - \lambda_{a-1})$, that can be enumerated by one integer a for simplicity. The conjugate of the representation λ_1 is equivalent to the representation λ_{N-1}, where the states have the weights $\lambda_{a-1} - \lambda_a$ and are enumerated by one index $\bar{a} = -a$.

The standard method for calculating the solutions $\hat{R}_q$ of the simple Yang-Baxter equation [7,8] is to expand it by the projection operators. In the direct product spaces of $\lambda_1\otimes\lambda_1$, $\lambda_{N-1}\otimes\lambda_{N-1}$, $\lambda_1\otimes\lambda_{N-1}$, and $\lambda_{N-1}\otimes\lambda_1$, the solutions of the simple Yang-Baxter equation are as follows, respectively:

$$\begin{aligned}
\left(\hat{R}_q^{\lambda_1\lambda_1}\right)^{ab}_{cd} &= q\ \left(\mathcal{P}^{\lambda_1\lambda_1}_{2\lambda_1}\right)^{ab}_{cd} - q^{-1}\ \left(\mathcal{P}^{\lambda_1\lambda_1}_{\lambda_2}\right)^{ab}_{cd} \\
\left(\hat{R}_q^{\lambda_{N-1}\lambda_{N-1}}\right)^{\bar{a}\bar{b}}_{\bar{c}\bar{d}} &= q\ \left(\mathcal{P}^{\lambda_{N-1}\lambda_{N-1}}_{2\lambda_{N-1}}\right)^{\bar{a}\bar{b}}_{\bar{c}\bar{d}} - q^{-1}\ \left(\mathcal{P}^{\lambda_{N-1}\lambda_{N-1}}_{\lambda_{N-2}}\right)^{\bar{a}\bar{b}}_{\bar{c}\bar{d}} \\
\left(\hat{R}_q^{\lambda_1\lambda_{N-1}}\right)^{\bar{a}b}_{c\bar{d}} &= q\ \left(\mathcal{P}^{\lambda_1\lambda_{N-1}}_{\lambda_1+\lambda_{N-1}}\right)^{\bar{a}b}_{c\bar{d}} + (-q)^{1-N}\ \left(\mathcal{P}^{\lambda_1\lambda_{N-1}}_{0}\right)^{\bar{a}b}_{c\bar{d}} \\
\left(\hat{R}_q^{\lambda_{N-1}\lambda_1}\right)^{a\bar{b}}_{\bar{c}d} &= q\ \left(\mathcal{P}^{\lambda_{N-1}\lambda_1}_{\lambda_1+\lambda_{N-1}}\right)^{a\bar{b}}_{\bar{c}d} + (-q)^{1-N}\ \left(\mathcal{P}^{\lambda_{N-1}\lambda_1}_{0}\right)^{a\bar{b}}_{\bar{c}d}
\end{aligned} \tag{2.1}$$

where the superscripts, for example $\lambda_1\lambda_1$, have been implied in the super- and sub-scripts ab and cd, and can be neglected.

We define the monoid $\epsilon^{a\bar{b}}$ as follows:

$$\begin{aligned}
\epsilon^{a\bar{b}} &= (-1)^{N-1}\ \epsilon_{a\bar{b}} = \delta_{ab}\ (-1)^{N-a}\ q^{a-(N+1)/2} \\
\epsilon_{\bar{b}a} &= (-1)^{N-1}\ \epsilon^{\bar{b}a} = \delta_{ab}\ (-1)^{N-a}\ q^{-a+(N+1)/2} \\
\epsilon^{a\bar{b}}\ \epsilon_{\bar{b}c} &= \delta^a_c; \qquad \epsilon_{\bar{a}b}\ \epsilon^{b\bar{c}} = \delta^{\bar{c}}_{\bar{a}}
\end{aligned} \tag{2.2}$$

where and hereafter, summation of the repeated indices is underatood.

Four $\hat{R}_q$ matrices can be related by $\epsilon^{a\bar{b}}$ matrices:

$$\begin{aligned}
\left(\hat{R}_q^{\pm 1}\right)^{c\bar{d}}_{\bar{a}b} &= q^{\pm 1}\ \epsilon_{\bar{a}r}\ \left(\hat{R}_q^{\mp 1}\right)^{rc}_{bs}\ \epsilon^{s\bar{d}} = q^{\pm 1}\ \epsilon^{c\bar{r}}\ \left(\hat{R}_q^{\mp 1}\right)^{\bar{d}\bar{s}}_{\bar{r}\bar{a}}\ \epsilon_{\bar{s}b} \\
\left(\hat{R}_q^{\pm 1}\right)^{\bar{a}b}_{c\bar{d}} &= q^{\pm 1}\ \epsilon^{\bar{a}r}\ \left(\hat{R}_q^{\mp 1}\right)^{bs}_{rc}\ \epsilon_{s\bar{d}} = q^{\pm 1}\ \epsilon_{c\bar{r}}\ \left(\hat{R}_q^{\mp 1}\right)^{\bar{r}\bar{a}}_{\bar{d}\bar{s}}\ \epsilon^{\bar{s}b}
\end{aligned} \tag{2.3}$$

$\mathcal{A}$ is a Hopf algebra, freely generated by non-commutating matrix entries $T^{\bar{a}}_{\ \bar{b}}$ of the fundamental representation of quantum group $SU_q(N)$, satisfying the relation:

$$\left(\hat{R}_q\right)^{ab}_{\ \ rs} T^r_{\ c} T^s_{\ d} = T^a_{\ r} T^b_{\ s} \left(\hat{R}_q\right)^{rs}_{\ \ cd} \tag{2.4}$$

where $\left(\hat{R}_q\right)^{ab}_{\ \ cd}$ is given in (2.1). The conjugate of the fundamental representation λ_1 is equivalent to the representation λ_{N-1}. Defining:

$$T^{\bar{a}}_{\ \bar{b}} = \epsilon_{\bar{b}r}\, \kappa(T^r_{\ s})\, \epsilon^{s\bar{a}}, \quad \kappa\left(T^b_{\ a}\right) = \epsilon^{b\bar{r}}\, T^{\bar{s}}_{\ \bar{r}}\, \epsilon_{\bar{s}a} \tag{2.5}$$

where κ denotes antipode, we are able to prove that $T^{\bar{a}}_{\ \bar{b}} \in \mathcal{A}$ belongs to the representation λ_{N-1}:

$$\begin{aligned} \left(\hat{R}_q\right)^{\bar{a}b}_{\ \ \bar{r}\bar{s}} T^r_{\ c} T^{\bar{s}}_{\ \bar{d}} &= T^{\bar{a}}_{\ \bar{r}} T^b_{\ s} \left(\hat{R}_q\right)^{\bar{r}s}_{\ \ c\bar{d}} \\ \left(\hat{R}_q\right)^{a\bar{b}}_{\ \ \bar{r}\bar{s}} T^{\bar{r}}_{\ \bar{c}} T^s_{\ d} &= T^a_{\ r} T^{\bar{b}}_{\ \bar{s}} \left(\hat{R}_q\right)^{r\bar{s}}_{\ \ \bar{c}d} \\ \left(\hat{R}_q\right)^{\bar{a}\bar{b}}_{\ \ \bar{r}\bar{s}} T^{\bar{r}}_{\ \bar{c}} T^{\bar{s}}_{\ \bar{d}} &= T^{\bar{a}}_{\ \bar{r}} T^{\bar{b}}_{\ \bar{s}} \left(\hat{R}_q\right)^{\bar{r}\bar{s}}_{\ \ \bar{c}\bar{d}} \end{aligned} \tag{2.6}$$

The direct product of $T^a_{\ c}$ and $T^{\bar{b}}_{\ \bar{d}}$ spans the mixed space of the adjoint and identity representations. The generalized q-Pauli matrices are introduced to separate the singlet and the adjoint components:

$$\begin{aligned} M^{\hat{I}}_{\ \hat{J}} &= (\sigma^{\hat{I}})_{a\bar{b}}\, T^a_{\ c} T^{\bar{b}}_{\ \bar{d}}\, (\sigma_{\hat{J}})^{c\bar{d}} = (\sigma^{\hat{I}})_a^{\ d}\, T^a_{\ c}\, \kappa(T^b_{\ d})\, (\sigma_{\hat{J}})^c_{\ b} \\ M^0_{\ 0} &= 1, \quad M^I_{\ 0} = M^0_{\ I} = 0 \\ \kappa^2(M^{\hat{I}}_{\ \hat{J}}) &= D^{\hat{I}}_{\ \hat{K}}\, M^{\hat{K}}_{\ \hat{L}}\, (D^{-1})^{\hat{L}}_{\ \hat{J}}, \quad D^0_{\ 0} = 1, \quad D^{(i\bar{j})}_{\ (i\bar{j})} = q^{2(i-j)} \end{aligned} \tag{2.7}$$

where D is a diagonal matrix. The explicit forms of σ matrices are listed in Ref. 2. In the present paper, if without special notification, the small Latin letter, except n and m, such as a and i, runs over 1, 2, $\cdots$, N, and a capital Latin, such as I, runs over $(i\bar{j})$ where $i \neq j$, and $(j\bar{j})$ where $j \geq 2$. In order to describe both singlet and adjoint components, a capital Latin with a hat, like $\hat{I}$, runs over 0 and I.

As usual, the linear functionals $(L^{\pm})^a_{\ b}$, belonging to the dual Hopf algebra $\mathcal{A}'$, are defined by their values on the entries $T^a_{\ b}$ [9]:

$$(L^+)^a_{\ b}(T^c_{\ d}) = q^{-1/N}\, (\hat{R}_q)^{ac}_{\ \ db}, \quad (L^-)^a_{\ b}(T^c_{\ d}) = q^{1/N}\, (\hat{R}_q^{-1})^{ac}_{\ \ db} \tag{2.8}$$

3. Bicovariant Differential Calculus and q-deformed BRST Algebra on Quantum Groups $SU_q(N)$

The bases $\eta^{\hat{I}}$ of the right-invariant elements in the bimodule Γ satisfy:

$$\begin{aligned} \Delta_R(\eta^{\hat{J}}) &= \eta^{\hat{J}} \otimes \mathbf{1}, \quad \Delta_L(\eta^{\hat{J}}) = M^{\hat{J}}_{\ \hat{K}} \otimes \eta^{\hat{K}} \\ \alpha\, \eta^{\hat{J}} &= \eta^{\hat{K}}(\alpha * L^{\hat{J}}_{\hat{K}}), \quad \alpha \in \mathcal{A}, \quad L^{\hat{J}}_{\hat{K}} \in \mathcal{A}' \end{aligned} \tag{3.1}$$

where Δ_L and Δ_R are the left and right action of the quantum group on Γ, and

$$\begin{aligned} L_{\hat{I}}^{\ \hat{J}} &= (\sigma^{\hat{J}})_{a\bar{r}} \left\{\epsilon^{d\bar{r}} \kappa'((L^+)^a{}_b)\ (L^-)^c{}_d\ \epsilon_{\bar{s}c}\right\}\ (\sigma_{\hat{I}})^{b\bar{s}} \\ (\rho\ *\ L_{\hat{I}}^{\ \hat{J}}) &= (L_{\hat{I}}^{\ \hat{J}}\ \otimes\ id)\ \Delta_L(\rho) \end{aligned} \tag{3.2}$$

Now, the quantum permutative operator can be calculated as follows:

$$\Lambda(\eta^{\hat{I}} \otimes \eta^{\hat{J}}) \;=\; \Lambda^{\hat{I}\hat{J}}_{\hat{K}\hat{L}}\ \eta^{\hat{K}} \otimes \eta^{\hat{L}}, \qquad \Lambda^{\hat{I}\hat{J}}_{\hat{K}\hat{L}} \;=\; L_{\hat{K}}^{\ \hat{J}}(M^{\hat{I}}{}_{\hat{L}}) \tag{3.3}$$

$$\begin{aligned} &(\sigma_{\hat{I}})^{a\bar{b}}\ (\sigma_{\hat{J}})^{c\bar{d}}\ \Lambda^{\hat{I}\hat{J}}_{\hat{K}\hat{L}}\ \left(\sigma^{\hat{K}}\right)_{i\bar{j}}\ \left(\sigma^{\hat{L}}\right)_{k\bar{l}} \;=\; \left(\sigma^{\hat{K}}\right)_{i\bar{j}}\ L_{\hat{K}}^{\ \hat{S}}(T^a{}_k)\ L_{\hat{S}}^{\ \hat{J}}(T^{\bar{b}}{}_{\bar{l}})\ (\sigma_{\hat{J}})^{c\bar{d}} \\ &= \left(\hat{R}_q^{-1}\right)^{ar}_{\ \ iu}\ \left(\hat{R}_q\right)^{u\bar{s}}_{\ \ \bar{j}k}\ \left(\hat{R}_q^{-1}\right)^{\bar{b}c}_{\ \ r\bar{v}}\ \left(\hat{R}_q\right)^{\bar{v}\bar{d}}_{\ \ \bar{s}\bar{l}} \end{aligned} \tag{3.4}$$

¿From the symmetry of the quantum Clebsch-Gordan coefficients (Ref. 8, P.156) we have:

$$\left(\Lambda^{-1}\right)^{(i\bar{j})(k\bar{l})}_{(a\bar{b})(c\bar{d})} \;=\; \Lambda^{(\ell\bar{k})(j\bar{i})}_{(d\bar{c})(b\bar{a})} \tag{3.5}$$

Through direct calculation for (3.4), we obtain the non-vanishing components of $\Lambda^{\hat{I}\hat{J}}_{\hat{K}\hat{L}}$ as follows:

$$\begin{aligned} \Lambda^{IJ}_{KL} &= \delta^I_K\ \delta^J_L \;+\; \bar{f}^{IJ}_P\ f^P_{KL} \;+\; \Phi^{IJ}_{KL}, \quad \Lambda^{00}{}_{00} \;=\; (\Lambda^{-1})^{00}{}_{00} \;=\; 1 \\ \Lambda^{I0}_{JK} &= (\Lambda^{-1})^{0I}_{JK} \;=\; \lambda\ f^I_{JK}, \quad \Lambda^{JK}_{0I} \;=\; (\Lambda^{-1})^{JK}_{I0} \;=\; \lambda\ \bar{f}^{JK}_I \\ \Lambda^{0I}_{J0} &= (\Lambda^{-1})^{I0}_{0J} \;=\; \delta^I_J, \quad \Lambda^{I0}_{0J} \;=\; (\Lambda^{-1})^{0I}_{J0} \;=\; (\lambda^2+1)\ \delta^I_J \end{aligned} \tag{3.6}$$

The non-vanishing components of Λ^{IJ}_{KL}, f^I_{JK} and $\bar{f}^{JK}_I$ are listed in Ref. 2.

Define some projection operators:

$$\begin{aligned} \mathcal{P}_S &\equiv \left(\mathcal{P}_{2\lambda_1}, \mathcal{P}_{2\lambda_{N-1}}\right) \;+\; \left(\mathcal{P}_{\lambda_2}, \mathcal{P}_{\lambda_{N-2}}\right) \\ \mathcal{P}_A &\equiv \left(\mathcal{P}_{2\lambda_1}, \mathcal{P}_{\lambda_{N-2}}\right) \;+\; \left(\mathcal{P}_{\lambda_2}, \mathcal{P}_{2\lambda_{N-1}}\right) \\ &\mathcal{P}_S \;+\; \mathcal{P}_A \;=\; \mathbf{1} \end{aligned} \tag{3.7}$$

where, for example,

$$\begin{aligned} &(\mathcal{P}_{2\lambda_1}, \mathcal{P}_{2\lambda_{N-1}})^{\hat{I}\hat{J}}_{\hat{K}\hat{L}} \\ &= (\sigma^{\hat{I}})_{a\bar{b}}\ (\sigma^{\hat{J}})_{c\bar{d}}\ \left(\hat{R}_q^{-1}\right)^{\bar{b}c}_{\ \ r\bar{v}}\ (\mathcal{P}_{2\lambda_1})^{ar}_{\ \ iu}\ (\mathcal{P}_{2\lambda_{N-1}})^{\bar{v}\bar{d}}_{\ \ \bar{s}\bar{l}} \left(\hat{R}_q\right)^{u\bar{s}}_{\ \ \bar{j}k}\ (\sigma_{\hat{K}})^{i\bar{j}}\ (\sigma_{\hat{L}})^{k\bar{l}} \end{aligned} \tag{3.8}$$

Now, from the definition for the exterior product of the elements in Γ:

$$\rho\ \wedge \rho' \;\equiv\; \rho\ \otimes \rho' \;-\; \Lambda(\rho\ \otimes \rho') \tag{3.9}$$

we have:

$$(\mathcal{P}_S)^{\hat{I}\hat{J}}_{\hat{K}\hat{L}}\ \left(\eta^{\hat{K}}\ \wedge\ \eta^{\hat{L}}\right) \;=\; 0 \tag{3.10}$$

The q-deformed exterior derivative δ is defined as a map:

$$\begin{aligned} &\delta : \mathcal{A} \;\rightarrow\; \Gamma, \qquad \delta : \Gamma^{\wedge n} \;\rightarrow\; \Gamma^{\wedge(n+1)} \\ &\delta\alpha \;=\; \{ig/\lambda\}\ (\eta^0\ \alpha \;-\; \alpha\ \eta^0), \qquad \alpha \in \mathcal{A} \\ &\delta\eta \;=\; \{ig/\lambda\}\ \{\eta^0\ \wedge\ \eta \;+\; \eta\ \wedge\ \eta^0\} \\ &\lambda \;=\; q \;-\; q^{-1} \end{aligned} \tag{3.11}$$

When the operator δ acts on the fields. Watamura [3] called it the BRST transformation operator, and the fields in Γ have the ghost number 1.

Introduce functionals $\chi_{\hat{J}} \in \mathcal{A}'$, that are the q-analogues of the tangent vectors at the identity element of the group:

$$\delta a = \eta^{\hat{J}} (a * \chi_{\hat{J}}), \qquad \chi_{\hat{J}} = \frac{ig}{\lambda} \left(\epsilon\, \delta^0_{\hat{J}} - L^{\ 0}_{\hat{J}} \right) \tag{3.12}$$

$\chi_{\hat{I}}(T^a_{\ b})$ are proportional to the q-deformed Pauli matrices:

$$\begin{aligned} \chi_I(T^a_{\ b}) &= -\, ig\, [N]^{-1/2}\, q^{1-N+2/N}\, (\sigma_I)^a_{\ b} \\ \chi_0(T^a_{\ b}) &= -ig\, q\, [N]^{1/2} \left\{ q^{-N+2/N}\, [N]^{-1} + \lambda^{-1} \left(1 - q^{2/N}\right) \right\} (\sigma_0)^a_{\ b} \end{aligned} \tag{3.13}$$

The q-deformed structure constants can be computed from (3.12) and (3.6):

$$\begin{aligned} \mathbf{C}_{\hat{J}\hat{K}}^{\ \hat{I}} &\equiv \chi_{\hat{J}}(M^{\hat{I}}_{\ \hat{K}}), \qquad & \mathbf{C}_{\hat{J}\hat{K}}^{\ 0} &= \mathbf{C}_{\hat{J}0}^{\ \hat{K}} = 0 \\ \mathbf{C}_{0K}^{\ J} &= -\, ig\lambda\, \delta^J_K, & \mathbf{C}_{JK}^{\ I} &= -\, ig\, f^I_{JK} \end{aligned} \tag{3.14}$$

The q-deformed Cartan-Maurer equation can be derived from (3.11):

$$\begin{aligned} \delta\eta^0 &= \{ig/\lambda\}\, \{\eta^0 \wedge \eta^0 + \eta^0 \wedge \eta^0\} = 0 \\ \delta\eta^I &= \{ig/\lambda\} \left\{\eta^0 \wedge \eta^I + \eta^I \wedge \eta^0\right\} \\ &= \mathbf{C}_{\hat{J}\hat{K}}^{\ I}\, \eta^{\hat{J}} \otimes \eta^{\hat{K}} = (\lambda^2+2)^{-1}\, \mathbf{C}_{JK}^{\ I} \left(\eta^J \wedge \eta^K\right) \end{aligned} \tag{3.15}$$

¿From the condition $\delta^2 a = 0$, the functionals χ_J span the "q-deformed Lie algebra":

$$\chi_{\hat{I}}\, \chi_{\hat{J}} - \Lambda^{\hat{K}\hat{L}}_{\ \hat{I}\hat{J}}\, \chi_{\hat{K}}\, \chi_{\hat{L}} = \mathbf{C}_{\hat{I}\hat{J}}^{\ \hat{K}}\, \chi_{\hat{K}} \tag{3.16}$$

Acting (3.16) on $T^a_{\ b}$ we obtain following relations:

$$\begin{aligned} &\chi_{\hat{I}}(T^a_{\ d})\, \chi_{\hat{J}}\left(T^d_{\ b}\right) - \Lambda^{\hat{K}\hat{L}}_{\ \hat{I}\hat{J}}\, \chi_{\hat{K}}(T^a_{\ d})\, \chi_{\hat{L}}\left(T^d_{\ b}\right) = \mathbf{C}_{\hat{I}\hat{J}}^{\ \hat{R}}\, \chi_{\hat{R}}(T^a_{\ b}) \\ &(\mathcal{P}_{Adj})^{KL}_{\ IJ}\, \chi_K(T^a_{\ d})\, \chi_L\left(T^d_{\ b}\right) = \xi\, (\lambda^2+2)^{-1}\, \mathbf{C}_{IJ}^{\ R}\, \chi_R(T^a_{\ b}) \\ &\xi = q^{2/N} \left\{1 - \lambda\, q^{-N}\, [N]^{-1}\right\} \end{aligned} \tag{3.17}$$

Watamura [3] investigated the q-deformed BRST algebra $\mathcal{B}$ for $SU_q(2)$. The investigation can be generalized into the quantum groups $SU_q(N)$ straightforwardly, if some important quanties in the bicovariant differential calculus on $SU_q(N)$ are known.

$\eta^{\hat{I}}$ in the bimodule Γ is defined as the ghost field in the BRST algebra, that has the ghost number 1, but the degree of form 0. The gauge potential $A^{\hat{I}}$ has the degree of form 1, but the ghost number 0. There are two nilpotent operators in the BRST algebra: The operator δ increases the ghost number by one, and the operator d increases the degree of form by one. Neglecting the matter field, that is irrelevant to our following discussion, we are only interested in four fields in the BRST algebra $\mathcal{B}$: η, $d\eta$, A, and dA, that satisfy the following algebraic relations.

Firstly, we introduce an index n that is equal to the difference between the degree of form and the ghost number. The indices n for η, $d\eta$, A and dA are -1, 0, 1, and 2, respectively. Both nilpotent operators δ and d satisfy the Leibniz rule in the graded sense for the index n, and are covariant for the left and right actions.

Secondly, the gauge potentials $A^{\hat{I}}$ are assumed [3] to have similar properties like $\eta^{\hat{I}}$. Hereafter, we neglect the wedge sign $\wedge$ for simplicity.

$$(\mathcal{P}_S)^{\hat{I}\hat{J}}_{\hat{K}\hat{L}} \left(A^{\hat{K}} A^{\hat{L}}\right) = 0 \tag{3.18}$$

¿From the consistent conditions [3], $d\eta^{\hat{J}}$ and $dA^{\hat{J}}$ have to satisfy another relation:

$$(\mathcal{P}_A)^{\hat{I}\hat{J}}_{\hat{K}\hat{L}} \left(d\eta^{\hat{K}} d\eta^{\hat{L}}\right) = 0, \quad (\mathcal{P}_A)^{\hat{I}\hat{J}}_{\hat{K}\hat{L}} \left(dA^{\hat{K}} dA^{\hat{L}}\right) = 0 \tag{3.19}$$

Thirdly, the covariant condition of the covariant derivative in the BRST transformation requires:

$$\delta A^0 = 0, \quad \delta A^I = d\eta^I + \frac{ig}{\lambda} \left(\eta^0 A^I + A^I \eta^0\right) \tag{3.20}$$

Fourthly, for two different fields $X^{\hat{J}}$ and $Y^{\hat{K}}$ in $\mathcal{B}$ with indices n_x and n_y, $n_x > n_y$, respectively, the consistent condition requires the following commutative relations:

$$(-1)^{n_x n_y} X^{\hat{I}} Y^{\hat{J}} = Y^{\hat{K}} \left(X^{\hat{I}} * L^{\hat{J}}_{\hat{K}}\right) = \Lambda^{\hat{I}\hat{J}}_{\hat{K}\hat{L}} Y^{\hat{K}} X^{\hat{L}} \tag{3.21}$$

It is proved that the commutative relations (3.21) can be rewritten as follows:

$$\begin{array}{l} (-1)^{n_x n_y} X^a{}_r \left(\hat{R}_q^{-1}\right)^{ri}{}_{sk} Y^s{}_t \left(\hat{R}_q^{-1}\right)^{tk}{}_{bj} = \left(\hat{R}_q^{-1}\right)^{ai}{}_{rk} Y^r{}_s \left(\hat{R}_q^{-1}\right)^{sk}{}_{tj} X^t{}_b \\ X^a{}_b = X^{\hat{I}} \left(\sigma_{\hat{I}}\right)^a{}_b, \quad Y^a{}_b = Y^{\hat{I}} \left(\sigma_{\hat{I}}\right)^a{}_b \end{array} \tag{3.22}$$

At last, the gauge fields F^J satisfy:

$$\begin{array}{rcl} F^{\hat{J}} &=& dA^{\hat{J}} + \{ig/\lambda\} \left(A^0 A^{\hat{J}} + A^{\hat{J}} A^0\right) \\ \delta F^{\hat{I}} &=& \{ig/\lambda\} \left(\eta^0 F^{\hat{I}} - F^{\hat{I}} \eta^0\right) = \eta^{\hat{J}} F^{\hat{K}} \mathbf{C}^{\hat{I}}_{\hat{J}\hat{K}} \\ dF^{\hat{I}} &=& -\{ig/\lambda\} \left(A^0 F^{\hat{I}} - F^{\hat{I}} A^0\right) = -A^{\hat{J}} dA^{\hat{K}} \mathbf{C}^{\hat{I}}_{\hat{J}\hat{K}} \end{array} \tag{3.23}$$

4. q-Deformed Chern Class

Define the "generalized q-deformed Killing forms" and the q-trace as follows:

$$\begin{array}{l} g_{I_1 I_2 \cdots I_m} = D^{a_0}{}_{a_1} \chi_{I_1}\left(T^{a_1}{}_{a_2}\right) \chi_{I_2}\left(T^{a_2}{}_{a_3}\right) \cdots \chi_{I_m}\left(T^{a_m}{}_{a_0}\right) \\ \langle X_1 , X_2 , \cdots , X_m \rangle = X_1^{I_1} X_2^{I_2} \cdots X_m^{I_m} g_{I_1 I_2 \cdots I_m} \end{array} \tag{4.1}$$

where X_i are fields η, $d\eta$, A or dA in the BRST algebra $\mathcal{B}$. In (4.1) the fields can also be replaced by, for example, XY^0, Y^0X or F. The m-th q-deformed Chern class P_m for the quantum group $SU_q(N)$ is defined as follows:

$$P_m = \langle F , F , \cdots , F \rangle \tag{4.2}$$

¿From (3.22) and (3.23) we have:

$$\begin{aligned} \delta P_m \; = \; & q^{2m/N} \left(-igq^{1-N}[N]^{-1/2}\right)^{m+1} \Big\{ D^{a_0}_{\ b}\, \eta^b{}_{a_1}\, F^{a_1}{}_{a_2}\, F^{a_2}{}_{a_3} \cdots F^{a_m}{}_{a_0} \\ & - D^b{}_{a_1}\, F^{a_1}{}_{a_2}\, F^{a_2}{}_{a_3} \cdots F^{a_m}{}_{a_0}\, \eta^{a_0}{}_b \Big\} \end{aligned} \tag{4.3}$$

Owing to [7] $\left(\hat{R}_q^{\pm 1}\right)^{ac}{}_{bd}\, D^d{}_c \; = \; q^{\pm N}\, \delta^a_b$, the second term cancels the first term:

$$\begin{aligned} & D^b{}_{a_1}\, F^{a_1}{}_{a_2}\, F^{a_2}{}_{a_3} \cdots F^{a_m}{}_{a_0}\, \eta^{a_0}{}_b \\ & = D^b{}_{a_1}\, F^{a_1}{}_{a_2}\, F^{a_2}{}_{a_3} \cdots F^{a_m}{}_{a_0} \left\{ D^i{}_j\, q^N \left(\hat{R}_q^{-1}\right)^{a_0 j}{}_{di} \right\} \eta^d{}_b \\ & = q^N\, D^b{}_{a_1}\, D^i{}_j\, F^{a_1}{}_{a_2}\, F^{a_2}{}_{a_3} \cdots F^{a_{m-1}}{}_{a_m}\, F^{a_m}{}_{a_0} \left(\hat{R}_q^{-1}\right)^{a_0 j}{}_{dk} \eta^d{}_c \left(\hat{R}_q^{-1}\right)^{ck}{}_{r\ell} \left(\hat{R}_q\right)^{r\ell}{}_{bi} \\ & = q^N\, D^b{}_{a_1}\, D^i{}_j\, F^{a_1}{}_{a_2}\, F^{a_2}{}_{a_3} \cdots F^{a_{m-1}}{}_{a_m} \left(\hat{R}_q^{-1}\right)^{a_m j}{}_{dk} \eta^d{}_c \left(\hat{R}_q^{-1}\right)^{ck}{}_{r\ell}\, F^r{}_{a_0} \left(\hat{R}_q\right)^{a_0\ell}{}_{bi} \\ & = q^N\, D^b{}_{a_1}\, D^i{}_j \left(\hat{R}_q^{-1}\right)^{a_1 j}{}_{dk} \eta^d{}_c \left(\hat{R}_q^{-1}\right)^{ck}{}_{r\ell}\, F^r{}_{a_2}\, F^{a_2}{}_{a_3} \cdots F^{a_{m-1}}{}_{a_m}\, F^{a_m}{}_{a_0} \left(\hat{R}_q\right)^{a_0\ell}{}_{bi} \\ & = q^N\, D^{a_0}{}_{a_1}\, D^i{}_j\, \eta^{a_1}{}_c \left(\hat{R}_q^{-1}\right)^{cj}{}_{ri} F^r{}_{a_2}\, F^{a_2}{}_{a_3} \cdots F^{a_{m-1}}{}_{a_m}\, F^{a_m}{}_{a_0} \\ & = D^{a_0}{}_{a_1}\, \eta^{a_1}{}_r\, F^r{}_{a_2}\, F^{a_2}{}_{a_3} \cdots F^{a_{m-1}}{}_{a_m}\, F^{a_m}{}_{a_0} \end{aligned}$$

Thus, $\delta P_m \; = \; 0$. The proof of $dP_m \; = \; 0$ can be performed analogously. Note that the components of the identity and the adjoint representations are separated in the q-deformed Chern class, although they are mixed in the commutative relations of BRST algebra.

5. q-Deformed Chern-Simons and Cocycle Hierarchy

In the classical case Zumino [10] introduced a homotopy operator k to compute the Chern-Simons. Generalizing his method we compute the m-th q-deformed Chern-Simons for $SU_q(N)$. Introduce a q-deformed homotopy operator k that is nilpotent and satisfies the Leibniz rule in the graded sense for the index n:

$$k^2 \; = \; 0, \qquad d\, k \; + \; k\, d \; = \; 1 \tag{5.1}$$

If k exsits, we are able to compute the q-deformed Chern-Simons $Q_{2m-1}(A)$ from the m-th q-deformed Chern class as follows:

$$\begin{aligned} P_m \; = \; (d\, k \; + \; k\, d)\, P_m \; = \; d(k\, P_m) \; = \; d\, Q_{2m-1}(A) \\ Q_{2m-1}(A) \; = \; k\, P_m \end{aligned} \tag{5.2}$$

where we used $dP_m = 0$.

In the following we are going to show the existence of k. Introduce a real parameter t. When t changes from 0 to 1, the gauge potentials A^J_t change from 0 to A^J:

$$A^J_t \; = \; tA^J, \qquad F^J_t \; = \; tdA^J + \{igt^2/\lambda\} \left(A^0 A^J + A^J A^0\right) \tag{5.3}$$

Define the q-deformed derivative along t [11], that satisfies the q-deformed Leibniz rule:

$$
\begin{aligned}
\frac{\partial}{\partial_q t} f(t) &= \frac{f(qt) - f(q^{-1}t)}{t\,(q - q^{-1})} \\
\frac{\partial}{\partial_q t} f(t)g(t) &= \frac{\partial f(t)}{\partial_q t}\, g(qt) + f(q^{-1}t)\,\frac{\partial g(t)}{\partial_q t}
\end{aligned}
\tag{5.4}
$$

The q-deformed integral is defined by:

$$
\int_0^{t_0} d_q t\; f(t) = t_0(1-q^2) \sum_{k=0}^{\infty} q^{2k}\, f(q^{2k+1}t_0)
\tag{5.5}
$$

Define the q-deformed operator ℓ_t that satisfies the q-deformed Leibniz rule in the graded sense for the index n:

$$
\ell_t A_t^J = 0, \quad \ell_t F_t^J = d_q t\; A^J
\tag{5.6}
$$

It is easy to check that for all formal polynomials (vanishing at $F_t^J = 0$ and $A_t^J = 0$):

$$
\ell_t\, \ell_t = 0, \quad \ell_t\, d + d\, \ell_t = \hat{\delta}_q \equiv d_q t\, \frac{\partial}{\partial_q t}
\tag{5.7}
$$

Comparing it with (5.1) we obtain:

$$
k = \int_0^1 \ell_t
\tag{5.8}
$$

The $(2m-1)$-th q-deformed Chern-Simons can be computed from (5.2) straightforwardly. For examples, when $m = 2$ we have:

$$
\begin{aligned}
Q_3(A) &= k\; (P_2)_t \\
&= \langle A\,.\, F\rangle - \{\xi[3]\}^{-1}\,\langle A\,;\, A\,;\, A\rangle
\end{aligned}
\tag{5.9}
$$

Just like those in the classical case [12], the gauge fields F^J are invariant under the transformation:

$$
A^J \to A^J - \eta^J, \quad d \to d + \delta
\tag{5.10}
$$

In fact,

$$
\begin{aligned}
F^J &\to \mathcal{F}^J \\
&= (d + \delta)\,(A^J - \eta^J) \\
&\quad + \frac{ig}{\lambda}\left\{(A^0 - \eta^0)(A^J - \eta^J) + (A^J - \eta^J)(A^0 - \eta^0)\right\} \\
&= F^J + \left\{\delta A^J - d\eta^J - \frac{ig}{\lambda}\,(\eta^0 A^J + A^J\eta^0)\right\} \\
&\quad - \left\{\delta\eta^J - \frac{ig}{\lambda}\left(\eta^0\eta^J + \eta^J\eta^0 - A^0\eta^J - \eta^J A^0\right)\right\} \\
&= F^J
\end{aligned}
$$

Now, transforming (5.2) and expanding it by the ghost number, we obtain:

$$\begin{aligned} Q_{2m-1}(A-\eta) &= \sum_{n=0}^{2m-1} \omega^n_{2m-n-1} \\ P_m &= d\omega^0_{2m-1}, \qquad \delta\omega_0^{2m-1} = 0 \\ \delta\omega^n_{2m-n-1} &+ d\omega^{n+1}_{2m-n-2} = 0. \qquad n = 0, 1, \cdots, (2m-2) \end{aligned} \tag{5.11}$$

where the subscripts denote the degrees of form of the quantities. and the superscripts denote the ghost numbers. For example, for $m = 2$ we have:

$$\begin{aligned} \omega^0_3 &= \langle A \, , \, dA \rangle + \frac{[4]}{\xi[3][2]} \langle A \, , \, A \, , \, A \rangle \\ \omega^1_2 &= - \langle \eta \, , \, dA \rangle \\ \omega^2_1 &= - \xi^{-1} \langle \eta \, , \, A \, , \, \eta \rangle \\ \omega^3_0 &= \{\xi[3]\}^{-1} \langle \eta \, , \, \eta \, , \, \eta \rangle \end{aligned} \tag{5.12}$$

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Dynamical Soluble Models of Quantum Measurement Process

YOU-QUAN LI AND YI-XIN CHEN

CCAST(World Lab), P.O.Box 8730 Beijing 100080 and

* *Zhejiang Institute of Modern Physics*

Zhejiang University, Hangzhou 310027, China

ABSTRACT

Quantum measurement process is studied by considering a quantum object and the measuring detector as a closed system. A method for to construct explicitly soluble models is suggested. Several models of an ultrarelativistic particle coupled with various measuring detectors are given and discussed. Meanwhile some ambiguities in some literature is pointed and overcame.

*mail address

Now, transforming (5.2) and expanding it by the ghost number, we obtain:

$$\begin{aligned} Q_{2m-1}(A-\eta) &= \sum_{n=0}^{2m-1} \omega^n_{2m-n-1} \\ P_m &= d\omega^0_{2m-1}, \qquad \delta\omega_0^{2m-1} = 0 \\ \delta\omega^n_{2m-n-1} &+ d\omega^{n+1}_{2m-n-2} = 0. \qquad n = 0, 1, \cdots, (2m-2) \end{aligned} \tag{5.11}$$

where the subscripts denote the degrees of form of the quantities. and the superscripts denote the ghost numbers. For example, for $m = 2$ we have:

$$\begin{aligned} \omega^0_3 &= \langle A , dA \rangle + \frac{[4]}{\xi[3][2]} \langle A , A , A \rangle \\ \omega^1_2 &= - \langle \eta , dA \rangle \\ \omega^2_1 &= - \xi^{-1} \langle \eta , A , \eta \rangle \\ \omega^3_0 &= \{\xi[3]\}^{-1} \langle \eta , \eta , \eta \rangle \end{aligned} \tag{5.12}$$

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Dynamical Soluble Models of Quantum Measurement Process

YOU-QUAN LI AND YI-XIN CHEN

CCAST(World Lab), P.O.Box 8730 Beijing 100080 and

* *Zhejiang Institute of Modern Physics*

Zhejiang University, Hangzhou 310027, China

ABSTRACT

Quantum measurement process is studied by considering a quantum object and the measuring detector as a closed system. A method for to construct explicitly soluble models is suggested. Several models of an ultrarelativistic particle coupled with various measuring detectors are given and discussed. Meanwhile some ambiguities in some literature is pointed and overcame.

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*mail address

In order to understand quantum mechanics completely, the measurement problem [1] in quantum mechanics is a sticky question, in which Von Neumann's or Lüder's projection postulate plays a curial role. The postulate says that if a physical quantity of a quantum object is measured twice in succession, then the same value is obtained each time. This means that once a determined value about the observable A is measured, the state vector of the quantum object Q (usually a certain superposition of some kind of basis in Hilbert space) must collapse into the corresponding eigenstate of the operator A. Simplified solvable models in quantum mechanics is undoubtedly very helpful for to comprehend the called measurement problem[1,2]. In Ref[3], the author multiply the evolution operator of his model only on one of the components of the initial state of the quantum object and measuring device system. So the discussion is only about a quantum phase shfiter instead of wave packet collapse. In present letter, we will analysis measure process formally in terms of quantum mechanics and formulate a condition of explicitly solvability. From this condition we construct several explicitly solvable models. they are ultrarelativistic particle interacting with fermionic or bosonic oscillators in one dimension. we also formulate models interacting with spin array and angular momentum array. One can find that the energy fluctuation of measuring device in the model of Ref[4] is divergent when the interaction between Q and M vanishes. This is overcame in the models of present letter.

As is known[5,6] that quantum mechanics is a fundamental theory governing the whole universe, we consider the quantum object Q to be measured and the measuring device M as a closed quantum system Q+M. Let $\mathcal{H}_Q$ and $\mathcal{H}_M$ stand for the Hilbert spaces of Q and M respectively. We assume that the operator A on $\mathcal{H}_Q$ representing the observable quantity under consideration has a spectrum $\{\lambda\}$. Let $|\psi(\lambda)> \in \mathcal{H}_Q$ be normalized eigenvectors corresponding to eigenvalue λ. Obviously if there are no interaction between Q and M, the evolutions of the states in Hilbert space $\mathcal{H}_Q$ is independent with that in $\mathcal{H}_M$. Thus the evolution of the state of the closed system Q + M is simply a tensor product of the independent evolution of the states in Hilbert spaces $\mathcal{H}_Q$ and $\mathcal{H}_M$.

$$\sum_{\lambda\mu} C_\lambda(t) D_\mu(t) |\psi_\lambda> \otimes |\phi_\mu> . \tag{1}$$

Von Neumann's postulate requests such final state of Q+M after measure process taking place that λ and μ are correlated i.e. $\mu := \mu(\lambda)$. Of cause the correlation do not happen unless the mutual interaction between Q and M in measure process is take into account. The models which we constructed exactly lead to such correlations.

In order to find dynamical models which can realize Von Neumann's postulate, we consider a total Hamiltonian for Q+M system

$$H = H_Q + H_M + H' \quad \text{(independent of time)}, \tag{2}$$

where H_Q and H_M is the free Hamiltonian of Q and M respectively, and H' the interaction Hamiltonian. Here the 'free' means that the Hamiltonian H_Q and H_M contain the operators of their own system only *i.e.* they commute each other. In this case, it is convenient to work in interaction picture and is easy to obtain the time evolution of state in Hilbert space $\mathcal{H}_{Q+M}$.

$$|\Psi(t)>= e^{-\frac{i}{\hbar}((H_Q+H_M)t}\; U(t,t')\; e^{\frac{i}{\hbar}((H_Q+H_M)t'}|\Psi(t')>, \tag{3}$$

where $|\Psi>\in \mathcal{H}_{Q+M}$ and the evolution operator satisfies

$$i\hbar\frac{\partial U(t,t')}{\partial t} = H_I'(t)U(t,t'), \tag{4}$$

with

$$H_I'(t) = e^{\frac{i}{\hbar}((H_Q+H_M)t}H'(t)e^{-\frac{i}{\hbar}((H_Q+H_M)t}, \tag{5}$$

and $U(t,t) = 1$, t' is a parameter due to there are infinite many interaction pictures.

From eq.(4) we can write out the evolution operator as an integral-integral

$$U(t,t') = \overrightarrow{exp}\left(\frac{1}{i\hbar}\int_{t'}^{t} H_I'(t'')dt''\right) \tag{6}$$

Evidently, If the commutator of the interaction Hamiltonians in interaction picture at different time vanishes, i.e.

$$[\, H_I'(t'),\ H_I'(t'')\,] = 0, \tag{7}$$

then the evolution operator has an explicit solution. Therefore eq.(7) is a start point to construct explicitly solvable models for quantum measure process.

