New Physics: Sae Mulli,
Vol. 65, No. 8, August 2015, pp. 825∼836

DOI: 10.3938/NPSM.65.825

Approximate Analytical Solutions of the Klein-Gordon Equation with an Exponential-type Potential

Akpan N. IKOT* · Hillary P. OBONG · Tamunoimi M. ABBEY
Theoretical Physics Group, Department of Physics,
University of Port Harcourt, Choba P.M.B 5323 Port Harcourt, Nigeria

Joy D. OLISA
Department of Mathematics and Statistics, University of Port Harcourt, Choba P.M.B 5323 Port Harcourt, Nigeria

(Received 13 February 2015 : revised 22 May 2015 : accepted 22 June 2015)

The approximate analytical bound and scattering state of the Klein-Gordon equation in D-dimensions for an arbitrary l-state for a seven-parameter exponential-type potential is presented. We obtain the bound-state solutions by using the supersymmetry quantum mechanics method, and the normalized wave function and the scattering phase-shift factors are determined by using the properties of the partial-wave analysis technique. Special cases of the seven-parameter exponential-type potential are discussed in detail.

PACS numbers: 03.65.Ge, 03.65.Ca, 03.65.Fd
Keywords: Klein-Gordon equation (KGE), Seven parameter exponential potential, Supersymmetric quantum mechanics

I. INTRODUCTION

The Klein-Gordon equation (KGE) is the well-known relativistic wave equation that describes spin-zero particles. It is also known that the analytic solutions of the KGE are possible only in a few cases such as harmonic and Coulomb potentials [1, 2]. However, for arbitrary l-states ($l \neq 0$), the Klein-Gordon does not admit an exact solution. Thus the KGE can be solved approximately using different approximation schemes [3–5]. Improved Greene and Aldrich approximation has been used by Chen *et al.* [6] to investigate the bound state solution of the Schrödinger equation with exponential-type model potentials. Quantum mechanical methods such as Nikiforov-Uvarov [7], exact quantization rule [8], Supersymmetric Quantum Mechanics (SUSYQM) [9] and asymptotic iteration method (AIM) [10] have been employed by different authors to obtain the bound state solutions and scattering state solutions of Schrödinger, Klein-Gordon and Dirac equations for different potentials [11, 12]. Interestingly, the study of the relativistic wave equation in the recent years particularly the Klein-Gordon and Dirac equation have attracted the attention of many authors because of the solutions that these equations play in getting the relativistic effect in nuclear physics and other areas [13]. With the introduction of the SUSYQM and the concept of shape invariance in physics [14], the study of the solvable potential models in both relativistic and non-relativistic quantum mechanics have received a great interest [15]. The concept of SUSYQM allows one to determine the eigenfunctions and eigenvalues analytically for solvable potential model using algebraic operator formulation without solving the Schrödinger-like differential equation by standard series method [16]. Recently, KGE in generalized D-dimensions for different potentials is getting the attention of researchers [17–20]. Liu *et al.* [21] investigated the molecular KGE with improved Tietz potential in D-dimensions. Zhang *et al.* [22] studied the KGE in D-dimensions for sodium dimer. Liu *et al.* [23] analyzed the

*E-mail: ndemikotphysics@gmail.com

This is an Open Access article distributed under the terms of the Creative Commons Attribution Non-Commercial License (http://creativecommons.org/licenses/by-nc/3.0) which permits unrestricted non-commercial use, distribution, and reproduction in any medium, provided the original work is properly cited.

properties of diatomic molecules with improved Rosen-Morse potential. Tang *et al.* studied the Schrödinger equation with improved Tietz potential [24]. Cesium and Sodium dimmers in D-dimension were investigated by Hu *et al.* [25] using SUSYQM. Jia *et al.* [26] studied the diatomic molecules with improved Manning-Rosen potential via SUSYQM. The D-dimensional energies for lithium dimer was analyzed by Hu *et al.* [27]. The improved expression for Schioberg potential energy models for diatomic molecules were examined by Wang *et al.* [28]. The equivalence of the Wei potential and Tietz potential for diatomic molecules had been investigated by Jia *et al.* [29]. Jia *et al.* [30] proposed five exponential type potential and Setare and Nazari [31] studied the Dirac equation with five-parameter exponential potential within the framework of spin and pseudospin symmetry. Following the work of Jia *et al.* [30], we proposed a seven parameter-type potential of the form,

$$V(r) = A + \frac{B}{(q+e^{2\alpha r})} + \frac{C}{(q+e^{2\alpha r})^2} + \frac{Fbe^{2\alpha r}}{(q+e^{2\alpha r})} + \frac{Gbe^{2\alpha r}}{(q+e^{2\alpha r})^2} \tag{1}$$

where A, B, C, F, G are the potential parameters, q is the deformation parameter , $b = e^{2\alpha r_e}$, r_e is the distance from equilibrium position and α is the screening parameter. Consequently, the choice A, B, C, F and G parameters leads to well-known exponential-type potential as a special case of Eq. (1).

II. BOUND STATE SOLUTION OF KLEIN-GORDON EQUATION IN D-DIMENSION

The KGE in higher dimension for spherically symmetric potential reads as [17–20],

$$-\hbar^2 c^2 \Delta_D \psi_{n,l,m}(r,\Omega_D) = \{[E_{n,l} - V(r)]^2 - [mc^2 + S(r)]^2\}\psi_{n,l,m}(r,\Omega_D) \tag{2}$$

where $E_{n,l}, m, V(r)$ and $S(r)$ are the relativistic energy, rest mass, the repulsive vector potential and the attractive scalar potential, respectively and Δ_D is defined as

$$\Delta_D = \nabla_D^2 = \frac{1}{r^{D-1}}\frac{\partial}{\partial}\left(r^{D-1}\frac{\partial}{\partial r}\right) - \frac{\Lambda_D^2(\Omega_D)}{r^2}. \tag{3}$$

