Aspects of the AdS/CFT correspondence

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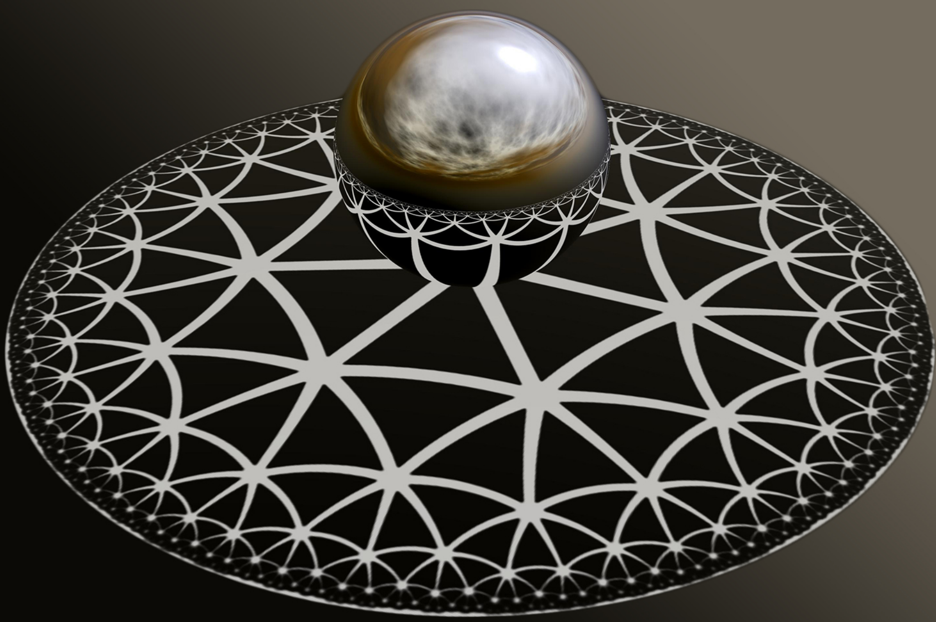

Ricardo Caldeira Costa

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Publications

This thesis is based on the following publications:

[1] R.N. Caldeira Costa and Marika Taylor
Holography for chiral scale-invariant models
JHEP **1102** (2011) 082, arXiv:1010.4800 [hep-th].

[2] R.N. Caldeira Costa
Holographic Reconstruction and Renormalization in Asymptotically Ricci-flat Spacetimes
JHEP **1211** (2012) 046, arXiv:1206.3142 [hep-th].

[3] R.N. Caldeira Costa
Aspects of the zero Λ limit in the AdS/CFT correspondence
Submitted to *Phys. Rev. D*, arXiv:1311.7339 [hep-th].

Contents

Overview

The past decade and a half witnessed a major breakthrough in our understanding of gravitational physics. Previous studies of black holes as quantum systems had suggested that gravity is fundamentally holographic. A semi-classical treatment of black holes by Hawking [4] revealed that these solutions carry an intrinsic notion of entropy and furthermore that this entropy is proportional to the area of the horizon. Together with the second law of thermodynamics, these results imply the Bekenstein bound $S_{\rm max} = A/4G_0$ relating the maximum entropy in a region of space that contains gravity to the area of the boundary of this region. This property is in sharp contrast with entropy bounds in local quantum field theories, where the number of degrees of freedom typically scales as the volume of the enclosing region, and shows that a quantum theory of gravity is not an ordinary field theory of a massless spin-2 particle. An interpretation of this bound naturally led to the holographic principle of 't Hooft and Susskind [5, 6, 7] according to which the states of any quantum gravity theory are in fact contained in a theory-without-gravity defined at the boundary of the space. One could stress at this point an apparent incompatibility of this principle with the Weinberg-Witten theorem [8] in quantum field theory. Apart from a few subtleties, this mainly states that a QFT with a conserved stress-energy tensor cannot have states for massless interacting particles of spin $j > 1$. This implies in particular that such QFTs cannot contain graviton states. The holographic principle evades the axioms of the theorem by placing the graviton in a space distinct from that of the boundary theory, in particular in a space with more dimensions.

This principle is presented to us as a fundamental property of quantum gravity but for several years it remained largely conceptual, mainly due to the lack of an exact framework where these ideas could be implemented at a computational level. The most promising candidate seemed to be string theory, where regions of space that contain gravity are described as ensembles of quantum states. In fact, in the late 90's Strominger and Vafa [9] were able to reproduce the Bekenstein-Hawking entropy of extremal black holes by engineering certain supersymmetric solutions in string theory and performing a statistical counting of their microstates. This procedure was possible only after the discovery of D-branes and their significance in string theory by Polchinski [10] and collaborators [11, 12]. This microscopic counting is a computation in the lower dimensional worldvolume theory of the black holes (or D-branes) and showed that string theory is capable of presenting gravity as a holographic theory. Earlier ideas on a possible lower dimensional description of gravity in string theory had been discussed by Thorn and collaborators [13].

A related but independent result by Brown and Henneaux [14] in the late 80's suggested that certain theories of gravity, more specifically those with Anti-de Sitter (AdS) asymptotics, are intimately connected with lower dimensional conformal field theories in a holographic fashion. Their analysis of three-dimensional Einstein gravity with AdS boundary conditions showed that the asymptotic symmetry group of AdS_3 acts at the boundary of the space as the two-dimensional conformal group. The algebra of the corresponding conserved charges is a centrally extended Virasoro algebra, originally derived in the context of string theory. This study recognised for the first time the importance of the asymptotic boundary of AdS spaces in a possible implementation of the holographic principle in the case of AdS gravity. In particular, it implied that any field theory defined at the boundary of Anti-de Sitter would be a conformally invariant QFT, but it didn't point towards any concrete proposal relating the boundary theory to gravitational physics in the interior and for several years it remained as an interesting curiosity. Nevertheless, based on these results, Strominger showed that the microstates of black holes with an AdS_3 near-horizon geometry are contained in a conformal field theory defined at the boundary of this geometry [15]. This was shown by using Cardy's formula [16] for the growth of states of the two-dimensional conformal field theory to reproduce the entropy of these black holes.

The analysis of Brown and Henneaux and of Strominger was perform without direct contact with string theory and therefore remained valid for any theory that reduces to Einstein gravity at low energies. However, the discovery of D-branes as solutions of supergravity suggested that string theory could finally realise the holographic ideas raised by these and related works. Supergravity is the low-energy limit of string theory and solutions of the former model the dynamics of massless closed string states at low energies. D-branes, on the other hand, are the surfaces where open strings end and their dynamics is described by lower dimensional worldvolume theories of Born-Infeld type. Closed string theories are essentially gravitational, while open string theories are essentially gauge theories that describe the dynamics of the D-branes. The discovery [10] that D-branes are the sources of electric and magnetic (Ramond-Ramond) flux in supergravity – in other words, that they are sources of closed strings – led to their identification with supersymmetric solutions of supergravity known as extremal black branes. These classical solutions therefore describe the backreaction of D-branes on the embedding geometry in a low-energy approximation and are black because they contain event horizons. The near-horizon geometries of these black D-branes consist in several cases of a product of an Anti-de Sitter space with a compact space. Simultaneously, the low-energy worldvolume theories on the branes are gauge theories (which are quantum field theories without gravity) with conformal symmetry. This raised the possibility that the lower-dimensional conformal field theories that live on the D-branes could be the holographic image of the gravitational theories that live in the corresponding near-horizon geometries. Significant evidence that this could the case followed from D-brane scattering calculations [17, 18, 19] which showed that the absorption rate of closed strings by D-branes could equally be computed using supergravity or the worldvolume theories.

The collection of these results pointed to the fact that the holographic aspects of gravity could probably be realised in string theory as a type of duality between open strings (or D-branes) and closed strings and it culminated in the late 90's with Maldacena's proposal [20] of a concrete equivalence between certain theories of closed strings in AdS spaces and conformally invariant gauge theories in less dimensions. In subsequent work, Witten and collaborators [21, 22] argued that these gauge theories (more exactly the fundamental theories, with no Wilsonian degrees of freedom integrated out) live at the boundary of the AdS spaces and further showed that string theory observables can be computed from the boundary theory. For these reasons, the proposal by Maldacena, also known as the AdS/CFT correspondence, is an exact realisation of the holographic principle in string theory, where all the gravitational physics is conjectured to be encoded at the boundary of the space. In fact, it was argued by Witten and Susskind [23] that the AdS/CFT correspondence saturates the Bekenstein bound characteristic of holographic theories by showing that the gravitational theory (which is equivalent to the boundary theory by the AdS/CFT duality) has precisely one degree of freedom per Planck boundary area.

The gauge theories involved in AdS/CFT are Yang-Mills or non-Abelian quantum field theories. These are the type of theories that describe the interactions of elementary particles in the standard model of particle physics (even though the field theories involved in AdS/CFT are an idealisation of these). The electroweak theory that describes weak interactions and quantum electrodynamics is a Yang-Mills theory based on a $SU(2)\times U(1)$ gauge group, while quantum chromodynamics (QCD) describes the strong interactions and is based on $SU(3)$. QCD is a particularly special type of gauge theory. While the strength of the interactions in the electroweak case weakens with decreasing energies involved in the processes, such as the momenta of the particles, the strength of the strong interactions increases at low energies where QCD becomes strongly coupled. Since most computations in quantum field theory are based on perturbation theory, this property prevents us from studying the low-energy regime of QCD and characteristic phenomena such as colour confinement with standard methods. It was the attempt to understand the strongly coupled physics of QCD that led for the first time to the idea that string theories could actually be gauge theories in disguise. In the decade of 1970 't Hooft suggested [24] that QCD could be approximated by a gauge theory with gauge group $SU(N) : N >> 1$. In this large N idealisation the theory simplifies considerably and is amenable to perturbation theory in $1/N$. It was then realised that the perturbative expansion of the gauge theory in Feynman diagrams is in fact an expansion in topologies of string theory worldsheets and therefore that this expansion could provide a definition of a string theory. We now know that this surprising relation between large N Yang-Mills theories and string theories is a particular case of the AdS/CFT correspondence, where the $1/N$ expansion of the gauge theory corresponds to string perturbation theory in the worlsheet coupling constant g_s.

A further property of the correspondence between gauge and string theories as determined by the AdS/CFT duality concerns the relationship between the ('t Hooft) coupling constant of the Yang-Mills theory – which determines the strength of the gauge theory interactions – and the string length scale, or inverse string tension, which determines in particular the strength of the gravitational field in the dual string theory. It turns out that this relationship is a strong/weak duality. This fact implies that when the gauge theory is in its strong coupling regime the dual string theory can be well-approximated by classical gravity. As discussed in the above example of QCD, strongly coupled field theories are very difficult to study and for this reason the AdS/CFT correspondence is an extremely useful tool to understand quantum field theories at strong coupling because it maps hard problems in the field theories to simple problems in classical gravity.

Due to this strong/weak property of the duality, soon after the discovery of AdS/CFT many authors proposed applications of the correspondence to condensed matter theory. Most systems in condensed matter physics are difficult to study using field theoretic methods alone. In particular, near quantum critical points – where a transition between different quantum states of matter takes places – the systems are typically strongly coupled, conformally invariant (or scale invariant) and strongly correlated. Via the AdS/CFT duality, different sectors of string theory have the potential to serve as holographic models of these systems at strong coupling and can be used in particular to gain some insight into critical phenomena in condensed matter theory.

This last aspect of the AdS/CFT correspondence is one of the main topics of this thesis and in chapter 2 we will explore applications of the duality to the holographic description of particular quantum field theories. The latter have the specific property of being non-relativistic and can be used to model physical systems in condensed matter theory with a certain type of anisotropy and scale-invariance. We will understand how the symmetries of such systems are realised in the field theory models and how to construct the gravitational duals. We will discuss the form of the correlation functions of these field theories and compute them holographically using specific gravity models.

In chapters 3 and 4 we will focus on a different aspect of the AdS/CFT correspondence. Currently, a central problem in holography is to understand how to formulate string theory in spacetimes with non-AdS asymptotics in terms of field theories in less dimensions. This direction of research has seen some success for the case of non-conformal brane backgrounds, and less successfully for spacetimes with de Sitter boundary conditions. Asymptotically flat spaces, however, remain the most important class of gravitational backgrounds in which string theory lacks a holographic formulation. The last two chapters focus on gaining some insight into this problem and consist of two approaches to flat space holography that follow different perspectives, one based on the flat space limit of AdS/CFT and another on the concept of holographic foliations. The first approach formulates the problem as a limit of AdS/CFT where the AdS curvature Λ vanishes. We will study the zero Λ limit of vacuum expectation values and correlation functions in AdS/CFT and address several of the necessary conditions for the correspondence between bulk and boundary physics to admit a well-behaved limit. We will find evidence that putative field theories dual to string theory in AdS in the limit of zero curvature Λ are essentially defined in two dimensions less, a property consistent with the fact that the asymptotic boundary of the AdS spaces becomes null in this limit.

The second approach is based on the observation that asymptotically Minkowski spaces can always be foliated by Euclidean AdS (or hyperbolic) hypersurfaces near null infinity. The foliation naturally converges asymptotically to a codimension two surface at the boundary of the space. Since each leaf of the foliation is an AdS space, we will explore this feature and conjecture that asymptotically Minkowski spaces admit a holographic description in terms of an infinite family of (conformally invariant) field theories that live at the degenerate boundary of the foliation. We will find that it is indeed possible to reconstruct the asymptotics of such spacetimes from observables belonging to a one-parameter family of conformal field theories in two dimensions less. In the case of two-dimensional field theories, this parameter is the central charge of the theories and it measures, on the gravity side, the gauge-invariant distances between the different AdS surfaces.

In the next chapter we begin by reviewing the AdS/CFT correspondence from first principles. We will start by discussing the large N limit of Yang-Mills theories and their relation with string theories as introduced above. We will then make a brief overview of string theory aspects relevant to our work and discuss in detail the original derivation of the correspondence from D-brane physics and supergravity. We will then devote a significant part of this first chapter to the correspondence between states and operators on each side of the duality. Finally, we will discuss the computation of quantum field theory correlation functions in string theory.

Chapter 1

Introduction to the AdS/CFT correspondence

1.1. Large N gauge theories

Consider the following Yang-Mills theory in four dimensional Minkowski space with internal symmetry group a $SU(N)$ gauge group:

$$\mathcal{L} \;=\; -\frac{1}{4}\,\delta_{AB}F^{A}_{\mu\nu}F^{B\mu\nu} + \sum_{n=1}^{N_F}\,\bar{\Psi}_{(n)a}\,(i\gamma^{\mu}D_{\mu} - \mathbb{1}m_f)^{a}{}_{b}\,\Psi^{b}_{(n)}\;, \tag{1.1}$$

where:

$$F^{A}_{\mu\nu} = \partial_{\mu}A^{A}_{\nu} - \partial_{\nu}A^{A}_{\mu} + ig_{YM}f^{A}_{BC}A^{B}_{\mu}A^{C}_{\nu}\;, \tag{1.2}$$

$$(D_{\mu})^{a}{}_{b} \;=\; \delta^{a}{}_{b}\partial_{\mu} + ig_{YM}A^{A}_{\mu}(T_{A})^{a}{}_{b}\;, \tag{1.3}$$

$$[T_B, T_C] = f^{A}_{BC}\,T_A\;. \tag{1.4}$$

The one-form gauge field $(A_{\mu})^{a}{}_{b} \;=\; A^{A}_{\mu}(T_{A})^{a}{}_{b}$ is an element of the Lie algebra $\mathfrak{su}(N)$.[1] The generators of the group are the $N^2 - 1$ traceless hermitian matrices

[1] Note that, unlike the gauge curvature $F^{A}_{\mu\nu}T_A$, the gauge field $A^{A}_{\mu}T_A$ does not transform as an element of the adjoint representation space under the action of the group (*i.e.* under a gauge transformation), but rather inhomogeneously as a connection form: $A^{A}_{\mu}T_A \xrightarrow{g} \left(\mathrm{Ad}(g)^{A}{}_{B}A^{B}_{\mu}\right)T_A = (gA_{\mu}g^{-1})^{A}T_A + ig^{-1}_{YM}(\partial_{\mu}g\,g^{-1})^{A}T_A$, with g an element of the group and Ad the adjoint representation. Strictly speaking, the gauge field A_{μ} is not Lie algebra valued in the same sense that a connection in differential geometry is not a tensor. One still says, however, that $A_{\mu} \in \mathfrak{su}(N)$ and that it transforms under the adjoint representation of the group.

$\{T_A\}$ that form a basis for $\mathfrak{su}(N)$ with structure constants f^A_{BC}. The theory contains N_F fermions $\Psi_{(n)}$ (also called quark flavours) with Ψ^a in the fundamental representation and $\bar{\Psi}_a$ in the anti-fundamental.[2]

The β-function for the dimensionless Yang-Mills coupling constant g_{YM} at one-loop order is given below. From this it follows that the theory is asymptotically free if $N_F < 11N/2$ and contains a Landau pole at some low energy. Standard QCD corresponds to this case, with group rank $N = 3$ (also called the number of colours) and $N_F = 6$. In particular, the theory is strongly coupled ($g_{YM} > 1$) at energies below a characteristic scale Λ. For this reason, the low energy regime of the theory and phenomena such as colour confinement cannot be studied using perturbation theory with g_{YM} the expansion parameter.

Since the rank N of the gauge group is a truly dimensionless parameter (*i.e.* does not run) and $1/N < 1$ for QCD, it was originally suggested by 't Hooft [24] that qualitative aspects of QCD at low energies could be derived be considering $SU(N)$ Yang-Mills theory at large N. The key observation is that the large N theory can in principle be studied over a broad range of energies using perturbation theory with $1/N$ the expansion parameter after an appropriate redefinition of the coupling constant. Approximate results for QCD would then be obtained from the large N theory by replacing $1/N$ by $1/3$ in the perturbation expansions.

The regime of QCD that cannot be studied with standard perturbation theory is the low energy, or confining, regime, so one may ask at this point whether low energy scales (*i.e.* of order $\sim \Lambda$) can be probed by approximating QCD by a large N theory with perturbation parameter $1/N$. We will see that such scales can indeed be probed by considering the strongly coupled regime of the large N theory in the new coupling constant. One may also ask whether QCD physics can be modelled by a large N theory since $1/3$ is not arbitrarily small. A positive answer to this question is justified by qualitative and quantitative results which show that large N gauge theories share many of the essential features expected of strongly coupled QCD [25, 26] and hence that they may provide a window to unknown aspects of its non-pertubative regime.

A fundamental feature that emerges from a detailed analysis of large N Yang-Mills theories is that they simplify considerably in this limit. If we start by expanding a given correlation function in Feynman diagrams with expansion parameter $1/N$, we find that the propagators and vertices themselves in each diagram contribute with powers of N. All diagrams that are not planar, as defined below, will

[2]Given a representation of a Lie group, it is common practice to refer to the representation *space* on which the group/algebra acts as *the representation* and we will adopt this terminology. This abuse of language is usually employed because a representation of a Lie group is essentially defined by the vector space on which it acts.

be suppressed by powers of N^{-2} in relation to the planar ones and therefore the correlation function will be dominated by the latter. For this reason, the large N limit is also called the planar limit. Furthermore, the subleading orders are organised as an expansion in topologies of compact two-dimensional surfaces (closed and orientable in the absence of matter in the fundamental), *i.e.* according to their Euler characteristic. An immediate consequence of this feature is that the large N theory can, in principle, be used to define a string theory if we identify $1/N$ with the string coupling constant as discussed below. This apparent relationship between the large N expansion of Yang-Mills theories and perturbative string theory is the strongest motivation for studying gauge theories at large N, even though they were originally proposed as an approximate model to non-pertubative QCD. This relationship suggests that gauge theories and string theories are in some way dual, *i.e.* equivalent, and that this duality is more easily seen in the large N limit [27].

We want to analyse the behaviour of the generating functional, or of correlation functions, of Yang-Mills theory (1.1) at large N. There is no explicit dependence of the Lagrangian on the group rank, but if we expand a given correlation function in Feynman diagrams with perturbation parameter g_{YM}, powers of N arise because the computation requires that we sum over adjoint $(A, B, ...)$ and fundamental $(a, b, ...)$ indices of $SU(N)$. If we then take the limit $N \to \infty$ for generic g_{YM}, we do not obtain any sensible results. This fact is reflected on the β-function of the theory. At one-loop order, this is given by:

$$\mu \frac{dg_{YM}}{d\mu} = \beta(g_{YM}) = -kNg_{YM}^3 + \mathcal{O}(g_{YM}^5)\ ,\ k = \frac{11}{48\pi^2}\left(1 - \frac{2}{11}\frac{N_F}{N}\right)\ . \tag{1.5}$$

From this expression it follows that the β-function is ill-defined in the limit $N \to \infty$ if g_{YM} is kept fixed. A necessary condition for correlation functions of some interacting QFT to be well-defined within a given range of energies is that the β-function for the coupling constant also be so. This implies that we need at least to redefine the coupling constant of the large N theory if we want to obtain any sensible correlation functions. From the analysis below of vacuum diagrams it follows that this is sufficient to guarantee that the generating functional of the large N theory is well-defined.

In order to understand how to scale g_{YM} with N as $N \to \infty$, we begin by solving the β-function equation (1.5) at one-loop:

$$g_{YM}^2(\mu) = \frac{1}{1 + 2kN\log(\mu/\Lambda)}\ , \tag{1.6}$$

where the integration constant Λ represents the value of the renormalization scale μ at which $g_{YM} = 1$. Note that perturbation theory is valid only for $g_{YM} << 1$, which requires $\mu >> \Lambda$. This gives the physical meaning to Λ as an IR cut-off in the perturbative theory.[3]

If we require that Λ remains fixed as $N \to \infty$,[4] we find that g^2_{YM} behaves asymptotically as $1/N$. This suggests that we introduce the 't Hooft coupling $\lambda := g^2_{YM} N$. Equation (1.5) then becomes:

$$\mu \frac{d\lambda}{d\mu} \quad = \quad \beta(\lambda) = -2k\,\lambda^2 + \mathcal{O}(\lambda^3) \;, \tag{1.7}$$

with solution:

$$\lambda(\mu) \quad = \quad \frac{1}{1 + 2k \log(\mu/\Lambda')} \;. \tag{1.8}$$

Comparing with (1.6) and using the definition $\lambda = g^2_{YM} N$, we obtain:

$$\Lambda' = \Lambda \exp\left(\frac{N-1}{2kN}\right) \;. \tag{1.9}$$

By construction, the solution $\lambda(\mu)$ is finite and the beta-function $\beta(\lambda)$ is well-defined (and independent of N_F) in the large N limit. If we take into account higher order terms in the β-function, one can show that the solution for $\lambda(\mu)$ remains well-behaved in the limit $N \to \infty$ still without the need to take a limit on the scale Λ [26]. These results suggest that we introduce the 't Hooft coupling as the coupling constant of the large N theory and study the limit $N \to \infty$ of Yang-Mills with λ kept fixed, known as the 't Hooft limit. Note that the limit theory with coupling λ is also an asymptotically free theory. Furthermore, it contains a Landau pole at $\mu = \Lambda$, so we can probe energy scales of the order Λ by considering the strongly coupled regime ($\lambda >> 1$) of the large N theory.

We then return to the Lagrangian (1.1), replace g_{YM} by $\sqrt{\lambda/N}$, and expand the generating functional, or a given correlation function, in Feynman diagrams with expansion parameter $1/N$.[5] Each diagram will have a specific dependence on N determined by the vertices and particularly by contractions of $SU(N)$ indices.

[3]The QCD, or confinement, scale $\Lambda_{QCD} \sim 200$ MeV is usually defined in the high-energy literature as the Landau pole $\mu = \Lambda \exp(-1/(2kN))$. Note, however, that this pole is not physically meaningful, as suggested by lattice simulations, and it only signals the breakdown of perturbation theory. On the other hand, the scale parameter Λ is meaningful and it represents the value of the RG scale μ below which perturbation theory is no longer reliable.

[4]Note that the requirement that the scale parameter Λ of the interaction be independent of N is equivalent to requiring that the masses of the particles of the theory remain fixed as $N \to \infty$.

[5]Here and in what follows we assume that a proper gauge-fixing condition has been implemented and ignore contributions from ghost fields.

In order to simplify the counting of powers of N in each diagram, it is convenient to rescale the fields as:

$$A_\mu \to \sqrt{\frac{N}{\lambda}}\, A_\mu \quad , \qquad \Psi \to \sqrt{N}\,\Psi \ , \tag{1.10}$$

which results in the Lagrangian:

$$\mathcal{L} \;=\; N\left(-\frac{1}{4\lambda}\,\delta_{AB}F^A_{\mu\nu}F^{B\mu\nu} + \sum_{n=1}^{N_F}\bar{\Psi}_{(n)a}\left(i\gamma^\mu D_\mu - \mathbb{1}m_f\right)^a{}_b\,\Psi^b_{(n)}\right) \ , \tag{1.11}$$

where now:

$$F^A_{\mu\nu} \;=\; \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + i f^A_{BC} A^B_\mu A^C_\nu \ , \tag{1.12}$$

$$\left(D_\mu\right)^a{}_b \;=\; \delta^a{}_b\partial_\mu + iA^A_\mu (T_A)^a{}_b \ . \tag{1.13}$$

Note that this redefinition is done for convenience only and physical results such as scattering amplitudes are independent of this rescalling. Each vertex is then proportional to N and each propagator to $1/N$.[6] A typical (internal) diagram of the theory is of the form:

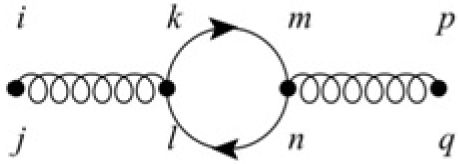


Figure 1.1: Internal diagram.

representing two gauge field (gluon) propagators, a quark momentum loop and four vertices. This diagram is associated with the amplitude:

$$\left[iN\gamma^\mu (T_A)^a_b\right]\langle A^A_\mu A^B_\nu\rangle \left[iN(T_B)^c_d\right]\mathrm{tr}\left\{\gamma^\nu\langle\bar{\Psi}_e\Psi^d\rangle\gamma^\alpha\langle\Psi^f\bar{\Psi}_c\rangle\right\}\left[iN(T_C)^e_f\right]\langle A^C_\alpha A^D_\beta\rangle\left[iN\gamma^\beta(T_D)^g_h\right]$$

$$= N^4\,\gamma^\mu\,\langle (A_\mu)^a_b (A_\nu)^c_d\rangle\ \mathrm{tr}\left\{\gamma^\nu\langle\bar{\Psi}_e\Psi^d\rangle\gamma^\alpha\langle\Psi^f\bar{\Psi}_c\rangle\right\}\langle (A_\alpha)^e_f (A_\beta)^g_h\rangle\,\gamma^\beta \ , \tag{1.14}$$

where the integral over the loop momentum is implicit in the trace over the Dirac indices. The structure of the adjoint and fundamental group/colour indices in the

[6] Schematically, from the Lagrangian it follows that the equation for a Green's function is of the form $N\Box G(x-y) = \delta(x-y) \Rightarrow G(x-y) \propto \frac{1}{N}$, with $G(x-y)$ the Green's function.

gluon and quark propagators is given by:

$$\begin{aligned} \langle A^A_\mu(x) A^B_\nu(y)\rangle &= \frac{1}{N}\,\delta^{AB}\,D_{\mu\nu}(x-y)\ , &(1.15)\\ \langle \Psi^a(x)\bar{\Psi}_b(y)\rangle &= \frac{1}{N}\,\delta^a_b\,S(x-y)\ , &(1.16)\end{aligned}$$

with $D_{\mu\nu}(x)$ and $S(x)$ the respective Green's functions with the N dependence factored out. If we use the identity:

$$\delta^{AB}\,(T_A)^a_b\,(T_B)^c_d \;=\; \frac{1}{2}\left(\delta^c_b\delta^a_d - \frac{1}{N}\delta^a_b\delta^c_d\right)\ , \qquad (1.17)$$

then each gluon propagator behaves at large N as a product of a quark and an antiquark propagators from a group theoretic point of view:

$$\gamma^\mu\,\langle (A_\mu)^a_b (A_\nu)^c_d\rangle\,\gamma^\nu = \frac{k}{N}\,\delta^c_b\delta^a_d + \mathcal{O}(1/N^2) \sim \langle \Psi^c\bar{\Psi}_b\rangle\langle\bar{\Psi}_d\Psi^a\rangle\ , \qquad (1.18)$$

with $k = 1/2\,\gamma^\mu\gamma^\nu D_{\mu\nu}$. This reflects the fact that an adjoint field transforms just as an element of the gauge group representation space given by the tensor product of a fundamental with an anti-fundamental representation spaces:[7]

$$\begin{aligned} (F_{\mu\nu})^a{}_b &\xrightarrow{g} (\mathrm{Ad}(g)F_{\mu\nu})^a{}_b = (gF_{\mu\nu}g^{-1})^a{}_b = (g)^a{}_c(F_{\mu\nu})^c{}_d(g^{-1})^d{}_b\ , &(1.19)\\ \Psi^a\bar{\Psi}_b &\xrightarrow{g} (g\Psi)^a(\bar{\Psi}g^\dagger)_b = (g\Psi)^a(\bar{\Psi}g^{-1})_b = (g)^a{}_c\,\Psi^c\bar{\Psi}_d\,(g^{-1})^d{}_b\ . &(1.20)\end{aligned}$$

This relation between the colour structure of the gluon and fermion propagators suggests that we represent the former as a double line of a quark and an antiquark propagators for the purpose of the N-counting:

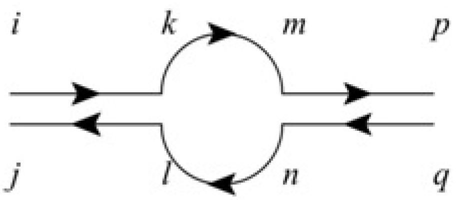


Figure 1.2: Internal diagram in double-line notation.

An amputated closed diagram (*i.e.* no external lines and all $SU(N)$ indices contracted) such as a gluon momentum loop with two three-point vertices is represented in the double line notation as three colour index loops:

[7]For a connection such as the gauge field, there is an extra inhomogeneous contribution as emphasized in footnote 1.

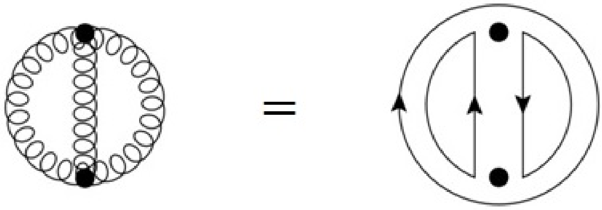


Figure 1.3: Closed diagram in double-line notation.

and is associated with the amplitude:

$$
\begin{aligned}
&N^2\langle A^A_\mu A^B_\nu\rangle\, k^\nu\, \langle A^C_\alpha A^D_\beta\rangle\, k^\beta\, \langle A^{E\alpha} A^{F\mu}\rangle \left(\delta_{AA'} f^{A'}_{DF}\right)\left(\delta_{CC'} f^{C'}_{BE}\right)\\
&= 4N^2\langle A^A_\mu A^B_\nu\rangle\, k^\nu\, \langle A^C_\alpha A^D_\beta\rangle\, k^\beta\, \langle A^{E\alpha} A^{F\mu}\rangle \left((T_A)^a_b (T_{A'})^b_a\, f^{A'}_{DF}\right)\left((T_C)^c_d (T_{C'})^d_c\, f^{C'}_{BE}\right)\\
&= 4N^2\langle A^A_\mu A^B_\nu\rangle\, k^\nu\, \langle A^C_\alpha A^D_\beta\rangle\, k^\beta\, \langle A^{E\alpha} A^{F\mu}\rangle \left((T_A)^a_b\, [T_D, T_F]^b_a\right)\left((T_C)^c_d\, [T_B, T_E]^d_c\right)\\
&\sim N^2\langle (A_\mu)^a_b (A_\nu)^d_{c'}\rangle\, k^\nu\, \langle (A_\alpha)^c_d (A_\beta)^b_{a'}\rangle\, k^\beta\, \langle (A^\alpha)^{c'}_c (A^\mu)^{a'}_a\rangle\\
&\sim N^2\left(N^{-1}\,\delta^d_b\,\delta^a_{c'} + \mathcal{O}(N^{-2})\right)\left(N^{-1}\,\delta^b_d\,\delta^c_{a'} + \mathcal{O}(N^{-2})\right)\left(N^{-1}\,\delta^{a'}_c\,\delta^{c'}_a + \mathcal{O}(N^{-2})\right)\\
&= \delta^b_b\,\delta^a_a\,\delta^c_{c'}\left(N^{-1} + \mathcal{O}(N^{-2})\right) \;=\; N^2 + \mathcal{O}(N)\,, \qquad (1.21)
\end{aligned}
$$

where we used the identity: $(T_A)^a{}_b (T_B)^b{}_a = \frac{1}{2}\delta_{AB}$. All contractions of adjoint indices can be rewritten in this way as traces over fundamental indices and the representation of gluon propagators by double lines of (anti-)fundamental indices expresses this property. The upshot of the double line notation is that the factors of N that arise from traces over $SU(N)$ indices are now easier to determine from the diagrams. In this notation, each trace over a fundamental (or anti-fundamental) colour index, say δ^a_a, becomes depicted by a colour index loop, resulting in a factor of N for each such loop in the diagram. Together with the fact that each vertex contributes with a factor of N and each propagator with $1/N$, this establishes the N-counting rules for each diagram.

The simplest case that we can study at large N is the expansion of the generating functional of pure Yang-Mills, *i.e.* the vacuum diagrams with all fermions switched off. The first few vacuum bubbles are given in Figure 1.4 and are represented in double line notation in Figure 1.5. From the above rules, each diagram with V vertices, E propagators and F colour index loops is proportional to a factor of N^{V-E+F}. It is then a simple matter to check that all diagrams that can be drawn on a plane (*i.e.* embedded in $\mathbb{R}^2$), called planar diagrams, scale as N^2,

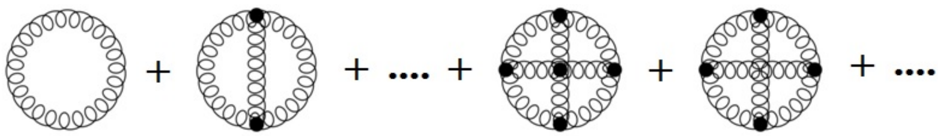


Figure 1.4: Vacuum bubbles.

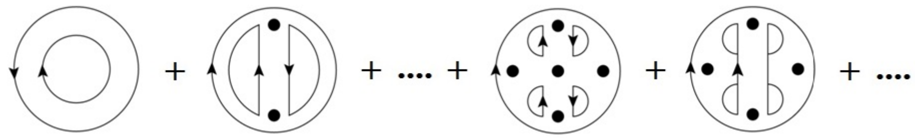


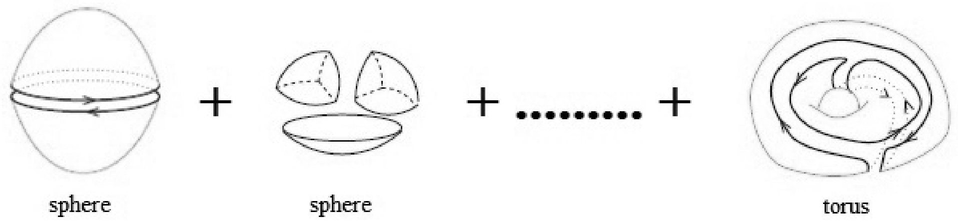
Figure 1.5: Vacuum bubbles in double-line notation.

whereas all diagrams that need to be embedded in $\mathbb{R}^3$ are suppressed by powers of N^{-2} in relation to the planar ones.

In topology, any connected closed 2-surface is completely characterised by its Euler characteristic $\chi = 2 - 2g$, with g its genus, and by whether it is orientable. From the classification theorem of closed 2-surfaces it then follows that every two-dimensional connected, closed and orientable surface is homeomorphic to a connected sum of g tori ($g = 0, 1, 2, ...$ respectively the 2-sphere, the torus, the double torus, etc). If we perform a one-point compactification of the surfaces in the above double line diagrams, then they will fall in this category since they will be closed (and connected) and the particle/anti-particle propagators induce an orientation on the surfaces, represented by the arrows.[8] Each diagram will in this way be homeomorphic to some n-fold torus and the different loops in the interior of the diagram will result in a triangulation of the respective surface. The planar diagrams will correspond to triangulations of a 2-sphere and, more generally, diagrams that scale as N^{2-2g} will correspond to triangulations of a g-fold torus. The asymptotic expansion of the generating functional with expansion parameter $1/N$ will in this way be a sum over different topologies:[9]

[8]Boundaries arise when we insert matter in the fundamental and the Euler characteristic is generalised to $\chi = 2 - 2g - b$, with b the number of boundaries. Also, orientability of the surfaces can be lost for certain gauge groups such as $SO(N)$ in which an adjoint field transforms as a product of two fundamental fields rather than a fundamental, anti-fundamental product [27].

[9]In the case of the S^2, for example, with the one-point compactification of the respective double line diagrams we are adding the point at infinity to obtain the south pole, as in an upside-down (triangulated) Riemann sphere. Note also that the surfaces are connected, we are segmenting each just for the purpose of emphasizing the triangulation.



and which is expressed as:

$$\mathcal{Z} \ = \ N^2 \sum_{g=0}^{\infty} N^{-2g} f_g(\lambda) \ , \tag{1.22}$$

with f_g some polynomial in the t Hooft coupling. This expansion is precisely the one that arises in string perturbation theory of closed oriented strings:

$$\mathcal{Z}_{\text{string}} \sim g_s^{-2} \sum_{g=0}^{\infty} g_s^{2g} \int [dX] \, e^{-S} \ , \tag{1.23}$$

with g_s the string coupling constant that determines the coupling to the different worldsheet topologies, X collectively the string embeddings, worldsheet metric and gravitino, and S the worldsheet action. If we identify $1/N$ with g_s we can see the expansion (1.22) as the definition of a string theory. The large N limit would then correspond on the string theory side to the limit in which all loop corrections are suppressed in relation to the tree-level diagrams, a classical limit. This connection, though derived at large N, suggests that one may be able to completely reformulate string theories in terms of Yang-Mills theories such that the full non-perturbative formulation of the former would be given by the latter for all N.

Even though non-Abelian gauge theories simplify considerably in the large N limit, it is still impossible to compute and sum all planar diagrams $\forall\lambda$ in a given correlation function. In fact, the large N expansion is an asymptotic expansion and may not be convergent. A further simplification would be to consider $\lambda << 1$ and do perturbation theory of the large N theory with expansion parameter λ. However, this is the weakly coupled regime of the theory and it excludes strongly coupled phenomena such as confinement and chiral symmetry breaking, which originally motivated the large N approach. The strongly coupled regime $\lambda >> 1$, on the other hand, cannot be studied using perturbation theory in λ.

The above apparent connection between gauge and string theories does not specify the relationship between the two remaining free parameters of each theory:[10] the 't Hooft coupling λ of the large N gauge theory and the string length scale (inverse string tension) of the string theory that determines the coupling to the target spacetime fields. If there is such a relationship, however, it must necessarily be a strong/weak one because the two theories are clearly different at the perturbative level and therefore, when one is weakly coupled, the other must be strongly coupled, so that perturbation theory in λ and in the string scale does not apply to both theories simultaneously. In the next section we will find that certain gauge theories, specifically those with a renormalization UV fixed-point, are in fact (in the strongest form of the correspondence) equivalent to string theory formulated in specific backgrounds, and that this equivalence is a strong/weak duality in the sense that λ is inversely proportional to the string scale. In this way, the strongly coupled regime of the gauge theory (at large N) can be probed by studying the dual string theory at lowest order approximation in the string scale, which simply corresponds to classical supergravity.

1.2. The AdS_5/CFT_4 Correspondence

1.2.1. $\mathcal{N}=4$ super Yang-Mills

Let us start with the pure $\mathcal{N}=1$ supersymmetric Yang-Mills Lagrangian with $U(N)$ gauge group (or a subgroup such as $SU(N)$) in ten dimensional flat space ($\eta_{mn}=\mathrm{diag}(-,+,..,+)$):

$$\mathcal{L} \;=\; -\frac{1}{2}\,(F_{mn})^a{}_b(F^{mn})^b{}_a + i(\bar{\psi})^a{}_b\,\Gamma^m(D_m\psi)^b{}_a\;, \tag{1.24}$$

where ψ is a Majorana-Weyl spinor (eight real degrees of freedom) and the Γ^m matrices are the 32×32 Dirac matrices in ten dimensions: $\{\Gamma^m,\Gamma^n\}=2\eta^{mn}$. Both ψ and F_{mn} are in the adjoint representation, *e.g.* $(F_{mn})^a{}_b = F^A_{mn}(T_A)^a{}_b$ with $\{T_A\}$ a basis for $\mathfrak{u}(N)$ normalized so that $\mathrm{Tr}(T_AT_B)=\frac{1}{2}\delta_{AB}$. The covariant derivative D_m is the adjoint derivative:

$$(D_m\psi)^b{}_a = \partial_m\psi^b{}_a + ig_{YM}\,[A_m,\psi]^b{}_a \;=\; \left(\partial_m\psi^A + ig_{YM}\,f^A_{BC}\,A^B_m\psi^C\right)(T_A)^b{}_a\;. \tag{1.25}$$

We will omit the $U(N)$ matrix indices ($a,b,...$ and $A,B,...$) from now onwards. Under the supersymmetry transformations:

$$\delta_\xi A_m = -i\bar{\xi}\,\Gamma_m\psi\;,\qquad \delta_\xi\psi = \frac{1}{2}\,F_{mn}\Gamma^m\Gamma^n\,\xi\;, \tag{1.26}$$

[10] In fact, λ is not really free, as determined by the β-function equation, and is dimensionally transmuted to the characteristic scale Λ' of the interaction, which represents the remaining (dimensionful) free parameter of the large N theory. Note also that the string coupling g_s is not actually an independent parameter, but rather corresponds to the asymptotic expectation value of the dilaton, which is a dynamical field.

with $\xi(x)$ an infinitesimal spinor, the Lagrangian changes by a total derivative and therefore the action remains invariant. In order to reduce the theory down to four dimensions, we write the ten-dimensional coordinates as $x^m = (x^\mu, x^a)$: $\mu = 0, .., 3$, $a = 4, .., 9$ and require that the fields be independent of x^a. We can similarly compactify the theory on a flat six torus and truncate to the massless sector (which is a consistent truncation). The Lagrangian then becomes:

$$\begin{aligned}\mathcal{L} \,=\, \mathrm{Tr}\Big\{ &-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - D_\mu A_a D^\mu A^a + \frac{1}{2}g_{YM}^2[A_a, A_b][A^a, A^b] \\ &+ i\bar{\psi}\Gamma^\mu D_\mu\psi - g_{YM}\bar{\psi}\Gamma^a[A_a, \psi]\Big\} \;. \end{aligned} \tag{1.27}$$

The Majorana-Weyl spinor ψ can be represented as:

$$\psi = \sum_{i=1}^{4}\left\{ v^i \otimes \begin{pmatrix} \lambda^i_\alpha \\ 0 \end{pmatrix} + v^{i+4} \otimes \begin{pmatrix} 0 \\ \bar{\lambda}^{i\dot{\alpha}} \end{pmatrix} \right\} \;, \tag{1.28}$$

where λ^i are four two-component Weyl spinors such that: $\bar{\lambda}^{i\dot{\alpha}} = (\lambda^{i\alpha})^*$, $\bar{\lambda}^i_{\dot{\alpha}} = (\lambda^i_\alpha)^*$, with the spinor indices raised and lowered with the antisymmetric tensor $\epsilon^{\alpha\beta} = -\epsilon_{\dot{\alpha}\dot{\beta}}$, and where v^j is a 8×1 column matrix such that:

$$(v^j)_{k1} = \begin{cases} 1 & \text{if} \quad k = j \\ 0 & \text{otherwise} \end{cases} \;. \tag{1.29}$$

The Dirac matrices can then be represented as:

$$\Gamma^\mu \,=\, \mathbb{1}_8 \otimes \gamma^\mu \;, \qquad \Gamma^a \,=\, \hat{\gamma}^a \otimes \gamma^5 \;, \tag{1.30}$$

$$\gamma^\mu \,=\, \begin{pmatrix} 0 & (\sigma^\mu_-)_{\alpha\dot{\beta}} \\ (\sigma^\mu_+)^{\dot{\alpha}\beta} & 0 \end{pmatrix} \;, \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_4 \;, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \;, \tag{1.31}$$

$$\hat{\gamma}^a \,=\, \begin{pmatrix} 0 & (\Sigma^a)_{ij} \\ (\bar{\Sigma}^a)_{ij} & 0 \end{pmatrix} \;, \quad \{\hat{\gamma}^a, \hat{\gamma}^b\} = 2\delta^{ab}\mathbb{1}_8 \;, \quad \Sigma^a\bar{\Sigma}^b + \Sigma^b\bar{\Sigma}^a = 2\delta^{ab}\mathbb{1}_4 \;, \tag{1.32}$$

where $\mathbb{1}_n$ is the $n \times n$ identity matrix, $\sigma^\mu_\pm = (1, \pm\vec{\sigma})$ are the Pauli matrices, and the six complex and antisymmetric constant matrices $\Sigma^a = (\bar{\Sigma}^a)^* = (\eta^1, \eta^2, \eta^3, i\bar{\eta}^1, i\bar{\eta}^2, i\bar{\eta}^3)$, with η^a the 't Hooft symbols [28] (see also [29]). The matrices γ^μ and $\hat{\gamma}^a$ are the Dirac matrices (*i.e.* satisfy the Clifford algebra) in four and six dimensions respectively.

With the six scalar fields A^a we then construct three complex scalars represented as ϕ^{ij} and defined by:

$$\phi^{ij} = \frac{1}{2} A_a(\Sigma^a)_{ij} \ , \qquad \bar{\phi}_{ij} = (\phi^{ij})^* = \frac{1}{2} A_a(\bar{\Sigma}^a)_{ij} = \frac{1}{2}\epsilon_{ijkl}\phi^{kl} \ , \tag{1.33}$$

where $i, j = 1, .., 4$ are the matrix indices of Σ^a such that $(\Sigma^a)_{ij} = -(\Sigma^a)_{ji}$. These are introduced so that the $SU(4)_R$ symmetry discussed below is manifest. We then have the identities ($\bar{\psi} = \psi^\dagger\Gamma^0$):

$$D_\mu A_a D^\mu A^a = -D_\mu\phi^{ij} D^\mu\bar{\phi}_{ij} \ , \tag{1.34}$$

$$[A_a, A_b][A^a, A^b] = [\phi^{ij}, \phi^{kl}][\bar{\phi}_{ij}, \bar{\phi}_{kl}] \ , \tag{1.35}$$

$$i\bar{\psi}\Gamma^\mu D_\mu\psi = 2i\,\bar{\lambda}^i_{\dot{\alpha}}(\sigma^\mu_+)^{\dot{\alpha}\beta} D_\mu\lambda^i_\beta \ , \tag{1.36}$$

$$\bar{\psi}\Gamma^a[A_a, \psi] = 2\lambda^{i\alpha}[\bar{\phi}_{ij}, \lambda^j_\alpha] - 2\bar{\lambda}^i_{\dot{\alpha}}[\phi^{ij}, \bar{\lambda}^{j\dot{\alpha}}] \ . \tag{1.37}$$

Note that:

$$\Gamma^0\Gamma^\mu = \mathbb{1}_8 \otimes \begin{pmatrix} (\sigma^\mu_+)^{\dot{\alpha}\beta} & 0 \\ 0 & (\sigma^\mu_-)_{\alpha\dot{\beta}} \end{pmatrix} \ , \qquad \Gamma^0\Gamma^a = \begin{pmatrix} 0 & (\Sigma^a)_{ij} \\ (\bar{\Sigma}^a)_{ij} & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -\delta^{\dot{\alpha}}_{\dot{\beta}} \\ \delta^\beta_\alpha & 0 \end{pmatrix} \ . \tag{1.38}$$

Finally, we replace these identities in (1.27), rescale $A_\mu \to g^{-1}_{YM} A_\mu$ and add the topological invariant $\theta_I/8\pi^2 \int F \wedge *F$ to obtain:

$$\begin{aligned} \mathcal{L} = \mathrm{Tr}\Big\{ &-\frac{1}{2g^2_{YM}} F_{\mu\nu}F^{\mu\nu} + \frac{\theta_I}{8\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} + D_\mu\phi^{ij} D^\mu\bar{\phi}_{ij} + \frac{g^2_{YM}}{2}[\phi^{ij}, \phi^{kl}][\bar{\phi}_{ij}, \bar{\phi}_{kl}] \\ &+ 2i\,\bar{\lambda}^i\sigma^\mu_+ D_\mu\lambda^i - 2g_{YM}\Big(\lambda^i[\bar{\phi}_{ij}, \lambda^j] - \bar{\lambda}^i[\phi^{ij}, \bar{\lambda}^j]\Big)\Big\} \ , \end{aligned} \tag{1.39}$$

where $\tilde{F} := \star F$. Further details on the dimensional reduction can be found in [30] (see also [31, 32]). All symmetries of the resulting theory in four dimensions follow by construction from the original ten-dimensional symmetries. Our final theory is invariant under $\mathcal{N} = 4$ supersymmetry transformations acting on $A_\mu, \phi^{ij}, \lambda^i$ and $\bar{\lambda}^i$ that follow from the transformations (1.26) (and are schematically given later in (1.77)). The maximum extended supersymmetry in four dimensions is $\mathcal{N} = 8$, but since $\mathcal{N} > 4$ necessarily requires fields of spin 3/2 and 2, it follows that the super Yang-Mills theory (1.39) is the maximally supersymmetric gauge theory in four dimensions. The theory has four supersymmetry generators (sixteen real supercharges) and the Lagrangian is invariant under a global $SU(4)_R \sim SO(6)_R$ R-symmetry (acting on the indices i, j). This symmetry group follows from the fact that the Lorentz group $SO(1, 9)$ of the original ten-dimensional theory decomposes as $SO(1, 9) \to SO(1, 3) \times SO(6)$. The Σ^a matrices then map tensors of

$SO(6)$ to tensors of $SU(4)$. The theory contains three complex (six real) scalar fields transforming under the two-index antisymmetric representation of $SU(4)_R$, four Weyl fermions in the fundamental of $SU(4)_R$, and the gauge field which is a scalar under the R-symmetry. All fields transform according to the adjoint representation of the $U(N)$ gauge group.

An important property of $\mathcal{N} = 4$ super Yang-Mills (SYM) is that the theory is invariant under the conformal group Conf($\mathbb{R}^{1,3}$) $\sim SO(2,4) \sim SU(2,2)$ consisting of the set of conformal transformations in flat space (see appendix A.3).[11] Because of supersymmetry, the symmetry group is enhanced to the maximal superconformal group $SU(2,2|4)$ in four dimensions consisting of the bosonic subgroup $SU(2,2) \times SU(4)_R$ together with the supersymmetries generated by the Poincaré and conformal supercharges.[12] This symmetry group is preserved at the quantum level at all orders in perturbation theory and also non-perturbatively [33, 34, 35, 36, 37]: the theory exhibits no UV divergences and therefore does not require renormalization. In this way, the β-functions are identically zero and the theory remains exactly conformal. Together, these features imply that $\mathcal{N} = 4$ SYM is the most symmetric gauge theory in four dimensions.

A further aspect of $\mathcal{N} = 4$ SYM is its conjectured invariance under the Montonen-Olive duality that acts on the coupling constants as the $SL(2,\mathbb{Z})$ group:

$$\tau \to \frac{a\tau + b}{c\tau + d} \quad : \; ad - bc = 1 \; , \quad a, b, c, d \in \mathbb{Z} \; , \tag{1.40}$$

where $\tau = 4\pi i/g_{YM}^2 + \theta_I/2\pi$. This is an example of a strong-weak S-duality. The quantum theory is invariant under the transformation $\theta_I \to \theta_I + 2\pi$ and it is conjectured that it is also invariant under the strong-weak transformation $\tau \to -1/\tau$ (together with the substitution of the gauge group by its Langlands dual). The combination of the two results in the above $SL(2,\mathbb{Z})$ symmetry. The existence of this symmetry group will be important for the consistency of the AdS/CFT duality as we will discuss in the next sections.

[11] The fields have mass dimensions $[A_\mu] = 1 = [\phi^{ij}]$, $[\lambda^i] = 3/2$, and the coupling constants are dimensionless.

[12] The conformal supercharges S^i_α are defined as the operators given by the commutators of the generators of special conformal transformations with the Poincaré supercharges Q^i_α. It should be further remarked that the full symmetry group is in fact the subgroup $PSU(2,2|4)$. In the algebra of $\mathfrak{su}(2,2|4)$, the anticommutator between the supercharges is given by: $\{Q^i_\alpha, S_{\beta j}\} = M_{\alpha\beta}\delta^i_j + \epsilon_{\alpha\beta}R^i_j + \epsilon_{\alpha\beta}\delta^i_j\,(D + C)$, where $M_{\alpha\beta} = \sigma^{\mu\nu}_{\alpha\beta}M_{\mu\nu}$ is the $SU(2)$ representation of the Lorentz generators, R^i_j are the R-symmetry generators, D is the dilatation generator and C is the central charge. In the case of $\mathcal{N} = 4$ SYM the central charge vanishes and the resulting algebra is the $\mathfrak{psu}(2,2|4)$ algebra.

1.2.2. String theory, Supergravity and D-branes

In the previous subsection we have analysed a highly symmetric gauge theory in four dimensions and discussed how its symmetries arise from the parent ten-dimensional SYM theory. In view of the results of the first section, we would like to explore how this gauge theory can be related to superstring theory.

Let us start with type IIB closed strings with supersymmetry implemented on the worldsheet, known as the Ramond-Neveu-Schwarz (RNS) formulation. Recall that the R/NS vacuum is the state annihilated by the positive modes of the bosonic and fermionic string embeddings with periodic/antiperiodic fermionic boundary conditions, also known as R/NS boundary conditions. The Hilbert space then corresponds to the Fock space generated by acting with the respective negative modes on the R and NS vacuum. Since the string embeddings decompose into left $(-)$ and right $(+)$ moving modes (*i.e.* holomorphic and anti-holomorphic components), the Hilbert space decomposes into four sectors. States in the X-Y sector are obtained by acting on the vacuum $(|0\rangle_X^+ \otimes |0\rangle_Y^-)$ with the negative modes of the left/right fermionic movers with X/Y boundary conditions and with the left/right bosonic movers, where X,Y $\in$ {R, NS}. The massless states of the closed string in the NS-NS sector are the first excited states and correspond to the particle states for the graviton $G_{\mu\nu}$, the 2-form B-field and the dilaton Φ (form a basis for the Hilbert spaces). The massless state in the R-R sector is the ground state, a tensor product of two spinor states, and decomposes into the particle states of the axion C_0, the 2-form C_2 and the 4-form C_4 with self-dual field strength. The remaining NS-R and R-NS sectors contain fermionic gravitinos' and dilatinos' particle states. All these states correspond to the lowest energy excitations of the target spacetime fields, with different configurations of the fields corresponding to different states, and couple to the IIB closed string worldsheet via specific vertex operators. From the one-loop β-functions of the closed string (leading order in α') it then follows that the dynamics of the spacetime fields is given by the IIB supergravity action [38, 39]:

$$\begin{aligned} S \;=\; & \frac{1}{(2\pi)^7 \ell_s^8} \int d^{10}x \sqrt{G}\, e^{-2\Phi} \left(R[G] + 4|\partial\Phi|^2 - \frac{1}{2}\,|H_3|^2 \right) \\ & - \frac{1}{2(2\pi)^7 \ell_s^8} \int d^{10}x \left[\sqrt{G} \left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}\,|\tilde{F}_5|^2 \right) + C_4 \wedge H_3 \wedge F_3 \right] + \text{fermions}\;, \end{aligned} \tag{1.41}$$

where the field strengths are given by ($|F_n|^2 = 1/n!\, F_{\mu\dots\nu} F^{\mu\dots\nu}$):

$$\begin{gathered} H_3 = dB\,, \quad F_1 = dC_0\,, \quad F_3 = dC_2\,, \quad F_5 = dC_4\,, \\ \tilde{F}_3 = F_3 - C_0\, H_3\,, \quad \tilde{F}_5 = *\tilde{F}_5 = F_5 - \tfrac{1}{2}\, C_2 \wedge H_3 + \tfrac{1}{2}\, B \wedge F_3\;, \end{gathered} \tag{1.42}$$

and where the string length scale $\ell_s := \sqrt{\alpha'}$ and is related to the ten-dimensional

Planck length as: $\ell_P^8 = 8\pi^6 g_s^2 \ell_s^8$ (which is the Newton constant G_{10} in the Einstein frame defined next) [40]. Note that IIB supergravity provides a good approximation to IIB closed strings only in the low-energy limit $\alpha' \to 0$, where the massive string states can be ignored (recall that the mass of the states $M^2 \sim 1/\alpha'$). Furthermore, the approximation can be trusted when the loop corrections in the string perturbation theory are suppressed, *i.e.* for $g_s << 1$.

IIB supergravity is invariant under $\mathcal{N} = 2$ supersymmetry, with 32 real supercharges (two generators in ten dimensions). The classical action is also invariant under an $SL(2,\mathbb{R})$ transformation. The latter symmetry is manifest if we rewrite the action in the Einstein frame by defining:

$$G^E_{\mu\nu} := e^{(\Phi_0 - \Phi)/2} G_{\mu\nu} \; , \tag{1.43}$$

where $e^{\Phi_0} = g_s$ is the asymptotic expectation value of the dilaton, together with: $\tau := C_0 + i\, e^{-\Phi}$ and $G_3 := (F_3 - \tau\, H_3)/\sqrt{\mathrm{Im}\ \tau}$. The symmetry transformation then acts as:

$$\tau \to \frac{a\tau + b}{c\tau + d} \; , \quad G_3 \to \frac{c\bar{\tau} + d}{|c\tau + d|}\, G_3 \quad : \; ad - bc = 1 \; , \quad a, b, c, d \in \mathbb{R} \; , \tag{1.44}$$

with the remaining fields fixed. However, the full IIB string theory is invariant only under the transformation $\tau \to \tau + 1$ together with the strong-weak transformation $\tau \to -1/\tau$ and therefore only the discrete subgroup $SL(2,\mathbb{Z})$ survives at the quantum level. This symmetry group is the S-duality of IIB closed strings and is related by the AdS/CFT correspondence to the S-duality of $\mathcal{N} = 4$ SYM discussed in the preceding section.

Above we have briefly discussed closed strings, but a similar analysis can be repeated for the open string. Recall that open strings end on Dp-branes with the bosonic string embeddings satisfying Dirichlet/Neumann boundary conditions in the directions transverse/parallel to the branes. In addition, appropriate Neveu-Schwarz and Ramond fermionic boundary conditions imposed at the ends of the string give rise to the NS and R sector of the open string Hilbert space corresponding to bosonic and fermionic states respectively. The massless states in the NS sector are the first excited states and decompose into the particle states of a gauge field A_a on the D-brane and of the D-brane bosonic embedding X^I in the transverse directions. The massless state in the R sector is the ground state (a spinor state) which decomposes into the particle states of the superpartners, *viz.* a fermion on the D-brane and the D-brane fermionic embedding in superspace. These states describe the lowest energy excitations of the D-brane; the massive states describe excitations of higher energy. Furthermore, a collection of N D-branes represents $N \times N$ different combinations of hyperplanes on which the open string can end. If the branes are coincident, the above fields become non-abelian and transform under the adjoint representation of a $U(N)$ gauge group.