Let us first consider an ultrarelativistic particle as the quantum object and consider an one-dimensional free bosonic or fermionic oscillator array as the measure detector. In this case, the free Hamiltonian for Q and M are

$$\begin{aligned} H_Q &= c\hat{P}, \\ H_M &= \hbar\omega\sum_{l=1}^{N}(a_l^+ a_l + \frac{1}{2}), \end{aligned} \tag{8}$$

where $\hat{P}$ is the momentum operator of the particle. a_l^+ or a_l is the creative or annihilative operator of the oscillator in lth site on the array. The interaction Hamiltonian must contain some of the operators of both Q and M. Considering the changes from H' to H_I' brought out by the free Hamiltonian eq.(8), we can write out the interaction Hamiltonian as the following

$$H' = \sum_{l=1}^{N} V(x - x_l) f(a_l e^{i\frac{\omega}{c}x}, a_l^+ e^{-i\frac{\omega}{c}x}), \tag{9}$$

where x is the position operator of the particle,x_l is a parameter indicating the site of lth oscillator, V is a real potential and f can be any analytical functions. This interaction Hamiltonian is transformed into the one in interaction picture via eq.(5). The change is that $V(x - x_n)$ becomes $V(x + ct - x_n)$ only. And it is easy to show that the condition eq.(7) is really satisfied. Thus the evolution operator is explicitly solvable.

Let us consider the one photon interaction case

$$H' = \sum_{l=1}^{N} V(x - x_l)(a_l e^{i\frac{\omega}{c}x} + a_l^+ e^{-i\frac{\omega}{c}x}). \tag{10}$$

It is also the general case for fermionic oscillators. Then the evolution operator is solved as following product,

$$S = \lim_{\substack{t\to\infty \\ t'\to-\infty}} U(t,t') = \prod_{l=1}^{N} S_{(l)}, \tag{11}$$

where

$$S_{(l)} = \exp(\frac{v^2\delta^2}{\hbar^2 c^2}) \exp(\frac{v\delta}{i\hbar c} e^{-i\frac{\omega}{c}x} a_l^+) \exp(\frac{v\delta}{i\hbar c} e^{i\frac{\omega}{c}x} a_l) \tag{12}$$

where $v\delta := \int_{-\infty}^{\infty} V(x)dx$. In deriving the above result, the known Baker-Compell-Hausdorff formula [7] has been used.

Suppose the initial state of Q is a plane wave $|p>$ (eigenstates of H_Q) and M is in its rest state (i.e. ground state) $|0>$ before interaction. Notice that $a_l|0>= 0$, the evolution of the initial state is obtained without much difficulty. For the case of M being bosonic oscillators, the result is

$$\begin{aligned} |\Psi_f> &= S|\Psi_i> \\ &= e^{N(v\delta/\hbar c)^2} \sum_{n=0}^{\infty} (\frac{N^n}{n!})^{\frac{1}{2}} (\frac{v\delta}{i\hbar c})^n \; |p - n\frac{\hbar\omega}{c}, n> . \end{aligned} \tag{13}$$

Here and the following $|p', n>$ stands for $|p'> \otimes |n>$ and $|n>$ is the symmetrized state of M. For the case of fermionic oscillator, the creative and annihilation operators are Grassmanian $(a^+)^2 = a^2 = 0$. Taking this in mind we obtained the following evolution

$$S|\Psi_i>= e^{N(v\delta/\hbar c)^2} \sum_{n=0}^{N} \begin{pmatrix} N \\ n \end{pmatrix}^{\frac{1}{2}} (\frac{v\delta}{i\hbar c})^n \; |p - n\frac{\hbar\omega}{c}, \; n> . \tag{14}$$

where $\begin{pmatrix} N \\ n \end{pmatrix}$ stands for the binomial coefficients. If M is a spin array, one can also construct an explicit solvable interaction Hamiltonian. The following is the total Hamiltonian of this case.

$$H = c\hat{P} + \frac{1}{2}\hbar\omega \sum_{l=1}^{N} \sigma_z^{(l)} + \sum_{l=1}^{N} V(x - x_l) \left[\sigma_+^{(l)} e^{-i\frac{\omega}{c}x} + \sigma_-^{(l)} e^{i\frac{\omega}{c}x}\right] . \tag{15}$$

This is nearly the Hamiltonian as in [4], a development of the Coleman -Hepp model [8]. One can solve the evolution operator, which is also a product of individual ones,

$$S^{(l)} = \cos(\frac{v\delta}{\hbar c}) - i\sin(\frac{v\delta}{\hbar c})(\sigma_+^{(l)} e^{-i\frac{\omega}{c}x} + \sigma_-^{(l)} e^{i\frac{\omega}{c}x}). \tag{16}$$

It leads to the following evolution of state

$$S|p, 0>= \sum_{j=0}^{N} \begin{pmatrix} N \\ j \end{pmatrix}^{\frac{1}{2}} [\; \cos(\frac{v\delta}{\hbar c}) \;]^{N-j} \; [\; -i\sin(\frac{v\delta}{\hbar c}) \;]^j \;\; |p - j\frac{\hbar\omega}{c}, \; j> . \tag{17}$$

where $|j>$ stands for the state of M in which there are j spin up and $N - j$ spin down states.

Similarly, one can find the solvable Hamiltonian of ultrarelativistic particle coupling with angular momentum array. There are two possibilities

$$H = c\hat{P} + \hbar\omega \sum_{l=1}^{N} J_z^{(l)} + \sum_{l=1}^{N} V(x - x_n) \left[J_+^{(l)} e^{-i\frac{\omega}{c}x} + J_-^{(l)} e^{i\frac{\omega}{c}x} + J_z \right]$$

or

$$H = c\hat{P} + \sum_{l=1}^{N} \frac{J_l^2}{2I_l} + \sum_{l=1}^{N} V(x - x_n)\, \mathbf{b} \cdot \mathbf{J}. \tag{18}$$

For the sake of saving space of the letter, We would like to omit further calculations of this case.

From the above explicit solvable models, we may find that the evolution caused by the interaction Hamiltonian really lead to the correlation of the eigenstates in $\mathcal{H}_Q$ and that in $\mathcal{H}_M$. Thus they do realize Von Neumann postulate. As we have obtained the evolution of the system Q + M for M being bosonic oscillators, fermionic oscillators and spin array respectively, we can calculate the energy fluctuation of M around the average of them. The results are

$$\begin{aligned} < \delta H_M > &= \hbar\omega\sqrt{N}\,(\frac{v\delta}{\hbar c}); \\ < \delta H_M > &= \hbar\omega\sqrt{N}\frac{(\frac{v\delta}{\hbar c})}{1 + (\frac{v\delta}{\hbar c})^2}; \\ < \delta H_M > &= \hbar\omega\sqrt{N}\sin(\frac{v\delta}{\hbar c})\cos(\frac{v\delta}{\hbar c}); \end{aligned} \tag{19}$$

for bosonic, fermionic and spin array cases respectively. The relative fluctuations are

$$\frac{< \delta H_M >}{< H_M >} = \frac{2}{\sqrt{N}} \frac{(\frac{v\delta}{\hbar c})}{1 + 2(\frac{v\delta}{\hbar c})^2}$$

$$\frac{<\delta H_M>}{<H_M>} = \frac{2}{\sqrt{N}} \frac{(\frac{v\delta}{\hbar c})}{1+3(\frac{v\delta}{\hbar c})^2}$$

$$\frac{<\delta H_M>}{<H_M>} = \frac{2}{\sqrt{N}} \frac{\sin(\frac{v\delta}{\hbar c})\cos(\frac{v\delta}{\hbar c})}{2\sin^2(\frac{v\delta}{\hbar c})-1} \qquad (20)$$

for bosonic, fermionic and spin array cases respectively. From eq.(19) we can see that for either of bosonic case, fermionic case or spin array, the energy fluctuations of them is proportional to $\hbar\omega\sqrt{N}$. Moreover, the relative fluctuation of them eq.(20) is reversely proportional to the square root of N. Furthermore we can find that both eq.(19) and eq.(20) are model independent in the limit of week coupling *i.e.* $v\delta/\hbar c$ becomes very small.

In above we discussed the quantum measurement process. The key point is that Q + M is considered as a closed system. Thus the total Hamiltonian is a sum of their free Hamiltonian $H_Q + H_M$ and the interaction Hamiltonian H' between Q and M. In order that the time evolution operator is explicitly solvable, the interaction Hamiltonians eq.(9), eq.(15) or eq.(18) was constructed by considering that the commutator of the interaction Hamiltonian at different time is desired to vanish in interaction picture (although do not commute in Schrödinger picture). It is also worthwhile to mention that the so called free Hamiltonian in [3] is not really a free one, because the Hamiltonian for the spin array director there contains functions of x which is the position operator of the ultrarelativistic particle. In [3] a discussion about decoherence of quantum phase shifter in the macroscopic limit was confused as wave packet reduction. It is also worthwhile to mention that the relative fluctuation in [4] is divergent when the interaction between quantum object and measuring detector vanishes. This is due to that the energy of ground state of the detector was defined as zero in [4]. However, this problem has been overcome in present letter.

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Cyclic Quantum Dilogarithm and Shift Operator [1]

Zhan-Ning Hu [2]

CCAST(World Lab.), P.O.Box 8730, Beijing 100080

and

Institute of Theoretical Physics, Academia Sinica,

P. O. Box 2735, Beijing 100080, China [3]

Bo-Yu Hou

Institute of Modern Physics, Northwest University, Xian 710069, China

Abstract

From the cyclic quantum dilogarithm the shift operator is constructed with q is a root of unit and the representation is given for the current algebra introduced by Faddeev *et al.* It is shown that the theta-function is factorizable also in this case by using the star-square equation of the Baxter-Bazhanov model.

[1]This research was partially supported by the China center of advanced science and technology.

[2]**email address: huzn@itp.ac.cn**

[3]**mail address**

1 Introduction

Recently people have pay more attention to the investigation of completely integrable discrete variants of conformal field theory such as lattice Wess-Zumino-Witten model [1], quantum Volterra model [2], the lattice Liouville model and the lattice sine-Gordon model [3]. Bruschi *et al* studied the properties and behaviors of integrable symplectic maps from the lattice evolution equation[4]. In order to providing the new way to construct the conservation laws and to discuss the nature of quantum integrabilit by using the discrete space-time method [5,6] Faddeev *et al* investigated the lattice Virasoro algebra and the corresponding hierarchy of conservation laws from the lattice current algebra where the shift operator with $|q| < 1$ is discussed in detail [7].

We know that the dilogarithm appeared firstly in the discussion of the XXZ-model at small magnetic field [8], then appeared in the 2-dimensional quantum filed theories and solvable lattice models [9]. The quantum dilogarithm identity introduced in Ref. [10] is equivalent to the restricted star-triangle relation of the three-dimensional Baxter-Bazhanov model [11,12] and more recently Kashaev built a connection between cyclic $6j$-symbol and the quantum dilogarithm.

In this letter, we will construct the shift operator from the cyclic quantum dilogarithm and give a representation of the lattice current algebra which combined with free discrete time dynamics. It is proved that the "theta-function" is also factorizable in the case of $|q| = 1$ by using the star-square equation [13,14] of the three-dimensional Baxter-Bazhanov model. In section 2 we give the cyclic quantum dilogarithm and the representation of the lattice current algebra. The shift operator is constructed and the "theta-function" is proved to be factorizable in section 3. Finally some remarks are given.

2 Cyclic Quantum Dilogarithm and the Lattice Current Algebra

By following the Refs.[11,13], define the function

$$w(a,b,c|l)=\prod_{j=1}^{l} b/(c-a\omega^{j}), \quad a^L+b^L=c^L, \quad l\geq 0, \tag{1}$$

with $w(a,b,c|0)=1$. For any operator A whose L-th power is the identity operator, the spectrum of the operator A is given by L distinct numbers

$$\omega^l, \quad l=0,1,\cdots,L-1. \tag{2}$$

We define the cyclic quantum dilogarithms $\Psi(A)$ and $\Phi(A)$ (See Ref. [10]) are the commuted operators which depend on the operator A and commute with A which has the spectrum (2). The spectrums of $\Psi(A)$ and $\Phi(A)$ have the following forms:

$$\Psi(\omega^l)=\Psi(1)w(a,b,c|l),$$
$$\Phi(\omega^l)=\Phi(1)w(c,\omega^{1/2}b,\omega a|l), \tag{3}$$

where $\Psi(1)$ and $\Phi(1)$ are the non-zero complex factors. And the "functional" relations of the cyclic quantum dilogarithms $\Psi(A)$ and $\Phi(A)$ can be written as

$$\Psi(\omega^{-1}A)\Psi(A)^{-1}=(c-aA)/b,$$
$$\Phi(\omega^{-1}A)\Phi(A)^{-1}=(\omega^{1/2}a-\omega^{-1/2}cA)/b, \tag{4}$$

which determine the operators $\Psi(A)$ and $\Phi(A)$ up to the complex factors. Furthermore, from the above relations, we have

$$\Psi(A)=\rho_1\sum_{l=0}^{L-1}A^l\prod_{j=1}^{l}\frac{a}{c-b\omega^{-j}}, \tag{5}$$

$$\Phi(A) = \rho_2 \sum_{l=0}^{L-1} A^l \prod_{j=1}^{l} \frac{c}{a\omega - b\omega^{1/2-j}}, \tag{6}$$

where ρ_1 and ρ_2 are also the non-zero complex factors. In this way, $\Psi(1)$ and $\Phi(1)$ can be expressed as

$$\Psi(1) = \rho_1 \frac{c - a\omega}{b} \sum_{l=0}^{L-1} \prod_{j=1}^{l} \frac{a\omega}{c - b\omega^{-j}},$$

$$\Phi(1) = \rho_2 \frac{\omega^{1/2}(a-c)}{b} \sum_{l=0}^{L-1} \prod_{j=1}^{l} \frac{c\omega}{a\omega - b\omega^{1/2-j}}. \tag{7}$$

By using the cyclic quantum dilogarithms (4) we will construct the shift operator with $|q| = 1$, in the following section, from the lattice current algebra

$$\begin{aligned} w_{n-1}w_n &= q^2 w_n w_{n-1}, \quad n = 2, 3, \cdots, 2N, \\ w_{2N}w_1 &= q^2 w_1 w_{2N}, \\ w_m w_n &= w_n w_m, 1 < |m-n| < 2N-1, \end{aligned} \tag{8}$$

which reduces to the periodic free field in the continuous limit [7]. Set

$$\begin{aligned} x_i &= \underbrace{I \otimes I \otimes \cdots \otimes I}_{i-1} \otimes x \otimes \underbrace{I \otimes I \otimes \cdots \otimes I}_{N-i-1}, \\ y_i &= \underbrace{I \otimes I \otimes \cdots \otimes I}_{i-1} \otimes y \otimes \underbrace{I \otimes I \otimes \cdots \otimes I}_{N-i-1}, \end{aligned} \tag{9}$$

with $i = 1, 2, \cdots, N-1$, where the L-by-L matrices x, y are given by

$$x = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \omega & 0 & \cdots & 0 \\ 0 & 0 & \omega^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \omega^{L-1} \end{bmatrix}, \tag{10}$$

with $\omega = exp(2\pi i/L)$ and I is the L-by-L unit matrix. When we fix that $w_n^L = 1$ $(n = 1, 2, \cdots, 2N)$ the representation of the lattice current algebra

(8) with $q^2 = \omega$ is given from the following relations:

$$\begin{array}{l} w_{2i+1} = x_i^{-1} x_{i+1}, \quad i = 0,1,2,\cdots,N-1, \\ w_{2j} = y_j, \quad j = 1,2,\cdots,N-1, \\ w_{2N} = \prod_{l=1}^{N-1} y_l^{-1}, \end{array} \tag{11}$$

where $x_0 = x_N = 1$ and $x_i, y_i (i = 1,2,\cdots,N-1)$ are the $N-1$ independent Weyl pairs satisfied the relations $x_i y_i = q^2 y_i x_i$.

3 Shift Operator and Factorized "Theta Function"

In the discrete space-time picture the equation of motion of the free field $p(x)$, $p_t(x,t) + p_x(x,t) = 0$, has the form $w_n(t+1) = w_{n-1}(t)$ with $w_n(0) = w_n$. Then it exists that the shift operator which satisfy

$$w_n U = U w_{n-1}. \tag{12}$$

The main aim in the rest of this letter is to discuss it by using the cyclic quantum dilogarithms. Let

$$h_n = \Psi(w_n)\Phi(w_n^{-1}) \tag{13}$$

with $w_n^L = 1$ for $n = 1,2,\cdots,2N$. From relation (4) we have

$$w_n h_{n-1} = -\omega^{1/2} h_{n-1} w_n w_{n-1}, \quad h_n w_{n-1} = -\omega^{1/2} w_n w_{n-1} h_n. \tag{14}$$

It can be obtained easily that

$$w_n h_{n-1} h_n = h_{n-1} h_n w_{n-1}, \quad n = 1,2,3,\cdots,2N, \tag{15}$$

with $w_0 = w_{2N}$. By considering that the two central elements $C_1 = \prod w_{odd}$ and $C_2 = \prod w_{even}$ are equal [4], from the above relation, we can express the

[4]We can always get the lattice current algebra with the two equal central elements C_1 and C_2 by choosing w_n properly.

shift operator U as

$$U = h_1 h_2 \cdots h_{2N-1}, \tag{16}$$

which satisfies the relation (12). And operators h_n give a representation of the braid group of the $A^{(1)}_{2N-1}$ type. It is interesting that the above relations hold also when we substitute operators h_n by $\bar{h}_n = \Phi(w_n)\Psi(w_n^{-1})$. In the other hand, the shift operator with $|q| \neq 1$ can be denoted by the theta functions [10,15] . Then we can ask the questions that what is the difference between the operators h_n and $\bar{h}_n$ and what happened about the theta functions when it denotes the shift operator with $|q| = 1$. The answers of them are given as the follows. We know that the star-square relation of the three dimensional Baxter-Bazhanov model can be written as [13,14]

$$\left\{ \sum_{\sigma \in Z_N} \frac{w(x_1, y_1, z_1|a+\sigma) w(x_2, y_2, z_2|b+\sigma)}{w(x_3, y_3, z_3|c+\sigma) w(x_4, y_4, z_4|d+\sigma)} \right\}_0$$

$$= \frac{(x_2 y_1/x_1 z_2)^a (x_1 y_2/x_2 z_1)^b (z_3/y_3)^c (z_4/y_4)^d}{\gamma(a-b)\omega^{(a+b)/2}}$$

$$\times \frac{w(\omega x_3 x_4 z_1 z_2/x_1 x_2 z_3 z_4|c+d-a-b)}{w\left(\frac{x_4 z_1}{x_1 z_4}|d-a\right) w\left(\frac{x_3 z_2}{x_2 z_3}|c-b\right) w\left(\frac{x_3 z_1}{x_1 z_3}|c-a\right) w\left(\frac{x_4 z_2}{x_2 z_4}|d-b\right)}, \tag{17}$$

with the constraint condition $y_1 y_2 z_3 z_4/(z_1 z_2 y_3 y_4) = \omega$ where the subscript "0" after the curly brackets indicates that the l. h. s. of the above equation is normalized to unity at zero exterior spins and the following notations are used:

$$w(x, y, z|l) = (y/z)^l w(x/z|l), \;\; \gamma(a-b) = \omega^{(a-b)(L+a-b)/2}. \tag{18}$$

From Eqs. (5), (6) and (7), the operators h_n can be denoted as

$$h_n = \rho_1 \rho_2 \sum_{k=0}^{L-1} a_k w_n^k \tag{19}$$

where

$$a_k = \sum_{l=0}^{L-1} \frac{1}{w(\omega^{-1}b, a, c|l) w(\omega^{-1/2}b, c, \omega a|k+l)} \tag{20}$$

g the above star-square relation it can be proved that

$$a_k = \gamma(k)a_0 \tag{21}$$

(k) is given in Eq. (18). Then we have

$$h_n = a_0\rho_1\rho_2\theta(w_n), \quad \theta(w_n) = \sum_{k=0}^{L-1}(-)^k\omega^{k^2/2}{w_n}^k. \tag{22}$$

more we can proved that $h_n = \bar{h}_n$ by using Eqs. (17) and (21). There- shift operator is constructed from the cyclic quantum dilogarithms h the "theta function" is factorized when $|q| = 1$.

'onclusions and Remarks

ıg the cyclic quantum dilogarithms (4) the shift operator of the peri- ː field in the discrete time-space picture is constructed from the lattice algebra for which the cyclic representation is given from $N-1$ inde- Weyl pairs. And we show that the "theta function" is also factoriz- en $|q| = 1$ by using the star-square relation of the three dimensional Bazhanov model.

.ct, the "shift" property can be connected to many domains of physics the quantum field theory, the solvable models of statistical mechanics : dynamical systems. Recently Faddeev and Volkov discussed the the saw S corresponding to the solution of Hirota Equation along :rete time axis as an example of an integrable symplectic map [16]. ft operator also appeared in the differential forms when we set $\partial_r f =$ ', $R_r f(k) = f(k+1)$, then $f(k+1)X = Xf(k)$ on the lattice where :orm and it will be useful in the lattice gauge theory [17]. Another he developments is in the domain of the supermanifolds and it is not ar about the structure of it when the relations similar to the lattice current algebra is intrduced. Due to the cyclic quantum dilogarithms have the deep connections with the solved lattice models we can discuss the shift properties of the two and three dimensional lattice model further.

* * *

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Solutions of n-simplex equation from Solutions of Braid Group Representation [1]

YOU-QUAN LI[a] ZHAN-NING HU[b, 2]

CCAST(World Lab.), P.O.Box 8730, Beijing 100080

[a] Zhejiang Institute of Modern Physics, Zhejiang University, Hangzhou 310027

[b] Institute of Theoretical Physics, Academia Sinica,

P. O. Box 2735, Beijing 100080, China [3]

Abstract

It is shown that a kind of solutions of n-simplex equation can be obtained from representations of braid group. The symmetries in its solution space are also discussed.

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[2]**email address: huzn@itp.ac.cn**

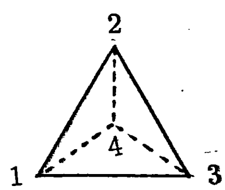
[3]**mail address**

Recently many interests have been paid on the investigations of the higher dimensional integrable systems in the quantum field theory [1] and in the statistical mechanics [2]. For the lower dimension case of them, the Yang-Baxter equation (YBE) plays a crucial role of which the structure is now fairly well understood. As a substitution of YBE the tetrahedron equation becomes a integrability condition of the exactly solved model in three dimensions [3], from which the community of the layer-to-layer transfermatrixes is preserved. One of the approaches is the n-simplex equation [4] and it is said that the case of $n = 3$ is corresponding to the tetrahedron equation. The aim of this letter is to expose some procedure for deriving solutions of n-simplex equation from braid group representations (ie. solutions of parameter independent YBE) [5]. Meanwhile we would like to derive some symmetry transformation in solution space of 3-simplex equation as an example.

The 3-simplex equation we will consider takes the following form

$$R_{123}R_{214}R_{341}R_{432} = R_{234}R_{143}R_{412}R_{321} \tag{1}$$

where the order of subscripts are chosen in such a way that the normal of each surface of the 3-simplex is always toward the inside of the 3-simplex (tetrahedron)



Certainly, the positive direction of the normal of a surface determined by a cycle (for example, (123), (341) etc.) following the right-hand helicity. The matrices in eq.(1) stands for the scattering of three strings, for example

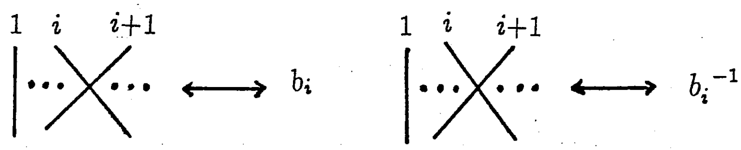
$$R_{214}|\mu_1, \mu_2, \mu_3, \mu_4 >= \sum_{\nu_1\nu_2\nu_3} R^{\nu_2\nu_1\nu_4}_{\mu_2\mu_1\mu_4}|\nu_1, \nu_2, \mu_3, \nu_4 > . \tag{2}$$

Solving solutions of eq.(1) is a complicated problem. It is known that many representations of braid group have been found in recently years. We will show that if one have a representation of braid group, one can obtain a kind of solutions of the 3-simplex equation. A braid group is a category of free group under the constraint of the following equivalence relations

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$$

$$b_i b_j = b_j b_i \quad for \quad |i-j| > 1. \tag{3}$$

It is called a braid group due to it has a simple realization on N-strings by identify



Then the equivalence relation (3) becomes an evident topological equivalence relation. If a representation of braid group takes as

$$\rho : b_i \rightarrow S_{i,i+1} = I^{(1)} \otimes \cdots I^{(i-1)} \otimes S \otimes I^{(i+2)} \otimes \cdots I^{(N)} \tag{4}$$

where $S \in End(V \otimes V)$ satisfying the following parameter independent Yang-Baxter equation

$$S_{12} S_{23} S_{12} = S_{23} S_{12} S_{23}. \tag{5}$$

If we define an operator

$$t := \prod_{i=1}^{n} \prod_{j=1}^{i-1} b_i$$

which is understood as an ordered product from right to left or vise versa. We can show that the following identity holds

$$t_1 t_2 t_1 t_2 \cdots = t_2 t_1 t_2 t_1 \cdots, \tag{6}$$

where the number of t's in alternative product is $n+1$. The case $n=2$ is exactly the elementary equivalence relations of braid group eq.(3). For $n=3$

we have

$$t_1 t_2 t_1 t_2 = t_2 t_1 t_2 t_1, \tag{7}$$

where $t_1 = b_1 b_2 b_1$, $t_2 = b_2 b_3 b_2$. Thus if we know a representation of braid group, we will have a solution of the following equation

$$\check{R}_{123}\check{R}_{234}\check{R}_{123}\check{R}_{234} = \check{R}_{234}\check{R}_{123}\check{R}_{234}\check{R}_{123} \tag{8}$$

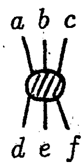
where $\check{R}_{123} := \check{R} \otimes I, \check{R}_{234} := I \otimes \check{R}$ and $\check{R} \in End(V \otimes V \otimes V)$. This is easily realized by

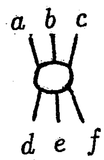
$$\rho : t_1 \to \check{R}_{123}$$

due to $t_1 = b_1 b_2 b_1$ etc., then the following identities holds

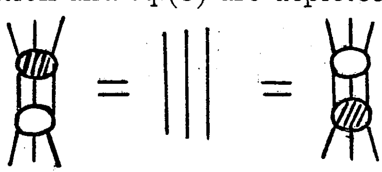
$$\check{R}_{123} = S_{12} S_{23} S_{12} \otimes I \quad etc. \tag{9}$$

As to eq.(8), one may find some symmetry transformation of it. If one write out eq.(8) into component form instead of matrix form, one can easily find that the equation can be symbolized by Kauffman diagram. That says if we denote

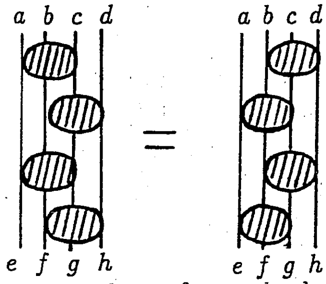
$\check{R}^{abc}_{def} \sim$ 

, $\check{R}^{-1}{}^{abc}_{def} \sim$ 

The inverse relation and eq.(8) are depicted respectively as



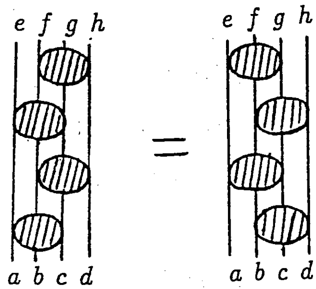
and



(10)

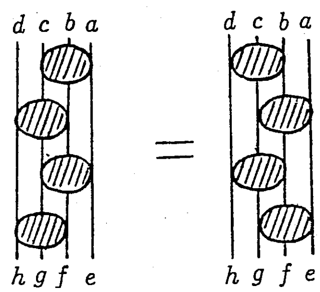
where the inner line connecting legs of two shadows implies the summation over the repeated labels on the legs, and a simple vertical line stands for a unit matrix. It is not difficult to find that the diagram eq.(10) has the following symmetries:

Flipping via a horizontal axis, denoted by H



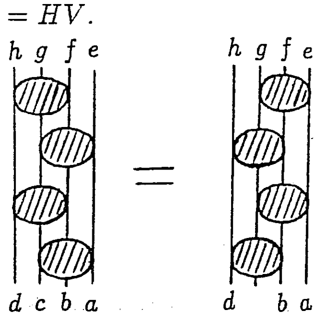
(11)

or flipping via a vertical axis denoted by V



(12)

or via both in term $VH = HV$.



(13)

Thus we have

$$\begin{array}{ccc} (7) & \xrightarrow{H} & (8) \\ V\downarrow & & \downarrow V \\ (9) & \xrightarrow{H} & (10) \end{array}$$

and $H^2 = id, V^2 = id, HV = VH$. All the four diagrams eq.(10), eq.(11), eq.(12), eq.(13) depict the same equation eq.(8). So the solution space of eq.(8) has a discrete group symmetry. $\{id, H, V, VH | H^2 = id, V^2 = id, HV = VH\}$. The action of this group brings one solution of) into other three new solutions. i.e. if $\check{R}^{abc}_{def}$ is a solution of eq.(8), then $\check{R}'^{abc}_{def} = \check{R}^{cba}_{fed}$, $\check{R}''^{abc}_{def} = \check{R}^{edf}_{abc}$ and $\check{R}'''^{abc}_{def} = \check{R}^{fed}_{cba}$ will be solutions of eq.(8).

Furthermore, if giving a direction to the Kauffnman diagram $\check{R}^{abc}_{def} \sim$ , and adding a minus sign to the labels on the tip of the arrow, we can find that the summation of such labels on both side of the diagram eq.(10) are equal. This brings about a contineous transformation from a solution of eq.(8) into another

$$\check{R}^{abc}_{def} \rightarrow \check{R}'^{abc}_{def} = t^{a+b+c-d-e-f}\,\check{R}^{abc}_{def}. \tag{14}$$

Starting from the matrix form of eq.(8), we can obtain two more contineous transformations in solution space. They are an overall factor transformation $\check{R} \rightarrow \tau\check{R}$; a similar transformation by a tensor product of matrices $\check{R} \rightarrow (\Lambda\otimes\Lambda\otimes\Lambda)\check{R}(\Lambda^{-1}\otimes\Lambda^{-1}\otimes\Lambda^{-1})$. Because eigenvalues of a matrix are invariant under a similar transformation, the latter is a transformation within the subset of solution space, which is specialfied by the eigenvalues of $\check{R}$.

In above we made much discussion on eq.(8), now we introduce a new R-matrix

$$\check{R} = RP \tag{15}$$

where P is defined as

$$P|\mu_1, \mu_2, \mu_3 > := |\mu_3, \mu_2, \mu_1 >.$$

Then we can show the R-matrix satisfing the following equation as long as the R-matrix satisfing eq.(8).

$$R_{123}R_{214}R_{341}R_{432} = R_{234}R_{143}R_{412}R_{321} \tag{16}$$

which is an variant of the FM 3-simplex equation we have introduced at the begining of our discussion.

In a similar way, one may discuss the case of 4-simplex equation and so on. The key point is that eq.(6) is an identity on braid group, then if one has a representation of braid group, one can write down a expression from the expression of t_i on the basis of S-matrix, which is supposed to be solutions of parameter independent Yang-Baxter equation.

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The Unitarity of the $W^{(3)}$ algebra from $SU(3)$ Parafermion

Xiang-Mao Ding, Heng Fan, Kang-Jie Shi, Pei Wang
CCAST(World Laboratory),P.O. Box 8730, Beijing,100080
and
Institute of Modern Physics, Northwest University
Xian 710069, **China** *

Abstract

We express the vacuum expectation value of the $SU(3)_k$ parafermion fields by that of two bosons and $SU(3)_k$ current algebra. When k=1,2,3, the later becomes an inner product of an unitary representation, and $T(z), W^{(3)}(z)$ are equivalent to "quasi-self adjoint" operators in this representation.

*Mailing address

1 Introduction

W algebra is an extension of the conformal algebra, which can be realized by several ways [1]. We can construct W algebra by parafermion [2]. In this construction, a crucial problem is the W algebra, which can be rigorously proven by the unitarity argument.

In ref [2], we formulate $W^{(3)}$ by $SU(3)$ parafermions. One can easily show that such $W^{(3)}$ is a primary field of dimension 3. Then the singular part of operator product expansion (OPE) of $W^{(3)}(z)W^{(3)}(w)$ contains a new primary field $W^{(4)}$, which must be null for the closure of the W_3 algebra. We can show that when $k = 3$, the two point function $< W^{(4)}(z)W^{(4)}(w) >$ equals to zero. If such construction is unitary and if $W^{(4)}$ is quasi-self adjoint, then one can conclude that $W^{(4)}$ is a null field. We only sketch the proof of this in ref.2. This paper is a detailed proof. The main idea is as follows.

We can realize the $SU(3)$ current algebra by parafermion and two boson field [3,4,2]. One can thus express the vacuum expectation value of parafermion fields by that of currents and boson fields [4,2]. From this relation, we can find a subset B of the operators representing the $SU(3)$ parafermions. We can use these operators of the subset to calculate the vacuum expectation value of parafermions fields. In the subset B, these is an ideal, the vacuum expectation value of which is zero. When k is a positive integer, the vacuum expectation value of currents can be expressed as an inner product of a unitary highest weight representation. In the composed representation of current and bosons, which is also unitary, the T, $W^{(3)}$ and $W^{(4)}$ of $SU(3)$ parafermions are expressed as essentially self adjoint operators t, $w^{(3)}$ and $w^{(4)}$ plus ideal operators. These ideas operators are equivalent to zero in the vacuum expectation value of fields of the subset B. We can thus use these quasi-self adjoint operators t, $w^{(3)}$ and $w^{(4)}$ to calculate the vacuum expectation values of T, $W^{(3)}$ and $W^{(4)}$ with all parafermion fields and prove that $W^{(4)}$ is null for $k = 3$.

2 W algebra from $SU(3)$ parafermion

Let a root system of $SU(3)$ be depicted by Fig.1, denote it of as Δ^+. The operator product of $SU(3)$ parafermion is [4,7,2]

$$\psi_\alpha(z)\psi_\beta(w)(z-w)^{\alpha\cdot\beta/k}$$
$$= \frac{\delta_{\alpha,-\beta}}{(z-w)^2} + \frac{K_{\alpha,\beta}}{z-w}\psi_{\alpha+\beta}(w) + \sum_{n=0}^{\infty}(z-w)^n \left(\psi_\alpha\psi_\beta(w)\right)_n(w) \qquad (1)$$

where

$$K_{\alpha,\beta} = \{ \begin{array}{cc} \epsilon_{\alpha,\beta}/\sqrt{k} & if\ \alpha+\beta\ \in \Delta \\ 0 & otherwise \end{array} \qquad (2)$$

and $\epsilon_{\alpha,\beta}$ are the structure constants of $SU(3)$,

$$\epsilon_{\alpha_1,\alpha_2} = -\epsilon_{\alpha_2,\alpha_1} = \epsilon_{-\alpha_1,-\alpha_2} = 1. \qquad (3)$$

For $\alpha = -\beta$,

$$\psi_\alpha(z)\psi_{-\alpha}(w)(z-w)^{-\alpha^2/k}$$
$$= \frac{1}{(z-w)^2} + \sum_{n=0}^{\infty}(z-w)^n \left(\tau^{(\alpha)}\right)_{-n}(w) \qquad (4)$$

We define T, $W^{(3)}$ as

$$T(z) = \frac{k}{2(k+3)}\sum_{i=1}^{3}\left(\tau_0^{(i)}(z) + \tau_0^{(-i)}(z)\right)$$
$$W^{(3)} = B_3\sum_{i=1}^{3}\left(\tau_{-1}^{(i)}(z) - \tau_{-1}^{(-i)}(z)\right)$$
$$B_3 = \left(k^3/6(k+1)(k+3)\right)^{1/2} \qquad (5)$$

$\tau^{(i)}$ corresponding to α_i, and we can calculate the OPE of $W^{(3)}(z)W^{(3)}(w)$ by

(1), from which we define $W^{(4)}(z)$

$$W^{(3)}(z)W^{(3)}(w) = (k-2)\{\frac{c/3}{(z-w)^6} + \frac{}{(.}$$

(6). 1.

$$
\begin{aligned}
&+\frac{1}{(z-w)^2}\left(2b^2\Lambda(w)+\frac{3}{10}\partial^2T(w)\right)\\
&+\frac{1}{z-w}\left(b^2\partial_w\Lambda(w)+\frac{1}{15}\partial^3T(w)\right)\}\\
+C_4^{33}&\left(\frac{1}{(z-w)^2}+\frac{1}{2(z-w)}\partial_w\right)W^{(4)}(w)+\ldots \qquad (6)
\end{aligned}
$$

where

$$
\begin{aligned}
\Lambda(w)&=(T\cdot T)_0(w)-\frac{3}{10}\partial^2T(w),\quad c=\frac{6(k-1)}{k+3}\\
b^2=\frac{16}{22+5c}&=\frac{4(k+3)}{13k+9},\quad C_4^{33}=\left(\frac{128(2k+3)(4k-3)}{3(k+3)(13k+9)}\right) \qquad (7)
\end{aligned}
$$

and $(T\cdot T)_0$ is defined through the OPE of $T(z)T(w)$, we can show that $T(z)$ is the stress tensor satisfying

$$
T(z)T(w)=\frac{c/2}{(z-w)^4}+\frac{2}{(z-w)^2}T(w)\frac{1}{z-w}\partial T(w)+(T\cdot T)_0(w)+\ldots; \qquad (8)
$$

due to (1). We find $W^{(3)}$, $W^{(4)}$ are primary fields, with dimension 3 and 4 respectively using (1) [2]

$$
\begin{aligned}
T(z)W^{(3)}&=\left(\frac{3}{(z-w)^2}+\frac{\partial}{z-w}\right)W^{(3)}+\ldots\\
T(z)W^{(4)}&=\left(\frac{4}{(z-w)^2}+\frac{\partial}{z-w}\right)W^{(4)}+\ldots \qquad (9)
\end{aligned}
$$

and the vacuum expectation value of $W^{(4)}(z)W^{(4)}(w)$ is

$$
< W^{(4)}(z)W^{(4)}(w) >=\frac{(k-2)(k-3)c/4}{(z-w)^8} \qquad (10)
$$

which is zero for $k=3$.

If $W^{(4)}$ is a null field then we have a closed W_3 algebra from (9), (8 and (6). This will be true if $W^{(4)}$ is self adjoint in a unitary framework.

$$\psi_\alpha(z)\psi_\beta(w)(z-w)^{\alpha\cdot\beta/k}$$
$$= \frac{\delta_{\alpha,-\beta}}{(z-w)^2} + \frac{K_{\alpha,\beta}}{z-w}\psi_{\alpha+\beta}(w) + \sum_{n=0}^{\infty}(z-w)^n \left(\psi_\alpha\psi_\beta(w)\right)_n (w) \tag{1}$$

where

$$K_{\alpha,\beta} = \{ \begin{matrix} \epsilon_{\alpha,\beta}/\sqrt{k} & if\ \alpha+\beta\ \in \Delta \\ 0 & otherwise \end{matrix} \tag{2}$$

and $\epsilon_{\alpha,\beta}$ are the structure constants of $SU(3)$,

$$\epsilon_{\alpha_1,\alpha_2} = -\epsilon_{\alpha_2,\alpha_1} = \epsilon_{-\alpha_1,-\alpha_2} = 1. \tag{3}$$

For $\alpha = -\beta$,

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$$= \frac{1}{(z-w)^2} + \sum_{n=0}^{\infty}(z-w)^n \left(\tau^{(\alpha)}\right)_{-n}(w) \tag{4}$$

We define T, $W^{(3)}$ as

$$\begin{aligned} T(z) &= \frac{k}{2(k+3)}\sum_{i=1}^{3}\left(\tau_0^{(i)}(z) + \tau_0^{(-i)}(z)\right) \\ W^{(3)} &= B_3\sum_{i=1}^{3}\left(\tau_{-1}^{(i)}(z) - \tau_{-1}^{(-i)}(z)\right) \\ B_3 &= \left(k^3/6(k+1)(k+3)\right)^{1/2} \end{aligned} \tag{5}$$

$\tau^{(i)}$ corresponding to α_i, and we can calculate the OPE of $W^{(3)}(z)W^{(3)}(w)$ by

(1), from which we define $W^{(4)}(z)$

$$W^{(3)}(z)W^{(3)}(w) = (k-2)\{\frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial_w T(w)}{(z-w)^3}$$

$$
\begin{aligned}
&+\frac{1}{(z-w)^2}\left(2b^2\Lambda(w)+\frac{3}{10}\partial^2 T(w)\right)\\
&+\frac{1}{z-w}\left(b^2\partial_w\Lambda(w)+\frac{1}{15}\partial^3 T(w)\right)\}\\
+C_4^{33}&\left(\frac{1}{(z-w)^2}+\frac{1}{2(z-w)}\partial_w\right)W^{(4)}(w)+\ldots
\end{aligned}
\tag{6}
$$

where

$$
\begin{aligned}
\Lambda(w) = (T\cdot T)_0(w) - \frac{3}{10}\partial^2 T(w), \quad c = \frac{6(k-1)}{k+3}\\
b^2 = \frac{16}{22+5c} = \frac{4(k+3)}{13k+9}, \quad C_4^{33} = \left(\frac{128(2k+3)(4k-3)}{3(k+3)(13k+9)}\right)
\end{aligned}
\tag{7}
$$

and $(T\cdot T)_0$ is defined through the OPE of $T(z)T(w)$, we can show that $T(z)$ is the stress tensor satisfying

$$
T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w)\frac{1}{z-w}\partial T(w) + (T\cdot T)_0(w) + \ldots; \tag{8}
$$

due to (1). We find $W^{(3)}$, $W^{(4)}$ are primary fields, with dimension 3 and 4 respectively using (1) [2]

$$
\begin{aligned}
T(z)W^{(3)} &= \left(\frac{3}{(z-w)^2}+\frac{\partial}{z-w}\right)W^{(3)}+\ldots\\
T(z)W^{(4)} &= \left(\frac{4}{(z-w)^2}+\frac{\partial}{z-w}\right)W^{(4)}+\ldots
\end{aligned}
\tag{9}
$$

and the vacuum expectation value of $W^{(4)}(z)W^{(4)}(w)$ is

$$
< W^{(4)}(z)W^{(4)}(w) >= \frac{(k-2)(k-3)c/4}{(z-w)^8} \tag{10}
$$

which is zero for $k=3$.

If $W^{(4)}$ is a null field then we have a closed W_3 algebra from (9), (8 and (6). This will be true if $W^{(4)}$ is self adjoint in a unitary framework.