The total wave function in D-dimension is written as,

$$\psi_{n,l,m}(r,\Omega_D) = R_{n,l}(r)Y_l^m(\Omega_D). \tag{4}$$

The term $\frac{\Lambda_D^2(\Omega_D)}{r^2}$ is the generalization of the centrifugal term for the higher dimensional space. The eigenvalues of $\Lambda_D^2(\Omega_D)$ are defined by the relation,

$$\Lambda_D^2(\Omega_D)Y_l^m(\Omega_D) = l(l+D-2)Y_l^m(\Omega_D) \tag{5}$$

where $Y_l^m(\Omega_D), R_{n,l}$ and l represent the hyper spherical harmonics, the hyper radial wave function and the orbital angular momentum quantum number, respectively. Now substituting ansatz $R_{n,l}(r) = r^{-\frac{(D-1)}{2}}F_{n,l}(r)$ for the wave function into equation (2) yields,

$$\left\{\hbar^2 c^2 \frac{d^2}{dr^2} + (E_{n,l} - V(r))^2 - (mc^2 + S(r))^2 - \frac{(D+2l-1)(D+2l-3)\hbar^2 c^2}{4r^2}\right\}F_{n,l}(r) = 0. \tag{6}$$

Considering equal scalar and vector potential as the seven parameter, $S(r) = V(r)$ we obtain the following second-order Schrödinger-like equation

$$\left[\frac{d}{dr^2} + \frac{E_{n,l}^2 - m^2c^4}{\hbar^2c^2} - 2\left(\frac{mc^2 + E_{n,l}}{\hbar^2c^2}\right)\left\{A + \frac{B}{(q+e^{2\alpha r})} + \frac{C}{(q+e^{2\alpha r})^2} + \frac{Fbe^{2\alpha r}}{(q+e^{2\alpha r})} + \frac{Gbe^{2\alpha r}}{(q+e^{2\alpha r})^2}\right\}\right]F_{n,l}(r) - \left\{\frac{(D+2l-1)(D+2l-3)}{4r^2}\right\}F_{n,l}(r) = 0 \quad (7)$$

where we have adopted the non-relativistic limit proposed by Alhaidari *et al.* [32] that the potential is $V(r)$ not $2V(r)$.

It is well-known that Eq. (7) cannot be solved exactly except for $l = 0$. In order to find the solutions of Eq. (7), we apply the Pekeris approximation for the centrifugal term [33]. This can be achieved by taking $x = (r - r_e)/r_e$ and expand the centrifugal term in power series around $x = 0$ as follows:

$$\frac{(D+2l-1)(D+2l-3)}{4r^2} \approx \frac{(D+2l-1)(D+2l-3)}{4r_e^2}\frac{1}{(1+x)^2} = \eta(1 - 2x + 3x^2 + \cdots) \quad (8)$$

where $\eta = \frac{(D+2l-1)(D+2l-3)}{4r_e^2}$.

Liu *et al.* [21] proposed a centrifugal term in the form, this approximation is valid for $\alpha r \ll 1$

$$\frac{\eta}{r^2} = \eta\left(d_0 + d_1\frac{1}{q+e^{2\alpha r}} + d_2\frac{1}{(q+e^{2\alpha r})^2}\right) \quad (9)$$

where d_0, d_1 and d_2 are constants coefficients obtained by expanding Eq. (9) in series around $x = 0$,with the transformation $x = (r - r_e)/r$ as

$$\begin{aligned}\frac{\eta}{r^2} &= \eta\left(d_0 + d_1\frac{1}{q+e^{2\alpha r_e}} + d_2\frac{1}{(q+e^{2\alpha r_e})^2}\right) - 2\alpha r_e e^{2\alpha r_e}\left(d_1\frac{1}{(q+e^{2\alpha r_e})^2} + d_2\frac{1}{(q+e^{2\alpha r_e})^3}\right)x \\ &= +4\eta\alpha^2 r_e^2 e^{2\alpha r_e}\left(\frac{1/2d_1(e^{2\alpha r_e} - q)}{(q+e^{2\alpha r_e})^3} + \frac{d_2(2e^{2\alpha r_e} - q)}{(q+e^{2\alpha r_e})^4}\right)x^2 + \cdots.\end{aligned} \quad (10)$$

Comparing Eqs. (8) and (10) up to second-order degrees in the series expansion, we obtain the following terms for the coefficients d_0, d_1 and d_2 as follows:

$$d_0 = 1 + \frac{1}{4\alpha^2 r_e^2}(3 - 6\alpha r_e + 6qe^{-2\alpha_e} + 3q^2e^{-4\alpha r_e} - 4q\alpha r_e e^{-2\alpha r_e} + 2q^2\alpha r_e e^{-4\alpha r_e} \quad (11)$$

$$d_1 = \frac{1}{2\alpha^2 r_e^2}(-9q + 6q\alpha r_e - 3e^{2\alpha r_e} + 4\alpha r_e e^{2\alpha r_e} - 9q^2e^{-2\alpha r_e} - 3q^3e^{-4\alpha r_e} - 2q^3\alpha r_e e^{-4\alpha r_e} \quad (12)$$

$$\begin{aligned}d_2 &= \frac{1}{4\alpha^2 r_e^2}(18q^2 + 12qe^{2\alpha r_e} + 3e^{4\alpha r_e} - 4q\alpha r_e e^{2\alpha r_e} - 2\alpha r_e e^{4\alpha r_e} + 12q^3e^{-2\alpha r_e} + 3q^4e^{-4\alpha r_e} \\ &= +4q^3\alpha r_e e^{-2\alpha r_e} + 2q^4\alpha r_e e^{-4\alpha r_e}.\end{aligned} \quad (13)$$