The action describing the dynamics of the low-energy excitations of the D-branes can be derived (at leading order in α') by coupling the string states to the worldsheet via vertex operators and deducing the one-loop β-functions; further methods involve T-duality and the BPS properties of D-branes. The dynamics of the fields is then described by the generalised DBI, or worldvolume action of N Dp-branes [41]:[13]

$$S_p = -\,T_p \int d^{p+1}\xi \,\mathrm{Tr}\Big\{ e^{-\Phi} \det{}^{1/2}\Big[G_{ab} + B_{ab} + 2\pi\alpha' F_{ab} + E_{aI}\left(Q^{-1} - \delta\right)^{IJ} E_{Jb}\Big] \det{}^{1/2}\Big[Q^I_J\Big]\Big\}$$

$$+\,T_p \int_{p+1} \Big[\sum_{n=0}^{2} C_{2n} + \sum_{n=3}^{4} C_{2n}\Big] \wedge \mathrm{Tr}\, e^{B_{ab}+2\pi\alpha' F_{ab}} + \text{ fermions}\,. \qquad (1.45)$$

Here we have decomposed the brane embedding $X^\mu(\xi)$ in spacetime into the directions parallel to the brane $X^a : a = 0, ..., p$ and transverse to the brane $X^I : I = p+1, ..., 9$. It is common practice to choose coordinates such that $X^a = \xi^a$, with ξ^a the coordinates on the brane worldvolume. The equations of motion are obtained by varying the action with respect to X^I and the gauge field A_a. The tension T_p of each brane (in the string frame) is given by [10, 42]:

$$T_p = (2\pi)^{-p}\,\ell_s^{-(p+1)}\,. \qquad (1.46)$$

The physical tension $T_p^E = g_s^{-1} T_p$ is obtained by moving to the Einstein frame (recall equation (1.43)). Note that the perturbative regime of string theory requires $g_s << 1$, which implies $T_p^E >> 1$ (in string units). Since the brane tension corresponds to its mass, the brane becomes infinitely heavy in string perturbation theory and therefore it is not visible in this sector of the theory.

The two-form $F_{ab}(\xi)$ is the field strength of the open string gauge field $A_a(\xi)$ and the remaining fields are given by the pullback of the closed string fields onto the brane worldvolume:

$$\Phi = \Phi(X)\,, \quad G_{ab} = \frac{dX^\mu}{d\xi^a}\frac{dX^\nu}{d\xi^b} G_{\mu\nu}(X)\,, \quad B_{ab} = \frac{dX^\mu}{d\xi^a}\frac{dX^\nu}{d\xi^b} B_{\mu\nu}(X)\,,$$

$$E_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}\,, \quad E_{aI} = \frac{dX^\mu}{d\xi^a} E_{\mu I}(X)\,, \qquad (1.47)$$

and analogous for the closed string R-R gauge fields C_n. The C_6 and C_8 potentials are the Hodge duals defined as $dC_6 = \star dC_2$ and $dC_8 = \star dC_0$. Note also that, in the last integral, when we replace the exponential by its power series, with products

[13] If the curvature of the target spacetime is non-vanishing, the Chern-Simons term (the last integral) also involves wedge-powers of the Riemann tensor pulledback to the worldvolume. Here we will be ignoring these contributions.

replaced by wedge products, only one term in the series is picked by each R-R field due to the integral over the worldvolume such that the resulting form is a $(p+1)$-form. The tensor Q_{IJ} is given by:

$$Q^I_J = \delta^I_J + i2\pi\alpha'[\phi^I, \phi^K]E_{KJ}(X) \quad : \quad \phi^I = \frac{1}{2\pi\alpha'}X^I \ , \tag{1.48}$$

with the I, J indices raised/lowered with the metric $E^{IJ} : E^{IK}E_{KJ} = \delta^I_J$. Finally, the traces Tr are taken over the fundamental indices of the gauge group $U(N)$.[14]

The worldvolume action contains interactions with the closed string modes. These interactions can be switched off by decomposing the target spacetime fields into their background configurations plus the α' corrections that represent the closed string excitations and taking the limit $\alpha' \to 0$. This is called a decoupling limit since the closed string modes decouple from the branes. In this limit, the worldvolume theory reduces to a Yang-Mills gauge theory with $U(N)$ gauge group. In the particular case of D3-branes, if the target space metric $G_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}(\alpha'^2)$, with $\eta_{\mu\nu}$ the Minkowski metric, the dilaton $e^\Phi = g_s + \mathcal{O}(\alpha')$, the axion $C_0 = k + \mathcal{O}(\alpha')$ (a constant), and the background configurations of the remaining target space fields vanish, the worldvolume theory in the decoupling limit becomes $\mathcal{N} = 4$ SYM in four dimensional flat space. In particular, the SYM field strength and the six real scalar fields in (1.39) are given by the open string field strength and the six transverse brane embeddings ϕ^I:[15]

$$\begin{aligned} S_p \ &= \ -\frac{T_p(2\pi\alpha')^2}{4g_s}\int d^4\xi \, \mathrm{Tr}\Big\{F_{ab}F^{ab} + 2D_a\phi^I D^a\phi_I - [\phi^I, \phi^J][\phi_I, \phi_J]\Big\} \\ &\quad + \frac{T_p(2\pi\alpha')^2 k}{2}\int d^4\xi \, \mathrm{Tr}\Big\{F_{ab}\tilde{F}^{ab}\Big\} + \text{ fermions} \ . \end{aligned} \tag{1.49}$$

If we replace for T_p and compare the result with the Yang-Mills Lagrangian (1.39), we find that the Yang-Mills coupling and the instanton angle θ_I are given by:

$$g^2_{YM} \ = \ \left(\frac{2g_s}{T_p(2\pi\alpha')^2}\right)_{p=3} \ = \ 4\pi g_s \, , \quad \theta_I \ = \ 8\pi^2\left(\frac{T_p(2\pi\alpha')^2 k}{2}\right)_{p=3} \ = \ 2\pi k \ . \tag{1.50}$$

A fundamental property of D-branes is the fact that they are the sources for the R-R gauge fields. Recall from Maxwell's theory that point charges couple to a one-form gauge potential via an interaction term of the form:

$$S_{\text{int}} = e\int A = e\int d\tau \frac{dx^\mu}{d\tau}A_\mu(x) \ , \tag{1.51}$$

[14] For a discussion of how the traces should be implemented, see references [43, 44, 45].

[15] Recall that the scalars ϕ^{ij} in (1.39) are in the two-index antisymmetric representation of $SU(4)_R$, whereas the scalars ϕ^I are in the fundamental of $SO(6)_R$. The isomorphism is given by the Σ^a matrices as in (1.33).

where A represents the pullback of the gauge field to the worldline of the particle with proper time τ and electric charge e. In the case of higher-order gauge fields, the sources for the $(p+1)$-form potentials are $(p+1)$-dimensional objects called p-branes which couple to the potentials as:

$$S_{\text{int}} = \mu_p \int_{p+1} A_{p+1} = \mu_p \int d^{p+1}\xi \, \frac{dX^{\mu_0}}{d\xi^0} ... \frac{dX^{\mu_p}}{d\xi^p} A_{\mu_0 ... \mu_p}(X) \; , \tag{1.52}$$

where μ_p is the charge of the p-brane, ξ^a are coordinates on the brane and $X^\mu(\xi)$ is the brane embedding into spacetime. p-branes backreact on the embedding spacetime just like point charges do, *i.e.* they are sources of closed strings, and in the case of the IIB R-R potentials, the backreaction of the corresponding p-branes is determined by the IIB supergravity action (1.41).

When we analyse the worldsheet of the closed and open strings with the vertex operators for the string states inserted, we find that the strings couple to the two-form B-field as above and therefore are the sources for the B-potential, but they do not carry R-R charge since the R-R vertex operator for the two-form C_2 field involves directly the field strength. On the other hand, in the case of Dp-branes, the interaction term with a R-R $(p+1)$-potential is of the form (1.52) (more clearly seen if we switch off the background B_{ab} and A_a fields) and therefore they are the sources for the R-R gauge fields with charge T_p. This implies in particular that the R-R p-branes that arise as solutions of IIB supergravity are the IIB Dp-branes with $p = \pm 1, 3$. This statement is not entirely correct, though. Dp-branes with odd $|p|$ are supersymmetric 1/2 BPS objects, *i.e.* their states are in representations of the superPoincaré group that saturate the BPS bound such that $M = |Z| = Q$, with M, Z and Q the mass, central charge and electric charge, and preserve half of the supersymmetry (the worldvolume theory is invariant under 16 supercharges).[16] On the other hand, the only supersymmetric p-brane solutions of supergravity are the extremal p-branes, which are also 1/2 BPS (admit 16 Killing spinors), so the identification between p-branes and supersymmetric Dp-branes only holds for the extremal case. D-branes at finite temperature are non-supersymmetric and in this case are identified with p-brane solutions of supergravity near extremality.

The discovery that D-branes are the R-R charge carriers led to the conjecture that their description as solutions of the worldvolume theory and as solutions of the supergravity equations of motion could be two equivalent descriptions. More generally, it led to the conjecture that the dynamics of open and closed string states could be described by physically equivalent theories and therefore that the open and closed string could be dual to each other. Note that the worldvolume

[16] Recall that IIB supergravity has $\mathcal{N} = 2$ supersymmetry, so there is only $\mathcal{N}/2 = 1$ central charge. Also, Dp-branes in IIB string theory with even p are non-BPS. They break all of the supersymmetry, do not carry conserved charges and are unstable.

description of D-branes is a gauge theory without gravity, so this open/closed string duality is in particular a gauge/gravity duality. Further, the theory on the D-brane is defined in less dimensions than the gravitational theory, so the degrees of freedom along the extra directions would have to be encoded (in a highly non-trivial way) on the D-brane. We will see that these extra degrees of freedom are encoded in the dynamics of the lower dimensional field theory description.

This conjectured relationship between the worldvolume theory on the D-brane and the gravitational theory in the bulk, describing the backreaction of the brane on the embedding spacetime, has its roots in the discovery of Hawking radiation and in the fact that black hole entropy scales with the area of the horizon. This fact led to the speculation that black holes (as bulk solutions) could equally be described by a lower dimensional field theory on the horizon in such a way that the Hilbert space of this theory would contain the black hole states. In the case of D-branes, this idea extrapolates to the statement that the bulk theory in the vicinity of the branes would admit an equivalent description in terms of the gauge theory on the worldvolume of the branes. This conjecture was originally supported by several results. The most important of these are the derivation of the Bekenstein-Hawking (BH) entropy of black holes obtained from intersecting D-branes by counting the degenerate states of the branes [9, 46] and the computation of absorption cross-sections of parallel D-branes [17, 18, 19]. In the first case, the calculation of the BH entropy is a bulk theory calculation, whereas the counting of the degenerate states is performed using statistical mechanics in the worldvolume theory, also known as the black hole microscopic theory. In the second case, the cross-section for infalling massless closed strings to be absorbed by the D-branes was computed using supergravity and the worldvolume theory (in which case the process corresponds to a computation of the decay rate of the closed strings into pairs of massless open strings on the branes [47]) and the results agree with one another; see [27] for a review. Since absorption cross-sections can be expressed in terms of correlation functions, this result suggested that worldvolume correlators could be computed from supergravity.

A hint of how this gauge/gravity duality could possibly work arises from M-theory. The different types of string theories are related by dualities and type IIA and the $E_8 \times E_8$ heterotic string, both formulated in ten dimensions, approach eleven-dimensional M-theory as $g_s \to \infty$. The low-energy limit of M-theory is supergravity in eleven dimensions, which reduces to 10D supergravity by Kaluza-Klein compactification. In the case of type IIA (resp. $E_8 \times E_8$), the argument to go up to eleven dimensions is that the higher-dimensional M-theory is compactified on a circle (resp. $S^1/\mathbb{Z}_2$ orbifold) of radius $r_{11} = g_s \ell_s$. Perturbative string theory therefore corresponds to the limit $r_{11} \to 0$, *i.e.* represents an expansion around

$r_{11} \sim 0$, and hence this dimension is not be visible in the perturbative regime of the string theory. However, as we go to the strong coupling regime, the extra dimension opens up. This relationship between strong coupling regimes and extra dimensions suggests that if the gauge theory on the brane admits an equivalent description in terms of a higher-dimensional theory, say one dimension extra, then maybe this equivalence could be more easily seen by going to the strong coupling regime of the gauge theory, where the extra dimension would become more visible. We will see that this is precisely how the duality is formulated.

1.2.3. AdS_5 Supergravity and $\mathcal{N} = 4$ SYM

The first proposal of a precise equivalence between the worldvolume and supergravity theories followed by studying a system of N parallel D3-branes in the decoupling, low-energy limit $\alpha' \to 0$, both from the point of view of the gauge theory (1.49) on the branes and their backreaction on the embedding spacetime [20], see also [27].

Suppose we have a set of N coincident D3-branes charged under the R-R C_4 potential. We will be assuming that $g_s << 1$ so that we can analyse this system using string perturbation theory. Furthermore, we will work in the lowest order approximation in α' so that the massive states of the open and closed strings can be ignored and the excitations of the D3-branes can be described by the worldvolume theory (1.45). In this approximation, the backreaction of the D3-branes is derived from the supergravity action (1.41). We want to look for solutions with constant R-R scalar C_0 and with vanishing B-field and R-R potential C_2 (and without fermionic degrees of freedom). If we further impose translational symmetry on the branes and rotational symmetry in the transverse space, we find the unique solution generated by the stack of D3-branes, also called the extremal black 3-brane:

$$
\begin{aligned}
ds_{10}^2 &= H(r)^{-1/2}\left(-dt^2 + d\vec{x}_3^{\,2}\right) + H(r)^{1/2}\left(dr^2 + r^2 d\Omega_5^2\right) \ , \\
F_5 &= \star F_5 = (1+\star)\, dH(r)^{-1} \wedge dt \wedge dx^1 \wedge dx^2 \wedge dx^3 \ , \\
e^{\Phi} &= g_s \ , \quad C_0 = k \ , \quad H(r) = 1 + \frac{L^4}{r^4} \ .
\end{aligned} \tag{1.53}
$$

In general, the solution for the dilaton is of the form $e^{2\Phi} = g_s^2 H(r)^{(3-p)/2}$ for the generic case of Dp-branes, but it must necessarily be a constant in the particular case $p = 3$ [48, 49]. Note that, since $e^{\Phi} = e^{\Phi_0} = g_s$, the metric in the Einstein or in the string frame is the same (recall (1.43)). The coordinates $(t, \vec{x})$ are the coordinates ξ^a on the worldvolume of the D-branes and (r, Ω^i) parametrise the transverse space. The parameter L with dimensions of length is the characteristic radius of the solution and is determined by a computation of the R-R flux over the S^5, which should be equal to the total charge of the N D3-branes:[17]

$$\int_{S^5} \star F_5 \quad = \quad 16\pi G_{10} T_p^E N \ . \tag{1.54}$$

Using the above formulas for T_p^E and for $\ell_P^8 = G_{10}$, we find that $16\pi G_{10} T_p^E N = (2\pi\ell_s)^{7-p} g_s N$. For $p = 3$ we then obtain ($\mathrm{Vol}(S^{n-1}) = 2\pi^{n/2}/\Gamma[n/2]$):

$$L^4 = 4\pi g_s N \ell_s^4 \ . \tag{1.55}$$

Note that the limit $\alpha' \to 0$ (with $g_s N$ fixed) implies $L \to 0$ and the spacetime reduces to Minkowski space everywhere but at $r = 0$ in this limit.

The extremal black 3-brane solution is completely regular, geodesically complete and free of essential singularities [53]. The region $r = 0$ where the D-branes are localised is a horizon; it may seem that this region is a curvature singularity, but in fact it is just a coordinate singularity. The easiest way to see this is to write the metric near $r = 0$:

$$ds_{10}^2 \sim \frac{r^2}{L^2}\left(-dt^2 + d\vec{x}_3^{\,2}\right) + \frac{L^2}{r^2} dr^2 + L^2 d\Omega_5^2 \qquad (r \sim 0) \ . \tag{1.56}$$

The metric reduces to $AdS_5 \times S^5$ in Poincaré coordinates near the centre of the Poincaré patch. In this form we see explicitly that $r = 0$ is the usual coordinate singularity of the Poincaré patch of AdS and a regular solution near $r = 0$ can be obtained by *e.g.* changing to global AdS coordinates.[18] The above parametrisation of the near-horizon geometry is incomplete because it is only valid for $r/L << 1$ and

[17] In the Einstein frame, the bulk plus Dp-brane action is given by:

$$S \;=\; \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{G^E_{\mu\nu}} \left(R_E - \frac{1}{2}\, e^{2(\Phi-\Phi_0)} |e^{\Phi_0} F_1|^2_E - \frac{1}{2} e^{\Phi-\Phi_0} |e^{\Phi_0} \tilde{F}_3|^2_E - \frac{1}{4} |e^{\Phi_0} \tilde{F}_5|^2_E + ... \right)$$

$$- T_p e^{-\Phi_0} \int_{p+1} d^{p+1}\xi \ \mathrm{Tr} \left(e^{(p-3)(\Phi-\Phi_0)/4} \sqrt{G^E_{ab}} + ... \right) + T_p \int_{p+1} C_{p+1} \ .$$

Recall that p-branes can be surrounded by $(d-p-2)$-spheres in d dimensions. For N Dp-branes and a constant dilaton, the flux is then given by [50, 51]:

$$\frac{1}{16\pi G_{10}} \int_{S^{8-p}} \star \left(e^{\Phi_0} F_{(p+2)} \right) = N T_p \ ,$$

which is equation (1.54). The Dirac quantization condition then reads [52]: $16\pi G_{10} N T_p^E T_{6-p}^E = 2\pi n \Leftrightarrow N = n$, where T_{6-p}^E is the magnetic charge of the D$(6-p)$-brane magnetic dual of the Dp-brane and $n \in \mathbb{N}$.

[18] Since the D-branes are localised at $r = 0$, it may seem that the pullback of the metric and C_4 potential onto the worldvolume of the branes will be ill defined in these coordinates, but in fact the divergences at $r = 0$ of the r-dependent pieces will cancel overall in the worldvolume action. An example of this can be found *e.g.* in [54].

a complete parametrisation can be obtained by starting from the original solution (1.53), performing the transformation of coordinates $r = \lambda\tilde{r}\,,\ t = \tilde{t}/\lambda\,,\ x^i = \tilde{x}^i/\lambda$ and in the end taking the limit $\lambda \to 0$. Note that we approach the near-horizon region $r = 0$ in this limit. The resulting metric will be the $AdS_5 \times S^5$ metric (1.56), now for all values of the new coordinates. Since we have just performed a reparametrisation and obtained a well-defined solution (both for the metric and the F_5 form), this result implies that $AdS_5 \times S^5$ is an exact solution of the same equations of motion satisfied by the black brane metric.

The duality originally proposed in [20] is a form of open/closed string duality and is based on the premise that the worldvolume description of the D3-branes and the supergravity description are equivalent. However, this equivalence is very hard to see even at the lowest order approximation in α', so it was suggested that we restrict to the low-energy limit $\alpha' \to 0$. Suppose we consider IIB string theory in the background (1.53) generated by the D-branes. Closed string states then describe excitations of target spacetime fields with vacuum expectation values (vevs) the background configurations (1.53). Open string states describe excitations of the fields on the D-branes with vevs certain background configurations obtained by explicitly finding a ground state solution of the worldvolume theory. We will argue that, under certain conditions, these two types of excitations are described by equivalent theories, *i.e.* that the dynamics of each type can be described both in terms of the worldvolume action and in terms of a gravitational theory. We will derive the regimes in which one description appears to be more natural than the other.

From the perspective of an observer in the bulk and far away from the D-branes, there are two kinds of low-energy closed string excitations: either the excitations are away from the D-branes and have low proper energy, or they have any proper energy but are located in the near-horizon region. This fact follows from a quick analysis of the spacetime metric. As $r \to \infty$ the metric reduces to Minkowski and therefore t is the time coordinate of inertial observers at infinity. The vector $k = \partial_t$ is a timelike Killing and the energy (per unit mass) $k_\mu u^\mu = G_{tt}\dot{t}$ is a constant of motion along geodesics with tangent vector $u^\mu\partial_\mu$. If we replace this constant of motion in the equation $-1 = u^2$, we find that static particles at position r satisfy:

$$\varepsilon \quad = \quad H(r)^{-1/4}\,\varepsilon_p\ , \tag{1.57}$$

where ε is the energy of the particle in string units as seen at infinity and ε_p is the proper energy of the particle (*i.e.* as measured at r).[19] If $r >> L$ we find

[19]String excitations have energies $E = \varepsilon/\sqrt{\alpha'}$, where increasingly excited states have increasingly higher values of ε. Low-energy excited states therefore satisfy $\sqrt{\alpha'}\,E = \varepsilon << 1$. The dimensionless energy ε is the relevant quantity to characterise excitations if we want to work in the limit $\alpha' \to 0$.

$\varepsilon = \big(1 + \mathcal{O}(\alpha'^2/r^4)\big)\,\varepsilon_p$ and therefore small ε requires low proper energy. On the other hand, if $r << L$ we obtain:

$$\varepsilon \quad \sim \quad \frac{r}{(\alpha')^{1/2}}\,\varepsilon_p \ . \tag{1.58}$$

In this case, the proper energy can be arbitrary while ε remains small as long as the particle is sufficiently close to the D-branes.

The first claim we make is that these two types of low-energy excitations in the bulk stop interacting with each other if the limit $\alpha' \to 0$ is taken. We will discuss this below. Note that, for the excitations in the near-horizon region at position r, the limit $\alpha' \to 0$ with $H(r)^{-1/4} << 1$ (so that (1.58) holds) requires that $r \to 0$ faster than $(\alpha')^{1/2}$. Just how fast we choose this to be is specified next, so for the moment we will simply write $r = (\alpha')^n\,\tilde{r}$, with $n > 1/2$, and take $\alpha' \to 0$ with the position of these excitations in the new coordinate $\tilde{r}$ fixed. In this limit, the closed string excitations that are away from the D-branes propagate in Minkowski space and are free strings, *i.e.* are described by free supergravity: the interaction terms in the action (1.41) are all switched off once we expand the fields into their background configurations plus the excitations and take the limit $\alpha' \to 0$ [55]. On the other hand, the closed string excitations that are near the D-branes propagate in the near-horizon geometry (parametrised by $\tilde{r}$) and are fully interacting even though we have taken the limit $\alpha' \to 0$. This feature will be shown more precisely below.

The second claim is that, if we take the same limit in the worldvolume action (1.45), the interaction of the D-branes with the closed strings is switched off and the theory on the branes reduces to $\mathcal{N} = 4$ SYM in four dimensions with a flat metric. We have argued that this is the case when we derived equation (1.49).[20]

Therefore, in both cases we have excitations that decouple from the (free) closed strings that are away from the D-branes in the limit $\alpha' \to 0$.[21] The conjecture is then that the dynamics of the open string excitations on the branes and

[20] We have derived this in the particular case of vanishing R-R potentials, but it also holds in the limit $\alpha' \to 0$ when we include the R-R fields. Furthermore, we have assumed that $G_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}(\alpha'^2)$. For the background metric (1.53) this is true except at $r = 0$, where the branes are localised. The fact that $G_{\mu\nu}$ is not of this form at $r = 0$ is the reason behind the fact that the closed strings near the D-branes are not described by free supergravity as $\alpha' \to 0$. If we pullback both C_4 and $G_{\mu\nu}$ in the coordinate system (1.53) onto the worldvolume of the D-branes, the divergences at $r = 0$ cancel and in the limit $\alpha' \to 0$ the theory still reduces to $\mathcal{N} = 4$ SYM. This can be seen more concretely by working directly with the metric (1.62) in the limit $\alpha' \to 0$. Divergences at $r = 0$ translate into divergences as $\alpha' \to 0$. However, there will be an overall factor of $L^4 \sim \alpha'^2$ in the worldvolume action that combines with the brane tension $T_p \sim \alpha'^{-2}$. See also footnote 18.

[21] Schematically, we write the target spacetime metric as $G_{\mu\nu} = \big(\eta_{\mu\nu} + \alpha'^2 h^{(1)}_{\mu\nu}\big) + \alpha'^2 h^{(2)}_{\mu\nu}$. The excitations $h_{\mu\nu}$ are closed strings generated by the D3-branes and the metric in parenthesis is the black-brane background. The excitations $h^{(2)}_{\mu\nu}$ are the closed string excitations in this background and decompose into far-away and near-horizon excitations. In the limit $\alpha' \to 0$, the

of the closed string excitations in the near-horizon region are determined by two equivalent theories. In the former case, the theory on the D3-branes in the limit $\alpha' \to 0$ is $\mathcal{N} = 4$ SYM, but we haven't seen yet how the theory for the closed strings survives the limit. To complete this correspondence we need to specify the parameter n that was introduced above.

The parameter n is chosen so that energies measured on the D-branes remain fixed as $\alpha' \to 0$. Recall from string theory that if we move a D-brane away from the stack of branes at $r = 0$, the gauge group $U(N)$ of the non-abelian gauge theory (1.49) is broken to $U(1)\times U(N-1)$ and the gauge field A_a decomposes as:

$$A_a = \begin{pmatrix} A_a^1 & W_a \\ W_a^\dagger & A_a^2 \end{pmatrix} \, , \tag{1.59}$$

where A_a^1 and A_a^2 are the $U(1)$ and $U(N-1)$ gauge fields transforming in the adjoint representation and W_a the W-boson in the fundamental representation of $U(1)\times U(N-1)$. From the relation (1.48) between the scalars ϕ^I and the position of the branes in transverse space, we have that giving a position $r = r_1$ to one D-brane corresponds to giving a configuration to the ϕ^I of the form:

$$\phi^I = \begin{pmatrix} \phi_1^I & 0 \\ 0 & \phi^I_{(N-1)\times(N-1)} \end{pmatrix} \quad : \quad \phi_1^I \partial_I = \frac{r_1}{2\pi\alpha'}\partial_r + \ldots \, , \quad \vec{\phi}_{(N-1)\times(N-1)} = 0 \, , \tag{1.60}$$

which minimizes the potential $[\phi^I, \phi^J][\phi_I, \phi_J]$. The Higgses ϕ^I therefore acquire a vacuum expectation value proportional to $r_1/\alpha' = (\alpha')^{n-1}\tilde{r}_1$ and particles such as the W-boson gain a mass given by the Higgs vev via the Higgs mechanism. If we want to keep masses fixed as we bring the D-brane closer and closer to the stack of D-branes, *i.e.* as $\alpha' \to 0$, we need to require that $n = 1$. Another way to see this is by recalling that the mass of the W-boson as measured by the field theory on the D-branes is equal to the mass M of an open string stretching between the branes at $r = 0$ and the D-brane at $r = r_1$ as measured at infinity. Since the tension $T = 1/(2\pi\alpha')$ of the string represents its mass per proper length l as measured locally, we obtain:

$$T = \frac{dM_p}{dl} = H(r)^{1/4}\frac{dM}{dl} = \frac{dM}{dr} \, , \tag{1.61}$$

where $dM = H(r)^{-1/4} dM_p$ is the mass of each point on the string with the red-shift factor derived in (1.57) and where $dl = H(r)^{1/4}dr$ is the infinitesimal proper

former decouple both from the latter and from the $h^{(1)}_{\mu\nu}$ excitations. However, the near-horizon excitations do not decouple from the $h^{(1)}_{\mu\nu}$ modes in this limit. Finally, the D3-branes decouple both from the $h^{(2)}_{\mu\nu}$ and from the $h^{(1)}_{\mu\nu}$ closed strings.

length of the string as deduced from the metric (1.53). Integrating the equation results in: $M = r_1/(2\pi\alpha') = (\alpha')^{n-1}\tilde{r}_1/(2\pi)$. The mass of the W-boson therefore remains fixed as $\alpha' \to 0$ if $n = 1$.

The near-horizon geometry in which the closed string excitations of arbitrary proper energy live is derived by parametrising the transverse space with the new coordinate $\tilde{r} : r = \alpha'\tilde{r}$ and taking the limit $\alpha' \to 0$. Note that in this limit we approach $r = 0\ \forall\tilde{r}$. In this case it is more convenient to define $z := \sqrt{4\pi g_s N}/\tilde{r}$. The spacetime metric (1.53) in the new coordinate z becomes:

$$\begin{aligned} ds^2_{10} &= L^2\left[\left(1+L^2/z^2\right)^{-1/2}\left(-\frac{dt^2}{z^2}+\frac{d\vec{x}_3^{\,2}}{z^2}\right)+\left(1+L^2/z^2\right)^{1/2}\left(\frac{dz^2}{z^2}+d\Omega_5^2\right)\right] \\ &\sim L^2\left[\frac{1}{z^2}\left(dz^2-dt^2+d\vec{x}_3^{\,2}\right)+d\Omega_5^2\right] \qquad (\alpha'\to 0)\ , \end{aligned} \tag{1.62}$$

which is $AdS_5 \times S^5$ (in the limit $\alpha' \to 0$) with radius $L \propto \sqrt{\alpha'}$. Note that, in the decoupling limit $\alpha' \to 0$, the radial coordinate z no longer parametrises the position of the D-branes in the transverse space, since these are localised at $r/L = 0$ and we have approached this region with this limit, so the closed strings do not see the D-branes localised at any specific region in the near-horizon geometry. We will discuss the role of this new radial coordinate in the next sections.

Due to the overall factor of L^2, the near-horizon space seems to reduce to zero size as $\alpha' \to 0$. This is indeed correct from the point of view of an observer away from the D-branes and explains the fact that the closed string excitations in the far-away region stop interacting with the excitations in the near-horizon region in the low-energy limit. The former excitations cannot probe this region in this limit since it reduces to zero size, or equivalently since their wavelengths become infinitely longer than the size of this region. As a consequence, the cross sections for the D-branes to absorb infalling particles from infinity reduces to zero [55]. Reciprocally, the closed string excitations near the D-branes see the Minkowski region infinitely far away. In other words, from the redshift equation (1.57) in the new coordinate z and in the limit $\alpha' \to 0$, we find that these excitations need an infinitely large proper energy to escape the gravitational potential and reach infinity with non-zero energy ε, so they do not propagate to this region.

From the point of view of the closed strings near the D-branes, the near-horizon region is $AdS_5 \times S^5$ with arbitrary radius ℓ (but α' independent!). If we write the background metric (1.62) in the limit $\alpha' \to 0$ as:

$$\begin{aligned} ds^2_{10} &= \frac{L^2}{\ell^2}\left[\frac{\ell^2}{z^2}\left(dz^2-dt^2+d\vec{x}_3^{\,2}\right)+\ell^2 d\Omega_5^2\right] \\ &= \frac{L^2}{\ell^2}\,\widetilde{G}_{\mu\nu}dx^\mu dx^\nu\ , \end{aligned} \tag{1.63}$$

and replace this background in the worldsheet sigma model of the closed strings, we obtain:

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{g}\left(G_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu g^{\alpha\beta}+...\right) \\ &= \frac{1}{4\pi\alpha''}\int d^2\sigma\,\sqrt{g}\left(\widetilde{G}_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu g^{\alpha\beta}+...\right) \end{aligned} \tag{1.64}$$

where:

$$\alpha'' \;=\; \ell^2\,\frac{\alpha'}{L^2} \;=\; \frac{\ell^2}{\sqrt{4\pi g_s N}}\;. \tag{1.65}$$

Even though we introduced ℓ for dimensional reasons, it is common practice to set $\ell = 1$, so we will adopt this convention unless specified otherwise.[22] From the point of view of the strings in the near-horizon geometry, the inverse string tension is effectively α'' rather than α' and the background metric is the $AdS_5 \times S^5$ metric $\widetilde{G}_{\mu\nu}$ with unit radius rather than radius L.[23] We may then safely take the limit $\alpha' \to 0$ and have a well-defined theory of interacting closed strings in the near-horizon region. In particular, at lowest order approximation in the new inverse string tension α'', the dynamics of the closed string excitations is described by the IIB supergravity action (1.41) in the $AdS_5 \times S^5$ background $\widetilde{G}_{\mu\nu}$ with $\ell_s = \sqrt{\alpha'}$ replaced by $\sqrt{\alpha''}$ (note that there is also a non-trivial background configuration for the R-R five form). The ten dimensional Newton constant $G_{10} = \ell_P^8 = 8\pi^6 g_s^2 \ell_s^8$ is also replaced by $\pi^4/(2N^2)$ (or by: $\ell^8\pi^4/(2N^2)$ if $\ell \neq 1$). It is never too much to emphasize that, in the case of strings propagating in the background (1.62), the inverse string tension is not α' because it cancels with the α' in the radius L^2 (and we have taken the limit $\alpha' \to 0$); in addition, the radius of the background is not L and it depends on how we define the inverse string tension α''. These issues reflect the fact that only the dimensionless ratio α'/L^2 is relevant, as opposed to α' and L separately (see also footnote 26 and the discussion below).

Recall now the relation (1.50) between the Yang-Mills coupling of the $\mathcal{N} = 4$ gauge theory on the D3-branes and the string coupling constant: $g_{YM}^2 = 4\pi g_s$. Furthermore, recall from the analysis in section 1.1 that the coupling constant of a non-abelian Yang-Mills theory with gauge group rank N is effectively $\lambda = g_{YM}^2 N$ rather than g_{YM} alone, where λ is the 't Hooft coupling. The inverse string tension and the 't Hooft coupling are then related as $\alpha'' = 1/\sqrt{\lambda}$. This implies

[22] Another common choice is $\ell = (4\pi g_s N)^{1/4}$ such that $\alpha'' = 1$. However, it doesn't really matter which convention one adopts because only the dimensionless ratio $\alpha'/L^2 = 1/\sqrt{4\pi g_s N}$ will appear in any final physical computation [27]. We will say more about this shortly.

[23] If $n \neq 1$, the background metric $\widetilde{G}_{\mu\nu}$ is ill defined in the limit $\alpha' \to 0$. Requiring that the background be well-defined is another approach to fixing the parameter n.

that the conjectured equivalence between the gauge theory on the D3-branes with coupling λ and IIB closed strings on $AdS_5 \times S^5$ with coupling α'' is a strong/weak duality: when one is weakly coupled the other is strongly coupled and vice-versa.[24] Once the low-energy limit $\alpha' \to 0$ is taken, it seems that we are free to adjust λ. Note, however, that the backreaction of the D3-branes was determined using supergravity. This approximation required both $g_s < 1$, which we have assumed throughout, and $\alpha' \sim 0$. If the curvature radius L is smaller than the string length scale, string corrections to supergravity become important. This can be seen by writing the α' corrections to the Einstein-Hilbert Lagrangian in the supergravity action as:

$$S = \frac{1}{16\pi G_{10}} \int d^{10}x\sqrt{G}\left(R[G] + \alpha' R^2 + ...\right) \ , \tag{1.66}$$

where R denotes the Riemann, or contractions thereof. Since $R \sim 1/L^2 \sim (g_s N)^{-1/2}/\alpha'$, we find that the string corrections are of the same order of the leading term unless $g_s N >> 1$. Note that this implies $N >> 1$. To suppress these corrections we then need to require that the 't Hooft coupling $\lambda = 4\pi g_s N$ be sufficiently large. This in turn implies that $\alpha'' \sim 0$. The gauge theory on the D-branes is therefore strongly coupled and cannot be studied by any known method that relies on perturbation theory in λ. However, the dual string theory is weakly coupled and therefore well-described by IIB supergravity on the $AdS_5 \times S^5$ background. We can then study the strong coupling regime of the gauge theory using a classical theory of gravity.

In section 1.1 we discussed the 't Hooft limit of the gauge theory: $N \to \infty$ with λ fixed. This is the large N limit under which the expansion of correlation functions in Feynman diagrams becomes an expansion in topologies of Riemann surfaces and reduces to a sum of planar diagrams at leading order in N as derived in equation (1.22). This expansion represented the first hint that gauge and string theories, as quantum theories, could be equivalent to one another. On the dual string theory side, the 't Hooft limit alone is also well-defined and corresponds to the limit $g_s \to 0$ with α'' fixed, under which all loop corrections in string perturbation theory are suppressed with respect to the tree level diagrams. Given this fact, it seems natural to formulate a stronger form of the correspondence and extend the conjecture to all orders in λ. This includes the weak coupling regime of the gauge theory which is amenable to perturbation theory in λ; on the string theory

[24]Note that we have chosen the radius ℓ so that this duality is explicit. For different choices of ℓ, in particular those that depend on λ such as $\ell = \lambda^{1/4} : \alpha'' = 1$, this duality can be seen by writing the supergravity approximation to string theory plus corrections as in equation (1.66) with α' replaced by α''. We then have that $R \sim 1/\ell^2 \sim 1/(\alpha''\sqrt{\lambda})$. Higher order curvature terms, or stringy corrections, are therefore increasingly subleading with respect to lower order ones iff $\lambda >> 1$. This analysis shows that the 't Hooft coupling is effectively the string tension regardless of the choice of ℓ.

side, α'' corrections to IIB supergravity then become successively more important than the leading terms and therefore the theory does not admit a description in terms of a classical theory of gravity.[25] The strongest form of the correspondence is obtained by extending the conjecture to all values of N, in addition to all λ. On the string theory side, we are extending the correspondence to all values g_s, in addition to all α''. We can then summarise the original AdS/CFT conjecture as follows:

Four dimensional $\mathcal{N} = 4$ SYM with gauge group $U(N)$ and 't Hooft coupling λ is equivalent to type IIB string theory with string coupling $g_s = \lambda/(4\pi N)$ and inverse string tension $\alpha'' = 1/\sqrt{\lambda}$ on an $AdS_5 \times S^5$ background, both spaces with unit radius.

An arbitrary but common radius ℓ for the AdS_5 and the compact S^5 spaces can be introduced as we did when we derived equation (1.65). We should further take into account the relationship (1.50) between the Yang-Mills instanton angle θ_I and the vacuum expectation value of the axion and supplement the conjecture with the identification between the two: $\langle C_0 \rangle = \theta_I/(2\pi)$.

As discussed above, the weakest form of the conjecture corresponds to a restriction to the regime $\alpha'' \sim 0$, $g_s \sim 0$. In this case the supergravity approximation is valid and we can consider perturbation theory in α''. We are then probing the regime of the (planar) gauge theory away from large λ. The mild form of the conjecture corresponds to a restriction to the regime $g_s \sim 0\ \forall \alpha''$. In this case we have to consider the full IIB string theory spectrum, but we still work in the classical approximation of the theory where the generating functional of string theory correlation functions is dominated by worldsheets with the topology of the sphere. We can then probe the non-perturbative regime of the string theory (in g_s) with the $1/N$ expansion of the dual gauge theory. The conjecture in its mild form is obtained by working first in the limit $N \to \infty$ such that λ/N is always

[25]We have argued above that λ needs to be large so that α' corrections to the equations of motion that govern the backreaction of the D3-branes are negligible and the black-brane background (1.53) is reliable. To be completely consistent with our reasoning, if we extend the conjecture away from $\lambda >> 1$, then not only we need to consider α'' corrections to the supergravity description of the dual string theory, but also α' corrections to the supergravity action (1.66) that governs the backreaction of the D3-branes. However, note that the distinction between the closed strings that generate the AdS background and the closed strings that propagate in this background is artificial and is helpful only conceptually. We can simply consider closed strings with inverse string tension α'' and with vacuum expectation values for the target spacetime fields given by the background AdS configurations. In this way, the α' corrections to the supergravity description of the former background-generating closed strings are part of the α'' corrections to the supergravity description of the latter strings with AdS vevs. We can then forget altogether about α' corrections and work solely with α''. Note that a double prime in α is simply indicating the fact that we have an AdS background, rather than a Minkowski background.

small. It then posits the existence of a single theory with a dimensionless coupling constant λ. The regime $\lambda << 1$ is most naturally described by a weakly coupled planar gauge theory. The opposite regime $\lambda >> 1$ admits a natural description in terms of a classical theory of gravity. Note that the latter is explicitly higher dimensional, effectively one dimension extra. Once we compactify the closed string fields on the S^5, as described in section 1.2.5, we are left with massive fields in five-dimensional AdS space. Therefore, as we move to the strong coupling regime, the extra dimension that is not apparent at weak coupling becomes more visible. This mimics the previous discussion about M-theory and string theory.

Finally, the strongest form of the conjecture corresponds to the regime of all g_s and all α''. In this case we have to consider the contribution of all worldsheet topologies to the generating functional and work with the full quantum theory of IIB string theory. On the gauge theory side, N is no longer necessarily large and all diagrams in the $1/N$ expansion contribute equivalently to the field theory generating functional.

It should be mentioned that an equivalent statement of the conjecture is commonly found in the literature that makes use of the auxiliary, but nevertheless redundant parameters g_{YM}, α' and L. These extra parameters are not visible to each theory and drop out of any final physical computations,[26] but are helpful conceptually. In this case, the gauge theory parameters are the group rank N and g_{YM}, while the string theory coupling $g_s = g_{YM}^2/(4\pi)$, the inverse string tension is α' and the radius of the background is $L^4 = 4\pi g_s \bar{N} \alpha'^2$, with $\bar{N} = N$ the number of D3-branes on the string theory side, or equivalently the flux of the self-dual R-R five-form in appropriate units. Note that the decoupling limit $\alpha' \to 0$ is implicit and the strong/weak duality relation is not explicit in this version.

It should also be mentioned that the gauge group $U(N)$ involved in the correspondence should be restricted more precisely to the subgroup $SU(N)$. The group product $SU(N) \times U(1)$ is an N-fold cover of $U(N)$ such that: $U(N) \sim (U(1) \times SU(N))/\mathbb{Z}_N$. If we perform this group decomposition, we find that the $U(1)$ fields are free and represent the degrees of freedom associated with the center of mass motion of the D-branes. Since there are no free fields on the gravity side, the dual string theory is rather describing the $SU(N)$ sector of the gauge theory [27, 56]. We will say more about this aspect in section 1.2.5.

A fundamental feature supporting the duality between the two theories is the equivalence of their symmetry groups, which are independent of the regimes of the

[26] One way to see this is by recalling that the gauge theory does not have any dimensionful parameter, so the dual string theory does not have either. In this way, the string theory does not see α' and L separately, but only the dimensionless ratio $\alpha'/L^2 = \alpha''$. Note that to introduce α' and L separately is effectively the same as to introduce ℓ.

parameters. If we analyse the global symmetries of string theory on $AdS_5 \times S^5$ (as opposed to local symmetries such as reparametrization or gauge invariance), we find that these match precisely with the global symmetries of the dual gauge theory discussed at the end of section 1.2.1. Note that we should restrict the discussion to the superconformal phase of $\mathcal{N} = 4$ SYM characterised by vanishing vacuum expectation values of the six scalars ϕ^I that represent the transverse positions of the D-branes. The moduli space of the gauge theory is the space of all commuting ϕ^I because the potential energy term is of the form $[\phi^I, \phi^J]^2$ and therefore is minimized by commuting scalars. It is then possible to have ground states with $\langle \phi^I \rangle \neq 0$ as long as the scalars commute. Such phase is called the Coloumb phase and the superconformal symmetry of the theory is spontaneously broken in that phase because the vevs introduce a scale.

The $SU(4)_R \sim SO(6)_R$ R-symmetry of the gauge theory matches precisely with the $SO(6)$ symmetry of the dual string theory corresponding to invariance under rotations on the S^5. Also, the conformal symmetry group $SU(2,2) \sim SO(2,4)$ of the gauge theory is the isometry group of AdS_5. However, as discussed at the end of section 1.2.1, the bosonic group $SU(2,2) \times SU(4)_R$ of the gauge theory is enhanced to the superconformal group $SU(2,2|4)$. On the gravity side, the D3-branes are 1/2 BPS and therefore preserve half of the $\mathcal{N} = 2$ supersymmetry of supergravity, which leaves us with 16 Poincaré supercharges. These are not enough to extend the symmetry group to $SU(2,2|4)$. However, in the near-horizon region we have the conformal symmetry group of AdS_5. The generators of the special conformal transformations do not commute with the Poincaré supercharges and these commutators represent 16 conformal supercharges. In this way, in the near-horizon geometry we have in total 32 supercharges and the symmetry group $SO(2,4) \times SO(6) \sim SU(2,2) \times SU(4)$ on the string theory side is lifted to the full superconformal group $SU(2,2|4)$,[27] such that all generators of the latter are either Killing vectors or Killing spinors of the near-horizon geometry. Finally, the Montonen-Olive S-duality of the Yang-Mills theory acts as in (1.40) where now $\tau = i/g_s + \langle C_0 \rangle$. This is precisely the S-duality of IIB string theory discussed in (1.44).

The matching of the symmetries on each side of the duality and the correspondence between the coupling constants represent the first entries in a dictionary that should describe how gauge theory quantities map to string theory ones and vice-versa. This is what is meant by equivalence between the two theories and a precise map should be established between the theories. In particular, there must be a correspondence between states as well as correlation functions. Since

[27] Just as on the gauge theory side, the central charge in the anticommutator of the Poincaré and conformal supercharges vanishes and therefore the full symmetry group is in fact the $PSU(2,2|4)$ subgroup as described in footnote 12.

the global symmetry group on each side of the duality is $SU(2,2|4)$, string theory states are in irreducible representation spaces for the superconformal algebra and the same for the gauge theory. One should then detail the mapping between the states. Closed string states correspond to excitations of the target spacetime fields, whereas gauge theory states are in one-to-one correspondence with local operators via the state-operator map,[28] so there should be a one-to-one correspondence between string theory fields and local operators on the dual gauge theory side, more exactly gauge invariant primary operators as we will now discuss.

1.2.4. Representations of the superconformal algebra

In order to derive this map, we need to discuss irreducible representations of $\mathfrak{su}(2,2|4)$. This algebra is derived in appendix A.3. Recall that the way we build irreducible representation spaces for the conformal and superconformal group is different from the way we do it for the Poincaré and superPoincaré group. In the latter case, we work with eigenstates of the generators P_μ of translations and start by restricting the representation space to the subspace of states with a given momentum $\mathring{p}_\mu$. We then find the subgroup of the Poincaré group that leaves $\mathring{p}_\mu$ invariant, called the stability/little group and which in this case is the spatial rotations group, find irreducible representations of this subgroup and then boost the states in such a representation by acting on them with the generators of Lorentz boosts. In this way we generate an (infinite dimensional) irreducible representation space for the whole Poincaré algebra and all states in such a space will be eigenstates of P_μ. Finally, we identify each such space with the Hilbert space of a fundamental particle. Since $P_\mu P^\mu$ and the square of the Pauli-Lubanski vector are Casimirs, we find that irreducible representations, or particles, are classified by the mass and by the spin/helicity of the particle, the latter given by the specific rotations representation that we have boosted.

If we include supersymmetry in the algebra, before boosting the states in a

[28]In a conformal field theory on the plane, each state is obtained by acting on the vacuum with a unique local operator (operator defined at a point), so there is an isomorphism between states and local operators, in particular between highest-weight states and primary operators (tensors under conformal transformations), and one rather speaks about the spectrum of operators – the primary and its descendants – associated with each representation space. Note that this isomorphism does not hold for non-conformal QFTs. In general, a state is obtained by acting on the vacuum with an operator at the infinite past to create an "in" state and then evolving it unitarily in time. For CFTs on the cylinder, the infinite past can be mapped to a single point on the plane by a conformal transformation and therefore each state is mapped to a unique local operator acting at the origin. In this sense, states in a CFT live at a point on the plane. On the other hand, states in non-conformal QFTs live over a whole spatial hypersurface at past infinity in the sense that each state is mapped to an infinite set of local operators corresponding to the infinite number of points on the hypersurface.

representation space for the rotations subgroup we act on such a space with the Poincaré supercharges Q_α. From the algebra of these charges with the generators of rotations we find that the Q_α lower and raise the spin of the states by 1/2, so the action of the supercharges results in several different irreducible representation spaces for rotations (the range of spins depends on the number $\mathcal{N}$ of supersymmetry and on the BPS property of the supermultiplet). This set of different irreducible spaces is called a pre-boosted supermultiplet. We then boost each such space as before to generate an irreducible representation space for the whole superPoincaré algebra. In this case, however, we don't associate the latter space with a single fundamental particle, but still classify particles according to the Poincaré representation. In other words, we rather associate each different Poincaré representation in the boosted supermultiplet with the Hilbert space of a different particle. In conclusion, a single irreducible representation space for the whole superPoincaré algebra consists of a multiplet of different particles, each classified according to the spin. This set of particles, each with its Hilbert space, is what we call a supermultiplet. The mass of the particles in a supermultiplet will be the same because P^2 is still a Casimir, so irreducible representation spaces for the whole superPoincaré algebra, or supermultiplets, are classified by the mass and by the highest spin in the representation (vector multiplet, hypermultiplet, etc.).

In the case of the conformal algebra, we start by noticing that each generator in the algebra is an eigenfunction of the dilatation generator D (*i.e.* $\mathrm{ad}(D)X \propto X$ for each generator X), so we build irreducible representations by working with eigenstates of D with eigenvalues Δ called conformal/scaling dimensions. Notice that $[D, P_\mu] \neq 0$, so these will not be eigenstates of P_μ (this and the fact that P^2 is not a Casimir are the main reasons one doesn't speak about particle states in a conformal field theory). On the other hand, D commutes with the Lorentz generators $M_{\mu\nu}$, so the states will transform in irreducible representations of the Lorentz group. In a fashion similar to the Poincaré group, we begin by restricting the representation space for the conformal algebra to the subspace of states with a given dilatation eigenvalue Δ_0, which we will fix below, and look for the subgroup that preserves Δ_0. The generators $M_{\mu\nu}$ form a subalgebra and are the only generators that commute with D, so the Lorentz group is in this case the stability subgroup and we start by finding irreducible representations of this group. Since $\mathfrak{so}(1,3) \sim \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, the Lorentz group is the tensor product of two spatial rotations groups and a basis for an irreducible representation space can be written as $\big\{|j_1, m_1\rangle \otimes |j_2, m_2\rangle : j_1 = s_1, j_2 = s_2\big\}$, where $\big\{|j, m\rangle : j = s\big\}$ is a basis for an irreducible representation of rotations. Each irreducible representation of the Lorentz group is labelled as (s_1, s_2), where s_1 and s_2 label the representation of the respective $SU(2)$ group. As a side remark, notice that the previous Poincaré group contains the Lorentz group as a subgroup, but in that case we don't work with representations of the full Lorentz group (only with its rotations subgroup) because the generators of boosts do not leave $\mathring{p}_\mu$ invariant.

Once we have chosen an irreducible representation space for the Lorentz algebra, say (m, n), we have that each state in the representation basis has the same dimension Δ_0. The generators we have left are P_μ and the generators K_μ of special conformal transformations. From the algebra of P_μ and K_μ with D we have that these raise and lower the dimension of the states by one unit, respectively. In an unitary representation space for the conformal group it follows that there is a lower bound on the possible values for the dimensions Δ of the states [57], so we fix the dimension Δ_0 we have started with to be that lowest possible one such that all the states in the basis of $(m, n)_{\Delta_0}$ are annihilated by K_μ. We then act on these states with all possible products of P_μ to generate a (infinite dimensional) set of states that form a basis for an irreducible representation space for the whole conformal algebra. All states in such a space are eigenstates of D, each classified by its dimension Δ. Irreducible representation spaces are labelled as $(\Delta_0; m, n)$ according to the lowest dimension Δ_0 in the representation and to the specific Lorentz representation (m, n) that we have "conformally boosted". Note that, from the algebra of P_μ and K_μ with $M_{\mu\nu}$, we have that P_μ and K_μ do not change the spin of the states and their action does not leave the representation (m, n) that we have started with: all states will have spin $m + n$.

Now, given an irreducible representation space $(\Delta_0; m, n)$ for the conformal group, it is almost universal in the literature to speak about a basis for a Lorentz representation space $(m, n)_\Delta$ inside $(\Delta_0; m, n)$ as a single "state" of conformal dimension Δ and to say that this "state" has Lorentz multiplicity $(2m + 1)(2n + 1)$, which is the degree/dimension of the (m, n) representation (the number of states in the basis). This terminology is employed for the sake of simplicity in the arguments. We will call such "state" a tensorstate $|\Delta; m \otimes n\rangle$ of dimension Δ and spin $m + n$. The tensorstate $|\Delta_0; m \otimes n\rangle$ that we have started with and which is annihilated by K_μ is called a primary state, or highest-weight state,[29] and all tensorstates obtained by acting with products of P_μ on $|\Delta_0; m \otimes n\rangle$ are called the descendants. The set of tensorstates given by a primary and its descendants is called a Verma module, which in our case is another name for an irreducible conformal representation space.

Given a conformal field theory, from the state-operator map we have that each tensorstate $|\Delta; m \otimes n\rangle$ corresponds to a unique field theory tensor operator $\mathcal{O}_\Delta^{\mu_1 \dots \mu_{(m,n)}}(x)$ of conformal dimension Δ in the (m, n) Lorentz representation and vice-versa (unlike its spin, the rank of the operator is not necessarily $m + n$ and it

[29] Note the misnomer: a highest-weight state is the tensorstate of lowest dimension in a representation space for the conformal algebra.

depends on the representation) and the isomorphism is given by:

$$|\Delta; m \otimes n\rangle \quad = \quad \lim_{x\to 0} \, \mathcal{O}_\Delta^{\mu_1 \dots \mu_{(m,n)}}(x)|0\rangle \ , \tag{1.67}$$

where the vacuum $|0\rangle$ is defined to be the state annihilated by all generators of the algebra, so one uses the notion of tensorstates and operators interchangeably. The field theory operator associated with a primary state is called a primary operator. Primary operators $\Phi(x)$ are therefore eigenfunctions of the dilatation generator at the origin such that $[D, \Phi(0)] \propto \Phi(0)$ and are annihilated by K_μ at the origin such that $[K_\mu, \Phi(0)] = 0$ (these expressions are evaluated on states, but also hold as operator identities). The action of the generators on a primary at a generic point can be derived from their action on the primary at the origin using the identity $\Phi(x) = e^{ix\cdot P}\Phi(0)e^{-ix\cdot P}$ and the result is given by:

$$[D, \Phi(x)] \quad = \quad -i\left(\Delta + x \cdot \partial\right)\Phi(x) \ , \tag{1.68}$$

$$[K_\mu, \Phi(x)] \quad = \quad \left[i\left(2x_\mu x \cdot \partial + 2x_\mu \Delta - x^2\partial_\mu\right) - 2x^\nu \Sigma_{\mu\nu}\right]\Phi(x) \ , \tag{1.69}$$

$$[P_\mu, \Phi(x)] \quad = \quad i\partial_\mu \Phi(x) \ , \tag{1.70}$$

$$[M_{\mu\nu}, \Phi(x)] \quad = \quad \left[i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right) + \Sigma_{\mu\nu}\right]\Phi(x) \ , \tag{1.71}$$

where the $\Sigma_{\mu\nu}$ matrices are the spin matrices that form the representation of the Lorentz algebra that acts on the Lorentz indices of $\Phi(x)$ (which we are omitting). These commutation relations imply that primary operators are tensors under conformal-Weyl transformations and therefore transform as:

$$\Phi^{\mu_1 \dots \mu_n}(x) \quad \to \quad \Omega^{-\Delta}(x)\, \frac{\partial \varphi^{\mu_1}}{\partial x^{\alpha_1}} \dots \frac{\partial \varphi^{\mu_n}}{\partial x^{\alpha_n}}\, \Phi^{\alpha_1 \dots \alpha_n}(x) \ , \tag{1.72}$$

under a conformal-Weyl transformation (A.10)–(A.11). Reciprocally, every operator that transforms as above creates at the origin a tensorstate which is annihilated by K_μ and therefore is a primary state.

The field theory operator associated with a descendant state is called a descendant or derivative.[30] Descendant operators are therefore eigenfunctions of the dilatation generator at the origin and hence obey (1.68) at a generic point and transform as in (1.72) under scale transformations. However, by definition they are not eigenfunctions of K_μ at the origin and therefore do not obey (1.69) and thus do not transform as tensors under a general conformal transformation. Reciprocally, every operator which is an eigenfunction of D at the origin but not of K_μ creates a descendant state. Any operator in a conformal field theory is an eigenfunction of D at the origin and therefore is either a primary or a descendant. The

[30] Since descendant states are obtained by acting on a primary state with products of P_μ, from the commutation relation (1.70) it follows that a descendant operator is some n$^{\text{th}}$-order partial derivative of a primary operator, hence the second terminology.

conformal dimension of an operator in the free theory coincides, by definition, with its physical dimension in units of mass, or inverse length. In the interacting theory, the dimensions in general will receive quantum corrections and are called renormalized dimensions. There are special operators which we discuss below, called BPS superprimary operators and their superdescendants, that are protected from quantum corrections and their dimensions remain the unrenormalized ones.

An operator is said to be invariant under a scale-Weyl transformation $x^\mu \to \bar{x}^\mu = \lambda x^\mu$ if it transforms as: $\Phi^{\mu_1\cdots\mu_n}(x) \to \bar{\Phi}^{\mu_1\cdots\mu_n}(\bar{x}) = \Phi^{\mu_1\cdots\mu_n}(\bar{x})$. Using the transformation law (1.72), we find that scale invariant operators satisfy:

$$\Phi^{\mu_1\cdots\mu_n}_{\nu_1\ldots\nu_m}(\lambda x) \quad = \quad \lambda^{-\Delta+n-m}\, \Phi^{\mu_1\cdots\mu_n}_{\nu_1\ldots\nu_m}(x)\ . \tag{1.73}$$

As an example of the relation between tensorstates and operators, consider the stress-energy tensor. Every conformal field theory in d dimensions has a stress-energy operator $T^{\mu\nu}$ of dimension $\Delta = d$ which is a tensor under general conformal transformations and is an example of a primary operator (in the case $d = 2$ it is a quasi-primary, a distinction relevant only in two dimensions). In the four-dimensional case, $T^{\mu\nu}$ is in the $(1,1)$ symmetric traceless representation of $SO(1,3)$. The tensorstate $|\Delta = 4; 1 \otimes 1\rangle = \lim_{x\to 0} T^{\mu\nu}(x)|0\rangle$ that it creates corresponds to the basis $\{T^{\mu\nu}(0)|0\rangle\}$. The vector space spanned by this basis coincides with the irreducible Lorentz representation space $(1,1)_{\Delta=4}$ spanned by the basis $\{|j = 1, m_1\rangle \otimes |j = 1, m_2\rangle\}$.[31]

As a final remark, note that primary or descendant operators away from the origin are not eigenfunctions of the dilatation generator as can be seen from (1.68) and their action on the vacuum results in a linear superposition of tensor eigenstates of D:

$$\begin{aligned}
\mathcal{O}^{\mu_1\cdots\mu_{(m,n)}}_{\Delta}(x)|0\rangle \quad &= \quad \left(e^{ix^\mu P_\mu}\, \mathcal{O}^{\mu_1\cdots\mu_{(m,n)}}_{\Delta}(0)\right)|0\rangle = e^{ix^\mu P_\mu}|\Delta; m\otimes n\rangle \\
&= \quad \sum_s \frac{i^s}{s!}\,(x^\mu P_\mu)^s\,|\Delta; m\otimes n\rangle\ ,
\end{aligned} \tag{1.74}$$

where we have used the Taylor expansion of the operator. In a similar fashion, the action of an operator on a generic tensorstate determines its operator product

[31] The correct procedure is to express each operator in the set $\{T_{\mu\nu}\}$ in terms of the tensor product of the irreducible tensor operators $\{\hat{T}_{1m}\} : m \in \{-1, 0, 1\}$ as the linear combination: $T_{\mu\nu} = \sum_{m,n}\left(a^{m,n}_{\mu\nu}\,\hat{T}_{1m}\otimes\hat{T}_{1n}\right)$, where $a^{m,n}_{\mu\nu}$ are coefficients. The action of an irreducible $\hat{T}_{jm}$ on the vacuum is given by: $\hat{T}_{jm}|0\rangle = \sum_{J,M}|J,M\rangle\langle j,m;0,0|J,M\rangle$, where $\{|J,M\rangle : J = J_0\}$ is a basis for an irreducible $SU(2)$ representation space and where the Clebsch-Gordan $\langle j,m;0,0|J,M\rangle = c_{j,m}\delta_{J,j}\delta_{M,m}$, with $c_{j,m}$ some constant. In this way: $T_{\mu\nu}|0\rangle\otimes|0\rangle = \sum_{m,n}\left(c^{m,n}_{\mu\nu}\,|j=1,m\rangle\otimes|j=1,n\rangle\right)$. Each state in $\{T_{\mu\nu}|0\rangle\otimes|0\rangle\}$ is therefore written in terms of the basis $\{|j = 1, m\rangle \otimes |j = 1, n\rangle\}$ and by inverting the above identity we have the reciprocal statement (recall that $T_{\mu\nu}$ has nine independent components). In this way we find that the representation space spanned by $\{T_{\mu\nu}|0\rangle\otimes|0\rangle\}$ coincides with that spanned by $\{|j = 1, m\rangle \otimes |j = 1, n\rangle\}$.

expansion (OPE) with the corresponding tensor operator:

$$\begin{aligned}\mathcal{O}_{\Delta_1}(x)\mathcal{O}_{\Delta_2}(0)|0\rangle &= \mathcal{O}_{\Delta_1}(x)|\Delta_2\rangle = \left(e^{ix^\mu P_\mu}\,\mathcal{O}_{\Delta_1}(0)e^{-ix^\mu P_\mu}\right)|\Delta_2\rangle \\ &= \sum_{\Delta} c_{\Delta}^{\Delta_1,\Delta_2}(x)|\Delta\rangle = \sum_{\Delta} c_{\Delta}^{\Delta_1,\Delta_2}\mathcal{O}_{\Delta}(0)|0\rangle\ , \end{aligned} \tag{1.75}$$

from which it follows the OPE:

$$\mathcal{O}_{\Delta_1}(x)\mathcal{O}_{\Delta_2}(0) = \sum_{\Delta} c_{\Delta}^{\Delta_1,\Delta_2}(x)\mathcal{O}_{\Delta}(0)\ , \tag{1.76}$$

and where we have omitted the Lorentz multiplicity of the operators and states for simplicity. The coefficients $c_{\Delta}^{\Delta_1,\Delta_2}(x)$ can be determined by expressing the operator $\left(e^{ixP}\mathcal{O}e^{-ixP}\right)$ in terms of commutators using the standard Hausdorff formula and then using the commutator (1.70) of $\mathcal{O}$ with P_μ (which also holds for descendants). A recent review of these topics in conformal field theory can be found in the lecture notes [58].