3 Unitarity of T $W^{(3)}$ and $W^{(4)}$

Introduce two boson fields $\phi_1(z)$, $\phi_2(z)$ and $su(3)_k$ currents $J_\alpha(z)$, $H_\alpha(z) \equiv \alpha \cdot H(z)$, with $\alpha^2 = 2$. They satisfy[5,6]

$$\phi_i(z)\phi_j(w) = -\delta_{ij} ln(z-w),$$

$$J_\alpha(z)J_{-\alpha}(w) = \frac{k}{(z-w)^2} + \frac{H_\alpha(w)}{z-w} + s_\alpha(w) + (z-w)u_\alpha + \dots,$$

$$J_\alpha(z)J_\beta(w) = \frac{1}{z-w}J_{\alpha+\beta}(w) + \dots, \ (\alpha+\beta \in \Delta), \tag{11}$$

$$H_\alpha(z)J_\beta(w) = \frac{(\alpha\cdot\beta)}{z-w}J_\beta(w) + \dots$$

$$H_\alpha(z)H_\beta(w) = \frac{k}{(z-w)^2}(\alpha\cdot\beta) + \dots$$

Let $a \cdot \phi \equiv a_1\phi_1 + a_2\phi_2$, we have

$$: e^{a\cdot\phi(z)} :: e^{b\cdot\phi(w)} :=: e^{a\cdot\phi(z)+b\cdot\phi(w)} : (z-w)^{-a\cdot b}. \tag{12}$$

The vacuum expectation value of boson fields can be expressed as the inner product of Fock spaces H_ϕ, which is unitary [6]. The vacuum expectation value of currents can also become the bilinear form of the highest weight representation, with the highest weight vector $|0>$, satisfying

$$H_i^0|0>= 0, \qquad c|0>= k|0> .$$

When k is a positive integer, such a representation is unitary [4]. The conjugate operator of the currents are

$$J_\alpha(z)^\dagger = (z')^2 J_{-\alpha}(z'), \qquad H_\alpha(z)^\dagger = (z')^2 H_{-\alpha}(z') \tag{13}$$

where $z' = 1/z^*$, and the conjugate of the vacuum states are

$$|0>_\phi^\dagger = {}_\phi < 0|, \qquad |0>_J^\dagger = {}_J < 0|.$$

Consider the direct product of the currents and the bosons as an algebra $A_{J\times\phi}$ with the highest weight vector

$$|0>=|0>_J\otimes|0>_\phi, \qquad |0>^+=<0|={}_\phi<0|\otimes{}_J<0|,$$

and

$$(A_J\otimes A_\phi)^\dagger=A_J^\dagger\otimes A_\phi^\dagger.$$

The composed representation of $A_{J\times\phi}$ is also unitary, and equation (13) still holds.

It is well known that the currents can be expressed by parafermions and bosons, giving [3,4,7,8]

$$\begin{aligned}&<J_{\alpha_1}(z_1)J_{\alpha_2}(z_2)\dots J_{\alpha_n}(z_n)>\\ &=k^{n/2}<\psi_{\alpha_1}(z_1)\psi_{\alpha_2}(z_2)\dots\psi_{\alpha_n}(z_n)>\textstyle\prod_{i<j}(z_i-z_j)^{\alpha_i\cdot\alpha_j/k},\end{aligned} \tag{14}$$

where α_i are roots of the algebra. The multiponit function of parafermion is related to that of currents by

$$\begin{aligned}&<\psi_{\alpha_1}(z_1)\psi_{\alpha_2}(z_2)\dots\psi_{\alpha_n}(z_n)>\\ &=k^{-n/2}<J_{\alpha_1}(z_1)J_{\alpha_2}(z_2)\dots J_{\alpha_n}(z_n)>\textstyle\prod_{i<j}(z_i-z_j)^{-\alpha_i\cdot\alpha_j/k},\end{aligned} \tag{15}$$

Consider a mapping of the parafermion algebra A_ψ to the algebra $A_{J\times\phi}$

$$M(\psi_\alpha(z))=k^{-1/2}:e^{\alpha\cdot\psi(z)/\sqrt{k}}: \tag{16}$$

with

$$M(aX+bY)=aM(X)+bM(Y), \quad M(XY)=M(X)M(Y) \tag{17}$$

For example

$$\begin{aligned}M\left(k\psi_\alpha(z)\psi_{-\alpha}(w)\right)(z-w)^{-\alpha^2/k}&=kM\left(\psi_\alpha(z)\right)M\left(\psi_{-\alpha}(w)\right)(z-w)^{-\alpha^2/k}\\ &=J_\alpha(z):e^{\alpha\cdot\phi(z)/\sqrt{k}}:J_{-\alpha}(w):e^{\alpha\cdot\phi(w)/\sqrt{k}}:(z-w)^{-\alpha^2/k}\\ &=\frac{k}{(z-w)^2}+\frac{1}{z-w}\left(\sqrt{k}\alpha\cdot\partial\phi(w)+H_\alpha(w)\right)\end{aligned}$$

$$+\frac{1}{2}\{(:(\alpha\cdot\partial\phi(w))^2:+\sqrt{k}\alpha\cdot\partial^2\phi(w)+\frac{2}{\sqrt{k}}\alpha\cdot\partial\phi(w)H_\alpha(w)+2s_\alpha(w)\}$$

$$+\frac{1}{2}(z-w)\{\frac{k}{3}\left(:\left(\frac{\alpha}{\sqrt{k}}\cdot\partial\phi(w)\right)^3:+\frac{3}{k}\alpha\cdot\partial\phi(w)\alpha\cdot\partial^3\phi(w):\right)$$

$$+H_\alpha(w)\left(:\left(\frac{\alpha}{\sqrt{k}}\cdot\partial\phi(w)\right)^2:+\frac{\alpha}{\sqrt{k}}\cdot\partial^2\phi(w)\right)+2s_\alpha(w)\frac{\alpha}{\sqrt{k}}\cdot\partial\phi(w)+u_\alpha(w)\}+\ldots$$

$$=\frac{k}{(z-w)^2}+\frac{1}{z-w}f_1^\alpha(w)+f_0^\alpha(w)+(z-w)f_{-1}^\alpha(w)+\ldots \quad (18)$$

When X_i are polynomials of generating parafermion ψ_{α_i}, this mapping obeys

$$< X_1(z_1)X_2(z_2)\ldots X_n(z_n) >=< M(X_1(z_1))M(X_2(z_2))\ldots M(X_n(z_n)) > \quad (19)$$

due to (15). We can thus express the vacuum expectation value of the elements of A_ψ by that of $A_{J\times\phi}$. The procedure is as follows. For an element of A_ψ, say a_ψ, we first choose an expansion of this element by the polynomials of generating parafermions. Then we obtain the image of this expansion via (16,17). The vacuum expectation value of this image equals to that of a_ψ. The choice of the expansion is not unique. Different expansion of the same element may give different images in $A_{J\times\phi}$. However, they must give the same vacuum expectation value (19) if X_i are elements of A_ψ. This is because the A_ψ algebra is in consistent with (15). Actually, the images of different expansions of the same element a_ψ form an equivalence class in $A_{J\times\phi}$. The difference of the elements in the same class has a preimage zero in A_ψ. For example the field $(\psi_\alpha\psi_{-\alpha})_1, \oint\psi_\alpha(z)\psi_{-\alpha}(w)(z-w)^{-\alpha^2/k}$ is null [4] in the PF algebra A_ψ [3,4,7], while $M(\psi_\alpha\psi_{-\alpha})=f_1^\alpha$ is not a null field in the algebra $A_{J\times\phi}$. However, inside a multipoint function of fields with the images of A_ψ, f_1^α always gives a zero factor(Note that MA_ψ is not the whole algebra $A_{J\times\phi}$). It is

$$< a_\psi(z_1,z_2,\ldots,z_n)f_1^\alpha >= 0 \quad (20)$$

where a_ψ is an element of $M[A_\psi]$. For convenience we define the following in the algebra $A_{J\times\phi}$:

(a) The image of A_ψ is called B field.

(b) inside the set of B field, we call those elements whose preimages are actually zero element of A_ψ as N fields. For example, $f_{-n}^\alpha(z)$ is a B field, because we have

$$f^{\alpha}_{-n}(w) = \frac{k}{(n+2)!}\left(\frac{d}{dz}\right)^{n+2}\left((z-w)^2 M\left(\psi_\alpha(z)\psi_{-\alpha}(w)\right)(z-w)^{-\alpha^2/k}\right)_{z=w},$$

The fields

$$S(x,w) \equiv k^{-2}\sum_{i=1}^{3} f_1^{\alpha_i}(x+w)\left(f_0^{\alpha_i}(w)+f_0^{\alpha_i}(w)\right),$$

$$F(x,w) \equiv k^{-3}\sum_{i=1}^{3} f_1^{\alpha_i}(w+2x) f_1^{\alpha_i}(w+x) f_1^{\alpha_i}(w),$$

and

$$R^{\alpha}(x,w) \equiv \frac{1}{2k^2}\left(\frac{d}{dz}\right)^2\left(x^2 f_1^{\alpha_i}(x+w) f_1^{\alpha_i}(w)\right),$$

are N fields. We see that B fields form a closed set under linear operation and multiplying, while N fields form and ideal of B fields. The expectation value of an N field with a B field is vanish

$$< BN >= 0 \tag{21}$$

Using the fact $\sum_i^3 \alpha_i = 0$, $\alpha^2 = 2$ and (11, 12), we can show that the limits of $S(x,w)$, $F(x,w)$ and $R^{\alpha}(x,w)$ exist when $x \to 0$. We denote them as $S(w)$, $F(w)$, $R^{\alpha}(w)$ respectively. From the definition of T, $W^{(3)}$ and $W^{(4)}$, we have

$$M(T) = C_2\sum_{i=1}^{3}\left(f_0^{(i)} + f_0^{(-i)}\right), \quad M(W^{(3)}) = C_3\sum_{i=1}^{3}\left(f_{-1}^{(i)} - f_{-1}^{(-i)}\right),$$

$$M(W^{(4)}) = C_4 M\left(\left(W^{(3)}W^{(3)}\right)_2 - 2b^2(TT)_0 - \frac{3}{10}(1-2b^2)\partial^2 T\right),$$

where $f^{(i)}$ corresponding to α_i as mentioned above, we will not explain this later, and $\left(W^{(3)}W^{(3)}\right)_2 = \frac{1}{4!}\left(\frac{d}{dz}\right)^4\left((z-w)^6 W^{(3)}(z)W^{(3)}(w)\right)$. From (18) we have

$$M(T(w)) = C_2\sum_{i=1}^{3}\left(:(\alpha_i\partial\phi(w))^2: + 2H_\alpha\frac{\alpha_i}{\sqrt{k}}\cdot\partial\phi(w) + s_{\alpha_i}(w) + s_{-\alpha_i}(w)\right)$$

$$M(W^{(3)}) = C_3 \sum_{i=1}^{3} \{ \frac{1}{3\sqrt{k}} : (\alpha_i \partial\phi(w))^3 : + H_\alpha : \left(\frac{\alpha_i}{\sqrt{k}} \cdot \partial\phi(w) \right)^2 :$$
$$+ (s_{\alpha_i}(w) + s_{-\alpha_i}(w)) \frac{\alpha_i}{\sqrt{k}} \cdot \partial\phi(w) + u_{\alpha_i}(w) - u_{-\alpha_i}(w) \} \qquad (22)$$

where They are not "quasi-self conjugate" however. This is because $(\partial\phi(z))^*$ does not fit the relation as (3). We then try to compensate these fields by some N fields. This is equivalent to choose different expansions of ψs for T. $W^{(3)}$ and $W^{(4)}$. We find $S(w)$, $F(w)$ and $R^\alpha(w)$ are good enough to do this. After some derivation, we have

$$F(x,w) = k^{-3} \sum_{i=1}^{3} f_1^{(i)}(w+2x) f_1^{(i)}(w+x) f_1^{(i)}(w)$$
$$= \sum_i \{ \frac{1}{\sqrt{k}} (\alpha_i \partial\phi(w+2x) + 2\alpha_i \partial\phi(w+x) + \alpha_i \partial\phi(w))$$
$$+ \frac{const}{k} \cdot \frac{\alpha_i^2}{x^2} (H_{\alpha_i}(w+2x) + 2H_{\alpha_i}(w+x) + H_{\alpha_i} H(w)) \}$$
$$+ k^{-3} : \sum_{i=1}^{3} f_1^{(i)}(w+2x) f_1^{(i)}(w+x) f_1^{(i)}(w) : \qquad (23)$$

we obtain

$$F(x.w) =: \ F(x,w)$$

and has a limit for $x \to 0$,

$$F(w) = lim_{x\to 0} F(x,w) = \sum_i : \{ \left(\frac{\alpha_i}{\sqrt{k}} \cdot \partial\phi(w) \right)^3 + 3 \left(\frac{\alpha_i}{\sqrt{k}} \cdot \partial\phi(w) \right)^2 \frac{1}{k} H_\alpha(w)$$
$$+ 3\frac{\alpha_i}{\sqrt{k}} \cdot \partial\phi(w) \left(\frac{1}{k} H_\alpha(w) \right)^2 + \left(\frac{1}{k} H_\alpha(w) \right)^3 \} :, (24)$$

Similarly

$$S(x,w) = k^{-2} \sum_{i=1}^{3} f_1^{(i)}(w+x) \left(f_0^{(i)}(w) + f_0^{(-i)}(w) \right)$$

$$
\begin{aligned}
&= \sum_i \left(\frac{\alpha_i}{\sqrt{k}} \cdot \partial\phi(w+x) + \frac{1}{k} H_\alpha(w+x) \right) \cdot \{ : \left(\frac{\alpha_i}{\sqrt{k}} \cdot \partial\phi(w) \right)^2 : \\
&\qquad + \frac{1}{k} H_{\alpha_i}(w) \frac{\alpha_i}{\sqrt{k}} \partial\phi(w) + \frac{1}{k}(s_{\alpha_i} + s_{-\alpha_i}) \} \\
&= \sum_i \frac{\alpha_i}{x^2}(a_1 \partial\phi(w) + a_2 H(w)) + \sum_i \frac{1}{\sqrt{k}} H_{\alpha_i}(w+x)(s_{\alpha_i} + s_{-\alpha_i}) \\
&+ \sum_i : \{ \frac{1}{k} \frac{\alpha_i}{\sqrt{k}} \cdot \partial\phi(w+x)(s_{\alpha_i} + s_{-\alpha_i}) + \frac{\alpha_i}{\sqrt{k}} \partial\phi(w+x) \left(\frac{\alpha_i}{\sqrt{k}} \partial\phi(w) \right)^2 \\
&+ \frac{2}{k} H_{\alpha_i}(w) \frac{\alpha_i}{\sqrt{k}} \partial\phi(w+x) \frac{\alpha_i}{\sqrt{k}} \partial\phi(w) + \frac{1}{\sqrt{k}} H_{\alpha_i}(w+x) \left(\frac{\alpha_i}{\sqrt{k}} \partial\phi(w) \right)^2 \\
&\qquad + \frac{2}{k^2} H_{\alpha_i}(w+x) H_{\alpha_i}(w) \frac{\alpha_i}{\sqrt{k}} \partial\phi(w) \} : \qquad (25)
\end{aligned}
$$

After summing over i, first term becomes zero, other terms are regular for $x \to 0$, thus

$$
\begin{aligned}
S(w) \equiv lim_{\to 0} S(x,w) &= \sum_i \frac{1}{k^2} H_{\alpha_i}(s_{\alpha_i}(w) + s_{-\alpha_i}(w)) \\
+ \sum_i : \{ &\frac{1}{k^{3/2}} \alpha_i \cdot \partial\phi(w)(s_{\alpha_i}(w) + s_{-\alpha_i}(w)) + \left(\frac{\alpha_i}{\sqrt{k}} \cdot \partial\phi(w) \right)^3 \\
&+ \frac{3}{k} H_{\alpha_i}(w) \left(\frac{\alpha_i}{\sqrt{k}} \partial\phi(w) \right)^2 + \frac{2}{k^2} \left(H_{\alpha_i}(w) \right)^2 \frac{\alpha_i}{\sqrt{k}} \partial\phi(w) \} : \qquad (26)
\end{aligned}
$$

Equation (24) and (26) are good for the compensation of (22). We also have

$$
R^\alpha =: (\alpha \cdot \partial\phi(w))^2 + \frac{2\alpha_i}{k^{3/2}} \partial\phi(w) \alpha \cdot H(w) + \frac{2}{k^2} (\alpha \cdot H(w))^2 :
$$

Which can compensate (22). Thus we can rewrite (22) as

$$
\begin{aligned}
M(T) = t + n^{(2)}, \quad & M(W^{(3)}) = t^{(3)} + n^{(3)}, \\
& M(W^{(4)}) = t^{(3)} + n^{(4)} \qquad (27)
\end{aligned}
$$

These $t^{(i)}$ are quasi-self adjoint

$$
\left(t^{(i)}(z) \right)^\dagger = (z')^{(2i)} t^{(i)}(z')|_{z' = \frac{1}{z^*}}
$$

$n^{(i)}$ are N fields. The explicit form of these fields are

$$t(w) = C_2 \sum_i^3 \left((s_{\alpha_i}(w) + s_{-\alpha_i}(w)) - \frac{1}{k} : H_i^2 w : \right)$$

$$n^{(2)}(w) = C_2 k \sum_i^3 R^{\alpha_i}(w)$$

$$w^{(3)}(w) = C_3 \sum_i^3 \{ (u_{\alpha_i}(w) - u_{-\alpha_i}(w)) - \frac{1}{k} H_{\alpha_i}(w+x)\,(s_{\alpha_i}(w) + s_{-\alpha_i}(w))$$
$$+ \frac{2}{3k^2} H_{\alpha_i}(w+2x) H_{\alpha_i}(w+x) H_{\alpha_i}(w) \}_{x \to 0}$$
$$n^{(3)}(w) = C_3 k\,(3F(w) - S(w))$$
$$w^{(4)}(w) = C_4 \{ \frac{1}{4!(k-2)} \left(\frac{d}{dz}\right)^4 \left((z-w)^6 t(z) t(w) \right)$$
$$- \left(2b^2 (t \cdot t)_0 + \frac{3}{10}(1 - 2b^2) \partial^2 t(w) \right) \}_{z=w} \tag{28}$$

$$n^{(4)}(w) = \frac{C_4}{4!(k-2)} \left(\frac{d}{dz}\right)^4 ((z-w)^6 (n^{(3)}(z) w^{(3)}(w)$$

$$+ w^{(3)}(z) n^{(3)}(w) + n^{(3)}(z) n^{(3)}(w)))_{z=w}$$

$$- \frac{2b^2}{4!} \left(\frac{d}{dz}\right)^4 \{ (z-w)^4 \left(n^{(2)}(z) t(w) + t(z) n^{(2)}(w) + n^{(2)}(z) n^{(2)}(w) \right) \}_{z=w}$$

$$- \frac{3}{10}(1 - 2b^2) \left(\frac{d}{dz}\right) n^2(z)|_{z=w}. \tag{29}$$

It is easy to see that $n^{(2)}$, $n^{(3)}$ are N fields. Since $M(T)$, $M(W^{(3)})$ are B fields. Thus t and $w^{(3)}$ are B fields. and $n^{(2)}$, $n^{(3)}$ are B fields. From (28, 29) we see that $w^{(4)}$ is a B field and $n^{(4)}$ is an N field. From (28) and

$$\left(\frac{d}{dz} X\right)^\dagger = \frac{d}{dz^\dagger} X^\dagger = z'^2 \frac{d}{dz} X^\dagger$$

we can check

$$(t(z))^\dagger = (z')^4 t(z'), \quad \left(w^{(3)}(z)\right)^\dagger = (z')^6 w^{(3)}(z'). \tag{30}$$

We then obtain

$$\left(w^{(4)}(z)\right)^\dagger = (z')^8 w^{(4)}(z'), \tag{31}$$

after some derivations. Thus the vector $w^{(4)}(z)|0>$ satisfies

$$\begin{aligned} <0|\left(w^{(4)}(z)\right)^\dagger w^{(4)}(z)|0> = (z')^8 <0|w^{(4)}(z')w^{(4)}(z)|0> \\ = (z')^8 < w^{(4)}(z')w^{(4)}(z)> \\ = (z')^8 < \left(w^{(4)}(z') + n^{(4)}(z)\right)\left(w^{(4)}(z) + n^{(4)}\right) > \\ = (z')^8 < W^{(4)}(z')W^{(4)}(z) >= 0, \end{aligned} \tag{32}$$

for $k = 2, 3$. This implies

$$< a_{J\times\phi}|w^{(4)} >= 0$$

for every element of $A_{J\times\phi}$. Finally we have for $a_\psi \in A_\psi$

$$< a_\psi W^{(4)} >=< M(a_\psi)w^{(4)} >= 0$$

due to (21).

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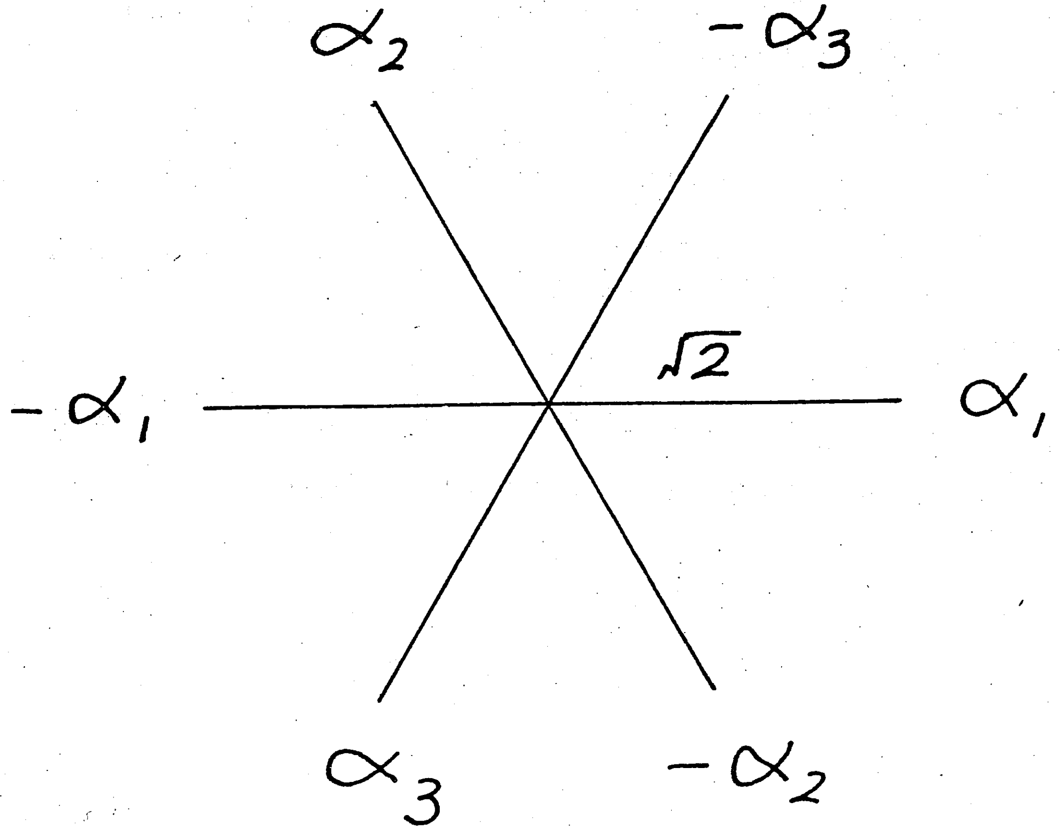

Fig. 1

$W_3^{(2)}$ algebra from the exchange algebra

Yanshen WANG, Liu CHAO

CCAST (World Laboratory), P. O. Box 8730, Beijing 100080, China
Institute of Modern Physics, Northwest University, 710069 Xian, China *

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Abstract

W_3^2 algebra is constructed from the exchange algebra.

*Mailling address

As extensions of the conformal algebra, W algebras have turned out to be important in two-dimensional field theory[1]. They are related not only to the classification of conformal field theory but also to some important physical problem, such as gauged WZW-models[2,3], Toda field theories[4], reductions of KP-hierarchy[5], Chern-Simons theory and string field theories[6] etc. Corresponding to different Hamiltonian reductions of $SL(n)$-gauge connections, there are different W algebras, of which the $W_n^{(l)}$ $(l < n)$ series[7] are particularly interesting. In particular, for $n = 3$, there are two inequivalent algebras in this series, $W_3^{(1)}$ and $W_3^{(2)}$. $W_3^{(1)}$ is standard W_3 algebra (i.e. the Zamolodchikov algebra) and $W_3^{(2)}$ contains a spin-1 field U and two spin-$\frac{1}{2}$ fields $G_\pm$ besides the stress tensor T.

On the other hand, the classical exchange algebras have appeared as important characteristics in many classical integrable systems, especially in various Toda contexts[8,9]. Such structures not only appeared in conformally invariant systems but also in some nonconformal systems. Therefore they have to be considered as a more "foundamental" structures compared to the W algebras. Moreover, provided the model is conformal symmetric, the exchange algebras will yield both the integrability and the conformality of the model [8]. It is therefore interesting to ask whether it is possible to construct various W algebras starting from the exchange algebra. This problem is answered affirmatively in the case of the standard W_n algebras in ref.[8], and we put it as the task for this letter to consider this problem in the case of $W_3^{(2)}$ algebra. The more general case of $W_n^{(l)}$ is still under investigation.

Let us start by considering following exchange algebra,

$$\begin{aligned}\{X_a(x)\otimes,\ X_b(y)\} &= \frac{1}{2}X_a(x)\otimes X_b(y)[\theta(x-y)r^+ + \theta(y-x)r^-], \quad (1)\\ (a,b &= 1,2)\end{aligned}$$

here, $X_1 = \sigma$, $X_2 = \rho$, $r^\pm$ are solutions of the classical Yang-Baxter equation,

$$
\begin{aligned}
r^+ &= \sum_{i,j=1}^{r} (K^{-1})^{ij} H_i \otimes H_j + 2\sum_{i>0} E_i \otimes F_i, \\
r^- &= -\left[\sum_{i,j=1}^{r} (K^{-1})^{ij} H_i \otimes H_j + 2\sum_{i>0} F_i \otimes E_i\right] \qquad (2).
\end{aligned}
$$

$\{H_i, E_i, F_i\}$ are Chevalley generators of $sl(n)$ and K Cardan matrix. If $n = 2$, σ and ρ have three exponents respectively. They satisfy the following Poisson brackets,

$$
\begin{aligned}
\{X_i^a(x),\ X_j^b(y)\} &= -\frac{1}{6} X_i^a(x) X_j^b(y)[\theta(x-y) - \theta(y-x)] \\
&+ X_i^b(x) X_j^a(y)[\theta(a-b)\theta(x-y) - \theta(b-a)\theta(y-x)] \qquad (3) \\
(a, b &= 1, 2, 3)
\end{aligned}
$$

Now,let us define a 3×3 matrix F,

$$
F = \begin{pmatrix} \sigma^1 & \sigma^2 & \sigma^3 \\ \rho^1 & \rho^2 & \rho^3 \\ -\partial\sigma^1 & -\partial\sigma^2 & -\partial\sigma^3 \end{pmatrix}, \qquad (4)
$$

and introduce the symbols $\Delta_i (i = 1, 2, 3)$, which are determinants of the $i \times i$ square submatrices of F taken from the northwest corner of F. We then define another 3×3 matrix Q such that $Q_{ij} = \frac{1}{\Delta_{i-1}} \sum_{l=1}^{i} \Delta_i(l, i) F_{lj}$, where $\Delta_i(l, i)$ is the algebraic co-minor of Δ_i with respect to the (l, i)-th entry, F_{lj} is the (l, j)-th entry of the matrix F. We have the following Drinfeld-Sokolov system,

$$
\partial Q = LQ \qquad (5)
$$

with

$$L = \begin{pmatrix} \partial k_1 & -p_2 & -1 \\ 0 & \partial(k_2 - k_1) & p_1 \\ 0 & 0 & -\partial k_2 \end{pmatrix}, \tag{6}$$

here,

$$\begin{aligned} k_1 &= \ln \sigma^1, \\ k_2 &= \ln(\sigma^1\rho^2 - \sigma^2\rho^1), \\ p_1 &= -\frac{\rho^1}{\sigma^1}, \\ p_2 &= \frac{\sigma^1\partial\sigma^2 - \sigma^2\partial\sigma^1}{\sigma^1\rho^2 - \sigma^2\rho^1}. \end{aligned} \tag{7}$$

Using (3), we have the following Poisson bracket satisfied by k_i and p_i,

$$\begin{aligned} \{k_i(x),\ k_i(y)\} &= \frac{1}{3}[\theta(x-y) - \theta(y-x)], \\ \{k_1(x),\ k_2(y)\} &= \frac{1}{6}[\theta(x-y) - \theta(y-x)], \\ \{p_i(x),\ p_i(y)\} &= 0, \\ \{p_1(x),\ p_2(y)\} &= \delta(x-y), \\ \{k_i(x),\ p_j(y)\} &= 0. \end{aligned} \tag{8}$$

Select a suitable $g \in SL(2)$,

$$g = \begin{pmatrix} 1 & 0 & 0 \\ p_1 & 1 & 0 \\ \frac{1}{2}[p_1p_2 - \partial(k_1 + k_2)] & p_2 & 1 \end{pmatrix}, \tag{9}$$

we have a gauge transformation to L,

$$\begin{aligned} L \mapsto L' &= gLg^{-1} + \partial g g^{-1} \\ &= \begin{pmatrix} \frac{1}{2}U & 0 & -1 \\ G_+ & -U & 0 \\ \frac{3}{4}U^2 + T & G_- & \frac{1}{2}U \end{pmatrix} \end{aligned} \tag{10}$$

T. $G_\pm$ and U may be expressed by k_i and p_i as follows,

$$
\begin{aligned}
U &= \partial k_1 - \partial k_2 + p_1 p_2, \\
G_+ &= -p_1 \partial k_2 + 2p_1 \partial k_1 + p_1^2 p_2 + \partial p_1, \\
G_- &= -p_2 \partial k_1 + 2p_2 \partial k_2 - p_1 p_2^2 + \partial p_2, \\
T &= -\partial k_1 \partial k_1 - \partial k_2 \partial k_2 + \partial k_1 \partial k_2 - \frac{1}{2}\partial(\partial k_1 + \partial k_2) + \frac{1}{2}(p_2 \partial p_1 - p_1 \partial p_2)
\end{aligned}
\quad (11)
$$

Using (8), we obtain

$$
\begin{aligned}
\{U(x),\ U(y)\} &= -\frac{2}{3}\delta'(x-y), \\
\{G_\pm(x),\ G_\pm(y)\} &= 0, \\
\{U(x),\ G_\pm(y)\} &= \mp G_\pm(x)\delta(x-y), \\
\{G_+(x),\ G_-(y)\} &= -[3U^2 - T]\delta(x-y) - \frac{3}{2}[U(x) + U(y)]\delta'(x-y) - \delta''(x-y), \\
\{T(x),\ T(y)\} &= [T(x) + T(y)]\delta'(x-y) + \frac{1}{2}\delta'''(x-y), \\
\{T(x),\ U(y)\} &= U(x)\delta'(x-y), \\
\{T(x),\ G_\pm(y)\} &= G_\pm(x)\delta'(x-y) + \frac{1}{2}G_\pm(y)\delta'(x-y).
\end{aligned}
\quad (12)
$$

T. $G_\pm$ and U are nothing but generators of $W_3^{(2)}$.

In the end of this letter let us mention that it is highly probable that the above construction may be extended to the case of arbitrary $W_n^{(l)}$ algebras. To achieve this the intermediate free fields k_i and p_i have to be replaced by appropriately chosen algebraic co-minors of the determinants which are analogous to Δ_i. This work is now undertaken.

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The qs-Hermite Polynomial and the Representations of Heisenberg and Two-Parameter Deformed Quantum Heisenberg Algebra

Xi-Wen Guan[1)2)] and Dian-Min Tong[1)3)]
1) CCAST (World Laboratory), P.O. BOX 8730, Beijing 100080, P.R. China
2) Department of Mechanics, Qingdao Staff and Workers' College, Qingdao 266021, P.R. China
3) Department of Physics, Shandong Normal University, Jinan 250014, P.R. China

Abstract

By introducing a pair of canonical conjugate two-parameter deformed operators D_{qs}, X_{qs}, we can naturally obtain the form of qs-analogous Taylor series of an arbitrary analytic function, and explicitly construct the realizations of Heisenberg and two-parameter deformed quantum Heisenberg algebra by means of the operators D_{qs} and X_{qs}, and show that the qs-analogous Hermite polynomials are the representations of Heisenberg and the quantum Heisenberg algebra.

I. Introduction

The development of the quantum inverse scattering method[1] has led to a very important concept, quantum group or quantum algebra. As a powerful tool to explore the mathematical structure of quantum groups, Macfarlane[2] and Biedenharn[3] independently proposed the concept of q-deformed oscillator. This may lead to a new kind of field theory. On the other hand, from the point of view of physical applicability in concrete physical problems, quantum algebra with multiparameter deformations have received much attention[4-9]. Many properties of multiparameter deformed quantum groups and algebra are quite similar to or richer than the ones of the usual Lie groups and Lie algebra with both the representation theory and the possible physical applications[10,11].

More recently, a lot of attention has been paid to the analogue of special function, because of their importance to integrable model, representations of quantum groups and Yang -Baxter equations. In particular, Chang, Guo and Yan[12] discussed q-series, q-Hermite polynomial and representations of Heisenberg and q-deformed quantum Heisenberg. Thus, a question naturally arises whether there exist similar constructions for the case of multiparameter deformation. This paper is addressed to this question. By introducing a pair of canonical conjugate two-parameter qs-deformed operators D_{qs}, X_{qs}, ,we discuss in detail the relations between the deformed operators D_{qs}, X_{qs} and qs-series, and explicitly construct the realizations of Heisenberg and quantum Heisenberg algebra by means of the operators D_{qs} and X_{qs}. Particularly, we show that the qs-analogous Hermite polynomials can be representations of the algebra.

II. qs-derivative and qs-series

Let us recall the definition of qs-derivative D_{qs}. For any continuous function $f(x)$, its qs-derivative is defined[13,14] by

$$D_{qs}f(x) \equiv \frac{f(s^{-1}qx)-f(s^{-1}q^{-1}x)}{(s^{-1}q-s^{-1}q^{-1})x} \quad . \tag{1}$$

Obviously, when $s \to 1$ the operator D_{qs} reduces to so-called q-derivative D_q, and $q \to 1, s \to 1$, the operators D_{qs} just is usual differential operator ∂. It follows easily that

$$D_{qs}x^n = [n]_{qs}x^{n-1} \quad , \tag{2}$$

where

$$[n]_{qs} = \frac{(s^{-1}q)^x-(s^{-1}q^{-1})^x}{(s^{-1}q-s^{-1}q^{-1})} \quad , \tag{3}$$

called qs-number. One finds a qs-Leibniz rule:

$$\begin{aligned} D_{qs}f(x)g(x) &= (D_{qs}f(x))\ g(s^{-1}qx)+f(s^{-1}q^{-1}x)\ (D_{qs}g(x)) \\ &= f(s^{-1}qx)\ (D_{qs}g(x))+(D_{qs}f(x))\ g(s^{-1}q^{-1}x). \end{aligned} \tag{4}$$

We may introduce the qs-exponential as follows:

$$\exp_{qs}(x) = \sum_{n=0}^{\infty}\frac{x^n}{[n]_{qs}!} \quad . \tag{5}$$

From Eq. (1) and (5), we have

$$D_{qs}e_{qs}(ax) = a\, e_{qs}(ax) \, . \tag{6}$$

Therefore the qs-exponential function is nothing but the eigenfunction of the qs-derivative D_{qs} just as the usual exponential function $\exp(x)$ is the eigenfunction of the differential operator ∂. To gain more information about the operator D_{qs}, it is necessary to know about its canonical conjugate X_{qs}. For this purpose, we introduce a function as follows

$$\xi_{qs}(x) \equiv \frac{(s^{-1}q)^x-(s^{-1}q^{-1})^x}{(s^{-1}q-s^{-1}q^{-1})x} = \frac{[x]_{qs}}{x} \quad . \tag{7}$$

It is obvious that $\xi_{qs}(x)$ and its inverse $(\xi_{qs}(x))^{-1}$ are both well defined for real number s and q. If let x take operator-value $x\partial$, one obtains the operator-valued function

$$\eta_{qs} \equiv \xi_{qs}(x\partial) = \frac{[x\partial]_{qs}}{x\partial} \quad . \tag{8}$$

Now, we introduce a pair of canonical conjugate two-parameter deformed operator D_{qs} and X_{qs},

$$D_{qs} \equiv \frac{1}{x}[x\partial]_{qs} = \partial \cdot \eta_{qs} \quad , \tag{9-1}$$

$$X_{qs} \equiv \eta_{qs}^{-1} \cdot x \quad , \tag{9-2}$$

obviously

$$D_{qs} \xrightarrow{q\to 1, s\to 1} \partial ,$$
$$X_{qs} \xrightarrow{q\to 1, s\to 1} x .$$

Notice that

$$D_{qs}X_{qs} = \partial \cdot x , \tag{10-1}$$
$$X_{qs}D_{qs} = x \cdot \partial , \tag{10-2}$$

and

$$[D_{qs} , X_{qs}] = [\partial , x] = 1 . \tag{10-3}$$

Therefore the algebra generated by the operators D_{qs} , X_{qs} and $X_{qs}D_{qs}$ is isomorphic to the Heisenberg algebra

$$\begin{aligned} &[a,a^{+}] = 1, \\ &[N,a] = -a, \qquad [N,a^{+}] = a^{+}, \\ &N = a^{+}a, \end{aligned} \tag{11}$$

generated by $a^{+} = x,\ a = \partial\ and\ N = a^{+}a = x\partial$. It is not difficult to see X_{qs} is a *qs*-analogous `coordinate' operator, the canonical conjugate of D_{qs} . From following discussion, we know X_{qs} is closely related to the *qs*-series.

For an analytic function $F(x)$ defined over *R*, it can be expanded into Taylor series :

$$F(x) = \sum_{n=0}^{\infty} \frac{c_n}{n!} x^n , \tag{12}$$

where c_n are coefficients in R. Similarly, we define the operator-valued function $F(X_{qs})$ as

$$F(X_{qs}) = \sum_{n=0}^{\infty} \frac{c_n}{n!} X_{qs}{}^n . \tag{13}$$

Let us observe the properties of the operator-valued function $F(X_{qs})$ from the action of $F(X_{qs})$ on an arbitrary analytic function *f(x)* defined on *R*

$$F(X_{qs}) \cdot f(x) \equiv \sum_{n=0}^{\infty} \frac{c_n}{n!} X_{qs}{}^n \cdot f(x) , \tag{14}$$

for special case of *f(x)=1* ,

$$F(X_{qs}) \cdot 1 = \sum_{n=0}^{\infty} \frac{c_n}{n!} X_{qs}^n \cdot 1 . \tag{15}$$

One finds that

$$X_{qs}^n \cdot 1 = \frac{n!}{[n]_{qs}!} x^n \tag{16}$$

Thus we naturally obtain the two-parameter analogous *qs*-Taylor series,

$$f_{qs}(x) \equiv F(X_{qs}) \cdot 1 = \sum_{n=0}^{\infty} \frac{c_n}{[n]_{qs}!} x^n , \tag{17}$$

by the actions of operator-valued function $F(X_{qs})$ on 1, where

$$[n]_{qs}!=[n]_{qs}[n-1]_{qs}\cdots[1]_{qs} \quad , \tag{18}$$

is qs-factorial with the convention $[0]_{qs}!=1$. It is easy to see that

$$\exp_{qs}(ax)\equiv\exp(aX_{qs})\cdot 1=\sum_{n=0}^{\infty}\frac{(ax)^n}{[n]_{qs}!} \quad , \tag{19-1}$$

$$\sinh_{qs}(ax)\equiv\sinh(aX_{qs})\cdot 1=\sum_{n=0}^{\infty}\frac{(ax)^{2n+1}}{[2n+1]_{qs}!} \quad , \tag{19-2}$$

$$\cosh_{qs}(ax)\equiv\cosh(aX_{qs})\cdot 1=\sum_{n=0}^{\infty}\frac{(ax)^{2n}}{[2n]_{qs}!} \quad . \tag{19-3}$$

III. Heisenberg Algebra and the qs-Hermite Polynomials

Let us consider a system with the following Hamiltonian

$$H_{qs}=-\frac{1}{2}D_{qs}^2+\frac{1}{2}X_{qs}^2 \quad , \tag{20}$$

obviously, when $q\to 1$, $s\to 1$, it reverts to standard a harmonic oscillator with the Harmiltonian

$$H=-\frac{1}{2}\frac{\partial^2}{\partial x^2}+\frac{1}{2}x^2 \quad . \tag{21}$$

If we introduce the operators

$$a\equiv\frac{1}{\sqrt{2}}(D_{qs}+X_{qs}) \,, \tag{22-1}$$

$$a^+\equiv\frac{1}{\sqrt{2}}(-D_{qs}+X_{qs}) \,, \tag{22-2}$$

$$N\equiv a^+a \,. \tag{22-3}$$

The Harmiltonian (20) can be written as

$$H_{qs}=N+\frac{1}{2} \quad . \tag{23}$$

It is easy to check that a, a^+ and N form a realization of the Heisenberg algebra satisfying relation (11). Here a and a^+ is annihilation operator and creation operator respectively, and N is the number operator. Therefore the Hatmiltonian can be diagonalized. We notice that this system is formally identical to the ordinary system of simple harmonic oscillator. Thus each state can be represented by operator-valued Hermite polynomial times operator-valued Gaussian function, i.e.

$$\Psi_n(X_{qs})=\frac{1}{\sqrt{2^n n!\pi^{1/2}}}e^{-X_{qs}^2/2}H_n(X_{qs}) \quad , \tag{24}$$

where

$$H_n(X_{qs}) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k n!}{k!(n-2k)!}(2X_{qs})^{n-2k} \quad , \tag{25}$$

and $\left[\frac{n}{2}\right]$ is the biggest positive integer less than $\frac{n}{2}$. Thus a formal time-independent Schrodinger equation reads

$$H_{qs}\Psi_n(X_{qs}) = E_n\Psi_n(X_{qs}) \ . \tag{26}$$

It should be noticed that $\Psi_n(X_{qs})$ is only formal solutions, because X_{qs} is not a true coordinate, but only an operator. Therefore $\Psi_n(X_{qs})\cdot f(x)$ are at all the solutions to the system and its hamiltonian, i.e.

$$H_{qs}\Psi_n(X_{qs})\cdot f(x) = E_n\Psi_n(X_{qs})\cdot f(x) \ . \tag{27}$$

Here we are only interest in the explicit form of qs-analogous Heimite polynomial, so we choose the simplest case of $f(x)=1$,

$$\Psi_n^{qs}(x) \equiv \Psi_n(X_{qs})\cdot 1 = \frac{1}{\sqrt{2^n n! \pi^{1/2}}} e^{-X_{qs}^2/2}\cdot H_n^{qs}(x) \ , \tag{28}$$

where

$$H_n^{qs}(x) \equiv \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k n!}{k![n-2k]_{qs}!}(2x)^{n-2k} \ , \tag{29}$$

is just the qs-analogous Hermite polynomial. It is not difficult to check that the qs-analogous Hermite polynomial satisfies the following qs-analogous recursive relations:

$$D_{qs}H_n^{qs}(x) = 2nH_{n-1}^{qs}(x) \ , \tag{30-1}$$

$$H_{n+1}^{qs}(x) = 2xH_n^{qs}(x) - 2nH_{n-1}^{qs}(x) \ , \tag{30-2}$$

$$H_n^{qs}(-x) = (-1)^n H_n^{qs}(x) \ . \tag{30-3}$$

IV. Quantum Heisenberg Algebra and Its Representations

Now let us turn to the study of qs-deformed oscillator with the Harmitonian

$$\overline{H}_{qs} = \frac{1}{2}(a_{qs}a_{qs}^{+} + a_{qs}^{+}a_{qs}) \ . \tag{31}$$

Recall that the creation and annihilation operators (denoted by a_{qs}^{+} and a_{qs} respectively) satisfy the following relations:

$$\begin{aligned} &[a_{qs} \, , \, a_{qs}^{+}] = [N+1]_{qs} - [N]_{qs} , \\ &[N \, , \, a_{qs}^{+}] = a_{qs}^{+} \ , \qquad [N \, , \, a_{qs}] = -a_{qs} \ . \end{aligned} \tag{32}$$

Here we notice that

$$a_{qs}^{+}a_{qs} = [N]_{qs} \ , \tag{33-1}$$

$$a_{qs}a_{qs}^{+} = [N+1]_{qs} \ . \tag{33-2}$$

Now let us consider a realization of two-parameter deformed quantum Heisenberg algebra as follows

$$a_{qs} = a\sqrt{\frac{[N]_{qs}}{N}} \ , \qquad (34\text{-}1)$$

$$a_{qs}^{+} = \sqrt{\frac{[N]_{qs}}{N}}a^{+} \ , \qquad (34\text{-}2)$$

$$N = a^{+}a = -\frac{1}{2}D_{qs}^{2} + \frac{1}{2}X_{qs}^{2} - \frac{1}{2} \ , \qquad (34\text{-}3)$$

which is a quantum algebra with the co-product, co-unit and antipode mapping well defined[13].