Substituting Eq. (9) into Eq. (7), we get

$$-\frac{d^2F_{n,l}}{dr^2}+\left(\frac{\frac{2(mc^2+E_{n,l})B}{\hbar^2c^2}+\eta d_1+\frac{2b(mc^2+E_{n,l})F}{\hbar^2c^2}e^{2\alpha r}}{q+e^{2\alpha r}}\right)F_{n,l}+\left(\frac{\frac{2(mc^2+E_{n,l})C}{\hbar^2c^2}+\eta d_2+\frac{2b(mc^2+E_{n,l})G}{\hbar^2c^2}e^{2\alpha r}}{(q+e^{2\alpha r})}\right)F_{n,l}(r)$$
$$=\tilde{E}_{n,l}F_{n,l}(r), \tag{14}$$

where

$$\tilde{E}_{n,l}=\frac{E_{n,l}^2-m^2c^4}{\hbar^2c^2}-\frac{2(mc^2+E_{n,l})A}{\hbar^2c^2}-\eta d_0. \tag{15}$$

In the SUSYQM formulation, the ground-state wave function $F_{0,l}(r)$ is given by [14,16,21]

$$F_{0,l}(r)=\exp\left(-\int W(r)dr\right), \tag{16}$$

where the integrand is called the super potential and the Hamiltonian is composed of the raising and lowering operators

$$H_- = \hat{A}^+\hat{A}=-\frac{d^2}{dr^2}+V_-(r) \tag{17}$$
$$H_+ = \hat{A}\hat{A}^+=-\frac{d^2}{dr^2}+V_+(r), \tag{18}$$

with

$$\hat{A} = \frac{d}{dr}-W(r), \tag{19}$$
$$\hat{A}^+ = -\frac{d}{dr}-W(r), \tag{20}$$
$$V_\pm(r) = W^2(r)\mp W'(r). \tag{21}$$

The super potential obeys the associated Riccati equation:

$$W^2(r)\mp W'(r)=V_{eff}(r)-\tilde{E}_{0,l}. \tag{22}$$

Based on the SUSYQM, we choose the super potential in the form,

$$W(r)=a+\frac{b'e^{2\alpha r}}{(q+e^{2\alpha r})}, \tag{23}$$

where

$$a = \frac{1}{2b'}\left(-b'^2+\frac{2Fb(mc^2+E_{n,l})}{\hbar^2c^2}-\left(\frac{2(mc^2+E_{n,l})B}{\hbar^2c^2q}+\frac{\eta d_1}{q}+\frac{2(mc^2+E_{n,l})C}{\hbar^2c^2q^2}+\frac{\eta d^2}{q^2}\right)\right) \tag{24}$$
$$b' = -\alpha\pm\alpha\sqrt{1+\frac{1}{\hbar^2c^2\alpha^2}\{2(mc^2+E_{n,l})\left(\frac{C}{q^2}-\frac{bG}{q}\right)+\frac{\hbar^2c^2\eta d_2}{q^2}\}} \tag{25}$$
$$\tilde{E}_{0,l} = \frac{2(mc^2+E_{n,l})B}{\hbar^2c^2q}+\frac{\eta d_1}{q}+\frac{2(mc^2+E_{n,l})C}{\hbar^2c^2q^2}+\frac{\eta d_2}{q^2}-a^2. \tag{26}$$

We construct the pair of supersymmetric partner potentials $V_+(r)$ and $V_-(r)$ as follows,

$$\begin{aligned}V_+(r) &= W^2(r)+\frac{dW(r)}{dr},\\ &= \frac{-qb'(b'-2\alpha)e^{2\alpha r}}{(q+e^{2\alpha r})^2}+(2ab'+b'^2)\frac{e^{2\alpha r}}{(q+e^{2\alpha r})}+a^2\\ &= \frac{-qb'(b'-2\alpha)e^{2\alpha r}}{(q+e^{2\alpha r})^2}+\left(\frac{2Fb(mc^2+E_{n,l})}{\hbar^2c^2}-\frac{2(mc^2+E_{n,l})B}{\hbar^2c^2q}-\frac{\eta d_1}{q}-\frac{2(mc^2+E_{n,l})C}{\hbar^2c^2q^2}-\frac{\eta d_2}{q^2}\right)\frac{e^{2\alpha r}}{(q+e^{2\alpha r})}+a^2\end{aligned} \tag{27}$$

$$\begin{aligned} V_-(r) &= W^2(r) - \frac{dW(r)}{dr}, \\ &= \frac{-qb'(b'+2\alpha)e^{2\alpha r}}{(q+e^{2\alpha r})^2} + (2ab'+b'^2)\frac{e^{2\alpha r}}{(q+e^{2\alpha r})} + a^2 \\ &= \frac{-qb'(b'+2\alpha)e^{2\alpha r}}{(q+e^{2\alpha r})^2} + \left(\frac{2Fb(mc^2+E_{n,l})}{\hbar^2c^2} - \frac{2(mc^2+E_{n,l})B}{\hbar^2c^2q} - \frac{\eta d_1}{q} - \frac{2(mc^2+E_{n,l})C}{\hbar^2c^2q^2} - \frac{\eta d_2}{q^2}\right)\frac{e^{2\alpha r}}{(q+e^{2\alpha r})} + a^2. \end{aligned} \tag{28}$$

It can be observed that the partner potential are shape invariant via mapping of the form $b' \to b' - 2\alpha$. Also, it is easy to check the shape-invariance condition

$$V_+(r,\rho_0) = V_-(r,\rho_i) + R(\rho_i) \tag{29}$$

which holds via the mapping $b' \to b' - 2\alpha$. In our study $\rho_0 = b'$ and ρ_i is a function of ρ_0, *i.e.*, $\rho_1 = f(\rho_0) = \rho_0 - 2\alpha$. Thus, $\rho_n = \rho_0 - 2\alpha n$ and from Eq. (29), we write

$$\begin{aligned} \tilde{E}^-_{n,l} &= \sum_{k=1}^{n} R(\rho_k) = \left[\frac{1}{2\rho_0}\left(-\rho_0^2 + \frac{2Fb(mc^2+E_{n,l})}{\hbar^2c^2} - \left(\frac{2(mc^2+E_{n,l})B}{\hbar^2c^2q} + \frac{\eta d_1}{q} + \frac{2(mc^2+E_{n,l})C}{\hbar^2c^2q^2} + \frac{\eta d_2}{q^2}\right)\right)\right]^2 \\ &= -\left[\frac{1}{2\rho_n}\left(-\rho_n^2 + \frac{2Fb(mc^2+E_{n,l})}{\hbar^2c^2} - \left(\frac{2(mc^2+E_{n,l})B}{\hbar^2c^2q} + \frac{\eta d_1}{q} + \frac{2(mc^2+E_{n,l})C}{\hbar^2c^2q^2} + \frac{\eta d_2}{q^2}\right)\right)\right]^2. \end{aligned} \tag{30}$$