Let us then turn to the representations of the superconformal group and extend the conformal algebra by including the Poincaré supercharges Q^i_α and the conformal supercharges S^j_α. Note that, in the case of $\mathcal{N}=4$ SYM, the indices $i,j=1,...,4$ on the supercharges transform in the fundamental of $SU(4)_R$, whereas the indices $I,J=1,...,6$ on the scalars ϕ^I transform in the fundamental of $SO(6)_R$. The latter are mapped to the $SU(4)_R$ indices via the Σ^a matrices discussed in (1.33) (see also footnote 15). Due to the Lorentz multiplicity of tensorstates, it is much easier to discuss irreducible representations in terms of tensor operators rather than states. From the algebra of the supercharges with the Lorentz generators $M_{\mu\nu}$ it follows that Q^i_α (and S^j_α) changes the spin of a state by 1/2, so by using tensor operators we avoid having to organize states in tensorstates, *i.e.* in irreducible representations of the Lorentz group, after acting with the Poincaré supercharges on the states in $(m,n)_{\Delta_0}$. As discussed above, a tensor operator creates directly a complete tensorstate: the set of states it creates forms a complete basis for a Lorentz representation space. So once we find the set of tensor operators that form an irreducible representation space for the superconformal group, the corresponding states in the representation automatically come organized in Lorentz representations. The same discussion applies to representations of the R-symmetry group which, together with the Lorentz group, forms the stability group $SO(1,3)\times SU(4)_R$ of the superconformal group since the R-symmetry and Lorentz generators are the only generators of the algebra that commute with D (and form a subalgebra). Besides carrying a Lorentz representation, or multiplicity, each operator also carries a representation of $SU(4)_R \sim SO(6)_R$, so the set of states the operator creates forms a basis for a representation space of the stability group. The fermionic generators Q^i_α and S^j_β carry the fundamental representation of $SU(4)_R$, so by acting with these generators on some initial operator to generate a representation space as discussed below, we obtain new operators that also carry some representation of the stability group. It should be emphasized that the action of the generators of the superconformal algebra on operators is via the adjoint representation of the generators, *i.e.* in terms of commutators (and anticommutators) as in (1.68)–(1.69).

In order to generate an irreducible representation space for the whole superconformal group, we start by noticing that, while P_μ and K_μ raise and lower the dimension of eigenfunctions of D by one unit, from the commutation relations with D it follows that the supercharges Q^i_α and S^j_α raise and lower the dimension by half a unit, respectively. Furthermore, from the anticommutation relations of the supercharges we have that $\{Q^i_\alpha, \bar{Q}^j_{\dot{\beta}}\} \sim P_\mu$ and $\{S^i_\alpha, \bar{S}^j_{\dot{\beta}}\} \sim K_\mu$, so certain products of Q's (and S's) correspond to products of P's (and K's) acting on operators. So we proceed as in the case of the conformal group and start with some operator $\mathcal{O}^{\mu_1\cdots\mu_{(m,n)}}_{\Delta_0}$ of dimension Δ_0 in the (m,n) Lorentz representation and in some representation of $SU(4)_R$ (representations of the latter are labelled according to the same Dynkin notation for $SO(6)$ that we will use in the next section; we omit these labels here, and will suppress the Lorentz indices, to avoid an excess of labels that are not necessary for the discussion). We choose the operator to be the one with the lowest possible dimension in the representation space for the superconformal group such that $\mathcal{O}_{\Delta_0}$ is annihilated by S^i_α and $\bar{S}^i_{\dot{\alpha}}$ and therefore also by K_μ.[32] We then act on $\mathcal{O}_{\Delta_0}$ with all possible products of Q's and $\bar{Q}$'s to generate an (infinite dimensional) irreducible representation space for the whole superconformal group. Since the conformal algebra is a subalgebra of the superconformal one, this representation space contains several irreducible representation spaces for the conformal algebra. The operator $\mathcal{O}_{\Delta_0}$ that we have started with and which is annihilated by the conformal supercharges S^i_α is called a superconformal primary operator, or superprimary. Those operators that we have derived from it by acting with the Poincaré supercharges can be divided into two sets. Those that are annihilated by K_μ are called superconformal descendants, or superdescendants. These superdescendants are the primary operators of the several irreducible conformal representation spaces. The remaining operators, those that are not annihilated by K_μ, can each be written as some product of P's acting on the superprimary or on a primary and therefore will be a descendant in the corresponding conformal representation space. The number of different primaries derived in this way from a superprimary will be finite, in general $2^N - 1$ for N real Poincaré supercharges, and each of the 2^N conformal representation spaces is classified by the Lorentz

[32] For the lower bounds on the possible conformal dimensions of operators in unitary superconformal representations, see ref. [59].

representation and conformal dimension of the corresponding primary as usual.

In conclusion, an irreducible representation space for the whole superconformal algebra consists of a superprimary and a finite set of primaries, each with its own (infinite dimensional) Verma module. This whole set is called a conformal supermultiplet and is classified by the dimension Δ_0 of the superprimary, which is the lowest one, by its $SU(4)_R$ representation and by the highest spin in the supermultiplet. The $\mathfrak{su}(2,2|4)$ algebra involves 16 real Poincaré supercharges, half with helicity $+1/2$ and half with $-1/2$, and each raises the dimension of an operator by half a unit, so the helicities in a conformal supermultiplet in general will range from $\lambda - 4$ to $\lambda + 4$, where $\lambda = m + n$ (the spins will range from λ to $\lambda + 4$), and the dimensions in general will range from Δ_0 to $\Delta_0 + 8$ (the dimensions in each Verma module will, of course, range from some Δ to ∞).

As in any algebra with supersymmetry generators, there will be special representations of the superconformal group that are shorter than generic ones. These are representations built from special superprimaries that are annihilated by some of the Poincaré supercharges. The number of supercharges we can effectively act with to obtain new operators is therefore reduced and for this reason the number of primaries in such special conformal supermultiplets is less than 2^N and the range of helicities will be shorter. If the superprimary is annihilated by N/n of the N real Poincaré supercharges, the corresponding representation is said to be $1/n$ BPS, or chiral.[33] In such representations of $SU(2,2|4)$ the spins will then range from λ to $\lambda + a$ and the dimensions of the primaries in general will be between Δ_0 and $\Delta_0 + 2a$, where $a = 4(1 - 1/n)$ (as we will see in the next section, there are specific 1/2 BPS representations where the range of the dimensions is narrower than this). Furthermore, the conformal dimension Δ_0 of a BPS superprimary operator is protected from quantum corrections: its dimension is uniquely fixed by the superconformal algebra, more precisely by the anticommutator of the Poincaré supercharges with the conformal supercharges. This can be derived by acting with the anticommutator on a BPS superprimary and using the fact that the latter is annihilated by all of the conformal supercharges and by some of the Poincaré

[33] The terminology BPS is employed by analogy with the superPoincaré case, where $1/n$ BPS representations are characterised by the fact that $1/n$ of the total number of anticommutators between Q and $Q^\dagger$ vanish on the representation space. In the superPoincaré case this happens whenever the mass of the particle states that form the representation is equal to the value of $\mathcal{N}/n$ central charges, *i.e.* when they saturate (part of) the BPS bound, and in that case only $(1-1/n)$ of the total number of $Q^\dagger$ are effective in building the representation space. However, the concept of BPS bound arises only when we consider representation spaces for the supersymmetry algebra where the states are eigenstates of the momentum operator P_μ. In the superconformal case, the states created by BPS operators are not eigenfunctions of P_μ and P^2 is not a Casimir, so there is no notion of mass and particles states and therefore one should not think of BPS representations in the conformal case in terms of mass/BPS bounds.

supercharges in order to express the dimension of the superprimary in terms of its Lorentz representation and R-symmetry representation [60] (the corresponding superdescendants will have dimension Δ_0 plus an integer or half-integer number and therefore are protected as well). In this way, BPS conformal supermultiplets can be uniquely classified by the R-symmetry and Lorentz representations of the superprimary operators. Since the supercharges are also in a representation of $SU(4)_R$, the R-symmetry representation of a primary in a supermultiplet is fixed by that of the superprimary and by the number of Poincaré supercharges we acted on it with to derive the primary.

1.2.5. Matching gauge theory operators with string states

In order to obtain the map between operators of the $\mathcal{N} = 4$ SYM theory and the string theory states, we need to organize all possible operators on the gauge theory side in irreducible representations of the superconformal group, *i.e.* to derive the spectrum of superprimary and primary operators that form conformal supermultiplets. Each (local) gauge invariant operator of the theory consists of some product of the elementary fields and their derivatives. Since the product needs to be gauge invariant, we need to trace over the gauge group indices. We can consider operators obtained by taking a single trace over a product of fields, or operators that are products of such single trace operators. The latter multi-trace operators are dual to multi-particle states or to bound states of single-particle states on the string theory side and we will not discuss them here. As discussed in the previous section, primary operators are derived from superprimaries by acting on the latter with the Poincaré supercharges. The action of the supercharges on the elementary fields is given by the $\mathcal{N} = 4$ supersymmetry transformations that leave the SYM Lagrangian invariant and which are schematically of the form:

$$\begin{aligned} \delta\phi &= [Q, \phi] \sim \lambda \ , \\ \delta\lambda &= \{Q, \lambda\} \sim F^+ + [\phi^I, \phi^J] \ , \\ \delta\bar{\lambda} &= \{Q, \bar{\lambda}\} \sim \mathcal{D}\phi \ , \\ \delta A &= [Q, A] \sim \bar{\lambda} \ , \end{aligned} \tag{1.77}$$

where F^+ is the self-dual gauge field strength and $\mathcal{D}$ the gauge covariant derivative. From this algebra it follows that superprimary operators, those of lowest dimension in their own supermultiplets, cannot contain products of fermions or gauge field strengths: operators with such products can always be written as a supercharge acting on some other operator and therefore the latter is of lower dimension. For this reason, superprimary operators in $\mathcal{N} = 4$ SYM always consist of products of scalars. Let us then consider a generic such product:

$$\mathcal{O}_k \quad := \quad \mathrm{Tr}\left(\phi^{I_1}\ldots\phi^{I_k}\right) \ . \tag{1.78}$$

Note that each scalar ϕ^I has unrenormalized conformal dimension $[\phi^I] = 1$, which is the physical dimension of the field in the free theory, and therefore $[\mathcal{O}_k] = k$. From the algebra (1.77) we have that antisymmetric products of scalars can be written as a supercharge acting on some operator, so if the $SO(6)_R$ indices in (1.78) are not completely symmetrised the operator will not be a superprimary. In this way, $\mathcal{N} = 4$ SYM superprimary operators always consist of symmetric products of Lorentz scalars ϕ^I. Then, the symmetric product of $SO(6)_R$ vectors ϕ^I (*i.e.* in the fundamental of $SO(6)$) is not an irreducible $SO(6)_R \sim SU(4)_R$ representation and therefore a representation space for the whole superconformal group derived from such a superprimary contains invariant subspaces (other than itself). To obtain irreducible representations, we perform a decomposition of the symmetric product of the scalars into those symmetric products that are totally traceless (the contraction of any two indices vanishes) and those that consist of the traces. Note that this is the standard irreducible decomposition of $SO(n)$ tensors of rank k, in this case symmetric ones. Finally, if the superprimary contains traces, the resulting supermultiplet will contain primary operators of spin higher than 2, in the above language the representation will not be 1/2 BPS.[34] Such operators correspond to string theory states obtained from the dimensional reduction of ten-dimensional massive string states, which are suppressed in the supergravity approximation $\alpha'' \to 0$, and for this reason we will not discuss them here. On the other hand, it can be shown that the totally traceless symmetric products that remain are annihilated by half of the Poincaré supercharges and therefore form 1/2 BPS representations [62]. From the above discussion, the corresponding conformal supermultiplets will be shortened and the spins will range from 0 to 2, as required for operators dual to supergravity multiplets. Furthermore, as discussed above, the conformal dimension of a BPS superprimary remains the unrenormalized one: $\Delta(\mathcal{O}_k) = [\mathcal{O}_k] = k$, and therefore the dimensions in a 1/2 BPS supermultiplet in general will range between k and $k + 4$. Since all superprimaries are in the $(0, 0)$ Lorentz representation and their dimensions are uniquely fixed by their R-symmetry representations, the symmetric traceless $SO(6)_R$ representations can be used to classify the BPS representations of the superconformal group.

In conclusion, the superprimary operators that are dual to supergravity fields are of the form:

$$\mathcal{O}_k \quad = \quad \mathrm{Tr}\left(\phi^{\{I_1}\ldots\phi^{I_k\}}\right) \quad : \quad 2 \leq k \leq N \ , \tag{1.79}$$

[34]In fact the representation will not be BPS at all since 1/4 and 1/8 BPS representations are derived from multitrace superprimary operators [61].

where N is the rank of the gauge group and $\phi^{\{I}\phi^{J\}}$ denotes the symmetric traceless product. The lower bound on k arises from the fact that we have a $SU(N)$ gauge group and the elementary fields are traceless in the gauge group indices (which would not be the case for $U(N)$), whereas the upper bound is due to the fact that $\mathcal{O}_{k>N}$ can always be expressed as a product of single-trace operators $\mathcal{O}_{k<N}$ (up to superdescendants) and therefore as a multi-trace operator. The primaries in a conformal supermultiplet are obtained by acting with products of the Poincaré supercharges on the respective superprimary $\mathcal{O}_k$ and their form follows from the algebra (1.77). These operators are given below in Table 1.1 after we discuss the string theory states as states in representations of the superconformal group. The descendants that form the Verma modules associated with each primary follow by acting with products of P_μ's on the corresponding primaries as discussed in the previous section (see also footnote 30).

On the string theory side, the right procedure to follow would be to determine the string theory spectrum on the $AdS_5 \times S^5$ background: to solve the worlsheet sigma-model in this background, to find the ground and excited states of the string, which are states obtained by acting on the vacuum state with products of the Fourier modes of the string embedding tensor operators, and then to verify that these operators (or the states) fit into conformal supermultiplets, *i.e.* that can be organized in irreducible representations of $SU(2,2|4)$ as discussed in the preceding section. We would then match the string supermultiplets obtained in this way with the gauge theory supermultiplets. Even though this programme can be followed in a Minkowski background, in which case the symmetry group is rather the superPoincaré group, it is not known how to find the string theory spectrum in curved backgrounds such as anti-de Sitter, though much progress has been made in this direction over the past years.[35] The only known states in the $AdS_5 \times S^5$ string spectrum are the massless string states we derived in flat space, the particle states of the ten-dimensional supergravity fields (note that the restriction of the full spectrum to this subspace of states is valid only for large values of the 't Hooft coupling λ). However, as such, the ten-dimensional supergravity fields, or the particle states associated with them,[36] are not explicitly organized in superconfor-

[35]For a review of integrability of string theory in $AdS_5 \times S^5$ see [63]. A way to deal with this question, though, is to trust the AdS/CFT correspondence from first principles. The string theory spectrum in $AdS_5 \times S^5$ would then correspond to the states obtained from the spectrum of $\mathcal{N} = 4$ SYM operators.

[36]For the sake of clarity, the string embedding tensor operators are the operator-version of the supergravity fields (in the supergravity approximation) and the states associated with each field are the states obtained by acting with the operators on the vacuum state. For example, in the case of a Minkowski target spacetime background, a specific state of the graviton is given by: $h_{\mu\nu}(p)\, \psi^{\{\mu}_{-1/2}|0,p\rangle^+_{NS} \otimes \tilde{\psi}^{\nu\}}_{-1/2}|0,p\rangle^-_{NS}$. Here, ψ^μ_r are the Fourier modes of the fermionic

mal irreducible representations but rather in a massless Poincaré supermultiplet. Since the global symmetry group on the string theory side is $SU(2,2|4)$, the supergravity fields necessarily must fit into conformal supermultiplets. The helicities of the fields range from -2 to 2, so from the discussion in the previous section it follows that they must form 1/2 BPS representations built on superprimaries that are Lorentz scalars.[37] We have seen that BPS representations are uniquely classified by the R-symmetry representation of the superprimaries and their Lorentz representations (which in this case is always the $(0,0)$) and that each primary is also in a specific R-symmetry representation. Since the R-symmetry group is the isometry group $SO(6)$ of the S^5, if we want to organize the supergravity fields in conformal supermultiplets we need to make explicit their $SO(6)$ representations. This means that we have to expand each supergravity field in spherical harmonics on the S^5, which is an expansion in elements of different irreducible representation spaces for $SO(6)$. For example, scalar spherical harmonics Y_k are in one-to-one correspondence with symmetric traceless $SO(n)$ tensors of rank k, which transform irreducibly under the action of $SO(n)$:

$$Y_k^{(m)}(\hat{x}) \quad = \quad T^{(m)}_{I_1 \dots I_k}\, \hat{x}^{I_1} \dots \hat{x}^{I_k} \quad : \quad \sum_{I=1}^{n} \hat{x}^I \hat{x}^I = 1 \ , \tag{1.80}$$

where $\{T^{(m)}\}$ forms a basis for the $SO(n)$ irreducible representation space V_k of symmetric traceless tensors of rank k such that $T^{(m)} \cdot T^{(m')} = \delta^{mm'}$ and: $m = 1, 2, ..., \dim(V_k)$, with $\dim(V_k)$ the dimension of the vector space given in terms of gamma functions by:[38]

$$\dim(V_k) = \frac{(n+2k-2)\,\Gamma(n+k-2)}{\Gamma(n-1)\Gamma(k+1)} \ . \tag{1.81}$$

string embedding acting on the NS left/right moving vacuum $|0,p\rangle_{NS}^{+/-}$, with $\psi^{\{\mu}\tilde{\psi}^{\nu\}}$ denoting the symmetric traceless part. The coefficients $h_{\mu\nu}$ are what we call a classical solution of the supergravity equations of motion around the flat background such that the metric $G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. The tensor operator $\widehat{G}^{\mu\nu} := \psi^{\{\mu}_{-1/2} \otimes \tilde{\psi}^{\nu\}}_{-1/2}$ is what we call the operator-version of the metric $G^{\mu\nu}$. Supergravity fields are in this way in a one-to-one correspondence with tensor operators and we will use the notion of fields, operators and the states they create interchangeably since this is common practice in the literature.

[37] In the supergravity limit $\alpha'' \to 0$ there are no other string states, so the supergravity fields cannot be in other BPS or non-BPS representations because these would require additional states of spin > 2.

[38] In the case $n=3$, for example, we have the standard identification between V_k of dimension $\dim(V_k) = 2k+1$ and the irreducible representation space for rotations spanned by the angular momentum basis $\{|j,m\rangle : j = k\}$. For $n=4$, V_k coincides with the $SO(4)$ irreducible representation spanned by $\{|j_1, m_1\rangle \otimes |j_2, m_2\rangle : j_1 = j_2 = k/2\}$, just as for the $(\frac{k}{2}, \frac{k}{2})$ representation of the Lorentz group. Note also that the notation in this case is different than the one in footnote 31 since $T_{\mu\nu}$ are operators, while each $T^{(m)}_{I\dots J}$ (for each m) is a matrix of c-numbers. So, while $T_{\mu\nu}$ represents nine independent components, each $T^{(m)}_{I\dots J}$ (for each m) represents one independent component.

The expansion of a Lorentz-scalar field in S^5 spherical harmonics is therefore an explicit expansion in symmetric traceless tensors of $SO(6)$ such that each mode corresponds to an element of a different symmetric traceless representation. More generally, each spherical harmonic mode in the expansion of a supergravity field will be in a specific $SO(6)$ irreducible representation labelled according to the Dynkin notation (a_1, k, a_2) for $SO(6)$, where $a_{1,2}$ depend in particular on the Lorentz representation of the spherical harmonic (determined by the Lorentz representation of the supergravity field) but are independent of the rank k of the spherical harmonic. Each mode will fit in a conformal supermultiplet derived from a superprimary which should be a spherical harmonic mode of a Lorentz scalar. Since the supercharges carry a $SU(4)_R \sim SO(6)_R$ representation (besides carrying a Lorentz representation) and each superdescendant in a supermultiplet is obtained by acting with some product of Poincaré supercharges on a scalar superprimary, the spherical harmonic modes that will be in a given supermultiplet are uniquely determined by the $SO(6)_R$ representation of the superprimary.

Therefore, in order to organize the supergravity fields (or the corresponding string states/operators) in conformal supermultiplets, we start by decomposing the ten-dimensional fields into their background configurations plus perturbations and then expanding the perturbations in S^5 spherical harmonics:[39]

$$\phi(x,\theta) \quad = \quad \sum_{k=0}^{\infty} \phi_k(x) Y_k(\theta) \ , \tag{1.82}$$

where x^μ are coordinates on AdS_5 and θ^a on S^5. Note that this expansion depends on the Lorentz representation of the ten-dimensional field $\phi(x)$. Explicit expressions for the expansions of each field are given in the original work [64]. In the case of the spacetime metric, we perform the decomposition into the $AdS_5 \times S^5$ background configuration (the metric $\widetilde{G}$ in (1.63) with $\ell = 1$) plus perturbations h_{MN} as:

$$\begin{aligned} ds_{10}^2 &= G_{MN} dx^M dx^N \\ &= G_{\mu\nu}^{AdS_5} dx^\mu dx^\nu + G_{ab}^{S^5} d\theta^a d\theta^b + h_{MN} dx^M dx^N \ , \end{aligned} \tag{1.83}$$

where:

$$G_{\mu\nu}^{AdS_5} dx^\mu dx^\nu = \frac{1}{z^2}\left(dz^2 - dt^2 + d\vec{x}_3^{\,2}\right) \ , \tag{1.84}$$

and expand the different components of $h_{MN}(x,\theta)$ in spherical harmonics. The S^5 components h_{ab} are Lorentz scalars on AdS_5, whereas the components $h_{\mu a}$ are

[39] Here we are omitting the inner product of each mode ϕ_k with the corresponding basis Y_k because the expansion depends on the Lorentz representation of ϕ. In the case of scalar harmonics (1.80), for example, we would have the usual sum over m.

$(\frac{1}{2}, \frac{1}{2})$ Lorentz vectors. If we introduce the S^5 expansions in the ten-dimensional supergravity action and dimensionally reduce it by integrating on the S^5, the Newton constant of the resulting five-dimensional action is determined as:

$$\begin{aligned}\frac{1}{16\pi G_{10}} \int d^{10}x\sqrt{G}\left(....\right) &= \frac{1}{16\pi G_{10}} \int d^5\theta\sqrt{G}_{S^5} \int d^5x\sqrt{G}_{AdS}\left(....\right) \\ = \frac{1}{16\pi G_5} \int d^5x\sqrt{G}_{AdS}\left(....\right) &\quad : \quad G_5 = G_{10}/\pi^3 = \pi/(2N^2) \ , \qquad (1.85)\end{aligned}$$

where π^3 =Vol(S^5) and we used the result of section 1.2.3 for G_{10} (if the $AdS_5 \times S^5$ radius $\ell \neq 1$ we have $G_5/\ell^3 = \pi/(2N^2)$).

After expanding a given field ϕ as in equation (1.82), we have that each spherical harmonic mode ϕ_k will be an eigenfunction of the dilatation generator in the following sense. On the string theory side, the generators of the conformal algebra are the Killing vectors of AdS_5. The dilatation generator D in this case corresponds to the derivative $z\partial_z + x^\alpha\partial_\alpha$ that generates scale transformations $z \to \lambda z, x^\alpha \to \lambda x^\alpha$, with z the AdS radial direction transverse to the D3-branes and $x^\alpha = (t, \vec{x})$ the coordinates on the branes. A supergravity mode $\phi_k(z, x)$ will be an eigenfunction of D if: $(z\partial_z + x \cdot \partial)\phi_k = (\Delta + x \cdot \partial)\phi_k$ (recall equation (1.68)),[40] which should be read asymptotically as $z \to 0$. Once we introduce the spherical harmonic expansions in the supergravity equations of motion (linearized around $AdS_5 \times S^5$), the five-dimensional modes of each ten-dimensional (massless) supergravity field will acquire a mass according to their $SO(6)$ and Lorentz representations. This mass in turn will determine the leading order asymptotics of the modes in the radial direction: $\phi_k = z^\Delta \left(\phi_{(\Delta)k} + O(z^{>0})\right)$, where $\Delta = \Delta(m)$ is a polynomial of the mass m associated with the mode, which in turn is uniquely fixed by the $SO(6)$ and Lorentz representation of ϕ_k. Since $\Delta(m)$ is the asymptotic eigenvalue of the mode with respect to $z\partial_z$, we conclude that each mode will be an eigenfunction of the dilatation generator with a specific dilatation eigenvalue according to its $SO(6)$ and Lorentz representation. We have restricted this analysis to the asymptotic regime (and to the normalisable solutions of the supergravity equations of motion) and the reason that this should be the case will be clear after the discussion in section 1.3. The relation between the eigenvalues Δ and

[40] In terms of states and operators as in section 1.2.4, this statement translates to: $D|\phi_k\rangle = -i\Delta|\phi_k\rangle$, where $|\phi_k\rangle = \lim_{z,x\to 0}\ \mathcal{O}_k^{\alpha_1\cdots\alpha_n}(z,x)|0\rangle$ is the tensorstate created by the x^α components of the string tensor operator $\mathcal{O}_k(z, x)$ associated with the five-dimensional supergravity field $\phi_k(z, x)$ such that: $[D, \mathcal{O}_k(0, x)] = -i(\Delta + x \cdot \partial)\mathcal{O}_k(0, x)$. The z-components can always be gauge-fixed near $z = 0$ and do not represent real degrees of freedom.

the masses of the modes is given by:

$$\begin{aligned} \text{scalars} &: \quad m^2 = \Delta(\Delta-4)\ , \\ \text{spin } 1/2,\, 3/2 &: \quad |m| = \Delta - 2\ , \\ p\text{-forms} &: \quad m^2 = (\Delta-p)(\Delta+p-4)\ , \\ \text{spin } 2 &: \quad m^2 = \Delta(\Delta-4)\ , \end{aligned} \tag{1.86}$$

where $\Delta = k + a$, with $a \geq 0$ an integer or half-integer number that depends on the Lorentz representation of the field [64] and is given below in Table 1.1 for each field.

1/2 BPS irreducible representations of the superconformal group built on Lorentz-scalar superprimaries were determined in [65] using oscillator representation methods, whereas the spectrum of the S^5 compactification of supergravity was determined in [64] (in the latter reference the procedure involves a diagonalisation of the supergravity equations of motion for the spherical harmonic modes in order to decouple all interactions and obtain equations for free fields with mass terms. Interactions can then be switched on after the spectrum has been determined). Given that each supergravity mode is in a specific $SO(6)_R$ representation (a_1, k, a_2) and Lorentz representation (s_1, s_2), we can organize the modes in conformal supermultiplets using the group theoretic analysis of [65].[41] The spectrum of superprimaries and corresponding superdescendants that form the gauge theory and string theory conformal supermultiplets is given below in Table 1.1, which was adapted from reference [61]. Here we are omitting the complex conjugates of the operators and fields. The fermionic supergravity fields ψ_M and χ correspond to the two gravitinos ($\mathcal{N}=2$) and the two dilatinos, respectively. The metric $G'_{\mu\nu} = G_{\mu\nu} - \frac{1}{3} G^{AdS_5}_{\mu\nu} G^a_a$,

[41] Note that this can be done particularly without knowing a priori the conformal dimension Δ discussed above associated with each supergravity mode. For example, from Table 1 in ref. [65] we have that the superprimary of lowest conformal dimension is a Lorentz scalar in the $(0,2,0)$, which corresponds to the **20** according to our formula (1.81) with $n=6, k=2$. Then, from the supergravity spectrum in Table III in ref. [64] we have that the only scalar modes in the **20** are the $k=2$ modes of the supergravity fluctuations $H := h^\alpha_\alpha - a_{\alpha\beta\gamma\delta}$ and B. However, from ref. [65] we see that the higher the dimension (or the rank k) of a $SO(6)_R$ representation in the expansion of a supergravity field, the higher its conformal dimension. For this reason, the lowest superprimary cannot be the **20** of the axion-dilaton field B because it has $SO(6)_R$ representations of lower dimension (namely the **1** and the **6**) and therefore of lower conformal dimension. On the other hand, the **20** of the H is the lowest dimensional $SO(6)_R$ representation in the spherical harmonic expansion of H, so the lowest superprimary will be the $(0,2,0)$ spherical harmonic mode of H. This reasoning can then be repeated in succession for all other supergravity modes to derive the conformal supermultiplets. The conformal dimensions in [65] will then coincide with the conformal dimensions Δ discussed above and this is an alternative way to derive the latter using group theory (note that in ref. [65] the conformal dimensions are given by $E_0/2$ because the dilatation generator D corresponds to the generator of $U(1)_E$ in L^0 multiplied by a factor of $1/2$, which can be seen by comparing equation (2) in this reference with the commutation relation between D and the Poincaré supercharges Q^i_α).

SYM Operator	superdesc	SUGRA	dim Δ	spin	$SO(6)_R$
$\mathcal{O}_k \sim \mathrm{Tr}\,\phi^k \;:\; k \geq 2$	$-$	$G^a_a\,, C_{4abcd}$	k	$(0,0)$	$(0,k,0)$
$\mathcal{O}^{(1)}_{1+k} \sim \mathrm{Tr}\,\lambda\phi^k \;:\; k \geq 1$	Q	ψ_a	$k+\frac{3}{2}$	$(\frac{1}{2},0)$	$(1,k,0)$
$\mathcal{O}^{(2)}_{2+k} \sim \mathrm{Tr}\,\lambda\lambda\phi^k$	Q^2	$B_{ab}+iC_{2ab}$	$k+3$	$(0,0)$	$(2,k,0)$
$\mathcal{O}^{(3)}_{2+k} \sim \mathrm{Tr}\,\lambda\bar{\lambda}\phi^k$	$Q\bar{Q}$	$G_{\mu a}\,, C_{4\mu abc}$	$k+3$	$(\frac{1}{2},\frac{1}{2})$	$(1,k,1)$
$\mathcal{O}^{(4)}_{1+k} \sim \mathrm{Tr}\,F_+\phi^k \;:\; k \geq 1$	Q^2	$B_{\mu\nu}+iC_{2\mu\nu}$	$k+2$	$(1,0)$	$(0,k,0)_1$
$\mathcal{O}^{(5)}_{2+k} \sim \mathrm{Tr}\,F_+\bar{\lambda}\phi^k$	$Q^2\bar{Q}$	ψ_μ	$k+\frac{7}{2}$	$(1,\frac{1}{2})$	$(0,k,1)$
$\mathcal{O}^{(6)}_{2+k} \sim \mathrm{Tr}\,F_+\lambda\phi^k$	Q^3	χ	$k+\frac{7}{2}$	$(\frac{1}{2},0)$	$(1,k,0)$
$\mathcal{O}^{(7)}_{3+k} \sim \mathrm{Tr}\,\lambda\lambda\bar{\lambda}\phi^k$	$Q^2\bar{Q}$	ψ_a	$k+\frac{9}{2}$	$(0,\frac{1}{2})$	$(2,k,1)$
$\mathcal{O}^{(8)}_{2+k} \sim \mathrm{Tr}\,F_+^2\phi^k$	Q^4	$C_0+i\Phi$	$k+4$	$(0,0)$	$(0,k,0)$
$\mathcal{O}^{(9)}_{2+k} \sim \mathrm{Tr}\,F_+F_-\phi^k$	$Q^2\bar{Q}^2$	$G'_{\mu\nu}$	$k+4$	$(1,1)$	$(0,k,0)$
$\mathcal{O}^{(10)}_{3+k} \sim \mathrm{Tr}\,F_+\lambda\bar{\lambda}\phi^k$	$Q^3\bar{Q}$	$B_{\mu a}+iC_{2\mu a}$	$k+5$	$(\frac{1}{2},\frac{1}{2})$	$(1,k,1)$
$\mathcal{O}^{(11)}_{3+k} \sim \mathrm{Tr}\,F_+\bar{\lambda}\bar{\lambda}\phi^k$	$Q^2\bar{Q}^2$	$C_{4\mu\nu ab}$	$k+5$	$(1,0)$	$(0,k,2)$
$\mathcal{O}^{(12)}_{4+k} \sim \mathrm{Tr}\,\lambda\lambda\bar{\lambda}\bar{\lambda}\phi^k$	$Q^2\bar{Q}^2$	$G_{(ab)}$	$k+6$	$(0,0)$	$(2,k,2)$
$\mathcal{O}^{(13)}_{4+k} \sim \mathrm{Tr}\,F_+^2\bar{\lambda}\phi^k$	$Q^4\bar{Q}$	χ	$k+\frac{11}{2}$	$(0,\frac{1}{2})$	$(0,k,1)$
$\mathcal{O}^{(14)}_{4+k} \sim \mathrm{Tr}\,F_+\lambda\bar{\lambda}\bar{\lambda}\phi^k$	$Q^3\bar{Q}^2$	ψ_a	$k+\frac{13}{2}$	$(\frac{1}{2},0)$	$(1,k,2)$
$\mathcal{O}^{(15)}_{3+k} \sim \mathrm{Tr}\,F_+F_-\lambda\phi^k$	$Q^3\bar{Q}^2$	ψ_μ	$k+\frac{11}{2}$	$(\frac{1}{2},1)$	$(1,k,0)$
$\mathcal{O}^{(16)}_{3+k} \sim \mathrm{Tr}\,F_+F_-^2\phi^k$	$Q^4\bar{Q}^2$	$B_{\mu\nu}+iC_{2\mu\nu}$	$k+6$	$(1,0)$	$(0,k,0)_2$
$\mathcal{O}^{(17)}_{4+k} \sim \mathrm{Tr}\,F_+F_-\lambda\bar{\lambda}\phi^k$	$Q^3\bar{Q}^3$	$G_{\mu a}\,, C_{4\mu abc}$	$k+7$	$(\frac{1}{2},\frac{1}{2})$	$(1,k,1)$
$\mathcal{O}^{(18)}_{4+k} \sim \mathrm{Tr}\,F_+^2\bar{\lambda}\bar{\lambda}\phi^k$	$Q^4\bar{Q}^2$	$B_{ab}+C_{2ab}$	$k+7$	$(0,0)$	$(0,k,2)$
$\mathcal{O}^{(19)}_{3+k} \sim \mathrm{Tr}\,F_+^2F_-\bar{\lambda}\phi^k$	$Q^4\bar{Q}^3$	ψ_a	$k+\frac{15}{2}$	$(0,\frac{1}{2})$	$(0,k,1)$
$\mathcal{O}^{(20)}_{4+k} \sim \mathrm{Tr}\,F_+^2F_-^2\phi^k$	$Q^4\bar{Q}^4$	$G^a_a\,, C_{4abcd}$	$k+8$	$(0,0)$	$(0,k,0)$

Table 1.1: Super Yang-Mills operators and the corresponding supergravity fields. The range of k is $k \geq 0$ unless otherwise specified.

whereas $F_{+/-}$ are the self-dual/antiself-dual SYM field strength. Since the gauge theory is defined in four dimensions, the Lorentz indices of the SYM operators are omitted to avoid confusion with the supergravity indices.

The conformal supermultiplets built on the superprimaries $\mathcal{O}_n$ of dimension $\Delta = n$ are given by the sets:

$$\left\{\mathcal{O}_2; \mathcal{O}_2^{(1)}, ..., \mathcal{O}_2^{(6)}, \mathcal{O}_2^{(8)}, \mathcal{O}_2^{(9)}\right\}\,, \tag{1.87}$$

$$\left\{\mathcal{O}_3; \mathcal{O}_3^{(1)}, ..., \mathcal{O}_3^{(11)}, \mathcal{O}_3^{(13)}, \mathcal{O}_3^{(15)}, \mathcal{O}_3^{(16)}\right\}\,, \tag{1.88}$$

$$\left\{\mathcal{O}_n; \mathcal{O}_n^{(1)}, ..., \mathcal{O}_n^{(20)}\right\} \;:\quad n \geq 4\,. \tag{1.89}$$

The string superprimaries and superdescendants are the $SO(6)_R$ modes of the perturbations of the corresponding SUGRA fields. A string superprimary of dimension k is a linear combination of the $(0,k,0)$ mode of the trace of the S^5 components of the metric perturbation with the $(0,k,0)$ mode of the S^5 components of the antisymmetric 4-form perturbation:

$$h_{(k)a}^{\;a} - 10(k+4)b_{(k)} \quad : \quad k \geq 2 \;, \tag{1.90}$$

where $a_{(k)4abcd} \sim b_{(k)}\epsilon_{abcd}$ are the modes of the perturbation of C_{4abcd}. The string superdescendant of highest dimension (in a supermultiplet built on a superprimary of dimension $4+k$) corresponds to a different linear combination of these modes: $h_{(k)a}^{\;a} + 10k\, b_{(k)}$. A similar thing happens in the other slots with two fields separated by a comma.[42] Note that a mode $(0,k,0)$ of the antisymmetric 2-form $B_{\mu\nu} + iC_{2\mu\nu}$ seems to lie in two different supermultiplets, one built on a superprimary of dimension $\Delta = 1+k$ and another on a superprimary of dimension $\Delta = 3+k$. What happens in this case is that the equation of motion for the antisymmetric 2-form factorizes into two field equations with different mass terms, so the 2-form effectively admits two different spherical harmonic expansions with modes $(0,k,0)_1$ and $(0,k,0)_2$.[43] Note also that there are supergravity modes with negative mass-squared: the $(0,2,0)$ and $(0,3,0)$ scalar superprimaries and the $(2,0,0)$ mode of the scalars $B_{ab} + iC_{2ab}$. However, in AdS spaces a scalar field can have negative mass-squared without being tachyonic: the AdS curvature provides a positive contribution to the energy of the field such that the total energy is non-negative if $m^2 \geq -4$ in five dimensions [66, 67]. This is the so-called BF bound, or stability bound, and is saturated by the lowest string superprimary.

It should also be mentioned that unitarity of $SU(2,2|4)$ representation spaces built on scalar superprimaries implies the unitarity bound $\Delta \geq 1$, which is less stringent than the bound in (1.79). For a $SU(N)$ gauge group, the elementary operators of the gauge theory are traceless as remarked above and the supermultiplet built on $\mathcal{O}_1$ is empty (for a $U(N)$ gauge group there is the so-called doubleton supermultiplet built on the superprimary $\mathcal{O}_1$). On the string theory side, the supermultiplet built on the $(0,1,0)$ superprimary consists of modes that are pure gauge and therefore it can be made empty by diffeomorphisms and gauge transformations. This result is consistent with the fact that the string theory side is describing the $SU(N)$ sector of the gauge theory.

[42] These linear combinations are obtained by diagonalizing the corresponding equations of motion. In the case of the $h_{(k)a}^{a}$ with $b_{(k)}$ modes this is given by equation (2.33) in ref. [64]. In the case of the $G_{\mu a}$ and $C_{4\mu abc}$ modes, the linear combinations are explicitly given in equation (2.28) in this reference.

[43] See equations (2.61)–(2.64) together with equation (2.48) in reference [64].

The representations built on the superprimaries of dimension $\Delta \geq 4$ have 20+1 primaries of dimension between Δ and $\Delta + 4$, but those built on superprimaries with $\Delta = 2$ and $\Delta = 3$ are even shorter and the dimensions have a range span of 2 and 3, respectively. The gauge theory supermultiplet built on the superprimary $\mathcal{O}_2$ contains the conserved currents of the theory (the stress-energy tensor, the R-symmetry currents and the supercharges) and for this reason this supermultiplet is called the currents supermultiplet. The highest dimension primary operator in this supermultiplet has conformal dimension 4 and is the Yang-Mills stress-energy tensor. On the string theory side this corresponds to the $(0, 0, 0)$ mode of the metric $G'_{\mu\nu}$, which is the only massless spin-2 mode as seen from equation (1.86) and therefore corresponds to the graviton. The string theory supermultiplet that contains this mode is therefore called the graviton supermultiplet and corresponds to the currents supermultiplet on the gauge theory side (in fact the names are used interchangeably).

The string theory field content of the graviton supermultiplet coincides with the field content of $\mathcal{N} = 8$ gauged supergravity in five dimensions. This theory can be obtained from the Kaluza-Klein reduction of ten-dimensional $\mathcal{N} = 2$ supergravity by working in the limit where the Kaluza-Klein radius of the compact manifold is small so that only the lightest mode of each field survives. All gauge fields in this theory are massless and for this reason the graviton supermultiplet is said to be massless (all scalars but one, all fermions and the 2-form in this supermultiplet are massive as seen from (1.86)). Note, however, that in the case of $AdS_5 \times S^5$ the radius of the S^5 is the same as that of AdS, so we cannot remove from the compactified theory all but the lightest modes by a similar argument. In this way, in our case the S^5 compactified theory does not correspond to an effective low-energy theory but remains the full theory since we are keeping the infinite tower of Kaluza-Klein modes from first principles. On the other hand, the *truncation* of the S^5 compactification of the ten-dimensional supergravity theory to the graviton supermultiplet is believed to be a consistent truncation in the sense that solutions of the resulting $\mathcal{N} = 8$ supergravity theory in five dimensions are exact solutions of ten-dimensional supergravity. So, by keeping only the field content of the graviton supermultiplet we are not working with an effective theory but rather restricting our analysis of the full ten-dimensional theory to a specific sector of the theory.

If we restrict the compactified theory to $\mathcal{N} = 8$ gauged supergravity in AdS_5, we have that each of the fields is dual to a gauge theory operator that lies in the supercurrent multiplet. To compute their correlation functions we deform the gauge theory by introducing sources for these operators. With the exception of the stress-energy tensor,[44] each of these operators has conformal dimension $\Delta < 4$ and

[44]In the case of the stress-energy operator, we compute its correlation functions by promoting the gauge theory flat metric to an arbitrary background metric $g_{(0)}$ and taking variations of the generating functional with respect to this source before setting back $g_{(0)}$ to be flat. Since

therefore is a relevant operator (see appendix A.4). While the undeformed gauge theory does not require renormalization, relevant (and marginal) deformations in general spoil the UV-finiteness of the theory such that the correlation functions of relevant operators will be UV-divergent.[45] Unlike the case of irrelevant operators, however, these divergences are renormalisable by a finite number of counterterms. These gauge theory UV-divergences have an exact analogue on the supergravity side, where they are mapped to divergences of the supergravity action associated with the infinite size of the *AdS* space. The latter arise from the behaviour of *AdS* gravity at long distances and therefore are called IR-divergences. This type of duality between the divergences of the two theories is called a UV/IR duality. The subsequent renormalization procedure on the gauge theory side also has an exact analogue on the string theory side. The systematic process of removing the supergravity divergences is called holographic renormalization and will be the subject of the next sections.

String supermultiplets other than the graviton one include fields dual to gauge theory operators of dimension $\Delta > 4$. Those operators are therefore irrelevant and their correlation functions contain UV divergences that are not renormalisable. This non-renormalizability of the gauge theory deformed by irrelevant operators is reflected on the string theory side, in which case the holographic renormalization procedure breaks down in the presence of the dual string theory fields and the corresponding effective action is non-renormalisable. Since the gauge theory supercurrent multiplet is the only supermultiplet with non-irrelevant operators, it follows that the string theory graviton supermultiplet is the only supermultiplet dual to a UV-conformal field theory (*c.f.* appendix A.4).

The matching between the gauge theory and string theory supermultiplets discussed in this section is a consistency check, albeit a very helpful one, and does not imply an equivalence between the two theories: this matching would necessarily take place between any two theories with the same symmetry groups (although it is hard to find inequivalent theories with the same symmetry groups). As mentioned at the end of section 1.2.2, the strongest suggestion that the two theories are describing the same aspects of the same system, just from different perspectives, follows from the fact that each theory seems to encode the observables of

the stress-energy tensor has conformal dimension 4, switching on an arbitrary metric can be interpreted as a deformation of the original theory by a marginal source.

[45] It should be emphasized that, if the undeformed theory is UV-finite, then the divergences in the correlators are always proportional to the sources and derivatives of it and therefore vanish if the sources are set to zero (and the metric to be flat if the correlator involves the stress-energy operator). This is just the statement that switching off the sources in a correlator should result in the same expression as that obtained by computing the correlator of an operator of the UV-finite theory. In this way, renormalization is only required if the sources are set to be coupling constants, or left arbitrary but non-vanishing functions.

the other. A given quantum field theory is completely characterised by the set of all its correlation functions and therefore by its generating functional. Following the original conjecture [20], and motivated in particular by the results of [17, 18, 19], it was proposed in [21, 22] how to use the string theory partition function with AdS boundary conditions to define the generating functional of a quantum field theory. The correlation functions computed via this prescription reproduce all expected results of a UV-conformal field theory. In the particular case of string theory approximated by $\mathcal{N} = 8$ gauged supergravity in AdS_5, these are the correlators of the $\mathcal{N} = 4$ SYM operators in the supercurrent multiplet that we discussed above.[46] Recall that this approximation is valid in the regime $\alpha'' \sim 0$, which corresponds to the strong coupling regime $\lambda >> 1$ of the gauge theory where perturbation theory in λ is not reliable. The prescription therefore allows us to study the non-perturbative sector of the gauge theory using the dual supergravity theory as we discussed in the previous sections. Even though the correlation functions at strong coupling cannot be computed perturbatively in the gauge theory, the conformal symmetry of the theory completely fixes the form of the $(n \leq 3)$-point correlators up to proportionality constants [68] as we will see in the next section, while $(n > 3)$-point functions are fixed up to a proportionality function of anharmonic ratios [69, 70]. These constraints imposed by conformal invariance are satisfied by the correlation functions derived from supergravity using the AdS/CFT prescription.

The computation of the gauge theory correlators from the dual string theory partition function requires detailing the precise way the supergravity fields are related to the super Yang-Mills operators. In this section we have shown how these fields are organized in the same conformal supermultiplets as these operators, but we have not discussed the identification between the Hilbert spaces of the two theories. The particle states associated with each supergravity field should be in one-to-one correspondence with the states created by the corresponding gauge theory operators, but this identification is highly non-trivial. Note that the SYM operators in Table 1.1 are defined in four dimensions, whereas the supergravity modes live in one dimension higher, with the radial coordinate z in the AdS_5 metric (1.84) parametrising the extra dimension. This implies in particular that

[46] Note that we compute correlation functions of the gauge theory operators that lie in the conformal supermultiplets. Correlation functions of the elementary fields themselves are not observables because the latter are not gauge invariant. To form gauge invariant quantities we need to trace over the gauge group indices, but then the trace of the elementary $SU(N)$ fields vanishes. In the $U(N)$ case the elementary fields correspond to the operator content of the doubleton supermultiplet: $\{\mathcal{O}_1; \mathcal{O}_1^{(1)}, \mathcal{O}_1^{(4)}\}$ and respective complex conjugates, and we can compute their correlation functions in the gauge theory (though this would be done at weak coupling). Since the dual string theory only describes the $SU(N)$ sector of the gauge theory, these correlators cannot be computed on the gravity side with the standard AdS/CFT correspondence.

the states created by the SYM operators do not correspond directly to the states created by the string operators associated with each five-dimensional supergravity mode.

We have seen that a fundamental piece in the duality between the two theories is the decomposition of each ten-dimensional supergravity field in S^5 harmonics so that the $SO(6)_R$ representations of the fields be explicit. The S^5 components and the Kaluza-Klein modes are then recovered on the gauge theory side from the infinite set of operators of different spins that form the supermultiplets. The precise correspondence between these operators and the modes requires in turn a decomposition of each mode along the radial direction that solves the supergravity equations of motion in the AdS_5 background. These equations are second order in the radial coordinate and therefore each mode admits two linearly independent solutions called the non-normalisable and normalisable solutions. The leading order coefficient in the radial expansion of each solution is then related to the dual SYM operator in a precise way. In the case of the normalisable solution, this is the coefficient $\phi_{(\Delta)k}$ that we introduced when we discussed the dilatation eigenvalue of each mode. The states created by the string operator associated with $\phi_{(\Delta)k}$ are then in one-to-one correspondence with the states created by the dual SYM operator (in fact, this is an equivalence between the operators). These results will be the main subject of the next section where we will further show that the classical coefficient $\phi_{(\Delta)k}$ is directly related to the vacuum expectation value of the dual operator.

1.2.6. The radius/energy-scale duality

The AdS radial direction plays a special role in the AdS/CFT correspondence that we haven't yet discussed. On string theory side, this direction is geometric and the theory behaves locally in z. The dual gauge theory is defined explicitly in one dimension less and for this reason this locality and the degrees of freedom along the extra direction are not apparent. If the two theories are equivalent, these degrees of freedom must be encoded in the observables of the gauge theory. The specific way the radial dimension is seen from the gauge theory perspective was originally discussed in [23] and is based on the observation that a scale transformation in the gauge theory, generated by the dilatation generator D of the $\mathfrak{su}(2,2|4)$ algebra, corresponds to a specific isometry of the AdS metric that rescales the radial coordinate z. Suppose we consider the gauge theory with a UV cut-off Λ such that the generating functional of the theory is a sum over all field configurations with momenta $|k| < \Lambda$. If the gauge theory is pure $\mathcal{N} = 4$ SYM no cut-off is needed because the theory is UV-finite, but once we deform the theory with relevant (or marginal) operators the cut-off is necessary to regulate the divergences.[47] As discussed more carefully in appendix A.4, a scale transformation $x^\alpha \to \lambda x^\alpha$ in the gauge theory results in a rescaling $\Lambda \to \Lambda/\lambda$ of the cut-off. If $\lambda > 1$, this in turn requires that we integrate over all field configurations with momenta above Λ/λ in the generating functional to obtain the effective theory at scales below Λ/λ. On the string theory side, such a transformation corresponds to the isometry $x^\alpha \to \lambda x^\alpha, z \to \lambda z$ of the AdS metric (1.84) generated by the Killing vector D. Note that this isometry results in motion of points of the manifold inwards ($\lambda > 1$) or outwards ($\lambda < 1$) from some region $z = z_\Lambda$ to some region $z = \lambda z_\Lambda$. This correspondence implies that the radial coordinate z behaves as an energy scale from the gauge theory perspective and it is natural to take $z_\Lambda \propto 1/\Lambda$ so that the UV limit $\Lambda \to \infty$ in the gauge theory corresponds to the infinity, or conformal boundary $z = 0$ of AdS space (see appendix B.4).

A similar way of seeing this result is to recall the above discussion about the UV-divergences of the gauge theory and the divergences of supergravity associated with the infinite size of AdS space. When using the string theory partition function in the supergravity approximation to define the generating functional of quantum field theory correlation functions, one faces divergences in the supergravity action as $z \to 0$ because supergravity solutions typically diverge as we approach infinity of AdS space. In order to remove these divergences, one regulates the action (or the area of the boundary) by replacing the true conformal boundary by a regulating boundary $z = z_\Lambda$. One then derives the appropriate counterterms that cancel the potentially divergent terms and in the end takes the limit $z_\Lambda \to 0$. This process mimics precisely the standard renormalization of quantum field theories. After computing the correlation functions, one finds that the regulator z_Λ plays exactly the role of a UV regulator $\Lambda \propto 1/z_\Lambda$ in the quantum field theory that was defined from the string theory.

An intuitive way of seeing this radius/energy-scale duality follows by considering lattice gauge theory, or statistical mechanics, in which case the UV regulator Λ of the theory is the inverse of the lattice spacing such that all wavelengths shorter than $1/\Lambda$ are suppressed. If the gauge theory is UV-regulated in this way, a scale transformation $x^\alpha \to \lambda x^\alpha$ rescales the lattice spacing as discussed in appendix A.4 and all wavelengths shorter than λ/Λ become suppressed. On the string theory side, we can consider a lattice as a hypersurface $z = z_\Lambda$. The scaling isometry

[47] In the Wilsonian sense, one can see pure $\mathcal{N} = 4$ SYM as a fundamental theory where no degrees of freedom have been integrated out. This is so because, as we start the process of integration of high-momenta degrees of freedom down to some low momenta, the theory continues to look the same and therefore is not associated with any particular energy scale. On the other hand, deformed theories are effective: they differ from the fundamental theory one starts with and are derived from it by integrating out degrees of freedom, hence are defined only below a certain energy scale.

$x^\alpha \to \bar{x}^a = \lambda x^\alpha, z \to \bar{z} = \lambda z$ of the AdS metric (1.84) preserves the induced metric on the hypersurfaces of constant z. The hypersurface $z = z_\Lambda$ is mapped by the isometry to a surface $\bar{z} = \lambda z_\Lambda$ farther ($\lambda > 1$) or closer ($\lambda < 1$) to the conformal boundary $\bar{z} = 0$. In addition, the lattice spacing increases ($\lambda > 1$) or decreases ($\lambda < 1$) because this isometry is a scale transformation that simultaneously leaves the induced metric fixed (see Figure A.1 in appendix A.4). Therefore, from the string theory point of view, successive changes in the lattice spacing – which is another way of seeing the process of successively integrating out gauge theory degrees of freedom – correspond to successive changes of the position of the lattice along the radial direction z such that approaching the boundary $z = 0$ corresponds to taking the zero size limit of the lattice spacing. From this picture one says that different hypersurfaces of constant z correspond to the gauge theory at different energy scales: the fundamental gauge theory where no degrees of freedom have been integrated out, the UV theory, is said to live at the boundary of AdS, whereas the gauge theory at different scales Λ describes supergravity in regions of AdS space bounded by different regulating sufaces $z \sim 1/\Lambda$ and therefore is said to live on such surfaces (see *e.g.* [71]). For these reasons, the AdS/CFT correspondence is said to be a holographic duality: the field theory dual to string theory in AdS is defined in one less geometric dimension and it encodes in its scaling behaviour the degrees of freedom, or dynamics, along the extra radial direction. In particular, the lower dimensional boundary theory contains all the gauge theory degrees of freedom and therefore encodes information about the gravitational physics of the entire bulk interior. It should be emphasized that the above discussion of scale transformations can be generalized to include the special conformal transformations of appendix A.3 that, together with translations and rotations, form the isometry group of AdS. These isometries result in conformal transformations at the boundary and for this reason any field theory defined at the boundary of AdS will enjoy the conformal group as its symmetry group.

The identification of the bulk radius as a gauge theory energy scale is at the core of the holographic renormalization group [72, 73],[48] where the renormalization flow of the coupling constants of the gauge theory translates into a radial flow of solutions of the gravity theory. The successive rescallings of the gauge theory cut-off and the corresponding integrations over high-momenta degrees of freedom can be transferred into the coupling constants, so that the resulting theory is defined at the same cut-off but the couplings undergo renormalization group (RG) flow according to the energy scales being observed (this is the flow in λ of the coupling constant in the specific example of appendix A.4). On the gravity side, this process results in changes of the regulating boundary and therefore in the radial position of the boundary configuration of the supergravity solutions. In the Hamilton-

[48]See also the review [74] and more recently the discussion in [75].

Jacobi formulation of the gravity theory, the Hamiltonian equations of motion together with the Hamilton-Jacobi equation result in first order 'flow' equations that determine the radial evolution of the fields and therefore the evolution of the boundary configurations with the radial position of the boundary. As we will discuss in the next section, these boundary configurations are mapped by the AdS/CFT correspondence to the couplings of the gauge theory (which can be constant or position dependent, in which case are called sources). The radial flow equations therefore correspond via the radius/energy-scale duality to the beta-function equations of the gauge theory couplings that dictate their RG flow.

1.3. AdS/CFT Correlators and Renormalization

As mentioned in the previous sections, the results on the computation of the absorption cross-sections of D-branes were the main precursor of the AdS/CFT correspondence and suggested that the observables of the worldvolume theory could be computed from supergravity in the near-horizon AdS region. In this section we are interested in following the proposal of [21, 22] to define the generating functional of a quantum field theory from the string theory partition function in order to compute these observables. This allows us in particular to compute the correlation functions of the SYM operators in the supercurrent multiplet dual to $\mathcal{N} = 8$ supergravity in five dimensions that we have been analysing, but we will not restrict our discussion to this specific case. The exact same analysis of the previous sections can be repeated to other p-brane solutions of supergravity and the respective worldvolume theories, such as the M-branes of eleven-dimensional supergravity and systems of D-branes in supergravity in ten dimensions, see *e.g.* [20, 27], all of these solutions free of essential singularities. The important point to retain is that the worldvolume gauge theories are conformal field theories and the near-horizon regions in which the dual closed strings live are products of AdS spaces with compact spaces. In each case, the conformal group of the gauge theory coincides with the symmetry group of the corresponding AdS space, and the R-symmetry with the symmetry group of the compact space. For this reason, we will keep the dimension of the AdS space arbitrary and we will assume that we have reduced our theory along the compact space as we did in the case of the S^5 to obtain the $SO(6)_R$ Kaluza-Klein modes and their masses.

As we briefly mentioned in the previous section, in the Wilsonian treatment of renormalization theory, quantum field theories are understood by starting with some field theory defined at a fundamental energy scale Λ and integrating out high-momenta degrees of freedom down to some observable scale Λ/λ. The Λ-theory we start from can in turn be understood as a deformation of a conformal field theory because CFTs are fixed points of the renormalization flow; in this sense, quantum field theories differ from CFTs because we have moved along a RG flow away from the fixed point. Following this viewpoint, we will use the string theory partition function to define a quantum field theory as a deformation of a CFT. If the deformation is irrelevant, such as deformations by those operators in all but the supercurrent multiplet of $\mathcal{N} = 4$ SYM, the quantum field theory is non-renormalisable as already discussed and therefore we will restrict to relevant (and marginally non-irrelevant) deformations. These deformations vanish as we approach the UV and therefore the gauge theory is UV-conformal; in the specific example of deformations by operators in the supercurrent multiplet, the theory will approach $\mathcal{N} = 4$ SYM at high-energies. On the string theory side, the spacetime in general will not be AdS everywhere because the closed string excitations dual to the deformations backreact on the geometry as in the case of equation (1.83).[49] On the other hand, since the gauge theory is UV-conformal, from the discussion on the radius/scale duality it follows that the bulk geometry is AdS asymptotically *i.e.* as we approach the conformal boundary. In fact, we can be more general than this and require that a neighbourhood of the boundary be AdS only locally in the sense of appendix B.1, such that the global properties of the near-boundary region such as its topology be relaxed. The most general way to describe deformations of CFTs is by starting from the UV theory and letting it flow to some observable scale, so the quantum field theory we define from the string partition function is the boundary theory perturbed by the deformation.[50]

Let us then start with the string theory partition function (1.23). The worldsheet action S is schematically of the form (1.64) and contains the target spacetime fields with AdS vevs (the string excitations in an AdS background) pulled-back to the worldsheet. In the partition function we sum over all possible configurations for the string embeddings. The target space fields, as functions of the embeddings and at leading order in g_s, satisfy the supergravity equations of motion with a negative cosmological constant plus all α'' corrections.[51] Since we are considering

[49] In the probe approximation, we consider the supergravity fields as infinitesimal and neglect their backreaction such that the geometry on which they propagate is exactly AdS. On the gauge theory side, this is equivalent to working in the vicinity of a CFT, where the deformations are infinitesimal.

[50] Note in particular that if we are interested in computing correlation functions of operators at a point such as those in the supercurrent multiplet of $\mathcal{N} = 4$ SYM, we must give a definition of the field theory generating functional in the UV. These operators are all defined at $x^\alpha = 0$. We localise gauge theory operators at a single point by performing a scale transformation $x^\alpha \to \lambda x^\alpha$ and taking $\lambda \to 0$. Since the energy scale $\Lambda \to \Lambda/\lambda$ and the bulk radius $z \to \lambda z$, this implies that correlation functions of these operators are computed at the boundary, *i.e.* in the UV.