Where a , a^{+} and N satisfy the commutation relations (11). Because the infinite dimensional representation of the qs-oscillator algebra is isomorphic to that of the simple harmonic oscillator algebra in Eq.(28), i.e.

$$\left|n\right\rangle_{qs} \equiv \overline{\Psi}_{n}(X_{qs})\cdot 1 = \frac{1}{\sqrt{2^{n}n!\pi^{1/2}}}e^{-X_{qs}^{2}/2}\overline{H}_{n}(X_{qs})\cdot 1 \ , \qquad (35)$$

where using $\left|n\right\rangle_{qs}$ to symbol the eigenstate of qs-deformed oscillator. Therodinger equation is

$$\overline{H}(X_{qs})\overline{\Psi}_{n}(X_{qs})\cdot 1 = \overline{E}_{n}\overline{\Psi}_{n}(X_{qs})\cdot 1 \qquad (36)$$

where the energy spectrum is

$$\overline{E}_{n} = \frac{1}{2}([n]_{qs} + [n+1]_{qs}) \ . \qquad (37)$$

We define the peseudo-vacuum state $\left|0\right\rangle_{qs} \equiv \Psi_{0}(X_{qs})\cdot 1$ and have

$$a_{qs}\Psi_{0}(X_{qs})\cdot 1 = 0 \ , \qquad (38\text{-}1)$$

$$\left|n\right\rangle_{qs} \equiv \overline{\Psi}_{n}(X_{qs})\cdot 1 = \frac{(a_{qs}^{+})^{n}}{\sqrt{[n]_{qs}!}}\overline{\Psi}_{0}(X_{qs})\cdot 1 \ . \qquad (38\text{-}2)$$

We verify that

$$\left|n\right\rangle_{qs} \equiv \frac{(a_{qs}^{+})^{n}}{\sqrt{[n]_{qs}!}}\overline{\Psi}_{0}(X_{qs})\cdot 1 = \frac{(a^{+})^{n}}{\sqrt{n!}}\Psi_{0}(X_{qs})\cdot 1 \equiv \left|n\right\rangle \ , \qquad (39\text{-}1)$$

$$a_{qs}\left|n\right\rangle_{qs} \equiv \sqrt{[n]_{qs}}\,\overline{\Psi}_{n-1}(X_{qs})\cdot 1 = \sqrt{[n]_{qs}}\left|n-1\right\rangle_{qs} \ , \qquad (39\text{-}2)$$

$$a^{+}{}_{qs}\left|n\right\rangle_{qs} = \sqrt{[n+1]_{qs}}\,\overline{\Psi}_{n+1}(X_{qs})\cdot 1 = \sqrt{[n+1]_{qs}}\left|n+1\right\rangle_{qs} \ , \qquad (39\text{-}3)$$

and

$$N\left|n\right\rangle_{qs} = n\overline{\Psi}_{n}(X_{qs})\cdot 1 = n\left|n\right\rangle_{qs} \ . \qquad (39\text{-}4)$$

Therefore the Hilbert space of the qs-deformed oscillator is

$$F = \left\{\ \left|n\right\rangle_{qs} \ , \quad n = 0,1,2,\cdots\right\} \ . \qquad (40)$$

V. Conclusion and Discussion

So far, we have explicitly constructed the realizations of Heisenberg and quantum Heisenberg algebra by means of a pair canonical-conjugate operators D_{qs} and X_{qs}, and have shown that the qs-analogous Hermite polynomials are representations of the algebra. In particular, a systematic way to qs-series is given. This is an useful tool to study multiparameter analogous special functions and the representations of multiparameter deformed quantum groups. Finally, we point out that the extension of the constructions to other multiparameter deformed models in quantum mechanics, as for qs-deformations of the Hydrogen atom is also possible. The details will be published elsewhere.

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Non-perturbative approach for non-local currents of the massive and integrable quantum field theories

YI-XIN CHEN AND HUAN-XIONG YANG

CCAST (World Lab.), P.O.Box 8730, Beijing 100080,
Zhejiang Institute of Modern Physics, Zhejiang University, Hangzhou,310027,
Institute of Theoretical Physics, Academia Sinica, Beijing 100080, China.

Abstract

In this paper, we review the recent developments of the non-local conserved currents in the massive quantum field theories. The necessity of the non-perturbative approach for non-local conserved currents is emphasized. we analyze the applicability of the non-local current method to sine-Gordon and Zhiber-Mikhailov-Shabat (ZMS) systems within the Chang and Rajaraman's non-perturbative framework. Based on the chiral quantization, finally, we develop a non-perturbative method for non-local conserved currents for Toda-type integrable systems.

I. Introduction.

Symmetry in quantum field theory is widely recognized as being of fundamental importance. In 3+1 space-time dimensions, the likely symmetries of the S-matrix are subject to the severe limitations of the Coleman-Mandula theorem [1]. The theorem states that any symmetry group is necessarily isomorphic to the direct product of an internal symmetry group and the Poincare group. These possible symmetries are normally not restrictive enough to allow a non-perturbative solution of the theory.

In lower-dimensional quantum field theory, some of the postulates of Coleman-Mandula theorem may be relaxed in a non-trivial way. There exist some new quantum symmetries in a wide variety of integrable quantum field theories in 1+1 dimensions. The conserved currents that generate the symmetries are non-local and further characterized by non-trivial equal-time commutation, or braiding, relations. These exceptional properties of the currents are responsible for the non-trivial comultiplication of the charges when acting on multi-particle states. Meanwhile, the assumption of the trivial comultiplication in the Coleman-Mandula theorem is violated.

Non-local current approach in 2-dimensional quantum field theories appeared originally in perturbed conformal field theories (perturbed CFTs)[2, 3]. Its physical background is that many integrable models of quantum field theories could be formulated as conformal field theories perturbed by certain relevant operators [2]. Some conformal field theories preserve their integrability under perturbation which breaks the initial conformal invariance. Zamolodchikov claimed that only the first order terms of coupling, in perturbed-CFTs, do the equations of conservation of the non-local currents involve in [2]. Bernard and LeClair (BL) further pointed out that such nonlocal currents are subject to so exceptional braiding relations (or operator product expansions (OPEs)) that they can be used to expose the quantum group symmetries that are hidden in the considered 2-dimensional integrable quantum field models [3,4]. Among these quantum integrable models, there exist a large class of interesting ones, such as sine-Gordon model, for which we can find out four nonlocal conserved charges in the framework of perturbed-CFTs. These four charges are proved to obey such equal-time commutation relations that turn out to form the quantum algebra of certain quantum group. It is also proved that the nontrivial quantum group comultiplication laws will arise when the nonlocal charges act on multi-soliton states, which would imply that the soliton S-matrix of the system under consideration is factorizable. Moreover, by some scaling arguments BL have justified that the charge algebra is exact to all orders in perturbation theory. Hence the quantum group symmetries are actually non-perturbative symmetries of the integrable systems.

In perturbed-CFTs, the quantization scheme is the so-called radial quantization. Although the radial quantization in conformal field theories (CFTs) and their perturbed versions is essentially equivalent to its canonical counterpart, the nonlocal currents obtained by BL in the perturbed-CFT framework for an off-conformal theory may be not valid, if one goes over to the traditional canonical quantization instead which is realized though equal-time commutators and Hamiltonian evolution in the usual time. Then it becomes an interesting issue to develop a non-perturbative approach for the nonlocal conserved currents themselves. The goal of this paper is to introduce some attempts had been made by Chang et al [5], and us [6] in this direction. The organization of the paper is as follows. Section 2 is a review of BL current method based on the version of perturbed CFT. The Zamolodchikov equatiion in perturbed CFT is the key point in the BL's method. The fundamental objects are introduced for us to describe the Zamolodchikov equation. In section 3, we analyze the applicability of the non-local current method to the ZMS system in the Chang and

Rajaraman's non-perturbative framework. In section 4, a non-perturbative method for non-local conserved currents for Toda-type integrable systems is developed based on the chiral quantization. A short remark is included in the last section.

II. Zamolodchikov equation and BL's current method in the perturbed CFT framework.

Conformal symmetry of CFT is generated by the left and right components $T = T^{zz}$ and $\bar{T} = T^{\bar{z}\bar{z}}$ of symmetric traceless stress-energy tensor $T^{\mu\nu}$. By assuming spatial reflection symmetry. one can discuss only the left chiral part of the conformal algebra. The component T satisfies the equation

$$\partial_{\bar{z}} T = 0, \tag{1}$$

which makes it possible to define infinte set of Virasoro generators L_n, $n = 0, \pm 1, \pm 2, \cdots$ acting in the space $\mathbf{A}$ of local fields of CFT

$$L_n A(z, \bar{z}) = \oint_z d\zeta (\zeta - z)^{n+1} T(\zeta) A(z, \bar{z}), \tag{2}$$

where $A \in \mathbf{A}$. According to (1), the operators L_n are integrals of motion. It is well known that they satisfy the Virasoro algebra. The identity operator $\mathbf{I}$ is the particular field in $\mathbf{A}$, satisfying the equations

$$L_n \mathbf{I} = 0 \qquad for \quad n \geq -1 \ . \tag{3}$$

The application of the operators L_n with $n \leq -2$ to $\mathbf{I}$ gives rise to infinite set of local fields. For instance

$$(L_{-n}\mathbf{I})(z) = \oint_z d\zeta T(\zeta)(\zeta - z)^{-1} = T(z). \tag{4}$$

The fields obtained by the successive applications of more than one operators L_{-n} with $n \geq -2$ can be identified with the composite fields made of $T(z)$ and its derivatives, i.e., $: T^2 :$, $: T^3 :$, $: (\partial_z T)^2 :$, etc. Let Λ be the infinite dimensional space spaned by these composite fields including the identity operator $\mathbf{I}$. By definition, the space Λ is an irreducible Virasoro module with the highest weight equal to zero. The space Λ admits the following decomposition

$$\Lambda = \bigoplus_{s=0}^{\infty} \Lambda_s \tag{5}$$

in terms of eigen-spaces of the operator L_0

$$L_0 \Lambda_s = s \Lambda_s; \qquad \bar{L}_0 \Lambda_s = 0. \tag{6}$$

All the fields belong to Λ_s have the conformal dimensions $(s,0)$ and therefore the spin s. The fields constituting Λ are all analytic, i.e., they satisfy the equations like (1)

$$\partial_{\bar{z}}\Lambda = 0. \tag{7}$$

Every field $T_s^{(\alpha)} \in \Lambda_s$ gives rise to an infinite set of operators

$$\oint_z d\zeta T_s^{(\alpha)}(\zeta)(\zeta - z)^{n+s-1}, \qquad n = 0, \pm 1, \pm 2, \cdots \tag{8}$$

which are integrals of motion. Clearly, these operators are not all linearly independent. This is because there are some fields in Λ which are total ∂_z derivatives. In order to seperate linearly independent set, one can take the factor space $\hat{\Lambda} = \Lambda/L_{-1}\Lambda$ instead of Λ, here $L_{-1}\Lambda \subset \Lambda$ is the subspace in Λ constituted by the total derivatives. Like Λ itself, the space $\hat{\Lambda}$ enjoys the following decomposition

$$\hat{\Lambda} = \bigoplus_{s=0}^{\infty} \hat{\Lambda}_s, \qquad L_0\hat{\Lambda}_s = s\hat{\Lambda}_s. \tag{9}$$

We denote basic vectors of $\hat{\Lambda}_s$ as $T_s^{(\kappa)}$. Furthermore, the operators

$$\mathcal{L}_{s,n}^{(\kappa)} A(z,\bar{z}) = \oint_z d\zeta T_s^{(\kappa)}(\zeta)(\zeta - z)^{n+s-1} A(z,\bar{z}) \tag{10}$$

with $n = 0, \pm 1, \pm 2, \cdots$ constitute an infinite set of linearly independent integrals of motion in any CFT.

In the following, we shall how some of these operators can survives as integrals of motion if the CFT is pertuebed by particular relevent field. A relevent primary field is defined as its conformal dimensions obeying the relations $\Delta = \bar{\Delta}$ and $\Delta < 1$. Now, let us restrict attention to the case where only one relevent primary field Φ is taken as perturbation. The action of perturbed CFT is read as

$$H_\lambda = H_{CFT} + \lambda \int \Phi(x) d^2x, \tag{11}$$

where H_{CFT} is the action of CFT (the fixed-point hamiltonian in statistical mechanics) and λ the "coupling constant". In order to compute the correlation fuctions of this thoery by means of techniques of the original CFT, Zamolodchikov proposed the assumption that *the space* $\mathbf{A}$ *of local fields in the perturbed field theory (11) has the same structure as that in the original CFT.* He [2] argued that his assumption is resonable. Hence, one can keep the same notation of the fields as in the original CFT. Moreover, the fields Φ_α and their "descendents in the perturbed theory (11) have exactly the same spins and scale dimensions as in CFT. Of course, the perturbed field theory is not scale invariant since (11) contains a dimensionful constant λ.

Let us consider the space $\hat{\Lambda}$ in perturbed theory (11). At $\lambda \neq 0$ the fields $T_s^{(\kappa)} \in \Lambda_s$ do not of course satisfy (7). Rather the $\partial_{\bar{z}}$ derivative of $T_s^{(\kappa)}$ has the form

$$\partial_{\bar{z}} T_s^{(\kappa)} = \lambda R_{s-1}^{(\kappa)1} + \cdots + \lambda^n R_{s-1}^{(\kappa)n} + \cdots \tag{12}$$

where $R_{s-1}^{(\kappa)n}$ are some local fields belonging to $\mathbf{A}$. The dimensions of each term in the r.h.s of (12) must be $(s, 1)$ to agree with those in l.h.s. Hence the dimensions of the field $R_{s-1}^{(\kappa)n}$ are $(s + n\Delta - n, 1 + n\Delta - n)$ where (Δ, Δ) are the dimensions of Φ. Now we concentrate attention on the first order contribution. The equation (12) is reduced to

$$\partial_{\bar{z}} T_s^{(\kappa)} = \lambda R_{s-1}^{(\kappa)} \tag{13}$$

Let V be the irreducible highest weight module over the left Virasoro algebra with the highest weight vector Φ. This is the space spanned by the vectors

$$L_{-n_1} L_{-n_2} \cdots L_{-n_N} \Phi \tag{14}$$

with $N \geq 0$ and $n_1 \geq n_2 \geq \cdots \geq n_N > 0$. In fact, the conformal class $[\Phi]$ is the direct product $V \otimes \bar{V}$. Like (5), the space V enjoys the decomposition

$$V = \bigoplus_{s=0}^{\infty} V_s, \qquad L_0 V_s = (\Delta + s) V_s, \qquad \bar{L}_0 V_s = \Delta V_s. \tag{15}$$

The dimensional counting explained above shows that $R_{s-1}^{(\kappa)} \in V_{s-1}$. Therefore, the symbol $\partial_{\bar{z}}$ in (13) can be considered as linear operator $\partial_{\bar{z}} : \hat{\Lambda}_s \rightarrow V_{s-1}$. The action of $\partial_{\bar{z}}$ extended to the whole space Λ and the above mapping is its restriction to the factor space $\hat{\Lambda} = \Lambda / L_{-1}\Lambda$. Since the operator $\partial_{\bar{z}}$ commutes with L_{-1}, the mapping carries all significant information about $\partial_{\bar{z}}$.

The first order correction to any correlation function involving the field $T_s^{(\kappa)}$ is given by the integral

$$\int d\zeta d\bar{\zeta} < \Phi(\zeta, \bar{\zeta}) T_s^{(\kappa)} \cdots >_0 \tag{16}$$

where $< \cdots >_0$ stands for the correlation function of unperturbed CFT. obviously, the contribution to $\partial_{\bar{z}} T_s^{(\kappa)}$ can come only from the vicinity of the singular point $(\zeta,\bar{\zeta}) \to (z,\bar{z})$, where we can use the operator product expansion

$$T_s^{(\kappa)}(z)\Phi(\zeta,\bar{\zeta}) = \sum_{n=0}^{\infty}(z-\zeta)^{n-s}(\mathcal{L}_{s,-n}^{(\kappa)}\Phi)(\zeta,\bar{\zeta}), \tag{17}$$

here $\mathcal{L}_{s,-n}^{(\kappa)}\Phi$ are the certain local fields. Using (17) and taking count of contribution of the perturbative term, one can show that the r.h.s. in (13) reduces to the integral

$$\partial_{\bar{z}}T_s^{(\kappa)} = \lambda \oint_z \frac{d\zeta}{2i\pi}\Phi(\zeta,\bar{z})T_s^{(\kappa)}(z) \tag{18}$$

taken over small closed contour surrounding z. Here the operator product expansion of unperturbed CFT is implied in the r.h.s. The contour in (18) is closed because in the CFT the fields $T_s^{(\kappa)}$ are analytic and local with respect to Φ. Recalling the fact that the contour integral in (18) is equivalent to commutator, we can rewrite (18) in the form

$$\partial_{\bar{z}}T_s^{(\kappa)}(z,\bar{z}) = [T_s^{(\kappa)}, H_{int}], \tag{19}$$

where

$$H_{int} = \lambda \int d\zeta\Phi(\zeta,\bar{z}). \tag{20}$$

The equation (18) or (19) is the so-called Zamolodchikov's equation.

The starting point of the BL's non-local current method is to formularize the Zamolodchikov's equation into the conserved form of current. in order to do this, they introduce chiral fields $F(z,\bar{z}), \bar{F}(z,\bar{z})$ satisfying $\partial_{\bar{z}}F(z,\bar{z}) = \partial_z\bar{F}(z,\bar{z}) = 0$ in the conformal limit. The Zamolodchikov's equation leads to equations of motion for the perturbed chiral fields which are local with respect to the perturbed field to first order in perturbation theory

$$\partial_{\bar{z}}F(z,\bar{z}) = \lambda \oint_z \frac{d\zeta}{2i\pi}\Phi(\zeta,\bar{z})F(z), \qquad \partial_z\bar{F}(z,\bar{z}) = \lambda \oint_{\bar{z}} \frac{d\bar{\zeta}}{2i\pi}\Phi(z,\bar{\zeta}\bar{F}(\bar{z}). \tag{21}$$

Let us now suppose that there are currents conserved to first order in perturbation theory

$$\partial_{\bar{z}}J^a(z,\bar{z}) = \partial_z H^a(z,\bar{z}), \qquad \partial_z\bar{J}^{\bar{a}}(z,\bar{z} = \partial_{\bar{z}}\bar{H}^{\bar{a}}(z,\bar{z}). \tag{22}$$

Furthermore, we assume that in the conformal theory these currents are chiral fields, i.e., when $\lambda = 0$ they satisfy $\partial_{\bar{z}}J^a = \partial_z\bar{J}^{\bar{a}} = 0$. The condition for the currents to be conserved to first order in perturbation theory is then a condition on the residue of the operator product expansion between them and the perturbing field. Namely, the conservation laws (22) hold if the residues of these operator product expansions are total derivatives. From the conserved currents (22) we can define the conserved charges

$$Q^a = \frac{1}{2i\pi}(\int dz J^a + \int d\bar{z} H^a), \qquad \bar{Q}^{\bar{a}} = \frac{1}{2i\pi}(\int d\bar{z} \bar{J}^{\bar{a}} + \int dz \bar{H}^{\bar{a}}). \qquad (23)$$

However, as emphasized by Bernard and LeClair, the currents can be non-local. Let us illustrate how the BL's method is used to construct the non-local currents for the sine-Gordon system.

The quantum sine-Gordon theory is described by the action

$$S = \frac{1}{4\pi}\int d^2z \partial_z \Phi \partial_{\bar{z}} \Phi + \frac{\lambda}{\pi}\int d^2 z : cos(\beta\Phi) : . \qquad (24)$$

The parameter β is a coupling constant, and the parameter λ defines the mass scale of the model. We treat the $\lambda cos(\beta\Phi)$ term as a perturbation of the conformal field theory corresponding to a single free boson. Following Zamolodchikov, one can suppose that all the operators $O(x,t)$ of the sine-Gordon theory have a smooth ultra-violet limit and that they are in correspondence with the fields of the ultra-violet CFT. We can thus label in a unique way the fields of the sine-Gordon theory by the corresponding fields in the ultra-violet limit. In the massless limit, the free boson can be expanded as $\Phi(z,\bar{z}) = \phi(z) + \bar{\phi}(\bar{z})$ with $< \phi(w)\phi(z) >= -log(w-z)$ and similarly for $\bar{\phi}(\bar{z})$. In the deep ultra-violet limit the chiral and anti-chiral components $\phi(x,t)$ and $\bar{\phi}(x,t)$ can be expressed in a non-local way in terms of the sine-Gordon field $\Phi(x,t)$, The relations are

$$\phi(x,t) = \frac{1}{2}(\Phi(x,t) + \int_{-\infty}^{x} dy \partial_t \Phi(y,t)), \qquad \bar{\phi}(x,t) = \frac{1}{2}(\Phi(x,t) - \int_{-\infty}^{x} dy \partial_t \Phi(y,t). \qquad (25)$$

Subsequently, we can derive the existence of conserved currents generated by fields of the form $J_\alpha = exp(i\alpha\phi)$ and $\bar{J}_\alpha = exp(i\alpha\bar{\phi})$. Using the operator product expansions of them, one can see that the condition for a conservation law for J_α (and similarly for the anti-holomorphic sector) amounts to a condition on α, namely $\alpha = \pm 2/\beta$. Therefore we find the following conserved currents

$$\partial_{\bar{z}} J_\pm = \partial_z H_\pm, \qquad \partial_z \bar{J}_\pm = \partial_{\bar{z}} \bar{H}_\pm, \qquad (26)$$

where

$$J_\pm(x,t) = exp(\pm\frac{2i}{\beta}\phi(x,t)), \quad H_\pm = \lambda\frac{\beta^2}{\beta^2-2}exp[\pm i(\frac{2}{\beta}-\beta)\phi(x,t) \mp i\beta\bar{\phi}(x,t)], \qquad (27)$$

$$\bar{J}_{\pm} = exp(\mp\frac{2i}{\beta}\bar{\phi}(x,t)), \quad \bar{H}_{\pm} = \lambda\frac{\beta^2}{\beta^2-2}exp[\mp i(\frac{2}{\beta}-\beta)\bar{\phi}(x,t)\pm i\beta\phi(x,t)]. \quad (28)$$

From these conserved currents we can define four conserved charges by means of (23). Such conserved charges are non-local due to the fact that the chiral and anti-chiral components, ϕ and $\bar{\phi}$, of the sine-Gordon field Φ are non-local. This non-locality is reflected in the relations (25). On the other hand, these non-local conserved charges together with the topological charge of the sine-Gordon theory build up a known infinite dimensional algebra, namely the q-deformation of the $sl(2)$ affine Kac-Moody algebra, denoted $\hat{sl}_q(2)$ with zero center. Since these conserved charges generate the infinite dimensional symmetries of the considered system, the S-matrix of the system must commute with the action of the non-local conserved charges. Such commutativity leads to a set of equations satisfied by the S-matrix, one can check that the solution of the equations automatically satisfies the Yang-Baxter equation, which is erquired for factorization of the multiparticle S-matrix. The minimal solution is the known sine-Gordon S-matrix.

Up to now, readers can see that the BL's method for the non-local currents is based on the perturbed CFT. Therefore, their method is called as the non-local current method in the perturbed CFT framework. It should be emphasized that although the fields of the perturbed CFT are in correspondence with those of the ultra-violet CFT, this does not mean that they are same. Therefore, it can not satisfying for us that the integrablity of the massive quantum field theories is charaterized by the non-local currents constructed by means of the chiral and anti-chiral components, ϕ and $\bar{\phi}$, in the massless limit.

III. On validity of the Chang and Rajaraman's method of non-local currents in non-perturbative framework.

Recently, Chang and Rajaranian (CR) have made some enlightening attempts in the direction of construction of non-local currents in perturbative framework [5]. They obtained new nonlocal and Lorentz covariant conserved currents for sine-Gordon model with the coupling constant $\frac{1}{2} < \beta^2 < \frac{3}{2}$, by non-perturbatively treating this off-conformal system in the traditional canonical quantization scheme. CR's nonlocal currents have different expressions from the currents given by BL. Nevertheless the corresponding charges are almost subject to the same commutation relations as BL's. Relying on this fact, CR declared that their method can be universally applied to studying the integrable properties of some other off-conformal integrable systems besides the quantum sine-Gordon model [5].

It should be pointed out that only when the coupling β falls into the interval $\frac{1}{2} < \beta^2 < \frac{3}{2}$ does CR's method lead to the similar results (for sine-Gordon model) to those deduced by BL from the perturbed-CFT point. That is to say, in CR's non-perturbative framework, the other two important coupling regions $0 \leq \beta^2 \leq \frac{1}{2}$ and $\frac{3}{2} \leq \beta^2 < 2$ where the sine-Gordon model is also well defined have been left out. Furthermore, as pointed by BL [3], when $\beta^2 = 2$ the sine-Gordon theory can be formulated as a current-current perturbation system of the $k = 1$ $SU(2)$ WZW model, and its hidden symmetry at this special coupling will limit to the Yangian symmetry [3, 7] with which the sine-Gordon soliton S-matrix will degenerate to a rational solution of the Yang-Baxter equation. This is of course interesting. Since CR's method could say nothing for the nonlocal currents of sine-Gordon model at this fascinating coupling, as well as could say nothing within the other two coupling regions aforesaid, it may be not appropriate to the description of the integrable properties of quantum sine-Gordon model when $0 \leq \beta^2 \leq \frac{1}{2}$ and $\frac{3}{2} \leq \beta^2 \leq 2$.

We feel anxious if there arise more serious troubles besides the coupling restrictions when the CR's method is applied to other off-conformal integrable models. To examine whether such worries and misgivings is true or not, in the present section we investigate the application of CR's method to the famous Zhiber- Mikhailov-Shabat (ZMS) model. After some careful analyses, we find out that in CR's non-perturbative framework the existence of nonlocal conserved currents will split the regions of coupling β for ZMS model into two separate ones, and in each region of β there exist two (not four) nonlocal conserved charges merely, which together with the topological charges do only satisfy a finite dimensional quantum algebra $sl_q(2)$. We have known from BL's perturbed-CFT framework that ZMS model is complete integrable and its S-matrix has an infinite dimensional symmetry $A^{(2)}_{q2}$ [1, 8, 9]. Unfortunately, the nonlocal charges with their commutation algebras obtained with CR's method are not sufficient to determine the factorizable soliton S-matrix of ZMS model.

Let us now demonstrate the application of CR's method to ZMS model in detail. Our starting point is the following Langrangian density

$$\mathcal{L} = \frac{1}{8\pi}\partial_\mu\phi\partial^\mu\phi + \frac{\lambda}{2}\ (\ e^{i\beta\phi} + e^{-i2\beta\phi}\ ). \qquad (29)$$

The system described by (29) is known as the Zhiber-Mikhailov-Shabat model with imaginary coupling constant ($\text{Im}\beta = 0$). One can easily verify that Langragian (29) is not invariant under conformal transformation. Whereas ZMS model is still completely integrable. In Ref.[8], with the introduction of the background charge ZMS model (29) has been associated with a complex Liouville theory describing the minimal models of CFTs perturbed by $\Phi_{1,2}$ field, which is just among the set of Zamolodchikov's relevant operators [2]. From the point of view of perturbed-CFT, the constant λ appearing in (29) plays a role of characterizing the order of perturbation expansion. So far, many of integrability data including the BL nonlocal conserved currents, the infinite dimensional quantum group symmetry and the factorizable soliton S-matrix, of the ZMS model, have been obtained in the perturbed-CFT framework. Hence it is convenient for us to compare BL nonlocal currents of ZMS model with those which we are going to find in the CR's non-perturbative framework, and judge whether CR's method could be used to study the integrability of quantum ZMS field theory.

For the above purpose, we are obliged to quantize the ZMS model (29) in the traditional canonical quantization scheme, just as what CR did. The canonical quantization means that we take the following equal-time commutation relation

$$[\, \phi(x), \pi_\phi(y)\,] = i\delta(x^1 - y^1)\,, \tag{30}$$

where $\pi_\phi(x) \equiv \frac{1}{4\pi}\partial_0\phi(x)$. Besides the Heisenberg equation is postulated as

$$\partial_\mu\partial^\mu\phi - 2\pi\lambda\frac{d}{d\phi}\,:(e^{i\beta\phi} + e^{-i2\beta\phi}): = 0. \tag{31}$$

The symbol : : stands for normal ordering with respect to the "massless free field"

$$\begin{aligned} \phi(x) &= \int_{-\infty}^{+\infty} \frac{dk}{\sqrt{|k|}}\,[\, a_k \exp(ikx^1) + a_k^\dagger \exp(-ikx^1)\,]\,, \\ \pi_\phi(x) &= -\frac{i}{4\pi}\int_{-\infty}^{+\infty} dk\sqrt{|k|}\,[\, a_k \exp(ikx^1) - a_k^\dagger \exp(-ikx^1)\,]\,. \end{aligned} \tag{32}$$

It is worthwhile to mind that because the field $\phi(x)$ obeys a high coupled nonlinear equation of motion, which is not a genuine free field. Thereby $\phi(x)$ can not be expanded in terms of the plane wave actually, against that in non-interaction case. (32) should be correctly understood as the expansions of the field operator $\phi(x)$ and its canonical momenta at an arbitrary given time, say x^0, in terms of their Fourier components [5]. Owing to (30), the operators a_k and $a_k^\dagger$ satisfy the commutation relations $[\, a_k, a_{k'}^\dagger\,] = \delta(k - k')$, $[\, a_k, a_{k'}\,] = [\, a_k^\dagger, a_{k'}^\dagger\,] = 0$, which naturally acquire the explanations of the annihilation and creation operators of k-soliton. Following Chang and

Rajaraman, we now introduce two chiral-like components of the field $\phi(x)$, defined as

$$\rho(x) \equiv \tfrac{1}{2}\,[\ \phi(x) + 4\pi \int_{-\infty}^{x^1} dy\pi_\phi(y)\ ] = \rho_+(x) + \rho_-(x)\ ,$$
$$\bar{\rho}(x) \equiv \tfrac{1}{2}\,[\ \phi(x) - 4\pi \int_{-\infty}^{x^1} dy\pi_\phi(y)\ ] = \bar{\rho}_+(x) + \bar{\rho}_-(x)\ . \tag{33}$$

The readers should be awoken that here the nonlocally defined operators $\rho(x)$ and $\bar{\rho}(x)$ do not obey the chiral conservation laws unless the interaction terms in (29) are vanishing, which is the most striking difference between the CR's non-perturbative method and the BL's method of perturbed-CFT. In (33) ρ_+ ($\bar{\rho}_+$) and ρ_- ($\bar{\rho}_-$) are made of the annihilation part and creation part of ρ ($\bar{\rho}$) respectively

$$\rho_+(x)\ (\ \bar{\rho}_+(x)\ ) = \tfrac{1}{2}\int_{-\infty}^{+\infty} dk a_k\,[\ \frac{1}{\sqrt{|k|}} e^{ikx^1} + \tfrac{1}{i}\,(\ i\ )\ \sqrt{|k|} \int_{-\infty}^{x^1} dy e^{iky^1}\ ]\ ,$$
$$\rho_-(x)\ (\ \bar{\rho}_-(x)\ ) = \tfrac{1}{2}\int_{-\infty}^{+\infty} dk a_k^\dagger\,[\ \frac{1}{\sqrt{|k|}} e^{-ikx^1} + i\,(\ \tfrac{1}{i}\ )\ \sqrt{|k|} \int_{-\infty}^{x^1} dy e^{-iky^1}\ ]\ . \tag{34}$$

From the annihilation and creation properties of the operators a_k and $a_k^\dagger$, we can easily derive the canonical commutators of these chiral-like operators. The results are

$$[\ \rho_+(x), \rho_-(y)\ ] = -\ln\,[\ ik_0\ (x^1 - y^1 - i\epsilon)\ ],$$
$$[\ \bar{\rho}_+(x), \bar{\rho}_-(y)\ ] = -\ln\,[\ -ik_0\ (x^1 - y^1 - i\epsilon)\ ], \tag{35}$$
$$[\ \rho_+(x), \bar{\rho}_-(y)\ ] = [\ \rho_-(x), \bar{\rho}_+(y)\ ] = -\tfrac{1}{2}\pi i,$$

with $[\ \rho_\pm, \rho_\pm\ ]$, $[\ \rho_\pm, \bar{\rho}_\pm\ ]$ and $[\ \bar{\rho}_\pm, \bar{\rho}_\pm\ ]$ vanishing. Note that the choice of normalization in (32) is very similar to that of massless field, which yields introducing an infrared cutoff $k_0 e^{-\gamma}$ (γ is the Euler constant, $k_0 \to 0$) into k-integrals when we calculate the commutation relation (35).

We further introduce some vertex operators in terms of the above chiral-like fields ρ and $\bar{\rho}$

$$W_{a,\ b}(x) \equiv\ :\ e^{ia\rho x + ib\bar{\rho}(x)}\ :\ = (-i)^{ab}\ W_{a,\ 0}(x)\, W_{0,\ b}(x) \tag{36}$$

which will be more convenient for us to derive the CR's nonlocal currents for ZMS model. It is evidently that the interaction potential energy $-\frac{\lambda}{2} :(e^{i\beta\phi} + e^{-i2\beta\phi}):$ in (29) can be recast as $-\frac{\lambda}{2}\ (\ W_{\beta,\ \beta} + W_{-2\beta, -2\beta}\ )$. Moreover, the quantum Poincare generators (translation P, Hamiltonian H and the Lorentz generator M) for the system can be expressed with such vertex operators and the chiral-like operators

$$P = \frac{1}{4\pi} \int_{-\infty}^{+\infty} dy^1 \; [\; (\partial_y \rho)^2 - (\partial_y \bar{\rho})^2 \;]$$

$$H = \frac{1}{4\pi} \int_{-\infty}^{+\infty} dy^1 \; [\; (\partial_y \rho)^2 + (\partial_y \bar{\rho})^2 - 2\pi\lambda \; (\; W_{\beta,\;\beta}(y) \; + \; W_{-2\beta,\;-2\beta}(y) \;) \;]$$

$$M = \frac{1}{4\pi} \int_{-\infty}^{+\infty} dy^1 \; [\; (y^0 + y^1) \; (\partial_y \rho)^2 - (y^0 - y^1) \; (\partial_y \bar{\rho})^2 - 2\pi\lambda y^1 \; (\; W_{\beta,\;\beta}(y)$$
$$+ \; W_{-2\beta,\;-2\beta}(y) \;) + \frac{2i}{\beta} y^1 (\partial_y^2 \rho) + \frac{2i}{\beta} y^1 (\partial_y^2 \bar{\rho}) \;] \; . \qquad (37)$$

Then it is the direct consequences of the canonical commutation relations (35) that

$$[\; W_{a,\;b}(x), \;\; P \;] \; = i\partial_1 W_{a,\;b}(x)$$
$$[\; W_{a,\;b}(x), \;\; H \;] \; = i\partial_0 W_{a,\;b}(x)$$
$$[\; W_{a,\;b}(x), \;\; M \;] \; = i \; [\; x^0 \partial_1 + x^1 \partial_0 + \frac{a^2 + b^2}{2} + \frac{a-b}{\beta} \;] \; W_{a,b}(x)$$
$$+ \frac{\lambda}{2} \int_{-\infty}^{+\infty} dy^1 (x^1 - y^1) \; [\; W_{a,\;b}(x), W_{\beta,\;\beta}(y) + W_{-2\beta,\;-2\beta}(y) \;] \qquad (38)$$

as well as

$$W_{a,\;b}(x) W_{c,\;d}(y) \; = \; (\; i \;)^{(a-b)(c+d)} [\; k_0(x^1 - y^1 - i\epsilon) \;]^{ac} \; [\; k_0(x^1 - y^1 + i\epsilon) \;]^{bd}$$
$$: \; e^{ia\rho(x) + ib\bar{\rho}(x) + ic\rho(y) + id\bar{\rho}(y)} \; : \; . \qquad (39)$$

The first two equations in (38) form the foundation of evaluating the CR's nonlocal conserved currents for ZMS model. Combining these equations and taking (37) in mind we get

$$\partial_- W_{a,\;0}(x) = i\tfrac{\lambda}{2} \int_{-\infty}^{+\infty} dy^1 \; [\; W_{a,\;0}(x), W_{\beta,\;\beta}(y) + W_{-2\beta,\;-2\beta}(y) \;] \; ,$$
$$\partial_+ W_{0,\;b}(x) = i\tfrac{\lambda}{2} \int_{-\infty}^{+\infty} dy^1 \; [\; W_{0,\;b}(x), W_{\beta,\;\beta}(y) + W_{-2\beta,\;-2\beta}(y) \;] \; . \qquad (40)$$

These results are much enlightening for deriving the nonlocal conserved currents of ZMS model (29). In fact, Eqs.(40) will become conservation equations of the currents consisting of vertex operators, for some special values of parameters a and b. To determine such a and b, let us first recall an useful mathematical formula

$$\lim_{\epsilon\to 0} [\,(x^1-y^1-i\epsilon)^{-n}-(x^1-y^1+i\epsilon)^{-n}\,] = 2\pi i\,\frac{(-1)^{n-1}}{(n-1)!}\,\delta^{(n-1)}(x^1-y^1). \quad (41)$$

$$n=1,2,3,\cdots$$

Thus, when $a,\ b=-\frac{2}{\beta}$ and $a,\ b=\frac{1}{\beta}$ the eqs.(40) might become the equations of conservation of some currents. As a matter of fact, the first one of (40) in $a=-\frac{2}{\beta}$ case yields

$$\begin{aligned}\partial_- W_{-2/\beta,\,0} = {} & i^{(2-\beta^2)}\tfrac{\pi\lambda}{2k_0^2}\,[\,\tfrac{\beta^2}{\beta^2-2}\partial_+(\,W_{\beta-2/\beta,\,0}\,W_{0,\,\beta}\,)-\partial_-(\,W_{\beta-2/\beta,\,0}\,W_{0,\,\beta}\,)\,]\\ & -i^{(2-\beta^2)}\tfrac{\pi\lambda}{2k_0^2}\tfrac{2}{\beta^2-2}\mathbf{X}(\lambda)\;.\end{aligned} \quad (42)$$

Contrary to the Efthimiou's corresponding equation obtained in perturbed-CFT framework, (42) includes an extra term $\mathbf{X}(\lambda)$ which appears because the chiral-like fields ρ and $\bar{\rho}$ are not true chiral-functions of space-time:

$$\begin{aligned}\mathbf{X}(\lambda) &= i\tfrac{\lambda}{2}\,\tfrac{\pi\lambda}{2k_0^2}\int_{-\infty}^{+\infty} dy^1\ [\,W_{\beta-2/\beta,\,\beta}(x), W_{\beta,\,\beta}(y)+W_{-2\beta,\,-2\beta}(y)\,]\\ &= \tfrac{\pi^2\lambda^2}{2k_0^2}\,\{\,k_0^{2(\beta^2-1)}\int_{-\infty}^{+\infty} dy^1\ (x^1-y^1)^{2\beta^2}\ \delta^{(1)}(x^1-y^1)\\ &\quad :\ \exp[\,i(\,\beta-2/\beta)\rho(x)+i\beta\bar{\rho}(x)+i\beta\rho(y)+i\beta\bar{\rho}(y)\,]\ :\\ &\quad +\tfrac{1}{3!}\,k_0^{-4(\beta^2-1)}\int_{-\infty}^{+\infty} dy^1\ (x^1-y^1)^{4(\,2-\beta^2\,)}\ \delta^{(3)}(x^1-y^1)\\ &\quad :\ \exp[\,i(\,\beta-2/\beta)\rho(x)+i\beta\bar{\rho}(x)-i2\beta\rho(y)-i2\beta\bar{\rho}(y)\,]\ :\,\}.\end{aligned} \quad (43)$$

The existence of such an extra term $\mathbf{X}(\lambda)$ has us not to be able to consider (42) as an equation of current-conservation unless $\mathbf{X}(\lambda)$ could be set to zero. Only in certain special cases is this requirement satisfied. In (43) the possible singularities respectively come from the factors $(x^1-y^1)^{2\beta^2}$ and $(x^1-y^1)^{4(2-\beta^2)}$ when $x^1\to y^1$. In view of this fact and taking the contributions of the derivatives of δ-function into account, we find out that $\mathbf{X}(\lambda)=0$ if and only if $1/2<\beta^2<5/4$. Namely, only in the region of $1/2<\beta^2<5/4$ does Eq.(42) become a equation of conservation of nonlocal current:

$$\partial_- j_+(x)\ +\ \partial_+ j_-(x)\ =0, \quad (44)$$

where the nonlocal conserved currents $j_\pm(x)$ are read as

$$\left\{\begin{aligned} j_+(x)\ &=\ W_{-2/\beta,\,0}(x)\ +\ \tfrac{\pi\lambda}{2k_0^2}\,W_{\beta-2/\beta,\,\beta}(x), \qquad \text{for}\ \ 1/2<\beta^2<5/4\\ j_-(x)\ &=\ \tfrac{\pi\lambda}{2k_0^2}\tfrac{\beta^2}{2-\beta^2}\ W_{\beta-2/\beta,\,\beta}(x).\end{aligned}\right. \quad (45)$$

Just as what we have indicated earlier, the first equation of (40) could also result in another conservation equation of current when $a = \frac{1}{\beta}$. It is directly followed from (40) that

$$\begin{aligned}\partial_- W_{1/\beta,\,0} = i^{(2-4\beta^2)} \tfrac{\pi\lambda}{2k_0^2} \, [\, \tfrac{4\beta^2}{4\beta^2-2} \partial_+ (\, W_{1/\beta-2\beta,\,0}\, W_{0,\,-2\beta}\,) \\ -\partial_-(\, W_{1/\beta-2\beta,\,0}\, W_{0,\,-2\beta}\,)\,] \;-\; i^{(2-4\beta^2)} \tfrac{\pi\lambda}{2k_0^2} \tfrac{1}{2\beta^2-1} \mathbf{Y}(\lambda)\,,\end{aligned} \tag{46}$$

where

$$\begin{aligned}\mathbf{Y}(\lambda) &= i\tfrac{\lambda}{2}\, \tfrac{\pi\lambda}{2k_0^2} \int_{-\infty}^{+\infty} dy^1 \, [\, W_{1/\beta-2\beta,\,-2\beta}(x), W_{\beta,\,\beta}(y) + W_{-2\beta,\,-2\beta}(y)\,] \\ &= -\tfrac{\pi^2\lambda^2}{2k_0^2} \{\, k_0^{(1-4\beta^2)} \int_{-\infty}^{+\infty} dy^1 \, (x^1-y^1)^{(2-4\beta^2)} \, \delta^{(0)}(x^1-y^1) \\ &\quad :\, \exp[\, i(\, 1/\beta - 2\beta)\rho(x) - i2\beta\bar{\rho}(x) + i\beta\rho(y) + i\beta\bar{\rho}(y)\,] \,: \\ &\quad - k_0^{2(4\beta^2-1)} \int_{-\infty}^{+\infty} dy^1 \, (x^1-y^1)^{(\,8\beta^2\,)} \, \delta^{(1)}(x^1-y^1) \\ &\quad :\, \exp[\, i(\, 1/\beta - 2\beta)\rho(x) - i2\beta\bar{\rho}(x) - i2\beta\rho(y) - i2\beta\bar{\rho}(y)\,] \,:\, \}.\end{aligned} \tag{47}$$

After a similar analysis to that for $\mathbf{X}(\lambda)$, we find that the vanishing of $\mathbf{Y}(\lambda)$ does also produce an restriction to the values of the coupling constant β: $1/8 < \beta^2 < 1/2$. Noticeably different from the case of sine-Gordon model where the potential energy possesses a symmetry under the transform $\beta \to -\beta$ [2, 5, 10], this region $1/8 < \beta^2 < 1/2$ does not coincide with the above one $1/2 < \beta^2 < 5/4$ determined by (43), and there is no intersection between them. The nonlocal conserved current derived from (46) and (47) turns out to be

$$\left\{\begin{array}{lcll} J_+(x) &=& W_{1/\beta,\,0}(x) + \frac{\pi\lambda}{2k_0^2} W_{1/\beta-2\beta,\,-2\beta}(x), & \text{for } 1/8 < \beta^2 < 1/2 \\ J_-(x) &=& \frac{\pi\lambda}{2k_0^2}\frac{4\beta^2}{2-4\beta^2} W_{1/\beta-2\beta,\,-2\beta}(x). & \end{array}\right. \tag{48}$$

Of course, we have, from the second one of eqs.(40), other two conservation equations moreover. To save the lengthy of the paper we had better list out the corresponding nonlocal conserved currents here:

$$\left\{\begin{array}{lcll} \tilde{j}_+(x) &=& \frac{\pi\lambda}{2k_0^2}\frac{\beta^2}{2-\beta^2} W_{\beta,\,\beta-2/\beta}(x), & \text{for } 1/2 < \beta^2 < 5/4 \\ \tilde{j}_-(x) &=& W_{0,\,-2/\beta}(x) + \frac{\pi\lambda}{2k_0^2} W_{\beta,\,\beta-2/\beta}(x). & \end{array}\right. \tag{49}$$

and

$$\left\{\begin{array}{lcll} \tilde{J}_+(x) &=& \frac{\pi\lambda}{2k_0^2}\frac{4\beta^2}{2-4\beta^2} W_{-2\beta,\,1/\beta-2\beta}(x), & \text{for } 1/8 < \beta^2 < 1/2 \\ \tilde{J}_-(x) &=& W_{0,\,1/\beta}(x) + \frac{\pi\lambda}{2k_0^2} W_{-2\beta,\,1/\beta-2\beta}(x). & \end{array}\right. \tag{50}$$

All these nonlocal currents are Lorentz covariant. With the help of Lorentz operator M given in (37), one can easily check that these currents $(j_+,\ j_-)$, $(\tilde{j}_+,\ \tilde{j}_-)$, $(J_+,\ J_-)$ and $(\tilde{J}_+,\ \tilde{J}_-)$ carry Lorentz weights $(0,\ 2)$, $(2,\ 0)$, $(\frac{3}{2\beta^2},\ \frac{3-4\beta^2}{2\beta^2})$ and $(\frac{4\beta^2-3}{2\beta^2},\ -\frac{3}{2\beta^2})$ respectively. In a sense, such "spin" spectra are compatible with the fact that $(\ \partial_+/\partial_-\ )$ carries Lorentz weight 2, which is just what we have anticipated.