Using Eqs. (15), (26) and (30), we obtained the energy spectrum for the seven parameter potential model for the KGE in D-dimension as follows,

$$\tilde{E}_{n,l} = \tilde{E}^-_{n,l} + \tilde{E}_{0,l} = -\left[\frac{1}{2\rho_n}\left(-\rho_n^2 + \frac{2Fb(mc^2+E_{n,l})}{\hbar^2c^2} - \left(\frac{2(mc^2+E_{n,l})B}{\hbar^2c^2q} + \frac{\eta d_1}{q} + \frac{2(mc^2+E_{n,l})C}{\hbar^2c^2q^2} + \frac{\eta d_2}{q^2}\right)\right)\right]^2 \tag{31a}$$

or in a more compact form we write Eq.(31a) as

$$\begin{aligned} &\frac{E_{n,l}^2 - m^2c^4}{\hbar^2c^2} - \frac{2(mc^2+E_{n,l})A}{\hbar^2c^2} - \eta d_0 = \\ &-\left[\frac{1}{2\rho_n}\left(-\rho_n^2 + \frac{2Fb(mc^2+E_{n,l})}{\hbar^2c^2} - \left(\frac{2(mc^2+E_{n,l})B}{\hbar^2c^2q} + \frac{\eta d_1}{q} + \frac{2(mc^2+E_{n,l})C}{\hbar^2c^2q^2} + \frac{\eta d_2}{q^2}\right)\right)\right]^2. \end{aligned} \tag{31b}$$

Equation (31b) is the relativistic energy equation for seven parameter exponential-type potential in the presence of equal scalar and vector potentials. This result can be used to study the relativistic rovibrational energy for a diatomic system in the presence of the Manning-Rosen system and others known potential models as special cases as discussed in section 4. In order to determine the corresponding wave function for the system, we make a change of variable in Eq. (14) by writing $z = qe^{-2\alpha r}$ to obtain

$$\frac{d^2F_{n,l}}{dz^2} + \frac{1}{z}\frac{dF_{n,l}}{dz} + \frac{1}{z^2(1-z)^2}(\chi_1 z^2 - \chi_2 z + \chi_3)F_{n,l}(z) = 0 \tag{32}$$

where

$$\chi_1 = \left(\frac{\tilde{E}_{n,l}}{4\alpha^2} + \frac{(mc^2+E_{n,l})B}{4\alpha^2 q\hbar^2 c^2} + \frac{\eta d_1}{4\alpha^2 q} + \frac{(mc^2+E_{n,l})C}{4\alpha^2 q^2\hbar^2 c^2} + \frac{\eta d_2}{4\alpha^2 q^2}\right) \tag{33}$$

$$\chi_2 = \left(\frac{\tilde{E}_{n,l}}{4\alpha^2} + \frac{(mc^2+E_{n,l})B}{4\alpha^2 q\hbar^2 c^2} + \frac{\eta d_1}{4\alpha^2 q} + \frac{(mc^2+E_{n,l})F}{4\alpha\hbar^2 c^2} + \frac{(mc^2+E_{n,l})G}{4\alpha^2 q\hbar^2 c^2}\right) \tag{34}$$

$$\chi_3 = \frac{\tilde{E}_{n,l}}{4\alpha^2} - \frac{(mc^2+E_{n,l})F}{4\alpha^2\hbar^2 c^2}. \tag{35}$$

The solution of Eq. (32) becomes

$$F_{n,l}(r) = N_{n,l}(e^{-2\alpha r})^{\sqrt{-\chi_3}}(1-e^{-2\alpha r})^{\sqrt{\frac{1}{4}-\chi_1-\chi_2-\chi_3}} P_n^{\left(2\sqrt{-\chi_3},2\sqrt{\frac{1}{4}-\chi_1-\chi_2-\chi_3}\right)}(1-2e^{2\alpha r}) \tag{36}$$

where $N_{n,l}$ is the normalization constant and $P_n^{(\mu\nu)}(x)$ is the Jacobi polynomial.

III. SCATTERING STATE SOLUTIONS

With the change of variable $y = 1 - qe^{2\alpha r}$ in Eq.(32), we get

$$y(1-y)F''(y) - yF'(y)\left(\frac{\gamma_1}{y} + \frac{\gamma_2}{1-y} - \gamma_3\right)F(y) = 0, \tag{37}$$

where

$$\begin{aligned}
\gamma_1 &= \frac{(mc^2+E_{n,l})}{4\alpha^2\hbar^2 c^2}\left(\frac{C}{q^2} - bF - bG\right) + \frac{\eta d_2}{4\alpha^2 q^2} \\
\gamma_2 &= \frac{\tilde{E}_{n,l}}{4\alpha^2} - \frac{(mc^2+E_{n,l})bF}{4\alpha^2\hbar^2 c^2} \\
\gamma_3 &= \left(\frac{\tilde{E}_{n,l}}{4\alpha^2} + \frac{(mc^2+E_{n,l})B}{4\alpha^2 q\hbar^2 c^2} + \frac{\eta d_1}{4\alpha^2 q} + \frac{(mc^2+E_{n,l})C}{4\alpha^2 q^2\hbar^2 c^2} + \frac{\eta d_2}{4\alpha^2 q^2}\right)
\end{aligned} \tag{38}$$

and with transformation $F_{n,l} = y^{\lambda_1}(1-y)^{\lambda_2}\varphi(z)$ in Eq. (37), we obtain the hypergeometric function in the form [34]

$$z(1-z)\varphi''(z) + (1+2\eta_1-(1+2\eta_1+2\eta_2)z)\varphi(z) - \eta_1\eta_2\varphi(z) = 0, \tag{39}$$

where

$$\begin{aligned}
\lambda_1 &= \frac{1}{2}\left(1+\sqrt{1-4\gamma_1}\right), \\
\lambda_2 &= -\frac{ik}{\alpha}, \quad k = \sqrt{\alpha^2\gamma_2}.
\end{aligned} \tag{40}$$