[51] Note that the cosmological constant follows from the dimensional reduction of the ten-dimensional supergravity theory along the compact space; it is given by the Ricci scalar of the

closed strings propagating in AdS, the string states satisfy AdS boundary conditions rather than the usual flat space ones. These boundary conditions are held fixed in the string path integral and therefore the latter is a functional of the boundary configurations of the fields. In this way, we write the string partition function as:

$$\mathcal{Z}_{\text{string}} = \mathcal{Z}_{\text{string}}\Big[\phi|_{\partial AdS} = \phi_0\Big] \ , \tag{1.91}$$

where ϕ is a generic target space field with configuration ϕ_0 at the conformal boundary. With the string partition function we compute correlation functions of string states (or their scattering amplitudes) by inserting the vertex operators of these states and evaluating the string path integral. Equivalently, we can compute such correlators by working with the partition function of supergravity with the same boundary conditions plus the α'' corrections that render the theory finite. Working at leading order in g_s in the string path integral is then equivalent to approximating the supergravity partition function by a saddle point (note that g_s behaves as $\hbar$ in the supergravity partition function) such that:

$$\mathcal{Z}_{\text{string}}\Big[\phi|_{\partial AdS} = \phi_0\Big] \sim \mathcal{Z}_{\text{sugra}(g_s,\,\alpha'')}\Big[\phi|_{\partial AdS} = \phi_0\Big] = \exp\Big(iI_{\text{sugra}(\alpha'')}\Big[\phi|_{\partial AdS} = \phi_0\Big]\Big) + \ldots$$

where $I_{\text{sugra}(\alpha'')}$ is the on-shell supergravity action with all α'' corrections and the ellipsis denote quantum corrections in g_s. We can then suppress all stringy correction in the saddle point approximation by working in the limit $\alpha'' \to 0$ to obtain our first result:

$$\mathcal{Z}_{\text{string}}\Big[\phi|_{\partial AdS} = \phi_0\Big] \quad \sim \quad \exp\Big(iI_{\text{sugra}}\Big[\phi|_{\partial AdS} = \phi_0\Big]\Big) \ . \tag{1.92}$$

1.3.1. Scalar field in AdS

In order to understand how to use (1.92) to define the generating functional of a field theory at the boundary, we particularise the formalism and evaluate the right-hand side of the equation by considering a specific case. The simplest example one can consider is a scalar field in AdS with a possible mass term while simultaneously neglecting any backreaction on the geometry by working in the probe approximation. In the case of $\mathcal{N}=8$ supergravity dual to the supercurrent multiplet of $\mathcal{N}=4$ SYM, this scalar can be any of the scalar modes in Table 1.1 and respective mass term, with the remaining supergravity modes switched off (*i.e.* with vanishing configurations). The (truncated) supergravity action in (d+1)-dimensional AdS space is then given in our case by:

$$S = \frac{1}{2\eta}\int d^{d+1}x\sqrt{G}\left(G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right) \ , \tag{1.93}$$

compact space and generically by the background value of the field strength of the $(p+1)$-form that sources the geometry.

where η is a normalization constant that depends on the parent ten-dimensional supergravity theory. In order to avoid many subtleties that arise in the case of a Lorentzian signature, we will work with Euclidean AdS, also known as the hyperbolic space. We should note that, while the Poincaré coordinate system that we have been using parametrises only half of AdS, the Wick rotation of this coordinate system results in a parametrisation – the upper half plane parametrisation – that covers the entire hyperbolic space. This can be seen by performing a Wick rotation of AdS in global coordinates and verifying that the upper half plane parametrisation covers the Wick rotated space. The main advantage of working with this parametrisation lies in the fact that the conformal transformations that form the isometry group of AdS are manifest. In this coordinate system, (Euclidean) AdS_{d+1} takes the form:

$$ds^2_{d+1} = \frac{\ell^2}{z^2}\left(dz^2 + d\vec{x}^{\,2}_d\right) \ . \tag{1.94}$$

Here we have reinstated the AdS radius ℓ that was introduced in section 1.2.3 for dimensional reasons. The equation of motion for the scalar field in this background is given by:

$$\partial_z^2\phi - \frac{d-1}{z}\,\partial_z\phi + \vec{\nabla}^{\,2}\phi \quad = \quad \frac{\ell^2 m^2}{z^2}\,\phi \ , \tag{1.95}$$

where $\vec{\nabla}^{\,2} = \delta^{ij}\partial_i\partial_j$. The equation is second order in z and therefore admits two linearly independent solutions. These can be obtained by working with the Fourier transform $\hat{\phi}(z,p) = (2\pi)^{-d/2}\int d^dx\, e^{-ip\cdot x}\phi(z,x)$ and are given by:

$$\hat{\phi}(z,p) \quad = \quad z^{d/2}\Big(\hat{A}(p)\,K_{k/2}(z|p|) + \hat{B}(p)\,I_{k/2}(z|p|)\Big) \ , \tag{1.96}$$

where $K_\alpha(x)$ and $I_\alpha(x)$ are the modified Bessel functions, $|p| = \sqrt{\vec{p}^{\,2}}$, $k = 2\Delta - d$ and Δ is the largest root of the polynomial:

$$\Delta(\Delta - d) = m^2\ell^2 \ , \tag{1.97}$$

and which is equation (1.86) for scalars in the case $\ell = 1$, $d = 4$. Also, $\hat{A}(p)$ and $\hat{B}(p)$ are arbitrary functions of the momenta. For generic values of these coefficients, the solution is not well-behaved in the interior of AdS. If we analyse the asymptotics of the Bessel functions as $z \to \infty$ we find that the solution diverges unless $\hat{B}(p) = 0$. In this way, the solution that is regular in the interior is given by:

$$\hat{\phi}(z,p) \quad = \quad \hat{A}(p)\,z^{d/2}K_{k/2}(z|p|) \ . \tag{1.98}$$

We can Fourier transform this solution back to position space by using the identity:

$$\frac{2^{\Delta-1}\Gamma(\Delta)}{(2\pi)^{d/2}}\int d^dy\,\frac{\epsilon^\Delta}{\left(\epsilon^2+\vec{y}^{\,2}\right)^\Delta}\,e^{ip\cdot y} = \quad \epsilon^{d/2}|p|^{k/2}K_{k/2}(\epsilon|p|) \quad : \quad k = 2\Delta - d \ , \ \forall\epsilon \ . \tag{1.99}$$

We obtain:

$$\phi(z,x) \;=\; \frac{\Gamma(\Delta)}{\pi^{d/2}\Gamma(\Delta-d/2)}\int d^dy\,\varphi_{(0)}(y)\,\frac{z^\Delta}{\left(z^2+|\vec{x}-\vec{y}|^2\right)^\Delta}\;, \qquad (1.100)$$

where $\varphi_{(0)}$ is an arbitrary function of y given by the Fourier transform of a term proportional to $\hat{A}(p)$. If we now use the asymptotics of the integrand given by [21, 76]:

$$\lim_{z\to 0}\,\frac{\Gamma(\Delta)}{\pi^{d/2}\Gamma(\Delta-d/2)}\,\frac{z^\Delta}{\left(z^2+|\vec{x}-\vec{y}|^2\right)^\Delta} \;=\; z^{d-\Delta}\left(\delta^d(\vec{x}-\vec{y})+\mathcal{O}(z^2)\right)\;, \qquad (1.101)$$

we obtain:

$$\phi(z,x) \;=\; z^{d-\Delta}\Big(\varphi_{(0)}(x)+\mathcal{O}(z^2)\Big) \;=\; \phi_0+\mathcal{O}(z^{d-\Delta+2})\;, \qquad (1.102)$$

where $\phi_0 = \phi|_{\partial AdS}$ is the configuration of the field at the conformal boundary as introduced in the previous section. For generic values of Δ, or m^2, this boundary configuration is singular at $z=0$. We therefore introduce the renormalized boundary value:

$$\varphi_{(0)} \;=\; \lim_{z\to 0}\, z^{\Delta-d}\phi(z,x)\;. \qquad (1.103)$$

The integrand in (1.100) is called the bulk-to-boundary propagator, it is a Green's function in AdS and results in a regular bulk solution for each boundary configuration $\varphi_{(0)}$.

We are now in position to evaluate the action (1.93) on-shell as required in the approximation (1.92). In order to do so, we start by introducing a regulating boundary $z=\varepsilon$ in our space. If we then integrate the first term in the action by parts and use the equation of motion $\Box\phi = m^2\phi$, we obtain:[52]

$$I = \frac{1}{2\eta}\,(\ell/\varepsilon)^{d-1}\int\limits_{z=\varepsilon} d^dx\,\phi\partial_z\phi\;, \qquad (1.104)$$

where the integration by parts resulted in a surface term because our space is a manifold-with-boundary. We can now use the solution (1.100) together with the asymptotic behaviour (1.101) to find:

$$I_{\text{sugra}}\Big[\phi\big|^{ren.}_{\partial AdS}=\varphi_{(0)}\Big] \;=\; \frac{\ell^{d-1}}{2\eta\,\pi^{d/2}}\,\frac{\Gamma(\Delta+1)}{\Gamma(\Delta-d/2)}\int\limits_{z=\varepsilon} d^dxd^dy\,\frac{\varphi_{(0)}(x)\,\varphi_{(0)}(y)}{\left(\varepsilon^2+|\vec{x}-\vec{y}|^2\right)^\Delta}\,\Big(1+\mathcal{O}(\varepsilon^2)\Big)\;. \qquad (1.105)$$

[52] Our convention for the integral in z is as follows:

$$S = 1/(2\eta)\int\limits_{\infty}^{\varepsilon} dz\int d^dx\sqrt{G}\,(...)\;.$$

With the limit $\varepsilon \to 0$ the regulating boundary approaches the true conformal boundary of the space. In this limit we see explicitly that the on-shell action is a functional of the boundary configuration of the field. We also find that the partition function (1.92) in this limit behaves as a generating functional of conformal field theory correlation functions, with the renormalized boundary configuration $\varphi_{(0)}$ playing the role of a source for a conformal scalar operator $\mathcal{O}_\Delta$ of dimension Δ. Indeed, if we define [21, 22]:

$$\left\langle e^{\int d^d x\, \varphi_{(0)} \mathcal{O}_\Delta} \right\rangle_{CFT} \quad := \quad \mathcal{Z}_{\text{string}}\left[\phi\big|^{ren.}_{\partial AdS} = \varphi_{(0)} \right] , \tag{1.106}$$

and use the approximation (1.92), we obtain:

$$\begin{aligned} \langle \mathcal{O}_\Delta(x) \mathcal{O}_\Delta(y) \rangle \; &= \; - \frac{\delta^2 \log \left\langle e^{\int \, d^d x\, \varphi_{(0)} \mathcal{O}_\Delta} \right\rangle}{\delta\varphi_{(0)}(x)\, \delta\varphi_{(0)}(y)} \bigg|_{\varphi_{(0)}=0} \; = \; - \frac{\delta^2 \log \mathcal{Z}_{\text{string}}}{\delta\varphi_{(0)}\, \delta\varphi_{(0)}} \bigg|_{\varphi_{(0)}=0} \\ &\sim \; \frac{\delta^2 I_{\text{sugra}}}{\delta\varphi_{(0)}\, \delta\varphi_{(0)}} \bigg|_{\varphi_{(0)}=0} \; = \; \frac{\ell^{d-1}}{2\eta\, \pi^{d/2}} \, \frac{\Gamma(\Delta+1)}{\Gamma(\Delta - d/2)} \, \frac{1}{|\vec{x}-\vec{y}|^{2\Delta}} \qquad (\varepsilon \to 0) \; . \end{aligned} \tag{1.107}$$

Note that we are taking variations with respect to the generator $-\log \mathcal{Z}$ of connected correlation functions. The result is exactly the 2-point correlator of a conformal operator of dimension Δ as dictated by conformal symmetry, as we review next. In this specific model, higher-order correlators vanish. To compute n-point correlation functions we need to consider n-point vertices and add to the Lagrangian (1.93) an interaction term proportional to ϕ^n. If we repeat the computation with such interactions, as well as for fields of different spin, we find that all observables derived in this way satisfy the axioms of a conformal field theory. Therefore, the definition (1.106) gives a prescription for computing conformal field theory observables from string theory. At the lowest order approximation in g_s and α'' it implies that the effective action of the conformal field theory that we define from string theory in AdS is the on-shell action of the corresponding supergravity theory. From the AdS/CFT conjecture [20] we then know that the latter approximation corresponds to the large N, large λ regime of the field theory. Note that, a priori, the definition (1.106) does not say anything about the coupling constants on each side of the duality. In fact, any quantum theory of gravity with AdS Einstein gravity as a low energy limit defines in this way a CFT at the boundary of AdS. However, in order to derive equivalences between specific theories such as the case of $\mathcal{N}=4$ SYM and $AdS_5 \times S^5$ supergravity one needs string theory. Ultimately, the AdS/CFT prescription (1.106) is justified by the fact that the AdS boundary configurations of the string states behave as sources for boundary correlators that satisfy the right properties that define a conformal field theory.

1.3.2. Conformal correlators and Ward identities

In this section we review very briefly the implications of conformal invariance to the correlation functions of a conformal field theory in flat space. Reviews of this topic can be found in [68, 27, 69] and references therein.

Correlators

Lorentz invariance implies that only scalar operators can have non-zero vevs. Together with scale invariance (1.73), this fixes the one-point functions to be of the form:

$$\langle \mathcal{O}_\Delta \rangle = \frac{C_\Delta}{|\vec{x}|^\Delta} \,, \tag{1.108}$$

where Δ is the conformal dimension and C_Δ a constant. However, invariance under special conformal transformations (1.69) fixes $C_\Delta \propto \delta_{\Delta,0}$. If the field theory is unitary, the unitarity bounds discussed in section 1.2.4 on the conformal dimensions of the operators imply in particular that $\Delta > 0$ and therefore the one-point function must vanish in unitary CFTs.

In the case of two-point functions, Poincaré and scale invariance implies that $\langle \mathcal{O}_{\Delta_1} \mathcal{O}_{\Delta_2} \rangle = C_{\Delta_1,\Delta_2}/|\vec{x}-\vec{y}|^{\Delta_1+\Delta_2}$. Invariance under special conformal transformations implies $\Delta_1 = \Delta_2$ and therefore fixes the correlator up to a Δ_1-dependent constant:

$$\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \rangle \;\; = \;\; \frac{C_{\Delta_1}}{|\vec{x}_1 - \vec{x}_2|^{2\Delta_1}} \, \delta_{\Delta_1,\Delta_2} \,. \tag{1.109}$$

Note that the correlation function (1.107) is exactly of this form. Finally, the form of three-point functions is fixed in a similar fashion by conformal symmetry to be of the form:

$$\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \mathcal{O}_{\Delta_3}(x_3) \rangle \; = \; \frac{C_{\Delta_1,\Delta_2,\Delta_3}}{|x_1 - x_2|^{\Delta - 2\Delta_3} |x_3 - x_1|^{\Delta - 2\Delta_2} |x_2 - x_3|^{\Delta - 2\Delta_1}} \,, \tag{1.110}$$

where: $\Delta = \Delta_1 + \Delta_2 + \Delta_3$. Higher-point correlators are also fixed by conformal symmetry, this time up to a proportionality function of anharmonic ratios [69].

Renormalized correlators

The correlation functions given above are singular at coincident points and therefore do not admit Fourier transforms and are not well-defined as distributions. Well-behaved expressions can be obtained by regularizing the correlators in position space, transforming to momentum space and then introducing counterterms that cancel the potential divergent terms before removing the regulator [77, 68]. One can then transform the renormalized expressions back to position space. This implies that the correlators given above are effective rather than exact: they coincide with the renormalized correlators in position space asymptotically away from coincident points, but are ill-defined at short distances. In the case of the two-point function, the renormalized expression can be obtained as follows. We start by ϵ-regularizing the short-distance divergences in the correlator and Fourier transform it by recalling the identity (1.99):

$$\int d^d y \, \frac{e^{ip\cdot y}}{\left(\epsilon^2 + \vec{y}^{\,2}\right)^{\Delta}} \;=\; \left(\frac{(2\pi)^{d/2}}{2^{\Delta-1}\Gamma(\Delta)}\right) \epsilon^{-k/2} |p|^{k/2} K_{k/2}(\epsilon|p|) \;, \quad k = 2\Delta - d \;. \tag{1.111}$$

The Bessel function admits the following asymptotics as $x \to 0$:

$$\begin{aligned} K_{k/2}(x) \;=\; & 2^{k/2-1}\,\Gamma(k/2)\,x^{-k/2}\Bigg(1 + \frac{(ix)^2}{2(k-2)} + \frac{(ix)^4}{2(k-2)4(k-4)} \\ & + \ldots + a_k\,x^k + \tilde{a}_k\,x^k \log x^2 + \mathcal{O}(x^{>k})\Bigg) \;, \end{aligned} \tag{1.112}$$

where the coefficient $\tilde{a}_k$ of the inhomogeneous term is non-vanishing only if $k/2 \in \mathbb{N}$ and in such case is given by:

$$\tilde{a}_k \;=\; -\frac{(-1)^{k/2}\;2^{-k}}{\Gamma(1+k/2)\Gamma(k/2)} \quad : \quad k/2 \in \mathbb{N} \;. \tag{1.113}$$

The coefficient a_k is in general non-vanishing, but its expression is only relevant when $k/2 \notin \mathbb{N}$ because otherwise it can be absorbed into the argument of the logarithm of the inhomogeneous term. Therefore, in such case it is given by:

$$a_k \;=\; 2^{-k}\,\frac{\Gamma(-k/2)}{\Gamma(k/2)} \quad : \quad k/2 \notin \mathbb{N} \;. \tag{1.114}$$

If we replace x by $\epsilon|p|$, we find that all terms of order below ϵ^0 in the expansion of $\epsilon^{-k/2}K_{k/2}(\epsilon|p|)$ diverge as the regulator $\epsilon \to 0$. We then introduce the corresponding counterterms that minimally subtract these divergences to obtain the renormalized expression:[53]

$$\left[\epsilon^{-k/2}|p|^{k/2}K_{k/2}(\epsilon|p|)\right]_{ren.} = \begin{cases} 2^{k/2-1}\,\Gamma(k/2)\,\tilde{a}_k\,|p|^k \log(|p|^2/\mu^2) + \mathcal{O}(\epsilon^{>0}) \;, & (k/2 \in \mathbb{N}) \\ 2^{k/2-1}\,\Gamma(k/2)\,a_k\,|p|^k + \mathcal{O}(\epsilon^{>0}) \;, & (k/2 \notin \mathbb{N}) \end{cases}$$

[53] For consistency, these counterterms should be added to the field theory Lagrangian in the generating functional used to compute the correlators. From the expansion (3.182) with x replaced by $\epsilon|p|$ we find that the divergent terms in the correlator (1.111) are proportional to powers of the momentum. In position space, these divergences are therefore derivatives of delta

where the scale μ in the first case was introduced for dimensional reasons and the coefficient a_k was absorbed into μ. In this way, we find:

$$\lim_{\epsilon\to 0}\left[\int d^d y\,\frac{e^{ip\cdot y}}{\left(\epsilon^2+\vec{y}^2\right)^{\Delta}}\right]_{ren.} = \begin{cases} \tilde{c}_k\,|p|^k\log(|p|^2/\mu^2)\ , & (k/2\in\mathbb{N}) \\ c_k\,|p|^k\ , & (k/2\notin\mathbb{N}) \end{cases} \tag{1.115}$$

where:

$$\begin{cases} \tilde{c}_k &= (-1)^{1+\frac{k}{2}}\,\dfrac{2^{-k}\,\pi^{d/2}}{\Gamma(\frac{k+d}{2})\,\Gamma(1+\frac{k}{2})}\ , \\ c_k &= \dfrac{\pi^{d/2}}{2^k}\,\dfrac{\Gamma(-\frac{k}{2})}{\Gamma(\frac{k+d}{2})}\ . \end{cases} \tag{1.116}$$

Note in particular that, due to the singularities of the gamma function, the coefficient c_k cannot be extended to integer values of $k/2$ and which is the regime of validity of the other branch. Finally, we can obtain the renormalized correlator in position space:

$$\left[\frac{1}{|\vec{x}|^{2\Delta}}\right]_{ren.} = \begin{cases} \dfrac{\tilde{c}_k}{(2\pi)^d}\displaystyle\int d^d p\, e^{-ip\cdot x}|p|^k\log(|p|^2/\mu^2)\ , & (k/2\in\mathbb{N}) \\ \dfrac{c_k}{(2\pi)^d}\displaystyle\int d^d p\, e^{-ip\cdot x}|p|^k\ , & (k/2\notin\mathbb{N})\ . \end{cases} \tag{1.117}$$

Ward identities

Ward identities are the quantum version of Noether's theorem and represent the conservation laws associated with the invariance of the generating functional under the classical symmetry group. This group is usually broken by quantum corrections because renormalization in general does not preserve part of the symmetries, so the renormalized generating functional should be invariant up to possible anomalies such as trace/Weyl/conformal anomalies or diffeomorphism anomalies. Since one is often interested in the expression of the correlators in the presence of the sources for the operators, the Ward identities are usually given with non-vanishing sources,

functions. If we add to the field theory Lagrangian counterterms of the form:

$$\mathcal{L}_0+\int d^d x\,\varphi_{(0)}\mathcal{O}\ \longrightarrow\ \mathcal{L}_0+\int d^d x\,\varphi_{(0)}\mathcal{O}+c_n\int d^d x\,\varphi_{(0)}\Box^n\varphi_{(0)}\ ,$$

and take variations of the generating functional with respect to the source $\varphi_{(0)}$ to derive the correlation functions, the counterterms will result in derivatives of delta functions that cancel the divergences in the correlators. Note that in the case $k/2\in\mathbb{N}$ the inhomogeneous term proportional to the logarithm in the expansion (3.182) is non-vanishing and will result in a divergence proportional to $\log\epsilon$. The corresponding counterterm is proportional to $\log(\epsilon\mu)$, with μ some energy scale, and therefore breaks scale invariance of the generating functional .

but which may in turn be switched off. Let us write the generating functional of the field theory as:

$$\mathcal{Z}[g, A, \varphi_{(0)}] \;=\; \left\langle \exp\left(-\int d^dx\,\sqrt{g}\left[A_\mu J^\mu + \varphi_{(0)}\mathcal{O}_\Delta\right]\right)\right\rangle_{CFT} , \quad (1.118)$$

where the metric $g_{\mu\nu}(x)$, the gauge field $A_\mu(x)$ and the scalar $\varphi_{(0)}(x)$ are respectively the sources for the stress-energy operator $T_{\mu\nu}$, a possible symmetry current J^μ and the conformal scalar operator $\mathcal{O}_\Delta$ such that:

$$\langle T_{\mu\nu}\rangle \;:=\; -\frac{2}{\sqrt{g}}\frac{\delta W}{\delta g^{\mu\nu}} , \quad (1.119)$$

$$\langle J^\mu\rangle \;:=\; -\frac{1}{\sqrt{g}}\frac{\delta W}{\delta A_\mu} , \quad (1.120)$$

$$\langle \mathcal{O}_\Delta\rangle \;:=\; -\frac{1}{\sqrt{g}}\frac{\delta W}{\delta \varphi_{(0)}} . \quad (1.121)$$

where $W = \log\mathcal{Z}$. We will assume that the current has vanishing conformal dimension, whereas $\mathcal{O}_\Delta$ has dimension Δ. Unlike the example in appendix A.4, the source $\varphi_{(0)}(x)$ is now an arbitrary function, so it can transform as $\varphi_{(0)}(x) \to \Omega^{-\alpha}\varphi_{(0)}(x)$ under a conformal-Weyl transformation (A.10)–(A.11) such that:

$$\begin{aligned}\int d^dx\sqrt{g}\,\varphi_{(0)}(x)\,\mathcal{O}_\Delta(x) &\overset{\text{conf.}}{\longrightarrow} \int d^d\bar{x}\sqrt{\bar{g}}\,\bar{\varphi}_{(0)}(\bar{x})\,\bar{\mathcal{O}}_\Delta(\bar{x}) \;=\; \int d^dx\sqrt{g}\,\varphi_{(0)}(x)\,\mathcal{O}_\Delta(x)\\ &\overset{\text{Weyl}}{\longrightarrow} \int d^dx\sqrt{g}\,\Omega^{d-\Delta-\alpha}\,\varphi_{(0)}(x)\,\mathcal{O}_\Delta(x) , \end{aligned}\quad (1.122)$$

where the first identity holds because the integrand is a Lorentz-scalar. If the unrenormalized generating functional is conformally invariant we find that $\alpha = d-\Delta$. If we then consider an infinitesimal Weyl transformation $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$ such that $\Omega^2 = e^{2\sigma} = 1 + 2\sigma + \mathcal{O}(\sigma^2)$, the generating functional transforms as:

$$\begin{aligned} W \to W + \delta_\sigma W \;=&\; W + \int d^dx\left(\frac{\delta W}{\delta g_{\mu\nu}}\,\delta_\sigma g_{\mu\nu} + \frac{\delta W}{\delta\varphi_{(0)}}\,\delta_\sigma\varphi_{(0)}\right)\\ =&\; W + \int d^dx\,\sigma\left(\frac{\delta W}{\delta g_{\mu\nu}}\,2g_{\mu\nu} - \frac{\delta W}{\delta\varphi_{(0)}}\,(d-\Delta)\varphi_{(0)}\right)\\ =&\; W + \int d^dx\sqrt{g}\,\sigma\left(\langle T^\mu_{\ \mu}\rangle + \langle\mathcal{O}_\Delta\rangle(d-\Delta)\varphi_{(0)}\right) . \end{aligned}\quad (1.123)$$

Weyl invariance of the renormalized generating functional for arbitrary $\sigma(x)$ can then be expressed as:

$$\langle T^\mu_{\ \mu}\rangle \;=\; -(d-\Delta)\varphi_{(0)}\langle\mathcal{O}_\Delta\rangle + \mathcal{A} , \quad (1.124)$$

and which is the trace Ward identity. The possible quantum anomaly $\mathcal{A}$ is called the Weyl anomaly. The same procedure can be repeated for the case of general diffeomorphisms. As derived in appendix A.3, under an infinitesimal diffeomorphism (A.13) generated by a vector field ξ^μ, the metric transforms at leading order as:

$$g^{\mu\nu}(x) \to (\varphi_\star g)^{\mu\nu}(\bar{x}) = g^{\mu\nu}(\bar{x}) - (\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu) = g^{\mu\nu}(\bar{x}) + \delta g^{\mu\nu}(\bar{x}) \ . \tag{1.125}$$

In addition, the scalar source and the gauge field transform at leading order as (see equation (A.4)):

$$\varphi_{(0)}(x) \to \ (\varphi_\star \varphi_{(0)})(\bar{x}) = \varphi_{(0)}(\bar{x}) - \xi \cdot \nabla \varphi_{(0)} = \varphi_{(0)}(\bar{x}) + \delta\varphi_{(0)}(\bar{x}) \ , \tag{1.126}$$

$$A_\mu(x) \to \ (\varphi_\star A)_\mu(\bar{x}) = A_\mu(\bar{x}) - (\xi \cdot \nabla A_\mu + A_\alpha \nabla_\mu \xi^\alpha) = A_\mu(\bar{x}) + \delta A_\mu(\bar{x}) \ . \tag{1.127}$$

The generating functional in this case transforms at leading order in ξ as:

$$\begin{aligned} W \to W + \delta_\xi W &= W + \int d^d\bar{x} \left(\frac{\delta W}{\delta g^{\mu\nu}}\, \delta_\xi g^{\mu\nu} + \frac{\delta W}{\delta A_\mu}\, \delta_\xi A_\mu + \frac{\delta W}{\delta \varphi_{(0)}}\, \delta_\xi \varphi_{(0)} \right) \\ &= W + \int d^d x \sqrt{g} \left(\langle T_{\mu\nu} \rangle \nabla^\mu \xi^\nu + \langle J^\mu \rangle \left(\xi \cdot \nabla A_\mu + A_\alpha \nabla_\mu \xi^\alpha \right) + \langle \mathcal{O}_\Delta \rangle \, \xi \cdot \nabla \varphi_{(0)} \right) \\ &= W - \int d^d x \sqrt{g}\, \xi^\nu \left(\nabla^\mu \langle T_{\mu\nu} \rangle + \langle J^\mu \rangle F_{\mu\nu} + A_\nu \nabla_\mu \langle J^\mu \rangle - \langle \mathcal{O}_\Delta \rangle \nabla_\nu \varphi_{(0)} \right) \ . \end{aligned} \tag{1.128}$$

Diffeomorphism invariance of the renormalized generating functional for arbitrary ξ can then be expressed as:

$$\nabla^\mu \langle T_{\mu\nu} \rangle + \langle J^\mu \rangle F_{\mu\nu} + A_\nu \nabla_\mu \langle J^\mu \rangle - \langle \mathcal{O}_\Delta \rangle \nabla_\nu \varphi_{(0)} \quad = \quad \mathcal{A}_\nu \ , \tag{1.129}$$

and which is the diffeomorphism Ward identity. The possible quantum anomaly $\mathcal{A}_\nu$ is called the diffeomorphism anomaly. This identity can be simplified further by using the Ward identity that arises in a similar fashion from invariance of the generating functional under gauge transformations $A_\mu \to A_\mu + D_\mu \phi$:

$$\nabla_\mu \langle J^\mu \rangle \quad = \quad 0 \ . \tag{1.130}$$

1.3.3. *AdS* renormalization of the scalar field

In the previous section we showed that the form of the correlation functions as directly fixed by conformal symmetry suffers from short-distance singularities and therefore cannot possibly represent the correlators of operators at coincident points, such as those SYM operators in Table 1.1 that we discussed before. We then showed that a derivation of the correlators with the required properties entails a regularization and renormalization of the CFT generating functional. The two-point function (1.107) that we derived by holographic methods matches the effective correlator and suffers in a similar way from short-distance divergences. Just as the left-hand side of the AdS/CFT prescription (1.106) should be improved with counterterms that render the correlation functions well-behaved, the gravity side of the correspondence should undergo a similar renormalization programme called holographic renormalization. In this case, the short-distance divergences in the holographic correlators are associated with (infra-red) divergences of the on-shell action due to the infinite area of the conformal boundary of AdS. Note that, even though we haven't yet discussed them, these divergences are indeed there: in the specific example of the previous section, if we use the asymptotics (1.101) in the on-shell action (1.105) we find that the latter behaves asymptotically as:

$$I \quad \sim \quad \int\limits_{z=\varepsilon} \varepsilon^{-k} \left(1 + \mathcal{O}(\varepsilon^2)\right) \ , \tag{1.131}$$

with ε the regulator that we introduced before. The correlation functions with the required properties at short distances are obtained by ensuring that the gravity action from which the correlators are extracted is free of these infra-red divergences. The procedure entails improving the regulated action with boundary terms called holographic counterterms that cancel the divergent terms. Since these counterterms are surface terms, the bulk equations of motion are not affected by the procedure. In this section we will exemplify the basics of the holographic renormalization programme by deriving the renormalized correlator (1.117) that we obtained from a quantum field theory computation. Further aspects on this topic can be found in the lecture notes [78] and references therein.

Let us return to the action (1.93) of the scalar field in AdS_{d+1} and the resulting equation of motion (1.95). Instead of finding an exact solution to the equation, let us rather solve it asymptotically as $z \to 0$. We start by defining:

$$\varphi(z,x) \quad = \quad z^{\Delta-d}\phi(z,x) \ , \tag{1.132}$$

with Δ the largest root of the polynomial (1.97) as before. In terms of φ, the equation of motion becomes:

$$\varphi'' - \frac{k-1}{z}\,\varphi' + \vec{\nabla}^2\varphi \quad = \quad 0 \ , \qquad k = 2\Delta - d \ , \ \varphi' = \partial_z\varphi \ . \tag{1.133}$$

If we solve asymptotically this equation in powers of z we obtain the solution:

$$\begin{aligned}\phi(z,x) = \ & z^{d-\Delta}\varphi(z,x) \\ = \ & z^{d-\Delta}\left(\varphi_{(0)}(x) + z^2\varphi_{(2)}(x) + ... + z^k\varphi_{(k)}(x) + z^k \log z\, \tilde{\varphi}_{(k)}(x) + ...\right) \ ,\end{aligned} \tag{1.134}$$

where the coefficients $\varphi_{(0)}$ and $\varphi_{(k)}$, called respectively the non-normalisable and the normalisable modes, are arbitrary functions of x. Note that $\varphi_{(0)}$ is the source term, or renormalized boundary configuration, that we identified in (1.103). The remaining coefficients $\varphi_{(n<k)}$ are local functionals of the source:

$$\varphi_{(n<k)} = \frac{1}{n(k-n)} \vec{\nabla}^{2} \varphi_{(n-2)} \ , \tag{1.135}$$

while the coefficient $\tilde{\varphi}_{(k)}$ of the inhomogeneous term is non-vanishing only if $k/2 \in \mathbb{N}$ and in that case is also a local functional of the source:

$$\tilde{\varphi}_{(k)} = -\frac{2^{-k+1}}{\Gamma(1+k/2)\Gamma(k/2)} \vec{\nabla}^{k} \varphi_{(0)} \quad : \quad k/2 \in \mathbb{N} \ . \tag{1.136}$$

Given this solution, we start by inspecting the regulated on-shell action (1.104):

$$\begin{aligned} I = & \ \frac{1}{2\eta} (\ell/\varepsilon)^{d-1} \int\limits_{z=\varepsilon} d^d x \left[\phi \partial_z \phi\right]_{\text{on-shell}} \\ = & \ \frac{\ell^{d-1}}{2\eta} \int\limits_{z=\varepsilon} d^d x \, \varepsilon^{-k} \Big((d-\Delta)\varphi^2 + \varepsilon\, \varphi\, \varphi'\Big) \\ = & \ \frac{\ell^{d-1}}{2\eta} \int\limits_{z=\varepsilon} d^d x \Bigg[(d-\Delta)\, \varepsilon^{-k} \Big(\varphi_{(0)}^2 + 2\varepsilon^2 \varphi_{(0)}\varphi_{(2)} + ... + 2\varepsilon^k \log\varepsilon\, \varphi_{(0)}\tilde{\varphi}_{(k)} + \mathcal{O}(\varepsilon^k)\Big) \\ & + \ \varepsilon^{-k} \Big(\varepsilon\varphi_{(0)}^2 + 3\varepsilon^3 \varphi_{(0)}\varphi_{(2)} + ... + k\varepsilon^k \log\varepsilon\, \varphi_{(0)}\tilde{\varphi}_{(k)} + \mathcal{O}(\varepsilon^k)\Big)\Bigg] \ . \end{aligned} \tag{1.137}$$

All terms proportional to $\varepsilon^{<0}$ will be divergent as the regulator $\varepsilon \to 0$ and therefore should be subtracted. Note that such terms do not involve coefficients $\varphi_{(n\geq k)}$. Since each coefficient (1.135) and (1.136) is a local functional of the source $\varphi_{(0)}$, the divergences can be removed by covariant boundary counterterms (up to a possible anomalous term):

$$S^{ct} := \frac{\ell^{-1}}{2\eta} \int\limits_{z=\varepsilon} d^d x \sqrt{\gamma}\, \phi \left(-(d-\Delta) - \frac{\ell^2\,\Box_\gamma}{k-2} - \frac{\left(\ell^2\,\Box_\gamma\right)^2}{(k-2)^2(k-4)} + ... + \alpha_k \left(\ell^2\,\Box_\gamma\right)^{k/2} \log\varepsilon\right) \phi \tag{1.138}$$

where $\gamma_{ij} = (\ell/\varepsilon)^2 \delta_{ij}$ is the induced metric on the regulating boundary. The coefficient α_k is non-vanishing only if $k/2 \in \mathbb{N}$ and in such case is given by: $\alpha_k = 2^{-(k-2)}\Gamma^{-2}(k/2)$. It can be checked by explicitly expanding the counterterms that these cancel exactly the divergences of the action. In this way, we define the renormalized action:

$$S^{ren} \ := \ S + S^{ct} \ , \tag{1.139}$$

where S is the covariant action (1.93). Note that the renormalized action depends explicitly on the regulator ε if $k/2 \in \mathbb{N}$. This dependence partly breaks invariance of the action under diffeomorphisms that involve the radial direction z, more specifically those bulk isometries that result in a conformal transformation at the boundary. This is the holographic counterpart of the breaking of symmetries by the renormalization procedure on the quantum field theory side and will result in quantum anomalies. As we will find in the late sections, if we take into account the backreaction of the field on the geometry, the anomalous term in the gravity action will be responsible for the Weyl anomaly $\mathcal{A}$ that we discussed in the previous section. Since the counterterms depend on the value of k, or the value of the mass m^2, in order to proceed and compute the renormalized correlators we will focus on the specific case $k = 2$. The renormalized action in such case is given by:

$$\begin{aligned} S^{ren} &= \frac{1}{2\eta}\int d^{d+1}x\sqrt{G}\left(G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right) \\ &+ \frac{\ell^{-1}}{2\eta}\int\limits_{z=\varepsilon} d^dx\sqrt{\gamma}\,\phi\Big(-d/2+1+\ell^2\log\varepsilon\,\Box_\gamma\Big)\phi\,. \end{aligned} \tag{1.140}$$

We will start by computing the one-point function of the operator dual to the scalar field. We do this by following AdS/CFT prescription (1.106) and taking the variation of the action with respect to the boundary configuration $\varphi_{(0)}$ before evaluating the result on-shell:

$$\delta S^{ren}\Big|_{on-shell} = \frac{1}{\eta}\int\limits_{z=\varepsilon} d^dx\sqrt{\gamma}\left(\frac{\varepsilon}{\ell}\,\phi' - \frac{d/2-1}{\ell}\phi + \ell\log\varepsilon\,\Box_\gamma\phi\right)\delta\phi \tag{1.141}$$

If we introduce the renormalized configuration of the boundary metric:

$$g_{(0)ij} = \lim_{z\to 0}\left(\frac{z^2}{\ell^2}\,\gamma_{ij}\right)\,, \tag{1.142}$$

and which in our case is $g_{(0)ij} = \delta_{ij}$, the vacuum expectation value (1.121) is then given by:

$$\begin{aligned} \langle\mathcal{O}\rangle = -\frac{1}{\sqrt{g_{(0)}}}\frac{\delta S^{ren}}{\delta\varphi_{(0)}} &= \lim_{\varepsilon\to 0}\left(\frac{\ell^d}{\varepsilon^\Delta}\frac{1}{\sqrt{\gamma}}\frac{\delta S^{ren}}{\delta\phi}\right) = \lim_{\varepsilon\to 0}\frac{\ell^d}{\eta\,\varepsilon^k}\left(\frac{\varepsilon}{\ell}\,\varphi' + \frac{\varepsilon^2}{\ell}\log\varepsilon\,\vec{\nabla}^2\varphi\right) \\ &= \frac{\ell^{d-1}}{\eta}\left(2\varphi_{(k)} - \frac{1}{2}\,\vec{\nabla}^2\varphi_{(0)}\right)\,, \end{aligned} \tag{1.143}$$

where we used the solution (1.134). The last term is scheme dependent: it can be subtracted by adding to the action (1.140) a finite boundary term proportional to: $\int d^dx\sqrt{\gamma}\phi\Box_\gamma\phi$, and therefore can be ignored. This fact reflects the renormalization scheme dependence that is also presence on the quantum field theory side. In this way, the vacuum expectation value is proportional to the normalisable mode $\varphi_{(k)}$. The isometry $z \to \lambda z, x^i \to \lambda x^i$ of the metric (1.94) induces a scale-Weyl transformation at the boundary. From the solution (1.134) we find that the source transforms as: $\varphi_{(0)} \to \lambda^{-(d-\Delta)}\varphi_{(0)}$ under this transformation, whereas the normalisable mode transforms as a field of conformal dimension Δ:

$$\varphi_{(k)} \to \lambda^{-\Delta}\varphi_{(k)} \ . \tag{1.144}$$

This implies that the dual operator $\mathcal{O}$ has dimension Δ as expected. In order to compute the renormalized two-point correlator, we take a second variation with respect to the source:

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = -\frac{1}{\sqrt{g_{(0)}}}\frac{\delta^2 S^{ren}}{\delta\varphi_{(0)}\delta\varphi_{(0)}} = 2\frac{\ell^{d-1}}{\eta}\frac{\delta\varphi_{(k)}}{\delta\varphi_{(0)}} \ . \tag{1.145}$$

In order to compute the response of the normalisable mode under a variation of the source, we need to have an exact solution of the equations of motion, in particular a physical solution that is regular in the interior of AdS. In section 1.3.1 we found the regular solution (1.98) in momentum space. Using the asymptotics (3.182), we find that the normalisable mode is related to the source as:

$$\varphi_{(k)}(p) \quad = \quad \tilde{a}_k |p|^k \log\left(|p|^2/\mu^2\right) \varphi_{(0)}(p) \ , \tag{1.146}$$

where the scale μ was introduced for dimensional reasons and we absorbed the coefficient a_k in the definition of μ. The two-point function is then given by:

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = \frac{2\,\tilde{a}_k\,\ell^{d-1}}{\eta\,(2\pi)^{d/2}}\int d^d x\, e^{ip\cdot(x-y)}|p|^k \log\left(|p|^2/\mu^2\right) \ , \tag{1.147}$$

and which agrees with the renormalized correlator (1.117). The case $k/2 \notin \mathbb{N}$ can be dealt with in a similar fashion and the result will agree with the second branch of (1.117).

Chapter 2

Holographic chiral scale-invariant models

2.1. Introduction

In the previous chapter we discussed how certain classical gravity theories can be used to compute observables of quantum field theories that are both UV-conformal and strongly coupled. The latter are difficult to study with standard quantum field theory techniques because their strongly coupled nature prevents calculations based on perturbation theory. The methods described thus far therefore represent an important tool to study the non-perturbative sector of these theories.

The original AdS/CFT correspondence has been generalised over the years to a broader class of field theories other than those worldvolume theories of the brane solutions in string theory that we have been discussing. The proposed dualities follow the same ingredients of AdS/CFT and map in a similar fashion a classical theory of gravity – in manifolds that can have non-AdS asymptotics – to a strongly coupled gauge theory at the boundary of the space.[1] For this reason they are generically called gauge/gravity or holographic dualities. The field theories involved are typically of some phenomenological relevance and the dualities usually have potential applications as holographic models of condensed matter systems at strong coupling. In many-body physics, the state of a system is captured in the expectation values of its operators and its correlations in its

[1]Recall that this is the UV theory. The interpretation of the coordinate transverse to the boundary as an energy scale on the field theory is usually retained in those models built on AdS/CFT ideas.

Green's functions. The main purpose of the dualities in these cases is to extract some physics from the non-perturbative sector of the system using holographic methods to compute these observables.

Condensed matter systems such as Fermi liquids, insulating quantum magnets and high temperature superconductors typically undergo quantum phase transitions and are completely characterized by their correlation functions. In a neighbourhood of their quantum critical points (QCP), these systems are conformally invariant and therefore should admit a gauge/gravity description where the observables of the systems are computed holographically.[2] However, despite the success of AdS/CFT technology applied to these and many other systems, several scale invariant condensed matter theories remain that cannot be described by gauge/gravity dualities, in particular by relativistic ones. Generically, gauge/gravity dualities are developed to model relativistic systems only and their further generalization to the non-relativistic case is desirable, not only from a phenomenological viewpoint but also from a pure theoretical perspective. Strange metal phases of high temperature superconductors have underlying non-relativistic scale invariant QCPs. Systems of interacting fermions in three spatial dimensions that saturate the unitarity bound have been realised using trapped cold atoms and are non-relativistic in nature; these systems become scale invariant in the limit when the range of the interacting potential is zero. Several heavy fermion metals exhibit non-relativistic but scale invariant QCPs with an associated non-Fermi liquid phase. Strongly coupled non-relativistic systems are common place in condensed matter physics and as such there would be many interesting applications had one had under control holographic dualities involving non-relativistic quantum field theories.

Motivated by such applications, references [79, 80] initiated a discussion of holography for $(d+1)$ dimensional spacetimes with metric[3],

$$ds^2 = \frac{\sigma^2 du^2}{r^{2z}} + \frac{2dudv + dx^i dx^i + dr^2}{r^2}\,, \tag{2.1}$$

[2]Zero-point fluctuations of a condensed matter system are quantum fluctuations at zero temperature of its physical quantities (e.g. the energy at a point) around their expectation values, allowing the existence of different states of matter at such temperature. Transitions between such quantum states are driven by variations of physical parameters such as an external magnetic field or the system's pressure. A quantum critical point is the locus in the phase diagram of a material where a second order (i.e. continuous) phase transition occurs between two quantum states of matter and is characterised by an infinite susceptibility of the material to transit from its disordered phase to an ordered one. At such particular combination of temperature and physical parameters, the fluctuations are scale invariant and extend over the entire system.

[3]An earlier approach to the geometric realization of non-relativistic symmetries can be found in [81] and the connection between this approach and holographic realizations is discussed in [82].

with $i \in \{1, \ldots, d-2\}$. The isometries of this metric form include

$$\begin{aligned} &\mathcal{H} : u \to u + a, \\ &\mathcal{M} : v \to v + a, \\ &\mathcal{D} : r \to (1-a)r, \qquad u \to (1-a)^z u, \qquad v \to (1-a)^{2-z} v, \qquad x^i \to (1-a)x^i \end{aligned} \tag{2.2}$$

along with rotations, translations and Galilean boosts in the x^i directions. Here $\mathcal{D}$ is the generator of non-relativistic scale transformations with dynamical exponent z. In the case of $z = 2$ the isometry group becomes the Schrödinger group, which includes the additional special conformal symmetry

$$\mathcal{C} : r \to (1-au)r, \qquad u \to (1-au)u, \qquad v \to v + \frac{a}{2}(x^i x^i + r^2), \qquad x^i \to (1-au)x^i \tag{2.3}$$

Much of the interest in such holographic models has centered around this case of $z = 2$, following the initial suggestion that the metric (2.1) could play the role of a background for the holographic study of critical non-relativistic systems with $z = 2$ in $(d-1)$ spacetime dimensions, for example fermions at unitarity, which have the same symmetry group.

The spacetime (2.1) solves the equations of motion for gravity coupled to a massive vector field for all $z > 0$. Working in the limit where σ^2 is small and treated as a perturbation around AdS, the standard AdS/CFT dictionary shows that the dual field theory is a deformation of the conformal field theory by a vector operator. More specifically, the dual conformal field theory is deformed by a constant null source for a vector operator $\mathcal{V}_v$ of scaling dimension $(d + z - 1)$

$$S_{\rm cft} \to S_{\rm cft} + \int du dv d^{d-2}x b \mathcal{V}_v. \tag{2.4}$$

With respect to the relativistic scaling dimension, this deformation is relevant for $z < 1$, marginal for $z = 1$ and irrelevant for $z > 1$. However, the deformation is exactly marginal with respect to the non-relativistic scaling symmetry for any z and in this chapter we will explore holographic duality for these models. In the context of two dimensional conformal field theories, such deformations have been previously considered by Cardy [83] and the resulting models were called chiral scale-invariant models, a terminology which we will adopt here[4].

In the case of $z = 2$ the original goal was to model holographically a dual non-relativistic $(d-1)$ dimensional theory, in a background with coordinates (u, x^i) where u plays the role of time. In this setup one considers operators $\mathcal{O}_{\Delta_s,m}(u, x^i)$ of definite scaling dimension Δ_s and of charge m under the symmetry $\mathcal{M}$. This

[4]Whilst such theories are often called non-relativistic, or Schr(z), this terminology is arguably somewhat misleading; the theory only becomes non-relativistic after compactification on a null direction.

charge m, which corresponds to momentum in the v direction, would then have to be identified with a discrete quantum number such as particle number. In order to discretize the possible values of m one therefore needs to compactify the v direction in the holographic realization. This procedure is however very nontrivial as in general quantum corrections become important and one cannot trust the metric (2.1) with a compact null direction, see the discussions in [84]. (The problems in compactifying any field theory along a null direction are discussed in, for example, [85] and would in particular apply to the field theories considered here.) Recent work aiming at obtaining Schrödinger solutions without such a compact direction can be found in [86]. For general z and $\sigma^2 > 0$ one can reduce along u (for $z < 1$) and v (for $z > 1$) to obtain a $(d-1)$-dimensional theory with non-relativistic scale invariance; in all cases the reduction is however null from the perspective of the dual quantum field theory.

For every value of z compactification of a null direction will be associated with problems at the quantum level and in this thesis we will consider (2.1) with both u and v non-compact. The effects of such a compactification may be considered afterwards but this issue will be for the most part suppressed. If the coordinates (u, v) are non-compact, holography relates the bulk spacetime to a d-dimensional theory which is not Lorentz invariant but which admits scaling symmetry. Theories of this anisotropic scale-invariant type can certainly model interesting physical systems and have appeared previously in the condensed matter literature. For example, the Z_N chiral Potts models were introduced to model systems with melting transitions [87, 88]. The isotropic Z_N models admit a continuum limit at criticality which is described by two-dimensional Z_N conformal field theories [89]. Since the chiral Z_N models are inherently anisotropic in their critical properties [90, 91, 92], they cannot be described by a conformal field theory in the continuum limit. Instead, as was shown in [83] for superintegrable chiral Potts models, their continuum limits correspond to deformations of conformal field theories of the type (2.4), which are anisotropic but respect scale invariance.

The case studied in [83] was the deformation of a specific two-dimensional conformal field theory by a vector operator of dimension 9/5, which corresponds to scale invariance with $z = 4/5$. As we show in section 2.3, anisotropic scale invariance constrains two point functions of scalar operators at zero temperature to be of the form

$$\langle \mathcal{O}_{\Delta_{\mathcal{D}}}(k_u, k_v) \mathcal{O}_{\Delta'_{\mathcal{D}}}(-k_u, -k_v) \rangle = k_u^{(\Delta_{\mathcal{D}} + \Delta'_{\mathcal{D}})/z} f(b k_\chi), \tag{2.5}$$

where (k_u, k_v) are the lightcone momenta, $\Delta_{\mathcal{D}}$ is the anisotropic scaling dimension and $f(bk_\chi)$ is an arbitrary function of the quantity

$$k_\chi = 2^{z/2} k_v^{z/2} k_u^{z/2-1}, \tag{2.6}$$

which is invariant under anisotropic scale transformations. In [83] two point functions in the deformed theory were computed to leading order in b using conformal perturbation theory; this amounts to computing the function $f(bk_\chi)$ to first order in the expansion in powers of (bk_χ).

Conformal perturbation theory is restricted to weak chirality, namely since b must be small, the theory must be close to the isotropic point. Since the deformation is exactly marginal with respect to anisotropic scaling, the chirality b can be arbitrarily large, and the holographic realizations allow correlation functions to be computed in a strongly coupled theory, at finite chirality. Quantities computed from the holographic models have certain universal features, as is typical for holography. For example, only certain functions $f(bk_\chi)$ are realized in these models and the ratio of η/s for black holes in these models is the expected $1/4\pi$, since the background solves relativistic two derivative equations of motion.

There are several other motivations for exploring these anisotropic backgrounds. The case of $z = 0$, which cannot be realized with massive vectors but can be realized by coupling gravity to a scalar field, is related to Lifshitz with dynamical exponent $Z_L = 2$ upon dimensional reduction. Embedding Lifshitz into string compactifications had proved elusive, but this kind of realization can be obtained in Sasaki-Einstein reductions [93]. Note that the $z = 0$ anisotropic geometry is asymptotically AdS, but the dimensionally reduced theory has Lifshitz symmetry; since holography for the former is well-understood, a holographic dictionary for the latter can be obtained straightforwardly by dimensional reduction. However, as we will discuss here, the dimensional reduction is on a null circle, and this DLCQ reduction introduces subtleties.

Another reason for studying general z is the following. The case of Schrödinger ($z = 2$) has been extensively studied in previous literature, but the encoding of the dual stress energy tensor in the asymptotics of the bulk geometry remains elusive. As shown in [94] there are several reasons for this subtlety. Firstly, the natural operator in the anisotropic dual theory couples not to the metric, but to the vielbein. Secondly, linearized sources for the dual stress energy tensor and deforming vector operator blow up near the boundary of the spacetime faster than the Schrödinger background. In [94] the general linearized solution of the metric and vector equations of motion about the Schrödinger background was presented; this solution consisted of certain independent 'T' and 'X' modes, which should relate to the stress energy tensor and deforming vector operator respectively. In this part of the thesis we will show how these 'T' and 'X' modes are related to the dual operators for $z < 1$ (when the spacetime is asymptotically locally anti-de Sitter) and explain what this implies for the holographic dictionary of Schrödinger. More generally, for $z > 2$, we demonstrate that the irrelevant nature of the deforming operator in the original CFT is reflected in the counterterm structure of the deformed theory: an infinite series of counterterms are required to compute correlation functions in the deformed theory.

The plan of this chapter is as follows. In section 2.2 we introduce the massive vector models used to engineer the anisotropic geometries, and discuss how they may be embedded into string theory. We also consider the special case of $z = 0$ which can be realized using gravity coupled to a scalar field. In section 2.3 the field theoretic description of these models is described, as well as the form of the correlation functions in the anisotropic theory. In section 2.5 holographic renormalization is carried out in the case of $d = 2$, resulting in a precise map between the asymptotic geometry and boundary data. In section 2.6 two point functions of the stress energy tensor and of the deforming vector operator in the scale invariant background are computed. In section 2.7 black hole solutions which are asymptotic to the anisotropic scale invariant background are explored.

2.2. Massive vector model

Consider the Lagrangian:

$$S = \frac{1}{2\kappa_{d+1}^2} \int d^{d+1}x\sqrt{-g}\left[R + \Lambda - \frac{1}{4}F_{mn}F^{mn} - \frac{1}{2}m^2 B_m B^m\right], \tag{2.7}$$

where $F_{mn} = 2\partial_{[m}B_{n]}$, $\Lambda = d(d-1)$ and $m^2 = z(z+d-2)$. The field equations are

$$\begin{aligned} R_{mn} &= -dg_{mn} - \frac{1}{4(d-1)}F^2 g_{mn} + \frac{1}{2}F_{mp}F_n{}^p + \frac{1}{2}m^2 B_m B_n; \\ D_m F^{mn} &= m^2 B^n, \end{aligned} \tag{2.8}$$

where in addition $D_m B^m = 0$.

These equations of motion admit both an AdS_{d+1} solution,

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}\eta_{ab}(x)dx^a dx^b, \tag{2.9}$$

in which $B_m = 0$ and a solution with anisotropic scale invariance:

$$\begin{aligned} ds^2 &= \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}\left(\sigma^2\rho^{1-z}(du)^2 + 2dudv + dx^i dx_i\right); \\ B_u &= b\rho^{-z/2}, \end{aligned} \tag{2.10}$$

where

$$b^2 = \frac{2\sigma^2(1-z)}{z}. \tag{2.11}$$

This solution is a special case of an AdS pp-wave solution and it becomes AdS_{d+1} when the parameter σ is zero whilst any finite σ can be rescaled to one via the rescalings $u \to \sigma^{-1}u$, $v \to \sigma v$. In addition to the rotations, translations and Galilean boosts in the $(d-2)$ spatial directions x^i, the isometry group of this background is:

$$\begin{aligned} M &: \quad v \to v + a, \qquad H : u \to u + a, \\ D &: \quad \rho \to (1-a)^2 \rho, \qquad x^i \to (1-a)x^i, \qquad v \to (1-a)^{2-z} v, \qquad u \to (1-a)^z u. \end{aligned} \tag{2.12}$$

Here D is the non-relativistic scaling (dilatation) symmetry. For general z these are the only symmetries, but at $z = 2$ the metric admits the Schrödinger symmetry group, which includes in addition a special conformal symmetry.

In the case of $z = 1$ the vector field vanishes. The metric

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}\left(\sigma^2 (du)^2 + 2dudv + dx^i dx_i\right) \tag{2.13}$$

solves the Einstein equations with negative cosmological constant for any constant value of σ^2. Here σ^2 acts as a constant source for the T_{vv} component of the stress energy tensor. If this source is zero, the metric is pure AdS_{d+1} whilst if σ^2 is non-zero the metric admits only non-relativistic scale invariance, as the rotational symmetry is broken.

The case of $z = 0$ is also special: the vector field is massless, dual to a conserved current, and adding a source for this current given by

$$B_u = b, \tag{2.14}$$

gives no contribution to the bulk stress energy tensor, so AdS_{d+1} with this vector field solves the bulk field equations for any value of b. For $z = 0$ and $d = 2$ the constant coefficient σ^2 can however be switched on arbitrarily, independently of b, and relates to the expectation value of the stress energy tensor. One can realize $z = 0$ in general dimensions by coupling gravity to a scalar field, as we will discuss below.

It is interesting to note that the solutions (2.10) also arise in topologically massive gravity (TMG) in three dimensions. The action for TMG is

$$S = \frac{1}{16\pi G_N}\int d^3x\,\sqrt{-g}\Big(R - 2\Lambda + \frac{1}{2\mu}\epsilon^{lmn}\big(\Gamma^r_{ls}\partial_m \Gamma^s_{rn} + \frac{2}{3}\Gamma^r_{ls}\Gamma^s_{mt}\Gamma^t_{nr}\big)\Big) \tag{2.15}$$

where Γ^l_{mn} are the connection coefficients associated to the metric g_{mn} and where we use the covariant ϵ-symbol such that $\sqrt{-g}\epsilon^{mnr} = 1$, with r the radial direction in (2.1). Variation of the action results in the equations of motion:

$$R_{mn} - \frac{1}{2}g_{mn}R + \Lambda G_{mn} + \frac{1}{2\mu}\Big(\epsilon_m{}^{rs}\nabla_r R_{sn} + \epsilon_n{}^{rs}\nabla_r R_{sm}\Big) = 0. \tag{2.16}$$

Spacetimes (2.10) with generic z can be realized as solutions of TMG: the spacetime solves the TMG field equations when $\mu = (2z-1)$. These TMG solutions were discussed in [95] and fit into the classification given in [96] as pp-waves. Solutions of this type with u compactified were recently discussed in [97].

In [98, 99] details of the holographic dictionary for TMG were presented, and this dictionary reflects the various problems of the theory: the theory is non-unitary and contains negative norm states. The most important feature of the dictionary for our purposes is that, since the equations of motion of TMG are third order in derivatives, we need to specify not only the boundary metric but also (a component of) the extrinsic curvature in order to find a unique bulk solution. When we apply gauge/gravity duality to TMG with a negative cosmological constant, the extra boundary data corresponds to the source of an extra operator. Therefore, besides the boundary energy-momentum tensor T_{ij}, which couples to the boundary metric $g_{(0)ij}$, we also have a new operator X_{vv} which couples to the leading coefficient of the radial expansion of the (uu) component of the extrinsic curvature. It was shown in [98] that this operator X_{vv} has weights $(h_L, h_R) = \frac{1}{2}(\mu+3, \mu-1)$.

In order to realize the scale invariant background with exponent z we need to work at $\mu = (2z-1)$ and switch on a constant source for the operator X_{vv}. However for $z < 1$, the case of primary interest in this thesis, the deforming operator X_{vv} has negative scaling weights in the conformal field theory. This pathology is related to the lack of unitarity of the dual theory, and in this thesis we work instead with the massive vector models which do not exhibit such problems.

2.2.1. Linearized equations of motion about AdS background

Let us first linearize the equations of motion about the AdS background by letting $g_{ab} = \eta_{ab} + h_{ab}$; we fix radial axial gauge for the metric fluctuations. The linearized Einstein equations decouple from the vector field equations, and the linearized vector field equations are solved by

$$B_a = B_{(-z)a}(x^c)\rho^{-z/2} + B_{(2-z)a}(x^c)\rho^{1-z/2} + \cdots + B_{(z+d-2)a}(x^c)\rho^{z/2+d/2-1} + \cdots \tag{2.17}$$

where $(B_{(-z)a}, B_{(z+d-2)a})$ are arbitrary d-dimensional 1-forms, and the other coefficients in the expansion are determined in terms of these functions. The radial component of the vector field is completely determined in terms of these coefficients via the divergence equation (2.94) and the vector field equations. Note that the relation between mass and CFT operator dimension Δ_v for a vector is

$$m^2 = (\Delta_v - 1)(\Delta_v + 1 - d), \tag{2.18}$$

which implies that

$$\Delta_v = (z + d - 1) \tag{2.19}$$

for the operator dual to the vector field. This relation means in particular that the vector operator is irrelevant for $z > 1$ and relevant for $z < 1$. When $z = 0$, the vector field becomes massless and is dual to a conserved current. When $z = 1$ the vector operator is marginal with respect to the relativistic scaling symmetry.