Up to now, we have obtained a set of nonlocal Lorentz covariant conserved currents for ZMS model in CR's non-perturbative framework. The charges corresponding to these currents are

$$\begin{aligned} Q^0 &\equiv \tfrac{1}{2} \int_{-\infty}^{+\infty} dx^1\ [\, j_+(x) + j_-(x)\,] \qquad \text{for } 1/2 < \beta^2 < 5/4 \\ \tilde{Q}^0 &\equiv \tfrac{1}{2} \int_{-\infty}^{+\infty} dx^1\ [\, \tilde{j}_+(x) + \tilde{j}_-(x)\,]\,, \end{aligned} \tag{51}$$

and

$$\begin{aligned} Q^1 &\equiv \tfrac{1}{2} \int_{-\infty}^{+\infty} dx^1\ [\, J_+(x) + J_-(x)\,] \qquad \text{for } 1/8 < \beta^2 < 1/2 \\ \tilde{Q}^1 &\equiv \tfrac{1}{2} \int_{-\infty}^{+\infty} dx^1\ [\, \tilde{J}_+(x) + \tilde{J}_-(x)\,]\,, \end{aligned} \tag{52}$$

Since the two regions of $1/2 < \beta^2 < 5/4$ and $1/8 < \beta^2 < 1/2$ have not any domains in common, only two CR's nonlocal charges exist there at most for a definite ZMS theory. Such an outcome is neither similar to the CR's discussion for sine-Gordon theory nor compatible with the corresponding BL's charges obtained by Efthimiou in perturbed-CFT scheme for ZMS model. Can these two charges (51) or (52) provided enough physical information for displaying the integrability of ZMS model? To answer this question let us examine the equal-time commutation relations obeyed by these charges in the following. Without losing generality, we would like study the algebraic and topologocal properties of the nonlocal charges (51) only. Following Bernard and LeClair we introduce a topological conserved charge

$$T = \frac{\beta}{2\pi} \int_{-\infty}^{+\infty} dx^1\ [\, (\frac{\partial \rho}{\partial x^1}) + (\frac{\partial \bar{\rho}}{\partial x^1})\,]\,. \tag{53}$$

Then a tedious but straightforward calculation lead

$$\begin{aligned} &[\, T,\ Q^0\,] = -2Q^0\,, \qquad [\, T,\ \tilde{Q}^0\,] = 2\tilde{Q}^0\,, \\ &Q^0\, \tilde{Q}^0 - q^{-2}\, \tilde{Q}^0\, Q^0 = \mathrm{a}\, (1 - q^{-2T})\,, \end{aligned} \tag{54}$$

which are based on the commutation relations (35) and (38). In (54),

$$\mathrm{a} \;=\; \frac{\pi\lambda}{2k_0^2}\,[\,2\pi i/q\,(2-\beta^2)^2\,]\,, \qquad q \;=\; \exp\,(-2\pi i/\beta^2)\,. \qquad (55)$$

The quantum charge algebra (54) is obviously the Chevalley basis of the finite-dimensional quantum algebra $sl_q(2)$. The same conclusion is also valid for the charge algebra of T, Q^1 and $\tilde{Q}^1$. However, the quantum algebra of the four BL-type nonlocal conserved charges for ZMS model obtained in perturbed-CFT framework has proved to be affine algebra $A_{q2}^{(2)}$[8]. The generators of $A_{q2}^{(2)}$ consist of the topological charge and the four BL-type charges, which can nontrivially carry a spectrum parameter relating to the Lorentz weights of the BL-type nonlocal conserved charges. This implies that the quantum group symmetry of the ZMS model is infinitely dimensional [8, 11]. The algebras obeyed by CR's charges (51) and (52) are merely its two finite-dimensional subalgebras. Comparing with the case of the infinitely dimensional quantum group, the number of the equations of the S-matrix governed by the above finitely dimensional charge algebras must decrease. Therefore, the physical S-matrix of the ZMS solitons, which has been obtained by Efthimiou [8], can not be covered by the corresponding charge algebras of the CR's charges. In this sense, the CR approach for nonlocal currents may be not as universal as the BL perturbed-CFT approach.

IV. A non-perturbative approach for non-local conserved currents based on the chiral quantization.

In our previous section, it has been shown that the Chang and Rajaraman's (CR's) non-perturbative method [8] in the traditional canonical quantization scheme is not universally appropriate for studying the quantum group symmetries, factorizable S-matrices and then the integrabilities of non-simply laced affine Toda systems, *e. g.*, the famous ZMS model. CR's non-perturbative framework can merely display partial quantum group symmetries of non-simply laced affine Toda theories, which are not enough to determine the physical soliton S-matrices for such systems. Taking account of this fact, we are necessary to pursue new non-perturbative methods to evaluate nonlocal conserved currents, which should be applicable to investigating the quantum integrabilities of both simply laced affine Toda systems and their non-simply laced analogues. In a sense, the present paper is a primary attempt on this direction. In the following context, we suggest a candidate for the desired non-perturbative nonlocal current approach. Our approach is based on the chiral quantization prescription, *i. e.*, the light cone coordinate $x_- = -\frac{1}{\sqrt{2}}\,(\,x^0 - x^1\,)$ or

$x_+ = \frac{1}{\sqrt{2}}$ ($x^0 + x^1$) being characterized as the time-evolution parameter. This quantization prescription is strictly different from both the radial quantization used by Bernard and LeClair to study nonlocal currents in the perturbation theories of the conformal field theories (perturbed-CFT) [3] and the canonical quantization used in CR's non-perturbative method. As a matter of fact, it is well known that in the conformal field theories (CFT) the radial quantization is intrinsically equivalent to the traditional canonical quantization. In below, one can see that the employment of the chiral quantization will lead to constructing in a non-perturbative way two nonlocal conserved currents for affine Toda systems, regardless they are simply laced or not, and these two currents will cover most of the physical information about these systems.

For definiteness let us consider the sine-Gordon and the ZMS models concretely. These two models are the most typical delegates to the simply-laced Toda and nonsimply-laced Toda systems respectively, of which the classical actions can be written into the following unified form

$$S = \int d^2x \, [\, \frac{1}{8\pi}\partial_\mu\phi\partial^\mu\phi + \frac{\lambda}{2} \, (\, e^{i\beta\phi} + e^{-is\beta\phi} \,) \,] \, . \tag{56}$$

where β is a real constant and the parameter "s " is assumed to be among the set { 1, 1/2, 2 }. When "s " is taken as *one*, the action (56) describes sine-Gordon model. Otherwise (56) describes the ZMS model. It is easily from (56) to read off the equation of motion in laboratory coordinates

$$\partial_\mu\partial^\mu\phi - 2\pi i\lambda\beta \, (\, e^{i\beta\phi} \; - \; se^{-is\beta\phi} \,) \; = \; 0. \tag{57}$$

By using the light cone coordinates $x_\pm$, the Eq.(2) can alternatively be expressed as

$$\partial_+\partial_-\phi + \pi i\lambda\beta \, (\, e^{i\beta\phi} \; - \; se^{-is\beta\phi} \,) \; = \; 0. \tag{58}$$

The purpose of this section is to investigate the nonlocal currents of the quantum versions of systems (56) in light cone quantization framework and to study the integrabilities of them. For more transparent, we choose the light cone coordinate x_- as the time-evaluation parameter. Then it is apparent that (58) is the Euler-Lagrange equation from the following Lagrangian density

$$\mathcal{L} = \frac{1}{4\pi}\partial_+\phi\partial_-\phi - \frac{\lambda}{2} \, (\, e^{i\beta\phi} + e^{-is\beta\phi} \,). \tag{59}$$

It deserves to be pointed out that the chiral quantization scheme does not simply describe the results of the conventional canonical quantization in the light cone coordinate system because the light cone coordinate x_- (rather than

x^0) will be regarded as "time" variable throughout the chiral quantization scheme we work. Keeping this fact in mind, we see from the definition of the canonical energy-momentum tensor that

$$T_{\mu\nu} \;=\; \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)}\,\partial_\nu \phi \;-\; \mathcal{L} g_{\mu\nu}, \qquad \mu,\ \nu = +,\ -, \tag{60}$$

of which the off-diagonal components are $T_{\pm,\,\mp} \;=\; 1/4\pi(\partial_\mp \phi)^2$. This means that the energy-momentum tensor for system (59) in chiral scheme is not symmetric, then and the angular momentum which generates the Lorentz transformation will not be conserved with respect to x_--"time". Hence $T_{\mu\nu}$ (60) is the non-physical energy-momentum tensor of the system (59) since both the sine-Gordon field theory and the ZMS theory are Lorentz invariant in the usual space-time.

Now we perform the chiral quantization for the considered system (59). In order to do so, let us first describe the classical dynamics in Hamiltonian formalism. According to the light cone time x_-, the canonical momentum conjugate to ϕ is $\pi_\phi = \frac{1}{4\pi}\partial_+\phi$. Thus "velocity" $\partial_-\phi$ can not be solved in terms of momentum π_ϕ, which means that there exist an infinite number of constraints with respect to the spatial points

$$\mathcal{C}(x) = \pi_\phi(x) - \frac{1}{4\pi}\,\partial_+\phi(x) \simeq 0. \tag{61}$$

Recalling the equal-lightcone Poisson bracket, defined by

$$\{\ A(x),\ B(y)\ \}_{x_-=y_-} \;=\; \int d\xi_+\ [\ \frac{\delta A(x)}{\delta\phi(\xi)}\,\frac{\delta B(x)}{\delta\pi_\phi(\xi)} \;-\; \frac{\delta A(x)}{\delta\pi_\phi(\xi)}\,\frac{\delta B(x)}{\delta\phi(\xi)}\ ],$$

we get the Poisson bracket between the constraints

$$\{\ \mathcal{C}(x),\ \mathcal{C}(y)\ \}_{x_-=y_-} \;=\; -1/2\pi\ \partial_+\delta(x_+ - y_+)\ , \tag{62}$$

which turn out to be a kernel rather than a matrix. Following Dirac's quantization prescription for the system involved the second-class constraints [12], the Poisson bracket must be modified for the system under consideration. Note that the inverse of the kernel (62) which is defined by $\int d\xi_+\Delta(x,\xi)\{\mathcal{C}(\xi),\mathcal{C}(y)\}_{\xi_-=y_-}= \delta(x_+ - y_+)$ is

$$\Delta(x,\ y) \;=\; -\pi\ \varepsilon(x_+ - y_+)\ , \tag{63}$$

where $\varepsilon(x)$ denotes $sign(x)$. Then the expected new Poisson bracket, called as Dirac bracket in usual, for the considered system (59) is defined as

$$\begin{aligned}\{\ A(x)\ ,\ B(y)\ \}^*_{x_-=y_-} \;&=\; \{\ A(x),\ B(y)\ \}_{x_-=y_-} \\ &+\ \pi\int\!\!\int d\xi_+ d\zeta_+\ [\ \{\ A(x),\ \mathcal{C}(\xi)\ \}_{x_-=\xi_-}\ \varepsilon(\xi_+ - \zeta_+)\ \{\ \mathcal{C}(\zeta),\ B(y)\ \}_{\zeta_+=y_+}\ ].\end{aligned}$$

In particular, we have

$$\{\ \phi(x),\ \phi(y)\ \}^*_{x_-=y_-} \ = \ -\pi\ \varepsilon(x_+ - y_+)\ . \tag{64}$$

With this Dirac bracket replacing the naive Poisson bracket, we acquire a well-defined Hamiltonian description for the system (59) in light cone coordinate framework. The evolution of our system in light cone "time" x_- will be governed by the above Dirac bracket and the following Hamiltonian quantity

$$H \ = \ \frac{\lambda}{2} \int dx_+ \ (\ e^{i\beta\phi} \ + \ e^{-is\beta\phi}\ )\ . \tag{65}$$

The chiral quantization for system (56) is carried out by regarding the field $\phi(x)$ and its conjugate momentum $\pi_\phi(x)$ as the Hermitian operators in Hilbert space, and postulating these operators obeying the following equal-lightcone commutation relation

$$[\ \phi(x),\ \phi(y)\ ]_{\,x_-=y_-} \ = \ -i\pi\ \varepsilon(x_+ - y_+)\ , \tag{66}$$

instead of Dirac bracket (64). [To save writing we will suppress the subscript $x_- = y_-$.] For the same reason as that indicated in our previous section, the field $\phi(x)$ can not be expanded in terms of the plane wave modes. Nevertheless it can be expanded at an arbitrary given "time", called as x_-, in terms of its Fourier components. In this sense, we divide $\phi(x)$ into its annihilation and creation parts as follows:

$$\phi(x) \ \equiv \ \phi_+(x) + \ \phi_-(x)\ ,$$

where $\phi_+(x)$ and $\phi_-(x)$ are nonlocally dependent upon the positive light-cone coordinate x_+

$$\begin{aligned} \phi_+(x) \ &= \tfrac{1}{2} \int_{-\infty}^{+\infty} dk a_k \ [\ \tfrac{1}{\sqrt{|k|}}\ e^{ikx_+} \ - \ i\sqrt{|k|} \int_{-\infty}^{x_+} dy_+ \ e^{iky_+}\ ]\ , \\ \phi_-(x) \ &= \tfrac{1}{2} \int_{-\infty}^{+\infty} dk a_k^\dagger \ [\ \tfrac{1}{\sqrt{|k|}} e^{-ikx_+} + i\sqrt{|k|} \int_{-\infty}^{x_+} dy_+ e^{-iky_+}\ ]\ . \end{aligned} \tag{67}$$

In accordance with the commutator (66), the annihilation operator a_k and creation operator $a_k^\dagger$ satisfy the standard commutation relations $[\ a_k, a_{k'}^\dagger\ ] = \delta(k-k')$, $[\ a_k, a_{k'}\ ] \ = \ [\ a_k^\dagger, a_{k'}^\dagger\ ] \ = 0$. Hence it follows directly from (67) that

$$\begin{aligned} [\ \phi_+(x), \phi_-(y)\ ] &= \ -\ln\ [\ ik_0\,(x_+ - y_+ - i\epsilon)\ ], \\ [\ \phi_\pm(x), \phi_\pm(y)\ ] &= \ 0\ . \end{aligned} \tag{68}$$

In (68) the factor k_0 ($k_0 \to 0$) comes from introducing an infra-red cut-off $k_0 e^{-\gamma}$ into k-integrals, where γ is the Euler constant.

An important ingredient of the method for non-perturbatively studying the nonlocal conserved currents of Toda-type systems is the concept of so-called vertex operators. In our case such operators are defined as

$$A_a(x) \equiv : e^{ia\phi(x)} : = e^{ia\phi_-(x)} \, e^{ia\phi_+(x)} \, . \qquad (69)$$

It is easily deduced that the above vertex operators satisfy the following operator product arithmetic

$$A_a(x) \, A_b(y) = i^{ab} \, [\, k_0(x_+ - y_+ - i\epsilon) \,]^{ab} \, : e^{ia\phi(x)+ib\phi(y)} : , \qquad (70)$$

and commutation relation

$$[\, (\frac{\partial\phi}{\partial y_+}), \, A_a(x) \,] = 2\pi a A_a(x)\delta(x_+ - y_+) \, . \qquad (71)$$

As a fundamental hypothesis, the Heisenberg equation of motion of vertex operator $A_a(x)$ is assumed as $i\partial_- A_a(x) = [A_a(x), \, H]$ in our chiral- quantization scheme. Due to this equation and (65) and (70), we see that the evolution of $A_a(x)$ in "x_--time" is governed by

$$\begin{aligned} i\partial_- A_a(x) &= [\, A_a(x), H \,] \\ &= i^{a\beta}\frac{\lambda}{2}k_0^{a\beta} \int dy_+ \, [\, (x_+ - y_+ - i\epsilon)^{a\beta} - (y_+ - x_+ - i\epsilon)^{a\beta} \,] \\ &\qquad : e^{ia\phi(x)+i\beta\phi(y)} : \\ &\quad + i^{-as\beta}\frac{\lambda}{2}k_0^{-as\beta} \int dy_+ \, [\, (x_+ - y_+ - i\epsilon)^{-as\beta} - (y_+ - x_+ - i\epsilon)^{-as\beta} \,] \\ &\qquad : e^{ia\phi(x)+i\beta\phi(y)} : \end{aligned} \qquad (72)$$

The equation (72) is very enlightening for constructing nonlocal conserved currents for system (56). Taking account of $\{s = 1/2, \, 1, \, 2\}$ and the mathematical formula

$$\lim_{\epsilon\to 0} [\, (\, x^1 - y^1 - i\epsilon \,)^{-n} - (\, x^1 - y^1 + i\epsilon \,)^{-n} \,] = 2\pi i \, \frac{(-1)^{n-1}}{(n-1)!} \, \delta^{(n-1)}(x^1 - y^1), \qquad (73)$$

$$n = 1, 2, 3, \cdots$$

we find out that Eq.(72) will become into the equation of current conservation

$$\partial_- j_+(x) + \partial_+ j_-(x) = 0 \, ,$$

when the parameter a is taken to be $-2/\beta$ or $2/s\beta$. In such a way we can obtain two nonlocal conserved currents altogether for sine-Gordon and ZMS models:

$$\left\{ \begin{array}{l} j_+^{(0)}(x) = A_{-2/\beta}(x) \\ j_-^{(0)}(x) = -\frac{\pi\lambda}{k_0^2}\frac{\beta^2}{2-\beta^2}A_{\beta-2/\beta}(x) \ , \end{array} \right. \tag{74}$$

and

$$\left\{ \begin{array}{l} j_+^{(1)}(x) = A_{2/s\beta}(x) \\ j_-^{(1)}(x) = -\frac{\pi\lambda}{k_0^2}\frac{(s\beta)^2}{2-(s\beta)^2}A_{2/s\beta-s\beta}(x) \ . \end{array} \right. \tag{75}$$

These conserved nonlocal currents do exist for all real coupling constant β on which the system (56) is well defined, which is just the reason why we develop non-perturbative method in chiral quantization scheme. As a price the above nonlocal currents have no longer the Lorentz covariance, relying on the fact that the action (56) does not possess the "Lorentz" invariance in view of the light cone coordinate x_- being "time".

It is worthwhile to stress that although there are only two nonlocal conserved currents acquired from the non-perturbative scheme based on the chiral quantization, rather than four currents as those appearing in perturbed-CFT framework, there is no missing of the main physical information and the quantum integrability of the system (56) can be displayed still. In order to make this argument transparent, let us focus our attention to the case of $s = 2$, *i. e.* the ZMS model firstly. Following (74) and (75), the two nonlocal charges of this system read

$$\begin{array}{rclcl} Q_0 & = & \int dx_+ \, j_+^{(0)}(x) & = & \int dx_+ \ :\exp(-i\frac{2}{\beta}\phi(x)): \\ Q_1 & = & \int dx_+ \, j_+^{(1)}(x) & = & \int dx_+ \ :\exp(i\frac{1}{\beta}\phi(x)): \end{array} \tag{76}$$

Moreover, there are two conserved topological charges T_0 and T_1 for the ZMS system as well

$$T_1 \equiv -\frac{1}{2}T_0 \equiv \frac{\beta}{\pi}\int dx_+ \, (\frac{\partial\phi}{\partial x_+}) \ . \tag{77}$$

Then it is a consequence of commutator (71) that the nonlocal charges defined above are subject to the algebra

$$[\, T_i, \, Q_j \,] \ = \ a_{ij} \, Q_j \qquad i, \, j \ = 0, \, 1 \ , \tag{78}$$

where

$$[\, a_{ij} \,] = \left[\begin{array}{cc} 8 & -4 \\ -4 & 2 \end{array} \right]$$

This tell us that the conserved charges frame the affine Lie algebra $A_2^{(2)}$ with zero center, which is a subalgebra of the quantum loop algebra $A_{q2}^{(2)}$ the symmetric algebra of ZMS system as well [8]. To confirm that the dominant data about the physical S-matrix of the ZMS solitons are preserved, we have to define the soliton fields besides. These fields will create the ZMS solitons when they act on the vacuum. In the procedure of constructing such fields, one should take the following requirements in mind: since the fundamental representation of the zero-center $A_2^{(2)}$ algebra is three dimensional, the fundamental soliton fields must appear as a triplet and carry topological charges $-2,\ 0,\ 2$ respectively. Such fields are found to be

$$\begin{aligned} \Psi_\pm(x) &= :\exp\left(\pm i\tfrac{1}{\beta}\phi(x)\right): \\ \Psi_0(x) &= :\partial_+ \exp\left(\ i\mathbf{a}\phi(x)\ \right)::\exp\left(\ -i\mathbf{a}\phi(x)\ \right): \end{aligned} \tag{79}$$

where $\mathbf{a}$ is an arbitrary real constant. It is a trivial thing to check that soliton fields (79) turn out to be the eigenvectors of the operator T_1 with the eigenvalues ± 2, 0 respectively

$$[\ T_1,\ \Psi_\sigma(x)\ ] \ = \ \tau_1(\sigma)\Psi_\sigma(x)\ , \qquad \sigma = \ +,\ -,\ 0, \tag{80}$$

where $\tau_1(+) = 2$, $\tau_1(-) = -2$ and $\tau_1(0) = 0$. Thus, by acting with Q_1 on these soliton fields one will find fields with topological charge τ_1 increased by 2 with respect to the initial ones. In fact, the Wick product formula (70) will result in a set of important braiding relations when one acts with the nonlocal conserved current components $j_+^{(0)}$ and $j_+^{(1)}$ on the soliton fields, given by

$$\begin{aligned} j_+^{(0)}(x)\ \Psi_\sigma(y) &= q^{\tau_0(\sigma)}\ \Psi_\sigma(y)\ j_+^{(0)}(x) \\ j_+^{(1)}(x)\ \Psi_\sigma(y) &= q^{\tau_1(\sigma)}\ \Psi_\sigma(y)\ j_+^{(1)}(x)\ , \end{aligned} \qquad \text{for } x_+ < y_+\ , \tag{81}$$

where $\tau_0(\sigma) \equiv -2\tau_1(\sigma)$ and $q \equiv \exp\left(\ -i\pi/2\beta^2\ \right)$. These braiding relations induce the following comultiplications for the conserved charges

$$\begin{aligned} \Delta(\ Q_i\ ) &= \ Q_i \otimes \mathbf{1}\ +\ q^{T_i} \otimes Q_i \\ \Delta(\ T_i\ ) &= \ T_i \otimes \mathbf{1}\ +\ \mathbf{1} \otimes T_i \end{aligned} \tag{82}$$

by the charges acting on the tensor products of two soliton fields. In (82) the second relation is deduced from the additivity of the topological charges T_i.

The commutators (78) tied with comultiplications (82) could set up the bridge between the charge algebra and the infinite dimensional 'Borel' subalgebra of the quantum algebra $A_{q2}^{(2)}$. It is not difficult to see that there is an isomorphism

$$Q_i \ = \ E_i q^{\frac{H_i}{2}}\ , \qquad T_i \ = \ H_i \tag{83}$$

between the charges and the generators of $A_{q2}^{(2)}$. In (83), H_i and E_i $i = 0,\ 1$ form the Chevalley basis of such a 'Borel' subalgebra, which can explicitly be expressed as the 3×3 matrices

$$H_0 = -4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad H_1 = 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$E_0 = 2\lambda\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad E_1 = 2\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

in the fundamental representation of $A_{q2}^{(2)}$. Note that an arbitrary λ-parameter appears in the generator E_0 as the loop parameter of this zero-center $A_{q2}^{(2)}$. With the generators H_i and E_i, the comultiplication laws (82) can alternately be recast as

$$\begin{aligned} \Delta(\ H_i\ ) &= H_i \otimes \mathbf{1} + \mathbf{1} \otimes H_i \\ \Delta(\ E_i\ ) &= E_i \otimes q_i^{-\frac{H_i}{2}} + q_i^{\frac{H_i}{2}} \otimes E_i \end{aligned} \tag{84}$$

where $q_i \equiv q^{a_{ii}/2}$. Since the above 'Borel' subalgebra has non-trivially carried the spectrum parameter λ, it describes an infinite dimensional symmetry of the ZMS model. As a result, the physical S-matrix of the two-soliton states must be commutative with the comultiplications of the generators H_i and E_i

$$[\ S,\ \Delta(\ H_i\ )\ ] = [\ S,\ \Delta(\ E_i\ )\ ] = 0. \tag{85}$$

Compared with the corresponding result obtained in perturbed-CFT approach [8], the commutators $[\ S,\ \Delta(\ F_i\ )\ ] = 0$, $(i = 0,\ 1)$ are suppressed here because there do only exist two nonlocal conserved charges in our scheme. Nevertheless this outcome provides the sufficient data for demonstrating the integrability of ZMS model. In terms of the 'proposition 2' proved by Jimbo in Ref.[12], a solution of the second equation in (85) is also the solution of the first one in (85), as well as that of $[\ S,\ \Delta(\ F_i\ )\ ] = 0$ $(i = 0,\ 1)$. This solution is unique, which is proportional to the well-known Izergin-Korepin R-matrix [13]. In view of this analysis, we see that the S-matrix of the ZMS model determined by Eq.(85) well be almost the same as that obtained by Efthimiou [8] except S-matrix will be associated with the so-called homogeneous gradation of $A_{q2}^{(2)}$, rather than with the spin gradation.

Secondly, we discuss the application of our method to sine-Gordon model for completeness. In this case, the non-perturbatively defined nonlocal charges and the corresponding topological charge are written as

$$
\begin{aligned}
Q_0 &= \int dx_+ \, j_+^{(0)}(x) = \int dx_+ \; :\exp(-i\tfrac{2}{\beta}\phi(x)): \\
Q_1 &= \int dx_+ \, j_+^{(1)}(x) = \int dx_+ \; :\exp(i\tfrac{2}{\beta}\phi(x)): \\
T &= \tfrac{\beta}{2\pi} \int dx_+ \, (\tfrac{\partial\phi}{\partial x_+}) \, ,
\end{aligned}
\tag{86}
$$

which gives

$$
[\, T, \, Q_0 \,] = -2Q_0, \qquad [\, T, \, Q_1 \,] = 2Q_1 \, . \tag{87}
$$

Corresponding to this commutator algebra, the fundamental soliton fields should be among the families of operators with topological charge ± 1: $[\, T, \, \Psi_\pm(x) \,] = \pm\Psi_\pm(x)$. These fields are defined as $\Psi_\pm(x) \; = : \exp\,(\pm i\frac{1}{\beta}\phi(x)\,)$:, which possess the following braiding properties with the nonlocal currents

$$
\begin{aligned}
j_+^{(0)}(x) \, \Psi_\pm(y) &= q^{\mp 1} \, \Psi_\pm(y) \, j_+^{(0)}(x) \\
j_+^{(1)}(x) \, \Psi_\pm(y) &= q^{\pm 1} \, \Psi_\pm(y) \, j_+^{(1)}(x) \, ,
\end{aligned}
\qquad \text{for } x_+ < y_+ \, , \tag{88}
$$

where $q \equiv e^{-2\pi i/\beta^2}$. Therefore, the comultiplication of the charges exhibits

$$
\begin{aligned}
\Delta(\, Q_0 \,) &= Q_0 \otimes \mathbf{1} \, + \, q^T \otimes Q_0 \\
\Delta(\, Q_1 \,) &= Q_1 \otimes \mathbf{1} \, + \, q^{-T} \otimes Q_1 \\
\Delta(\, T \,) &= T \otimes \mathbf{1} \, + \, \mathbf{1} \otimes T \, .
\end{aligned}
\tag{89}
$$

Associating with this comultiplication, we see that (87) is actually an infinite dimensional subalgebra of the q-deformation $\widetilde{sl_q(2)}$ of the $\widetilde{sl(2)}$ Kac-Moody algebra. Let $E_i, \; H_i \; (i = 0, \, 1)$ denote the Chevalley basis for this centerless quantum algebra, we have

$$
Q_i \; = \; E_i \, q^{\frac{H_i}{2}} \;\; (i = 0, \, 1), \qquad T \; = \; - \, H_0 \; = \; H_1 \, , \tag{90}
$$

This is in fact the isomorphic relation between the conserved charges and the generators of quantum loop algebra $\widetilde{sl_q(2)}$ (or $A_{q1}^{(1)}$ with zero center). Along the lines of the discussions below Eq.(84), one can see that the S-matrix determined by means of the symmetric algebra (87) and (89) is indeed in accord with that obtained by Bernard and LeClair in the perturbed-CFT framework [3].

V. Remark.

In summary, we have established a non-perturbative framework for nonlocal conserved currents in sine-Gordon and ZMS models based on the chiral quantization. Although sine-Gordon and ZMS models, considered as Hamiltonian systems in light cone coordinates, are quite dissimilar to those in laboratory coordinates, the equivalent amounts of information could be obtained from the either. This fact was noticed in Ref.[14] earlier at the level of soliton solutions of classical sine-Gordon equation. Now we have got the same conclusion at the level of multi-soliton S-matrix for quantum ZMS and sine-Gordon fields. Opposed to appearance of four nonlocal charges from the perturbed-CFT in laboratory coordinate, in our scheme only two nonlocal charges arise and they are not Lorentz covariant quantities. Fortunately, such two charges together with the topological charges generate infinite dimensional subalgebras of the quantum algebras $A_{q2}^{(2)}$ and $A_{q1}^{(1)}$, respectively for ZMS and sine-Gordon systems. In addition, these charges satisfy some nontrivial comultiplication laws which turn out to be the part ingredients of algebras $A_{q2}^{(2)}$ and $A_{q2}^{(2)}$. The soliton S-matrices obtained in this way are evidently in agreement with those given by BL for sine-Gordon system and Efthimiou for ZMS model. Relying on these facts, we have reason regarding the present method here as a reasonable candidate of the non-perturbative counterpart of the Bernard and LeClair's approach.

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Hawking evaporation for the 2 + 1 dimensional radiating black hole

Cai Ronggen, Zhang Yijun, Su Rukeng

Department of Physics, Fudan University, Shanghai 200433

CCAST (World Laboratory) P. O. Box 8730, Beijing 100080, China

Abstract

The Hawking evaporation for the 2+1 dimensional radiating black hole is investigated. The Hawking temperature of this radiating black hole is given.

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Since the discovery of the black hole solution to $2+1$ dimensional (3D) general relativity with a negative cosmological constant by Banados, Teitelboin and Zanelli (BTZ) [1], much effort has been devoted to studying the properties of this black hole, including geometry [2], gauge formulation [3], supersymmetry [4], back reaction [5], thermodynamics and statistical mechanics [6]. More recently, Husain [7] has obtained a 3D radiating black hole solution by adding a pure radiational field. His metric is a counterpart of 4D Vaidya metric

$$ds^2 = -(-m(v) + \frac{r^2}{l^2})dv^2 + 2dvdr + r^2 d\varphi^2 \tag{1}$$

where l^{-2} is the negative cosmological constant and $m(v)$ the mass. The objective of this paper is to use the Husain's metric to investigate the Hawking evaporation of the 3D black hole. A remarkable property of black hole is that it can radiate quantum particles like a blackbody with a temperature proportional to its surface gravity. Hereafter we will use the method which was introduced by Damour and Ruffini [8] for discussing the static and stationary black holes first and generalized by Zhao et. al [9] for the evolutional black hole to calculate the Hawking temperature of the radiating 3D black hole.

To specify the location of the horizon r_H, using the condition

$$g^{\mu\nu}\frac{\partial f}{\partial x^\mu}\frac{\partial f}{\partial x^\nu} = 0 \tag{2}$$

and considering the symmetry, we find the equation of r_H from eqs.(1) and (2) as

$$r_H^2 - ml^2 - 2l^2\dot{r}_H = 0\ . \tag{3}$$

The horizon's location is

$$r_H = l\sqrt{m + 2\dot{r}_H} \ , \tag{4}$$

here a dot over r_H denotes the derivative with respect to v.

Considering a massless scalar field in the black hole background

$$g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 0 \tag{5}$$

and setting $\phi = R(v,r)e^{in\varphi}$, $(n = 0, \ \pm 1, \ \pm 2, ...)$, we find $R(v,r)$ satisfies

$$2r\frac{\partial^2 R}{\partial v \partial r} + \frac{\partial R}{\partial v} + r(-m + \frac{r^2}{l^2})\frac{\partial^2 R}{\partial r^2} + (-m + \frac{3r^2}{l^2})\frac{\partial R}{\partial r} - \frac{n^2 R}{r} = 0 \ . \tag{6}$$

Introducing the generalized tortoise coordinate transformation [9],

$$\begin{cases} r_* = r + \dfrac{1}{2\kappa}\ln(r - r_H) \ , \\ v_* = v \end{cases} \tag{7}$$

where κ denotes the surface gravity, we can rewrite eq.(6) as

$$\begin{aligned} &\left[(-m + \frac{r^2}{l^2})(1 + \frac{1}{2\kappa(r - r_H)}) - \frac{2\dot{r}_H}{2\kappa(r - r_H)}\right]\frac{\partial^2 R}{\partial r_*^2} + 2\frac{\partial^2 R}{\partial v_* \partial r_*} \\ &+ \left[r(1 + \frac{1}{2\kappa(r - r_H)})\right]^{-1}\left\{\left[\frac{r\dot{r}_H}{\kappa(r - r_H)^2} - \frac{\dot{r}_H}{2\kappa(r - r_H)} - \frac{r(-m + r^2/l^{\,2})}{2\kappa(r - r_H)}\right.\right. \\ &\left.\left. +(-m + \frac{3r^2}{l^2})(1 + \frac{1}{2\kappa(r - r_H)})\right]\frac{\partial R}{\partial r_*} + \frac{\partial R}{\partial v_*} - \frac{n^2}{r}R\right\} = 0 \ . \end{aligned} \tag{8}$$

Using the condition that eq.(8) will change to a standard wave equation near the horizon, one has

$$\lim_{r \to r_H} \frac{(-m + \frac{r^2}{l^2})(2\kappa(r - r_H) - 2\dot{r}_H)}{2\kappa(r - r_H)} = 1 \ , \tag{9}$$

i.e.,

$$\kappa \simeq \frac{r_H}{l^2 + ml^2 - r_H^2} \ . \tag{10}$$

Near the horizon, eq.(8) has the standard solutions of the ingoing wave $R^{\rm in}$ and outgoing wave $R^{\rm out}$, respectively,

$$R^{\rm in} = e^{-i\omega v} \ , \tag{11}$$

$$R^{\rm out} = e^{2i\omega r_*} e^{i\omega v} \ . \tag{12}$$

At the horizon, $R^{\rm out}$ is singular. We must continue $R^{\rm out}$ to the inside of the horizon analytically and get

$$R^{\rm out} = e^{-i\omega v} e^{2i\omega r_*} \theta(r - r_H) + e^{\pi\omega/\kappa} e^{-i\omega v} e^{2i\omega r_*} \theta(r_H - r) \ . \tag{13}$$

According to ref. [8] and with the help of eq.(11)-(13), we obtain the particle spectrum,

$$< N_\omega > = \frac{\Gamma_\omega}{e^{\omega/T} - 1} \tag{14}$$

where

$$T = \frac{\kappa}{2\pi} = \frac{r_H}{2\pi(l^2 + ml^2 - r_H^2)} \tag{15}$$

is the Hawking temperature of the radiating 3D black hole, r_H is determined by eq.(3) or eq.(4).

In order to compare our result with that given by ref. [1] and [5], let us consider the condition: $\dot{m}$ is small. In this case eq.(15) reduces to

$$T \simeq \frac{\sqrt{m}}{2\pi l}(1 + \frac{l}{\sqrt{m}}\dot{m} + \frac{l}{2m}\dot{m}) \ . \tag{16}$$

If $\dot{m} = 0$, eq.(16) reduces to

$$T = \frac{\sqrt{m}}{2\pi l} . \tag{17}$$

This is just the Hawking temperature of a 3D BTZ black hole which was first given by BTZ [1].

In summary, by using the method of ref. [8] and [9], we have obtained the Hawking temperature of 3D radiating black hole. When $\dot{m}$ is zero, our result reduces to that given by ref. [1]. When $\dot{m}$ is small, an approximate expression for the Hawking temperature is given.

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Thermo Field Theory and Chiral $\sigma - \omega$ Model

Su Ru-Keng, Gao Song, Zhang Yi-Jun

Department of physics, Fudan University, Shanghai 200433, China

China Center of Advanced Science and Technology (World Laboratory),

P. O. Box 8730, Beijing 100080, China

Abstract

In the framework of thermo field dynamics, the temperature and density effects on nucleon-nucleon interaction for chiral $\sigma - \omega$ model are investigated. The effective masses of nucleon, pion, σ-meson and ω-meson at finite-temperature and -density are calculated. We have found that the potential well of the nucleon-nucleon interaction becomes shallow as the density increases. At a critical density ρ_c, the potential well disappears and the nuclear matter becomes a hadron gas.

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I. INTRODUCTION

It is generally believed that in future relativistic heavy ion collision experiments, a very hot and dense hadronic matter and/or quark gluon plasma will be produced. The behaviour of a nuclear matter under high temperature and high density condition has attracted much attention nowadays [1-12]. Since the nucleon-nucleon (NN) interaction plays a very important role in nuclear physics and hadron physics, it is of interest to study the temperature and density effects on NN interaction.

In a previous paper [12], employing the imaginary-time Green's function method, we have extended the chiral $\sigma - \omega$ model to finite temperature and studied the temperature dependence of NN interaction. We have found two interesting results:

1. Under one-loop approximation, the potential well of the NN interaction becomes shallow as the temperature increases. The temperature plays a "repulsive" role for NN interaction.

2. At a critical temperature T_C, the potential well disappears and no bound states can exist when $T \geq T_C$.

Since the extreme condition including not only high temperature effect, but also high *density* effect, it is of interest to extend our study to finite density regions. In particular, if we want to extend our investigation from NN system to nuclear matter, we must consider the density effect. To extend our previous study of chiral $\sigma - \omega$ model to finite density is the objective of the present paper.