The solutions of Eq.(39) is the hypergeometric function given by

$$\varphi(y) =_2 F_1(\eta_1,\eta_2,\eta_3,y), \tag{41}$$

where

$$\begin{aligned}
\eta_1 &= \frac{1}{2}\left(1+\sqrt{1-4\gamma_1}\right) - \frac{ik}{\alpha} + \sqrt{\gamma_3}, \\
\eta_2 &= \frac{1}{2}\left(1+\sqrt{1-4\gamma_1}\right) - \frac{ik}{\alpha} - \sqrt{\gamma_3}, \\
\eta_3 &= 1+\sqrt{1-4\gamma_1}.
\end{aligned} \tag{42}$$

We can write the complete wave function as,

$$F_{n,l}(r) = N_{n,l}(-qe^{-2\alpha r})^{\eta_1}(1-qe^{-2\alpha r})^{\eta_2}\ {}_2F_1(\eta_1,\eta_2\eta_3;1-qe^{2\alpha r}). \tag{43}$$

In order to obtain the normalized wave function and the phase shift of the scattering state, we apply the following properties of the hypergeometric functions [34–36]

$$\begin{aligned} {}_2F_1(a,b,c;0) &= 1, \\ {}_2F_1(a,b,c;z) &= \frac{\Gamma(a)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\ {}_2F_1(a;b;a+b-c+1;1-z) \\ &= +(1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\ {}_2F_1(c-a;c-b;c-a-b+1;1-z). \end{aligned} \tag{44}$$

Thus, the term ${}_2F_1(\eta_1,\eta_2,\eta_3;1-qe^{-2\alpha r})$ for $r\to\infty$ in Eq. (41) becomes,

$$\begin{aligned} &{}_2F_1(\eta_1,\eta_2,\eta_3;1-qe^{-2\alpha r}) \\ &= \Gamma(\eta_3)\left[\frac{\Gamma(\eta_3-\eta_1-\eta_2)}{\Gamma(\eta_3-\eta_1)\Gamma(\eta_3-\eta_2)} + \left[\frac{\Gamma(\eta_3-\eta_1-\eta_2)}{\Gamma((\eta_3-\eta_1))\Gamma(\eta_3-\eta_2)}\right]^* q^{(\eta_3-\eta_1-\eta_2)}e^{-2\alpha(\eta_3-\eta_1-\eta_2)r}\right], \\ &= \Gamma(\eta_3)q^{i(\eta_3-\eta_1-\eta_2)}\left[\frac{\Gamma(\eta_3-\eta_1-\eta_2)q^{-i(\eta_3-\eta_1-\eta_2)}}{\Gamma(\eta_3-\eta_1)\Gamma(\eta_3-\eta_2)} + \left[\frac{\Gamma(\eta_3-\eta_1-\eta_2)q^{i(\eta_3-\eta_1-\eta_2)}}{\Gamma(\eta_3-\eta_1)\Gamma(\eta_3-\eta_2)}\right]^* e^{-2i\alpha(\eta_3-\eta_1-\eta_2)r}\right]. \end{aligned} \tag{45}$$

Using the following relations in Eq. (43),

$$\begin{aligned} \eta_3-\eta_1-\eta_2 &= 2ik = (-\eta_3+\eta_1+\eta_2)^*, \\ \eta_3-\eta_1 &= \eta_2{}^* \\ \eta_3-\eta_2 &= \eta_1{}^* \end{aligned} \tag{46}$$

we obtain

$${}_2F_1(\eta_1,\eta_2,\eta_3;1-qe^{-2\alpha r}) = \Gamma(\eta_3)q^{ik}\left[\frac{\Gamma(\eta_3-\eta_1-\eta_2)q^{-ik}}{\Gamma(\eta_3-\eta_1)\Gamma(\eta_3-\eta_2)} + \left[\frac{\Gamma(\eta_3-\eta_1-\eta_2)q^{-ik}}{\Gamma(\eta_3-\eta_1)\Gamma(\eta_3-\eta_2)}\right]^* e^{-2ik\alpha r}\right]. \tag{47}$$

By defining

$$\frac{\Gamma(\eta_3-\eta_1-\eta_2)}{\Gamma(\eta_3-\eta_1)\Gamma(\eta_3-\eta_2)} = \left|\frac{\Gamma(\eta_3-\eta_1-\eta_2)}{\Gamma(\eta_3-\eta_1)\Gamma(\eta_3-\eta_2)}\right| e^{i\delta}, \tag{48}$$

$$q^{-ik} = \left|q^{-ik}\right| e^{i\delta'} \tag{49}$$

and inserting it into Eq.(45), we have

$$\begin{aligned} {}_2F_1(\eta_1,\eta_2,\eta_3;1-qe^{-2\alpha r}) &= \Gamma(\eta_3)q^{i\alpha k}\left|\frac{\Gamma(\eta_3-\eta_1-\eta_2)q^{-i\alpha k}}{\Gamma(\eta_3-\eta_1)\Gamma(\eta_3-\eta_2)}\right|\left[e^{i(\delta+\delta')}+e^{-2\alpha ikr}e^{-i(\delta+\delta')}\right] \\ &= \Gamma(\eta_3)q^{i\alpha k}\left|\frac{\Gamma(\eta_3-\eta_1-\eta_2)q^{-i\alpha k}}{\Gamma(\eta_3-\eta_1)\Gamma(\eta_3-\eta_2)}\right| e^{-i\alpha kr}\left[e^{i(\delta+\delta'+i\alpha kr)}+e^{-i(\delta+\delta'+i\alpha kr)}\right] \\ &= \Gamma(\eta_3)q^{i\alpha k}\left|\frac{\Gamma(\eta_3-\eta_1-\eta_2)q^{-i\alpha k}}{\Gamma(\eta_3-\eta_1)\Gamma(\eta_3-\eta_2)}\right| e^{-i\alpha kr}\sin\left(\alpha kr+\delta+\delta'+\frac{\pi}{2}\right). \end{aligned} \tag{50}$$