Consider now the non-relativistic background (2.10). Suppose the parameter b is small and one retains only terms linear in b, so the metric is purely AdS_{d+1}. The linearized AdS/CFT dictionary then implies that there is a constant null source for the dual vector operator of dimension Δ_v. When the latter is irrelevant, deforming the theory in this way changes the UV structure. The corresponding holographic statement is that at finite b the spacetime ceases to be asymptotically AdS_{d+1}; its asymptotic structure is modified and holography is extremely subtle. However, when the deforming operator is relevant the spacetime remains asymptotically AdS_{d+1} and the standard AdS/CFT dictionary can be developed. It is this latter case that we will mostly focus on here, although we will extend our results to $z > 1$ wherever possible.

2.2.2. Global structure of the spacetime for $z < 1$

In this section we will briefly describe the global structure of the spacetime for $z < 1$, which is analogous to that of the corresponding spacetimes with $z > 1$. Since we are only interested in the case where $b \neq 0$, it is convenient to absorb the parameter b in the rescaling $u \to \sigma^{-1}u$, $v \to \sigma v$, and also change the radial coordinate to $\rho = r^2$. The background metric and the vector field are then

$$\begin{aligned} ds^2 &= g_{mn}dx^m dx^n = \frac{1}{r^2}\left(dr^2 + 2dudv + r^{2(1-z)}du^2 + dx^i dx_i\right); \\ B &= \frac{b}{\sigma}\, r^{-z}\, du. \end{aligned} \tag{2.20}$$

In order to infer geodesic incompleteness, it is useful to consider the equation for null geodesics, which satisfy the equation

$$\dot{r}^2 + 2\,\dot{v}\,\dot{u} + 2\dot{x}^i\dot{x}_i + r^{2(1-z)}\,\dot{v}^2 = 0. \tag{2.21}$$

This equation can be written in terms of the constants of motion associated with the Killing vectors $k_u = \partial_u$, $k_v = \partial_v$ and $k_i = \partial_x^i$:

$$P_u = k_u^a \dot{x}_a = \frac{\dot{v}}{r^2}; \qquad P_v = -k_v^a \dot{x}_a = \frac{\dot{u}}{r^2} + \frac{\dot{v}}{r^{2z}}; \qquad P_i = \frac{\dot{x^i}}{r^2}, \tag{2.22}$$

resulting in

$$\int_{r_0}^{r(\lambda)} \frac{dr}{r\sqrt{P_u^2\, r^{2(2-z)} + 2\, r^2\, P_u P_v + r^2 P_i P^i}} = \pm \int_{\lambda_0}^{\lambda} d\tau. \tag{2.23}$$

Provided that $2P_uP_v + P_iP_i > 0$, null geodesics reach $r = \infty$ in real, finite affine parameter and hence the spacetime is geodesically incomplete. However, this geodesic incompleteness will not prevent us from computing correlation functions unambiguously in this background, as we will see later; the situation is analogous to that found in Lifshitz spacetimes [100]. Moreover, in section 2.7 we will see that the geometries can be blackened, with a horizon cloaking the geodesic incompleteness. A singularity is considered acceptable according to the commonly applied holographic criteria discussed in [101] if correlation functions can be computed unambiguously and the singularity can be cloaked by a horizon. Precisely this criterion was used in [100] to argue that holography for Lifshitz spacetimes made sense, despite the geodesic incompleteness. Applying the same criteria here, one can sensibly discuss holography for these spacetimes but it would of course still be desirable to understand the resolution of this singularity at the quantum level, for example, by embedding these geometries into string theory.

Next let us consider whether there is a well-defined time function in the spacetime. Reinstating the parameter b the metric is

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}\left(\frac{z}{2(1-z)}b^2\rho^{1-z}du^2 + 2dudv + dx^i dx_i\right) \tag{2.24}$$

where $b^2 > 0$ in the massive vector model. Thus $g_{uu} > 0$ (for finite ρ) for $z < 1$ and $g_{uu} < 0$ (for finite ρ) for $z > 1$, but note that for all z hypersurfaces of constant u are null. In the case of $z > 1$, the u coordinate has been treated as a time coordinate and real-time physics has been defined with respect to this coordinate [102, 103]. However, the fact hypersurfaces of constant u are actually null is symptomatic of a larger issue: there is no global time function in these spacetimes and the spacetimes are said to be causally non-distinguishing, which in turn implies subtleties in treating modes of zero lightcone momentum [104].

In the case where $z < 1$, one might similarly suppose that the u coordinate should be treated as spacelike, but note that hypersurfaces of constant u are still null and u is a null coordinate in the background for the dual quantum field theory. Unlike the $z > 1$ case, there is a global time function: the spacetime is asymptotically anti-de Sitter, and the coordinate defined as

$$t = \frac{1}{\sqrt{2}}(v - u) \tag{2.25}$$

is everywhere timelike for $b^2 > 0$ and $z < 1$, since hypersurfaces of constant t are everywhere spacelike. Unlike the case of $z > 1$, there are no subtleties in addressing real-time physics, and real-time issues will be suppressed here.

2.2.3. Embedding of massive vector models into string theory

One may next wonder whether these massive vector models can be realized in string theory compactifications. In the case of $z > 1$, various embeddings into string theory have been found, with the massive vector actions arising as consistent truncations of type II supergravities, see for example [84, 105, 106, 107, 108, 109, 110, 111, 112, 113]. Note that in these cases the truncation to a graviton and massive vector suffices for the zero temperature background, but additional scalar fields are switched on in the corresponding black hole solutions. From the consistent truncation perspective, it is only consistent for the scalar fields to vanish when the vector field is null.

A necessary condition for an embedding of $0 < z < 1$ into string theory to exist would be that there is a vector of mass squared $0 < m^2 < (d-1)$ (in AdS units) in the spectrum around an AdS_{d+1} solution, corresponding to a vector operator in the dual CFT_d of dimension $(d-1) < \Delta < d$. However, in spherical compactifications, the dimensions of all vector operators dual to supergravity modes are necessarily integral; this follows from the eigenvalue spectra of operators on the sphere, see for example [64] for the S^5 compactification of type IIB. Whilst spherical compactifications includes vectors dual to symmetry currents of dimension $(d-1)$ and can include vectors of dimension d also, the chiral spectrum does not include non-integral dimension vectors.

Irrational values for the conformal dimensions of operators dual to supergravity modes in Sasaki-Einstein compactifications are however generic. As an example, let us consider the $T^{1,1}$ compactification of type IIB supergravity, whose spectrum was computed in [114, 115]. Since $T^{1,1}$ is a rank one $SU(2)^2/U(1)$ coset, all differential operators can be expressed in terms of the Laplace-Beltrami operator, which is the only functionally independent differential operator. This property allows one to compute the complete KK spectrum in this case, whilst for generic Sasaki-Einstein compactifications only a subset of the KK spectrum is known. All masses are expressible in terms of the scalar Laplacian eigenvalue:

$$H_0(j,l,r) = 6[j(j+1) + l(l+1) - \frac{1}{8}r^2], \tag{2.26}$$

where (j,l,r) refer to the $SU(2)^2$ and R-symmetry quantum numbers. The supergravity compactification consists of graviton multiplet, four gravitino multiplets and four vector multiplets, for which the conformal dimensions of the dual operators are expressible in terms of the function H_0. These conformal dimensions are generically irrational. In particular, considering one of the vector multiplets, the corresponding dual operator to the vector is of dimension

$$\Delta = -1 + \sqrt{4 + H_0(j,l,r)}. \tag{2.27}$$

In special cases where the square root assumes a rational value the dual operator will have a rational conformal dimension, and will form part of a shortened multiplet. Generically, however, the dimension will be irrational and the operator will be part of a massive long multiplet. For the chiral model to be realized, we would need the spectrum to contain a vector operator of dimension $3 < \Delta < 4$. From [114, 115], one finds that vector operators with protected dimensions do not realize any operators with dimension $3 < \Delta < 4$, although both $\Delta = 3$ and $\Delta = 4$ do occur. This is in agreement with the fact that $0 < z < 1$ solutions were not found in the systematic explorations of [110, 111]. However, since for general compactifications there is no supersymmetry or unitarity obstruction to such operators being contained in the spectrum, it would be interesting to explore further whether embeddings of these models into such string compactifications exist.

2.2.4. Realization of $z = 0$ with scalar fields

In general dimensions, the case of $z = 0$ realized with gravity coupled to a gauge field is special, since the gauge field corresponding to a constant null source for the dual current does not backreact on the metric. However, $z = 0$ can also be realized by coupling gravity and a cosmological constant to a massless scalar field; as we will now discuss, this case is related to the supergravity solutions found recently in [93].

Consider first the Lagrangian:

$$S = \frac{1}{2\kappa_{d+1}^2} \int d^{d+1}x \sqrt{-g} \left[R + \Lambda - \frac{1}{2}(\partial\Phi)^2 \right], \tag{2.28}$$

where $\Lambda = d(d-1)$. The field equations are

$$R_{mn} = -d g_{mn} + \partial_m \Phi \partial_n \Phi; \qquad \Box\Phi = 0. \tag{2.29}$$

As well as the AdS_{d+1} solution with constant scalar field they also admit a solution with non-relativistic scale invariance $z = 0$:

$$\begin{aligned} ds^2 &= \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}\left(\sigma^2 \rho (dF)^2 + 2du dv + dx^i dx_i\right); \\ \Phi &= \sqrt{(d-2)}\sigma F(u), \end{aligned} \tag{2.30}$$

where $F(u)$ is an arbitrary function of u. The scalar field vanishes in $d = 2$, where an arbitrary value of σ^2 satisfies the Einstein equations with negative cosmological constant. In this case σ^2 corresponds to a non-vanishing expectation value of T_{vv}, and the geometry is dual to a specific state in the conformal field theory, rather than to a non-relativistic deformation of the original conformal field theory.

A massless field Φ is dual to a marginal scalar operator $\mathcal{O}_\Phi$ in the conformal field theory. A non-vanishing σ implies that there is a u-dependent source for the dual operator, so the deformed theory is:

$$S_{\rm CFT} \to S_{\rm CFT} + \sqrt{(d-2)}\sigma \int dudvd^{d-2}xF(u)\mathcal{O}_\Phi. \tag{2.31}$$

A priori it is not obvious that such deformations are exactly marginal with respect to the $z = 0$ scaling symmetry. When the function $f(u)$ is constant, the deformation does not break Lorentz symmetry, but the marginal scalar operator is not generically exactly marginal with respect to the relativistic scaling symmetry. For general $f(u)$ the deformation respects $z = 0$ symmetry under which $v \to \lambda^2 v$, $u \to u$, $x^i \to \lambda x^i$, given that the scalar operator has non-relativistic scaling dimension equal to the relativistic scaling dimension d. One would however still need to show that the scaling dimension remains exactly marginal under the deformation and hence that the deformed theory remains scale invariant; this proof will be discussed in the next section.

This system is particularly interesting for the following reason. If one considers the case where $dF = du$, the metric can be written as

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}\left(dx_i dx^i - \sigma^{-2}\frac{dv^2}{\rho}\right) + \sigma^2(du + \rho^{-1}\sigma^{-2}dv)^2. \tag{2.32}$$

Dimensionally reducing along the u direction results in a d-dimensional metric with vector field A,

$$ds_d^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}\left(dx_i dx^i - \sigma^{-2}\frac{dv^2}{\rho}\right); \tag{2.33}$$

$$A = \frac{dv}{\sigma^2\rho}, \tag{2.34}$$

which exhibits Lifshitz symmetry with dynamical exponent $Z_L = 2$ and corresponds to the massive vector model used to obtain Lifshitz solutions in [116]. The Lifshitz symmetry group with dynamical exponent Z_L includes a dilatation symmetry

$$\rho \to \lambda^2\rho; \qquad v \to \lambda^{Z_L}v; \qquad x^i \to \lambda x^i, \tag{2.35}$$

and v is a time coordinate. Note however that strictly speaking the scalar field in (2.30) cannot be dimensionally reduced along the u direction, as $F(u) = u$. Whilst the d-dimensional vector and metric, together with the constraint that $d\Phi = \sqrt{d-2}\sigma$, are sufficient to solve the $(d+1)$-dimensional equations of motion, it would be desirable to find an explicit realization of a $z = 0$ system in string theory and, if possible, a consistent truncation to $(d+1)$-dimensional equations of motion.

Such families of solutions were found in Sasaki-Einstein compactifications in [93]. In particular, compactifications of type IIB on Sasaki-Einstein manifolds E_5 admit solutions in which the ten-dimensional metric is

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}\left(f\rho(du)^2 + 2dudv + dx^i dx_i\right) + ds^2(E_5) \tag{2.36}$$

where f is in general a function of both Sasaki-Einstein coordinates and of u. The corresponding five-form F_5, the complex three-form G and the complex one-form P are respectively

$$\begin{aligned} F_5 &= du \wedge dv \wedge d(\rho^2) \wedge dx_1 \wedge dx_2 + 4\mathrm{Vol}_{E)5}; \\ G_3 &= du \wedge W; \qquad P = gd\sigma, \end{aligned} \tag{2.37}$$

where W and g are a three-form and a function defined on E_5 which may also depend on u. The equations of motion imply that

$$\begin{aligned} du \wedge dW &= d_{*_E} W = 0; \\ -\Box_{E_5} f + 4f &= 4|g|^2 + |W|^2. \end{aligned} \tag{2.38}$$

In general the function f depends both on u and on the Sasaki-Einstein coordinates. There is a simpler subclass of solutions in which f is constant and the metric becomes the product of (2.32) with a Sasaki-Einstein space. We can furthermore consider the case where the axion and dilaton is trivial, and so $g = 0$ In this case the solutions require that

$$4f = |W|^2, \tag{2.39}$$

with W a harmonic form on the Sasaki-Einstein. To make contact with the discussion above it is useful to let $f = \sigma^2$, so that

$$\begin{aligned} ds^2 &= \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}\left(\sigma^2\rho(du)^2 + 2dudv + dx^i dx_i\right) + ds^2(E_5); \\ F_5 &= du \wedge dv \wedge d(\rho^2) \wedge dx_1 \wedge dx_2 + 4\mathrm{Vol}_{E5}; \\ G_3 &= 2\sigma du \wedge \tilde{W}, \end{aligned} \tag{2.40}$$

where $W = 2\sigma\tilde{W}$ and hence $|\tilde{W}|^2 = 1$. In the limit where σ is small, we may analyze the interpretation of the solution using the standard AdS/CFT dictionary. From the form of G_3 one can see that at order σ it indeed corresponds to switching on a u dependent source for a dimension four scalar operator in the dual four-dimensional CFT. Moreover, suppose one considers the reduction

$$\begin{aligned} ds^2 &= ds^2(M_5) + ds^2(E_5); \\ F_5 &= 4(\mathrm{Vol}(M_5) + \mathrm{Vol}(E_5)); \\ H &= \sqrt{2} d\Phi \wedge \tilde{W}. \end{aligned} \tag{2.41}$$

The equations of motion for the metric on the five-dimensional non-compact manifold M_5 and the scalar Φ are precisely those given in (2.29), but in order to satisfy the ten-dimensional equations of motion, one needs to impose the additional constraint

$$\partial^m \Phi \partial_m \Phi = 0, \tag{2.42}$$

and thus the reduction is not technically a consistent reduction. A similar issue was found in [93] in reducing the system further to four dimensions, retaining only the four-dimensional metric and massive vector. A consistent truncation to four dimensions involving additional fields was presented in [93].

To summarize, for the cases described in [93] where the function f is independent of the Sasaki-Einstein, the corresponding $(d-1)$-dimensional holographic theory should be the dimensional reduction along the u direction of a d-dimensional CFT deformed by an operator respecting $z = 0$ scale invariance. Note that the dual d-dimensional field theory is in a flat Minkowski background, with coordinates (u, v, x^i) and the reduction is along a null direction, which would be expected to produce the standard problems and subtleties of DLCQ. In the general case in which f depends on the Sasaki-Einstein coordinates, a similar correspondence should hold. Decomposing f into harmonics of the Sasaki-Einstein, one could infer which chiral primary operators are sourced in the dual four-dimensional conformal field theory.

From the bulk perspective, one can see immediately implications of reducing the $z = 0$ geometry along a compact u direction. Any asymptotically locally anti-de Sitter geometry reduced along a spacelike circle will result in a geometry which is conformally asymptotically locally anti-de Sitter in lower dimensions. This fact was used to analyze holography for non-conformal branes in [117, 118]. In the present case, reduction along the circle does not produce a geometry which is conformal to anti-de Sitter, and the reason is that the reduction being carried out is not along a spacelike circle: the u circle becomes null at infinity, corresponding to the fact that u is a null coordinate in the dual quantum field theory. Therefore one needs to carry out a DLCQ of the deformed theory to obtain a Lifshitz theory in one lower dimension.

2.3. Field theory analysis

The chiral backgrounds for general z originate from deforming the dual conformal field theory by operators which respect a non-relativistic scaling invariance. In this section we will discuss these deformations in more detail from the field theory perspective.

Consider first a conformal field theory in d spacetime dimensions, with coordinates (u, v, x^i). The conformal group $SO(2, d)$ contains the group of non-relativistic conformal symmetries with arbitrary z, which we will denote S_z. The embedding is the following. Choosing lightcone coordinates u, v, the relativistic momentum generators P_u and P_v are identified with $\mathcal{H}$ and $\mathcal{M}$, respectively, the non-relativistic scaling generator $\mathcal{D}$ is a linear combination of the relativistic scaling generator and a boost in the uv direction and $\mathcal{C}$. Translations, rotations and Galilean boosts and related to translations and rotations in the relativistic theory. More details can be found, for example, in [79] or [84]. Note in particular that the non-relativistic dilatation $\mathcal{D}$ is given in terms of the relativistic D and the boost M_{uv} (normalized so that the eigenvalues of (u, v) are $(1, 1)$ respectively) as

$$\mathcal{D} = D + (z-1)M_{uv}. \tag{2.43}$$

Thus any conformal field theory also admits the non-relativistic symmetry S_z.

2.4. Marginal deformations respecting anisotropic scaling symmetry

One can next pose the question as to what deformations preserve S_z but break the relativistic conformal symmetry. Such deformations should be marginal with respect to S_z, and thus the non-relativistic scaling dimension of the deforming operator should be $\Delta_{\mathcal{D}} = d$. The deforming operator should also break Lorentz invariance, by breaking rotational symmetry in the (uv) plane. The simplest possibility is a vector operator $\mathcal{V}_\mu$ of relativistic scaling dimension $\Delta = d + (z-1)$. Using (2.43) we note that

$$\Delta_{\mathcal{D}}(\mathcal{V}_v) = d; \qquad \Delta_{\mathcal{D}}(\mathcal{V}_u) = d + 2(z-1), \tag{2.44}$$

and thus $\mathcal{V}_v$ is marginal with respect to the non-relativistic symmetry. It is this case which is modeled holographically by gravity coupled to massive vector fields,

$$S_{\text{CFT}} \to S_{\text{CFT}} + b \int d^d x \mathcal{V}_v + \cdots \tag{2.45}$$

In the specific case of two dimensions, the dual two-dimensional CFT is deformed by the right-moving component of a vector operator, namely $\mathcal{V}_{(1+z/2, z/2)}$ with holomorphic and anti-holomorphic dimensions $(h_v, h_u) = (1 + z/2, z/2)$, so that

$$S_{\text{CFT}} \to S_{\text{CFT}} + b \int dv du \mathcal{V}_{(1+z/2, z/2)} + \cdots \tag{2.46}$$

with b constant and where the ellipses denote terms higher order in b. This deformation is manifestly consistent with the non-relativistic scaling symmetry

$$v \to \lambda^{2-z} v; \qquad u \to \lambda^z u, \tag{2.47}$$

along with translational symmetries in the (u, v) direction. Note that the combination:

$$\chi^2 \equiv v^z u^{z-2} \tag{2.48}$$

is invariant under the non-relativistic scaling symmetry, whilst the (Lorentz-invariant) combination $(2uv)$ scales as λ^2.

It is interesting to note that such deformations by vector operators are only one special case of a more general situation in two dimensions, in which one deforms a 2d CFT by a (p, q) operator $\mathcal{Y}_{p,q}$ where (p, q) are the CFT scaling weights corresponding to (v, u) respectively,

$$S_{\text{CFT}} \to S_{\text{CFT}} + b_{p,q} \int d^2x \mathcal{Y}_{p,q}. \tag{2.49}$$

As discussed in [94] such a deformation respects anisotropic scale invariance with exponent z under which $u \to \lambda^z u$ and $v \to \lambda^{2-z} v$ provided that

$$(p-1)(z-2) = (q-1)z. \tag{2.50}$$

Vector deformations in which $p = q \pm 1$ are just one special case. Another interesting case is that of strictly chiral deformations of conformal field theories, by which we mean

$$S_{\text{CFT}} \to S_{\text{CFT}} + b_{p,0} \int d^2x \mathcal{Y}_{p,0}, \tag{2.51}$$

where $\mathcal{Y}_{p,0}$ is a holomorphic field of arbitrary integral spin. The dynamical exponent in this example is

$$z = 2\left(1 - \frac{1}{p}\right). \tag{2.52}$$

The case of $p = 1$ corresponds to deformation by a conserved current, which as we saw earlier is trivial from the bulk perspective; that of $p = 2$ corresponds to $z = 1$ anisotropic symmetry and could be realized by deforming with the holomorphic component of the stress energy tensor. Such chiral deformations of CFTs have arisen previously in many contexts, from two-dimensional large N QCD to Kodaira-Spencer theory, see for example [119], but the existence and implications of the anisotropic scaling symmetry have not been discussed. From the form of (2.50) one can see that a theory with exponent z can also be viewed as a theory with exponent $z' = (2 - z)$ upon exchanging the rôles of u and v.

Returning to the case of vector deformations, while such deformations are manifestly marginal, one also needs to show that they are exactly marginal. A priori, one might not have expected such deformations to be exactly marginal with respect to the non-relativistic symmetry group. However, holographic duals for such deformations (at strong coupling) exist generically, and this implies that such operator deformations do indeed remain exactly marginal. Using conformal perturbation theory, the correction to the two point function of the deforming operator itself, in the deformed theory, is expressed in terms of higher point functions in the conformal theory as

$$\delta\langle\mathcal{V}_v(x)\mathcal{V}_v(0)\rangle = \sum_{n\geq 1}\frac{1}{n!}\langle\mathcal{V}_v(x)\prod_{a=1}^{n}\int dx_a(b\mathcal{V}_v(x_a))\mathcal{V}_v(0)\rangle. \tag{2.53}$$

This expression can be rewritten in momentum space as

$$\delta\langle\mathcal{V}_v(k)\mathcal{V}_v(-k)\rangle = \sum_{n\geq 1}\frac{1}{n!}\langle\mathcal{V}_v(k)(b\mathcal{V}_v(0))^n\mathcal{V}_v(-k)\rangle. \tag{2.54}$$

If the deformation is to be exactly marginal, at zero momentum, the anomalous dimension of the operator must vanish at zero momentum. A simple argument why this is true was given in [94] for the case of $z=2$ and follows from (relativistic) conformal invariance, which implies that

$$\langle\mathcal{V}_v(k)(b\mathcal{V}_v(0))^n\mathcal{V}_v(-k)\rangle = (bk_v)^n\langle\mathcal{V}_v(k)\mathcal{V}_v(-k)\rangle f^{(n)}\left(\ln(k^2/\mu^2)\right), \tag{2.55}$$

where the function $f^{(n)}$ can depend at most logarithmically on the scale. The right-hand side always vanishes for $k_v \to 0$, and therefore the deforming operator itself cannot acquire an anomalous dimension.

For general values of z (excluding the cases where $z/2$ is an integer) the argument is even simpler because the integrals appearing in (2.53) are not scale invariant. This implies, following section 4.4 of [94], that for generic values of z no operators acquire anomalous scaling dimensions in the deformed theory (again, except when $z/2$ is an integer). Instead the corrections to the two point function of the deforming operator are simply of the form

$$\langle\mathcal{V}_v(k)(b\mathcal{V}_v(0))^n\mathcal{V}_v(-k)\rangle = (bk_v^{z/2}k_u^{z/2-1})^n\langle\mathcal{V}_v(k)\mathcal{V}_v(-k)\rangle, \tag{2.56}$$

where no logarithmic term can appear on the right hand side. The quantity $(k_v^z k_u^{z/2-1})$ is, according to (2.48), invariant under the anisotropic scaling symmetry and therefore the deformation corrects only the normalization of the operator but not its non-relativistic scaling dimension. Thus the operator indeed remains marginal in the deformed anisotropic theory.

Note that an analogous simple argument cannot be made for deformations by marginal scalar operators. In such a case the deformation of the scalar two point function is

$$\delta\langle\mathcal{O}(x)\mathcal{O}(0)\rangle = \sum_{n\geq 1}\frac{1}{n!}\langle\mathcal{O}(x)\prod_{a=1}^{n}\int dx_a(\alpha\mathcal{O}(x_a))\mathcal{O}(0)\rangle, \tag{2.57}$$

where α is a scalar parameter. Conformal invariance implies that

$$\langle\mathcal{O}(k)(\alpha\mathcal{O}(0))^n\mathcal{O}(-k)\rangle = \alpha^n\langle\mathcal{O}(k)\mathcal{O}(-k)\rangle f^{(n)}\left(\ln(k^2/\mu^2)\right), \tag{2.58}$$

and if any of the $f^{(n)}$ are non-zero the operator acquires an anomalous dimension. Generically the $f^{(n)}$ are indeed non-zero, and one needs to use additional structure such as supersymmetry to determine when operators are exactly marginal.

2.4.1. Deformations with $z = 0$

Scaling symmetry with $z = 0$ cannot be realized non-trivially with by vector operator deformations. The vector operator which would respect $z = 0$ has relativistic dimension $(d - 1)$ and is a conserved current. The deformation by a constant null source for this operator introduces chemical potentials in the dual theory and breaks the relativistic invariance in a trivial way; correspondingly the bulk metric remains AdS_{d+1} after the deformation. In section 2.2.4 we showed that $z = 0$ bulk solutions could be obtained by coupling gravity to a massless scalar, and switching on a profile for the scalar field which depends on the lightcone coordinate u. Let us now discuss the corresponding field theory deformations.

Working to leading order around the AdS_{d+1} background, the solution (2.30) corresponds to a deformation of the CFT,

$$S_{\mathrm{CFT}} \rightarrow S_{\mathrm{CFT}} + \int du F(u) \int dv d^{d-2}x \mathcal{O}_d, \tag{2.59}$$

where the operator $\mathcal{O}_d$ is a marginal scalar operator dual to the bulk field Φ. Recalling that the scaling symmetry with $z = 0$ acts as

$$u \rightarrow \lambda^0 u; \qquad v \rightarrow \lambda^2 v; \qquad x \rightarrow \lambda x, \tag{2.60}$$

and that the marginal scalar operator with scale as $\mathcal{O}_d \rightarrow \lambda^{-d}\mathcal{O}_d$, one notes that the deformation indeed respects $z = 0$ symmetry for any choice of the function $F(u)$. The question next arises as to whether this deformation is exactly marginal, since as we discussed above, marginal scalar operators are generically not exactly marginal. However, in the bulk realization, the scalar operator is a chiral primary which is exactly marginal. In the holographic realizations, therefore, the deformation is indeed exactly marginal for any choice of $F(u)$, with the case of constant $F(u)$ being a special case in which relativistic symmetry remains unbroken.

2.4.2. Correlation functions in the deformed theory

Next let us consider the behavior of correlation functions under such deformations, focusing on the case of two dimensions. Suppose that in the original CFT the stress energy tensor is T_{ab}, the vector operator of relativistic dimension $(1 + z)$ is V_a and let $\mathcal{O}_{h,\bar{h}}$ be generic chiral operators of relativistic dimension $(h, \bar{h})$. Here v corresponds to the holomorphic coordinate, scaling weight h, and u corresponds to the anti-holomorphic coordinate, scaling weight $\bar{h}$. The corresponding non-relativistic scaling dimension for the operator $\mathcal{O}_{h,\bar{h}}$ is

$$\Delta_{\mathcal{D}} = h(2-z) + \bar{h}z. \tag{2.61}$$

Non-relativistic scale invariance generically constrains the two point functions to be of the form

$$\langle \mathcal{O}_{\Delta_{\mathcal{D}}}(u,v)\mathcal{O}_{\Delta'_{\mathcal{D}}}(0)\rangle = \frac{1}{u^{(\Delta_{\mathcal{D}}+\Delta'_{\mathcal{D}})/z}} f(\chi), \tag{2.62}$$

where $f(\chi)$ is an arbitrary function of the scale invariant quantity χ defined in (2.48). The relativistic two-point function is of the required form noting that

$$\frac{1}{v^{2h}u^{2\bar{h}}} \equiv \frac{1}{u^{2\Delta_{\rm nr}/z}}\chi^{-2h/z} \equiv \frac{1}{(uv)^{\Delta_{\rm nr}}}\chi^{\bar{h}-h}. \tag{2.63}$$

One should note that for generic z operators of different scaling dimension can have non-vanishing two point functions. In the cases of $z=1$ and $z=2$ the additional special conformal symmetry imposes the further restriction that $\Delta_{\rm nr} = \Delta'_{\rm nr}$.

Using conformal perturbation theory one can derive the corrections to the two point function at non-zero b. To leading order this results in (see section 4.4 of [94] for a detailed analysis),

$$\langle \mathcal{O}(u,v)\mathcal{O}(0)\rangle = \frac{1}{v^{2h}u^{2\bar{h}}}\left(c_0 + c_1 b(\chi)^{-1/2}\right), \tag{2.64}$$

where c_0 denotes the operator normalization in the CFT and c_1 is a computable numerical constant, proportional to the structure constant of the three point function between these operators and the deforming vector operator. When $(z/2)$ is an integer the corresponding expression involves logarithms and is instead of the form,

$$\langle \mathcal{O}(u,v)\mathcal{O}(0)\rangle = \frac{1}{v^{2h}u^{2\bar{h}}}\left(c_0 + c_1 b\chi^{-1/2}\ln(m^2(uv))\right). \tag{2.65}$$

The appearance of logarithms reflects the fact that operators acquire anomalous scaling dimensions in the deformed theory; this is only possible when $z/2$ is an integer.

Returning to the generic case where $z/2$ is not an integer, the corrections are organized in powers of $b\chi^{-1/2}$ since the deformed action remains invariant under the original dilatation symmetries provided that the coupling b is also transformed. Working to higher orders in b in the case where $z/2$ is non-integral gives

$$\langle \mathcal{O}(u,v)\mathcal{O}(0)\rangle = \frac{1}{v^{2h}u^{2\bar{h}}}\sum_n c_n b^n(\chi)^{-n/2} \tag{2.66}$$

The corrections in b hence change the χ dependent normalization of the correlator, but do not the scaling dimension of the operator. By contrast in the case where

$z/2$ is integral the logarithmic terms in the expansion in b indicate that the scaling dimension is also modified at non-zero b; the case of Schrödinger symmetry, $z = 2$, was the main focus of [94].

In the holographic realizations considered here, there are no three-point couplings between the metric and the vector field in the bulk action. This implies that the leading corrections to their two point functions occur at order b^2, and they are related to four point functions at the conformal point. More generally, since all odd couplings vanish in the bulk, their corrected two point functions involve functions of (b^2/χ). For generic z the stress energy tensor and the vector operator can have non-zero two point functions with each other, at non-zero b, and indeed as we will show the Ward identities do force these two point functions to be non-zero.

2.4.3. Counterterms and renormalizability

In this section we will consider what counterterms are needed in computing the two point functions in conformal perturbation theory. Explicit expressions for the corrections to correlation functions at leading order in b were obtained in [94] using the method of differential regularization [77]. Counterterms in this method are implicit, although they can be constructed explicitly as in [120]. In the case at hand we would like to explore the structure of the required counterterms and compare it with the counterterms obtained in holographic renormalization.

Following [94], the leading order correction (2.64) is determined by the three point function between the deforming operator and the other two operators. Analytically continuing to Lorentzian signature via $v \to w$ and $u \to \bar{w}$ the correction behaves as

$$\delta\langle\mathcal{O}(w,\bar{w})\mathcal{O}(0)\rangle \sim \frac{b}{w^{2h-1-z/2}\bar{w}^{2\bar{h}-z/2}}\int \frac{d^2y}{(w-y)^{z/2+1}y^{z/2+1}(\bar{w}-\bar{y})^{z/2}\bar{y}^{z/2}} \tag{2.67}$$

$$\sim \frac{b}{w^{2h-1-z/2}\bar{w}^{2\bar{h}-z/2}}\partial_w^2\int \frac{d^2y}{|y-w|^z|y|^z}. \tag{2.68}$$

When $2z$ is not an integer, then $|y|^{-z}$ is well-defined as a distribution, and its Fourier transform is

$$\int dw d\bar{w} e^{-ikw-i\bar{k}\bar{w}}|w|^{-z} = \pi 2^{2-z}\frac{\Gamma(1-z/2)}{\Gamma(z/2)}|k|^{z-2}. \tag{2.69}$$

Noticing that the integral (2.68) is a convolution, the integral may be computed via the inverse Fourier transform of the products of the two Fourier transforms. This results in a leading correction to the two point function of the form (2.64).

Whilst the correct finite contribution to the two point function is obtained in this way, note that the integrals being computed are in general divergent and additional counterterms are required relative to $b = 0$. One way to see this is remove small circles of radius Λ^{-1} around points where the vertices coincide; with this regulator the integral will have power divergences which can be cancelled by adding contact terms. Let us consider the case where the operator is the deforming operator itself. The new counterterms are then precisely the same counterterms needed in computing the three point function of this (for $z > 1$, irrelevant) operator in the CFT. The counterterms at order b^n will similarly be related to the counterterms that arise in computing $(n+2)$-point functions, and the latter must on general grounds be local, covariant functionals of the vector operator sources.

Let us restrict to the case where the conformal field theory is treated within the flat background. The leading order counterterms at $2n$-th order in the vector field sources diverge as

$$S_{\rm ct} \sim \Lambda^{2n(z-1)+2} \int d^2x (b^a b_a)^n + \cdots , \tag{2.70}$$

and we have used the fact that the counterterm is necessarily covariant. The degree of divergence is computed using the known dimensionality of the operator source, of the metric and of derivatives. Since counterterms must be scalars, any additional derivatives acting on the sources will reduce the degree of divergence,

$$S_{\rm ct} \sim \Lambda^{2n(z-1)+2-2m} \int t^{a_1 \cdots a_{2n} c_1 \cdots c_{2m}} \prod_i^{2m} D_{c_i} \prod_j^{2n} b_{a_j}, \tag{2.71}$$

where $t^{a_1 \cdots a_{2n} c_1 \cdots c_{2m}}$ is a tensor, which must include $(m+n)$ inverse metrics, since the counterterm is a scalar. Compared to (2.70), these terms are indeed more divergent when $m > 0$. The actual tensors which arise need to be obtained by explicit computation, but note that when $m = 0$ the tensor needs to be completely symmetric and built out of the (flat) metric, with (2.70) being the only possibility.

Now let us suppose one has computed the counterterms to arbitrarily high order in the vector field sources and then let

$$b_a = b\delta_{au} + a_a, \tag{2.72}$$

where b is constant and finite whilst a_a is treated perturbatively. The $2n$-point correlation functions in the deformed theory may then be computed by working to order $2n$ in the source a_a. Consider which counterterms can contribute to this calculation: when $a_a = 0$, all of the counterterms vanish, since there are no covariant scalar invariants which can be formed from a null vector field. The absence of such scalar invariants is related to the exactly marginal nature of the anisotropic deformation.

In computing the two point functions in the deformed theory, one needs to retain only terms quadratic in the source a_a. For $z < 2$ this implies that only a finite number of counterterms are needed. This follows from (2.71): since the background source is null, at order n we need to include at least $m = (n-2)$ v derivatives to form a scalar invariant. The leading non-vanishing counterterms have the structure

$$S_{\rm ct} \sim \Lambda^{2n(z-2)+6-2\tilde{m}} \int d^2k k^{2\tilde{m}} (bk_v)^{2n-4} (ba_v)^2, \tag{2.73}$$

where we work in momentum space, k_v is the lightcone momentum and k schematically denotes all momenta. Clearly for $z < 2$ there are only a finite number of divergent counterterms. However, for $z > 2$, counterterms of arbitrarily high order in the finite source b can contribute. In the holographic computation we will find the same analytic structure, and we will argue in addition that for non-rational values of z the counterterms cannot give finite contributions to the renormalized two point functions.

2.4.4. Stress energy tensor and deforming vector operator

Let us next consider the stress energy tensor and the deforming vector operator, focusing on the case of $z < 1$ where the latter is a relevant operator. Our starting point is a two-dimensional conformal field theory which is invariant under diffeomorphisms and Weyl rescalings (up to the usual conformal anomaly). If the generating functional of the field theory is W the stress energy tensor $\mathcal{T}_{ab}$ may be defined as[5]:

$$\mathcal{T}_{ab} = \frac{2i}{\sqrt{-g_{(0)}}} \frac{\delta W}{\delta g^{ab}_{(0)}}, \tag{2.74}$$

where $g_{(0)ab}$ is the background metric for the field theory. The vector operator $\mathcal{V}_a$ of scaling dimension $(1+z)$ which couples to a source b^a is correspondingly defined as:

$$\mathcal{V}_a = \frac{i}{\sqrt{-g_{(0)}}} \frac{\delta W}{\delta b^a}. \tag{2.75}$$

Diffeomorphisms act as

$$\delta g^{ab}_{(0)} = -(D^a \zeta^b + D^b \zeta^a); \qquad \delta b_a = \zeta^c D_c b_a + D_a \zeta^c b_c, \tag{2.76}$$

with D_a the covariant derivative. Weyl transformations act as

$$\delta g^{ab}_{(0)} = -2\lambda g^{ab}_{(0)}, \qquad \delta b_a = z\lambda b_a. \tag{2.77}$$

Imposing invariance of the generating functional under diffeomorphisms and Weyl transformations gives the following Ward identities:

$$D^b \langle \mathcal{T}_{ab} \rangle_J - b_a D^b \langle \mathcal{V}_b \rangle_J + F_{ba} \langle \mathcal{V}^b \rangle_J = 0; \tag{2.78}$$
$$\langle \mathcal{T}^a_a \rangle_J - z b^a \langle \mathcal{V}_a \rangle_J = \mathcal{A}[g_{(0)}, b]. \tag{2.79}$$

[5]Note that we are working here in Lorentzian signature.

where $\langle \mathcal{O} \rangle_J$ denotes the expectation value of an operator $\mathcal{O}$ in the presence of sources J, and $\mathcal{A}$ denotes the conformal anomaly. Here F_{ab} is the curvature of the vector field source b_a, $F_{ab} = 2\partial_{[a} b_{b]}$. The anomaly can in principle depend covariantly both on the background metric $g_{(0)}$ and on the source b. Since the anomaly must have dimension two, for generic values of z there is no covariant quantity of the right weight that can be formed out of b and the anomaly will be given entirely in terms of the Ricci scalar of the metric $g_{(0)}$ and the central charge c of the CFT:

$$\mathcal{A}(g_{(0)}) = \frac{c}{24\pi} R[g_{(0)}] \tag{2.80}$$

(The additional factor of 2π on the righthand side relative to usual CFT conventions follows from the absence of the 2π factor in our normalization of the stress energy tensor.) For specific values of z where a quantity of the form $\partial^n b^p$ can dimension two there are additional contributions to the conformal anomaly, as will be discussed in section 2.5.

The Ward identities imply an infinite number of relations for correlation functions in the deformed theory, which are obtained by differentiating with respect to the sources and then setting $g_{(0)ab}$ and b_a to their background values. In particular, the identities for two point functions can be completely solved, up to the two point functions of the vector operators. For notational convenience let us denote $T \equiv T_{vv}$, $\bar{T} \equiv T_{uu}$ and $\theta \equiv T_{uv}$. The dilatation Ward identity implies that

$$\langle \theta(u,v) \mathcal{V}_v(0) \rangle = \tfrac{1}{2} z b \langle \mathcal{V}_v(u,v) \mathcal{V}_v(0) \rangle; \qquad \langle \theta(u,v) \mathcal{V}_u(0) \rangle = \tfrac{1}{2} z b \langle \mathcal{V}_v(u,v) \mathcal{V}_u(0) \rangle, \tag{2.81}$$

whilst

$$\langle \theta(u,v) \theta(0) \rangle = \frac{1}{4} z^2 b^2 \langle \mathcal{V}_v(u,v) \mathcal{V}_v(0) \rangle + \cdots, \tag{2.82}$$

where the ellipses denote local terms. Solving the v component of the diffeomorphism identity then results in

$$\begin{aligned}
\langle T(u,v) T(0) \rangle &= \frac{c}{8\pi^2 v^4} + \frac{1}{4} z^2 b^2 \frac{\partial_v^2}{\partial_u^2} \left(\langle \mathcal{V}_v(u,v) \mathcal{V}_v(0) \rangle \right); \\
\langle T(u,v) \theta(0) \rangle &= -\frac{1}{4} z^2 b^2 \frac{\partial_v}{\partial_u} \left(\langle \mathcal{V}_v(u,v) \mathcal{V}_v(0) \rangle \right). \\
\langle T(u,v) \bar{T}(0) \rangle &= \frac{b^2 z}{4} (z-2) \langle \mathcal{V}_v(u,v) \mathcal{V}_v(0) \rangle - \frac{b^2 z^2}{2} \frac{\partial_v}{\partial_u} \left(\langle \mathcal{V}_u(u,v) \mathcal{V}_v(0) \rangle \right); \\
\langle T(u,v) \mathcal{V}_v(0) \rangle &= -\tfrac{1}{2} z b \frac{\partial_v}{\partial_u} \left(\langle \mathcal{V}_v(u,v) \mathcal{V}_v(0) \rangle \right); \\
\langle T(u,v) \mathcal{V}_u(0) \rangle &= -\tfrac{1}{2} z b \frac{\partial_v}{\partial_u} \left(\langle \mathcal{V}_u(u,v) \mathcal{V}_v(0) \rangle \right),
\end{aligned} \tag{2.83}$$

where local terms have been suppressed. Real-time issues and contact terms in the correlators have also been suppressed, since they do not play a rôle in the discussions here. Solving the u component of the diffeomorphism identity results in

$$\begin{aligned}
\langle\bar{T}(u,v)\bar{T}(0)\rangle &= \frac{c}{8\pi^2 u^4} + b^2(2-z)\left(\frac{\partial_u^2}{2\partial_v^2}\langle\mathcal{V}_v(u,v)\mathcal{V}_v(0)\rangle + \frac{\partial_u}{\partial_v}\langle\mathcal{V}_v(u,v)\mathcal{V}_u(0)\rangle\right) \\
&\quad + b^2\langle\mathcal{V}_u(u,v)\mathcal{V}_u(0)\rangle; \\
\langle\bar{T}(u,v)\theta(0)\rangle &= \frac{1}{4}zb^2(2-z)\frac{\partial_u}{\partial_v}\left(\langle\mathcal{V}_v(u,v)\mathcal{V}_v(0)\rangle\right) + \frac{b^2 z}{2}\langle\mathcal{V}_u(u,v)\mathcal{V}_v(0)\rangle; \quad (2.84) \\
\langle\bar{T}(u,v)\mathcal{V}_v(0)\rangle &= b(1-\tfrac{1}{2}z)\frac{\partial_u}{\partial_v}\left(\langle\mathcal{V}_v(u,v)\mathcal{V}_v(0)\rangle\right) + b\langle\mathcal{V}_u(u,v)\mathcal{V}_v(0)\rangle; \\
\langle\bar{T}(u,v)\mathcal{V}_u(0)\rangle &= b(1-\tfrac{1}{2}z)\frac{\partial_u}{\partial_v}\left(\langle\mathcal{V}_u(u,v)\mathcal{V}_v(0)\rangle\right) + b\langle\mathcal{V}_u(u,v)\mathcal{V}_u(0)\rangle.
\end{aligned}$$

Thus in the deformed theory the relativistic stress energy tensor is no longer conserved and has non-trivial two point functions, determined in terms of the conformal anomaly of the original theory and the correlation functions of the vector operator.

In the deformed theory the relativistic stress energy tensor is no longer conserved. However, when b is covariantly constant and has zero curvature $F \equiv 0$, there is a non-symmetric stress-energy tensor t_{ab} defined such that

$$t_{ab} = \mathcal{T}_{ab} - b_a\mathcal{V}_b, \tag{2.85}$$

which is covariantly conserved. As discussed in [94], the components of t_{ab} are related to the Noether charges, including those associated with the translational symmetries $u \to u + a$, $v \to v + b$, and the fact that t_{ab} is non-symmetric follows from the general result that a conserved stress tensor in any field theory in a Minkowski background which breaks Lorentz invariance cannot be symmetric.

The conserved stress energy tensor t_{ab} is obtained by coupling the CFT to a vielbein $e^a_{\hat{a}}$, and then defining

$$t_{ab} = e^{\hat{b}}_b\frac{i}{|e|}\frac{\delta W_{\rm cft}}{\delta e^{a\hat{b}}}, \tag{2.86}$$

where $\hat{a}$ denotes tangent space indices. The vector operator deformation is then given by

$$S_{\rm cft} \to S_{\rm cft} + i\int d^2x|e|b^a e^{\hat{a}}_a\mathcal{V}_{\hat{a}}. \tag{2.87}$$

If we now consider the behavior of the generating functional under Lorentz transformations, diffeomorphisms and Weyl transformations, respectively, we can derive the following Ward identities for t_{ab}:

$$\begin{aligned}
\langle t_{[ab]}\rangle + b_{[a}\langle\mathcal{V}_{b]}\rangle &= 0; \quad (2.88) \\
D^{\hat{a}}\langle t_{\hat{b}\hat{a}}\rangle + D^b b_{\hat{b}}\langle\mathcal{V}_b\rangle + F_{b\hat{b}}\langle\mathcal{V}^b\rangle &= 0; \\
\langle t^a_a\rangle + (1-z)b^a\langle\mathcal{V}_a\rangle_J &= \mathcal{A}(e),
\end{aligned}$$

where we again assume that the only anomaly is the conformal anomaly $\mathcal{A}(e)$, which depends only on the scalar curvature. The operator t_{ab} is indeed conserved when b is covariantly constant and has zero curvature.

For $z < 1$ deformations, which are relevant with respect to the conformal symmetry group, both the relativistic stress energy tensor $\mathcal{T}_{ab}$ and the (conserved) anisotropic stress energy tensor t_{ab} are natural well-defined operators to consider. The relativistic stress energy tensor $\mathcal{T}_{ab}$ is natural when we are treating the theory as a deformation of a CFT, whilst the tensor t_{ab} is natural if we view the theory as an intrinsically anisotropic scale-invariant theory, which acquires additional symmetries in the UV. Correlation functions of the tensors are non-locally related to each other, but may be obtained straightforwardly by the defining relation (2.85). For $z > 1$ deformations, however, which are irrelevant with respect to the conformal symmetry, it is rather less natural to work with the operator $\mathcal{T}_{ab}$, since it is neither conserved nor is the theory conformal in the UV. However, for generic values of z such that $z > 1$, two point functions around the scale invariant vacuum, including correlation functions of $\mathcal{T}_{ab}$, are reconstructable from those of the deforming vector operator, using the Ward identities, and we thus evade having to work in a vielbein formalism.

2.5. Holographic renormalization for $d = 2$

In this section we will derive general expressions for the renormalized holographic one point functions of dual operators in terms of coefficients in the near boundary expansions of bulk solutions. We will focus first on the case of $z < 1$ in two dimensions, and then comment on the case of $z > 1$ which is no longer asymptotically AdS. We discuss holographic renormalization using two methods. The first uses the general asymptotic solution of the bulk field equations to regulate the volume divergences of the on-shell action with covariant counterterms being obtained by inverting these expansions. This method is the most familiar approach to holographic renormalization, see the review [78], but becomes increasingly cumbersome as the number of counterterms required increases. Since this method works with the asymptotic expansion of the bulk metric and vector, it allows us to appreciate the roles of different terms in the asymptotic expansions, which we will exploit in section 2.6.

The second method of holographic renormalization exploits the Hamiltonian approach developed in [121, 122], which uses covariant expansions in terms of eigenfunctions of a dilatation operator. This approach is much more efficient and powerful; the main advantage here is that renormalized correlation functions can, in favorable cases, be determined without explicit computation of the counterterms. In cases where many counterterms are needed, and the inversion of the asymptotic expansions of the bulk fields is cumbersome, this methodology is the more appropriate one to use.

2.5.1. Asymptotic expansions and their inversion

We begin by analyzing the most generally asymptotically locally AdS solutions of the bulk field equations. In the neighborhood of the conformal boundary at $\rho \to 0$, the metric and vector field can be expressed as:

$$\begin{aligned} ds^2 &= \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{ab}(x,\rho) dx^a dx^b; \\ B_a &= \rho^{-z/2} b_a(x,\rho); \\ B_\rho &= \rho^m b_\rho(x,\rho), \end{aligned} \tag{2.89}$$

where the power m will be determined below. Given this coordinate choice, the Einstein equations can be written as:

$$\begin{aligned} &R_{ab}[g_{cd}] + (d-2)g'_{ab} + \mathrm{tr}(g^{-1}g')g_{ab} - \rho(2g'' - 2g'g^{-1}g' + \mathrm{tr}(g^{-1}g')g')_{ab} \\ &= \frac{1}{2}m^2 B_a B_b - \frac{1}{4(d-1)}\rho(g^{ce}g^{df}F_{cd}F_{ef} + 8\rho g^{cd}F_{c\rho}F_{d\rho})g_{ab} + \frac{1}{2}\rho(g^{cd}F_{ac}F_{bd} + 4\rho F_{a\rho}F_{b\rho}); \end{aligned} \tag{2.90}$$

$$D_a(\mathrm{tr}(g^{-1}g')) - D^b g'_{ab} = -(m^2 B_a B_\rho + \rho g^{cd} F_{ac} F_{\rho d}); \tag{2.91}$$

$$\begin{aligned} \frac{1}{4}\mathrm{tr}(g^{-1}g'g^{-1}g') - \frac{1}{2}\mathrm{tr}(g^{-1}g'') = \frac{1}{2}m^2 B_\rho B_\rho - \frac{1}{16(d-1)} g^{ce}g^{df}F_{cd}F_{ef} \\ + \frac{(d-2)}{2(d-1)}\rho g^{cd} F_{c\rho} F_{d\rho}. \end{aligned} \tag{2.92}$$

The vector field equations in this coordinate system become:

$$\begin{aligned} \partial_a\left(\sqrt{-g}g^{ab}F_{b\rho}\right) &= \frac{m^2}{\rho}\sqrt{-g}B_\rho; \\ \partial_a\left(\sqrt{-g}g^{ab}g^{cd}F_{bd}\right) + 4\rho^{d/2-1}\partial_\rho\left(\sqrt{-g}\rho^{2-d/2}g^{cd}F_{\rho d}\right) &= \frac{m^2}{\rho}\sqrt{-g}g^{cd}B_d. \end{aligned} \tag{2.93}$$

The divergence equation for the vector field is:

$$\partial_a\left(\sqrt{-g}g^{ab}B_b\right) + 4\rho^{d/2}\partial_\rho\left(\sqrt{-g}\rho^{1-d/2}B_\rho\right) = 0. \tag{2.94}$$

Here for future convenience the equations are written for general d, although in this section we will consider only $d = 2$. The leading order terms in these equations as $\rho \to 0$ imply that:

$$g_{ab}(x,0) = g_{(0)ab}(x); \qquad b_a(x,0) = b_{(-z)a}(x), \tag{2.95}$$

for arbitrary (non-degenerate) metric and 1-form respectively. By the usual rules of AdS/CFT, $g_{(0)}$ acts as a source for the stress energy tensor in the dual theory, whilst $b_{(-z)a}$ acts a source for the dual vector operator of dimension $(d + z - 1)$.

The leading term in the expansion of B_ρ (including the polynomial power m) is determined by the divergence equation, and does not therefore represent an additional independent source. Indeed, using the leading order terms in the equations of motion one finds that the power of ρ in the leading order term of B_ρ is the same as in the leading term of B_a:

$$\begin{aligned} B_\rho &= \rho^{-z/2}\left[b_{(-z)\rho} + \cdots\right]; \\ b_{(-z)\rho} &= \frac{1}{2z} D^a_{(0)} b_{(-z)a}, \end{aligned} \tag{2.96}$$

where $D_{(0)}$ is the covariant derivative associated with the metric $g_{(0)ab}$. Therefore the value of m in (2.89) is $-z/2$.

The first step in holographic renormalization is to determine the general asymptotic expansion near the boundary, namely the radial expansion of the fields. We thus expand the fields in Fefferman-Graham form as:

$$\begin{aligned} g_{ab}(x,\rho) &= g_{(0)ab}(x) + \cdots + \rho g_{(2)ab}(x) + \cdots; \\ b_a(x,\rho) &= b_{(-z)a}(x) + \cdots + \rho^z b_{(z)a}(x) + \cdots; \\ b_\rho(x,\rho) &= b_{(-z)\rho}(x) + \cdots + \rho^z b_{(z)\rho}(x) + \cdots. \end{aligned} \tag{2.97}$$

The radial expansion only needs to be calculated to sufficient order to determine the divergences in the on-shell action; in practice this means up to the order at which coefficients are undetermined or only partially determined by the asymptotic analysis.Since the coefficients in the field equations (2.90) are polynomials in ρ this system of equations admits solutions with $(g_{ab}(x,\rho), b_a(x,\rho), b_\rho(x,\rho))$ regular functions of ρ. To solve these equations, one may successively differentiate the equations w.r.t. ρ and set $\rho = 0$. In pure gravity, the metric is expanded in integral powers of ρ, with additional logarithmic terms generically needed to solve the equations of motion in odd dimensions. In the case of gravity coupled to the massive vector, the powers of ρ that occur in the expansions need to be determined from the equations of motion, and should not a priori be assumed to be integral.

Explicit solution of the equations of motion for $0 < z < 1/2$ determines that the first subleading term in the metric is actually of order ρ^{1-z}. The form of the asymptotic expansions for $0 < z < 1/2$ can be summarized as follows. The only terms required in determining the counterterms and renormalized one point functions are

$$\begin{aligned} g_{ab} &= g_{(0)ab} + \rho^{1-z} g_{(2-2z)ab} + \rho g_{(2)ab} + \tilde{h}_{(2)ab}\rho \log\rho + \cdots; \\ b_a &= b_{(-z)a} + \rho^z b_{(z)a} + \cdots; \\ b_\rho &= b_{(-z)\rho} + \rho^z b_{(z)\rho} + \cdots, \end{aligned} \tag{2.98}$$

where the ellipses denote subleading terms. The following coefficients are completely determined in terms of the non-normalizable modes:

$$\begin{aligned} g_{(2-2z)ab} &= \frac{z}{2(1-z)}\left[b_{(-z)a}b_{(-z)b} - \tfrac{1}{2}\mathrm{Tr}(b_{(-z)}g^{-1}_{(0)}b_{(-z)})g_{(0)ab}\right], \\ b_{(-z)\rho} &= \frac{1}{2z}D^{a}_{(0)}b_{(-z)a}, \\ \tilde{h}_{(2)ab} &= \tfrac{1}{2}\left(R_{(0)ab} - \tfrac{1}{2}g_{(0)ab}R_{(0)}\right) = 0. \end{aligned} \tag{2.99}$$

In the latter expression the identity relating Ricci curvature and Ricci scalar in two dimensions has been imposed.

In the vector field, the coefficient $b_{(z)a}$ is totally undetermined, whilst

$$b_{(z)\rho} = -\frac{1}{2z}D_{(0)a}b^{a}_{(z)}. \tag{2.100}$$

The metric coefficient $g_{(2)ab}$ is undetermined, but subject to the following constraints:

$$\begin{aligned} \mathrm{Tr}g_{(2)} &= -\tfrac{1}{2}R_{(0)} + z^2\mathrm{Tr}(b_{(-z)}g^{-1}_{(0)}b_{(z)}); \\ D^{b}_{(0)}g_{(2)ba} &= \partial_a\left(\mathrm{Tr}(g_{(2)})\right) + \frac{z}{2}\left(b_{(-z)a}b_{(z)\rho} + b_{(-z)\rho}b_{(z)a}\right) \\ &\quad + \frac{z}{2}\left(F_{(-z)ac}b^{c}_{(z)} - F_{(z)ac}b^{c}_{(-z)}\right), \end{aligned} \tag{2.101}$$

where $F_{(-z)ab}$ is the curvature of the field $b_{(-z)a}$.

For generic $0 < z < 1$, the asymptotic expansion of the metric has the form

$$g_{ab} = \sum_{m,n} g_{(2m+2n(1-z))}\rho^{m+n(1-z)} + \cdots \tag{2.102}$$

with (m, n) integral and coefficients of terms with

$$m + n(1-z) < 1 \tag{2.103}$$

contribute to the on-shell divergences. For $1/2 < z < 1$ this implies that an increasing number of coefficients can contribute to the on-shell divergences, and the Hamiltonian approach to renormalization is more efficient. Note also that the coefficient $g_{2n(1-z)}$ is of order b^{2n}_{-z}, and whenever $(1-z) = 1/p$, with p an integer, logarithmic terms will arise, corresponding to conformal anomalies.

Next one can proceed to renormalize the on-shell action for $0 < z < 1/2$ as follows. One substitutes these expansions into the regulated on-shell action:

$$S = \frac{1}{2\kappa^2_{d+1}}\int_{\mathcal{M}} d^{d+1}x\sqrt{-g}\left[R + \Lambda - \frac{1}{4}F_{mn}F^{mn} - \frac{1}{2}m^2 B_m B^m\right] - \frac{1}{\kappa^2_{d+1}}\int_{\partial\mathcal{M}} d^d x\sqrt{-h}K \tag{2.104}$$

with the boundary regulated at $\rho = \epsilon$ and we now let $\kappa^2 \equiv \kappa_3^2$ in the case of interest, $d = 2$. The Gibbons-Hawking boundary term is included to ensure that the Dirichlet variational problem is well-defined on the surface of fixed radius; note that K is the trace of the second fundamental form. This procedure results in a regulated action of the form:

$$S_{\text{reg}} = \frac{1}{2\kappa^2} \int_{\rho=\epsilon} d^2x \sqrt{-g_{(0)}} \left[\epsilon^{-1} a_{(0)} + \epsilon^{-z} a_{2(1-z)} + \tilde{a}_2 \log \epsilon + \mathcal{O}(\epsilon^0) \right] \tag{2.105}$$

which involves a finite number of terms that diverge as $\epsilon \to 0$. Here all coefficients $(a_{(k)}, \tilde{a})$ of divergent terms are local functions of the sources $(g_{(0)ab}(x), b_{(-z)a}(x))$:

$$\begin{aligned} a_{(0)} &= 2 \qquad a_{2(1-z)} = -\frac{z}{2} b_{(-z)} g_{(0)}^{-1} b_{(-z)}, \\ \tilde{a}_2 &= \text{Tr} g_{(2)} - z^2 b_{(-z)} g_{(0)}^{-1} b_{(z)} = -\tfrac{1}{2} R_{(0)}. \end{aligned} \tag{2.106}$$

These divergences can be removed using the following covariant counterterm action:

$$S_{\text{ct}} = \frac{1}{2\kappa^2} \int d^2x \sqrt{-\gamma} \left(-2 + \frac{z}{2} B^a B_a + \frac{1}{2} R[\gamma] \log \epsilon \right), \tag{2.107}$$

where γ is the induced metric. From the renormalized action, $S_{\text{ren}} = S + S_{\text{ct}}$, one can define the following renormalized one point functions:

$$\langle V_a \rangle = -\frac{1}{\sqrt{-g_{(0)}}} \frac{\delta S_{ren}}{\delta b^a_{(-z)}} = \lim_{\epsilon \to 0} \left[\frac{\epsilon^{-z/2}}{\sqrt{-\gamma}} \frac{\delta S_{ren}}{\delta B^a} \right] = -\frac{z}{\kappa^2} b_{(z)a}, \tag{2.108}$$

and for the stress energy tensor:

$$\begin{aligned} \langle T_{ab} \rangle &= -\frac{2}{\sqrt{-g_{(0)}}} \frac{\delta S_{\text{ren}}}{\delta g^{ab}_{(0)}} = -\lim_{\epsilon \to 0} \left[\frac{2}{\sqrt{-\gamma}} \frac{\delta S_{\text{ren}}}{\delta \gamma^{ab}} \right] \\ &= \frac{1}{\kappa^2} \Bigg[g_{(2)ab} + \frac{1}{2} R(g_{(0)}) g_{(0)ab} - \frac{z}{2} \left(b_{(-z)a} b_{(z)b} + b_{(z)a} b_{(-z)b} \right) \\ &\quad - z(z - \frac{1}{2}) \left(b_{(-z)c} b^c_{(z)} \right) g_{(0)ab} \Bigg]. \end{aligned} \tag{2.109}$$

Note that the answer for pure Einstein gravity, i.e. when $B_m = 0$, agrees with that given in [123]. Since the generating functional of the dual field theory W in Lorentzian signature is related to the renormalized on-shell action as $W = iS_{ren}$, these definitions for the operators agree with those given in section 2.4.4, as do the dilatation and diffeomorphism Ward identities, which are respectively:

$$\begin{aligned} \langle T^a_a \rangle &= -\frac{1}{\kappa^2} \text{Tr}(g_{(2)}) = \frac{1}{\kappa^2} R_{(0)} + z b^a_{(-z)} \langle V_a \rangle; \\ D^b \langle T_{ab} \rangle &= \left(b_{(-z)a} D^b \langle V_b \rangle + F^b_{(-z)a} \langle V_b \rangle \right). \end{aligned} \tag{2.110}$$

The relation between the bulk Newton constant G_3 and the central charge c of the dual two-dimensional CFT is

$$\frac{1}{\kappa^2} = \frac{1}{8\pi G_3} = \frac{c}{24\pi}, \tag{2.111}$$

as derived in [124].