As a working framework, we will employ the thermo field dynamics (TFD) theory which was suggested by Umezawa et al. many years ago [13,14]. TFD is a real-time formalism of finite temperature quantum field theory. It can be proved that under one-loop approximation, the self-energy given by TFD will equal to that by real-time Green's function method [10,14]. In ref.[11], we have employed TFD to study the NNω interaction at finite-temperature and -density.

There are nucleon, pion, σ-meson and ω-meson in the chiral $\sigma - \omega$ model. The masses of nucleon and mesons will be modified by the corrections of the temperature and density dependent self-energy in hot and dense medium. After summing the self-energy Feynman diagrams (Fig. 1), we can obtain effective masses of the corresponding particles. The NN potential can be obtained by calculating the scattering amplitude diagrams [7-12] in which the propagators of mesons have been corrected by the corresponding effective masses. All these calculations can be done in the framework of TFD.

As in ref. [12], we will calculate the self-energy under one-loop approximation. We will prove that the effective masses of mesons and nucleon are almost linear dependent on density for a fixed temperature. The NN potential well will becomes shallow as the density increases. To a critical density $\rho_c = 0.41 \text{ fm}^{-3}$ ($T = 0$), the NN potential well disappears, the bound state of nucleon-nucleon will be dissolved, and the phase transition of nuclear matter to hadron gas will take place. The density plays the same role as the temperature under the extreme condition.

The organization of this paper is as follows. We will summarize the Feynman propagators of mesons and nucleon in TFD in section II. In section III, we will investigate the propagators and the effective masses of mesons and nucleon in a hot and dense medium. In section IV, the density effects on NN interaction are calculated. Our numerical results and discussions will be summarized in the last section.

II. THE FEYNMAN PROPAGATORS IN THERMO FIELD DYNAMICS

In order to investigate the behaviour of chiral $\sigma-\omega$ model in hot and dense nuclear matter, we employ the thermo field dynamics, which is a powerful framework for describing many-body systems at finite-temperature and -density. In this theory, the ground state is identified as the thermal vacuum state which depends on temperature. All statistical averages are calculated as the thermal vacuum expectation values. In this framework, almost all operator formalisms in quantum field theory at zero temperature can be extended to finite temperature and density directly [13,14].

In TFD, each dynamical degree of freedom has double components and they lead to 2×2 matrix propagator. The propagator of fermion field in TFD is

$$iS_F(k) = \begin{pmatrix} i\Delta^{11}(k) & i\Delta^{12}(k) \\ i\Delta^{21}(k) & i\Delta^{22}(k) \end{pmatrix} \tag{1}$$

where

$$\begin{aligned} \Delta^{11}(k) &= (\not{k} + m)\left\{\frac{1}{k^2 - m^2 + i\epsilon} + 2\pi i[\theta(k_0)n_F(k) + \theta(-k_0)\overline{n}_F(k)]\delta(k^2 - m^2)\right\} \\ &= -\Delta^{22*}(k) , \end{aligned} \tag{2a}$$

$$\Delta^{12}(k) = 2\pi i e^{-\beta\mu/2}(\not{k}+m)[\theta(k_0)e^{\beta(|k_0|-\mu)}n_F(k) - \theta(-k_0)e^{\beta(|k_0|+\mu)}\overline{n}_F(k)]\delta(k^2-m^2)$$
$$= -e^{-\beta\mu/2}\Delta^{21}(k) \tag{2b}$$

where $\theta(k_0)$ is step function, m is the mass of fermion, $n_F(k)$ and $\overline{n}_F(k)$ are, respectively, fermion and antifermion distributive function

$$n_F(k) = \frac{1}{e^{\beta(|k_0|+\mu)}+1}, \quad \overline{n}_F(k) = \frac{1}{e^{\beta(|k_0|-\mu)}+1}, \tag{3}$$

and $\beta = T^{-1}$ is the inverse temperature, where we have chosen unit k_B=1. The chemical potential μ is determined by

$$\rho = \frac{\gamma}{(2\pi)^3}\int \mathrm{d}^3k[n_F(k) - \overline{n}_F(k)] \tag{4}$$

where γ is the spin-isospin degeneracy, and for nuclear matter, $\gamma = (2s+1)(2\tau+1) = 4$.

The propagator of scalar field with mass m_B in TFD is

$$iD(k) = \begin{pmatrix} iD^{11}(k) & iD^{12}(k) \\ iD^{21}(k) & iD^{22}(k) \end{pmatrix} \tag{5}$$

where

$$D^{11}(k) = \frac{1}{k^2 - m_B^2 + i\epsilon} - 2\pi i n_B(k)\delta(k^2 - m_B^2) = -D^{22*}(k)\ , \tag{6a}$$

$$D^{12}(k) = -2\pi i\delta(k^2 - m_B^2)e^{\beta|k_0|/2}n_B(k) = D^{21}(k)\ , \tag{6b}$$

and n_B is the Boson distributive function

$$n_B(k) = \frac{1}{e^{\beta|k_0|}-1}\ . \tag{7}$$

For the massive vector boson field, for example, the ω-meson, the propagator can be obtained from scalar boson by adding a prefactor $(-g_{\mu\nu}+k_\mu k_\nu/m_\omega^2)$ to each terms of Eqs. (6a) and (6b).

We would like to point out that the topological structure of the Feynman diagrams in TFD is the same as that of the quantum field theory at zero temperature, and then their contribution can be separated into T=0 and $T \neq 0$ two parts.

III. THE CHIRAL $\sigma - \omega$ MODEL AT FINITE-TEMPERATURE AND -DENSITY

The Lagrangian density of chiral $\sigma - \omega$ model is [4,12]

$$\begin{aligned} \mathcal{L} &= \overline{\psi}[i\,\partial\!\!\!/ - g(\phi + i\vec{\pi}\cdot\vec{\tau}\gamma_5) - g_\omega\gamma_\mu\omega^\mu]\psi + \frac{1}{2}[(\partial_\mu\phi)^2 + (\partial_\mu\vec{\pi})^2] \\ &\quad -\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}\mu^2(\phi^2+\vec{\pi}^2) - \frac{\lambda^2}{4}(\phi^2+\vec{\pi}^2)^2 + \frac{1}{2}m_\omega^2\omega_\mu\omega^\mu + \mathcal{L}_1 \ , \qquad (8) \\ \mathcal{L}_1 &= c\phi \ , \end{aligned}$$

where ψ, ϕ, $\vec{\pi}$ and ω^μ are the fields of nucleon, σ-meson, pion and ω-meson respectively, $G_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu$, and g, g_ω, λ are the coupling constants, $\mathcal{L}_1$ is the chiral symmetry breaking term, which leads to partially conserved axial vector current (PCAC), and $\mu^2 > 0$. Since this model has spontaneous symmetry breaking, the expectation value of ϕ is non-zero. Shifting the field ϕ as

$$\phi = \sigma + v \ , \qquad (9)$$

and taking $<0|\phi|0>= v$, $<0|\sigma|0>=0$, $<0|\vec{\pi}|0>=0$, we have

$$\sigma^2 + \vec{\pi}^2 = \mu^2/\lambda^2 = v^2 \ . \tag{10}$$

Substituting Eqs. (9) and (10) into (8), we get

$$\begin{aligned}
\mathcal{L}' &= \overline{\psi}[i\,\partial\!\!\!/ - gv - g(\sigma + i\vec{\pi}\cdot\vec{\tau}\gamma_5) - g_\omega\gamma_\mu\omega^\mu]\psi + \frac{1}{2}[(\partial_\mu\sigma)^2 - 2\lambda^2v^2\sigma^2] \\
&+\frac{1}{2}(\partial_\mu\vec{\pi})^2 - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} - \frac{\lambda^2}{4}(\sigma^2+\vec{\pi}^2)^2 - \lambda^2 v\sigma(\sigma^2+\vec{\pi}^2) \\
&+\frac{1}{2}m_\omega^2\omega_\mu\omega^\mu + \frac{\lambda^2}{4}v^4 + \mathcal{L}_1' \ , \qquad (11) \\
\mathcal{L}_1' &= c(\sigma + v) \ .
\end{aligned}$$

This model can be renormalized and the detail procedure of the renormalization has been discussed in refs. [4,12]. Following [4,12], we choose the masses of nucleon and mesons as free parameters and define the coupling constants g and λ as

$$\begin{aligned}
m_N &= gv \ , \\
m_\sigma^2 - m_\pi^2 &= 2\lambda^2 v^2
\end{aligned} \tag{12}$$

where m_N, m_σ, m_π and m_ω are the physical masses of nucleon, σ, π and ω mesons respectively.

In order to extend the chiral $\sigma - \omega$ model to finite- temperature and -density, we will calculate the self-energies of mesons and nucleon at finite-temperature and -density by using TFD method under one loop-approximation.

In the framework of TFD, the propagator has 2×2 matrix structure, but only the 1-1 component contributes to the real part of the self-energy. Using the Dyson's equation in TFD [14], the full 1-1 component propagators of scalar mesons can be found as

$$D_\alpha(q) = \frac{1}{q^2 - m_\alpha^2 - \Sigma_\alpha + i\epsilon}\,, \qquad (\alpha = \pi\,, \sigma) \tag{13}$$

where Σ_α ($\alpha = \pi$, σ) are the self-energies of the pion and σ-meson at finite-temperature and -density under one-loop approximation. According to the calculation rules [14] of TFD, the self-energies of pion and σ-meson illustrated in Fig. 1 can be written as

$$\Sigma_\pi = ig^2 \int \frac{d^4k}{(2\pi)^4} Tr[\gamma_5 \tau_i \Delta^{11}(k) \gamma_5 \tau_j \Delta^{11}(k-q)] + 4i\lambda^4 v^2 \int \frac{d^4k}{(2\pi)^4} D_\sigma^{11}(k) D_\pi^{11}(k-q)$$
$$+ i\lambda^2 \int \frac{d^4k}{(2\pi)^4} [D_\sigma^{11}(k) + 5D_\pi^{11}(k)]\,, \tag{14}$$

$$\Sigma_\sigma = -ig^2 \int \frac{d^4k}{(2\pi)^4} Tr[\Delta^{11}(k)\Delta^{11}(k-q)] + 3i\lambda^2 \int \frac{d^4k}{(2\pi)^4} [D_\sigma^{11}(k) + D_\pi^{11}(k)]$$
$$+ 6i\lambda^4 v^2 \int \frac{d^4k}{(2\pi)^4} [3D_\sigma^{11}(k) D_\sigma^{11}(k-q) + D_\pi^{11}(k) D_\pi^{11}(k-q)]\,. \tag{15}$$

The treatment of the vector ω-meson self-energy has a slight difference from that of the scalar meson. Under the condition of baryon current conservation [4] $q_\mu \Sigma_\omega^{\mu\nu} = 0$, the general expression of the vector ω-meson self-energy at finite- temperature and -density can be expressed as [15,16]

$$\Sigma_\omega^{\mu\nu} = \Sigma_L^\omega P_L^{\mu\nu} + \Sigma_T^\omega P_T^{\mu\nu} \tag{16}$$

where

$$\begin{aligned} \Sigma_L^\omega &= -\frac{q^2}{\vec{q}^{\,2}} u_\mu u_\nu \Sigma_\omega^{\mu\nu}\,, \\ \Sigma_T^\omega &= \frac{1}{2}\left(\frac{q^2}{\vec{q}^{\,2}} u_\mu u_\nu - g_{\mu\nu}\right) \Sigma_\omega^{\mu\nu}\,, \end{aligned} \tag{17}$$

and u_μ is the four-velocity of the medium. In the rest frame of the medium, $u_\mu = (1, \vec{0})$. $P_L^{\mu\nu}$ and $P_T^{\mu\nu}$ are the projection tensors defined as

$$
\begin{aligned}
&P_T^{00} = P_T^{0i} = P_T^{i0} = 0 \, , \\
&P_T^{ij} = \delta^{ij} - q^i q^j / \vec{q}^{\,2} \, , \\
&P_T^{\mu\nu} + P_L^{\mu\nu} = -g^{\mu\nu} + q^\mu q^\nu / q^2 \, .
\end{aligned}
\tag{18}
$$

Therefore, the full propagator of ω-meson in the hot and dense medium can be obtained as

$$
D_\omega^{\mu\nu}(q) = -\frac{P_L^{\mu\nu}}{q^2 - m_\omega^2 - \Sigma_L^\omega} - \frac{P_T^{\mu\nu}}{q^2 - m_\omega^2 - \Sigma_T^\omega} - \frac{q^\mu q^\nu}{m_\omega^2 q^2} \, , \tag{19}
$$

and the self-energy of ω-meson at finite-temperature and -density in one loop-approximation be

$$
\Sigma_\omega^{\mu\nu} = -i g_\omega^2 \int \frac{d^4 k}{(2\pi)^4} Tr[\gamma^\mu \Delta^{11}(k) \gamma^\nu \Delta^{11}(k-q)] \, . \tag{20}
$$

Substituting the TFD Feynman propagators into Eqs. (14), (15) and (20) respectively, we can prove that the self-energies of π, σ and ω mesons can be divided into two parts: the vacuum part which is independent of temperature and density and the part of finite- temperature and -density. The vacuum part is divergent, and can be absorbed in the renormalized masses by renormalization [4,12]. Hereafter we focus our attention on the part of finite-temperature and -density only.

In the limit $q_0 = 0$, $\vec{q} \to 0$, the temperature and density dependent part of self-energy of pion $\Sigma_\pi(q_0, \vec{q}; T, \rho)$ can be obtained

$$
\Sigma_\pi(0, \vec{q} \to 0; T, \rho) = \frac{2g^2}{\pi^2} I_2 + \frac{3\lambda^2}{2\pi^2}(I_2^\pi + I_2^\sigma) \tag{21}
$$

where I_2, I_2^π and I_2^σ are the convergent integrals, their expressions are shown in the appendix.

Therefore, we can define the "effective" mass of pion at finite-temperature and -density from Eq. (13) in the above limit [7,8,11,12]

$$\begin{aligned} m_\pi^* &= [m_\pi^2 + \Sigma_\pi(0, \vec{q} \to 0; T, \rho)]^{1/2} \\ &= \left[m_\pi^2 + \frac{2g^2}{\pi^2} I_2 + \frac{3\lambda^2}{2\pi^2}(I_2^\pi + I_2^\sigma)\right]^{1/2} . \end{aligned} \tag{22}$$

Similarly, the effective mass of σ-meson is

$$\begin{aligned} m_\sigma^* &= [m_\sigma^2 + \Sigma_\sigma(0, \vec{q} \to 0; T, \rho)]^{1/2} \\ &= \left[m_\sigma^2 + \frac{g^2}{\pi^2}(I_2 - m_N^2 I_{3/2}) - \frac{3\lambda^4 v^2}{2\pi^2}(3I_{3/2}^\sigma + I_{3/2}^\pi) + \frac{3\lambda^2}{2\pi^2}(I_2^\pi + I_2^\sigma)\right]^{1/2} . \end{aligned} \tag{23}$$

The convergent integrals $I_{3/2}$, $I_{3/2}^\pi$ and $I_{3/2}^\sigma$ are also given in the appendix. It can easily be seen that the Feynman propagators of the pion and σ-meson in above limit are

$$D_\alpha = \frac{1}{\vec{q}^{\,2} + m_\alpha^{*2}} , \quad (\alpha = \pi, \sigma) . \tag{24}$$

For the vector ω-meson, In the limit $q_0 = 0$, $\vec{q} \to 0$, using Eqs. (17) and (20), we find that the longitudinal and transverse temperature and density dependence self-energies are

$$\begin{aligned} \Sigma_L^\omega(0, \vec{q} \to 0; T, \rho) &= 0 , \\ \Sigma_T^\omega(0, \vec{q} \to 0; T, \rho) &= \frac{g_\omega^2}{2\pi^2}(2I_2 + m_N^2 I_{3/2}) \end{aligned} \tag{25}$$

respectively. By using Eqs. (18) and (19), we obtain that the Feynman propagator of ω-meson in the hot and dense medium are

$$\begin{aligned} D_\omega^{00} &= \frac{-1}{\vec{q}^{\,2} + m_\omega^2} , \quad D_\omega^{i0} = D_\omega^{0i} = 0 , \\ D_\omega^{ij} &= \frac{1}{\vec{q}^{\,2} + m_\omega^{*2}}(\delta^{ij} - q^i q^j / \vec{q}^{\,2}) \end{aligned} \tag{26}$$

where m_ω^* is

$$m_\omega^* = [m_\omega^2 + \Sigma_T^\omega(0, \vec{q} \to 0; T, \rho)]^{1/2} \\ = \left[m_\omega^2 + \frac{g_\omega^2}{\pi^2}(2I_2 + m_N^2 I_{3/2})\right]^{1/2} . \tag{27}$$

The numerical results for the effective masses of π, σ and ω mesons will be given in the last section.

The effective mass of nucleon in the hot and dense medium can also be calculated [3,4,12] under one-loop approximation. We can prove that the effective mass of nucleon is

$$m_N^* = m_N \left\{1 - \frac{g^2}{4\pi^2}\left[\frac{I_2 + 2I_2^\sigma}{m_N^2 - m_\sigma^2} + \frac{8I_2}{m_\sigma^2} + \frac{3(I_2 + 2I_2^\pi)}{m_N^2 - m_\pi^2} + 18\left(1 - \frac{m_\pi^2}{m_\sigma^2}\right) I_2^\pi\right] \right. \\ \left. + \frac{g_\omega^2}{4\pi^2}\left[\frac{3(I_2 + 2I_2^\omega)}{m_N^2 - m_\omega^2} - \frac{I_2}{m_\omega^2}\right]\right\} . \tag{28}$$

IV. NN POTENTIAL IN NUCLEAR MATTER

Using the same procedures as our previous works [7-12], the NN interaction of chiral $\sigma - \omega$ model due to exchange π, σ, and ω mesons at finite-temperature and -density in the coordinate space can be found as

$$V(r) = V_\pi(r) + V_\sigma(r) + V_\omega(r) , \tag{29}$$

$$V_\pi(r) = \frac{g^2}{4\pi}\frac{m_\pi^{*3}}{12m_N^2}[Z(x_\pi)S_{12} + Y(x_\pi)(\vec{\sigma}_1 \cdot \vec{\sigma}_2)](\vec{\tau}_1 \cdot \vec{\tau}_2) , \tag{30}$$

$$V_\sigma(r) = -\frac{g^2}{4\pi}m_\sigma^*\left[\left(1 - \frac{m_\sigma^{*2}}{4m_N^2}\right)Y(x_\sigma) - \frac{m_\sigma^{*2}}{2m_N^2}\vec{S} \cdot \vec{L}\frac{1}{x_\sigma}\frac{\mathrm{d}}{\mathrm{d}x_\sigma}Y(x_\sigma)\right] , \tag{31}$$

$$V_\omega(r) = \frac{g_\omega^2}{4\pi}m_\omega\left[\left(1 + \frac{m_\omega^2}{4m_N^2}\right)Y(x_\omega') + \frac{m_\omega^2}{2m_N^2}\vec{S} \cdot \vec{L}\frac{1}{x_\omega'}\frac{\mathrm{d}}{\mathrm{d}x_\omega'}Y(x_\omega')\right]$$

$$+\frac{g_\omega^2}{4\pi}m_\omega^*\left\{\frac{1}{4}\left(\frac{m_\omega^*}{m_N}\right)^2 Y(x_\omega)+\left(\frac{m_\omega^*}{m_N}\right)^2 \vec{S}\cdot\vec{L}\frac{1}{x_\omega}\frac{\mathrm{d}}{\mathrm{d}x_\omega}Y(x_\omega)\right.$$
$$\left.-\frac{1}{12}\left(\frac{m_\omega^*}{m_N}\right)^2[Z(x_\omega)S_{12}+Y(x_\omega)(\vec{\sigma}_1\cdot\vec{\sigma}_2)]+\frac{1}{4}\left(\frac{m_\omega^*}{m_N}\right)^2 Y(x_\omega)(\vec{\sigma}_1\cdot\vec{\sigma}_2)\right\} \quad (32)$$

where

$$x_\pi = m_\pi^* r\ ,\ x_\sigma = m_\sigma^* r\ ,\ x_\omega = m_\omega^* r\ ,\ x'_\omega = m_\omega r\ ,$$
$$Y(x) = e^{-x}/x\ ,\quad Z(x) = (1+3/x+3/x^2)Y(x)\ , \quad (33)$$
$$S_{12} = \frac{3(\vec{\sigma}_1\cdot\vec{r})(\vec{\sigma}_2\cdot\vec{r})}{r^2} - \vec{\sigma}_1\cdot\vec{\sigma}_2\ ,$$

and σ_i (τ_i) $(i = 1,2)$ are the spin (isospin) of nucleon, $\vec{S} = \frac{1}{2}(\vec{\sigma_1}+\vec{\sigma_2})$, $\vec{L}$ is the angular momentum in coordinate space, and r is the distance between two nucleons.

The numerical results of the NN potential will be given in the next section.

V. RESULTS AND DISCUSSION

The numerical results of the effective masses of nucleon, pion, σ-meson and ω-meson and the NN potential are shown in Figs. 2—4, where we choose the parameters as $m_N = 939$ MeV, $m_\sigma = 600$ MeV, $m_\pi = 139$ MeV, $m_\omega = 783$ MeV, $g = 10$, and $g_\omega^2/4\pi = 10.7$ as in refs. [4] and [12]. Since the temperature dependence of the chiral σ-ω model has been investigated in our previous study in details, hereafter we will discuss the density dependence mainly.

The effective masses of π, σ and ω-mesons change with density at zero temperature are shown in Fig. 2. We see from Fig. 2 that the effective masses of π , σ and ω-mesons all almost linearly depend on density but their slopes are different.

The effective mass of nucleon vs. density for different temperature are shown in Fig. 3a where curves A, B and C refer to $T = 0$, 100, 120 MeV respectively. We see that the slopes of curves A, B and C almost equal each other. It means that the slope of m_N^* vs. ρ curve is independent of temperature. In order to show the temperature and density dependence of m_N^* more transparently, we show m_N^* vs. T curve for different densities in Fig. 3b, where curves A, B and C refer to $\rho = 0$, 0.1, 0.17 fm^{-3} respectively.

The numerical results of the NN potential V(r) for the I=0 and S=1 state are shown in Fig. 4. We show that the NN potential curves of zero temperature for $\rho = 0$ (curve A), 0.3 fm^{-3} (curve B) and 0.41 fm^{-3} (curve C) in Fig. 4 respectively. We see that the potential well of V(r) becomes shallower as the density increases. At a critical density $\rho_c = 0.41$ fm^{-3} ($T = 0$), the potential well disappears. It means that NN bound states will not be exist above ρ_c in nuclear matter. When $\rho > \rho_c$ and/or $T > T_C$ [12], the nuclear matter will become hadron gas. The density as well as the temperature plays the same "repulsive" role in NN interaction. When density and/or temperature increases higher and higher, the attractive interaction between nucleon and nucleon become weaker and weaker. When density approaches to ρ_c and/or temperature to T_C, the attractive interaction disappears. This result is in agreement qualitatively with that given by ref. [8] for one pion exchange potential.

In summary, we would like to point out that the effect of density on NN interaction is important. The density plays the same role as the temperature in nuclear force. Based on TFD, we have obtained the density and temperature dependence of NN potential for the chiral $\sigma - \omega$ model. We have found that the potential well becomes shallower when density increases. When density approaches to the critical density ρ_c, the bound state of nucleons disappears and the nuclear matter to hadron gas phase transition takes place.

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FIGURES

FIG. 1. Feynman diagrams for the self-energies of pion, σ-meson and ω-meson. The full line denotes nucleon, dashed line: pion, dash-dotted line: σ-meson, wavy line: ω-meson.

FIG. 2. The density dependence of effective masses vs. density curves for pion, σ-meson and ω-meson at zero temperature.

FIG. 3. The density and temperature dependence of effective nucleon mass m_N^*. (a) m_N^* vs. density curves for various temperature, A: $T = 0$, B: $T = 100$ MeV, C: $T = 120$ MeV. (b) m_N^* vs. temperature curves for various density, A: $\rho = 0$, B: $\rho = 0.1$ fm^{-3}, C: $\rho = 0.17$ fm^{-3}.

FIG. 4. The NN potential V(r) for various densities at zero temperature. A: $\rho = 0$, B: $\rho = 0.3$ fm^{-3}, C: $\rho = 0.41$ fm^{-3}.

APPENDIX

The integrals in section III are defined as follows

$$I_2 = \int_0^\infty \frac{\mathrm{d}x\ x^2}{\sqrt{x^2+m_N^2}} \left(\frac{1}{\exp(\beta\sqrt{x^2+m_N^2}-\beta\mu)+1} + \frac{1}{\exp(\beta\sqrt{x^2+m_N^2}+\beta\mu)+1} \right) \tag{A1}$$

$$I_{3/2} = \int_0^\infty \frac{\mathrm{d}x\ x^2}{(x^2+m_N^2)^{3/2}} \left(\frac{1}{\exp(\beta\sqrt{x^2+m_N^2}-\beta\mu)+1} + \frac{1}{\exp(\beta\sqrt{x^2+m_N^2}+\beta\mu)+1} \right) \tag{A2}$$

$$I_2^\alpha = \int_0^\infty \frac{\mathrm{d}x\ x^2}{\sqrt{x^2+m_\alpha^2}} \frac{1}{\exp(\beta\sqrt{x^2+m_\alpha^2})-1} \ . \ (\alpha = \pi, \sigma, \omega) \tag{A3}$$

$$I_{3/2}^\alpha = \int_0^\infty \frac{\mathrm{d}x\ x^2}{(x^2+m_\alpha^2)^{3/2}} \frac{1}{\exp(\beta\sqrt{x^2+m_\alpha^2})-1} \ . \ (\alpha = \pi, \sigma) \tag{A4}$$

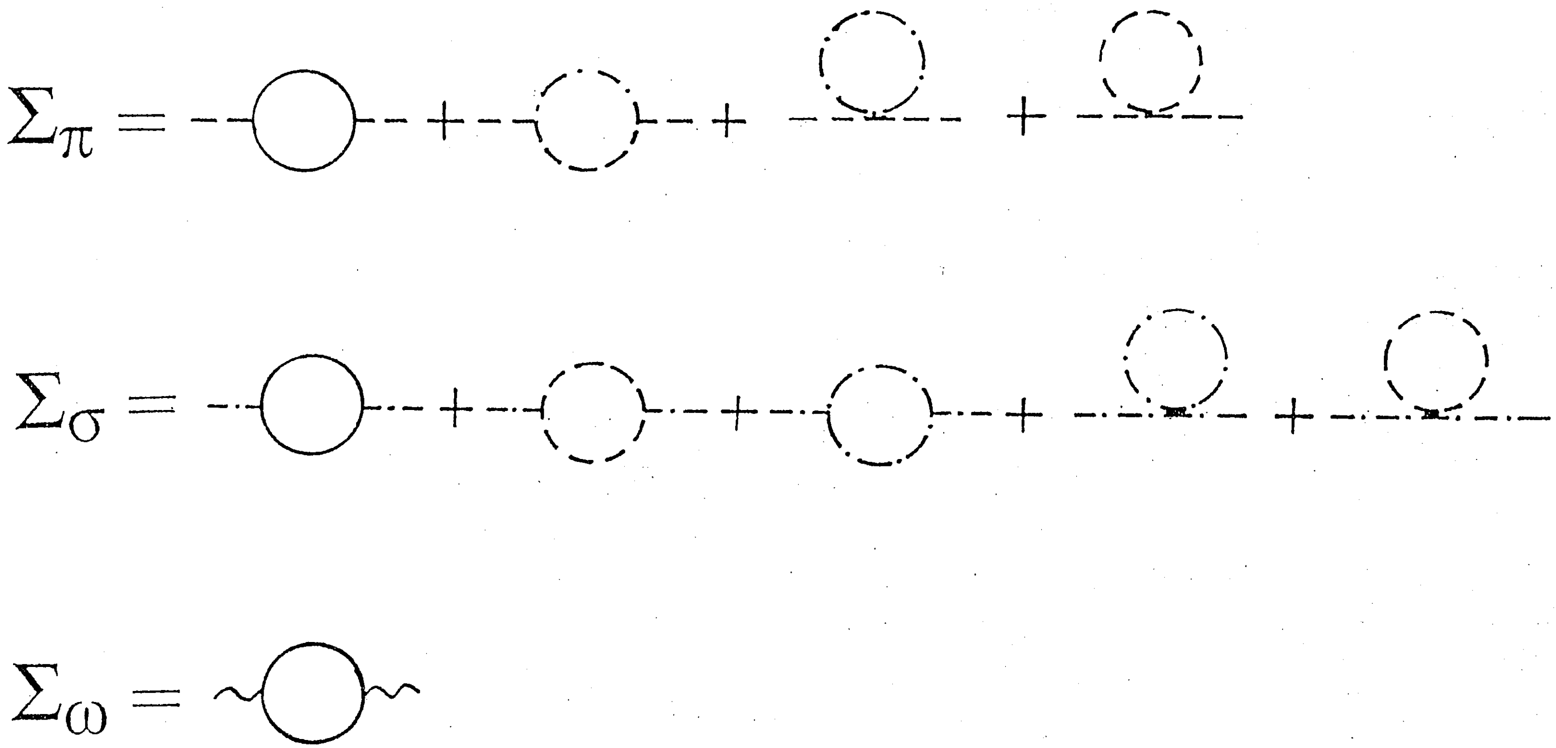
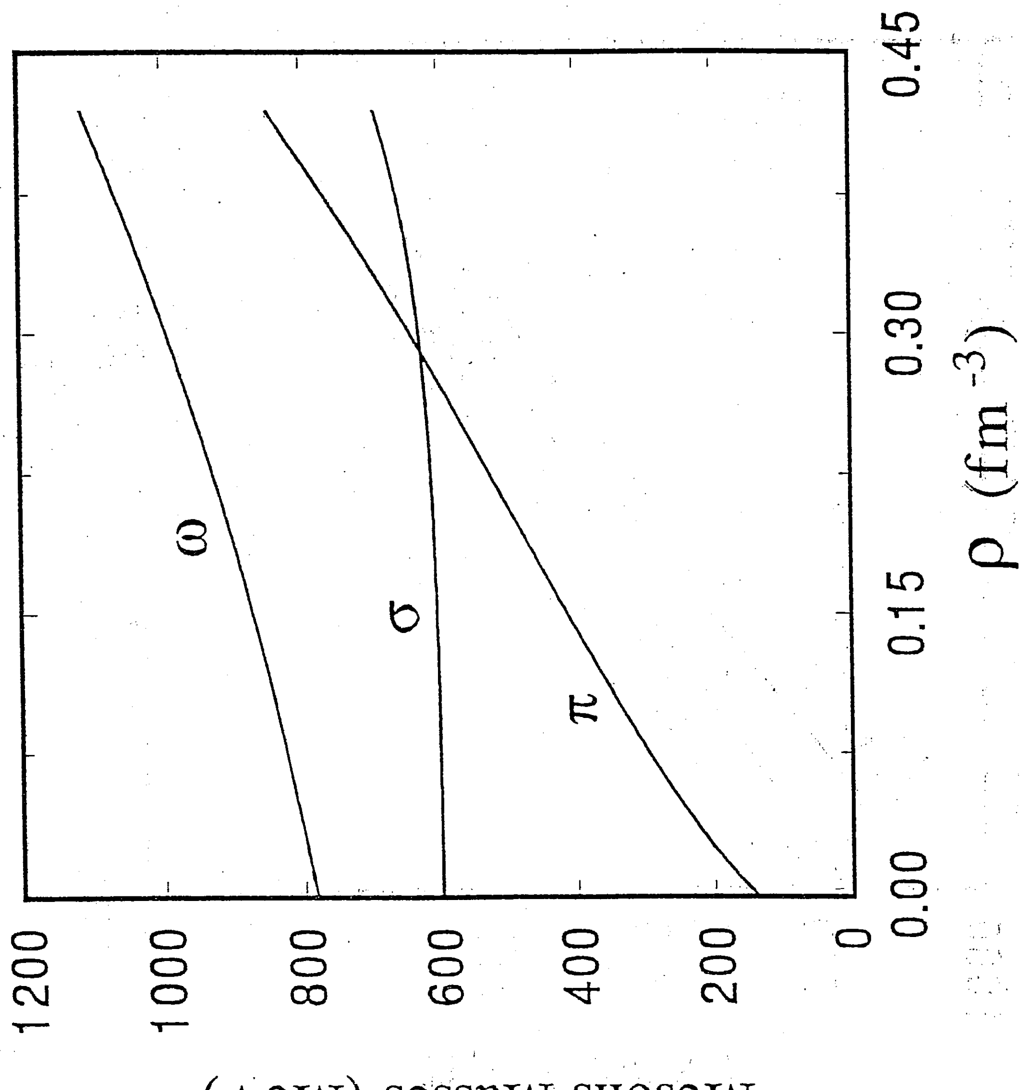


Fig. 1



Mesons Masses (MeV)
1200
1000
800
600
400
200
0
0.00
0.15
0.30
0.45
ρ (fm^{-3})
ω
σ
π

Fig. 2

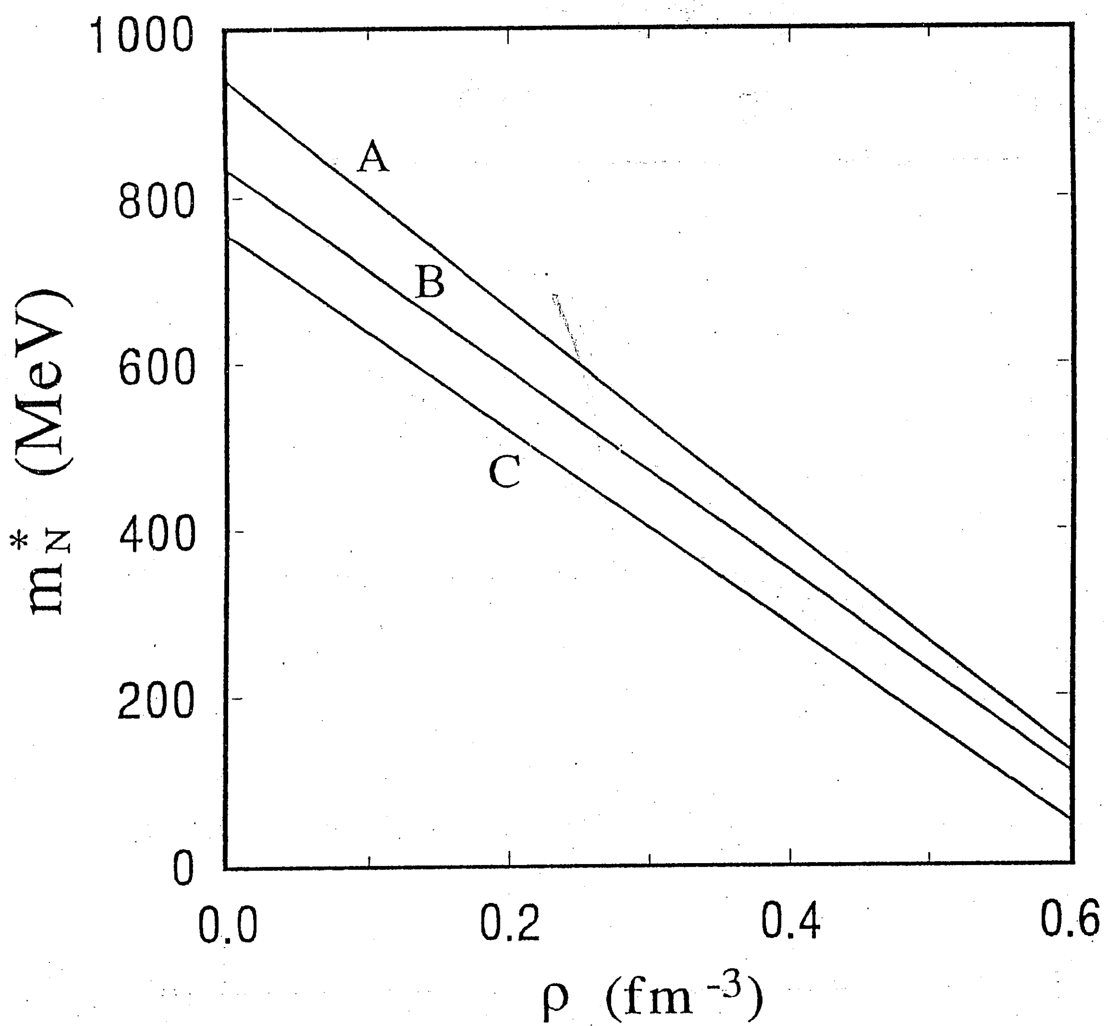


Fig. 3a

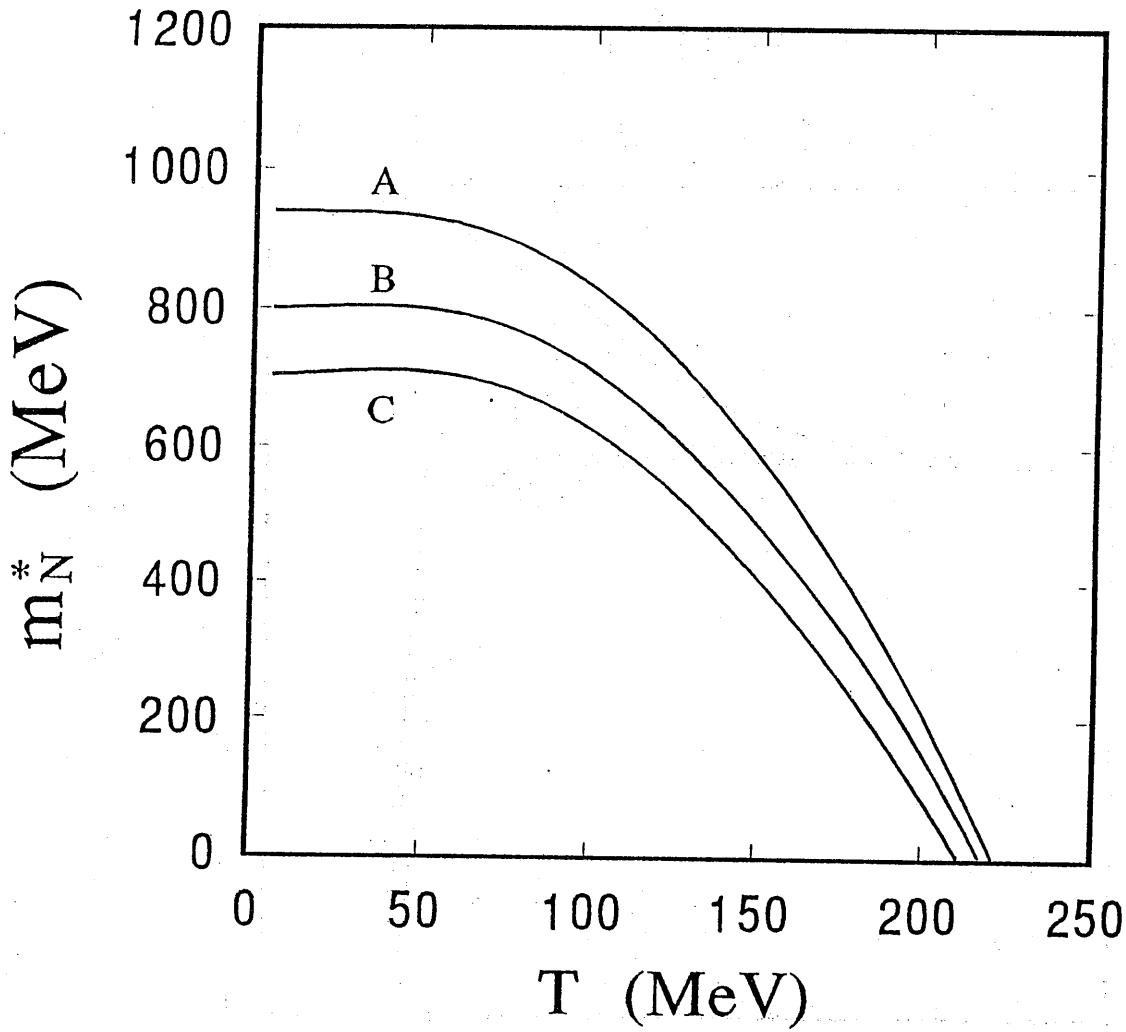


Fig. 3b

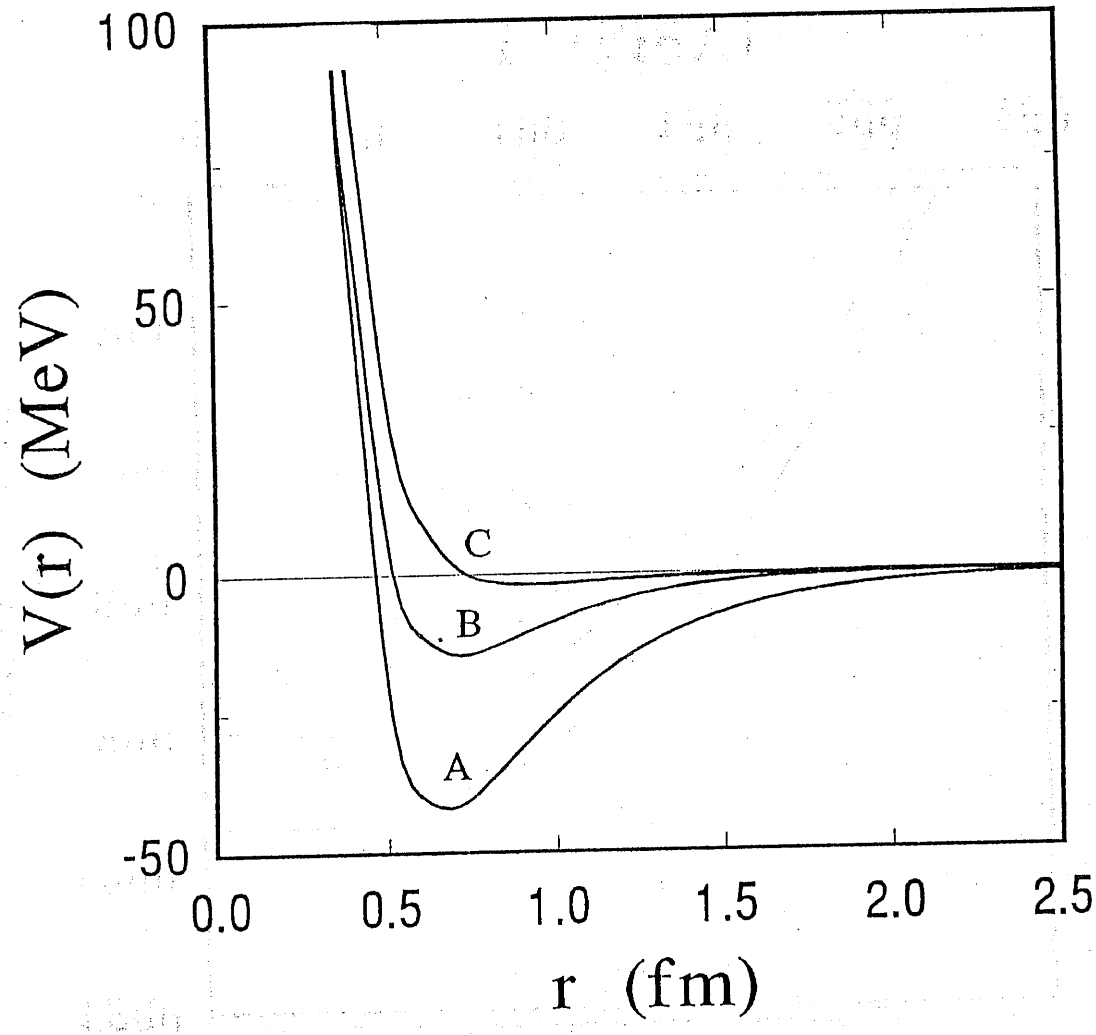


Fig. 4

Q-deformed Geometric Phase in Terms of the Invariant Operator Method*

Meng Xin-He, Li Xue-Qian, Lu Jing-Fa

CCAST (World Lab.), P.O.Box 8730, Beijing 100080, China

and

Physics Department , Nankai University, Tianjin 300071 ,

P.R.China †

Liu Yong

Physics Department , Nankai University, Tianjin 300071 , P.R.China

Abstract

Employing the Lewis-Riesenfield invariant operator method, the q-deformed geometric phase is presented with its explicit dependence on adjustable parameters and the physical significance is also discussed.