Therefore, the asymptotic form of Eq. (41) for $r \to \infty$ becomes

$$F_{n,l} = 2N_{n,l}\Gamma(\eta_3)\left|\frac{\Gamma(\eta_3-\eta_1-\eta_2)q^{-i\alpha k}}{\Gamma(\eta_3-\eta_1)\Gamma(\eta_3-\eta_2)}\right| e^{-i\alpha kr}\sin\left(\alpha kr+\delta+\delta'+\frac{\pi}{2}\right). \tag{51}$$

Now comparing Eq. (48) with the boundary conditions [34,35],

$$\lim_{r\to\infty} F(\infty) = 2\sin\left(kr-\frac{1}{2}l_D\pi+\delta_{L_D}\right),$$

we get the phase shift and the normalization constant as follows:

$$\begin{aligned}\delta_{l_D} &= \frac{\pi}{2}(l_D+1)+\delta+\delta' \\ &= \frac{\pi}{2}(l_D+1)+\arg\Gamma(\eta_3-\eta_1-\eta_2)-\arg\Gamma(\eta_3-\eta_1)-\arg\Gamma(\eta_3-\eta_2)+\arg\Gamma(q^{-i\alpha k})\end{aligned} \tag{52}$$

$$N_{n,l} = \frac{1}{\Gamma(\eta_3)}\left|\frac{\Gamma(\eta_3-\eta_1)\Gamma(\eta_3-\eta_2)}{\Gamma(\eta_3-\eta_1-\eta_2)q^{-i\alpha k}}\right|. \tag{53}$$

IV. DISCUSSIONS AND SPECIAL CASES

In this section we investigated the energy eigenvalues of Hulthen, Manning-Rosen, Eckart, Deng-Fan and Woods-Saxon potentials as special cases of seven parameter exponential-type potential.

1. Hulthen potential

The Hulthen potential is very important in atomic and molecular fields [37]. This potential has been used extensively in nuclear and plasma physics [38]. In this special case, we choose $A = C = G = F = 0, B = -Ze^2\delta, q = -1, \alpha \to \frac{\delta}{2}$, where δ is the screening parameter and the seven parameter exponential-type potential turns into the Hulthen potential as,

$$V(r) = \frac{-Ze^2\delta e^{-\delta r}}{1-e^{-\delta r}}. \tag{54}$$

The energy spectra of the Hulthen potential is obtained from the energy equation of the seven parameter exponential-type potential Eq. (28) by substituting the above parameters as,

$$E_{n,l}^2 - m^2c^4 = \hbar^2c^2\eta d_0 - (mc^2+E_{n,l})Ze^2\delta + \eta\hbar^2c^2(d_2-d_1) - \frac{\hbar^2c^2}{4}\left(\frac{4\beta^{HT}+\gamma^{HT}}{\delta(n+\sigma^{HT})}+\frac{\delta(n+\sigma^{HT})}{4}\right)^2, \tag{55}$$

where

$$\begin{aligned}\beta^{HT} &= \frac{(mc^2+E_{n,l})Ze^2\delta}{\hbar^2c^2} \\ \gamma^{HT} &= \eta(d_2-d_1), \\ \sigma &= \frac{1}{2}\left(1+\sqrt{1-\eta d_1-\eta(d_2-d_1)}\right).\end{aligned} \tag{56}$$

2. Manning-Rosen potential

Manning-Rosen potential is one of the short range potential and it has been used to describe the diatomic molecular vibration [39]. The Manning-Rosen potential has been one of the most useful and elegant potential model for studying the energy eigenvalues of diatomic molecules [40]. As an empirical potential, the Manning-Rosen potential gives an excellent description of the interaction between two atoms in a diatomic molecule, and it is very good for describing such interactions close to the surface [41]. The special case of Manning-Rosen potential is obtained from the seven parameter exponential-type potential by considering, $A = G = F = 0$, $B = -\frac{V_0}{b'^2}$, $C = \frac{\beta'(\beta'-1)}{b'^2}$, $q = -1, \alpha \to \frac{2}{b'}$. Thus, the Manning-Rosen potential becomes,

$$V(r) = \frac{1}{b'^2}\left(\frac{\beta'(\beta'-1)e^{-\frac{2r}{b'}}}{\left(1-e^{-\frac{r}{b'}}\right)^2} - \frac{V_0 e^{-\frac{r}{b'}}}{1-e^{-\frac{r}{b'}}}\right). \tag{57}$$

The energy level of the Manning-Rosen is obtained from the energy equation of the multiparameter exponential-type potential by putting the values of A, B and C given above as,

$$E_{n,l}^2 - m^2c^4 = \hbar^2c^2\eta d_0 + \frac{(mc^2+E_{n,l})(V_0+\beta'(\beta'-1))}{b'^2} + \eta\hbar^2c^2(d_2-d_1) - \frac{\hbar^2c^2}{4}\left(\frac{b'\beta^{MR}+\gamma^{MR}}{(n+\sigma^{MR})} + \frac{(n+\sigma^{MR})}{b'}\right)^2, \tag{58}$$

where

$$\begin{aligned}
\beta^{MR} &= \frac{(mc^2+E_{n,l})(V_0+\beta'(\beta'-1))}{\hbar^2c^2b'^2},\\
\gamma^{MR} &= \eta(d_2-d_1),\\
\sigma^{MR} &= \frac{1}{2}\left(1+\sqrt{1-\eta d_1 - \frac{(mc^2+E_{n,l})\beta'(\beta'-1)}{\hbar^2c^2b'^2} - \eta(d_2-d_1)}\right).
\end{aligned} \tag{59}$$