2.5.2. Hamiltonian analysis

In the previous section we showed that an increasing number of counterterms are needed for $z > 1/2$. The renormalized one point functions and counterterms are in such cases more conveniently computed using the Hamiltonian formulation of holographic renormalization. In this section we will analyze holographic renormalization using the methods developed in [121, 122]. These will allow us to compute renormalized correlation functions for generic values of $z > 1/2$.

We begin by expressing the metric as

$$ds^2 = g_{mn}dx^m dx^n = (N^2 + N_a N^a)dr^2 + 2N_a dx^a dr + \gamma_{ab}dx^a dx^b, \tag{2.112}$$

where N is the lapse and N_a is the shift. The choices of $N = 1$ and $N^a = 0$ make r a Gaussian normal coordinate, related to the Fefferman-Graham coordinate ρ as $\rho = e^{-r}$. In order to provide a Hamiltonian description of the dynamics one first expresses the curvature part of the action in terms of quantities on hypersurfaces Σ_r, of constant r:

$$S = \frac{1}{2\kappa^2}\int d^3x\sqrt{\gamma}N\left[\hat{R} + K^2 - K_{ab}K^{ab} + \Lambda - \frac{1}{4}F_{mn}F^{mn} - \frac{1}{2}m^2 B_m B^m\right], \tag{2.113}$$

where $\hat{R}$ is the Ricci scalar of Σ_r and K_{ab} is its second fundamental form. After using the gauge freedom to fix $N = 1$ and $N_a = 0$ the Einstein equations of motion become

$$\begin{aligned} K^2 - K^b_a K^a_b &= \hat{R} + 2\kappa^2 T_{rr} \\ D_a K^a_b - D_b K &= \kappa^2 T_{br} \\ \dot{K}^b_a + K K^b_a &= \hat{R}^b_a - \kappa^2(T^b_a - \delta^b_a T) \end{aligned} \tag{2.114}$$

where $\dot{a}$ denotes $\partial_r a$ and

$$\kappa^2 T_{mn} = (1 - \frac{1}{8}F^2 - \frac{1}{4}m^2 B^2)g_{mn} + \frac{1}{2}F_{mp}F_n{}^p + \frac{1}{2}z^2 B_m B_n, \tag{2.115}$$

with $T = T^m_m$. Note that the (ra) and (rr) Einstein equations are the momentum and Hamilton constraints, which enforce that the momenta conjugate to the lapse and shift functions vanish identically. The momentum conjugate to B_r also vanishes (corresponding to the divergence equation for the vector field) and the non-trivial canonical momenta are

$$\begin{aligned}\pi_{ab} &= \pi_{ab}[\gamma, B_c] = \frac{\delta L}{\delta \dot{\gamma}^{ab}} = \frac{\delta I_r}{\delta \gamma^{ab}} = -\frac{1}{2\kappa^2}\sqrt{\gamma}(K\gamma_{ab} - K_{ab}), \quad (2.116)\\ \pi_a &= \pi_a[\gamma, B_b] = \frac{\delta L}{\delta \dot{B}^a} = \frac{\delta I_r}{\delta B^a} = -\frac{1}{2\kappa^2}\sqrt{\gamma}F_{ra} = -\frac{1}{2\kappa^2}\sqrt{\gamma}(\dot{B}_a - \partial_a B_r), \quad (2.117)\end{aligned}$$

where $I_r = \left[\int drL\right]_{\text{on-shell}}$ is the on-shell action. This implies that the extrinsic curvature K_{ab} and the momenta of the vector field B_a are themselves functionals of the induced fields on Σ_r. Note that the extrinsic curvature is given by

$$K_{ab} = \frac{1}{2}n^m\partial_m\gamma_{ab} = \frac{1}{2}\dot{\gamma}_{ab}, \tag{2.118}$$

where n is the normal to Σ_r. In the Hamiltonian version of holographic renormalization one uses the equations of motion to determine the asymptotic form of the momenta as functionals of the induced fields. This method has the key advantage of maintaining covariance at all stages, thus ensuring that Ward identies are manifest and it also shortens the computation of counterterms.

In the method of holographic renormalization used in the last section the asymptotic analysis begins by expanding the bulk fields in the ρ coordinate. In the Hamiltonian method one notes that the non-normalizable modes of the induced fields behave asymptotically as

$$\begin{aligned}&\gamma_{ab} \sim e^{2r}g_{(0)ab}, \qquad \dot{\gamma}_{ab} \sim 2\gamma_{ab}, \quad (2.119)\\ &B_a \sim e^{zr}b_{(-z)a}, \qquad \dot{B}_a \sim zB_a.\end{aligned}$$

Note that the field B_r is entirely determined by these fields, using the vector divergence equation. The dilatation operator, identified with the functional form of the asymptotic r-derivative in the solution space, is found to be:

$$\partial_r = \int d^2x\left(\dot{\gamma}_{ab}\frac{\delta}{\delta\gamma_{ab}} + \dot{B}_a\frac{\delta}{\delta B_a}\right) \sim \int d^2x\left(2\gamma_{ab}\frac{\delta}{\delta\gamma_{ab}} + zB_a\frac{\delta}{\delta B_a}\right) \equiv \delta_D. \tag{2.120}$$

Since $K_{ab}, \dot{B}_a$ and B_r are functionals of the induced fields, each can be written asymptotically as an expansion in eigenfunctions of the dilatation operator, (2.120). Furthermore, the leading terms in the asymptotic radial expansions coincide with those in the asymptotic expansions in eigenfunctions of the dilatation operator. This allows one to write:

$$\begin{aligned}&{K_a}^b = {K_{(0)a}}^b + {K_{(\alpha_1)a}}^b + {K_{(\alpha_2)a}}^b + \cdots + {K_{(2)a}}^b + {\tilde{K}_{(2)a}}^{\ b}\log e^{-2r} + \cdots, \quad {K_{(0)a}}^b = \delta_a^b\\ &\dot{B}_a = \dot{B}_{(-z)a} + \dot{B}_{(\beta_1)a} + \dot{B}_{(\beta_2)a} + \cdots, \qquad \dot{B}_{(-z)a} = zB_a \quad (2.121)\\ &B_r = B_{(2-z)r} + B_{(\sigma_1)r} + \cdots,\end{aligned}$$

where the dilatation weights are such that

$$\begin{aligned} \delta_D K_{(n)a}{}^b &= -nK_{(n)a}{}^b, \quad n < 2 \\ \delta_D \dot{B}_{(n)a} &= -n\dot{B}_{(n)a}, \qquad \delta_D B_{(n)r} = -nB_{(n)r}. \end{aligned} \tag{2.122}$$

with the logarithmic terms similarly transforming homogeneously. Note that $\delta_D K_{(n)ab} = -(n-2)K_{(n)ab}$ and $[\delta_D, \partial_a] = 0$ but $[\delta_D, \partial_r] \neq 0$. The term $K_{(2)a}{}^b$ transforms as

$$\delta_D K^b_{(2)a} = -2K^b_{(2)a} - 2\tilde{K}^b_{(2)a}. \tag{2.123}$$

This inhomogeneous transformation is obtained by requiring firstly that δ_D does not act on coordinates (i.e, on the logarithm) and secondly that the action of ∂_r on K^b_a provides asymptotically the same result as the action of δ_D, where $\partial_r K_{(d)a}{}^b \sim -dK_{(d)a}{}^b$. Using the vector field equations and divergence equation, one can show that

$$D_a B^a = -\frac{1}{\sqrt{\gamma}}\partial_r(\sqrt{\gamma}B_r), \qquad \dot{B}_r = zB_{(2-z)r} - KB_r, \tag{2.124}$$

and hence the expansion for $\dot{B}_r$ can indeed be written in terms of the expansion for K and B_r.

The expansions of the momenta in eigenfunctions of the dilatation operator can be determined iteratively by solving the field equations. One can now deduce immediately the first subleading term $K_{(\alpha_1)a}{}^b$ of K^b_a by looking at the leading order terms in the Einstein equations. The (ra) equation implies that

$$D_b K^b_a - D_a K = \kappa^2 T_{ar} = \frac{1}{2}F_{ab}\gamma^{cb}(\dot{B}_c - \partial_c B_r) + \frac{1}{2}z^2 B_a B_r. \tag{2.125}$$

Since $D_b K_{(0)a}{}^b - \partial_a K_{(0)} = 0$, the lowest order terms contributing are

$$D_b K_{(\alpha_1)a}{}^b - \partial_a K_{(\alpha_1)} = \frac{z}{2}F_{ab}\gamma^{cb}B_c + \frac{z^2}{2}B_a B_{(2-z)r}. \tag{2.126}$$

This implies that $\alpha_1 = 2(1-z)$. One can then use this fact in the (ab) Einstein equations to find $K_{(\alpha_1)a}{}^b$, resulting in

$$K_{(2-2z)a}{}^b = -\frac{z}{2}\big(B_a B^b - \frac{1}{2}(B\gamma^{-1}B)\delta^b_a\big). \tag{2.127}$$

Note that $K_{(2-2z)} := K_{(2-2z)i}{}^i = 0$. One can derive similar equations for further coefficients in (2.121) but the ordering of the weights $(\alpha_{(n)}, \beta_{(n)})$ depends on the value of z. For example, when $z < 1/2$, the coefficient $\beta_1 = z$ is the first subleading term in the vector field expansion, as we showed in the previous section, whilst for $z > 1/2$, the first subleading term is instead $B_{(2-3z)a}$ since $(2-3z) < z$. At

$z = 1/2$ one needs to include logarithmic terms, related to the conformal anomalies, to satisfy the field equations.

Before solving for further coefficients, let us discuss how this information will be used to determine the renormalized on-shell action and one point functions. Starting from (2.113) one can differentiate the on-shell action with respect to r to obtain

$$\dot{S}_{\mathrm{on-shell}} = \frac{1}{\kappa^2}\int_{\Sigma_r} d^2x\sqrt{\gamma}\left(\hat{R} + 1 - \frac{1}{4}F_{ab}F^{ab} - \frac{1}{2}z^2 B_a B^a\right). \tag{2.128}$$

One can then write the regulated action as

$$I_r = \frac{1}{\kappa^2}\int_{\S_r} d^dx\sqrt{\gamma}(K - \lambda), \tag{2.129}$$

where λ satisfies

$$\dot{\lambda} + K\lambda - \kappa^2(2 + \frac{1}{4}F^2 + \frac{1}{2}z^2B^2) = 0. \tag{2.130}$$

The variable λ admits an expansion in dilatation eigenfunctions:

$$\lambda = \lambda_{(\epsilon_0)} + \lambda_{(\epsilon_1)} + ... + \lambda_{(2)} + \tilde{\lambda}_{(2)}\log e^{-2r} + \cdots \tag{2.131}$$

where each term transforms homogeneously, namely $\delta_D\lambda_{(n)} = -n\lambda_{(n)}$ except for $\lambda_{(2)}$. The transformation law for the latter is obtained in a similar fashion as that for $K_{(2)a}{}^b$. Terms in the on-shell action are divergent as $r \to \infty$ only for $n < 2$, along with the logarithmic term, and thus the counterterm action is formally given by

$$I_{\mathrm{ct}} = -\frac{1}{\kappa^2}\int_{\S_r} d^2x\sqrt{\gamma}\left(\sum_{n<2}(K_{(n)} - \lambda_{(n)}) - \tilde{\lambda}_{(2)}\log e^{-2r}\right). \tag{2.132}$$

The terms in the dilatation expansion of λ can be obtained by iteratively solving the above first order equation defining λ, but a more efficient procedure is the following. Note first that

$$\begin{aligned} 2\gamma_{ab}\pi^{ab} + zB_a\pi^a &= 2\gamma_{ab}\frac{\delta I_r}{\delta\gamma_{ab}} + zB_a\frac{\delta I_r}{\delta B_a} \\ &= \frac{1}{\kappa^2}\int_{\S_r} d^2x\sqrt{\gamma}\left[2\gamma_{ab}\frac{\delta}{\delta\gamma_{ab}} + zB_a\frac{\delta}{\delta B_a}\right](K - \lambda). \end{aligned} \tag{2.133}$$

Hence,

$$2\gamma_{ab}\pi^{ab} + zB_a\pi^a = \frac{1}{\kappa^2}\delta_D(\sqrt{\gamma}(K-\lambda)) \Leftrightarrow (1+\delta_D)K + \frac{z}{2}B_a\gamma^{ab}(\dot{B}_b - \partial_b B_r) = (2+\delta_D)\lambda \tag{2.134}$$

where one has used (2.116) and (2.117), together with: $\delta_D\sqrt{\gamma} = 2\sqrt{\gamma}$ which follows from the definition of δ_D. This last equation then allows the iterative determination of the expansion of λ. For example, looking at the leading order term one finds that

$$K_{(0)} = (2+\delta_D)\lambda_{(0)} \qquad \rightarrow \qquad \lambda_{(0)} = 1. \tag{2.135}$$

The first subleading term has weight $2(1-z)$ and is given by

$$\frac{z^2}{2}(B\gamma^{-1}B) = (2+\delta_D)\lambda_{(2-2z)} \leftrightarrow \lambda_{(2-2z)} = \frac{z}{4}(B\gamma^{-1}B). \tag{2.136}$$

As mentioned already above, the question of which terms appear at subsequent order depends on the value of z. For $z < 1/2$, the only other divergent term is the logarithmic term, which follows from solving (2.134) at weight two. Using the expression for $K_{(2)}$ given in (2.145) one finds that

$$(2+\delta_{\mathcal{D}})(\lambda_{(2)} + \tilde{\lambda}_{(2)}\mathrm{log}e^{-2r}) = -2\tilde{\lambda}_{(2)} = -\frac{1}{2}R. \tag{2.137}$$

For $z < 1/2$ this suffices to determine explicitly the counterterm action

$$I_{\mathrm{ct}} = \frac{1}{2\kappa^2}\int_{\S_r} d^2x\sqrt{\gamma}\big(-2+\frac{z}{2}(B\gamma^{-1}B)+\frac{1}{2}R\log e^{-2r}\big), \tag{2.138}$$

in agreement with that found in the previous section. Further counterterms are needed for $z \geq 1/2$ but, as we will see, the explicit form is not needed to compute renormalized correlation functions for non-rational values of z.

In general, the renormalized on-shell action is given by

$$I_{\mathrm{ren}} = \lim_{r\to\infty}(I_r + I_{\mathrm{ct}}) = \frac{1}{\kappa^2}\int d^2x\sqrt{\gamma}(K_{(2)} - \lambda_{(2)}) \ . \tag{2.139}$$

The one-point functions can be determined by using the Hamilton-Jacobi relations, which can be written as:

$$\pi^{ab}\delta\gamma_{ab} + \pi^a\delta B_a = \frac{1}{\kappa^2}\int_{\S_r} d^2x\delta[\sqrt{\gamma}(K-\lambda)]. \tag{2.140}$$

Taking $r \to \infty$, one expands the momenta and the integrand in eigenfunctions of the dilatation operator and matches terms with the same weight. The procedure implies in particular that:

$$\delta I_{\mathrm{ren}} = \frac{1}{\kappa^2}\int d^2x\delta\left[\sqrt{\gamma}(K_{(2)}-\lambda_{(2)})\right] = \left[\pi_{ab}\gamma^{ac}\gamma^{be}\delta\gamma_{ce}\right]_{(0)} + \left[\pi_a\,\gamma^{ab}\delta B_b\right]_{(0)} \ , \tag{2.141}$$

where the subscript represents the overall terms with zero dilatation weight. Since, by the definition of δ_D, the vector field has weight z, the induced metric weight 2 and its inverse weight -2, the renormalized one-point functions are then found to be:

$$\langle T_{ab}\rangle = -\frac{2}{-\sqrt{g_{(0)}}}\frac{\delta I_{ren}}{\delta g^{ab}_{(0)}} = \lim_{r\to\infty}\left[\frac{2}{-\sqrt{\gamma}}\frac{\delta I_{\rm ren}}{\delta\gamma^{ab}}\right] = \frac{1}{\kappa^2}\left[K_{(2)}\gamma_{ab} - K_{(2)a}{}^{c}\gamma_{cb}\right]; \tag{2.142}$$

$$\langle V_a\rangle = -\frac{1}{\sqrt{-g_{(0)}}}\frac{\delta I_{ren}}{\delta b^a_{(-z)}} = \lim_{r\to\infty}\left[\frac{e^{zr}}{\sqrt{-\gamma}}\frac{\delta I_{ren}}{\delta B^a}\right] = \frac{1}{2\kappa^2}\lim_{r\to\infty}\left[e^{zr}\dot{B}_{(z)a}\right]. \tag{2.143}$$

It should be emphasized that these expressions for the renormalized one point functions hold for general values of $z<1$, as does the form (2.139) for the renormalized action. However, one still needs to determine the relation between the momenta coefficients and coefficients in the asymptotic expansions of the fields, which in general can involve both the normalizable modes and local functionals of the sources.

When $z \neq (1-\frac{1}{n})$, with n an integer, the map between momenta coefficients and terms in the asymptotic expansions is particularly simple. Let us express the asymptotic expansions as in the previous section as

$$\begin{aligned}\gamma_{ab} &= g_{(0)ab} + \cdots + e^{-2r}g_{(2)ab} + \cdots \\ B_a &= e^{zr}(b_{(-z)a} + \cdots) + e^{-zr}(b_{(z)a} + \cdots),\end{aligned} \tag{2.144}$$

where $\rho = e^{-2r}$. Then,

$$\begin{aligned} K_{(2)} &= \frac{1}{2}\left(R[g_{(0)}] - 2z^2 b^a_{(-z)}b_{(z)a}\right); \\ K_{(2)ab} &= -g_{(2)ab} + \frac{z}{2}(b_{(-z)a}b_{(z)b} + b_{(z)a}b_{(-z)b}) - \frac{z}{2}b_{(-z)c}b^c_{(z)}g_{(0)ab}; \\ \left[e^{zr}\dot{B}_{(z)a}\right] &= -2zb_{(z)}. \end{aligned} \tag{2.145}$$

and substituting into the renormalized one point functions (2.142) results in the same expressions as (2.108) and (2.109).

When $z = (1-\frac{1}{n})$, with n an integer, functionals of the vector operator source can have the required dilatation weight to contribute to the one point functions. In such cases there are additional contributions to the map between momenta coefficients and terms in the asymptotic expansions, and one has to compute the one point functions on a case-by-case basis. For example, in the case of $z=1/2$

$$K_{(2)} = \frac{1}{2}\left(R[g_{(0)}] - 2z^2 b^a_{(-z)}b_{(z)a}\right) + \frac{1}{2}K^{\ b}_{(2-2z)a}K^{a}_{(2-2z)b}, \tag{2.146}$$

where, using (2.127),

$$K^{\ b}_{(2-2z)a} = -\frac{z}{2}\left(b_{(-z)a}b^b_{(-z)} - \frac{1}{2}(b_{(-z)c}b^c_{(-z)})\delta^b_a\right). \tag{2.147}$$

This implies that the conformal anomaly is given by

$$\langle T^a_a \rangle = \frac{1}{\kappa^2}\left(R_{(0)} - z^2 b^a_{(-z)} b_{(z)a} + \frac{z^2}{16}(b^a_{(-z)} b_{(-z)a})^2 \right), \tag{2.148}$$

and thus involves a local functional of the vector field source.

2.5.3. Analysis for $z > 1$

Let us now discuss the issues that arise when $z > 1$ and the vector field is dual to an irrelevant operator in the conformal field theory. Since irrelevant operators modify the UV behavior of the quantum field theory, their sources can only be treated perturbatively, which allows their correlation functions to be computed. The holographic analogue can be seen in (2.98): even for $z > 1$ the data $(g_{(0)ab}, g_{(2)ab}, b_{(-z)a}, b_{(z)a})$ supplies the independent integration constants for the bulk equations, but when $z > 1$ the limit of $g_{ab}(\rho \to 0)$ is no longer finite. In fact, using (2.102), one sees that the metric

$$g_{ab} = \sum_{m,n} g_{2m+2n(1-z)} \rho^{m+n(1-z)} \tag{2.149}$$

contains terms for $m = 0$ and $n \geq 0$ which behave as

$$g_{-2n(z-1)} \rho^{-n(z-1)} \sim b^{2n}_{(-z)} \rho^{-n(z-1)}, \tag{2.150}$$

and thus terms which are higher order in the vector operator source diverge faster as $\rho \to 0$, as expected. Working at finite $b_{(-z)}$ an infinite number of counterterms would thus in general be needed. A well-defined problem is obtained by working perturbatively with small $b_{(-z)a}$ such that

$$|b_{(-z)}|^2 \ll \epsilon^{z-1}, \tag{2.151}$$

where $\rho = \epsilon$ is the cutoff. To compute an n-point function of the dual vector operator, one should only retain terms to order $b^n_{(-z)}$ and thus only a finite number of counterterms are needed. Logarithmic terms in the on-shell action related to conformal anomalies can arise whenever

$$z = 1 + \frac{p}{q}, \tag{2.152}$$

where (p, q) are integers. Except in such cases, where z is rational, the renormalized one point functions are just as for $z < 1$, i.e.

$$\langle V_a \rangle = -\frac{z}{\kappa^2} b_{(z)a}; \tag{2.153}$$

$$\begin{aligned}\langle T_{ab} \rangle = \frac{1}{\kappa^2}\Big[& g_{(2)ab} + \frac{1}{2} R(g_{(0)}) g_{(0)ab} - \frac{z}{2}\left(b_{(-z)a} b_{(z)b} + b_{(z)a} b_{(-z)b} \right) \\ & - z(z - \frac{1}{2})\Big(b_{(-z)c} b^c_{(z)} \Big) g_{(0)ab} \Big].\end{aligned} \tag{2.154}$$

To prove this, one can use the Hamiltonian method of the previous section: provided that the source is treated perturbatively, the dilatation operator is well-defined and the momenta admit expansions in eigenfunctions of this dilatation operator. The general expressions for the renormalized one-point functions in terms of the momenta coefficients given in (2.142) can then immediately be rewritten in terms of coefficients in the asymptotic expansion when z is not rational, as terms involving only the vector field sources cannot have the correct dilatation weight. For rational values of z the map between the momenta and asymptotic coefficients can indeed involve polynomials in the vector field sources, and it needs to be worked out iteratively on a case by case basis.

Note that in the Hamiltonian method one does not actually need to explicitly compute the counterterms $\lambda_{(n)}$ to derive the correlation functions, although they would be needed to compute the on-shell value of the action. Formally, at least, one can work to arbitrarily high perturbative order in the operator source $b_{(-z)a}$, with corresponding counterterms of increasing order of divergence. If however the source $b_{(-z)a}$ is treated as finite, then there is no well-defined asymptotic, or equivalently dilatation, expansion and the counterterm action (2.132) is not a priori valid. This corresponds to the fact that switching on a generic finite deformation by the dual vector operator makes the dual quantum field theory non-renormalizable.

In the case of interest here, however, the source $b_{(-z)a}$ is finite but null: just as in the field theory discussion earlier, we can compute correlators of the vector operator in the deformed theory by setting

$$g_{(0)ab} = \eta_{ab}; \qquad b_{(-z)a} \equiv b\delta_{au} + a_{(-z)a}, \tag{2.155}$$

where the source $a_{(-z)a}$ is treated perturbatively. The existence of a dilatation symmetry is preserved at finite b and all bulk fields still admit an asymptotic expansion in terms of eigenfunctions of the dilatation operator, even though the metric g_{ab} does not have a finite limit as $\rho \to 0$.

Now consider the following: treating $b_{(-z)a}$ perturbatively first derive the counterterm action (2.132), working recursively in powers of the source. Then

$$I_{\rm ct} = -\frac{1}{\kappa^2}\int_{\S_r} d^2x\sqrt{\gamma}\left(\sum_{n<2}(K_{(n)} - \lambda_{(n)}) - \tilde{\lambda}_{(2)}\log e^{-2r}\right), \tag{2.156}$$

where in addition to the counterterms $\lambda_{(0)}$, $\tilde{\lambda}_{(2)}$ and $\lambda_{-2(z-1)}$ computed explicitly earlier there are an infinite number of counterterms at $z > 1$. For example, polynomials of the vector field occur,

$$\lambda_{-2n(z-1)} = c_{2n(z-1)}(B^a B_a)^n \tag{2.157}$$

where the coefficients $c_{2n(z-1)}$ may be determined iteratively in n, working perturbatively in the source. This counterterm is the holographic analogue of (2.70) and counterterms involving further derivatives and curvatures will also occur. If these counterterms are evaluated on the anisotropic background itself (2.155) in which $a_{(-z)a} = 0$, then, since the source is both null and constant, all counterterms apart from $\lambda_{(0)}$ vanish. This is the holographic analogue of the deformation being exactly marginal with respect to the anisotropic symmetry. To compute the two point function of the vector operator in the deformed theory we will need to retain terms in the action to order $a^2_{(-z)}$, and following the arguments of section (2.4.3) there will be a finite number of terms for $z < 2$.

2.6. Linearized analysis around chiral background

In this section we will consider the linearized equations of motion around the chiral background for generic values of z in two dimensions, and the corresponding two point functions of the stress energy tensor and vector operator in the deformed theory. We should note that the analysis excludes those values of z for which the deforming operator itself acquires an anomalous dimension; the case of $z = 2$, Schrödinger, is one such example, which was analyzed in detail in [94].

2.6.1. Linearized equations

Let us perturb the fields around the background as:

$$\begin{aligned} B_m &= \rho^{-z/2} b_m(x,\rho) = b\,\rho^{-z/2}\delta^u_m + a_m(\rho,u,v), \\ ds^2 &= \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{ab}(x,\rho) dx^a dx^b, \qquad g_{ab} = \bar{h}_{ab}(\rho) + h_{ab}(\rho,u,v), \end{aligned} \tag{2.158}$$

where

$$\bar{h}_{ab} dx^a dx^b \equiv (2dudv + \sigma^2 \rho^{1-z} du^2). \tag{2.159}$$

The linearized Einstein equations can then be written as:

$$\begin{aligned} & R_{ab}[h] + \mathrm{tr}(\bar{h}^{-1}h')\bar{h}_{ab} - \rho\left(2\,h''_{ab} - 2\,z\,b^2 \delta^u_{(a} h'_{b)v}\,\rho^{-z} + \frac{z}{2} b^2\,\mathrm{tr}(\bar{h}^{-1}h')\delta^u_a\delta^u_b\,\rho^{-z}\right) \\ & = \frac{1}{2}(1+z)z\,b^2 h_{vv}\bar{h}_{ab}\,\rho^{-z} + \left(\frac{z}{2}\,b^2\right)^2 h_{vv}\delta^u_a\delta^u_b\,\rho^{1-2z} + z^2\,b\,a_{(a}\delta^u_{b)}\,\rho^{-z/2} \\ & \quad + 2\,z\,b\left(\delta^u_{(a} f_{b)\rho} - f_{v\rho}\,\bar{h}_{ab}\right)\rho^{1-z/2}; \end{aligned} \tag{2.160}$$

$$\begin{aligned} & \partial_a\left(\mathrm{tr}(\bar{h}^{-1}h')\right) - \bar{h}^{bc}\partial_c h'_{ab} - \frac{1}{4} z\,b^2 \partial_a h_{vv}\,\rho^{-z} + \frac{1}{2} z\,b^2 \delta^u_a\,\rho^{-z}\,\partial_c\left(h^c_v - \frac{1}{2}\mathrm{tr}(h)\delta^c_v\right) \\ & \quad = \frac{1}{2} z\,b\,f_{av}\,\rho^{-z/2} - z^2 b\,a_\rho\,\delta^u_a\,\rho^{-z/2}\ ; \end{aligned} \tag{2.161}$$

$$\frac{1}{4} z\,b^2\,\partial_\rho\left(\rho^{-z} h_{vv}\right) = \frac{1}{2}\mathrm{tr}(\bar{h}^{-1}h'') \tag{2.162}$$

$$R_{ab}[h] - \frac{1}{2}\bar{h}_{ab} R[h] = 0. \tag{2.163}$$

The last equation is the linearization of the two-dimensional identity $R_{ab}[g] - \frac{1}{2}g_{ab}R[g] = 0$. Note also that we define $f_{ab} := \partial_a a_b - \partial_b a_a$ as the curvature of the vector fluctuation a_b.

The linearized vector field equations are

$$\partial_a \left(\bar{h}^{ab} f_{b\rho}\right) - \frac{z}{2}\, b\, \rho^{-1-z/2}\partial_a \left(h^a_v - \frac{1}{2}\,\mathrm{tr}(h)\delta^a_v\right) = \frac{z^2}{\rho}\, a_\rho; \tag{2.164}$$

$$\partial_a \left(\bar{h}^{ab}\bar{h}^{cd} f_{bd}\right) + 4\partial_\rho \left(\rho \bar{h}^{ac} f_{\rho a}\right) + 2\, z\, b\, \rho^{-z/2}\partial_\rho \left(h^c_v - \frac{1}{2}\,\mathrm{tr}(h)\delta^c_v\right) = \frac{z^2}{\rho}\, \bar{h}^{ca} a_a. \tag{2.165}$$

whilst the linearized divergence equation is:

$$\partial_a \left(\bar{h}^{ac} a_c\right) + 4\rho\, a'_\rho - b\, \rho^{-z/2}\partial_a \left(h^a_v - \frac{1}{2}\,\mathrm{tr}(h)\delta^a_v\right) = 0. \tag{2.166}$$

It is also useful to write:

$$\mathrm{tr}(h) = -\sigma^2 h_{vv}\rho^{1-z} + 2h_{uv}; \qquad \partial_a \left(h^a_v - \frac{1}{2}\mathrm{tr}(h)\delta^a_v\right) = -\frac{1}{2}\sigma^2 h_{vv,v}\rho^{1-z} + h_{vv,u}. \tag{2.167}$$

One now begins with the identity (2.163), which implies that $R_{vv} = 0$, $R_{uu} = \sigma^2\rho^{1-z}R_{uv}$, where the components of the Ricci tensor are:

$$\begin{aligned} R_{uv}[h] &= \frac{1}{2}h_{vv,uu} + \frac{1}{2}h_{uu,vv} - h_{uv,uv}; \qquad (2.168)\\ R_{uu}[h] &= \sigma^2\rho^{1-z} \left(\frac{1}{2}h_{vv,uu} + \frac{1}{2}h_{uu,vv} - h_{uv,uv}\right). \end{aligned}$$

Using these identities, the (vv) component of the Einstein equations is solved by

$$h''_{vv} = 0 \qquad \to \qquad h_{vv} = h_{(0)vv} + \rho h_{(2)vv}, \tag{2.169}$$

where both $h_{(0)vv}$ and $h_{(2)vv}$ are arbitrary functions of (u, v). The (v) component of (2.161) together with (2.162) lead to:

$$h'_{uv,v} - h'_{vv,u} - \frac{1}{4}\, z\, b^2\, h_{vv,v}\, \rho^{-z} = 0; \qquad h''_{uv} = \frac{1}{4}\, z\, b^2\, \partial_\rho \left(\rho^{-z}\, h_{vv}\right). \tag{2.170}$$

Integrating the second of these equations gives:

$$h_{uv} = h_{(0)uv} + \rho h_{(2)uv} + \frac{b^2 z}{4}\left(\frac{1}{(1-z)}\rho^{1-z}h_{(0)vv} + \frac{1}{(2-z)}\rho^{2-z}h_{(2)vv}\right), \tag{2.171}$$

whilst the first equation implies that:

$$\partial_v h_{(2)uv} = \partial_u h_{(2)vv}. \tag{2.172}$$

The other Einstein equations do not decouple from the vector field fluctuations. One can however use equation (2.163) to express the remaining graviton fluctuation as

$$
\begin{aligned}
h_{uu} &= \tilde{H}_{(0)uu} + \rho\tilde{H}_{(2)uu} - \frac{z}{4(1-2z)}\sigma^4 h_{(0)vv}\rho^{2-2z} \\
&\quad - \frac{z}{4(3-2z)}\sigma^4\rho^{3-2z}h_{(2)vv} + \frac{\sigma^2}{(2-z)}\rho^{2-z}h_{(2)uv} + h^V_{uu}, \quad (2.173)
\end{aligned}
$$

where $(\tilde{H}_{(0)uu}, \tilde{H}_{(2)uu})$ are integration constants and h^V_{uu} is defined as the solution to

$$
\partial^2_\rho h^V_{uu} = -\frac{1}{2}zb\sigma^2\rho^{1-z}\partial_\rho(\rho^{-z/2}a_v) + zb\partial_\rho(\rho^{-z/2}a_u) + zb\rho^{-z/2}(\frac{1}{2}\sigma^2\rho^{1-z}a_{\rho,v} - a_{\rho,u}). \quad (2.174)
$$

In order to solve the remaining Einstein equations, the graviton fluctuation must in addition satisfy the u component of (2.161) and the (uv) component of (2.160), which requires

$$
\begin{aligned}
h'_{uu,v} &= \frac{1}{2}zb^2\rho^{-z}(h_{(0)vv,u} + \rho h_{(2)vv,u}) + \partial_u h_{(2)uv} + X; \quad (2.175)\\
X &= \frac{1}{2}zb\rho^{-z/2}\left(a_{u,v} - a_{v,u} + 2za_\rho\right); \\
h_{uu,vv} &= 2h_{uv,uv} - h_{vv,uu} - 4h_{(2)uv} + 2z\sigma^2\rho^{1-z}h_{(2)vv} + Y; \quad (2.176)\\
Y &= 2zb\rho^{1-z}\left(\partial_\rho\left(\rho^{z/2}\,a_v\right) - \rho^{z/2}\,a_{\rho,v}\right).
\end{aligned}
$$

As we will show below, these constraints impose a restriction on the integration constant $\tilde{h}_{(2)uu}$, related to the diffeomorphism Ward identity. These equations are automatically satisfied when the vector field equations are solved. We use the notation $(\tilde{H}_{(0)uu}, \tilde{H}_{(2)uu})$ to denote the integration constants anticipating the fact that h^V_{uu} could also contribute terms at order ρ^0 and ρ in the asymptotic expansion as $\rho \to 0$.

The linearized vector field equations can be written in terms of the metric fluctuations as follows. The divergence equation (2.166) becomes

$$
\begin{aligned}
4\rho a'_\rho &= -a_{v,u} - a_{u,v} + \sigma^2 a_{v,v}\rho^{1-z} + b\rho^{-z/2}\Bigg(-\frac{1}{2}\sigma^2\partial_v(h_{(0)vv} + \rho h_{(2)vv})\rho^{1-z} \\
&\quad + \partial_u(h_{(0)vv} + \rho h_{(2)vv})\Bigg). \quad (2.177)
\end{aligned}
$$

Equation (2.164) becomes

$$
\begin{aligned}
z^2\rho^{-1}a_\rho &= \partial_v\left(a_{\rho,u} - a'_u\right) + \partial_u\left(a_{\rho,v} - a'_v\right) - \sigma^2\rho^{1-z}\partial_v\left(a_{\rho,v} - a'_v\right) \quad (2.178)\\
&\quad - \frac{1}{2}zb\rho^{-1-z/2}\left(-\frac{1}{2}\sigma^2\partial_v(h_{(0)vv} + \rho h_{(2)vv})\rho^{1-z} + \partial_u(h_{(0)vv} + \rho h_{(2)vv})\right).
\end{aligned}
$$

Equations (2.165) become

$$\rho^{z/2}\partial_\rho\left[\rho^{1-z}\,\partial_\rho\left(\rho^{z/2}\,a_v\right)\right] = \partial_\rho\left(\rho\,a_{\rho,v}\right) - \frac{1}{4}\partial_v\left(a_{v,u} - a_{u,v}\right) - \frac{1}{2}zb\rho^{-z/2}h_{(2)vv}; \tag{2.179}$$

$$\begin{aligned}\rho^{z/2}\partial_\rho\left[\rho^{1-z}\,\partial_\rho\left(\rho^{z/2}\,a_u\right)\right] = & -\frac{1}{4}\partial_u\left(a_{u,v} - a_{v,u}\right) + \sigma^2\rho^{1-z}\rho^{z/2}\partial_\rho\left[\rho^{1-z}\partial_\rho\left(\rho^{z/2}a_v\right)\right] \\ & + \partial_\rho\left(\rho a_{\rho,u}\right) - \sigma^2\partial_\rho\left(\rho^{2-z}a_{\rho,v}\right) + (1-z)\sigma^2\rho^{1-z}a_v' \\ & + \frac{1}{4}z(1-z)b\sigma^2\rho^{-3z/2}h_{(0)vv} + \frac{1}{4}z(2-z)\sigma^2\rho^{1-3z/2}h_{(2)vv}.\end{aligned} \tag{2.180}$$

These field equations can be diagonalized to give fourth order differential equations. To show this, let us first define the differential operator

$$\Delta := \rho\partial_\rho^2 + \partial_\rho - \frac{z^2}{4}\rho^{-1} + \frac{1}{2}\partial_u\partial_v - \frac{\sigma^2}{4}\rho^{1-z}\partial_v^2. \tag{2.181}$$

We then define

$$a_v^V \equiv a_v - \frac{b}{2}\rho^{-z/2}(h_{(0)vv} + \rho h_{(2)vv}), \tag{2.182}$$

as well as

$$a_\rho^V \equiv a_\rho - \frac{1}{2}b\partial_u^{-1}h_{(2)uv}\rho^{-z/2}, \tag{2.183}$$

where the inverse derivative is abbreviated notation such that

$$A = \partial^{-1}B \qquad \rightarrow \qquad \partial A = B. \tag{2.184}$$

(In practice the solutions are expressed in momentum space, where the inverse derivative acts by division of momenta.) By differentiating (2.177) with respect to v and inserting into (2.179) one obtains

$$\Delta a_v^V = \partial_v a_\rho^V. \tag{2.185}$$

Differentiating (2.177) with respect to ρ and subtracting it from (2.178) one obtains

$$\Delta a_\rho^V = \frac{\sigma^2}{4}(1-z)\rho^{-z}\partial_v a_v^V. \tag{2.186}$$

Combing these equations, one finds that a_ρ^V satisfies the fourth order equation

$$\rho^z\Delta(\rho^z\Delta a_\rho^V) = \frac{\sigma^2}{4}(1-z)\rho^z\partial_v^2 a_\rho^V, \tag{2.187}$$

whilst a_v^V also satisfies a fourth order equation

$$\Delta^2 a_v^V = \frac{\sigma^2}{4}(1-z)\rho^{-z}\partial_v^2 a_v^V. \tag{2.188}$$

Given the solutions for a^V_v and a^V_ρ one can then determine a_u using the remaining vector field equations; one first determines a_u using (2.177) and then checks that the remaining equations are solved.

The general solution of the linearized equations of motion can hence be expressed in terms of solutions to coupled second order equations or, equivalently, the fourth order equations as

$$\begin{aligned} a_v &= \frac{b}{2}\rho^{-z/2}(h_{(0)vv} + \rho h_{(2)vv}) + a^V_v\,; \\ a_\rho &= \frac{1}{2}b\partial_u^{-1}h_{(2)uv}\rho^{-z/2} + a^V_\rho\,. \end{aligned} \tag{2.189}$$

Since (a^V_v, a^V_ρ) satisfy coupled second order equations, the general solution involves four independent integration constants. The other fluctuations can be formally expressed in terms of (a^V_v, a^V_ρ) as

$$\partial_v a^V_u = -4\rho\partial_\rho a^V_\rho - \partial_u a^V_v + \sigma^2\rho^{1-z}\partial_v a^V_v\,; \tag{2.190}$$

$$\partial_v a^V_u \equiv \partial_v a_u - bz\rho^{-z/2}\partial_u^{-1}h_{(2)uv} - \frac{1}{2}b\rho^{-z/2}(\partial_u h_{(0)vv} + \rho\partial_v h_{(2)uv})$$

$$\begin{aligned} \partial^2_\rho h^V_{uu} &= \partial^2_\rho\Big(\rho^{1-z}\sigma^2\partial_v^{-2}(zh_{(2)vv} + \partial_u\partial_v h_{(0)vv}) \\ &\quad +\rho^{2-z}\frac{b^2 z}{2(2-z)}h_{(2)uv} \\ &\quad +2zb\rho^{1-z}\partial_v^{-2}(\partial_\rho(\rho^{z/2}a^V_v) - \rho^{z/2}a^V_{\rho,v})\Big)\,. \end{aligned} \tag{2.191}$$

Now let us consider the differential equations satisfied by the vector fluctuations in more detail. If one Fourier transforms to momentum space so that for every field $\phi(r,u,v)$

$$\tilde\phi(r,k_u,k_v) = \int dudv e^{ik_u u + ik_v v}\phi(r,u,v), \tag{2.192}$$

then the operator Δ acts on $\tilde\phi$ as

$$\Delta\tilde\phi = (\rho\partial^2_\rho + \partial_\rho - \frac{z^2}{4\rho} - \frac{1}{2}k_u k_v + \frac{\sigma^2}{4}\rho^{1-z}k_v^2)\tilde\phi. \tag{2.193}$$

It is then natural to introduce a new dimensionless coordinate x

$$x = (2k_u k_v)\rho \equiv k^2\rho, \tag{2.194}$$

such that

$$\Delta\tilde\phi = k^2(x\partial^2_x + \partial_x - \frac{z^2}{4x} - \frac{1}{4} + \sigma^2 x^{1-z}k_\chi)\tilde\phi \equiv k^2\Delta_x\tilde\phi, \tag{2.195}$$

where

$$k_\chi \equiv 2^{z-4}k_v^z k_u^{z-2}. \tag{2.196}$$

Then the fourth order equation for a_v^V is

$$\Delta_x^2 a_v^V = (z-1)\sigma^2 k_\chi x^{-z} a_v^V \,. \tag{2.197}$$

Since this equation only depends on the dimensionless coordinate x and the quantity $\sigma^2 k_\chi$, the exact solution for a_v^V can only depend on these quantities, as discussed earlier. Regularity throughout the spacetime will, as we show below, impose two conditions on the four independent solutions of this equation. In what follows we will solve the equations at weak chirality, namely perturbatively in σ^2, in which case it is more convenient to use the coupled second order equations rather than the fourth order equation.

2.6.2. 'T' and 'X' modes of solution

One can summarize the general solution of the linearized equations of motion as follows. The metric fluctuations are

$$\begin{aligned} h_{vv} =& \; h_{(0)vv} + \rho h_{(2)vv}; \\ h_{uv} =& \; h_{(0)uv} + \rho h_{(2)uv} + \frac{b^2 z}{4}\left(\frac{1}{(1-z)}\rho^{1-z}h_{(0)vv} + \frac{1}{(2-z)}\rho^{2-z}h_{(2)vv}\right); \end{aligned} \tag{2.198}$$

$$\begin{aligned} h_{uu} =& \; \tilde{h}_{(0)uu} + \rho\tilde{h}_{(2)uu} + \rho^{1-z}\sigma^2\partial_v^{-2}(zh_{(2)vv} + \partial_u\partial_v h_{(0)vv}) \\ &+ \rho^{2-z}\frac{b^2 z}{2(2-z)}h_{(2)uv} \\ &+ 2zb\rho^{1-z}\partial_v^{-2}(\partial_\rho(\rho^{z/2}a_v^V) - \rho^{z/2}a_{\rho,v}^V). \end{aligned}$$

The vector fluctuations are

$$\begin{aligned} a_v &= \frac{b}{2}\rho^{-z/2}(h_{(0)vv} + \rho h_{(2)vv}) + a_v^V; \\ a_\rho &= \frac{1}{2}b\partial_u^{-1}h_{(2)uv}\rho^{-z/2} + a_\rho^V; \\ \partial_v a_u^V &= \partial_v a_u - bz\rho^{-z/2}\partial_u^{-1}h_{(2)uv} - \frac{1}{2}b\rho^{-z/2}(\partial_u h_{(0)vv} + \rho\partial_v h_{(2)uv}); \\ \partial_v a_u^V &= -4\rho\partial_\rho a_\rho^V - \partial_u a_v^V + \sigma^2\rho^{1-z}\partial_v a_v^V \,. \end{aligned} \tag{2.199}$$

The propagating modes a_m^V solve the coupled differential equations:

$$\Delta a_v^V = \partial_v a_\rho^V; \qquad \Delta a_\rho^V = \frac{\sigma^2}{4}(1-z)\rho^{-z}\partial_v a_v^V \,, \tag{2.200}$$

where the second order differential operator Δ is given in (2.181).

Let us express the source and normalizable modes in the asymptotic expansion of h_{uu} as $\rho \to 0$ as

$$h_{uu} = h_{(0)uu} + \rho h_{(2)uu} + \cdots \,, \tag{2.201}$$

respectively. These are given in terms of the integration constants $(\tilde{h}_{(0)uu}, \tilde{h}_{(2)uu})$ as

$$\begin{aligned} h_{(0)uu} &= \tilde{h}_{(0)uu} + 2z^2 b \partial_v^{-2} a^V_{(z)v}; \\ h_{(2)uu} &= \tilde{h}_{(2)uu} + 2zb(z+1)\partial_v^{-2} a^V_{(z+2)v} - 2zb\partial_v^{-1} a^V_{(z)\rho}, \end{aligned} \tag{2.202}$$

where $a^V_{(m)a}$ is the coefficient of the term at order $\rho^{m/2}$ in the asymptotic expansion of a^V_a as $\rho \to 0$. Note that it is $h_{(0)uu}$ which is the source for dual operator, and the (u), (v) and (uv) components of Einstein equations at order ρ^0 enforce the linearized Ward identities

$$\begin{aligned} \partial_v h_{(2)uv} &= \partial_u h_{(2)vv}; \\ \partial_v h_{(2)uu} &= \partial_u h_{(2)uv} - zb\partial_u a^V_{(z)v}, \\ h_{(2)uv} &= -\frac{1}{4} R[h_{(0)}] + \frac{z^2}{2} b a^V_{(z)v}, \end{aligned} \tag{2.203}$$

and

$$R[h_{(0)}] = \partial_u^2 h_{(0)vv} + \partial_v^2 h_{(0)uu} - 2\partial_u \partial_v h_{(0)uv}. \tag{2.204}$$

We have used the fact that the coupled differential equations for a^V_m can be solved asymptotically as $\rho \to 0$. The resulting solutions have the structure expected from the previous (non-linear) analysis, namely

$$\begin{aligned} a^V_a &= a^V_{(-z)a}\rho^{-z/2} + \cdots + a^V_{(z)a}\rho^{z/2} + \cdots, \\ a^V_\rho &= \frac{1}{2z}(\partial_v a^V_{(-z)u} + \partial_u a^V_{(-z)v})\rho^{-z/2} + \cdots \\ &\quad - \frac{1}{2z}(\partial_v a^V_{(z)u} + \partial_u a^V_{(z)v})\rho^{z/2} + \cdots \end{aligned} \tag{2.205}$$

where we have isolated the terms corresponding to the operator source $(a^V_{(-z)a})$ and operator expectation value $(a^V_{(z)a})$ respectively. The analogous terms in the radial component of the vector field are completely determined in terms of these components. The case of $z = 2$ is special, as the radial powers in the independent solutions depend explicitly on b^2, see [94], because the dimension of the dual operator is modified at non-zero b.

Recall that for $z < 1$ the holographic one point functions at the linearized level are given in terms of coefficients in the asymptotic expansion as

$$\begin{aligned} \langle T_{vv} \rangle &= \frac{1}{\kappa^2} h_{(2)vv}; \qquad \langle T_{uv} \rangle = -\frac{1}{\kappa^2} h_{(2)uv}; \\ \langle T_{uu} \rangle &= \frac{1}{\kappa^2}\left(h_{(2)uu} - bza_{(z)u}\right); \\ \langle V_a \rangle &= -\frac{z}{\kappa^2} a_{(z)a}. \end{aligned} \tag{2.206}$$

Combining the propagating solution for a^V with the other modes results in the following source and vev terms in the asymptotic expansions for the metric and vector fluctuations

$$
\begin{aligned}
h_{vv} &= h_{(0)vv} + \rho h_{(2)vv}; \\
h_{uv} &= h_{(0)uv} + \rho h_{(2)uv} + \cdots \\
h_{uu} &= h_{(0)uu} + \rho h_{(2)uu} + \cdots \\
a_v &= \rho^{-z/2}\left(a^V_{(-z)v} + \frac{b}{2}h_{(0)vv}\right) + \cdots + a^V_{(z)v}\rho^{z/2} + \cdots \\
a_u &= \rho^{-z/2}\left(a^V_{(-z)u} + \frac{b}{2}\partial_v^{-1}\partial_u h_{(0)vv} + bz\partial_v^{-1}\partial_u^{-1}h_{(2)uv}\right) \\
&\quad + a^V_{(z)u}\rho^{z/2} + \cdots
\end{aligned}
\tag{2.207}
$$

These expressions imply that the stress energy tensor sources are $g_{(0)ab} = \eta_{(0)ab} + h_{(0)ab}$ and the vector operator sources are given by

$$
\begin{aligned}
b_{(-z)v} &= a_{(-z)v} = \left(a^V_{(-z)v} + \frac{b}{2}h_{(0)vv}\right); \\
b_{(-z)u} &= b + a_{(-z)u} = b + \left(a^V_{(-z)u} + \frac{b}{2}\partial_v^{-1}\partial_u h_{(0)vv} + bz\partial_v^{-1}\partial_u^{-1}h_{(2)uv}\right).
\end{aligned}
\tag{2.208}
$$

Thus in particular the fluctuation $h_{(0)ab}$ sources not just the stress energy tensor but also the vector operator. To compute two point functions of the stress energy tensor one should set the vector source to zero, by switching on appropriate $a^V_{(-z)a}$, whilst to compute the two point functions of the vector operator one should set to zero $h_{(0)ab}$. Note that switching off the sources for either set of operators does not switch off their expectation values, since the two point functions in the deformed theory are non-diagonal.

In [94] the general linearized solution for $z = 2$ was given in terms of independent solutions of the equations of motion, the 'T' and 'X' modes. The 'T' mode solution is the $z \to 2$ limit of the solution given above with $a^V = 0$, which involves only the integration constants $(h_{(0)ab}, h_{(2)ab})$. The limit of $z \to 2$ requires

$$
\frac{1}{(2-z)}\rho^{2-z} \to \ln(\rho). \tag{2.209}
$$

This 'T' mode solution is non-dynamical, in that there is no bulk differential equation satisfied by these modes. From a field theoretic perspective, these correspond to quantities which are completely determined by Ward identities. From the bulk perspective the corresponding statement is that the 'T' mode solution is equivalent to a bulk diffeomorphism. To show this, let us consider a bulk diffeomorphism generated by a vector field ζ_m such that

$$
\delta g_{mn} = (D_m\zeta_n + D_n\zeta_m). \tag{2.210}
$$

Restricting to diffeomorphisms which respect the Fefferman-Graham form of the metric requires $\delta g_{rr} = \delta_{ra} = 0$ and hence

$$\begin{aligned} \zeta_\rho &= \frac{\zeta}{\rho}; \qquad \zeta_v = \frac{\zeta_{(0)v}}{\rho} - \partial_v \zeta; \\ \zeta_u &= \frac{\zeta_{(0)u}}{\rho} - \partial_v \zeta + \sigma^2 \rho^{-z} \zeta_{(0)v} \\ &\quad + \frac{\sigma^2 (z-1)}{(2-z)} \rho^{1-z} \partial_v \zeta, \end{aligned} \tag{2.211}$$

where $(\zeta, \zeta_{(0)a})$ are independent arbitrary functions of (u, v). The metric variations are then

$$\begin{aligned} \delta g_{vv} &= \frac{1}{\rho} \left(2\partial_v \zeta_{(0)v} - 2\partial_v^2 \zeta \rho \right); \\ \delta g_{uu} &= \frac{1}{\rho} \left(2\partial_u \zeta_{(0)u} - 2\partial_u^2 \zeta \rho \right) \\ &\quad - \frac{4\sigma^2}{(z-1)} \rho^{-z} \zeta + \frac{2\sigma^2}{(2-z)} \rho^{1-z} \partial_u \partial_v \zeta; \\ \delta g_{uv} &= \frac{1}{\rho} \left((\partial_u \zeta_{(0)v} + \partial_v \zeta_{(0)u}) - 4\zeta - 2\partial_u \partial_v \zeta \rho \right) \\ &\quad + \sigma^2 \rho^{-z} \partial_v \zeta_{(0)v} + \frac{\sigma^2 (z-1)}{(2-z)} \rho^{1-z} \partial_v^2 \zeta, \end{aligned} \tag{2.212}$$

and using the analog of (2.76) the vector field fluctuations are

$$\begin{aligned} \delta b_\rho &= -b \partial_v \zeta \rho^{-z/2}; \\ \delta b_v &= b \rho^{-z/2} (\partial_v \zeta_{(0)v} - \rho \partial_v^2 \zeta); \\ \delta b_u &= -b \rho^{-z/2} (\partial_u \zeta_{(0)v} - \rho \partial_v \partial_u \zeta) - 2bz \rho^{-z/2} \zeta. \end{aligned} \tag{2.213}$$

Noting that $\delta g_{mn} = h_{mn}/\rho$ this agrees with the 'T' mode fluctuations, under the identifications

$$h_{(0)vv} = 2\partial_v \zeta_{(0)v}; \qquad h_{(0)uu} = 2\partial_u \zeta_{(0)u}; \qquad h_{(0)uv} = (\partial_u \zeta_{(0)v} + \partial_v \zeta_{(0)u}) - 4\zeta, \tag{2.214}$$

with all other modes determined in terms of these quantities. The 'X' mode solution of [94] corresponds to our propagating solution a^V. In the limit of $z \to 2$ the coupled differential equations (2.200) remain well-defined, but the asymptotic solutions of these equations depend explicitly on b^2, since the corresponding vector operator picks up an anomalous dimension at non-zero b [94].

One of the puzzling features in [94] was that in the 'T' mode solution the vector field is expressed non-locally in terms of the "source" data $h_{(0)ab}$. In the case of $z < 1$, where the relationship between asymptotics of the fluctuations and operator data is known, the reason for this feature is now clear: $h_{(0)ab}$ does not source just T_{ab}, but it also sources the vector operator V_a. Moreover, from (2.208), one sees that the source $a_{(-z)u}$ is non-locally expressed in terms of $h_{(0)ab}$. Note however that the 'T' mode solution is manifestly local when expressed in terms of the vector ζ^m generating the bulk diffeomorphism. From the boundary perspective ζ parameterizes a Weyl rescaling, whilst $\zeta_{(0)a}$ generates a boundary diffeomorphism.

Since the asymptotic expansion is local in the ζ_m it would be natural to set up a variational problem in terms of these quantities. Such a vector field formalism for the case of $z > 1$ will be explored elsewhere. In the case of $z < 1$, it is not necessary to use such a formalism as one can exploit the fact that the spacetime is asymptotically locally anti-de Sitter to set up the variational problem and holographic renormalization in terms of the usual data $(g_{(0)ab}, b_{(-z)a})$. In this case one can compute the two point functions as follows: using the Ward identities the only undetermined information is the two point functions of the vector operator. These can be computed by setting to zero the sources $g_{(0)ab}$, and solving the differential equations for the propagating modes a^V. The sources for the vector operator will induce expectation values for the stress energy tensor, corresponding to the cross correlators between the stress energy tensor and the vector operators. These two point functions are also completely determined by the Ward identities, and therefore do not give additional information. By the arguments given in the previous section, the same procedure may be carried out for generic $z > 1$, when z is not rational, and when the deforming vector operator does not acquire an anomalous dimension. What remains to be done, therefore, is to find the regular solutions for a^V.

2.6.3. Solution around AdS_3

Let us first solve the vector field equations for $\sigma^2 = 0$. Using (2.185) and (2.185), one can show that the regular solutions are:

$$\begin{aligned} a_v(\rho,k) &= a_{(-z)v}(k)\tilde{K}_z(k\sqrt{\rho}) + \frac{2ik_v}{k^{1-z}}a_{(-z)\rho}\frac{2^{1-z}\sqrt{\rho}}{\Gamma(z)}K_{(z-1)}(k\sqrt{\rho}); \\ a_\rho(\rho,k) &= a_{(-z)\rho}\tilde{K}_z(k\sqrt{\rho}), \end{aligned} \tag{2.215}$$

where $\tilde{K}_z(k\sqrt{\rho})$ represents the modified Bessel function with a specific normalization such that

$$\begin{aligned} \tilde{K}_z(k\sqrt{\rho}) &= \frac{2^{(1-z)}}{\Gamma(z)}k^z K_z(k\sqrt{\rho}) \\ &= \rho^{-z/2}(1 + \frac{k^2\rho}{4(1-z)} + \cdots) + \frac{\Gamma(-z)}{2^{2z}\Gamma(z)}k^{2z}\rho^{z/2}(1 + \frac{k^2\rho}{4(1+z)} + \cdots), \end{aligned}$$

and the latter is the expansion as $\rho \to 0$. In solving the equations recurrence relations for modified Bessel functions are useful:

$$K_{z+1}(x) = K_{z-1}(x) + \frac{2z}{x} K_z(x); \qquad \partial_x K_z(x) = -K_{z-1}(x) - \frac{z}{x} K_z(x). \quad (2.216)$$

Using (2.177) one can then show that the solution

$$\begin{aligned} a_u(\rho, k) &= a_{(-z)u}(k) \tilde{K}_z(k\sqrt{\rho}) + \frac{2ik_u}{k^{1-z}} a_{(-z)\rho} \frac{2^{1-z}\sqrt{\rho}}{\Gamma(z)} K_{(z-1)}(k\sqrt{\rho}) \\ a_{(-z)\rho} &= \frac{1}{2z}(\partial_v a_{(-z)u} + \partial_u a_{(-z)v}), \end{aligned} \quad (2.217)$$

satisfies all remaining equations. As a consistency check note that the asymptotic expansions of all vector field components as $\rho \to 0$ agree with those given in section 2.5, with the normalizable modes determined in terms of the non-normalizable modes as follows

$$\begin{aligned} a_v(\rho, k) &= a_{(-z)v}(k)\rho^{-z/2} + \cdots \frac{k_v k^{2z}}{k_u} \frac{\Gamma(-z)}{2^{2z}\Gamma(z)} a_{(-z)u}\rho^{z/2} + \cdots \quad (2.218) \\ a_u(\rho, k) &= a_{(-z)u}(k)\rho^{-z/2} + \cdots \frac{k_u k^{2z}}{k_v} \frac{\Gamma(-z)}{2^{2z}\Gamma(z)} a_{(-z)v}\rho^{z/2} + \cdots \end{aligned}$$

The two point functions are computed using

$$\langle V_a(k) V_b(-k) \rangle = \frac{z}{\kappa^2} \frac{\delta b_{(z)a}}{\delta b^b_{(-z)}} = \frac{z}{\kappa^2} \frac{\delta a_{(z)a}}{\delta a^b_{(-z)}} + \cdots, \quad (2.219)$$

where the ellipses denotes contact terms, and thus

$$\langle V_v(k) V_v(-k) \rangle = -\frac{1}{\kappa^2} \frac{k_v k^{2z}}{k_u} \frac{\Gamma(1-z)}{2^{2z}\Gamma(z)}; \qquad \langle V_u(k) V_u(-k) \rangle = -\frac{1}{\kappa^2} \frac{k_u k^{2z}}{k_v} \frac{\Gamma(1-z)}{2^{2z}\Gamma(z)},$$

with the cross correlation function vanishing, as it should, since the operators have different scaling weights. These expressions can be written in position space as follows. Recall that the general expression for the Fourier transform of a polynomial in d dimensions is

$$\frac{1}{(2\pi)^d} \int d^d k e^{-i\vec{k}\cdot\vec{x}} (k^2)^\lambda = \pi^{-d/2} 2^{2\lambda} \frac{\Gamma(d/2 + \lambda)}{\Gamma(-\lambda)} (|x|^2)^{-\lambda - d/2}, \quad (2.220)$$

which is valid when $\lambda \neq -(d/2 + n)$, where n is zero or a positive integer. Using this Fourier transform, and its derivatives with respect to x, one obtains

$$\langle V_v(x) V_v(0) \rangle = \frac{z(z+1)}{4\pi\kappa^2} \frac{1}{|x|^{2z} v^2}; \qquad \langle V_u(x) V_u(0) \rangle = \frac{z(z+1)}{4\pi\kappa^2} \frac{1}{|x|^{2z} u^2}, \quad (2.221)$$

which is of the expected form for operators of these scaling dimensions.