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†Mailing address

It is well known that q-deformation, as a realization of the quantum algebra[1], has provided us a practical way to approach the real physical world. However, it is always puzzling whether the quantum q-deformation with a free parameter q is only sort of parametrization or indeed refers to practical physics processes. It motivates us to apply this obscure theory to study the observable geometric phase (GP). [2]

For a quantum system in an adiabatic cyclic evolution, the GP is just the famous Berry phase[3], which can be interpreted as a holonomy associated with the parallel transport around a circuit in a parameter space[4]. In a generalization of Berry's idea, Aharonov and Anandan (AA)[5] removed the adiabatic restriction and studied the geometric AA phase for any cyclic evolution, which have been observed in optical and NMR experiments[6]. The Aharonov-Bohm effect[7] can be regarded as a special realization of the AA phase. Further, it is pointed that GP also appears even when the motion of a quantum system is neither unitary nor cyclic[8].

In this letter we mainly concentrate on the q-deformed GP for a time dependent harmonic oscillator by means of the Lewis-Riesenfield invariant operator method[9] and discuss its parameter dependence, which eliminate the discrepancy between the experimental value and theoretical one for GP in certain quantum system[10].

A GP exists in a coherent-state system and bears a close resemblance to AA phase[11]. The problem of how a coherent state in evolution can be preserved as a coherent state was answered by the invariant operator theorem[9]. Given a time dependence Hamiltonian

$$H(t) = \omega(t)N + d(t) \tag{1}$$

where N is the particle number operator, $\omega(t)$ and d(t) are two non-singular functions of time t respectively.

A coherent state will remain in its evolution as a coherent state up to a phase. We choose the invariant operator as

$$I(t) = a^+ a - z a^+ - z^* a + g(t) \tag{2}$$

Then by the invariant operator method

$$\frac{dI(t)}{dt} = \frac{\partial I}{\partial t} + \frac{[I, H]}{i\hbar} = 0, \tag{3}$$

we have the following solutions

$$z(t) = c.exp(-i \int \omega(t) dt) \tag{4}$$

$$g(t) = const. \tag{5}$$

with an arbitrary function d(t) and constant c undetermined. They can be specified in a concrete physical model with appropriate initial conditions.

The GP then can be written as

$$g(c) = \int \langle z | i \frac{d}{dt} | z \rangle dt \tag{6}$$

where $|z\rangle$ is a coherent state as

$$|z\rangle = exp(-|c|^2/2) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle \tag{7}$$

and$|n\rangle$are the eigenstates of the Hamiltonian. Similarly, if the Hamiltonian (1) is q-deformed and a deformed invariant operator of I(t) in Eq.(2) is taken to the form

$$I(t)_q = (a_q^+ - z^*)(a_q - z) - z^* z + g(t) \tag{8}$$

where a_q^+ and a_q are the commonly adopted form of the q-deformed annihilation and creation operators, and

$$a_q^+ a_q - q a_q a_q^+ = q^{-N_q} \tag{9}$$

where N_q is the q-deformed particle-number operator. We can have the q-deformed GP as

$$g(q,c) = \int_q \langle z| i\frac{d}{dt}|z\rangle_q dt \tag{10}$$

Inserting the coherent state's Fock representation

$$|z\rangle_q = exp_q(-|c|^2/2)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{[n]!}}|n\rangle_q \tag{11}$$

into Eq.(7), where

$$exp_q(a) = \sum_{n=0}^{\infty}\frac{a^n}{\sqrt{[n]!}} \tag{12}$$

$$[n]! = [n][n-1]...[2][1] \tag{13}$$

$$[x] = \frac{q^{-x} - q^x}{q^{-1} - q} \tag{14}$$

$$|n\rangle_q = \frac{a_q^{+n}}{\sqrt{[n]!}}|0\rangle_q \tag{15}$$

then, we can derive the q-deformed GP as

$$g(q,c) = exp_q(-|c|^2)\sum_{n=1}^{\infty}\frac{n}{[n]!}\int \omega(t)dt \tag{16}$$

It can be easily checked that when $q \to 1, g(q,c) = g(c)$.

To manifest the physical significances of parameters q and c, we define a function of

$$f(q,c) = \frac{g(q,c)}{g(c)} \tag{17}$$

One can imagine that for some ranges of q with fixed c, the function f(q,c) may have values more than one. We have drawn figures of the dependence of the function f(q, c) on q with some c, from which one can observe that the function f(g,c) is multi-valued. As one can see, in certain regions of the parameter q with a fixed c, the q-deformed

GP is larger than the undeformed one. This might present us an access to remedy the difference between experimental data and theoretical value for GP of certain quantum system in evolution. The figures and more details of the calculation will be published in our forthcoming work[12]. Moreover, a straightforward consideration is the q-deformed gauge potential induced from the q-deformed GP of the harmonic oscillator. Their meanings and possible results will be detailed in our forthcoming paper too.

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Screening effect at finite temperature and chiral transition in a 2+1 dimensional chiral four-fermion model

Kun Shen

CCAST (World Laboratory), P.O.Box 8730,

Beijing 100080,China

Department of Physics, Tsinghua University

Beijing, 100084 China

December 2, 1994

Abstract

The chiral Ward-Takahashi identities with composite fields at finite temperature are utilized to study the mechanism of chiral phase transition in the presence of a current mass in a 2+1 dimensional chiral four-fermion model. The mass spectra of fermion and bound states at finite temperature are obtained in terms of these identities. By evaluating the vertex correction in next to the leading $1/N$ order, thermal screening effect and chiral transition are discussed. It turns out that when temperature reaches a critical temperature T_c, the screening effect will make dynamical breaking restored. This shows that screening effect plays an important role in chiral transition.

Lattice Quantum Chromodynamics provides us a non-perturbative method to study the low-energy hadron physics on the basis of the first principles. Many Monte Carlo simulations show that at high temperature and /or density QCD vacuum undergoes structural changes and there is a deconfinement transition. These results are verified in some models.

At finite temperature, there are two aspects of thermal effects. One is to excite fermion pair condensate. This makes the fermion pair condensate gradually melted as temperature arises. The other is to produce a screening effect and make an influence on the interaction. However, in the mean field approximation, the screening effect is neglected[1][2]. In this paper, we study the screening effect and it's influence in chiral phase transition.

In order to study the screening effect, it is necessary to calculate the vertex correction. As the vertex correction is neglected in the mean-field approach,we adopt chiral Ward-Takahashi identities to study the screening effect. It was shown that it is more convenient to investigate the high order effects on phase structure than the Schwinger-Dyson equation[3]. In the point of view that four-fermion couplings are not renormalizable in 3+1 dimensions but renormalizable in 2+1 dimensions in $1/N$ expansion[4], in this section we take a 2+1 dimensional chiral four-fermion model as an example to study dynamical mass generation, the screening effect and the chiral phase transition at finite temperature.

At finite temperature, the quantum statistical partition function $Z_\beta[J]$ is[5]

$$Z_\beta[J] = \mathrm{Tr} e^{-\beta H_J}$$
$$= \int \mathcal{D}[\bar{\psi},\psi] \exp\Big(- \int_0^\beta d\tau \int d^2x [\mathcal{L} + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)$$

$$+K(x)\bar{\psi}(x)\psi(x) + K_5(x)\bar{\psi}(x)i\gamma_5\psi(x)]\Big)$$
$$\equiv e^{-W_\beta[J]}, \quad (1)$$

where composite external sources $K(x), K_5(x)$ are introduced in the partition function $Z_\beta[J]$ to describe dynamical breaking, and J denotes the abbreviation of $(\bar{\eta}, \eta; K, K_5)$. The Lagrangian density is

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_S - m_0\bar{\psi}\psi, \\ \mathcal{L}_S &= -\bar{\psi}\gamma\cdot\partial\psi + \tfrac{g^2}{2N}[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2]. \end{aligned} \quad (2)$$

where g^2 is positive, and $\mathcal{L}_S$ possesses chiral $U_V(1)\otimes U_R(1)$ symmetry.

Since a current mass term can be regarded as an external source[6], it is easy to see

$$W_\beta[\bar{\eta}, \eta; K, K_5] = W_\beta^S[\bar{\eta}, \eta; K - m_0, K_5], \quad (3)$$

where a physical meaningless constant term has been omitted.

At finite temperature, the Gibbs average of fields are defined as

$$\begin{aligned} \frac{\delta W_\beta[J]}{\delta\eta(x)} &= -\bar{\psi}_\beta(x), \\ \frac{\delta W_\beta[J]}{\delta\bar{\eta}(x)} &= \psi_\beta(x), \\ \frac{\delta}{\delta\eta(x)}\frac{\delta}{\delta\bar{\eta}(x)}W_\beta[J] &= G_\beta(x), \\ \frac{\delta}{\delta\eta(x)}i\gamma_5\frac{\delta}{\delta\bar{\eta}(x)}W_\beta[J] &= G_{5\beta}(x), \end{aligned} \quad (4)$$

Similar to zero temperature field theory, one can obtain the effective action $\Gamma[\phi_\beta]$ by performing the Legendre transformation. From the fact that the connected partition function $W_\beta^S[J]$ remains invariant under chiral transformation, we can obtain the finite temperature chiral Ward-Takahashi identities[3].

After shifting the external source $K \to K - m_0$, we obtain chiral Ward-Takahashi identities in the presence of the current mass

$$\int_0^\beta d\tau \int d^2x \quad \left[\bar{\psi}_\beta(x)\frac{\delta\Gamma[\phi_\beta]}{\delta\bar{\psi}_\beta(x)} + \frac{\delta\Gamma[\phi_\beta]}{\delta\psi_\beta(x)}\psi_\beta(x)\right] = 0, \tag{5}$$

$$\begin{aligned}\int_0^\beta d\tau \int d^2x \quad & \left[\bar{\psi}_\beta(x)\frac{i}{2}\gamma_5\frac{\delta\Gamma[\phi_\beta]}{\delta\bar{\psi}_\beta(x)} - \frac{\delta\Gamma[\phi_\beta]}{\delta\psi_\beta(x)}\frac{i}{2}\gamma_5\psi_\beta(x)\right. \\ & +[\bar{\psi}_\beta(x)i\gamma_5\psi_\beta(x) + G_5(x)]m_0 \\ & \left.+\frac{\delta\Gamma[\phi_\beta]}{\delta G_\beta(x)}G_{5\beta}(x) - \frac{\delta\Gamma[\phi_\beta]}{\delta G_{5\beta}(x)}G_\beta(x)\right] = 0. \end{aligned} \tag{6}$$

Differentiating eqs. (6) several times with respect to fields $\bar{\psi}_\beta, \psi_\beta, G_\beta$ and $G_{5\beta}$, we can get some Ward-Takahashi identities for proper vertexes at finite temperature.

With the aid of the Ward-Takahashi identities for two-point vertexes, we can obtain the mass spectra of fermion and bound states. In $1/N$ expansion, four-fermion couplings are renormalizable. In order to obtain physical results, it is necessary to introduce a fermion wavefunction renormalization constant Z_ψ, a coupling renormalization constant Z_{g^2}, a composite field renormalization constant $Z_{\bar{\psi}\psi}(= Z_\sigma^{1/2})$ and a current mass renormalization constant Z_{m_0}[8].

When the broken direction is chosen as

$$\langle\bar{\psi}\psi\rangle_0 \neq 0, \tag{7}$$

$$\langle\bar{\psi}i\gamma_5\psi\rangle_0 = 0, \tag{8}$$

and $\langle\bar{\psi}i\gamma_5\psi\rangle_\beta$ remains zero, the finite temperature mass spectra of fermion and the bound states are

$$M_f(\beta) = m_0^{ren} + \Gamma^{(3)ren}_{\bar{\psi}_\beta,\psi_\beta;\sigma_\beta}(p,-p;0)|_{p^2=0}\langle\sigma_{ren}\rangle_\beta, \tag{9}$$

$$m_\pi^2(\beta) = -m_0^{ren}\langle\bar\psi\psi\rangle_\beta^{ren}/\langle\sigma_{ren}\rangle_\beta^2, \quad (10)$$

$$m_\sigma^2(\beta) = m_\pi^2(\beta) + \Gamma^{(3)ren}_{\sigma_\beta,\pi_\beta;\pi_\beta}(p,-p;0)|_{p^2=0}\langle\sigma_{ren}\rangle_\beta, \quad (11)$$

where $p = (i\omega, p_1, p_2)$. The bound states are defined as[3]

$$\sigma(x) = a\bar\psi(x)\psi(x), \quad (12)$$

$$\pi(x) = a\bar\psi(x)i\gamma_5\psi(x), \quad (13)$$

$$a = \frac{\langle\sigma\rangle_0}{\langle\bar\psi\psi\rangle_0}, \quad (14)$$

where the subscript '0' denotes zero temperature.

At zero temperature, the Feynman rules in 1/N expansion are given in ref.[7]. The propagators are

$$G(p) = \frac{i}{\hat p - M_f}\delta_{ij}, \quad (15)$$

$$D_\sigma(p) = \frac{2\pi}{iN}\frac{1}{2(M_f - M) + \frac{-p^2+4M_f^2}{\sqrt{-p^2}}\tan^{-1}\sqrt{-p^2}/2M_f}, \quad (16)$$

$$D_\pi(p) = \frac{2\pi}{iN}\frac{1}{2(M_f - M) + \sqrt{-p^2}\tan^{-1}\sqrt{-p^2}/2M_f}, \quad (17)$$

Here M_f and M are

$$M_f = \frac{1}{2}[M + \sqrt{M^2 + m^2}], \quad (18)$$

$$\frac{1}{g^2} = \int\frac{d^3p}{(2\pi)^3}\frac{1}{-p^2 + M^2}, \quad (19)$$

where m is a parameter to describe the explicit breaking, which is related to the current mass m_0 by the relation $m^2 = 4\pi m_0 Z_{m_0}/g^2 Z_{g^2}$. We assume that the explicit breaking is very small, $m/M \ll 1$. As the fermion mass and σ meson mass are the order of M , the corrections resulted from m can be omitted in the mass spectra of the fermion and σ meson. However, it is quite different for π meson. In the low momentum limit, eq. (17) shows that m is the π meson mass and the residue of $D_\pi(p)$ isn't 1, but $4\pi M/N$. When π field is taken as the same form in ref.[8], one can see that at the leading order the π mass spectrum derived from Ward-Takahashi identities is in agreement with that in $1/N$ expansion. Here, we use the Feynman rules in ref.[7] to discuss the vertex correction of fermions and σ meson.

In terms of the relation between field theory and imaginary time temperature field theory[5], the Feynman rules at finite temperature in 1/N expansion can be easily obtained. According to the Feynman rules at finite temperature, in the non-relativistic limit the vertex function beyond the leading order is

$$i\Gamma^{(3)}_{\psi_\beta,\bar{\psi}_\beta,\sigma_\beta}(p,-p;0)|_{p^2=0} = -i + i\Lambda^\beta_\sigma(p,-p;0)|_{p^2=0} + i\Lambda^\beta_\pi(p,-p;0)|_{p^2=0}, \quad (20)$$

where

$$\begin{aligned} i\Lambda^\beta_\sigma(p,-p;0)|_{p^2=0} = \quad & i\Lambda^0_\sigma(p,-p;0)|_{p^2=0} - i\frac{2}{N}\Big[-\frac{1}{3\ln 3}\frac{1}{e^{\beta M}+1} \\ & +\frac{4(2\ln 3-1)}{9\ln^2 3}\frac{1}{\beta M}\ln(e^{-\beta M}+1)\Big], \quad (21) \\ i\Lambda^\beta_\pi(p,-p;0)|_{p^2=0} = \quad & i\Lambda^0_\pi(p,-p;0)|_{p^2=0} + i\frac{2}{N}\Big[\frac{4}{3\ln^2 3\beta M}\ln(e^{-\beta M}+1) \\ & -\frac{1}{\beta M}\ln(1-e^{-\beta m_\pi}) + \frac{1}{\ln 3}\frac{1}{e^{\beta M}+1}\Big], \quad (22) \end{aligned}$$

where $\Lambda^0_\pi, \Lambda^0_\sigma$ represent the vertex corrections caused by exchanging π, σ at zero temperature. As a 2+1 dimensional theory is infrared safety, the very small π meson mass can be ignored at zero temperature.

Applying eqs. (21) and (22), one finds that the effective coupling between fermions and σ meson in the next leading order is

$$g_{eff}(\beta) = \quad -1 + \tfrac{0.182}{N} + \frac{2}{N}\Big[-\frac{1}{\beta M}\ln(1-e^{-\beta m_\pi}) + \frac{4}{3\ln 3}\frac{1}{e^{\beta M}+1)}$$

$$+\frac{8(2\ln 3-1)}{9\ln^2 3}\frac{1}{\beta M}\ln(e^{-\beta M}+1)\Big], \tag{23}$$

where M is the function of temperature. From eq. (23), one can see that the vertex correction at finite temperature results from two part contributions: one is the correction at zero temperature, which comes from quantum fluctuation in the vacuum; the other is thermal fluctuation, which is classical thermal effects. As ultraviolet divergences result from quantum fluctuation and usually exist in the vertex correction at zero temperature, they can be removed by the renormalization procedure at zero temperature. Note that if $m_\pi = 0$, the thermal fluctuation of π meson will cause an infrared divergence. The small explicit breaking gets rid of the infrared divergence at finite temperature.

From eq. (19), it is easy to get

$$M - M_0 + \frac{2}{\beta}\ln(1+e^{-\beta M}) = 0. \tag{24}$$

where M_0 denotes the dynamical generation mass at zero temperature, which is related to the fermion pair condensate.

From eq. (23), one can see that with increasing temperature, $-g_{eff}(\beta)$ decreases gradually; thus the attractive interaction between fermions will decrease as temperature increases. So there is a temperature T_D to make the effective coupling vanish, i.e.

$$g_{eff}(\beta_D) = 0. \tag{25}$$

We call T_D decoupling transition temperature. When temperature exceeds T_D, the attractive interaction between fermions will be completely screened by thermal fluctuation and the interaction becomes replusive.

At the next leading order, the fermion mass correction results from two parts: i) vertex correction, which is inverse to M_f; ii) the melting effect $\langle\sigma\rangle$ at $1/N$ order, which is proportional to M_f. As temperature increases, M_f decreases. Thus, the melting effect can be neglected. Using eq. (9) and in above approximation, we can be expressed the next leading order fermion mass M_f' as

$$M_f'(\beta) = g_{eff}(\beta)M_f + (1 - g_{eff}(\beta))m_0^{ren} \tag{26}$$

where M_f is the leading order fermion mass. Eq. (26) shows that the fermion mass M_f' decreases with temperature arising, which is shown. When it reaches T_D, $M_f'(\beta)$ is equal to the current mass. If it continuously increases, $M_f'(\beta)$ becomes less that the current mass. There is a temperature T_c to make

$$M_f'(T_c) = 0. \tag{27}$$

As the fermion pair condensate in the next leading order is proportional to $M_f'(\beta)$, so we have

$$\langle\bar{\psi}\psi\rangle_{\beta_c} = 0. \tag{28}$$

When temperature exceeds T_c, the fermion mass $M_f'(\beta)$ becomes negetive, which indicates that the vacuum is unstable. So the bound states can't be formed and are dissociated as high temperature. These correspond to the picture of chiral phase transition. We call T_c the critical temperature of chiral phase transition. It should be emphasized that not only the melting effect but also the screening effect play an important role in the mechanism of chiral phase transition.

As infrared properties will affect phase structure and phase transition, whether above mechanism holds in 3+1 dimensions needs further investigation.

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The definitions of the renormalized quantities are different from those.
The relations are $Z_1 = Z_\psi$, $Z_2 = Z_\psi^{1/2} Z_{\bar{\psi}\psi}^{1/2}$, $Z_3 = Z_{g^2}^{-1}$, and $Z_4 = Z_{m_0}^{-1}$.

Brown-Rho conjecture and EMC effect

Feng-Cai Ma

Department of Physics, Jinzhou Normal College, Jinzhou 121003, China

Ru-Keng Su

China Center of Advanced Science and Technology (world Laboratory)

P. O. Box 8730, Beijing 100080, China

Department of Physics, Fudan University, Shanghai 200433, China

Abstract

Based on the Brown-Rho conjecture, the nuclear density effect of the nucleon structure function is investigated. A explaination for EMC effect from Brown-Rho conjecture is given.

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The experiment of deep inelastic scattering of charged lepton with nucleons of European Muon Collaboration (EMC) [1] have showed that, the difference between the structure function F_2 of the bound nucleons and that of free nucleons is significant. After then several precise experiments [2-4] have been carried out to measure the ratio $R = F_2^A/F_2^D$ and the results confirm that $R \neq 1$.

The structure function of a nucleon is related to the momentum distribution of partons. In the quark-parton model, F_2 depends on the Fermi motion only. But the experiment data of EMC shows that the finite-density background of nucleon is important. A nucleon in a heavy nucleus is surrounded by many other nucleons, the effect of the surrounding nucleons may be explained as putting a nucleon in a finite-density medium.

Based on the scaling invariance of QCD, Brown and Rho argued that, the masses of the nucleon, ω-meson ρ-meson, and σ-meson in a zero temperature finite-density medium satisfy [5,6]

$$\frac{M_N^*(\rho)}{M_N} \simeq \frac{M_\omega^*(\rho)}{M_\omega} \simeq \frac{M_\rho^*(\rho)}{M_\rho} \simeq \frac{M_\sigma^*(\rho)}{M_\sigma} \simeq \frac{f_\pi^*}{f_\pi} \simeq 1 - \frac{\lambda\rho}{2\rho_0} \qquad (1)$$

where f_π is the pion decay constant, the masses and f_π with asterisks stand for the finite-density values of the corresponding quantites, ρ is the density of the nucleon background, and ρ_0=0.17 fm^{-3} is the saturation density.

According to ref. [7], the result of f_π^*/f_π at saturation density is

$$f_\pi^*(\rho_0)/f_\pi = 0.91 \qquad (2)$$

From Eqs. (1) and (2), we obtain λ=0.18, and

$$\frac{M_N^*}{M_N} = 1 - 0.09\frac{\rho}{\rho_0} \qquad (3)$$

Since the nucleon mass will shift with the density of the medium, the structure function of a nucleon in a heavy nucleus must be affected by this shift. The EMC effect may be a result of this shift.

The cross section of charged leptons deep inelastic scattering on nucleons can be written as

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\frac{F_2(x,Q^2)}{x}\left[1 - y - \frac{xyM}{2E} + \frac{y^2}{2}\frac{1 + 4M^2x^2/Q^2}{1 + R(x,Q^2)}\right] \quad (4)$$

where $Q^2 = -q^2$, q^2 is the squre of the four momentum transfer from the lepton to the target nucleon, $\nu = E - E'$ is the virtual photon energy, $y = \nu/E$, M stands for nucleon mass and

$$x = \frac{Q^2}{2M\nu} \quad (5)$$

is Bjorken scaling variable.

To the first order approximation, it was shown that [2] F_2 is independent of Q^2, we can consider F_2 as a function of x only. The nucleon in a deuterium can be treated as a free nucleon approximately, but the nucleon in a heavy nucleus as a nucleon in a finite-density medium. We see from Eq. (4) that the reduction of nucleon mass implies a shift of x. The structure function will becomes

$$F_2^A(x) = F_2^N(x') \quad (6)$$

where F_2^A stands for the structure function of a nucleon bounded in a nucleus with A nucleons, F_2^N is that of free nucleons, and

$$x' = \frac{Q^2}{2M^*\nu} \quad (7)$$

where M^* is the effective mass of nucleon in A nucleons background. The ratio of the structure function of a bound nucleon F_2^A to that of a free nucleon F_2^N reads

$$R(x) = F_2^A(x)/F_2^N(x) = F_2^N(x')/F_2^N(x) \tag{8}$$

The numerical results of the ratio $R(x)$ for four different nuclei are shown in Fig. 1. We choose the function of a free nucleon as [8]:

$$F_2^N(x) = 0.59\sqrt{x}(1-x)^{2.8} + 0.33\sqrt{x}(1-x)^{0.38} + 0.49(1-x)^8 \tag{9}$$

In the large x region, the effect of Fermi-motion [9] is taken into account. We see from Fig. 1 that our results are in good agreement with the experiment data [3].

In summary, the correction of nucleon structure function by nuclear medium can be calculated by Brown-Rho conjecture and the result of the effective mass of nucleon given by Brown-Rho conjecture can successfully explain EMC effect. The effect of x-rescaling may be come from the density dependence of nucleon mass.

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Figure Caption

Fig. 1 The predictions of Brown-Rho Conjecture results compared with data SLAC E139 [3]. The solid curves are the results calculated by Eq. (8).

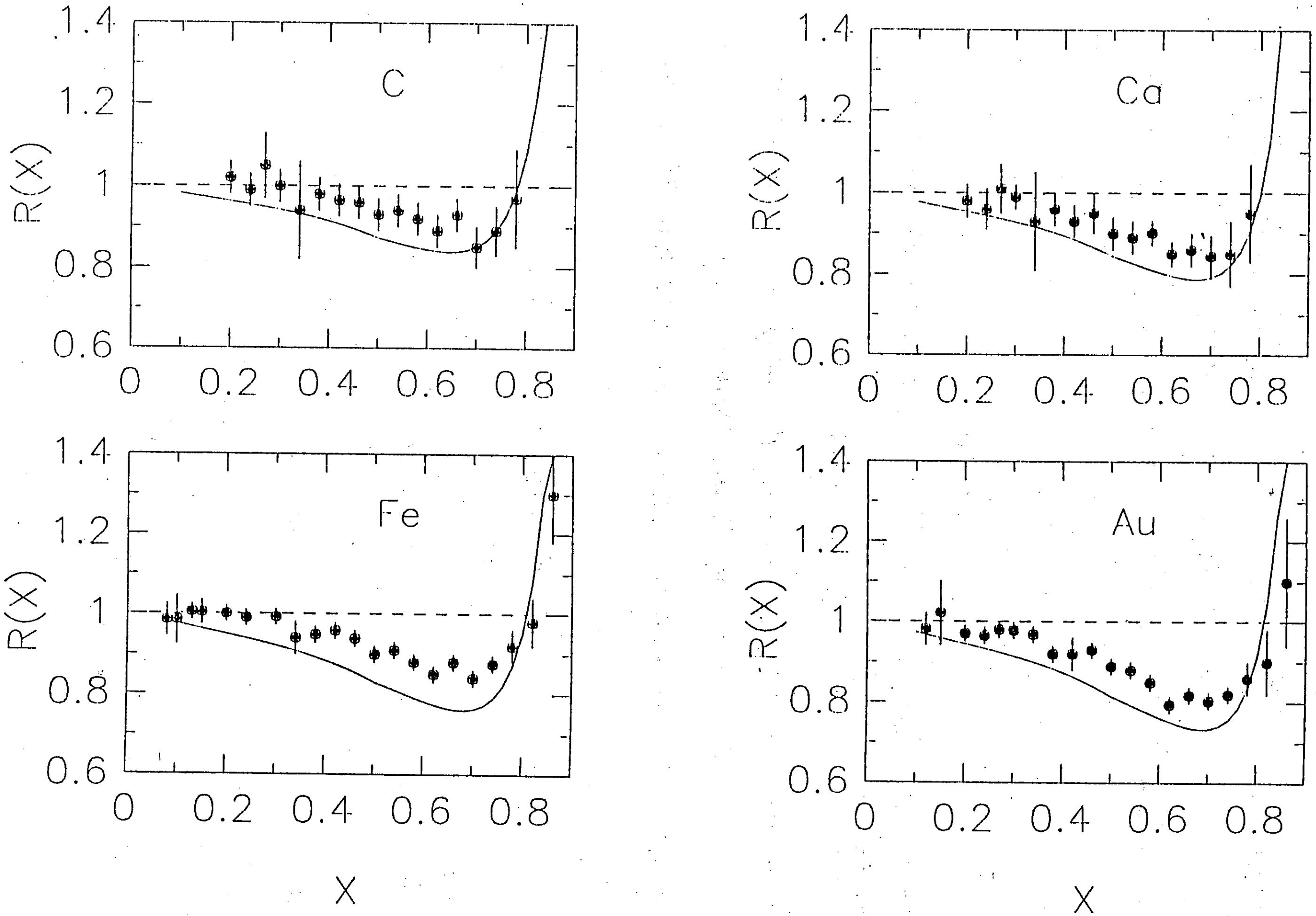

Fig. 1

Improved finite temperature QCD sum rules and J/Ψ suppression in hot hadronic matter and quark-gluon plasma

Song Gao

Department of Physics, Fudan University, Shanghai 200433, China

Ru-Keng Su

China Center of Advanced Science and Technology (world Laboratory)

P. O. Box 8730, Beijing 100080, China

Department of Physics, Fudan University, Shanghai 200433, China

Abstract

An improved finite temperature QCD sum rules has been employed to calculate the J/Ψ suppression in hot hadronic matter and quark-gluon plasma separately. We have found that the J/Ψ suppression will occur in both hadronic matter and quark-gluon plasma, but a mass shift about 0.49 GeV of J/Ψ will exist in QGP background only. In hadronic matter background, no mass shift exists. The results obtained by QCD sum rules and by potential model are compared.

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I. INTRODUCTION

It is generally believed that the phase transition from hadronic matter (HM) to quark-gluon plasma (QGP) will take place at high temperature and/or high density, and the relativistic heavy ion collision (RHIC) [1] will provide this extreme condition. The key for the QGP formation problem is to find a clear signal which can give rise to a sharp distinction between predictions from QGP and HM, and in particular, this signal can be measured clearly [2]. J/Ψ suppression may be one of such candidates.

A strong and systematic suppression of J/Ψ in the dilepton spectrum in QGP background was first suggested by Matsui and Satz [3]. By using the Debye screening potential model, many workers [3,4] have calculated the mass of J/Ψ in QGP background and found that the J/Ψ formation will be prevented by the plasma Debye screening effect and then a strong suppression of J/Ψ peak in the dilepton spectrum will be occurred. This result seems to be confirmed by the RHIC experiment of NA38 collaboration [5]. But unfortunately, in a recent high energy proton-nucleus collision experiment under its condition that the QGP has no opportunity to be produced, the J/Ψ suppression has also been found [6,7]. Therefore, can J/Ψ suppression be a signature of QGP formation? It is still an open question.

The objective of this paper is to study this question. The basical importance for answer this question is to employ an effective tool to calculate the J/Ψ suppression in QGP and in HM separately and find their distinctions. The QCD sum rules at finite temperature is one of these effective tools.

The method of QCD sum rules was first introduced at zero temperature to predict features of low-energy hadronic physics in the absence of an exact solution of QCD at large distance. This method is built upon the working hypothesis that there is a kinematic region, in which the correlation function may be evaluated at the quark-gluon level by using perturbative QCD, but augmented with non-perturbative gluon and quark condensates to replace the large distance confinement effect. Results given by QCD sum rules are compared with those obtained from dispersion relations at the hadronic level. Some applications of QCD sum rules have been found to be phenomenologically successful and can provide an insight to the properties of hadrons [8-11].

In order to study the temperature and density dependences of hadron properties and QGP, one must extend the QCD sum rules to finite temperature. Although much effort has been devoted for this extension [12-16], but as was pointed out by Hatsuda et al. [17], none of them contains a satisfactory and consistent formulation at finite temperature. The essential drawbacks of the previous works are [17,18]:

(1) The finite temperature correlation function for massless quarks has been used to calculate the short-distance properties of the correlations even in the hadronic phase at low temperature. This procedure introduces an unnecessary and erroneous mixture of the short and long distance dynamics, and leads to temperature dependent Wilson coefficients. But in the hadronic phase, since the thermal average of the local operator O_n takes care of all the temperature dependences of $\Pi(q,T)$ in the operator product expansion (OPE) $\Pi(q,T) = \sum_n c_n(q) < O_n >_T$, the Wilson coefficients $c_n(q)$ should be temperature independent.

(2) The OPE for the current product should contain all the possible non-scalar operators as well as scalar operators, because Lorentz invariance is broken at finite temperature.

Because of these difficulties, some results given in the previous papers, including the J/Ψ discussion in ref. [14], are not reliable.

To overcome above difficulties, Hatsuda and his coworkers suggested an improved QCD sum rules (IQSR) at finite temperature, in which the Wilson coefficients are temperature independent, and used them to study the light meson [17,18]. In this paper, we will employ IQSR to study the heavy quark system. We will calculate the mass, the resonance strength and the continuum threshold of J/Ψ meson in hot HM and QGP background separately and compare their results carefully, we will indicate that although the J/Ψ suppression will occur in both HM and QGP backgrounds, but there are several distinctions between these two backgrounds, we will prove that the J/Ψ suppression in HM with no mass shift, but in QGP, the J/Ψ suppression will be accompanied with a J/Ψ mass shift 0.49 GeV. Besides, we will prove the critical temperatures in HM and QGP at which the J/ψ bound state dissolves are different. The former is 0.16 GeV but the later is 0.22 GeV. Due to this difference, the variations of J/Ψ resonance strength of HM and QGP with temperature are also different.

II. FORMALISM

We take the current of J/Ψ meson as

$$J_\mu = \bar{c}\gamma_\mu c \tag{1}$$

where c denotes the charm-quark field. Since the J/Ψ meson is a heavy quark system, the heavy quark condensates can be neglected [8-11,14], we consider the gluon condensate $< 0|G_{\mu\nu}G^{\mu\nu}|0 >$ only [14]. In the hot medium, the thermal average of the time-order vector current-current correlation function can be written as

$$\Pi_{\mu\nu}(\omega, \vec{q}, T) = i \int \mathrm{d}^4 x e^{iq\cdot x} < 0|J_\mu(x)J_\nu(0)|0 >_T \tag{2}$$

where T is the temperature of the hot medium, and $q_\mu = (\omega, \vec{q}\,)$ the transfer four-momentum. One can prove that the correlation function of a conserved vector current contains two independent invariants which correspond to the transverse and longitudinal polarization Π_T and Π_L in the medium respectively. Because no specific spatial direction can be existed in the rest frame of the medium, therefore we have [12,17]

$$\begin{aligned} &\Pi_T = \omega^2 \Pi_L \ , \ \Pi_L(\omega, \vec{q} = 0) = \Pi_L(\omega^2) \ , \\ &\Pi_L(\omega^2) = \lim_{\vec{q}\to 0} [\Pi^\mu_\mu(\omega, \vec{q})/(-3\omega^2)] \ . \end{aligned} \tag{3}$$

The longitudinal part $\Pi^R_L(\omega)$ of the retarded correlation function at $\vec{q} = 0$ satisfies the following standard dispersion relation in medium,

$$\Pi^R_L(\omega) = \int_{-\infty}^{\infty} \mathrm{d}s \frac{\rho(s)}{s - \omega - i\epsilon} \tag{4}$$

where $\rho(s)$ is the spectral density at finite temperature and can be treated phenomenologically. Considering the OPE of the retarded correlation function and taking the thermal average, we get [17]

$$\mathrm{Re}\Pi^R_L(\omega) = \sum_n c_n(\omega^2, \mu^2) < O_n(\mu^2) >_T \tag{5}$$

where μ^2 is a renormalization point of the local operator O_n, $c_n(\omega^2, \mu^2)$ are the temperature independent Wilson coefficients and can be calculated by naive perturbative theory. The non-perturbative dynamics and the temperature effects are included in the thermal average of the local operator

$< O_n(\mu^2) >_T$. It means that the condensates of quarks $< \overline{q}q >$ and gluons will be temperature dependent in the hot medium. Applying the Borel transformation

$$L_M \Pi_L^R(Q^2) = \lim_{\substack{n, Q^2 \to \infty, \\ Q^2/n \equiv M^2}} \frac{1}{(n-1)!}(Q^2)^n(-\frac{\mathrm{d}}{\mathrm{d}Q^2})^n \Pi_L^R(Q^2) \tag{6}$$

to the dispersion relation Eq. (4), we obtain the Borel sum rules as

$$L_M \Pi_L^R(Q^2) = \frac{1}{M^2}\int \mathrm{d}s \exp(-s/M^2)\rho(s) \tag{7}$$

where M is the Borel mass parameter.

Now we are in a position to use this sum rule to study the behaviour of J/Ψ meson. Hereafter we will calculate the J/Ψ suppression in HM and QGP background separately.

1. Hot hadronic matter background

Since the temperature of hot HM is less than the deconfinement temperature, using the same arguments of [9,10], to a good approximation, the theoretical side of the QCD sum rules for J/Ψ meson can be obtained similarly as that of zero temperature, we get

$$M^2 L_M \Pi_L^R(Q^2) = e^{-4m_c^2/M^2}\pi A(M^2)[1 + \alpha_s a(M^2) + \phi b(M^2)] \tag{8}$$

where m_c is the mass of charm quark, α_s is the QCD coupling constant. In Eq. (8), $\pi A(M^2)$ is the contribution of free quarks, $\alpha_s a(M^2)$ corresponds to the first perturbative gluon correction and $\phi b(M^2)$ represents the non-perturbative correction. The expressions of $\pi A(M^2)$, $a(M^2)$ and $b(M^2)$ are given in the Appendix. The gluon condensate parameter ϕ is defined as

$$\phi = \frac{4\pi^2}{9(4m_c^2)^2} < \frac{\alpha_s}{\pi} G_{\mu\nu} G^{\mu\nu} > \tag{9}$$

As was pointed by Hatsuda et al. [17], the temperature dependent part of gluon condensate is quite small numerically: $< (\alpha_s/\pi)G^2 >_T$ is at most 0.5% of its corresponding value of zero temperature even through temperature increases to 0.20 GeV. To a good accuracy, instead of $< (\alpha_s/\pi)G^2 >_T$, we can safely use the value of $< (\alpha_s/\pi)G^2 >_0$ for our J/ψ spectrum calculation in HM background.

The spectral density in Eq. (4) for J/Ψ at hot HM has the structure

$$\rho(\omega^2) = f m_{J/\Psi}^2 \delta(\omega^2 - m_{J/\Psi}^2) + \frac{1}{8\pi^2}(1 + \frac{\alpha_s}{\pi})\theta(\omega^2 - S_0) + (\rho_D + \rho_{D_s})\delta(\omega^2) \tag{10}$$

where $m_{J/\psi}$ is the mass of J/Ψ meson, f the resonance strength and $\sqrt{S_0}$ the continuum threshold. In the right hand side of Eq. (10), the first term is the contribution of the resonance part, the second the continuum term and the third the Landau damping. The Landau damping term $\rho_D + \rho_{D_s}$ comes from the scattering of J/Ψ with thermal D mesons in the hot medium. The contributions due to the thermal charm D mesons can be obtained as [14]

$$\rho_i(\omega^2) = \int_{4m_i}^{\infty} \mathrm{d}\omega^2 \, n_B\left(\frac{\omega}{2T}\right) \frac{v(3 - v^2)}{8\pi^2} \, , \qquad (i = D, D_s) \tag{11}$$

where $v = (1 - 4m_i^2/\omega^2)^{1/2}$, $n_B(x) = (e^x - 1)^{-1}$ is the boson distribution function.

Substituting Eqs. (8) and (10) into Eq. (7), we have

$$\begin{aligned} & f m_{J/\Psi}^2 e^{-m_{J/\Psi}^2/M^2} + \frac{1}{8\pi^2}(1 + \frac{\alpha_s}{\pi}) \int_{S_0}^{\infty} \mathrm{d}\omega^2 e^{-\omega^2/M^2} + \rho_D + \rho_{D_s} \\ & = e^{-4m_c^2/M^2} \pi A(M^2)[1 + \alpha_s a(M^2) + \phi b(M^2)] \end{aligned} \tag{12}$$

A straightforward calculation of Eq. (12) shows the mass of J/Ψ as

$$m_{J/\Psi}^2 = \left[e^{-4m_c^2/M^2}\pi A(1+\alpha_s a+\phi b) - \frac{1}{8\pi^2}(1+\frac{\alpha_s}{\pi})M^2 e^{-S_0/M^2} - (\rho_D+\rho_{D_s})\right]^{-1}$$
$$\left\{4m_c^2 e^{-4m_c^2/M^2}\pi A(1+\alpha_s a+\phi b)\left[1+\frac{M^4}{4m_c^2}(\frac{A'}{A}+\frac{\alpha_s a'+\phi b'}{1+\alpha_s a+\phi b})\right]\right.$$
$$\left.-\frac{1}{8\pi^2}(1+\frac{\alpha_s}{\pi})M^4(1+\frac{S_0}{M^2})e^{-S_0/M^2}\right\} \qquad (13)$$

and the resonance strength as

$$8\pi^2 f m_{J/\Psi}^2 = e^{m_{J/\Psi}^2/M^2}[8\pi^2 e^{-4m_c^2/M^2}\pi A(1+\alpha_s a+\phi b)$$
$$-(1+\frac{\alpha_s}{\pi})M^2 e^{-S_0/M^2} - 8\pi^2(\rho_D+\rho_{D_s})] \; . \qquad (14)$$

where A', a' and b' denote the corresponding M^2 derivatives, for example, $A' \equiv dA/dM^2$. By using Eqs. (13) and (14), we can study the behaviour of J/Ψ in the hot HM background.