3. Eckart potential

The Eckart potential is one of the solvable exponential-type potential in quantum mechanics since its introduction by Eckart [42] in 1930. Eckart potential is one of most important potential model in physics and chemical physics [43] and the bound state solution of the Schrödinger equation [44] and the scattering states [45] of this potential has been investigated. The Eckart potential is obtained from the seven parameter exponential type potential by the setting $A = C = G = 0, q = -1, B = -\alpha', \alpha \to \frac{2}{a}, F = \beta'$ and we get

$$V(r) = \frac{\alpha' e^{-\frac{r}{a}}}{1-e^{-\frac{r}{a}}} + \frac{\beta e^{-\frac{r}{a}}}{\left(1-e^{-\frac{r}{a}}\right)^2}. \tag{60}$$

The energy eigenvalues for the Eckart potential from Eq.(28) becomes,

$$E_{n,l}^2 - m^2c^4 = \hbar^2c^2\eta d_0 - (mc^2+E_{n,l})B + \eta\hbar^2c^2(d_2-d_1) - \frac{\hbar^2c^2}{4}\left(\frac{a\beta^{EK}+\gamma^{EK}}{(n+\sigma)} + \frac{(n+\sigma^{EK})}{a}\right)^2, \tag{61}$$

where

$$\begin{aligned}
\beta^{EK} &= -\frac{(mc^2+E_{n,l})}{\hbar^2c^2}[\alpha'+b\beta'],\\
\gamma^{EK} &= \eta(d_2-d_1),\\
\sigma^{EK} &= \frac{1}{2}\left(1+\sqrt{1+\frac{(mc^2+E_{n,l})}{\hbar^2c^2}(\alpha'+b\beta') - \eta d_1 - \frac{(mc^2+E_{n,l})}{\hbar^2c^2}b\beta' - \frac{(mc^2+E_{n,l})}{\hbar^2c^2}\alpha' - \eta(d_2-d_1}\right).
\end{aligned} \tag{62}$$

4. Deng-Fan potential

The Deng-Fan potential [46,47] discovered more than 50 years ago is the simplest modified form of Morse potential and is related to the Manning-Rosen and Eckart potentials. This potential is used to describe diatomic molecular energy spectra and electromagnetic transition and is usually regarded as the true inter nuclear potential in diatomic molecules. In this case, the choice $A = G = F = 0, B = -2bD_e, C = D_e b^2, q = -1$ and $\alpha = \frac{\alpha}{2}$, where D_e is the dissociation energy. With these parameters, we obtained the Deng-Fan potential from Eq. (6) as

$$V(r) = \frac{-2bD_e e^{-\alpha r}}{1 - e^{-\alpha r}} + \frac{D_e b^2 e^{-2\alpha r}}{(1 - e^{-2\alpha r})^2} \tag{63}$$

and from Eq. (28), the energy spectra for the Deng-Fan potential becomes

$$E_{n,l}^2 - m^2c^4 = \hbar^2 c^2 \eta d_0 + (mc^2 + E_{n,l}) D_e b(b+2) + \eta \hbar^2 c^2 (d_2 - d_1) - \frac{\hbar^2 c^2}{4} \left(\frac{2\beta^{DF} + \gamma^{DF}}{\alpha(n + \sigma^{DF})} + \frac{\alpha(n + \sigma^{DF})}{2} \right)^2, \tag{64}$$

where

$$\begin{aligned} \beta^{DF} &= \frac{(mc^2 + E_{n,l}) D_e b(b+2)}{\hbar^2 c^2}, \\ \gamma^{DF} &= \eta(d_2 - d_1). \end{aligned} \tag{65}$$

$$\sigma^{DF} = \frac{1}{2}(1 + \sqrt{1 - \eta d_1 - \eta(d_2 + d_1 q)}. \tag{66}$$

5. Woods-Saxon potential

The Woods-Saxon is a short range and reasonable potential for nuclear shell models and it attracts much attention in nuclear physics and is used to represent the distribution of nuclear densities [48–50]. The Woods-Saxon potential has been used extensively in numerous problems in nuclear and particles physics, atomic physics, condensed matter and chemical physics. In this special case, we choose $A = C = G = F = 0$, $B = -V_0$, $q = 1$ and the seven parameter exponential-type potential turns into the Woods-Saxon potential as,

$$V(r) = -\frac{V_0}{1 + e^{2\alpha r}} \tag{67}$$

$$E_{n,l}^2 - m^2c^4 = \hbar^2 c^2 \eta d_0 + (mc^2 + E_{n,l}) V_0 + \eta \hbar^2 c^2 (d_2 - d_1) - \frac{\hbar^2 c^2}{4} \left(\frac{2\beta^{WS} + \gamma^{WS}}{\alpha(n + \sigma^{WS})} + \frac{\alpha(n + \sigma^{WS})}{2} \right)^2, \tag{68}$$

where

$$\begin{aligned} \beta^{WS} &= \frac{(mc^2 + E_{n,l}) V_0}{\hbar^2 c^2}, \\ \gamma^{WS} &= \eta(d_2 - d_1), \\ \sigma^{WS} &= \frac{1}{2}\left(1 + \sqrt{1 - \eta d_1 - \eta(d_2 - d_1)}\right). \end{aligned} \tag{69}$$

V. CONCLUSIONS

In this paper, we have investigated the KGE with seven parameter exponential-type potential and obtain the bound state energy eigenvalues and the scattering state phase factor. Special cases of the potential are deduced in details.

REFERENCES

[1] S. H. Dong, *Factorization Method in Quantum Mechanics* (Dordreht, Springer, 2007).

[2] L. D. Landau and E. M .Lifshitz, *Quantum Mechnaics, Non-Relativistic Theory* (New York, Pergamon, 1977).