2.6.4. General solution

We now consider the case of $\sigma \neq 0$ with the sources for the dual stress-energy tensor switched off. Let us first express the asymptotic expansions of the solutions to the dynamical vector field equation as

$$a_a^V = \rho^{-z/2} a^V_{(-z)a} + \cdots + X_a{}^b(\sigma, k) a^V_{(-z)b} \rho^{z/2} + \cdots , \tag{2.222}$$

where the matrix $X_a{}^b(\sigma, k)$ is to be determined by solving the inhomogeneous differential equations exactly and imposing regularity conditions. The asymptotic expansion of the vector field is then written in terms of this data as

$$a_a = \rho^{-z/2}(a^V_{(-z)a} + b\, z\, \delta_{au} \partial_v^{-1} \partial_u^{-1} h_{(2)uv}) + \cdots + X_a{}^b(\sigma, k) a^V_{(-z)b} \rho^{z/2} + \cdots , \tag{2.223}$$

Note that the source for the vector operator includes another term involving $h_{(2)uv}$, as the latter is not automatically set to zero by setting $h_{(0)ab} = 0$. Indeed from the linearized Ward identity (2.203) one knows that

$$h_{(2)uv} = \frac{z^2}{2} b a^V_{(z)v} \equiv \frac{z^2}{2} b X_v{}^b(\sigma, k) a^V_{(-z)b}. \tag{2.224}$$

The true vector operator sources are thus defined in terms of the asymptotic solutions to the dynamical equations as

$$a_{(-z)a} = a^V_{(-z)a} - \frac{b^2 z^3}{k^2} \delta_{au} X_v{}^b(\sigma, k) a^V_{(-z)b}, \tag{2.225}$$

and therefore

$$\begin{aligned} a_{(-z)v} &= a^V_{(-z)v}; \\ a_{(-z)u} &= (1 - \frac{b^2 z^3}{k^2} X_{vv}) a^V_{(-z)u} - \frac{b^2 z^3}{k^2} X_u{}^v a^V_{(-z)v}. \end{aligned} \tag{2.226}$$

These relations allow one to rewrite the modes $a^V_{(-z)a}$ in terms of the true sources, and thence one can also obtain the relationship between the normalizable modes $a^V_{(z)a}$ and the sources. Functionally differentiating the linearized one point functions with respect to the sources one then finds that

$$\begin{aligned} \langle V_v V_v \rangle &= \frac{z}{\kappa^2} \left(1 - \frac{b^2 z^3}{k^2} X_{vv}\right)^{-1} X_{vv}; \\ \langle V_u V_v \rangle &= \frac{z}{\kappa^2} X_{vu} \left(1 - \frac{b^2 z^3}{k^2} X_{vv}\right)^{-1}; \\ &\equiv \frac{z}{\kappa^2} X_{uv} \left(1 - \frac{b^2 z^3}{k^2} X_{vv}\right)^{-1}; \\ \langle V_u V_u \rangle &= \frac{z}{\kappa^2} \left(X_{uu} + \frac{b^2 z^3}{k^2} X_{uv} X_{vu} (1 - \frac{b^2 z^3}{k^2} X_{vv})^{-1}\right). \end{aligned} \tag{2.227}$$

The fact that $\langle V_v V_u \rangle = \langle V_u V_v \rangle$ must then follow from the symmetry of the matrix X_{ab} that arises in solving the differential equations.

Next let us consider the vector field equations at $\sigma \neq 0$. It is useful to write the equations (2.185) and (2.186) in the form:

$$\begin{aligned} \Delta_0 a_v^V - \partial_v a_\rho^V &= \frac{1}{4}\sigma^2 \rho^{1-z} \partial_v^2 a_v^V; \\ \Delta_0 a_\rho^V &= \frac{1}{4}\sigma^2 (1-z)\rho^{-z}(\rho \partial_v^2 a_\rho^V + \partial_v a_v^V), \end{aligned} \tag{2.228}$$

where Δ_0 is the restriction of the differential operator Δ to $\sigma^2 = 0$. It is interesting to note that the corrections to the differential equation at $\sigma^2 \neq 0$ vanish when the lightcone momentum $k_v = 0$. Working in conformal perturbation theory we noted that corrections were organized in powers of $\sigma^2 k_v^2$, and the same behavior is found holographically. Let us try to solve the equations perturbatively in σk_v at $k_v \neq 0$ by looking for solutions of the form

$$\begin{aligned} a_v^V &= \sum_{n>0} \sigma^{2n} (a_v^V)_n; \\ a_\rho^V &= \sum_{n>0} \sigma^{2n} (a_\rho^V)_n, \end{aligned} \tag{2.229}$$

where the $n = 0$ solutions are given by (2.215). The coupled differential equations then reduce to pairs of inhomogeneous differential equations generating a recurrence relation

$$\begin{aligned} \Delta_0 (a_v^V)_{n+1} - \partial_v (a_\rho^V)_{n+1} &= \frac{1}{4}\rho^{1-z} \partial_v^2 (a_v^V)_n; \\ \Delta_0 (a_\rho^V)_{n+1} &= \frac{1}{4}(1-z)\rho^{-z}(\rho \partial_v^2 (a_\rho^V)_n + \partial_v (a_v^V)_n). \end{aligned} \tag{2.230}$$

For generic values of z the corrections $(a_a^V)_n$ are bounded as $\rho \to \infty$, since the differential operator Δ_0 has an essential singularity as $\rho \to \infty$ and so the regular $n = 0$ solutions decay exponentially there:

$$a_\rho(\rho, k) = a_{(-z)\rho} e^{-k\sqrt{\rho}} \left(\frac{(k/2)^{z-1/2}\sqrt{\pi}}{\Gamma(z)\rho^{1/4}} + \mathcal{O}(\rho^{-3/4}) \right). \tag{2.231}$$

Solving for the inhomogeneous contributions to the corrections as $\rho \to \infty$ one finds that they also behave as

$$(a_a^V)_n \sim e^{-k\sqrt{\rho}} \rho^{-1/4}, \tag{2.232}$$

and are hence exponentially small.

Once we have established that the inhomogeneous contributions to the corrections are finite everywhere, we need to solve the inhomogeneous differential equations to extract the asymptotic coefficients $(a^V_{(-z)a}, a^V_{(z)a})$. This could be carried out numerically for finite chirality b^2, and can be done perturbatively in b^2 at small chirality, using the Green function for the differential operator Δ_0, which is given in the appendix. This results in the following correlation functions

$$\begin{aligned}
\langle V_v(k)V_v(-k)\rangle &= -\frac{1}{\kappa^2}\frac{k_v k^{2z}}{k_u}\frac{\Gamma(1-z)}{2^{2z}\Gamma(z)}\left(1+c_{vv}k_\chi^2\sigma^2\right); \\
\langle V_v(k)V_u(-k)\rangle &= -\frac{1}{\kappa^2}c_{vu}k_\chi^2\sigma^2; \\
\langle V_u(k)V_u(-k)\rangle &= -\frac{1}{\kappa^2}\frac{k_u k^{2z}}{k_v}\frac{\Gamma(1-z)}{2^{2z}\Gamma(z)}\left(1+c_{uu}k_\chi^2\sigma^2\right),
\end{aligned} \tag{2.233}$$

where the constant numerical coefficients c_{ab} are given in the appendix.

To summarize, these correlation functions are sufficient to reconstruct all two point functions of the stress energy tensor and vector operator, to leading order in b^2. The functional form of the correlation functions is as anticipated from anisotropic scale invariance and conformal perturbation theory. As we will emphasize in the conclusions, the holographic models for scale invariance with exponent z always include fields dual to the deforming, Lorentz symmetry breaking, operators. These operators must therefore necessarily play an important rôle in the physics of the condensed matter system being modeled. At small chirality, the correlation functions of these operators are given by the above formulae and these should match the features of the system under consideration. One would also like the finite temperature holographic realization to match the behavior of the physical system under consideration, and we will next turn to modeling finite temperature physics with black holes.

2.7. Black holes

It would be interesting to find black hole solutions of the gravity-vector system, in order to probe the phase structure of the anisotropic theory. Again there will be a qualitative difference between the cases of $z < 1$ and $z > 1$. In the former case, the deformation is relevant with respect to the conformal symmetry and one would only expect to retain the effects of the deformation at temperatures which are small compared to the deformation parameter:

$$T \ll b^{1/z}. \tag{2.234}$$

Let us start by considering the following black hole solution in three dimensions

$$\begin{aligned}
ds^2 = {} & \frac{dr^2}{r^2(1-(r/r_+)^{2(2-z)})} + \\
& \frac{1}{r^2}\left(-\left(2-(r/r_+)^{2(1-z)}-(r/r_+)^2\right)d\eta^2 + 2\left(1-(r/r_+)^2\right)d\eta dx + (r/r_+)^2 dx^2\right).
\end{aligned} \tag{2.235}$$

This geometry describes a black hole with Killing horizon $r = r_+$ and generator $K = \partial_\eta$. The critical exponent is restricted to $z < 2$, otherwise $g_{rr} \not\to 1/r^2$ as $r \to 0$. Under the coordinate transformation: $x := \xi + \eta$, the metric becomes:

$$ds^2 = \frac{1}{r^2}\left(\frac{dr^2}{1-(r/r_+)^{2(2-z)}} + (r/r_+)^{2(1-z)}d\eta^2 + 2\,d\xi\,d\eta + (r/r_+)^2 d\xi^2\right). \tag{2.236}$$

In these coordinates, the black hole is manifestly asymptotic to the chiral scale-invariant background as $r \to 0$ for $z < 2$. Next one lets $\eta = \sigma r_+^{1-z}u$ and $\xi = r_+^{z-1}v/\sigma$ so that

$$ds^2 = \frac{dr^2}{r^2(1-(r/r_+)^{2(2-z)})} + \sigma^2 r^{-2z}du^2 + \frac{2}{r^2}dudv + \frac{dv^2}{\sigma^2 r_+^{4-2z}}. \tag{2.237}$$

The anisotropic scale invariant background can be obtained as the zero temperature limit of the black hole, corresponding to $r_+ \to \infty$ with σ finite.

Einstein equations admitting such solutions can be constructed as follows. Writing the scale invariant geometry as

$$ds^2 = \frac{1}{r^2}\left(dr^2 + \sigma^2 r^{2(1-z)}du^2 + 2\,d\xi\,du\right) \tag{2.238}$$

note that the Einstein tensor G_{ab} satisfies

$$G_{ab} = g_{ab} + z^2 B_a B_b, \qquad B = br^{-z}du \qquad b^2 = 2\frac{1-z}{z}\sigma^2. \tag{2.239}$$

The contravariant components are correspondingly

$$G^{ab} = g^{ab} + z^2 B^a B^b, \qquad B = -br^{2-z}\partial_\xi. \tag{2.240}$$

For the above black hole solution, the Einstein tensor satisfies

$$G^{ab} = g^{ab} + z^2 B^a B^b, \qquad B = -br^{2-z}\partial_\xi, \qquad b^2 = 2\frac{1-z}{z}r_+^{-2(1-z)}. \tag{2.241}$$

This means that the contravariant energy tensor is actually exactly the same in both cases. However, while for the scale invariant background one can write a pure Proca action generating the required field equations, for the black hole solution one cannot. Note that the case of $z = 1$ is exceptional: the above black hole reduces to the BTZ black hole which satisfies the Einstein equations without matter. The case of $z = 0$ in three dimensions is also special, as the spacetime is Einstein.

The fact that the black hole solution does not follow from a Proca action suggests that a string theory embedding may give rise to consistent truncations involving not just vectors, but vectors coupled to scalars. This is indeed known to be the case for Schrödinger ($z = 2$) in five bulk dimensions, see the consistent truncation found in [84]. It is interesting however to note that the black hole solution can be supported by dust or by a perfect fluid; appropriate actions for dust solutions can be found in [125] and for perfect fluids in [126].

Starting from (2.235), the normal to the hypersurfaces of constant r is null at the horizon as well as the Killing vector ∂_η: $||dr||^2 = 0 = ||\partial_\eta||$ at r_+. To see that the Killing is normal to $r = r_+$, one rewrites (2.235) in the form:

$$ds^2 = \frac{1}{r^2}\left[-\left(2-(r/r_+)^{2(1-z)}-(r/r_+)^2\right)\left(d\eta^2-dr^{*2}\right)+2\left(1-(r/r_+)^2\right)d\eta dx+(r/r_+)^2dx^2\right], \tag{2.242}$$

where

$$dr^* = \frac{dr}{\sqrt{1-(r/r_+)^{2(2-z)}}\sqrt{2-(r/r_+)^{2(1-z)}-(r/r_+)^2}}. \tag{2.243}$$

Then define: $\eta = U + r^*$

$$\begin{aligned} ds^2 = \ & \frac{1}{r^2}\Bigg[-\left(2-(r/r_+)^{2(1-z)}-(r/r_+)^2\right)dU^2 - 2\sqrt{\frac{2-(r/r_+)^{2(1-z)}-(r/r_+)^2}{1-(r/r_+)^{2(2-z)}}}dUdr \\ & +2\left(1-(r/r_+)^2\right)(dUdx+dr^*dx)+(r/r_+)^2dx^2\Bigg]. \end{aligned} \tag{2.244}$$

In this coordinate system, the metric is well behaved at the horizon with the metric close to the horizon being

$$ds^2 = \frac{1}{r^2}\left(-2\,dUdr + \frac{1}{2-z}\,dxdr + dx^2\right), \tag{2.245}$$

which is well behaved everywhere near the horizon. One can further define $U = \frac{1}{2}y + \frac{x}{2(2-z)}$ with $r = \frac{1}{R}$ to obtain

$$ds^2 = dydR + R^2dx^2, \qquad (r \to r_+). \tag{2.246}$$

This is exactly the same metric as that of the non-rotating BTZ black hole near the horizon in Eddington-Finkelstein coordinates as long as one compactifies the coordinate x with period 2π. From (2.244), the Killing vector $k = \partial_\eta$ in this coordinate system becomes ∂_U. This means that

$$\bar{k} = g_{ac}k^c dx^a = g_{UU}dU + g_{rU}dr + g_{xU}dx. \tag{2.247}$$

At the horizon: $\bar{k} = \frac{1}{r_+^2}\,dr$, which implies that the horizon is indeed a Killing horizon with respect to ∂_η.

The temperature of the black hole (2.235) is

$$T_L = \frac{\kappa}{2\pi} = \frac{2-z}{2\pi r_+}, \tag{2.248}$$

where κ is the surface gravity, whilst the entropy is given by

$$S = \frac{\beta_x}{4G_3 r_+}, \tag{2.249}$$

where β_x is the periodicity of the x direction and G_3 is the Newton constant. The entropy density $s = S/\beta_x$ can be expressed as

$$s = \frac{\pi}{2G_3(2-z)} T_L, \tag{2.250}$$

the form of which is determined on dimensional and scaling grounds.

In the absence of a complete solution involving appropriate fields, one cannot directly interpret the black hole (2.235) in terms of finite temperature behavior of the deformed chiral theory. However, one can make interesting preliminary observations: the temperature is associated with the periodicity of the Euclidean coordinate $\bar{u} = iu$. Note however that u is a null coordinate in the quantum field theory, and therefore the temperature T_L relates to that in the left moving sector of the field theory, hence the notation used. It would be interesting to find an explicit embedding of this black hole into string theory, and thence its interpretation as a thermal state in the dual field theory.

2.8. Appendix: Solution of vector equations

The homogeneous equation

$$\left(\rho \partial_\rho^2 + \partial_\rho - \frac{z^2}{4\rho} - \frac{k^2}{4}\right) \phi(\rho) = 0, \tag{2.251}$$

admits modified Bessel functions as solutions

$$\phi = \alpha I_z(k\sqrt{\rho}) + \beta K_z(k\sqrt{\rho}). \tag{2.252}$$

The solution which is regular as $\rho \to \infty$ is the second, so $\alpha = 0$.

Let us next consider a generic inhomogeneous equation

$$\left(\rho \partial_\rho^2 + \partial_\rho - \frac{z^2}{4\rho} - \frac{k^2}{4}\right) \phi(\rho) = g(\rho). \tag{2.253}$$

By defining $x = k\sqrt{\rho}$, it becomes:

$$\Delta_x \phi(x) = \left[\partial_x \left(x\, \partial_x\right) - \left(x + \frac{z^2}{x}\right)\right] \phi(x) = x \left(\frac{2}{k}\right)^2 g([x/k]^2) := h(x). \tag{2.254}$$

The general solution to this equation is

$$\phi(x) = \phi_0(x) + \int_0^\infty dx' G(x,x')h(x'), \tag{2.255}$$

where $\phi_0(x)$ satisfies the homogeneous equation (the regular solution throughout the bulk being $K_z(x)$) and the Green's function is defined by

$$\Delta_x G(x,x') = \delta(x'-x). \tag{2.256}$$

Then, the solution for the Green's function for $x \neq x'$ is:

$$G(x',x) = \begin{cases} A(x')K_z(x) & : \quad x > x' \\ B(x')I_z(x) & : \quad x < x' \end{cases} \tag{2.257}$$

For $x > x'$, one chooses the $K_z(x)$ so that the Green's function is regular as $x \to \infty$. For $x < x'$, one chooses the $I_z(x)$ so that the results for the case $\sigma^2 = 0$ are recovered. Note that $I_z(x)$ does not contain the x^{-z} power, only the x^z one. In order to find the coefficients, one imposes continuity in the Green's function and integrating the equation for $G(x',x)$ between $x' - \epsilon$ and $x' + \epsilon$ with $\epsilon \to 0$, one obtains the second condition. Hence:

$$\begin{aligned} A(x')K_z(x') &= B(x')I_z(x'), \\ A(x')\,K_z'(x') - B(x')\,I_z'(x') &= 1/x'. \end{aligned} \tag{2.258}$$

The two above conditions have a unique solution if the Wronskian is non-vanishing, which is indeed the case as

$$K_z I_z' - I_z K_z' = 1/x'. \tag{2.259}$$

With $A(x') = -I_z(x')$ and $B(x') = -K_z(x')$, all conditions on the Green's function are satisfied.

Using this Green's function to solve the vector field equations iteratively gives that the v-component of the normalizable mode is

$$a_{(z)v} = a^{(0)}_{(-z)u} \frac{k_v k^{2z}}{k_u} \frac{\Gamma(-z)}{2^{2z}\,\Gamma(z)} \left(1 + \sigma^2 k_\chi^2 c_{vv}\right) + a^{(0)}_{(-z)v}\, \sigma^2 k_\chi^2 c_{uv}. \tag{2.260}$$

The u component of the normalizable mode is then

$$a_{(z)u} = a^{(0)}_{(-z)v} \frac{k_u\, k^{2z}}{k_v} \frac{\Gamma(-z)}{2^{2z}\Gamma(z)} \left(1 - \sigma^2 k_\chi^2 c_{uu}\right) + a^{(0)}_{(-z)u}\, \sigma^2 k_\chi^2 c_{uv}. \tag{2.261}$$

In these expressions,

$$\begin{aligned}
c_{vv} &= \frac{1-z}{2z^2\Gamma(-z)\Gamma(z)}\left[\left((1-z)-2(3-2z)z^2\right)\frac{\sqrt{\pi}\,\Gamma(1-z)\,\Gamma(2-2z)}{4\,\Gamma(5/2-z)} + S_1 + S_4 - 2(S_3+S_6)\right];\\
c_{uv} &= \frac{1-z}{2^{1+2z}\Gamma(1+z)^2}\left[\frac{\sqrt{\pi}\Gamma(1-z)\Gamma(2-2z)}{4\Gamma(5/2-z)} + S_1 + S_4 - 2(S_3+S_6) - 4z(S_2+S_5)\right];\\
c_{uu} &= \frac{1-z}{z^2\Gamma(-z)\Gamma(z)}\left[\frac{1-z(4+z-2z^2)}{2^{2z}(3+4z(-2+z))}\Gamma(1-z)^2 + S_1 + S_4 - 2(S_3+S_6) - 4z(S_2+S_5)\right].
\end{aligned}$$

The constants c_{ab} relate to the numerical constants appearing in the two point functions in (2.233). The constants S_a are given in terms of integral over Bessel functions as,

$$\begin{aligned}
S_1 &= \int_0^\infty dy\, y\, K_z(y)\, K_z(y) \int_0^y dy'\, (y')^{3-2z}\, I_z(y')\, K_z(y')\\
S_2 &= \int_0^\infty dy\, y\, K_z(y)\, K_z(y) \int_0^y dy'\, (y')^{1-2z}\, I_z(y')\, K_z(y')\\
S_3 &= \int_0^\infty dy\, y\, K_z(y)\, K_z(y) \int_0^y dy'\, (y')^{2-2z}\, I_z(y')\, K_{(z-1)}(y')\\
S_4 &= \int_0^\infty dy\, y\, K_z(y)\, I_z(y) \int_y^\infty dy'\, (y')^{3-2z}\, K_z(y')\, K_z(y')\\
S_5 &= \int_0^\infty dy\, y\, K_z(y)\, I_z(y) \int_y^\infty dy'\, (y')^{1-2z}\, K_z(y')\, K_z(y')\\
S_6 &= \int_0^\infty dy\, y\, K_z(y)\, I_z(y) \int_y^\infty dy'\, (y')^{2-2z}\, K_z(y')\, K_{(z-1)}(y')
\end{aligned}$$

Chapter 3

Aspects of Ricci-flat Holography - I

3.1. Introduction

In the previous two chapters we have discussed gauge/gravity dualities and explored possible applications to condensed matter theories realized in nature. The geometries used to study holographically the quantum field theories were always asymptotically AdS, but we have also briefly mentioned the possibility that geometries with non-AdS asymptotics have holographic duals. The Schrödinger and Lifshitz spacetimes with dynamical exponent $z > 1$, for example, represent two types of geometries that do not approach AdS at infinity but may admit dual field theories that are typically non-relativistic. In the next two chapters we will depart from geometries with AdS asymptotics and analyse possible holographic descriptions of gravitational theories with asymptotically flat boundary conditions.

Soon after the discovery of the AdS/CFT correspondence, several gauge/gravity proposals were constructed by analogy with AdS/CFT that relate string theory on spacetimes with non-AdS asymptotics to field theories formulated at the boundary. For the case of de Sitter gravity, and motivated by studies of the asymptotic symmetry group of de Sitter in a fashion similar to that of AdS [14], it has been conjectured that the bulk theory can be described by an Euclidean field theory defined at the spacelike conformal boundary [127, 128, 129, 130, 131, 132, 133]. A further motivation lies in the fact that every solution of AdS gravity is mapped to a solution of de Sitter's by an analytic continuation, leading to a possible dS/CFT correspondence. In the context of AdS/CFT, quantum field theory correlation functions are determined by computing string theory correlators and vice-versa, and the bulk/boundary dictionary is well established. Statements in dS/CFT can then be worked out from the AdS counterpart by analytically continuing the solutions with AdS boundary conditions to de Sitter signature.[1] In particular, the near-boundary asymptotics of AdS spaces admits an analytic continuation to dS asymptotics (see *e.g.* [78]), leading to a well-defined mapping between asymptotic data in the bulk and boundary data in the case of a positive cosmological constant Λ.

Despite many interesting results, a holographic description of de Sitter space remains unclear, mainly because string theory in dS is not well understood. Even though de Sitter vacua exists in string theory [135], unlike the case of flat or AdS vacua they are unstable and decay to vacua of different Λ signature. Another problem in a dS/CFT formulation is the fact that the conformal weights of the QFT operators are imaginary and the boundary theory is non-unitary. Nevertheless, one can still work out the details of such a correspondence and point to those ingredients that do not work.

The case of Ricci-flat gravity is substantially different. At the classical level, setting Λ to zero is just a fine-tuning problem and asymptotically flat spacetimes are the best controlled backgrounds in string theory to compute correlation functions. However, the mechanism in string theory by which the cosmological constant vanishes is not clear (see *e.g.* the discussion in [136]). More particularly in the context of AdS/CFT, the zero Λ limit of the correspondence in general is not well-understood. The limit taken on correlators and vacuum expectation values generically does not lead to any sensible results, mainly because most AdS solutions do not map to asymptotically flat ones. The conformal weigths of the QFT operators dual to massive bulk fields diverge in this limit, a problem associated with the fact that the conformal boundary is null in the zero Λ limit. The limit taken on the near-boundary asymptotics of AdS spaces in general does not result in Ricci-flat asymptotics, unless specific constraints are imposed, and a bulk/boundary mapping has not been established. Furthermore, and unlike the case of de Sitter gravity, holographic renormalization does not extend in a straigthforward manner to flat gravity, essentially because the asymptotics of bulk fields are non-local in this case [137, 138, 78, 139]. Nevertheless, quantum gravity in asymptotically flat spacetimes can be characterised by a unitary and analytic S-matrix and it is believed that a holographic description of the flat space S-matrix can be derived from the zero Λ limit of AdS/CFT. Indeed, explicit constructions for extracting S-matrix elements from boundary correlators have been proposed in [140, 141, 142, 143, 144, 145, 146] (see also the discussions in [147, 129]).

[1] Note, however, that to compute correlation functions in this way one has to take into account the global properties of asymptotically de Sitter spaces [134, 131].

A different approach to flat space holography formulated as a limit of AdS/CFT is based on studies of the asymptotic symmetry group of asymptotically Minkowski spacetimes, the BMS group. In four dimensions the symmetry algebra was originally derived in [148, 149, 150] and more recently investigated in [151] in general dimensions (see also [152, 153]). In the three dimensional case, the $\mathfrak{bms}_3$ algebra consists of diffeomorphisms on the circle and supertranslations and is isomorphic to the two-dimensional Galilean conformal algebra (GCA) consisting of a contraction of two copies of the Virasoro algebra. The Poisson algebra of the surface charges was found to admit a central extension with central charge $c = 3$ [151, 154],[2] representing a generalisation to the flat space case of those results originally obtained by Brown and Henneaux [14] for AdS_3 and which predated the AdS/CFT correspondence. In the four dimensional case, the $\mathfrak{bms}_4$ algebra is also isomorphic to a class of GCAs [155]. Based on these results, a possible connection between string theory on asymptotically flat spacetimes and non-relativistic conformal field theories defined at null infinity was proposed in [155, 156, 157, 158, 154]. In the same spirit, the authors in [154, 158] were able to reproduce the Bekenstein-Hawking entropy of three-dimensional flat cosmological horizons by counting states in a two-dimensional Galilean conformal field theory defined at null infinity. However, these studies leave open the question of how to compute string theory correlation functions from the boundary theory, as well as the precise form of the correspondence.

Returning to the context of AdS/CFT, and more specifically to the correspondence analysed in section 1.2, the mapping between the free parameters on each side of the duality was determined by Maldacena [20] as we reviewed in the first chapter. Recalling the definition of the 't Hooft coupling $\lambda := g_{YM}^2 N$ of the gauge theory, the relationship is given by:

$$g_s = \lambda/(4\pi N) \ , \tag{3.1}$$

$$\ell_s = \ell/\lambda^{1/4} \ , \tag{3.2}$$

where $\ell_s^2 := \alpha''$ and we have reinstated the AdS radius ℓ introduced in equation (1.65). We assume that the decoupling limit $\alpha' \to 0$ has already been taken. As we discussed in the first chapter, the 't Hooft limit $N \to \infty$, $\lambda = constant$ corresponds on the gravity side to the classical limit $g_s \to 0$ under which the partition function of string theory, or supergravity, is approximated by classical supergravity plus all α'' corrections. We can then work in the limit $\lambda \to \infty \Leftrightarrow \alpha'' \to 0$ under which all these stringy corrections are suppressed and the classical supergravity approximation to string theory is valid. Once this is done, we are left with a free

[2] The central charge c_{LM} in reference [154] is related to ours as: $c_{LM} = c/12$ since we follow the convention of formula (1) in this reference.

parameter ℓ that is not fixed by the Maldacena correspondence. In order to take the limit of a dimensionless quantity, we introduce some characteristic length scale $\ell_{\rm o}$ and rewrite the AdS radius as a multiple of $\ell_{\rm o}$ with proportionality constant α:

$$\ell \;=\; \alpha\, \ell_{\rm o}\,. \tag{3.3}$$

Since the only length scale of the gravity theory is the Planck length $\ell_P \sim \ell_s$, the characteristic scale $\ell_{\rm o}$ must be proportional to ℓ_P. The zero Λ limit in AdS/CFT then corresponds to taking $\alpha \to \infty$ with $\ell_{\rm o}$ fixed, but the nature of this limit on each side of the duality and an understanding of α are not clear.[3] We will not have an answer to this question, but we will make use of the relation (3.3) throughout this chapter to study the zero Λ limit of vacuum expectation values and specific correlation functions in AdS/CFT. This will be done formally, and in a fashion somewhat similar to the way vevs and boundary correlators in dS/CFT are derived from corresponding AdS results. The main difference, however, is that not every bulk solution with AdS boundary conditions is mapped to a solution in asymptotically flat space in the zero Λ limit. We will discuss this aspect in the next sections. This implies that we need to restrict the space of solutions of AdS gravity to the subspace of those that admit the limit, in the sense that they result in solutions of the bulk equations of motion with $\Lambda = 0$ once the limit $\alpha \to \infty$ is taken. Since gravity solutions are dual to QFT states, this corresponds to restricting the Hilbert space of the field theory to some subspace, say $\tilde{\mathcal{H}}$. Furthermore, since the limit $\alpha \to \infty$ is taken over solutions, on the QFT side this should correspond to some limit taken over $\tilde{\mathcal{H}}$. The objective is then to derive the correspondence between the resulting states in $\tilde{\mathcal{H}}$ and those bulk solutions of asymptotically flat gravity that result from the limit $\alpha \to \infty$. This will be done mainly by working out the mapping between QFT observables and the asymptotics of such solutions. We will find that well-definedness of this limit seems to be a statement about states and sources on the field theory side.

If the bulk field is in particular the spacetime metric, the choice of possible coordinate systems is constrained by the requirement that the solution be smooth in the zero Λ limit. Taking this limit on the metric must correspond to switching off the boundary lapse function so that the timelike conformal boundary of the asymptotically AdS solution becomes null as $\alpha \to \infty$. To some extent, it is a gauge-dependent condition the requirement that the solution be mapped to an asymptotically flat one in this limit and this fact will have an interesting implication to the holographic renormalization of the bulk theory as discussed below. This restriction to the subspace of solutions with a well-defined limit implies in

[3] Note that a different limit has been discussed in [140, 147]. Another perspective is that the limit corresponds to taking $\ell E \to \infty$ for all bulk energies E [146].

particular that the standard Fefferman-Graham coordinate system used in the near-boundary analysis of asymptotically AdS and dS spaces cannot be extended to derive the asymptotics of those solutions that are smooth in α.

The choice of coordinates we will then make near the asymptotic boundary are the well-known Gaussian *null* coordinates. This gauge is closely related to Bondi coordinates and was initially introduced by Isenberg and Moncrief [159] in order to prove the existence of a Killing vector field in any spacetime that contains a compact null surface with closed generators. It was further elaborated in [160] in order to generalise Isenberg and Moncrief's results, as well as Hawking's rigidity theorems, to non-analytic spacetimes (see also [161]) and it has been extensively used in the literature in order to study the near horizon geometry of black holes (see [162, 163] and references therein). This gauge choice is also motivated by those investigations of the asymptotic symmetries of asymptotically flat gravity discussed above.[4] In this coordinate system, the Einstein field equations decompose into a set of dynamical and constraint equations that are very tedious to solve asymptotically and increase in complexity with the spacetime dimension. For this reason we will focus specifically on the case of three and four bulk dimensions, but it is straightforward to extend the procedure to any dimension. From this analysis we will obtain in particular the unique asymptotics at null infinity of all those Ricci-flat metrics that result from the zero Λ limit of Einstein metrics.

As a final remark, it should be emphasized that, unlike the case of dS/CFT, holographic renormalization does not admit a straighforward extension to the asymptotically flat case. In general, the holographic counterterms introduce divergences in α that spoil the zero Λ limit of the renormalized on-shell gravity action. If one insists that the action be finite in this limit, further counterterms are needed to restore the well-definedness of the limit. The latter are finite in the holographic regulator and therefore are associated with a choice of renormalization scheme on the field theory side. These finite counterterms are covariant with respect to diffeomorphisms that preserve the spacelike foliation induced at the boundary by the bulk theory, but break invariance of the renormalized action with respect to diffeomorphisms that are not foliation-preserving. This reflects the fact that the well-definedness of the limit is a gauge-dependent requirement. We will analyse the effect of these anomalous counterterms on the holographic Ward identities of the field theory in the case of four bulk dimensions. A pathological aspect of this type of counterterms is that they introduce divergent contact terms in the two-point correlators of scalar operators. We will derive this result in section 3.6.3.

[4] See also [164] for a brief overview in three dimensions.

In the next section we introduce our coordinate system and determine the unique asymptotics of the bulk spacetime metric by solving the vacuum Einstein equations with a negative cosmological constant in a neighbourhood of the asymptotic boundary. We will then discuss the zero Λ limit of the solution and briefly compare the spacetime asymptotics in this limit with the standard definitions of asymptotic flatness at null infinity.

Section 3.3 contains the main results of this chapter. We will holographically renormalize the bulk gravity action in three and four dimensions and use the AdS/CFT prescription to compute the vacuum expectation value of the QFT energy tensor. The objective will be to analyse the correspondence between the metric asymptotics and the boundary data in the zero Λ limit and to address the issues associated with this limit. The three dimensional case is the best controlled setting and no major problems arise. The holographic Weyl anomaly in the zero Λ limit will be of particular interest in this case. The integrated anomaly is still a topological invariant and we will be able to obtain in this limit the Virasoro central charge that arises in the central extension of $\mathfrak{bms}_3$ as the proportionality constant between the anomaly and a geometric invariant. Still in three dimensions, we will find that it is possible to define an improved holographic energy tensor which is always traceless if the QFT metric is static. We will then apply our results to the zero Λ limit of the BTZ solution, which represents a three-dimensional flat cosmological solution, and find a matching between the energy and momentum of the QFT and those of the bulk theory.

In the case of four bulk dimensions we will find that the holographic renormalization spoils the zero Λ limit of the gravity action, as described above, by terms that are finite in the regulator and which can only be subtracted by a finite counterterm that partially breaks diffeomorphism invariance of the action. We will then compute the holographic energy tensor and address the issues associated with its zero Λ limit. Of particular interest will be the holographic Ward identities and the way they are affected by the anomalous counterterm. In the absence of the latter, the trace of the QFT energy tensor vanishes, but it is modified by a total derivative in the presence of the anomalous counterterm. We will also find that it is possible to introduce an improved energy tensor in which this total derivative vanishes if the boundary metric is static. As an application of our results, we will derive specifically the asymptotics of the Kerr solution and find a matching between the energy and momentum of this solution and those of the dual state of the field theory. At the end of this section we will address and solve the issues associated with the presence of null boundaries in the spacetime in addition to the asymptotic conformal boundary.

Finally, in section 3.6 we analyse the case of a non-backreacting massive bulk field propagating in AdS in a coordinate system appropriate to the zero Λ limit. We renormalize holographically the bulk action for the field, address its zero Λ limit and compute the vacuum expectation value and the renormalized two-point correlator of the dual scalar operator. As in the case of the spacetime metric, the objective will be to analyse the zero Λ limit taken on the vev and correlator. For "large" values of the conformal weights, contact terms associated with the anomalous counterterms arise in the two-point function, but vanish away from coincident points in time. In general, the two-point functions will be consistent with that of a conformal operator in two dimensions less in this limit.

3.2. Spacetime asymptotics

3.2.1. Choice of coordinates

We start with the action for the spacetime metric in $d+2$ dimensions written in the form:

$$16\pi G_0\, S \;=\; \int_{\mathcal{M}} d^{d+2}x\,\sqrt{G}\left(\frac{d(d+1)}{\alpha^2\ell_{\rm o}^2}+R[G]\right)+2\int_{\partial\mathcal{M}} d^{d+1}x\sqrt{q}\,Q\;, \tag{3.4}$$

where the cosmological constant $2\Lambda=-d(d+1)/(\alpha\ell_{\rm o})^2$ and where q_{ab} and Q_{ab} are the induced metric and extrinsic curvature of the boundary. As discussed in the previous section, we have rewritten the AdS radius ℓ as in (3.3) so that Λ is switched off by taking the limit $\alpha\to\infty$ of the dimensionless parameter α.

In order to solve asymptotically the Einstein field equations we introduce Gaussian *null* coordinates $x^\mu=(r,x^a)=(r,u,x^i)$ near the boundary $r=\infty$ of the manifold. In such gauge, the spacetime metric has the form [159, 160]:

$$\begin{aligned} ds^2_{d+2} &= G_{\mu\nu}dx^\mu dx^\nu \\ &= -\phi\, du^2+2dudr+\gamma_{ij}(dx^i+\sigma^i du)(dx^j+\sigma^j du)\;. \end{aligned} \tag{3.5}$$

where the metric components depend on all the coordinates, the spatial metric γ_{ij} is positive-definite and the function ϕ is positive by definition. The vector $\phi^{-1/2}(\partial_u-\sigma^i\partial_i)$ is future-directed timelike with unit norm. The manifold is defined to be foliated by a family of timelike hypersurfaces labelled by the coordinate r and by a family of null surfaces of constant u. Each submanifold $\{r=\mathit{constant}\}$ is foliated by spacelike surfaces of constant time coordinate u. All the above statements hold asymptotically. In appendix 3.7 we briefly deduce this coordinate system via an ADM analysis of the metric, but it all comes down to using diffeomorphisms in order to bring the metric to the desired form. In the case of asymptotically flat metrics in Gaussian null coordinates, the metric components behave asymptotically as [165, 166, 167, 168, 153]:

$$\gamma_{ij}(r,u,x) \;=\; r^2\left(\gamma_{(0)ij}(u,x)+\mathcal{O}(r^{<0})\right) , \tag{3.6}$$
$$\phi(r,u,x) \;=\; \phi_{(0)}(u,x)+\mathcal{O}(r^{<0}) , \tag{3.7}$$
$$\sigma^i(r,u,x) \;=\; \mathcal{O}(r^{<0}) , \tag{3.8}$$

with null infinity given by $r=+\infty$, so we will be interested in solving the field equations around $1/r=0$ with Λ switched on and in the end analyse the limit $\alpha\to\infty$.

Before doing so, we introduce a new coordinate $z:=\ell_{\rm o}^2/r$ and also define $g_{ij}:=(z/\ell_{\rm o})^2\gamma_{ij}$ and $\varphi:=(z/\ell_{\rm o})^2\phi$ such that:

$$ds^2_{d+2} \;=\; \frac{\ell_{\rm o}^2}{z^2}\left(-\varphi\, du^2-2dudz+g_{ij}(dx^i+\sigma^i du)(dx^j+\sigma^j du)\right) . \tag{3.9}$$

The decomposition of the Ricci tensor $R_{\mu\nu}[G]$ in terms of the metric components φ, g_{ij} and σ^i is given in appendix 3.8. If we solve the field equations $R_{\mu\nu}=-(d+1)/(\alpha\ell_{\rm o})^2G_{\mu\nu}$ around $z=0$ at leading and first subleading order, we find:

$$\varphi(z,u,x) \;=\; \frac{1}{\alpha^2}+z\,\varphi_{(1)}+\mathcal{O}(z^2) , \tag{3.10}$$
$$g_{ij}(z,u,x) \;=\; g_{(0)ij}+z\,g_{(1)ij}+\mathcal{O}(z^2) , \tag{3.11}$$
$$\sigma^i(z,u,x) \;=\; \sigma^i_{(0)}+\mathcal{O}(z^2) , \tag{3.12}$$

where the coefficients $\varphi_{(1)}(u,x)$, $g_{(0)ij}(u,x)$ and $\sigma^i_{(0)}(u,x)$ are completely arbitrary (*i.e.* integration constants) and where $g_{(1)ij}(u,x)$ obeys the equation:

$$\frac{1}{\alpha^2}\,g_{(1)ij}=(\partial_u-\pounds_{\sigma_{(0)}})g_{(0)ij}+\varphi_{(1)}g_{(0)ij} , \tag{3.13}$$

with $\pounds$ the Lie derivative. The asymptotic behaviour of the metric components therefore implies that the metric (3.9) is (at least C^2) conformally compact,[5] with defining function $z/\ell_{\rm o}$ and conformal boundary $z=0$. For $\alpha^{-2}>0$ the boundary is timelike and it becomes null in the zero Λ limit. We also find from (3.13) that the leading order term $g_{(0)ij}$ becomes constrained in the case $\alpha^{-1}=0$.

We will now use the freedom in the choice of defining function and introduce a more judicious one. We define a new coordinate $\bar z:=zN_{(0)}$, with $N_{(0)}(u,x)$

[5] See appendix B.1.

an arbitrary but positive smooth function of u and x^i. Under this change of coordinates the spacetime metric becomes:

$$ds^2_{d+2} = \frac{\ell_o^2}{\bar{z}^2}\Big(-\bar{\varphi}N_{(0)}du^2 - 2N_{(0)}dud\bar{z} + \bar{g}_{ij}\left(dx^i + \bar{\sigma}^i du\right)\left(dx^j + \bar{\sigma}^j du\right)\Big) , \quad (3.14)$$

where:

$$\bar{\varphi}N_{(0)} = \varphi N^2_{(0)} - 2\bar{z}\,(\partial_u - \pounds_\sigma)N_{(0)} + \bar{z}^2|\nabla_g \log N_{(0)}|^2 , \quad (3.15)$$

$$\bar{\sigma}^i = \sigma^i + \bar{z}\,N^{-2}_{(0)}g^{ij}\partial_j N_{(0)} , \quad (3.16)$$

$$\bar{g}_{ij} = N^2_{(0)}g_{ij} . \quad (3.17)$$

The metric component $\bar{\varphi}$ therefore has the asymptotics:

$$\bar{\varphi} = \bar{\varphi}_{(0)} + \bar{z}\,\bar{\varphi}_{(1)} + \mathcal{O}(\bar{z}^2) : \quad (3.18)$$

$$\bar{\varphi}_{(0)} = \frac{1}{\alpha^2}N_{(0)} , \quad (3.19)$$

$$\bar{\varphi}_{(1)} = \varphi_{(1)} - 2(\partial_u - \pounds_{\sigma_{(0)}})\log N_{(0)} . \quad (3.20)$$

We then choose our function $N_{(0)}(u,x)$ such that:[6]

$$(\partial_u - \pounds_{\sigma_{(0)}})\log N^2_{(0)} = \varphi_{(1)} , \quad (3.21)$$

which results in the asymptotics: $\bar{\varphi} = \bar{\varphi}_{(0)} + \mathcal{O}(\bar{z}^2)$. Recall that the coefficient $\varphi_{(1)}$ was an integration constant and therefore $N_{(0)}$, or $\bar{\varphi}_{(0)}$, remains arbitrary, *i.e.* undetermined by the field equations.

From this choice of defining function $\bar{z}/\ell_o$ and the requirement that the metric components be well-defined in the limit $\alpha \to \infty$, it follows from equation (3.13) that:

$$(\partial_u - \pounds_{\bar{\sigma}_{(0)}})\bar{g}_{(0)ij} = 0 \qquad (\alpha \to \infty) , \quad (3.22)$$

and therefore the timelike vector $\partial_u - \bar{\sigma}^i\partial_i$ is an asymptotic Killing vector of the spatial metric $\bar{g}_{ij}$ in this limit. Furthermore, with such defining function, the normal to the boundary $m^\mu = \widetilde{G}^{\mu\nu}\partial_\nu\bar{z}$ in the conformal embedding $\widetilde{G}_{\mu\nu} = (\bar{z}/\ell_o)^2 G_{\mu\nu}$ is shear, expansion and vorticity free in the zero Λ limit, and therefore totally geodesic:

$$\lim_{\alpha\to\infty} \widetilde{\nabla}_\nu m^\mu = \mathcal{O}(\bar{z}). \quad (3.23)$$

This is the standard gauge used in the study of asymptotically flat spacetimes (see *e.g.* [169]). More importantly, with our choice of coordinates the boundary metric

[6] Note that if we write: $N_{(0)} := N_{(0)1}N_{(0)2}$ such that $(\partial_u - \pounds_{\sigma_{(0)}})\log N_{(0)2} = 0$, we still have the freedom of choosing $N_{(0)2}(u,x)$ in the space orthogonal to the vector $\partial_u - \sigma^i_{(0)}\partial_i$.

(with components $N_{(0)}, \bar{g}_{(0)ij}$ and $\bar{\sigma}^i_{(0)}$) is completely unconstrained for finite α. In the next sections this feature will allow us to take the variations of the on-shell action with respect to all components of the boundary metric in order to derive the holographic energy tensor. The metric in the originial Gaussian null coordinates (3.9) therefore corresponds to the metric (3.14) with the lapse function $\frac{1}{\alpha}N_{(0)}$ of the boundary fixed by diffeomorphisms to a constant.

We will drop the bar notation from now on and work with the spacetime metric in the final form:

$$\begin{aligned} ds^2_{d+2} &= G_{\mu\nu}dx^\mu dx^\nu \\ &= \frac{\ell_o^2}{z^2}\Big(- \varphi N_{(0)}du^2 - 2N_{(0)}dudz + g_{ij}\left(dx^i + \sigma^i du\right)\left(dx^j + \sigma^j du\right)\Big) \ , \end{aligned} \tag{3.24}$$

where:

$$\varphi = \frac{1}{\alpha^2}N_{(0)} + \mathcal{O}(z^2) \ , \tag{3.25}$$

$$g_{ij} = g_{(0)ij} + \mathcal{O}(z) \ , \tag{3.26}$$

$$\sigma^i = \sigma^i_{(0)} + \mathcal{O}(z) \ . \tag{3.27}$$

The induced metric q_{ab} of the surfaces of constant z near the boundary $z = 0$ is given by:

$$\begin{aligned} ds^2_{d+1} = q_{ab}dx^a dx^b &= \frac{\ell_o^2}{z^2}\left(-\frac{1}{\alpha^2}N^2_{(0)}du^2 + g_{(0)ij}\big(dx^i + \sigma^i_{(0)}du\big)\big(dx^j + \sigma^j_{(0)}du\big) + \mathcal{O}(z)\right) \\ &:= \frac{\ell_o^2}{z^2}\Big(q_{(0)ab} + \mathcal{O}(z)\Big)dx^a dx^b \ , \end{aligned} \tag{3.28}$$

where $q_{(0)ab}$ represents the metric tensor of the conformal boundary and is the source for the energy tensor of the dual quantum field theory. From the determinant $\sqrt{q_{(0)}} = \frac{1}{\alpha}N_{(0)}\sqrt{g_{(0)}}$ we see clearly that the timelike boundary becomes null in the zero Λ limit.

3.2.2. Asymptotic solution

The decomposition of the Ricci tensor $R_{\mu\nu}[G]$ in our coordinate system (4.41) is given in appendix 3.8. If we solve the Einstein field equations around $z = 0$ with the cosmological constant switched on, we find that the asymptotics of the metric is uniquely determined:

$$g_{ij} \;=\; g_{(0)ij} + z\, g_{(1)ij} + z^2 g_{(2)ij} + \ldots + z^{d+1} g_{(d+1)ij} + z^{d+1} \log z\, \tilde{g}_{(d+1)ij} + \ldots \;, \tag{3.29}$$

$$\varphi \;=\; \varphi_{(0)} + z^2 \varphi_{(2)} + z^3 \varphi_{(3)} + \ldots + z^{d+1} \varphi_{(d+1)} + z^{d+1} \log z\, \tilde{\varphi}_{(d+1)} + \ldots \;, \tag{3.30}$$

$$\sigma^i \;=\; \sigma^i_{(0)} + z\, \sigma^i_{(1)} + z^2 \sigma^i_{(2)} + \ldots + z^{d+1} \sigma^i_{(d+1)} + z^{d+1} \log z\, \tilde{\sigma}^i_{(d+1)} + \ldots \tag{3.31}$$

Note that the expansions in z are not predetermined but uniquely fixed by the equations.[7] The coefficients $g_{(0)ij}$, $\varphi_{(0)}$ and $\sigma^i_{(0)}$, which we will denote collectively by $G_{(0)\mu\nu}$, are integration constants and therefore completely arbitrary functions of u and x^i. These are the standard non-normalizable modes, or sources, of asymptotically AdS metrics.[8] The coefficients $g_{(d+1)ij}$, $\varphi_{(d+1)}$ and $\sigma^i_{(d+1)}$, denoted collectively by $G_{(d+1)\mu\nu}$, are arbitrary up to specific constraints and are the standard normalizable modes. These will be associated to the different components of the holographic energy tensor and the constraints to its Ward identities. The coefficients of the logarithms, which we will denote by $\tilde{G}_{(d+1)\mu\nu}$, are only non-vanishing for odd values of $d > 1$, and in such case are local functionals of the sources $G_{(0)\mu\nu}$. The remaining coefficients $G_{(n)\mu\nu}$, as well as the constraints on the $G_{(d+1)\mu\nu}$, are all local functionals of the sources.

The expressions for the coefficients at first and second subleading orders are given by:

$$\frac{1}{2\,\alpha^2}\, g_{(1)ij} \;=\; k_{(0)ij} \;, \tag{3.32}$$

$$\begin{aligned} \frac{d-1}{\alpha^2}\, g_{(2)ij} \;=\; & (d-2)\left(k_{(1)ij} - \frac{1}{d}\, g_{(0)ij}\mathrm{Tr}[g^{-1}_{(0)} k_{(1)}]\right) - \left(R_{(0)ij} - \frac{1}{d} g_{(0)ij} R_{(0)}\right) \\ & + \frac{1}{4\alpha^2}\left(-g_{(1)ij}\mathrm{Tr}[g^{-1}_{(0)} g_{(1)}] + \frac{1}{d} g_{(0)ij}\mathrm{Tr}^2[g^{-1}_{(0)} g_{(1)}] + 2\big(g_{(1)}\cdot g_{(1)}\big)_{ij} + \frac{d-3}{d}\, g_{(0)ij}\mathrm{Tr}[g_{(1)}\cdot g_{(1)}]\right) , \end{aligned} \tag{3.33}$$

$$\varphi_{(0)} \;=\; \frac{1}{\alpha^2} N_{(0)} \;, \tag{3.34}$$

[7] An arbitrary term $z\,\varphi_{(1)}$ in the expansion of φ can always be cancelled by a choice of $N_{(0)}$ as described above. See, however, the discussion in section 3.4.1. There is also the possibility of including terms proportional to $\delta_{\Lambda,0}$ that vanish for all finite values of α, but we discard such terms since we are only interested in solutions for which the limit $\alpha \to \infty$ exists.

[8] See [123, 78] for a review of the asymptotics of such metrics in Fefferman-Graham coordinates.

$$\frac{d(d-1)}{N_{(0)}}\,\varphi_{(2)} \;=\; -2(d-1)\mathrm{Tr}[g^{-1}_{(0)}k_{(1)}] + R_{(0)} + \frac{1}{4\alpha^2}\left(\mathrm{Tr}^2[g^{-1}_{(0)}g_{(1)}] + (2d-3)\mathrm{Tr}[g_{(1)}\cdot g_{(1)}]\right), \tag{3.35}$$

$$\sigma_{(1)i} \;=\; \partial_i N_{(0)}\ , \tag{3.36}$$

$$\frac{2(d-1)}{N_{(0)}}\,\sigma_{(2)i} \;=\; -\,{}^{(0)}\nabla_j\big(g^{-1}_{(0)}g_{(1)}\big)_i^{\ j} + \partial_i\mathrm{Tr}[g^{-1}_{(0)}g_{(1)}] - (d-1)g_{(1)ij}\,{}^{(0)}\nabla^j\log N_{(0)}\ , \tag{3.37}$$

where $R_{(0)ij} := R_{ij}[g_{(0)}]$ and ${}^{(0)}\nabla_i g_{(0)jk} := 0$, and where the indices are raised and lowered with $g_{(0)ij}$ and the inner product taken with respect to the latter. It is also useful to emphasize that in three and four bulk dimensions the coefficient $g_{(2)ij}$ simplifies as:[9]

$$g_{(2)ij} \;=\; \frac{1}{4}\big(g_{(1)}\cdot g_{(1)}\big)_{ij} \qquad (d=1,2)\ . \tag{3.38}$$

In appendix 3.8 where the decomposition of the Ricci tensor is given we introduced the tensor k_{ij} defined as:

$$k_{ij} := \frac{1}{2N_{(0)}}\left(\partial_u - \pounds_\sigma\right)g_{ij}\ . \tag{3.39}$$

This tensor is proportional to the extrinsic curvature of the surfaces of constant time on each submanifold $\{z = \textit{constant}\}$. From the metric asymptotics, k_{ij} admits the expansion:

$$k_{ij} \;=\; k_{(0)ij} + z\,k_{(1)ij} + \ldots + z^{d+1}k_{(d+1)ij} + z^{d+1}\log z\,\tilde{k}_{(d+1)ij} + \ldots \tag{3.40}$$

Each coefficient $k_{(n<d+1)ij}$ can be written in terms of quantities defined at the boundary. For the first and second subleading orders we find:[10]

$$k_{(0)ij} = \frac{1}{2N_{(0)}}\left(\partial_u - \pounds_{\sigma_{(0)}}\right)g_{(0)ij} \;=\; \frac{1}{\alpha}\,K_{(0)ij}\ , \tag{3.41}$$

$$\begin{aligned} k_{(1)ij} = \;& \frac{1}{2N_{(0)}}\Big[\left(\partial_u - \pounds_{\sigma_{(0)}}\right)g_{(1)ij} - \pounds_{\sigma_{(1)}}g_{(0)ij}\Big] \;=\; \pounds_{n_{(0)}}K_{(0)ij} - {}^{(0)}\nabla_i a_{(0)j} - a_{(0)i}a_{(0)j} \\ =\;& R_{ij}[q_{(0)}] - R_{ij}[g_{(0)}] + 2\left(K_{(0)}\cdot K_{(0)}\right)_{ij} - K_{(0)}K_{(0)ij}\ , \end{aligned} \tag{3.42}$$

[9]For $d = 1$ the coefficient $g_{(2)ij}$ is totally determined by the trace constraint: $\mathrm{Tr}[g^{-1}_{(0)}g_{(2)}] = \frac{1}{4}\mathrm{Tr}[g^{-1}_{(0)}g_{(1)}g^{-1}_{(0)}g_{(1)}]$ that follows from equations (3.253) and (3.256). For $d = 2$ we use the matrix identity (3.259) to simplify equation (3.33).

[10]In the final expression for $k_{(1)}$ we made use of equations (3.32) and (3.36) and of the standard Gauss-Codazzi identities.

where $K_{(0)ij}$ is the extrinsic curvature of the surfaces of constant time at the boundary, $n^a_{(0)}\partial_a = \alpha N^{-1}_{(0)}\left(\partial_u - \sigma^i_{(0)}\partial_i\right)$ is the unit normal to these surfaces and $a_{(0)i} = \partial_i \log N_{(0)}$ the acceleration. Also, $R_{ij}[q_{(0)}]$ are the spatial components of the Ricci tensor $R_{ab}[q_{(0)}]$ of the boundary metric and we will see in section 3.3.2 that its trace will represent the holographic Weyl anomaly in three bulk dimensions in the zero Λ limit.

Let us start by discussing the solutions for the coefficients $G_{(n)\mu\nu}$. If the cosmological constant is non-vanishing, from the expressions (3.32)–(3.37) it follows that these coefficients are indeed local functionals of the sources $G_{(0)\mu\nu}$. On the other hand, in the case $\alpha^{-1} = 0$ we find that the algebraic equation for a given coefficient $g_{(n)ij}$ becomes a differential equation for the coefficient $g_{(n-1)ij}$ and therefore the coefficients $G_{(n)\mu\nu}$ become non-local functionals of the sources. This feature is responsible for the fact that holographic renormalization cannot be extended in a straightforward way to Ricci-flat spacetimes (see *e.g.* [78, 138]) and we will discuss this aspect in the next sections. The asymptotic expansions (3.29)–(3.31) together with the equations for the coefficients with $\alpha^{-1} = 0$ represent the unique asymptotics at null infinity of all Ricci-flat metrics that result from Einstein metrics in the zero Λ limit.

In the case of α finite, the sources $G_{(0)\mu\nu}$ are arbitrary functions, so we may have solutions of the equations of motion with Λ switched on that diverge as $\alpha \to \infty$. The same applies to the normalisable modes $G_{(d+1)\mu\nu}$. We are interested in those Ricci-flat metrics that result from the zero Λ limit, so we need to restrict our space of solutions of Einstein metrics to the subspace of those that admit the limit. For this purpose we require that the coefficients in the expansions (3.29)–(3.31) be non-divergent as $\alpha \to \infty$. For the normalisable modes, it is sufficient to restrict to those configurations that satisfy: $G_{(d+1)\mu\nu} = \mathcal{O}(\alpha^0)$. On the other hand, since the coefficients $G_{(n)\mu\nu}$ and $\tilde{G}_{(d+1)\mu\nu}$ are all functionals of the sources up to order z^{d+1}, this requirement imposes specific behaviours in α of the time derivatives of $g_{(0)ij}$. From equations (3.32) and (3.33) for example it follows that:

$$(\partial_u - \pounds_{\sigma_{(0)}})g_{(0)ij} \;=\; \mathcal{O}(\alpha^{-2}) \;, \tag{3.43}$$

$$(d-2)\left(k_{(1)ij} - \frac{1}{d}\,g_{(0)ij}\mathrm{Tr}[g^{-1}_{(0)}k_{(1)}]\right) - \left(R_{(0)ij} - \frac{1}{d}g_{(0)ij}R_{(0)}\right) \;=\; \mathcal{O}(\alpha^{-2}) \;, \tag{3.44}$$

with $k_{(1)ij}$ expressed in terms of $(\partial_u - \pounds_{\sigma_{(0)}})^2 g_{(0)ij}$ by using equations (3.32), (3.36) and the first identity in equation (3.42). From a holographic perspective, well-definedness of the gravity solutions in the zero Λ limit then translates into a statement about the sources and states on the QFT side. We will find another example of such a correspondence between the existence of the zero Λ limit of bulk solutions and the time behaviour of the sources when we discuss non-backreacting matter in AdS in section 3.6.1.