2. Quark-gluon plasma background

There are two major differences of QCD sum rules in HM background and QGP background. Firstly, in QGP, instead of considering the contributions of the scattering of J/Ψ and thermal charm D mesons, we must take the effects the charm quark with the thermal QGP bath into account, because the temperature of QGP is higher than the deconfinement temperature. The spectral function of c quark in a thermal bath has been calculated in [12,14]. The result is

$$\rho_s = \frac{1}{4\pi^2}\int_{4m_c^2}^{\infty} d\,\omega^2 n_F(\frac{\omega}{2T})(1-4m_c^2/\omega^2)^{1/2}(1+\frac{2m_c^2}{\omega^2}) \qquad (15)$$

where $n_F = (e^x+1)^{-1}$ is the fermion distribution function. When we use the QCD sum rules formula Eq. (13) to calculate the mass of J/Ψ meson in QGP at first, we must replace $\rho_D+\rho_{D_s}$ by ρ_s of Eq.(15).

Secondly, when temperature arrives at the deconfinement temperature, the gluon condensate, as a parameter of confinement, will change with temperature and vanish at the deconfinement temperature. It means that in QGP background, we cannot take the approximation $\phi_T \cong \phi_0$. By using the thermo field dynamics method, the gluon condensate at finite-temperature and -density had been considered by Mishra et al. [19] in details. We will use their results in our QCD sum rules calculations. Instead of $\phi \cong \phi_0$ of HM background, we put the temperature dependent gluon condensate ϕ_T of ref. [19] into Eq. (13).

After replacement $\rho_D + \rho_{D_s}$ by ρ_s and ϕ_0 by ϕ_T of [19], we can use Eq. (13) to calculate J/Ψ suppression in QGP numerically.

III. RESULTS AND DISCUSSIONS

In hot medium, the mass $m_{J/\Psi}$, the continuum threshold $\sqrt{S_0}$ and the resonance strength $8\pi^2 f$ of J/Ψ are temperature dependent. Using the formulae of Sec. II, we can calculate $m_{J/\Psi}$, $\sqrt{S_0}$ and $8\pi^2 f$ of J/Ψ in HM and QGP separately. The parameters for the charmonium system in our calculation are chosen as [10]: $m_c = 1.42$ GeV, $\alpha_s = 0.27$, $\phi_0 = 1.23 \times 10^{-3}$, $m_D = 1.87$ GeV, $m_{D_s} = 1.97$ GeV. At zero temperature, the continuum threshold $\sqrt{S_0} = 3.55$ GeV. In the hot medium, the Borel mass window and the continuum threshold will become temperature dependent. Since the physical quantities would not depend on the unphysical Borel mass window parameter M, we can use the condition that making $m_{J/\Psi}(M^2)$ least sensitive to M at each temperature for a given Borel mass window $M_{min} < M < M_{max}$ to search $\sqrt{S_0(T)}$

in our calculations [17-18]. Once the threshold $\sqrt{S_0(T)}$ determined, we can calculate the other physical quantities such as $m_{J/\Psi}$ and resonance strength.

The temperature dependences of the mass of J/Ψ and the continuum threshold $\sqrt{S_0}$ in hot HM are shown in Fig. 1. We see from Fig. 1 that $m_{J/\Psi}$ almost unchanges with temperature but $\sqrt{S_0}$ decreases monotonously as temperature increases. When temperature increase to a critical temperature $T_c = 0.16$ GeV, at which $m_{J/\Psi}$ equals to $\sqrt{S_0}$, the bound state of J/Ψ meson will dissolve and the suppression of J/Ψ peak in dilepton spectrum will happen. This result is in agreement with the experiment that J/Ψ suppression can be occurred in HM [6].

The curves of $m_{J/\Psi}$ vs. temperature and $\sqrt{S_0}$ vs. temperature in QGP background are shown in Fig. 2. We find that $m_{J/\Psi}$ and $\sqrt{S_0}$ all decrease as temperature increases. In the high temperature regions, $m_{J/\Psi}$ and $\sqrt{S_0}$ decrease with temperature rapidly. The critical temperature at which $m_{J/\Psi}$ equals to $\sqrt{S_0}$ is $T_c = 0.22$ GeV. Above critical temperature T_c, $m_{J/\Psi} > \sqrt{S_0}$, the J/Ψ bound state will dissolve and J/Ψ suppression will also be occurred in QGP.

Comparing Fig. 1 and Fig. 2 we find a basical distinction of J/Ψ suppression in HM and QGP. In HM, $m_{J/\Psi}$ is almost independent with temperature and no mass shift exists. But in QGP, a clear mass shift is exhibited. When temperature increases to T_c, we find that the mass shift $m_{J/\Psi}(T_c) - m_{J/\Psi}(0)$ of $m_{J/\Psi}$ be 0.49 GeV. It is a very large value and may be measured from experiment. The mass shift of J/Ψ had been found by Hashimoto et al. in [5]. They employed a potential model and supposed a temperature dependent string tension to calculate $m_{J/\Psi}$. It seems so artificial because the temperature dependence of the string tension is very arbitrary. In this paper, we employ an improved finite temperature QCD sum rules and study the QGP background carefully for $m_{J/\Psi}$ calculation. In order to compare our result with that given by potential model, we redraw the $m_{J/\Psi}$ vs. temperature curve of ref. [4] in Fig. 2 by dashed line. We find that the mass shift of $m_{J/\Psi}$ in QGP given by IQSR is in consistent with that by potential model qualitatively.

The resonance strength vs. temperature curves for HM (curve A) and QGP (curve B) are shown in Fig. 3. The curves of Fig. 3 indicate that the resonance strength will decrease in both HM and QGP as temperature increases. Near critical temperature T_c, the resonance strength decreases suddenly. The difference of curve A and curve B is very natural because the critical temperature T_c, the mass of J/Ψ and the continuum threshold $\sqrt{S_0}$ at T_c are very different for different background.

In summary, based on the improved finite temperature QCD sum rules, we have calculated the J/Ψ suppression in hot HM and QGP background separately. We have found that the J/Ψ suppression will exist in both HM and QGP. This result is in agreement with, respectively, the RHIC experiment [5] and high energy proton-nucleus collision experiment [6]. Even though the J/Ψ suppression will occur both in HM and QGP, but we have found that a mass shift of 0.49 GeV will accompany with J/Ψ suppression in QGP. This result is in agreement with that given by potential model qualitatively. In hot HM, there is almost no mass shift when J/Ψ bound state dissolves. As was discussed in ref. [4], if one can measure this mass shift, the J/Ψ suppression may be a candidate of signatures for QGP formation.

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Figure Captions

Fig. 1. The J/Ψ mass $m_{J/\Psi}$ and the continuum threshold $\sqrt{S_0}$ vs. temperature T in hot hadronic matter.

Fig. 2. Same as Fig. 1 but in quark-gluon plasma. The dashed line denotes the result given by potential model (taken from ref. [4], $\alpha = 0.2$).

Fig. 3. The resonance strength vs. temperature T. A: in hadronic matter; B: in quark-gluon plasma.

Appendix

The expression of $\pi A(M^2)$, $a(M^2)$ and $b(M^2)$ derived in ref. [10] are

$$\pi A(M^2) = \frac{3}{16\sqrt{\pi}} \frac{4m_c^2}{\omega} G(\frac{1}{2}, \frac{5}{2}, \omega) \ , \tag{A1}$$

$$a(M^2) = \frac{4}{3\sqrt{\pi}} G^{-1}(\frac{1}{2}, \frac{5}{2}, \omega) \left[\pi - c_1 G(1,2,\omega) + \frac{1}{3} c_2 G(2,3,\omega)\right] \\ -c_2 - \frac{4\ln 2}{\pi} h(\omega) \tag{A2}$$

where

$$\begin{aligned} c_2 &= \frac{\pi}{2} - \frac{3}{4\pi} \ , \\ c_1 &= \frac{\pi}{3} + \frac{1}{2} c_2 \ , \\ h(\omega) &= \omega\, G(\frac{1}{2}, \frac{3}{2}, \omega)\, G^{-1}(\frac{1}{2}, \frac{5}{2}, \omega) \end{aligned} \tag{A3}$$

and

$$b(M^2) = -\frac{1}{2} \omega^2\, G(-\frac{1}{2}, \frac{3}{2}, \omega)\, G^{-1}(\frac{1}{2}, \frac{5}{2}, \omega) \tag{A4}$$

where $\omega = 4m_c^2/M^2$, and $G(b,c,\omega)$ is the Whittaker function,

$$G(b,c,\omega) = \frac{1}{\Gamma(c)} \int_0^\infty e^{-x} x^{c-1} (\omega + x)^{-b} \mathrm{d}x \ . \tag{A5}$$

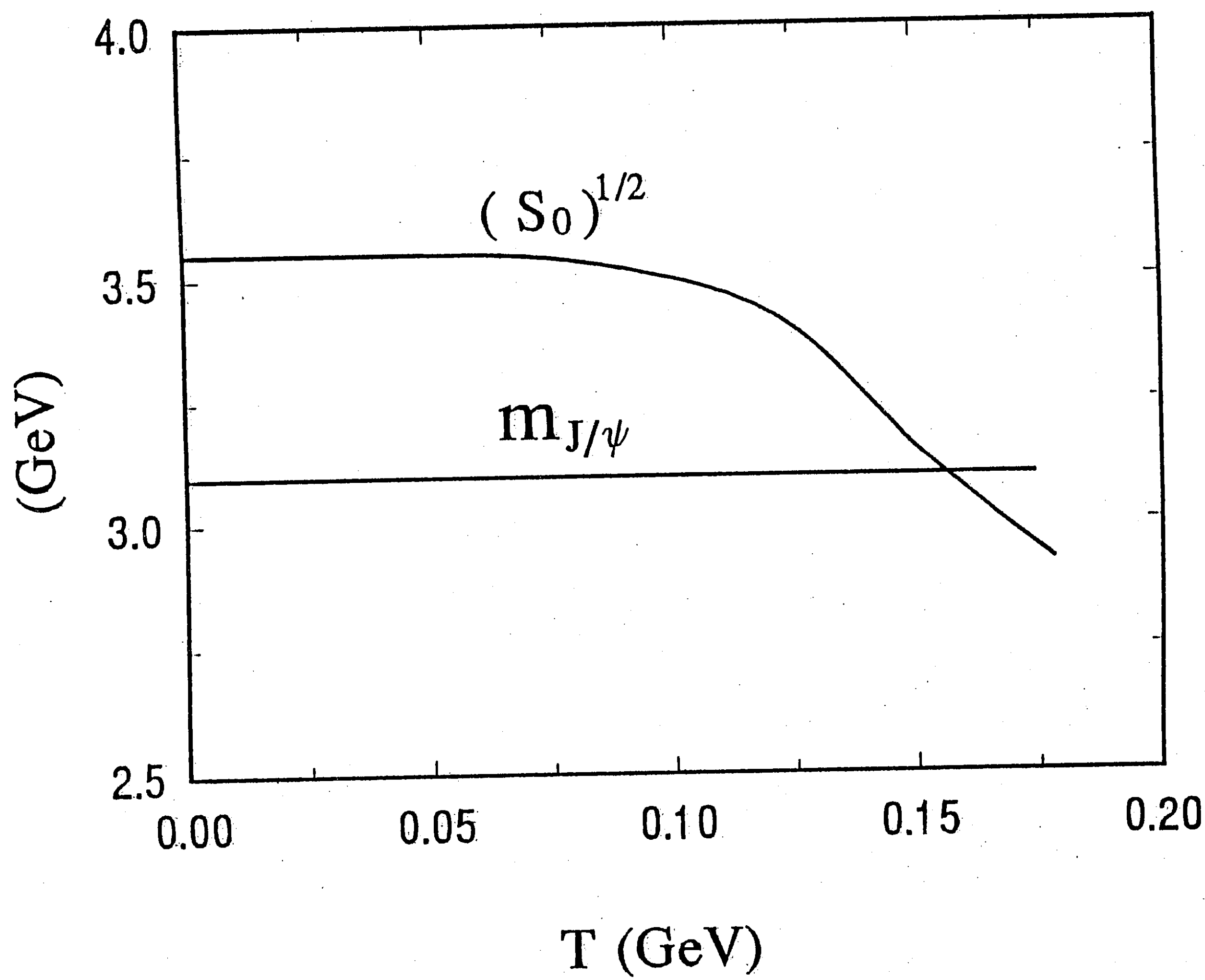

Fig.1

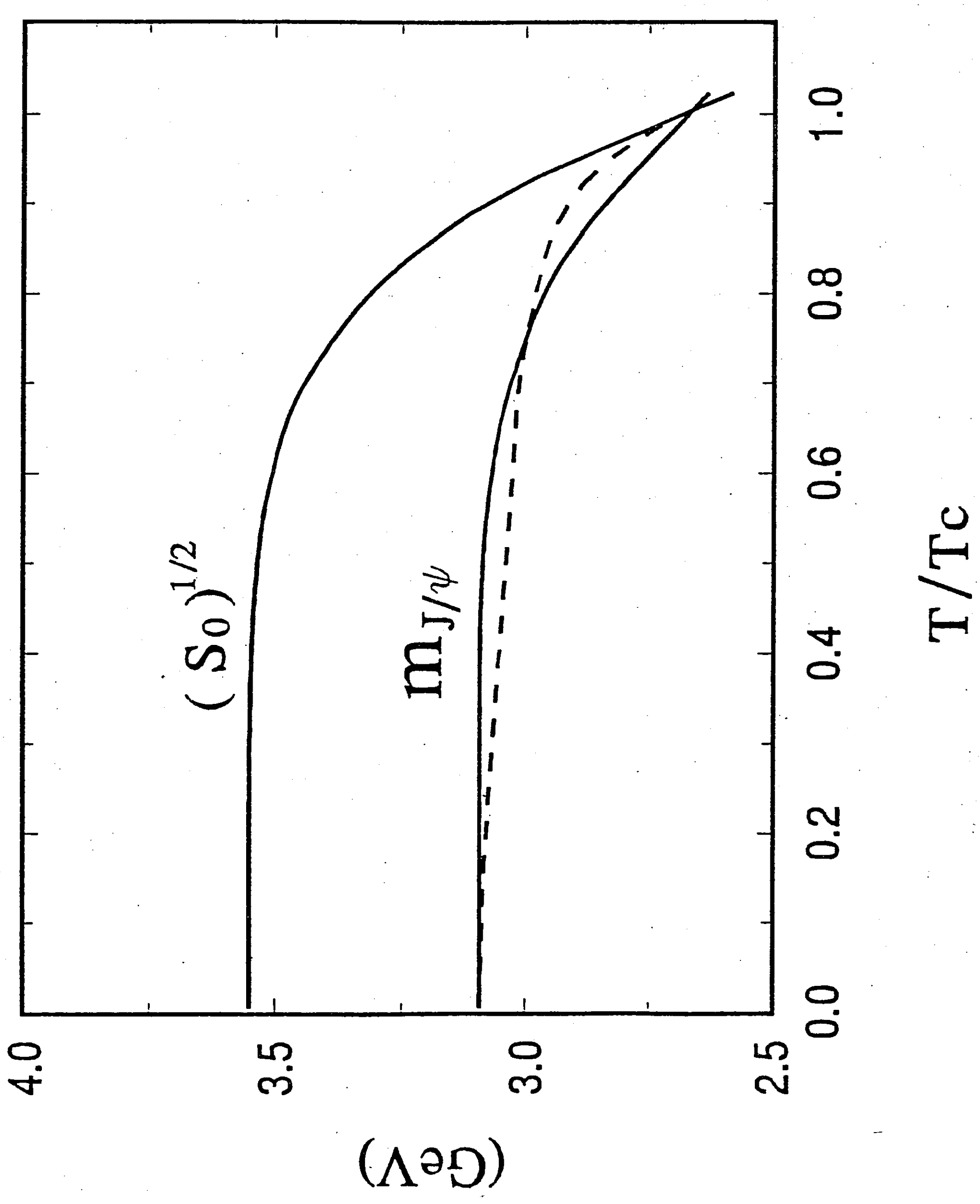

Fig. 2

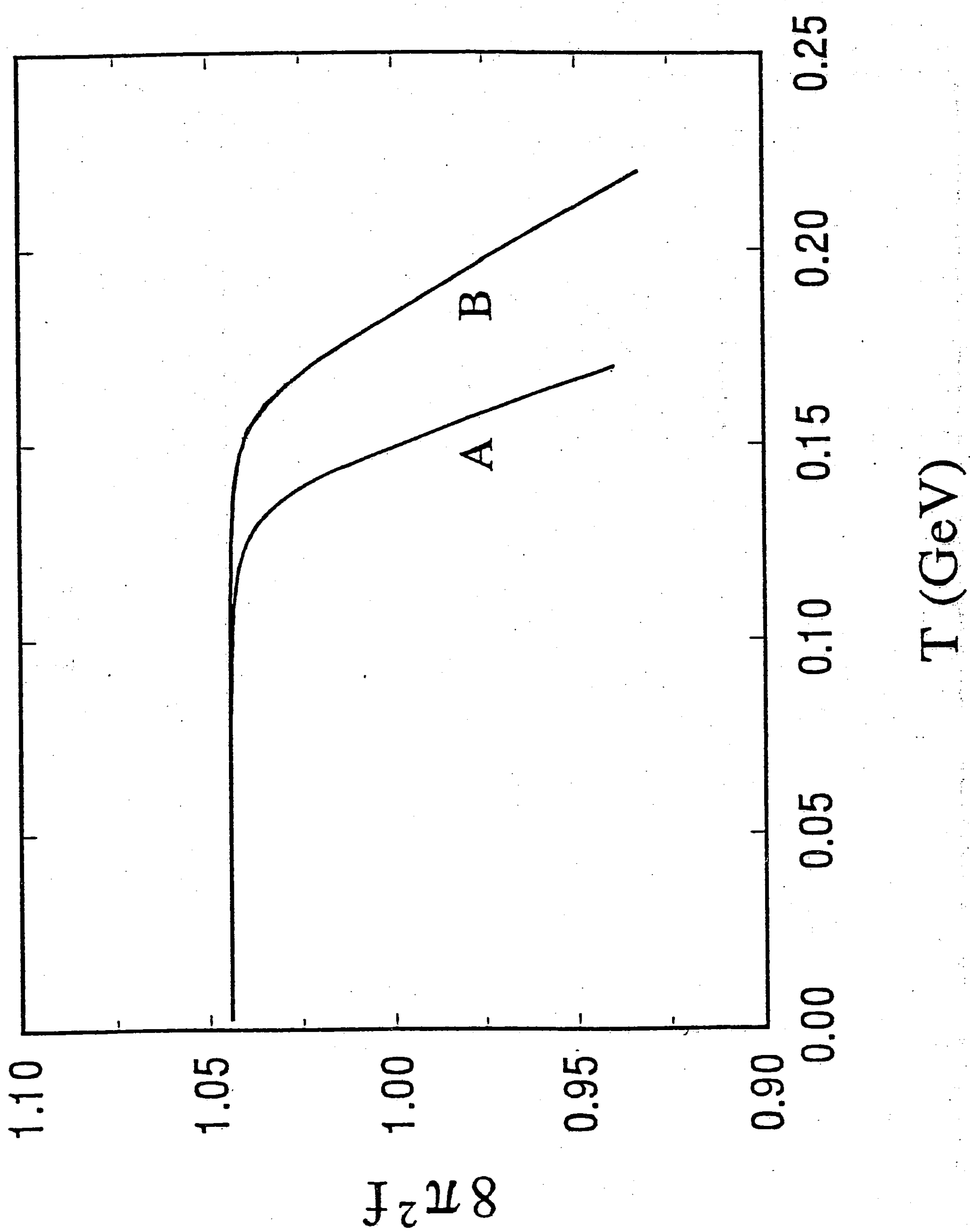

Fig. 3

Introduction to Neutrino Physics of Finite Temperature and Density [1]

Xue-Qian Li and Xin-He Meng

China Center of Advanced Science and Technology
(World Laboratory)
P.O.Box 8730, Beijing 100080, China

Department of Physics, Nankai University
Tianjin 300071, China

Abstract

Here we introduce some fundamental concepts and calculations concerning neutrino physics of finite temperature and density, based on the works of Pal and Nieves et al. We mainly stress on the difference between the physical consequences of the zero temperature and finite temperature, density at neutrinos.

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1 INTRODUCTION

From the $\beta-$decay, people warned that there was a very light fermion which was neutrino and partly participates in weak interaction, therefore a direct measurement on it was extremely difficult. But due to conservation law of angular momentun and energy-momentum one trillions that neutrino must exist and very light. Later, it is found that a which energies to gather truth a muon is different from the electron-neutrino, then the third generation $\tau-$ also was discovered. Lee and Yang established the two- component neutrino theory where the neutrino are assumed massless. If neutrino are indeed massless, there must not be a Cabibbo-Kabayashi-Maskawa matrix like quarks, because the thus neutrino we degenerate.

However, recently, people ask what happens if neutrinos are massive. Thus, just as quarks, there is a mixing matrix between neutrinos. Experimental data set upper bounds to neutrino masses. It is known, that the electron neutrino must be very light and can only have a few eV, but $\mu-$ or $\tau-$ neutrino can be must heavier as ν_τ has an upper bound of 31 MeV. If there is mixing between the neutrinos, they can oscillates into each other as they travel a fine distance. Of course, since they interact very weakly with matter, say, a neutrino can penetrate whole earth easily, observation on their reaction is difficult and misaccurate measurement of the oscillation is even harder. So far there is as evidence such oscillation yet. One reason is the mixing is small or the detecting technique are facility are not sufficient for measurements.

There are several anjectures concerning some phenomena observed in experiments. The print is the solar neutrino. The measured neutrino flux is less than estimated by the standard model about the S_{wh} (see below). The second is the dark matter. If the dark matter is nohadronic, a very possible candidate is neutrino (neutrinos), but there is a tringent restriction in the neutrino mass as the dark matter, but on other side, it must be massive. The third is the experiment about the double-β decay, if such decay occurs, there must be massive Majorana neutrinos.

As long as the neutrinos are massive, some such the stage of solar neutrino can be ensured somehow, but accompanying problems emerge. That is mainly from the experiments on the earth. No neutrino oscillation has ever been observed on the earth, as discussed above. Being dark matter, neutrino should massive, but must be less than a few eV, the universe would blow up, by the weight. If the neutrino, say, $\tau-$neutrino is more massive, if should be of a lifetime shorter than 10^6 years, otherwise it did have enough time to decay out, since the birth of the universe to present time. However, as $M_{\nu_\tau} \leq 31MeV$, the calculated lifetimes is greater than 10^{10} years which is the life of own universe. There must be a way out. The cosmological restriction to the neutrino mass is a question which an answer for any available models. Later, we will introduce the solar neutrino and then the MSW mechanism for it.

2 The solar neutrino problem

Based on the Standard Solar model, the dominant chain of reactions taking place in the sun can be summarized by the equation

$$4p \rightarrow^4 He + ze^+ + z\ nu_e + 25MeV$$

This reaction takes place in many steps. The energy released manifests mainly as photons. Therefore, as every 25 MeV energy, we is emitted, 2 neutrino are foregoned. The total luminosity of the sun is $L_\odot = 4\times10^{33}erg/s$. Thus the number of neutrinos produced per second is $2L_\odot/25MeV$. Dividing by $4\pi d^2$ where D is the distance from the Sun to the earth, we get the neutrino flux of about $6\times10^{10}cm^{-2}s^{-1}$. The reactions are listed in the following table.

Reaction	*Nameof reaction*	*E inMeV*	*Flux* $(10^{10}cm^{-2}s^{-1}$
$p+p \to^2 H + e^+ + \nu_e$	pp	≤ 0.42	$6.0 \times (1 \pm 0.02)$
$p+e^- + p \to^2 H + \nu_e$	pep	1.44	$0.014 \times (1 \pm 0.05)$
$^2 + p \to^3 He + \gamma$	-	-	-
$^3He +^3 He \to^4 He + p + p$	-	-	-
$^3He + p \to^4 He + e^+ + \nu_e$	Hep	≤ 18.77	8×10^{-7}
$^3He +^4 He \to^7 Be + \gamma$	-	-	-
$^7Be + e^- \to^7 Li + \nu_e$	^7Be	0.861	$0.47 \times (1 \pm 0.15)$
$^7Li + p \to^4 He +^4 He$	-	-	-
$^7Be + p \to^8 B + \gamma$	-	-	-
$^8B \to^8 B^* + e^+ + \nu_e$	^8B	≤ 14.06	$5.8 \times 10^{-4}(1 \pm 0.37)$
$^8B^* \to^4 He +^4 He$	-	-	-

Table I. Reactions in the pp chain.

Besides, there is CNO cycle. On the earth, the sector which can measure the neutrino flux can be divided into there categories, the radio chemical detectors, geochemical detector and electron scattering defectors. Mainly, they are ^{37}Cl and ^{71}Ge experiments. The data show obvious shortage of neutrino flux.

There are two possible ways to solve the puzzle. the first, if the electron neutrino oscillates to another type neutrino, such as $\mu-$ or $\tau-$ neutrino, this the detector which is made of electrons an protons cannot defect if, since neutrinos only participate in weak interactions. The neutrino produced in the Sun is the electron neutrino (almost), so the shortage requires a sufficient oscillation from ν_e to ν_μ or ν_τ. However, on the earth, no oscillation has ever been observed, so the oscillation effect in vacuum must be very small. In contrary, in the sun, there is a high temperature and density environment, the MSW mechanism confirms that at such a situation, oscillation can reach a resonance, so in the sun, possibly large fraction of ν_e can turn to ν_μ or ν_τ.

Another mechanism was suggested by Okun et al. They assumed that ν_e has a remarkable magnetic moment, so that in the magnetic field of the sun, the helicity of the neutrino can be flipped. Since neutrinos are very light, its is almost the chirality (if m=0, if holds exactly), the neutrinos are produced through weak interaction, so that they are left-handed. If the helicity is flipped, the chirality status would charge accordingly, namely, the left-handed neutrino would turn to right-handed. The EW bosns $W^{\pm}, Z^{\pm}$ are left-handed $(SU_L(2) \times U_Y(1))$, so the detector cannot detect right-handed neutrinos (in the framework of the standard Model), thus this fraction of neutrinos would "avoid" detecting at escaped out to make a shortage. However, neutrino is structureless neutral fermion, so that does not directly interact with electromagnetic field, i.e. has no magnetic moment at the tree level. But looking at the Feynman diagrams, it indeed interacts with photon via an EW loop. Calculation shows that at the vacuum magnetic moment of neutrino is too small to make the shortage. But in the medium, there is another strong, the magnetic moment indeed b loops can be much larger.

In the vacuum, the Hamiltonian for two generation neutrinos can be written as

$$H = |\vec{p}| + \frac{1}{2|\vec{p}|} \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} \tag{1}$$

with $\Delta \equiv m_1^2 - m_2^2$. Along the way from the sun to the earth, the evolution Schrödivger equation is

$$i\frac{d}{dx}(U^+\nu^{(f)}) = HU^+\nu^{(f)} \tag{2}$$

where. U is the Cabibbo-Kabayashi-Maskanwa matrix and the $v \approx c$.

$$H' = UHU = |\vec{p}| + \frac{m_1^2 + m_2^2}{4|\vec{p}|} + \frac{\Delta}{4|\vec{p}|} \begin{pmatrix} -cos2\theta & sin2\theta \\ sin2\theta & cos2\theta \end{pmatrix} \tag{3}$$

Thus solving the equation, one obtains

$$\begin{aligned} P_{\nu_e\nu_\mu}(x) &= | < \nu_\mu(0)\nu_e(x) > |^2 = sin^2 2\theta sin^2(\frac{\Delta}{2|\vec{p}|}x) \\ P_{\nu_e\nu_\mu}(x) &= 1 - P_{\nu_e\nu_\mu}(x) \end{aligned} \tag{4}$$

When P_{12} is the transition probability. The data obtained on the earth indicate small angle θ, and $|\vec{p}| \sim E$ ug extreme relativistic neutrino. Therefor the $P_{\nu_e\nu_\mu}$ is too small to explain the solar neutrino shortage.

But in the solar matter, the effective Leff for νe scattering is

$$
\begin{aligned}
Leff &= \frac{4G_F}{\sqrt{2}}[\bar{e}_L(p_1)\gamma^\alpha \nu_{eL}(p_2)][\bar{\nu}_{eL}(p_3)\gamma_\alpha e_L(p_4)] \\
&= \frac{4G_F}{\sqrt{2}}[\bar{e}_L(p_1)\gamma^\alpha e_L(p_4)][\bar{\nu}_{eL}(p_3)\gamma_\alpha \nu_{eL}(p_2)]
\end{aligned}
\tag{5}
$$

by the Fierz transformation, then by a thermal average, if can be written as

$$
Leff = \sqrt{2}G_F v_e \bar{\nu}_{eL}\gamma_0 \nu_{eL} \tag{6}
$$

Where n_e is the electron density in the sun. A more elegant way to achieve the same result is from the finite temperature field theory.

With the neutrino self-energy, the propagator can be written as

$$
\frac{1}{p - m\sum(p)} \tag{7}
$$

Where

$$
i\sum(p) = i(\frac{g}{\sqrt{2}})^2 \int \frac{d^4K}{(2\pi)^4}\gamma_\rho P_L \frac{ig^{\lambda\rho}}{M_W^2} \tag{8}
$$

since W is very heavy so its live can be shrunk into a point. The electron propagator is

$$
iS_e(k) = (k+m)[\frac{i}{k^2 - m_e^2} - 2\pi\delta(k^2 - m_e^2)f_F(k,n)] \tag{9}
$$

and

$$
f_{F(x)} = \frac{\theta(x)}{e^{\beta(x-\mu)}+1} + \frac{\theta(-x)}{e^{-\beta(x-\mu)}+1} \tag{10}
$$

the second term gives

$$
\sqrt{2}G_F(n_e^- - n_e^+)r_0 \tag{11}
$$

and it is the result given above.

With this correction term, the hamiltonian would change was

$$
\tilde{H} = H' + \begin{pmatrix} \sqrt{2}G_F(n_e - \frac{n_n}{2}) & 0 \\ 0 & -\frac{1}{\sqrt{2}}G_F n_n \end{pmatrix} \tag{12}
$$

where H' is the vacuum part, n_n is the neutron density which takes part in via the neutral current.

$$\tilde{H} = E + \frac{m_1^2 + m_2^2}{4E} - \frac{1}{\sqrt{2}} G n_n + \frac{1}{2E}\tilde{M}^2 \tag{13}$$

while the new oscillation angle

$$tan\tilde{2}\theta = \frac{\Delta sin2\theta}{\Delta cos2\theta - A} \tag{14}$$

when

$$A = 2\sqrt{2} G_F n_e E \tag{15}$$

$$\Delta cos2\theta = A, \qquad tan2\tilde{\theta} \to \infty, \tilde{\theta} \to \frac{\pi}{4}$$

and

$$\tilde{m}_{1,2}^2 = \frac{1}{2}[(m_1^2 + m_2^2 + A) \mp \sqrt{(\Delta cos2\theta - A)^2 + \Delta^2 sin^2 2\theta}] \to \frac{1}{2}[m_1^2 + m_2^2 + A \mp \Delta sin2\theta] \tag{16}$$

Thus a transmutation from ν_e to ν_μ occurs substantially.

For the Breit-Wigner resonance form with a width Γ, it turns to be

$$\frac{constant}{(A - A_R)^2 + \Gamma^2} \tag{17}$$

where $A_R = \Delta cos2\theta, \Gamma = \Delta sin2\theta$. A peak for the transmutation appears.

3 The Electronic Properties

In here, we mainly introduce J.Nieves and P.Pal's works. For a massive neutrino, the self energy has a form

$$R \sum L \tag{18}$$

In vacuum $\sum = ak$ where K is the neutrino momentum, but in medium

$$\sum = ak + bv \tag{19}$$

where u is the four velocity of the background, i.e., we have a new 4-vector in the problem. Compelling to the electromagnetic field, then

$$< \nu(k')|j_\alpha^{EM}|\nu(k) > \equiv \bar{u}(k')\Gamma_\alpha(k,k',v)u(k) \tag{20}$$

and the conservation of electromagnetic current demands

$$q^\alpha \Gamma_\alpha(k,k',v) = 0. \tag{21}$$

So in general, in vacuum, Γ_α can be written as

$$\Gamma_\alpha = (q^2 r_\alpha - q_\alpha q)(R + r r_5) + i\delta_{\alpha\beta} q^\beta (D_M + D_E r_5). \tag{22}$$

For a Majorana neutrino

$$R = D_M = D_E = 0 \tag{23}$$

only r can be non-zero, it means that in the vacuum, a Majorana neutrino can have an axial charge radio, but neither charge radio, and nor magnetic and electric dipole moments. However in the background

$$\Gamma'_\alpha(k,k',v) = id'_E(k,k',v)(r_\alpha v_\beta - r_\beta v_\alpha)q^\beta r_5 + iD'_M(k,k',v)(\epsilon_{\alpha\beta\lambda\rho} r^\beta r_5 q^\lambda v^\rho \tag{24}$$

D'_E and D'_M are new form factors. One has

$$O_E = D_E \bar{v}\sigma_{\alpha\beta} v \tilde{F}^{\alpha\beta}, \qquad O_M = D_M \bar{v}\sigma_{\alpha\beta} v F^{\alpha\beta} \tag{25}$$

and

$$O'_E = D'_E \bar{v}\sigma_{\alpha\beta}\bar{v} r_\alpha r_s \nu v_\beta F^{\alpha\beta}, \qquad O'_M = D'_M \bar{v} r_\alpha r_5 \nu v_\beta \tilde{F}^{\alpha\beta} \tag{26}$$

In the non-relativistic limit,

$$O'_E \to d'_E \phi^+ \sigma \phi \vec{E}, \qquad O'_M \to d'_m \phi^+ \sigma \phi \vec{B} \tag{27}$$

It is interesting to notice that for non-relativistic particles and antiparticles, $O_E + O_M$ reduces

$$d_E(\vec{s} - \bar{s}) \cdot \vec{E} + d_M(\vec{s} - \bar{s}) \cdot \vec{B} \tag{28}$$

where $\vec{s}$ and $\bar{s}$ are spin of ν and $\bar{\nu}$. For Majorana neutrino $\nu_M = \bar{\nu}_M$, so $d_E = d_M = 0$, but for $O'_E + O'_M$, it is

$$d'_E(\vec{s} - \bar{s}) \cdot \vec{E} - d'_M(\vec{s} - \bar{s}) \cdot \vec{B} \tag{29}$$

It is not zero for Majorana neutrino, for Divac neutrinos, it is $d + d'$, but for antineutrinos, becomes $-d + d'$, so that they are not exactly opposite of each other in a medium.

One can calculate d'_E and d'_M in the medium by the Standard Model. There are several Feynman diagrams which can contribute to the effective compelling $\Gamma_\mu(k, k', v)$. Since W is very heavy, so to the leading order, only two of need to be accounted.

They are

$$\begin{aligned} -i\Gamma_\mu^{(W)} &= \frac{1}{2} e g^2 \int \frac{d^4 p}{(2\pi)^4} r^\alpha L_i S_F(p-q) r_\mu i S_F(p) r_\alpha L \frac{1}{(k-p)^2 - m_n^2} \\ -i\Gamma_\mu^{(Z)} &= -\frac{e g_Z^2}{q^2 - M_Z^2} r^\alpha L \int \frac{d^4 p}{(2\pi)^4} T_r [i S_F(p-q) r_\mu i S_F(\rho) r_\alpha (a_Z + b_Z r_5)] \end{aligned} \tag{30}$$

After a long and difficult calculation, one obtains

$$\Gamma_\mu^{(W,Z)} = T_{\mu v}^{(W,Z)} r^v L \tag{31}$$

and

$$J_\mu^{(Z)} = \frac{4 e g_Z^2}{M_Z^2} \int \frac{d^3 p}{(2\pi)^3 2E} [a_E (f_- + f_+)[$$

Once one writes

$$\Gamma'_\mu = [F_1 \tilde{g}_{\mu v} r^v + F_2 \tilde{v}_\mu + i F_3 (r_\mu v_\nu - r_v v_\mu) q^v + i F_4 \epsilon_{\mu v \alpha \beta} r^\nu q^\alpha v^\beta] L \tag{33}$$

where the prime means dropping out the zero-temperature part, and

$$\begin{aligned} F_1 &= J_T + \frac{\Omega^2}{Q^2}(J_L - J_T) \\ F_2 &= \frac{1}{\tilde{v}^2}(J_L - J_T) \\ iF_3 &= -\frac{\Omega}{Q^2}(J_L - J_T) \\ F_4 &= \frac{J_P}{Q} \end{aligned} \tag{34}$$

Further, one has $iD'_E(r_\mu v_\nu - r_\nu v_\mu)q^\nu r_5 + iD'_M\epsilon_{\mu\nu\alpha\beta}r^\nu r_5 q^\alpha v^\beta$, so

$$\begin{aligned} D'_E &= -\frac{i\Omega}{2Q}(J_L - J_T) \\ D'_M &= -\frac{J_P}{2Q}. \end{aligned} \tag{35}$$

The $P_\mu^{(W)}$ evaluation is similar, but only a_E and b_E are replaced by appropriate setting $\frac{g_E^2}{M_E^2} \rightarrow, a_E \rightarrow \frac{1}{2}, b_E = -\frac{1}{2}$.

For a Majorana neutrino

$$\Gamma_\mu^{Majorana}(k, k', v) = \Gamma_\mu(k, k', v) + \Gamma_\mu^C(-k', -k, v) \tag{36}$$

So the electric and magnetic dipole moments are not zero.

2. In the medium the decay rate would be completely different from that in vacuum.

As discussed above, the cosmological constraint to the neutrinos whose mass may be larger than ten eV is that it lifeline must be shouter than the lifetime of our universe, otherwise the dark matter constituted by such neutrinos would be too heavy and blow up. Therefore as believed, $M_{\nu_\mu} \leq 150KeV$, and $M_{\nu_\tau} \leq 31MeV$ set upper bounds for ν_μ and ν_τ, if they are close to the bounds, they must decay fast enough.

In the framework of the Standard Model, heavy neutrinos only have two important channels, namely $\nu_1 \rightarrow \nu_2 + \gamma$ and $\nu_1 \rightarrow \nu_2 + e^+ + e^-$. There have been some alternative models, for example, $\nu_1 \rightarrow \nu_2 + x$ is an auxin. Since auxin has never been observed, this channel is less reliable, even not completely excluded. However, by the Standard Model, in the vacuum ν_τ of 31 MeV decays very slowly, so that recently Babu et al claimed that the window for heavy neutrino is chased, because they could not have sufficient time to decay before the universe reaches present stage.

But Nieve et al's work gives another way out, that in the early universe, temperature as matter density are very high, so neutrino decay occurred in a background which is very different from the vacuum. D'Olivo, Nieves and Pal calculated the radiative neutrino decay rate in the medium.

The formulae are similar to that given in (1). but there is a real photon emitted, and the two fermion external legs represent different type neutrinos. The Feynman diagram is the kenguin diagram.

$$iM' = -i\bar{u}(k')\Gamma'_\alpha u(k)\epsilon^{\alpha *}(q) \tag{37}$$

where $q = k - k'$ is the momentum of the emitted photon. and M' denotes that only background-dependent part is considered.

$$\Gamma'_\alpha = U^*_{e\nu} U_{e\nu'} J_{\alpha\beta} r^\beta L \tag{38}$$

where $U_{e\nu}$ is the Cabibbo-Kobayashi-Maskawa matrix entries. similar to the manipulation part (1), one has

$$|M'|^2 = m^2 |U^*_{e\nu} U_{e\nu'}|^2 |J_T|^2 [\frac{(k+k')v}{\Omega} - \frac{m^2}{2\Omega}] \tag{39}$$

where the background is supposed to attain electron and positronsonly.

$$d\Gamma' = \frac{1}{2k_0}(2\pi)^4 \delta^4(kk' - g)|M'|^2 \frac{d^3k}{(2\pi)2k_0} \frac{d^3q}{(2\pi)^3 2q^2} \tag{40}$$

Thus

$$\Gamma' = \frac{m}{16\pi}|U^*_{e\nu} U_{e\nu'}|^2 |J_T|^2 F(v) \tag{41}$$

where $F(v) = (1 - v^2)^{\frac{1}{2}}[\frac{2}{v} ln(\frac{1+v}{1-v}) - 3]$ and v in the magnitude of three-velocity of the decaying neutrino in the rest frame of the medium. Thus at non-relativistic background

$$\Gamma'^{(NR)} = \frac{1}{2}\alpha G_F^2 |U^*_{e\nu} U_{e\nu'}|^2 F(v) \frac{m n_e^2}{m_e^2} = (8 \times 10^{21} s)^{-1} (U_{e\nu} U_{e\nu'})^2 \frac{m}{1KeV} [\frac{n_e}{10^{24} cm^{-3}}]^2 \tag{42}$$

and for the extra-relativistic background.

$$\Gamma'^{(ER)} = \frac{1}{2}\alpha G_F^2 |U^*_{e\nu} U_{e\nu'}|^2 F(v) \frac{m T^4}{36} = (5 \times 10^4 s)^{-1} (U_{e\nu} U_{e\nu'})^2 \frac{m}{1MeV} [\frac{T}{1MeV}]^4 \tag{43}$$

In the vacuum

$$\Gamma = \frac{1}{2}\alpha F_F^2 (\frac{3}{32\pi^2})^2 m^5 | \sum_{l=e,\mu,\tau} \frac{m_e^2}{m_W^2} U^*_{e\nu} U_{e\nu'}|^2 \tag{44}$$

If τ is the dominate contribution,

$$\frac{\Gamma'^{(NR)}}{\Gamma} \simeq 1.3 \times 10^{19} r F(v) (\frac{n_e}{10^{24} cm_{-3}})^2 (\frac{1ev}{m})^4 \quad (45)$$

$$\frac{\Gamma'^{(ER)}}{\Gamma} \simeq 1.5 \times 10^{9} r F(v) (\frac{T}{m})^4$$

where

$$r = |U^*_{e\nu} U_{e\nu'}|^2 / |U^*_{\tau\nu} U_{\nu\tau'}|^2 \quad (46)$$

If $\nu = \nu_\tau, \nu' = \nu_e$, it is easy to see that both the C-K-M entries at the denominator and numerator eve double Cabibbo suppressed, so it is believed that ν is not far from unity. $F(v)$ is between $1 \sim 1.55$. Therefore $\frac{\Gamma'^{(ER)}}{\Gamma}$ could be very large.

If can be realized in the hot and dense background, such as the sun and the early universe.

For $\rightarrow \nu' + e^+ + e^-$, since the imaginary part of the penguin diagram exists, so that a more interesting situation stands. This is under work.

4 CONCLUSION

So far we have seen that the properties of neutrinos in medium can be quite different from that in vacuum. This investigation is motivated by the phenomena of solar neutrino flux shortage dark matter constraints and the experiments on the earth (vacuum). To solve all the problems, one must consider that most of the intriguing problems can find promising answers by introducing the medium effects.

Therefore it is worth to study it in more detail. The futural relativistic heavy ion collision RHIC can provide an ideal high temperature and density region or even the quark-gluon-plasma (QGP) can be realized, so there the neutrino (heavy) production can be a good spot for a direct observation of neutrino in medium. But since the electrons and positrons in RHIC are not very rich, one cannot be too optimistic unless there is something beyond the Standard Model.

Anyhow, this study is valuable for understanding the early universe and cosmology.