[3] S. H. Dong, Commun. Theor. Phys. **55**, 969 (2011).

[4] A. N. Ikot, O. A. Awoga, H. Hassanabdi and E. Maghsoodi, Commun. Theor. Phys. **61**, 457 (2014).

[5] R. L. Greene and C. Aldrich, Phys. Rev. A **14**, 2363 (1976).

[6] Z. Y. Chen, M. Li and C. S. Jia, Mod. Phys. Lett. A **24**, 1863 (2009).

[7] A. F. Nikiforov and V. B. Uvarov, *Special Functions of Mathematical Physics* (Birkhauser, Basel, 1988).

[8] Z. Q. Mai, A. Gonzalez-Cisneros, B. W. Xu and S. H. Dong, Phys. Lett. A **371**, 180 (2007).

[9] C. S. Jia, X. P. Li and L. H. Zhang, Few-Body Syst. **52**, 11 (2012).

[10] H. Ciftci, R. L. Hall and N. Saad, J. Phys. A: Math. Gen. **38**, 1147 (2005).

[11] A. N Ikot, E. Maghsoodi, S. Zarrinkamar and H. Hassanabadi, Ind. J. Phys. **88**, 283 (2014).

[12] J. Y. Liu, G. D. Zhang and C. S. Jia, Phys. Lett. A **377**, 1444 (2013).

[13] A. N. Ikot, S. E. Etuk, H. Hassanabadi, E. Maghsoodi and S. Zarrinkamar, In. J. Phys. **89**, 289 (2015).

[14] E .Witten, Nucl. Phys. B **188**, 513 (1981).

[15] R. Dutt, A. Gangopadhyaya and U. P. Sukhatme, Am. J. Phys. **65**, 400 (1997).

[16] F. L. Lu, C. Y. Chen and D. S. Sun, Chin. Phys. **14**, 463 (2005).

[17] L. L. Lu, B. H. Yazarloo, S. Zarrinkamar, G. Liu and H. Hassanabadi, Few-Body Syst. **53**, 573 (2012).

[18] S. M. Ikhdair and R. Sever, J. Math. Chem. **42**, 461 (2007).

[19] K. J. Oyewumi, F. O. Akinpelu and A. D. Agboola, Int. J. Theor. Phys. **47**, 1039 (2008).

[20] S. H. Dong, C. Y. Chen and M. L. Cassou, J. Phys. B **38**, 2211 (2005).

[21] J. Y. Liu, J. F. Du and C. S. Jia, Euro. Phys. J. Plus **58**, 139 (2012).

[22] G. D. Zhang, W. Zhou, J. Y. Liu, L. H. Zhang and C. S. Jia, Chem. Phys. **439**, 79 (2014).

[23] J. Y. Liu, X. T. Hu and C. S. Jia, Can. J. Chem. **92**, 40 (2014).

[24] H. M. Tang, G. C. Liang, L. H. Zhang, F. Zhao and C. S. Jia, Can. J. Chem. **92**, 201 (2014).

[25] X. T. Hu, L. H. Zhang and C. S. Jia, Can. J. Chem. **92**, 386 (2014).

[26] C. S. Jia, T. Chem and S. He, J. Phys. Lett. A **377**, 682 (2013).

[27] X. T. Hu, L. H. Zhang and C. S. Jia, J. Mol. Spectr. **297**, 21 (2014).

[28] P. Q. Wang, J. Y. Liu, L. H. Zhang, S. Y. Cao and C. S. Jia, J. Mol. Spectr. **278**, 23 (2012).

[29] C. S. Jia, Y. F. Diao, X. J. Liu, P. Q. Wang and J. Y. Liu *et al.*, J. Chem. Phys. **137**, 014181 (2012).

[30] C. S. Jia, X. L. Zheng and L. T. Sun, Phys. Lett. A **294**, 185 (2012).

[31] M. R. Setare and Z. Nazari, Acta Polo. B **40**, 2809 (2009).

[32] A. D. Alhaidari, H. Bahlouli and A. Al-Hasan, Phys. Lett. A **349**, 87 (2006).

[33] C. L. Pekeris, Phys. Rev. **45**, 98 (1934).

[34] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (New York, Dover, 1965).

[35] C. C. Yuan, L. F. Lin and Y. Yuan, Chin. Phys. B **21**, 030302 (2012).

[36] W. G. Feng, C. W. Li, W. H. Ying and L. Y. Yuan, Chin. Phys. B **18**, 3663 (2009).

[37] S. Flugge, *Practical Quantum Mechanics* (Birlin, Springer, 1974).

[38] F. Dominguez-Adame, Phys. Lett. B **136**, 175 (1989).

[39] M. F. Manning and N. Rosen, Phys. Rev. **44**, 953 (1933).

[40] P. Q. Wang, L. H. Zhang, C. S. Jia and J. Y. Liu, J. Mol. Spectrosc. **274**, 5 (2012).

[41] J. Y. Liu, G. D. Zhang and C. S. Jia, Phys. Lett. A **377**, 1444 (2013).

[42] C. Eckart, Phys. Rev. **35**, 1303 (1930).

[43] J. J. Weiss, J. Chem. Phys. **41**, 1120 (1964).

[44] X. Zou, L. Z. Yi and C. S. Jia, Phys. Lett. A **346**, 54 (2005).

[45] G. W. Wei, C. Y. Long, X. Y. Duan and S. H. Dong, Phys. Scr. **77**, 035001 (2008).

[46] Z. H. Deng and Y. P. Fan, Shandong Univ. J. **7**, 16 (1957).

[47] A. N. Ikot and O. A. Awoga, Arab. J. Sci. Eng. **39**, 467 (2014).

[48] R. D. Woods and D. S. Saxon, Phys. Rev. **95**, 577 (1954).

[49] X. L. Peng, H. Y. Liu and C. S. Jia, Phys. Lett. A **352**, 478 (2006).

[50] A. Tas, S. Alpogan and A. Havare, Adv. High Energ. Phys. **2014**, 619241 (2014).