It is worth comparing the asymptotic behaviour (3.29)–(3.31) of the spacetime metric in the limit $\alpha \to \infty$ with the standard definitions of asymptotic flatness at null infinity. For vacuum spacetimes in odd bulk dimensions higher than four, half integer powers in the asymptotics of the metric (starting at order $z^{d/2}$ in the conformal embedding $\widetilde{G}_{\mu\nu}$) are postulated in the definitions of asymptotic flatness so that linearized pertubations of the metric preserve the definition when the spacetime contains gravitational radiation [165, 166, 170] (see also [167, 168, 171]). The absence of half integer powers in the asymptotics (3.29)–(3.31) seems to indicate that vacuum, radiating spacetimes cannot be obtained from the zero Λ limit of Einstein metrics in five or higher odd dimensions. This subject will be analysed in more detail elsewhere. It is also worth emphasizing the presence of the inhomogeneous logarithmic terms in the asymptotics of the metric,[11] as well as the fact that the first subleading terms in the asymptotic expansions start at order z. The logarithmic terms are usually absent in the definitions of asymptotic flatness (see, however, the discussion in [172]) and the first subleading terms are usually postulated to start at order $z^{d/2}$ both in even and odd bulk dimensions.

As discussed above, the integration constants of the dynamical equations of motion for the metric are the modes $G_{(0)\mu\nu}$ and $G_{(d+1)\mu\nu}$ for non-vanishing Λ. On the other hand, we have also seen that the algebraic equation for a given $g_{(n)ij}$ is of the form:

$$\frac{d+1-n}{\alpha^2}\, g_{(n)ij} \quad = \quad \omega\left(\partial_u - \pounds_{\sigma_{(0)}}\right) g_{(n-1)ij} + \ldots \tag{3.45}$$

where the ellipsis denote lower order terms and ω is some proportionality factor. In the limit $\alpha \to \infty$ the algebraic equation for $g_{(n)}$ therefore results in the differential equation that defines the coefficient $g_{(n-1)}$. However, from the dynamical equation (3.252) and (3.256) for the metric component g_{ij} we find that ω is always proportional to $2(n-1)-d$.[12] This implies that the coefficient $g_{(d/2)ij}$, or more precisely $k_{(d/2)ij}$, becomes the integration constant in the limit $\alpha \to \infty$ instead of $g_{(d+1)ij}$. For odd values of d, $d/2$ is half-integer, so there is no coefficient $g_{(d/2)ij}$ in the expansion (3.29). This would be the leading mode that spoils smoothness of the metric in the definitions of asymptotic flatness at null infinity in odd dimensions as discussed above. On the other hand, the coefficient $g_{(d/2)ij}$ is non-vanishing

[11] The fact that the logarithmic terms are non-vanishing in five or higher odd bulk dimensions is associated to the fact that the conformal anomaly of the dual field theory is non-vanishing in even $(d+1)$ boundary dimensions.

[12] This fact follows from the two terms $4k'_{ij} - 2d/z\, k_{ij}$ in the last line of (3.252).

for even bulk dimensions. Just as the integration constants $G_{(d+1)\mu\nu}$ are associated to the different components of the holographic energy tensor for the case of non-vanishing Λ, the coefficient $g_{(d/2)ij}$, or $k_{(d/2)ij}$, will be related to the spatial components of the QFT energy tensor in even dimensions in the limit $\alpha \to \infty$. We will derive this result for the case $d = 2$ in section 3.4.2.

As a final remark, we will not discuss here the asymptotic symmetry group BMS_{d+2} of the metric (4.41) in the limit $\alpha \to \infty$ and the associated asymptotic charges, but we would still like to point out that our gauge-fixed metric is not invariant under boundary diffeomorphisms (*i.e.* transformations of the form $u \to \tilde{u}(u,x), x^i \to \tilde{x}^i(u,x)$) that do not preserve the foliation in surfaces of constant u. In fact, there is no gauge one can choose – where the gauge freedom has been completely fixed – that simultaneously admits a well-behaved zero Λ limit and is invariant under general boundary diffeomorphisms. This is so because the asymptotic boundary should approach a null manifold in the limit $\alpha \to \infty$ and therefore any gauge admitting a well-behaved zero Λ limit necessarily singles out the timelike direction as a preferred direction over the remaining boundary coordinates.

This observation implies in particular that the subgroup of the asymptotic symmetry group of the metric consisting of boundary diffeomorphisms must be foliation-preserving:[13]

$$\begin{cases} u \;\to\; \tilde{u}(u) \;, \\ x^i \;\to\; \tilde{x}^i(u,x) \;. \end{cases} \tag{3.46}$$

Furthermore, since full covariance, or gauge invariance, is weakened by the requirement that the limit $\alpha \to \infty$ be well-defined, the spectrum of possible holographic counterterms that we can have in the counterterm action is widened. We will see in the next sections that the canonical, fully covariant counterterms [123] are sufficient to render the on-shell gravity action finite once the regulator is removed, but if we also require that the action be finite in the limit $\alpha \to \infty$, further counterterms are needed. The latter preserve invariance of the action under all but those diffeomorphisms that are not foliation-preserving.

Finally, it should be emphasized that the asymptotic symmetry group of the metric contains a subgroup that generates conformal transformations at the boundary. This consists of the transformation $z \to \bar{z} = z\,\Omega(u,x^i)$ together with $x^a \to \bar{x}^a = X^a(u,x^i) + z\,Y^a(u,x^i) + \mathcal{O}(z^2)$, where the functions X^a are defined to satisfy: $q_{(0)ab}dX^a dX^b = \Omega^2 q_{(0)ab}dx^a dx^b$, and where the functions $Y^a(u,x^i)$ can be chosen so that the transformation is asymptotically a symmetry.[14]

[13] See [173] and references therein for a review of foliation preserving diffeomorphisms.

[14] See also [174] about the relation between bulk diffeomorphisms and conformal transformations at the boundary in the context of AdS/CFT.

3.3. Holographic energy tensor

3.3.1. Preliminaries

In order to compute the vacuum expectation value of the dual QFT energy tensor via the AdS/CFT prescription, we need to evaluate the gravitational action (3.4) on-shell and subtract the divergences via holographic renormalization [123, 78]. In the previous section we found that the coefficients in the asymptotic solution for the metric become non-local functionals of the sources in the limit $\alpha \to \infty$ and emphasized that this feature prevents the holographic renormalization of the action in the case of a vanishing cosmological constant. Indeed, if we attempt to renormalize the gravity action (3.4) with $\alpha^{-1} = 0$, we find that the divergent terms are functionals of the coefficients $G_{(n)\mu\nu}$. In this way, the divergences are not local functionals of the sources and therefore cannot be subtracted by local, covariant counterterms. On the other hand, it is possible to renormalize the action with the cosmological constant switched on and in the end analyse the limit $\alpha \to \infty$, so this is the procedure we will follow.

The induced metric and extrinsic curvature q_{ab} and Q_{ab} of the surfaces of constant z are given by:

$$q_{ab}dx^a dx^b = \frac{\ell_o^2}{z^2}\Big(-\varphi N_{(0)} du^2 + g_{ij}\left(dx^i + \sigma^i du\right)\left(dx^j + \sigma^j du\right)\Big) , \tag{3.47}$$

$$\begin{aligned}
Q_{ab} &= \frac{1}{2\beta}\left(\partial_z - \pounds_{\beta n}\right) q_{ab} \\
&= \frac{n_a n_b}{2\beta\varphi}\left[-\varphi' + \frac{2}{z}\,\varphi + \frac{1}{N_{(0)}}\left((\sigma_i\sigma^i)' - \frac{2}{z}\,\sigma_i\sigma^i \right) - (\partial_u - \pounds_\sigma)\log\frac{\varphi}{N_{(0)}} - 2\pounds_\sigma \log\varphi - \frac{2}{\varphi}\sigma^i\sigma^j k_{ij}\right] \\
&- n_{(a}\partial_{b)}x^i \frac{1}{N_{(0)}}\left[\sigma_i' - \frac{2}{z}\sigma_i - N_{(0)}\partial_i \log\varphi - \frac{2}{\varphi}\, N_{(0)}\sigma^j k_{ij}\right] \\
&+ \partial_a x^i \partial_b x^j \frac{\beta\varphi}{2N_{(0)}}\left[g_{ij}' - \frac{2}{z}\, g_{ij} - \frac{2}{\varphi} N_{(0)} k_{ij}\right] ,
\end{aligned} \tag{3.48}$$

where $\beta := (\ell_o/z)\sqrt{N_{(0)}/\varphi}$ is the lapse function of the surfaces of constant z, $\sigma_i := g_{ij}\sigma^j$, and the prime denotes differentiation with respect to z. Also: $n_a = -\varphi\beta\partial_a u$, $n^a\partial_a = \varphi^{-1}\beta^{-1}(\partial_u - \sigma^i\partial_i)$ represents the future-directed unit normal to the surfaces of constant time on each hypersurface of constant z. The on-shell action is then given by:

$$16\pi G_0\, S^{on-shell} \;=\; \int d^d x\, du \int^{\epsilon} dz\, \frac{\ell_{\rm o}^{d+2}}{z^{d+2}}\, N_{(0)} \sqrt{g} \left(-2\,\frac{d+1}{\alpha^2 \ell_{\rm o}^2}\right)$$
$$+ \int\limits_{z=\epsilon} d^d x\, du\, \frac{\ell_{\rm o}^d}{\epsilon^d} \sqrt{g} \Big(-2(d+1)\,\epsilon^{-1}\varphi + \partial_\epsilon \varphi + \varphi\, {\rm Tr}[g^{-1} g'] + (\partial_u - \pounds_\sigma) \log\left(\varphi/N_{(0)}\right) - 2N_{(0)} k \Big), \tag{3.49}$$

where $k = g^{ij} k_{ij}$. In the above we have replaced the asymptotic boundary $\{z = 0\}$ by a regulating surface $\{z = \epsilon\}$ and once the vevs are computed we will remove the regulator by taking the limit $\epsilon \to 0$. Note also that the last term in (3.49) is a total derivative and therefore can be removed from the on-shell action:[15]

$$-2\frac{\ell_{\rm o}^d}{\epsilon^d} \int\limits_{z=\epsilon} d^d x\, du\, \sqrt{g} N_{(0)} k = -2\frac{\ell_{\rm o}^d}{\epsilon^d} \int\limits_{z=\epsilon} d^d x\, du\, (\partial_u - \pounds_\sigma)\sqrt{g}\ . \tag{3.50}$$

The next step in determining the divergences of the action is to use our asymptotic solutions (3.29)–(3.31) for the fields φ, σ^i and g_{ij} and replace the expressions in (3.49). We then look for all the terms that are proportional to negative powers of ϵ, as well as to factors of $\log \epsilon$, and rewrite the respective coefficients in terms of the sources $G_{(0)\mu\nu}$ using (3.32)–(3.37). These terms are those that diverge if the limit $\epsilon \to 0$ is taken. Then, we invert the asymptotic expansions (3.29)–(3.31) in order to express the sources $G_{(0)\mu\nu}$ order by order in ϵ in terms of the fields φ, σ^i and g_{ij}, and then replace the inverted expansions $G_{(0)\mu\nu} = G_{(0)\mu\nu}[\varphi, \sigma^i, g_{ij}]$ in the coefficients of the $\epsilon^{<0}$ divergent terms (as well as the $\log \epsilon$ terms) in the on-shell action. The process results in the set of terms that contribute to the divergences of the on-shell action if the regulator ϵ is sent to zero. The divergent terms obtained in this way are written in a covariant form (except possible anomalous terms depending explicitly on the regulator via a factor of $\log \epsilon$) and can then be subtracted from the action by a counterterm action S_{ct} consisting of minus such terms. The renormalized gravity action S_{ren} will then consist of the original action (3.4) plus the counterterm action S_{ct} derived in this way.

As the spacetime dimension increases, the number of covariant boundary counterterms increases, so we will focus separately on the cases of three and four bulk dimensions. For each case, these counterterms must nevertheless coincide with the canonical counterterms originally obtained in [175, 176, 177, 178, 123]. Although

[15] If we also consider null boundaries $\{u = u_\pm\}$ in the spacetime, such term results in a corner integral $-2\int d^d x \sqrt{\gamma}$ at $\{z = \epsilon, u = u_\pm\}$, with γ_{ij} the induced metric on these codimension two surfaces. Corner terms will be analysed in section 3.5.

the latter were derived in a different coordinate gauge near the asymptotic boundary, these counterterms are covariant and therefore independent of the coordinate system we use. The possible exception are the anomalous counterterms in [123] that depend explicitly on the regulator and therefore that are not invariant under the full diffeomorphism group.

Apart from the canonical counterterm action, we are always free to add finite boundary terms to the renormalized gravity action S_{ren} that do not contribute with divergences in the limit $\epsilon \to 0$ and that provide a non-vanishing contribution to the finite piece once the regulator is removed. These terms are dual to a choice of renormalization scheme in the quantum field theory. In our case, once S_{ren} has been determined by the above procedure, we will have to take care of the zero Λ limit $\alpha \to \infty$. This is done by evaluating S_{ren} on-shell, taking the limit $\epsilon \to 0$ and looking for all those terms that diverge if the limit $\alpha \to \infty$ is taken. Such terms will always be proportional to positive powers of $\alpha\,\epsilon^0$ and the respective coefficients will always be local functionals of the sources $G_{(0)\mu\nu}$. These α-divergent terms can then be subtracted by adding a finite boundary action S_{finite} to S_{ren} (finite in ϵ) consisting of minus such terms. The subtraction of divergences associated to the zero Λ limit is therefore related in this way to a choice of scheme in the dual QFT. As emphasized at the end of section 3.2, however, these finite boundary terms will be invariant under spacetime diffeomorphisms that preserve our foliation, but will break invariance of the gravity action $S_{ren} + S_{finite}$ under those diffeomorphisms that are not foliation-preserving. This fact implies that the renormalization of quantum field theories with gravity duals that admit a well-defined zero Λ limit must involve renormalization schemes that break invariance of the QFT under transformations that do not preserve the spacelike foliation at the boundary.

3.3.2. Three bulk dimensions

3.3.3. Renormalization

If we follow the procedure described above for the case $d+2=3$, we find that the counterterm action is the canonical one in standard AdS_3 holographic renormalization:

$$16\pi G_0\, S_{ren} \;=\; \int d^3x\,\sqrt{G}\left(\frac{d(d+1)}{\alpha^2\ell_o^2} + R[G]\right) + 2\int\limits_{z=\epsilon} d^2x\sqrt{q}\,Q \;+ \frac{2\,d}{\alpha\,\ell_o}\int\limits_{z=\epsilon} d^2x\,\sqrt{q}\;, \tag{3.51}$$

Note, however, the absence of the anomalous topological invariant:

$$\alpha\ell_o \int_{z=\epsilon} d^2x\,\sqrt{q}\,R[q]\log\epsilon\;, \tag{3.52}$$

that arises in the canonical holographic counterterm action. Although such term does not contribute to the variations of the action, it plays an important role in the holographic correspondence: it represents the fact that we cannot renormalize the generating functional Z of the dual QFT and preserve all its symmetries. Such term breaks invariance of the gravity action under bulk diffeomorphisms that result in a conformal transformation at the boundary and it is dual to those counterterms in the renormalization of Z that do not preserve the conformal symmetry.

This term is absent in the present case because we have been careless about possible corner terms in the renormalized gravity action, *i.e.* about integrals on the codimension two surfaces $\{z = \epsilon, u = \pm\infty\}$. Note that in the case of a two dimensional manifold, the Ricci-scalar $R[q]$ can always be written as a total derivative (though not necessarily as an exact form). This is so because we can always imagine some hypersurface, say spacelike, in the two dimensional manifold and use the Gauss-Codazzi identities to express $R[q]$ as:

$$R[q] = R[\gamma] - K^2 + K \cdot K + 2\,\mathcal{D}_a\left(n^a K - a^a\right) \;, \tag{3.53}$$

where γ_{ij} is the metric on the hypersurface, K_{ij} its extrinsic curvature, and n^a and a^a the unit normal and acceleration of the surface. Also, $\mathcal{D}_c\, q_{ab} := 0$. Since the hypersurface is one-dimensional, then $R[\gamma]$ vanishes, and the terms K^2 and $K \cdot K$ cancel one another, leaving us with a total derivative. In our case, if we choose such hypersurface to be a surface of constant time u, we find:

$$\alpha\ell_{\rm o} \int_{z=\epsilon} dx\, du\, \sqrt{q}\, R[q] \log\epsilon \;= -2\alpha\ell_{\rm o}\left(\int_{z=\epsilon} dx\sqrt{\gamma}\, K \log\epsilon\right)_{u=-\infty}^{u=+\infty} \;, \tag{3.54}$$

which is a corner term. Such type of terms do not contribute to the computations of the vev of the QFT energy tensor and we will defer a detailed analysis of the possible corner terms until section 3.5. There we will find that the holographic renormalization of the gravity action indeed requires the term (3.54) as a counterterm.

Given the renormalized action (3.51) we now proceed as discussed at the end of section 3.3.1 and analyse whether the zero Λ limit of the on-shell action was spoiled by the counterterm. We evaluate (3.51) on-shell, take the limit as the regulator $\epsilon \to 0$ and, within the set of terms that survive the limit, we look for those that are proportional to positive powers of α. In three dimensions no such terms exist, which means that the canonical counterterm simultaneously renormalizes the gravity action and preserves the well-definedness of the zero Λ limit.

3.3.4. Vacuum expectation values

Now that we have guaranteed that the on-shell gravity action is free of divergences, we are in position to compute the holographic energy tensor. The variations of the renormalized on-shell action are given by:

$$16\pi G_0\,\delta S_{ren}^{on-shell} = \int\limits_{z=\epsilon} d^2x\sqrt{q}\,(Q_{ab} - q_{ab}Q)\,\delta q^{ab} - \frac{d}{\alpha\ell_{\rm o}}\int\limits_{z=\epsilon} d^2x\sqrt{q}\,q_{ab}\delta q^{ab}\ . \tag{3.55}$$

The renormalized Brown-York tensor [179] is then given by:

$$\begin{aligned} T_{ab} &:= \frac{2}{\sqrt{q}}\frac{\delta S_{ren}^{on-shell}}{\delta q^{ab}(z=\epsilon)} \\ &= \frac{1}{8\pi G_0}\left(Q_{ab} - q_{ab}Q - \frac{1}{\alpha\ell_{\rm o}}\,q_{ab}\right)\ . \end{aligned} \tag{3.56}$$

Using the expression (3.47) for the induced metric q_{ab} we now decompose the variations δq^{ab} in terms of the variations of the lapse, shift and spatial metric:

$$\delta q^{ab} = \left(2n^a n^b/N\right)\delta N + \left(2\,n^{(a}\gamma_i^{\,b)}/N\right)\delta\sigma^i + \gamma_i^a\gamma_j^b\,\delta\gamma^{ij}\ , \tag{3.57}$$

where: $N = (\ell_{\rm o}/z)\sqrt{\varphi N_{(0)}}\,, \gamma_{ij} = (\ell_{\rm o}/z)^2 g_{ij}$ and: $\gamma^{ab} = q^{ab} + n^a n^b$. Following [179] we then define the spatial stress tensor density s_{ij}, and the momentum and energy densities j_i and ε as:

$$s_{ij} := \gamma_i^a\gamma_j^b\,T_{ab} = \frac{2}{N\sqrt{\gamma}}\frac{\delta S_{ren}^{on-shell}}{\delta\gamma^{ij}}\ , \tag{3.58}$$

$$j_i := -n^a\gamma_i^b\,T_{ab} = -\frac{1}{\sqrt{\gamma}}\frac{\delta S_{ren}^{on-shell}}{\delta\sigma^i}\ , \tag{3.59}$$

$$\varepsilon := n^a n^b\,T_{ab} = \frac{1}{\sqrt{\gamma}}\frac{\delta S_{ren}^{on-shell}}{\delta N}\ . \tag{3.60}$$

We also define the trace density T as:

$$T := q^{ab}\,T_{ab} = \left(\gamma^{ab} - n^a n^b\right)T_{ab} = \gamma^{ij}s_{ij} - \varepsilon\ . \tag{3.61}$$

Using the AdS/CFT prescription and recalling the leading order behaviour (3.28), the expectation value of the dual field theory energy tensor is given by:

$$\langle T_{ab}\rangle = \frac{2}{\sqrt{q_{(0)}}}\frac{\delta S_{ren}^{on-shell}}{\delta q_{(0)}^{ab}} = \lim_{\epsilon\to 0}\frac{\ell_{\rm o}^{d-1}}{\epsilon^{d-1}}\,T_{ab}\ . \tag{3.62}$$

In terms of the above decomposition of T_{ab}, the spatial and time components of the holographic energy tensor are then given by:

$$\langle s_{ij}\rangle \;:=\; g_{(0)i}^{\;a} g_{(0)j}^{\;b}\,\langle T_{ab}\rangle \;=\; \frac{2}{\frac{1}{\alpha}N_{(0)}\sqrt{g_{(0)}}}\,\frac{\delta S_{ren}^{on-shell}}{\delta g_{(0)}^{ij}} \;=\; \lim_{\epsilon\to 0}\left(\frac{\ell_{\text{o}}^{d-1}}{\epsilon^{d-1}}\,s_{ij}\right)\;, \tag{3.63}$$

$$\langle j_i\rangle \;:=\; -n_{(0)}^{\;a} g_{(0)i}^{\;b}\,\langle T_{ab}\rangle \;=\; -\frac{1}{\sqrt{g_{(0)}}}\,\frac{\delta S_{ren}^{on-shell}}{\delta \sigma_{(0)}^{i}} \;=\; \lim_{\epsilon\to 0}\left(\frac{\ell_{\text{o}}^{d}}{\epsilon^{d}}\,j_i\right)\;, \tag{3.64}$$

$$\langle \varepsilon\rangle \;:=\; n_{(0)}^{\;a} n_{(0)}^{\;b}\,\langle T_{ab}\rangle \;=\; \frac{1}{\sqrt{g_{(0)}}}\,\frac{\delta S_{ren}^{on-shell}}{\delta(\frac{1}{\alpha}N_{(0)})} \;=\; \lim_{\epsilon\to 0}\left(\frac{\ell_{\text{o}}^{d+1}}{\epsilon^{d+1}}\,\varepsilon\right)\;, \tag{3.65}$$

where the induced metric $g_{(0)}^{ab} = q_{(0)}^{ab} + n_{(0)}^{a} n_{(0)}^{b}$. The vev of the trace of the QFT energy tensor is also given by:

$$\langle T\rangle \;:=\; q_{(0)}^{ab}\langle T_{ab}\rangle \;=\; g_{(0)}^{ij}\langle s_{ij}\rangle - \langle\varepsilon\rangle \;=\; \lim_{\epsilon\to 0}\left(\frac{\ell_{\text{o}}^{d+1}}{\epsilon^{d+1}}\,T\right)\;. \tag{3.66}$$

Now, by construction, the above vacuum expectation values cannot admit a well-behaved zero Λ limit because the lapse $\frac{1}{\alpha}N_{(0)}$ vanishes in this limit. For the vev of the stress tensor we have:

$$\langle s_{ij}\rangle \;=\; \alpha\left(\frac{2}{N_{(0)}\sqrt{g_{(0)}}}\,\frac{\delta S_{ren}^{on-shell}}{\delta g_{(0)}^{ij}}\right) \to \infty \quad (\alpha\to\infty)\;. \tag{3.67}$$

Similarly, for the vev of the energy density:

$$\langle \varepsilon\rangle \;=\; \alpha\left(\frac{1}{\sqrt{g_{(0)}}}\,\frac{\delta S_{ren}^{on-shell}}{\delta N_{(0)}}\right) \to \infty \quad (\alpha\to\infty)\;. \tag{3.68}$$

What we need to do is to work with the quantities that are well-defined in both cases $\Lambda\neq 0$ and $\Lambda = 0$ and these are represented by the tensor densities:

$$\sqrt{q_{(0)}}\,\langle s_{ij}\rangle \;=\; 2\,\frac{\delta S_{ren}^{on-shell}}{\delta g_{(0)}^{ij}}\;, \tag{3.69}$$

$$\sqrt{q_{(0)}}\,\langle \varepsilon\rangle \;=\; N_{(0)}\,\frac{\delta S_{ren}^{on-shell}}{\delta N_{(0)}}\;, \tag{3.70}$$

where: $\sqrt{q_{(0)}} = \frac{1}{\alpha} N_{(0)} \sqrt{g_{(0)}}$. A straightforward computation using (3.56) and (3.48) then leads to the following one-point functions:

$$\sqrt{q_{(0)}}\,\langle s_{ij}\rangle = \frac{\ell_{\rm o}}{8\pi G_0}\,N_{(0)}\sqrt{g_{(0)}}\left(-\frac{\varphi_{(2)}}{2\,N_{(0)}}\,g_{(0)ij}\right)\,, \tag{3.71}$$

$$\sqrt{q_{(0)}}\,\langle \varepsilon\rangle = \frac{\ell_{\rm o}}{8\pi G_0}\,N_{(0)}\sqrt{g_{(0)}}\left(\frac{1}{\alpha^2}\,{\rm Tr}[g^{-1}_{(0)}g_{(2)}] - \frac{\varphi_{(2)}}{2\,N_{(0)}} - {\rm Tr}[g^{-1}_{(0)}k_{(1)}]\right)\,, \tag{3.72}$$

$$\langle j_i\rangle = -\frac{\ell_{\rm o}}{8\pi G_0}\,N^{-1}_{(0)}\left(\sigma_{(2)i} + \frac{1}{2}\,(g^{-1}_{(0)}g_{(1)})_i^j\partial_j N_{(0)}\right)\,. \tag{3.73}$$

We therefore find as usual that the normalisable modes $G_{(d+1)\mu\nu}$ are directly associated to the vacuum expectation values [123]. Note that these expressions admit a well-behaved limit $\alpha \to \infty$.

3.3.5. Weyl anomaly

For the vev of the trace we find:

$$\sqrt{q_{(0)}}\,\langle T\rangle = \frac{\ell_{\rm o}}{8\pi G_0}\,N_{(0)}\sqrt{g_{(0)}}\left(-\frac{1}{\alpha^2}\,{\rm Tr}[g^{-1}_{(0)}g_{(2)}] + {\rm Tr}[g^{-1}_{(0)}k_{(1)}]\right)\,. \tag{3.74}$$

Notice now that if we perform a decomposition of the Ricci scalar of the QFT metric as in (3.53) we obtain:

$$R[q_{(0)}] = R[g_{(0)}] - K^2_{(0)} + K_{(0)}\cdot K_{(0)} + 2\,{}^{(0)}\mathcal{D}_a\left(n^a_{(0)}K_{(0)} - a^a_{(0)}\right)\,, \tag{3.75}$$

where ${}^{(0)}\mathcal{D}_c q_{(0)ab} := 0$. A quick computation using (3.41), (3.32) and (3.38) then reveals that:

$$R[q_{(0)}] = 2\,{}^{(0)}\mathcal{D}_a\left(n^a_{(0)}K_{(0)} - a^a_{(0)}\right) \tag{3.76}$$

$$= 2\left(-\frac{1}{\alpha^2}{\rm Tr}[g^{-1}_{(0)}g_{(2)}] + {\rm Tr}[g^{-1}_{(0)}k_{(1)}]\right)\,. \tag{3.77}$$

Replacing in (3.74) results in the standard holographic Weyl anomaly:

$$\begin{aligned}\sqrt{q_{(0)}}\,\langle T\rangle &= \frac{\ell_{\rm o}}{16\pi G_0}\,N_{(0)}\sqrt{g_{(0)}}\,R[q_{(0)}]\\ &= \frac{\alpha\ell_{\rm o}}{16\pi G_0}\,\sqrt{q_{(0)}}\,R[q_{(0)}]\\ &= \frac{c}{24\pi}\,\sqrt{q_{(0)}}\,R[q_{(0)}]\,,\end{aligned} \tag{3.78}$$

where $c = 3\alpha\ell_{\text{o}}/(2G_0)$ is the standard central charge in the $\text{AdS}_3/\text{CFT}_2$ correspondence. Note that the anomaly admits a well-behaved zero Λ limit. Using equation (3.74) we find:

$$\begin{aligned} \lim_{\alpha\to\infty} \sqrt{q_{(0)}}\,\langle T\rangle &= \frac{\ell_{\text{o}}}{8\pi G_0} N_{(0)}\sqrt{g_{(0)}}\ \text{Tr}[g_{(0)}^{-1}k_{(1)}] \\ &= \frac{\ell_{\text{o}}}{8\pi G_0}\,\partial_a\Big[N_{(0)}\sqrt{g_{(0)}}\left(n^a_{(0)}K_{(0)} - a^a_{(0)}\right)\Big] , \end{aligned} \tag{3.79}$$

where:

$$\begin{aligned} &\partial_a\Big[N_{(0)}\sqrt{g_{(0)}}\left(n^a_{(0)}K_{(0)} - a^a_{(0)}\right)\Big] \\ &\quad = \partial_u\left(\frac{1}{2}\sqrt{g_{(0)}}\ \text{Tr}[g_{(0)}^{-1}g_{(1)}]\right) - \partial_i\left(\sqrt{g_{(0)}}\left(\frac{1}{2}\ \text{Tr}[g_{(0)}^{-1}g_{(1)}]\,\sigma^i_{(0)} + g^{ij}_{(0)}\partial_j \log N_{(0)}\right)\right) . \end{aligned} \tag{3.80}$$

The anomaly is still a total derivative and the integrated anomaly a topological invariant. Using equation (3.42) we also find that $\text{Tr}[g_{(0)}^{-1}k_{(1)}] = g^{ij}_{(0)}R_{ij}[q_{(0)}]$ in the zero Λ limit. Since the anomaly admits a well-defined zero Λ limit, a central charge can be introduced in this limit by using the identity (3.77) to rewrite the right-hand side of equation (3.79) as a geometric invariant and introducing the limit:

$$\lim_{\alpha\to\infty}\left(\frac{G_0}{\alpha\ell_{\text{o}}}\sqrt{q_{(0)}}\,\langle T\rangle\right) = \frac{1}{8\pi}\sqrt{q_{(0)}}\ {}^{(0)}\mathcal{D}_a\left(n^a_{(0)}K_{(0)} - a^a_{(0)}\right) . \tag{3.81}$$

The proportionality constant between the trace and the total derivative is then:

$$\frac{1}{8\pi} = \frac{c}{24\pi} , \tag{3.82}$$

where $c = 3$ is the Virasoro central charge in the central extension of the asymptotic symmetry group $\mathfrak{bms}_3$ of three dimensional flat gravity [151, 154].[16]

3.3.6. Improved energy tensor

If we return to the full Weyl anomaly (3.74) or (3.78) for generic Λ and use equations (3.38), (3.42) and (3.36), we can rewrite it in terms of the coefficient $g_{(1)ij}$ as:

$$\begin{aligned} \sqrt{q_{(0)}}\ \langle T\rangle = \frac{\ell_{\text{o}}}{8\pi G_0} N_{(0)}\sqrt{g_{(0)}}\,\Bigg(&-\frac{1}{4\alpha^2}\,(g_{(1)}\cdot g_{(1)}) + \frac{1}{2N_{(0)}}\,g^{ij}_{(0)}(\partial_u - \pounds_{\sigma_{(0)}})g_{(1)ij} \\ &- \frac{1}{N_{(0)}}\,{}^{(0)}\Box N_{(0)}\Bigg) . \end{aligned} \tag{3.83}$$

[16] The central charge c_{LM} in reference [154] is related to ours as: $c_{LM} = c/12$ since we follow the convention of formula (1) in this reference as can be seen by comparing the central charge in this formula with that in the *AdS* case (3.78).

Notice now from equation (3.32) that a non-vanishing coefficient $g_{(1)ij}$ represents the fact that the QFT metric is time dependent. The boundary shift $\sigma^i_{(0)}$ can always be fixed to any configuration by boundary diffeomorphisms; in particular, we can fix $\sigma^i_{(0)}$ to zero by the transformation $x^i \to x^i - \int du\, \sigma^i_{(0)}$. In such coordinates, equation (3.32) becomes: $N_{(0)}/\alpha^2\, g_{(1)ij} = \partial_u g_{(0)ij}$. Therefore, if the QFT metric (3.28) is static the Weyl anomaly becomes:

$$\begin{aligned} \sqrt{q_{(0)}}\ \langle T\rangle &= -\frac{\ell_o}{8\pi G_0}\sqrt{g_{(0)}}\ {}^{(0)}\Box N_{(0)} \\ &= -\frac{\alpha\ell_o}{8\pi G_0}\sqrt{q_{(0)}}\ {}^{(0)}\mathcal{D}_b\, a^b_{(0)}\ , \end{aligned} \tag{3.84}$$

where the acceleration $a^a_{(0)} = g^{ab}_{(0)}\partial_b \log N_{(0)}$ as before. Using the definition of $g^{ab}_{(0)}$ this can be rewritten as:

$$\sqrt{q_{(0)}}\ \langle T\rangle = -\frac{\alpha\ell_o}{8\pi G_0}\sqrt{q_{(0)}}\left({}^{(0)}\mathcal{D}_a{}^{(0)}\mathcal{D}^a \log N_{(0)} + {}^{(0)}\mathcal{D}_a\left(n^a_{(0)}n^b_{(0)}\partial_b \log N_{(0)}\right)\right)\ . \tag{3.85}$$

Since the boundary metric is static, the second total derivative vanishes:

$$n^b_{(0)}\partial_b \log N_{(0)} = \alpha/N_{(0)}(\partial_u - \sigma^i_{(0)}\partial_i)\log N_{(0)} = 0\ .$$

The first total derivative that remains is unphysical in the sense that it can be absorbed in an improved energy tensor Θ^{ab} defined in terms of the QFT energy tensor T^{ab} and covariant derivatives of the acceleration $a^a_{(0)}$, or of the lapse $\log N_{(0)}$ (see *e.g.* [180]). The conformal Ward identity then becomes:

$$\sqrt{q_{(0)}}\ \langle \Theta^a{}_a\rangle = 0\ , \tag{3.86}$$

for a static metric $q_{(0)ab}$. In section 3.4.2 we will find another example where an improved energy tensor can be defined such that staticity of the boundary metric restores conformal invariance of the field theory.

3.3.7. Diffeomorphism Ward identity

In order to verify that the holographic energy tensor is conserved we need to solve the constraint equations (3.254)–(3.255) using (3.256) for the normalisable modes.

At first subleading order with $d = 1$ these two equations result in the constraints:

$$\frac{1}{\sqrt{g_{(0)}}}\left(\partial_u - \pounds_{\sigma_{(0)}}\right)\left(\sqrt{g_{(0)}}\; N_{(0)}^{-1}\left(\sigma_{(2)i} + \frac{1}{2}\left(g_{(0)}^{-1} g_{(1)}\right)_i{}^j \partial_j N_{(0)}\right)\right)$$
$$= -\frac{1}{2}\,\partial_i \varphi_{(2)} + \left(\frac{1}{\alpha^2}\mathrm{Tr}[g_{(0)}^{-1} g_{(2)}] - \frac{\varphi_{(2)}}{2N_{(0)}} - \mathrm{Tr}[g_{(0)}^{-1} k_{(1)}]\right)\partial_i N_{(0)}\ , \tag{3.87}$$

$$\frac{1}{\sqrt{g_{(0)}}}\left(\partial_u - \pounds_{\sigma_{(0)}}\right)\left(\sqrt{g_{(0)}}\left(-\frac{1}{\alpha^2}\,\mathrm{Tr}[g_{(0)}^{-1} g_{(2)}] + \frac{\varphi_{(2)}}{2N_{(0)}} + \mathrm{Tr}[g_{(0)}^{-1} k_{(1)}]\right)\right)$$
$$= \frac{1}{2}\,\varphi_{(2)}\, k_{(0)} - \frac{1}{\alpha^2 N_{(0)}}\ {}^{(0)}\nabla_i\left(N_{(0)}\left(\sigma_{(2)}^i + \frac{1}{2} g_{(1)}^{ij}\partial_j N_{(0)}\right)\right)\ . \tag{3.88}$$

These constraints result in the conservation equations for the QFT energy tensor:

$$\begin{aligned} 0 =&\ {}^{(0)}\mathcal{D}_a\left(\sqrt{q_{(0)}}\,\langle T^a{}_i\rangle\right) \\ =&\ \left(\partial_u - \pounds_{\sigma_{(0)}}\right)\left(\sqrt{g_{(0)}}\,\langle j_i\rangle\right) + {}^{(0)}\nabla_j\left(\sqrt{q_{(0)}}\,\langle s^j{}_i\rangle\right) + \sqrt{q_{(0)}}\,\langle\varepsilon\rangle\,\partial_i \log N_{(0)}\ , \end{aligned} \tag{3.89}$$

$$\begin{aligned} 0 =&\ \frac{1}{\alpha}\, n_{(0)}^b\ {}^{(0)}\mathcal{D}_a\left(\sqrt{q_{(0)}}\,\langle T^a{}_b\rangle\right) \\ =&\ -\left(\partial_u - \pounds_{\sigma_{(0)}}\right)\left(N_{(0)}^{-1}\sqrt{q_{(0)}}\,\langle\varepsilon\rangle\right) - N_{(0)}^{-1}\ {}^{(0)}\nabla_i\left((N_{(0)}/\alpha)^2\sqrt{g_{(0)}}\,\langle j^i\rangle\right) + \sqrt{q_{(0)}}\,\langle s^{ij}\rangle\, k_{(0)ij}\ . \end{aligned} \tag{3.90}$$

3.3.8. BTZ and 3-dimensional cosmology

In this section we would like to make a brief application of the results obtained so far to a particular bulk metric and the spacetime we are interested in is the BTZ black hole and its zero Λ limit, which represents a cosmological solution [181]. In Eddington-Finkelstein coordinates the BTZ metric is given by:

$$ds^2 \ = -\left(-8MG_0 + \frac{r^2}{\ell^2} + \left(\frac{4aG_0}{r}\right)^2\right)du^2 + 2dudr + r^2\left(d\theta - \frac{4aG_0}{r^2}\,du\right)^2\ , \tag{3.91}$$

where the cosmological constant $\Lambda = -\ell^{-2}$, M is the mass of the spacetime and a the angular momentum. Also, the angular coordinate $\theta \in [0, 2\pi[$. In order to bring the metric to the form (4.41) we introduce a coordinate $z := \ell_{\mathrm{o}}^2/r$:

$$ds^2 \ = \frac{\ell_{\mathrm{o}}^2}{z^2}\left(-\left(\frac{1}{\alpha^2} - \frac{8MG_0}{\ell_{\mathrm{o}}^2}\,z^2 + \left(\frac{4aG_0}{\ell_{\mathrm{o}}^3}\right)^2 z^4\right)du^2 - 2dudz + \ell_{\mathrm{o}}^2\left(d\theta - \frac{4aG_0}{\ell_{\mathrm{o}}^4}\,z^2 du^2\right)^2\right)\ . \tag{3.92}$$

Note that the metric is well-defined in the limit $\alpha \to \infty$. For this solution the holographic energy tensor reads:

$$\sqrt{q_{(0)}}\,\langle s_{ij}\rangle \;=\; \sqrt{g_{(0)}}\left(\frac{M}{2\pi\ell_{\text{o}}}\,g_{(0)ij}\right)\,, \tag{3.93}$$

$$\sqrt{q_{(0)}}\,\langle \varepsilon\rangle \;=\; \sqrt{g_{(0)}}\left(\frac{M}{2\pi\ell_{\text{o}}}\right)\,, \tag{3.94}$$

$$\langle j_i\rangle \;=\; \frac{a}{2\pi\ell_{\text{o}}}\,, \tag{3.95}$$

where the spatial metric $g_{(0)ij}dx^i dx^j = \ell_{\text{o}}^2 d\theta^2$. In this case, the characteristic length ℓ_{o} represents the radius of the boundary cylinder. If we then introduce the average energy $\langle E\rangle$ over a time interval $2T$ we obtain:

$$\langle E\rangle := \frac{1}{2T}\int_{-T}^{T} du \int d^d x\sqrt{q_{(0)}}\,\langle\varepsilon\rangle \;=\; M\,. \tag{3.96}$$

Also, for the angular momentum we find:

$$\langle J_i\rangle := \int d^d x\sqrt{g_{(0)}}\,\langle j_i\rangle \;=\; a\,. \tag{3.97}$$

Note that these results can be extended to the zero Λ limit of the solution and coincide with those obtained in [158, 154] via a thermodynamics analysis of the respective three-dimensional cosmological solution.

3.4. Four bulk dimensions

3.4.1. Renormalization

In the case of $d+2=4$ dimensions the renormalized action is given by:

$$\begin{aligned} 16\pi G_0\, S_{ren} \;=\; & \int d^4x\,\sqrt{G}\left(\frac{d(d+1)}{\alpha^2\ell_{\text{o}}^2}+R[G]\right)+2\int_{z=\epsilon} d^3x\sqrt{q}\,Q \\ & +\;\frac{2\,d}{\alpha\ell_{\text{o}}}\int_{z=\epsilon} d^3x\,\sqrt{q}+\frac{\alpha\ell_{\text{o}}}{d-1}\int_{z=\epsilon} d^3x\sqrt{q}\,R[q]\,, \end{aligned} \tag{3.98}$$

where the counterterms again coincide with the canonical ones in four bulk dimensions. The next step is to determine whether the renormalization spoils the zero Λ limit of the action. In the present case, if we evaluate S_{ren} on-shell and take the limit $\epsilon \to 0$ as described in section 3.3.1, we find again that no terms proportional to positive powers of α survive and therefore that the limit $\alpha \to \infty$ is still wel-defined in the presence of the counterterm action. However, this feature is peculiar to our particular choice of boundary lapse function $N_{(0)}$. In section 3.2.1 and 3.2.2 we found that, in general, the metric component φ admits an arbitrary term $z\varphi_{(1)}$ in the asymptotic expansion. We then argued that it is always possible to redefine the coordinate z and choose some new function $N_{(0)}$ as in (3.21) such that the term $\varphi_{(1)}$ is removed from the asymptotics. On the other hand, if we choose to decouple $N_{(0)}$ from $\varphi_{(1)}$ by requiring that equation (3.21) for $N_{(0)}$ does not hold, then the asymptotic solution (3.30) for φ will admit a term $z\varphi_{(1)}$, with $\varphi_{(1)}(u, x)$ an arbitrary function. In such case, the solution (3.32) is modified to:

$$\frac{1}{2\alpha^2}\, g_{(1)ij} = k_{(0)ij} + \frac{\varphi_{(1)}}{2N_{(0)}}\, g_{(0)ij} \ . \tag{3.99}$$

Although the renormalization in the three dimensional case analysed in the previous section remains unaffected, if we switch on the coefficient $\varphi_{(1)}$ by allowing the lapse $N_{(0)}$ to be independent, the canonical counterterm action in the four dimensional case will spoil the zero Λ limit via the term:

$$\lim_{\epsilon \to 0}\ 16\pi G_0\, S_{ren}^{on-shell} \ = \ \frac{\alpha^2 \ell_{\rm o}^2}{2} \int\limits_{z=0} d^3x\, \sqrt{g_{(0)}}\, \varphi_{(1)} R[g_{(0)}] + \mathcal{O}(\alpha^{\leq 0}) \ , \tag{3.100}$$

where $\mathcal{O}(\alpha^{\leq 0})$ denotes terms proportional to non-positive powers of α. From equation (3.99) it follows that the (finite) counterterm that restores the well-definedness of the limit is given by:

$$\begin{aligned} 16\pi G_0\, S_{ren} \ &= \ \int d^4x\, \sqrt{G} \left(\frac{d(d+1)}{\alpha^2 \ell_{\rm o}^2} + R[G] \right) + 2 \int\limits_{z=\epsilon} d^3x \sqrt{q}\, Q \\ &+ \ \frac{2\,d}{\alpha \ell_{\rm o}} \int\limits_{z=\epsilon} d^3x\, \sqrt{q} + \frac{\alpha \ell_{\rm o}}{d-1} \int\limits_{z=\epsilon} d^3x \sqrt{q}\, R[q] \\ &+ \ \frac{\alpha^2 \ell_{\rm o}^2}{2} \int\limits_{z=\epsilon} d^3x\, \sqrt{q}\, K R[\gamma] \ , \end{aligned} \tag{3.101}$$

with γ_{ij} and K_{ij} the induced metric and extrinsic curvature of the surfaces of constant time at the boundary as defined in section 3.3.3. This last counterterm is covariant with respect to diffeomorphisms that preserve our foliation of the spacetime, but breaks invariance of the action under those transformations that are not foliation-preserving, as discussed in section 3.3.1. The latter include those bulk diffeomorphisms that result in a conformal transformation at the boundary and therefore the trace Ward identity will be affected by such term as discussed in the next section. This counterterm is also finite in the regulator ϵ and therefore must be related to a choice of renormalization scheme in the dual field theory. In particular, it signals the fact that the scheme cannot preserve the invariance of the QFT under those transformations that are not foliation-preserving at the boundary if the gravity dual has a well-defined zero Λ limit. We will find more examples of counterterms of this type in section 3.6. With our choice of $N_{(0)}$, however, the canonical action (3.98) is well-defined, so we will ignore for now this extra counterterm and discuss its necessity and implications in the next section.

3.4.2. Vacuum expectation values and the Ward identities

The variations of the renormalized on-shell action (3.98) are given by:

$$\begin{aligned}16\pi G_0\, \delta S_{ren}^{on-shell} =& \int\limits_{z=\epsilon} d^3x \sqrt{q}\,(Q_{ab} - q_{ab}Q)\,\delta q^{ab} \\ &- \frac{d}{\alpha\ell_{\mathrm{o}}} \int\limits_{z=\epsilon} d^3x\sqrt{q}\, q_{ab}\delta q^{ab} + \frac{\alpha\ell_{\mathrm{o}}}{d-1} \int\limits_{z=\epsilon} d^3x\,\sqrt{q}\left(R_{ab}[q] - \frac{1}{2}\, q_{ab}\, R[q]\right)\delta q^{ab}\ .\end{aligned} \tag{3.102}$$

The spatial and time components of the Brown-York tensor as defined in section 3.3.4 are then given by:

$$\begin{aligned} s_{ij} =&\ \gamma_i^a\gamma_j^b \left(\frac{2}{\sqrt{q}}\,\frac{\delta S_{ren}^{on-shell}}{\delta q^{ab}(z=\epsilon)}\right) \\ =&\ \frac{1}{8\pi G_0}\left(Q_{ij} - \gamma_{ij}Q - \frac{d}{\alpha\ell_{\mathrm{o}}}\,\gamma_{ij} + \alpha\ell_{\mathrm{o}}\left(R_{ij}[q] - \frac{1}{2}\,\gamma_{ij}R[q]\right)\right)\ ,\end{aligned} \tag{3.103}$$

$$\begin{aligned} j_i =&\ -n^a\gamma_i^b \left(\frac{2}{\sqrt{q}}\,\frac{\delta S_{ren}^{on-shell}}{\delta q^{ab}(z=\epsilon)}\right) \\ =&\ \frac{1}{8\pi G_0}\left(-n^a Q_{ai} - \alpha\ell_{\mathrm{o}}\, n^a R_{ai}[q]\right)\ ,\end{aligned} \tag{3.104}$$

$$\begin{aligned} \varepsilon =&\ n^a n^b \left(\frac{2}{\sqrt{q}}\,\frac{\delta S_{ren}^{on-shell}}{\delta q^{ab}(z=\epsilon)}\right) \\ =&\ \frac{1}{8\pi G_0}\left(\gamma^{ij}Q_{ij} + \frac{d}{\alpha\ell_{\mathrm{o}}} + \frac{\alpha\ell_{\mathrm{o}}}{2}\left(n^a n^b R_{ab}[q] + \gamma^{ij}R_{ij}[q]\right)\right)\ .\end{aligned} \tag{3.105}$$

A lengthy computation using the prescriptions (3.69)–(3.70) and (3.64) for the vacuum expectation values of the components of the dual QFT energy tensor in

$d+2=4$ dimensions results in the following one-point functions:

$$\begin{aligned}\sqrt{q_{(0)}}\,\langle s_{ij}\rangle = \; & \frac{\ell_{\rm o}^2}{8\pi G_0}\,N_{(0)}\sqrt{g_{(0)}}\,\Bigg[\frac{3}{2\alpha^2}\,g_{(3)ij}-\frac{\varphi_{(3)}}{2N_{(0)}}\,g_{(0)ij}\\ & +\alpha^2\left(\left(\partial_u-\pounds_{\sigma_{(0)}}\right)k_{(1)ij}-\frac{1}{2}\,g_{(0)ij}\mathrm{Tr}\Big[g^{-1}_{(0)}\left(\partial_u-\pounds_{\sigma_{(0)}}\right)k_{(1)}\Big]\right)\\ & +\partial_{(i}\log N_{(0)}\partial_{j)}\mathrm{Tr}[g^{-1}_{(0)}g_{(1)}]-{}^{(0)}\nabla_k\big(g^{-1}_{(0)}g_{(1)}\big)^k{}_{(i}\,\partial_{j)}\log N_{(0)}\\ & +X_{ij}-\frac{1}{2}\,g_{(0)ij}\mathrm{Tr}[g^{-1}_{(0)}X]\Bigg]\;, \end{aligned} \tag{3.106}$$

$$\begin{aligned}\sqrt{q_{(0)}}\,\langle \varepsilon\rangle = \; & \frac{\ell_{\rm o}^2}{8\pi G_0}\,N_{(0)}\sqrt{g_{(0)}}\,\Bigg[-\frac{\varphi_{(3)}}{N_{(0)}}\\ & +\partial_i\log N_{(0)}\,{}^{(0)}\nabla^i\mathrm{Tr}[g^{-1}_{(0)}g_{(1)}]-{}^{(0)}\nabla^i g_{(1)ij}\,{}^{(0)}\nabla^j\log N_{(0)}\Bigg]\;, \end{aligned} \tag{3.107}$$

$$\begin{aligned}\langle j_i\rangle = \; & \frac{\ell_{\rm o}^2}{8\pi G_0}\Bigg[-\frac{3}{2N_{(0)}}\,\sigma_{(3)i}\\ & -\alpha^2\,{}^{(0)}\nabla^j\left(k_{(1)ij}-\frac{1}{2}\,g_{(0)ij}\mathrm{Tr}[g^{-1}_{(0)}k_{(1)}]\right)+\frac{\alpha^2}{4}\,\partial_i R[g_{(0)}]+X_i\Bigg]\;. \end{aligned} \tag{3.108}$$

From the trace constraint equation (3.253) using (3.256) it follows that the normalisable mode $g_{(3)ij}$ is traceless: $\mathrm{Tr}[g^{-1}_{(0)}g_{(3)}]=0$. The trace (3.66) of the holographic energy tensor is then given by:

$$\langle T\rangle \;=\; g^{ij}_{(0)}\langle s_{ij}\rangle-\langle\varepsilon\rangle = 0\;. \tag{3.109}$$

This is the expected result for a conformal field theory in three dimensions. From the above one-point functions for finite α, we find that the normalisable modes $G_{(d+1)\mu\nu}$ are again mapped to the vacuum expectation values. The expressions for the terms X_{ij} and X_i are given in appendix 3.9 and consist in a set of terms in $g_{(1)ij}$ proportional to non-positive powers of α. These terms are scheme dependent in the sense that they can be subtracted by a choice of finite counterterms of the form:

$$\alpha^2\ell_{\rm o}^2\int_{z=\epsilon}d^3x\sqrt{q}\left(a_1K^3+a_2K(K\cdot K)+a_3(K\cdot K\cdot K)+a_4KR[\gamma]+a_5\Box_\gamma K+...\right)\;. \tag{3.110}$$

As discussed in section 3.3.6, a non-vanishing coefficient $g_{(1)ij}$ represents the fact that the QFT metric is time dependent. It follows that the terms X_{ij} and X_i are possibly non-vanishing only if the boundary metric is not static.

Let us then discuss the terms in the second line of (3.106) and (3.108) that depend on α^2. The last of these, $(\alpha^2/4)\,\partial_i R[g_{(0)}]$, diverges in the limit $\alpha \to \infty$. Note, however, that if we preserve the counterterm introduced in (3.101), it will contribute to the variations of the on-shell action as:

$$\begin{aligned}\delta\left(\frac{\alpha^2\ell_{\rm o}^2}{2}\int\limits_{z=\epsilon} d^3x\sqrt{q}\,KR[\gamma]\right) &= \frac{\alpha^2\ell_{\rm o}^2}{2}\int\limits_{z=\epsilon} d^3x\,\sqrt{g}\,\partial_i R[g]\,\delta\sigma^i \\ &+ \frac{\alpha^2\ell_{\rm o}^2}{2}\int\limits_{z=\epsilon} d^3x\,\sqrt{g}\left(g_{ij}\nabla^k\nabla^l\left(N_{(0)}k_{kl}\right) - \nabla_i\nabla_j\left(N_{(0)}k\right)\right)\delta g^{ij}\ ,\end{aligned} \tag{3.111}$$

where $\nabla_i g_{jk} := 0$. This result implies that the one-point functions will be modified to:[17]

$$\langle j_i\rangle \to \langle j_i\rangle_{new} = \langle j_i\rangle - \frac{\ell_{\rm o}^2}{8\pi G_0}\left(\frac{\alpha^2}{4}\,\partial_i R[g_{(0)}]\right)\ , \tag{3.112}$$

$$\sqrt{q_{(0)}}\,\langle\varepsilon\rangle \to \sqrt{q_{(0)}}\,\langle\varepsilon\rangle_{new} = \sqrt{q_{(0)}}\,\langle\varepsilon\rangle\ , \tag{3.113}$$

$$\begin{aligned}\sqrt{q_{(0)}}\,\langle s_{ij}\rangle \to \sqrt{q_{(0)}}\,\langle s_{ij}\rangle_{new} &= \sqrt{q_{(0)}}\,\langle s_{ij}\rangle \\ + \frac{\ell_{\rm o}^2}{32\pi G_0}\sqrt{g_{(0)}}&\left(g_{(0)ij}\,{}^{(0)}\nabla^k\,{}^{(0)}\nabla^l\left(N_{(0)}g_{(1)kl}\right) - {}^{(0)}\nabla_i\,{}^{(0)}\nabla_j\left(N_{(0)}\mathrm{Tr}[g_{(0)}^{-1}g_{(1)}]\right)\right)\ .\end{aligned} \tag{3.114}$$

The anomalous counterterm therefore provides a contribution to $\langle j_i\rangle$ that cancels the α-divergence proportional to the gradient of the Ricci scalar without introducing further divergences. This is done, however, at the expense of modifying the conformal Ward identity (3.109) by a total derivative:

$$\begin{aligned}\sqrt{q_{(0)}}\,\langle T\rangle \to \sqrt{q_{(0)}}\,\langle T\rangle_{new} &= 0 + \frac{\ell_{\rm o}^2}{16\pi G_0}\sqrt{g_{(0)}}\,{}^{(0)}\nabla^i\,{}^{(0)}\nabla^j\left[N_{(0)}\left(g_{(1)ij} - \frac{1}{2}\,g_{(0)ij}\mathrm{Tr}[g_{(0)}^{-1}g_{(1)}]\right)\right] \\ &= \frac{\ell_{\rm o}^2}{8\pi G_0}\sqrt{g_{(0)}}\,{}^{(0)}\nabla^i\,{}^{(0)}\nabla^j\left[\alpha N_{(0)}\left(K_{(0)ij} - \frac{1}{2}\,g_{(0)ij}K_{(0)}\right)\right]\ ,\end{aligned} \tag{3.115}$$

[17] As a technical point, if the coefficient $\varphi_{(1)} \neq 0$ then the last integral in (3.111) will contribute with terms $\alpha^2\varphi_{(1)}$ to $\langle s_{ij}\rangle_{new}$. However, the previous spatial stress $\langle s_{ij}\rangle$ will contain the symmetric of such terms if $\varphi_{(1)} \neq 0$ such that they cancel overall.

which is finite in the limit $\alpha \to \infty$. Note that if we define:

$$v_{ij} \;:=\; \alpha N_{(0)} \left(K_{(0)ij} - \frac{1}{2}\, g_{(0)ij} K_{(0)} \right) \,, \tag{3.116}$$

$$v^{ab} \;:=\; g^{ai}_{(0)} g^{bj}_{(0)} v_{ij} \,, \tag{3.117}$$

and use the standard identities from the theory of embedded hypersurfaces, we obtain that:

$$\begin{aligned}
\alpha \sqrt{q_{(0)}}\, {}^{(0)}\mathcal{D}_a \left(N^{-1}_{(0)}\, {}^{(0)}\nabla_b v^{ab} \right) &= \sqrt{g_{(0)}}\; {}^{(0)}\nabla^i\, {}^{(0)}\nabla^j v_{ij} \,, \\
N^{-1}_{(0)}\, {}^{(0)}\nabla_b v^{ab} &= {}^{(0)}\mathcal{D}_b L^{ab} - \frac{1}{\alpha}\, n^a_{(0)} \left(L_{ij} L^{ij} \right) \,,
\end{aligned} \tag{3.118}$$

with ${}^{(0)}\nabla_a$ the covariant derivative induced on the surfaces of constant time at the boundary manifold, associated to the induced metric $g_{(0)ab} = q_{(0)ab} + n_{(0)a} n_{(0)b}$, and where:

$$L^{ab} \;:=\; N^{-1}_{(0)} v^{ab} = \alpha\, g^{ai}_{(0)} g^{bj}_{(0)} \left(K_{(0)ij} - \frac{1}{2}\, g_{(0)ij} K_{(0)} \right) . \tag{3.119}$$

The modified trace Ward identity can then be rewritten as:

$$\sqrt{q_{(0)}}\, \langle T \rangle_{new} = \frac{\ell_{\mathrm{o}}^2}{8\pi G_0} \sqrt{q_{(0)}} \left[\alpha\, {}^{(0)}\mathcal{D}_a {}^{(0)}\mathcal{D}_b\, L^{ab} - {}^{(0)}\mathcal{D}_a \left(n^a_{(0)} \big(L \cdot L \big) \right) \right] . \tag{3.120}$$

The first total derivative is unphysical because it can be absorbed in an improved energy tensor Θ^{ab} defined in terms of the QFT energy tensor T_{ab} and covariant derivatives of L_{ab} [180], but the second term remains. The Ward identity in such case becomes:

$$\sqrt{q_{(0)}}\, \langle \Theta^a{}_a \rangle \;=\; \frac{\ell_{\mathrm{o}}^2}{8\pi G_0} \sqrt{q_{(0)}}\; {}^{(0)}\mathcal{D}_a v^a \,, \tag{3.121}$$

where:

$$\begin{aligned}
\sqrt{q_{(0)}}\; {}^{(0)}\mathcal{D}_a v^a &= -\sqrt{q_{(0)}}\; {}^{(0)}\mathcal{D}_a \left(n^a_{(0)} \big(L \cdot L \big) \right) \\
&= -\partial_u \left(\sqrt{g_{(0)}}\, \frac{1}{4} \left(g_{(1)} \cdot g_{(1)} - \frac{1}{2}\, \mathrm{Tr}[g^{-1}_{(0)} g_{(1)}]^2 \right) \right) \\
&\quad + \partial_i \left(\sqrt{g_{(0)}}\, \sigma^i_{(0)}\, \frac{1}{4} \left(g_{(1)} \cdot g_{(1)} - \frac{1}{2}\, \mathrm{Tr}[g^{-1}_{(0)} g_{(1)}]^2 \right) \right) \,,
\end{aligned} \tag{3.122}$$

which is finite in the limit $\alpha \to \infty$. This result is expected because the anomalous counterterm in (3.101) breaks, in particular, invariance of the renormalized gravity action under bulk diffeomorphisms that result in a conformal transformation at the boundary. The generating functional of the dual QFT therefore will not be conformally invariant unless the QFT metric is static (which requires $g_{(1)ij} = 0$). As in section 3.3.6, we find here another relation between metric staticity and conformal invariance. Scale invariance of the dual field theory is, however, preserved because the anomaly is a total derivative. Recall that the breaking of conformal symmetry follows from the requirement that the renormalized gravity action be finite in the zero Λ limit. The dimensionless and positive proportionality constant between the trace and the total derivative is in this case given by: $\ell_o^2/(8\pi G_0)$. Below we will still discuss the implications of the anomalous counterterm to the diffeomorphism Ward identity.

With the divergent term $(\alpha^2/4)\partial_i R[g_{(0)}]$ subtracted in this way, the terms proportional to α^2 that remain in the expressions for the vacuum expectation values represent derivatives of the traceless part of the coefficient $k_{(1)ij}$:[18]

$$\begin{cases} \alpha^2\left(\partial_u - \pounds_{\sigma_{(0)}}\right)\left(k_{(1)ij} - \tfrac{1}{2}\, g_{(0)ij}\mathrm{Tr}[g_{(0)}^{-1}k_{(1)}]\right)\ , \\ \alpha^{2\ (0)}\nabla^j\left(k_{(1)ij} - \tfrac{1}{2}\, g_{(0)ij}\mathrm{Tr}[g_{(0)}^{-1}k_{(1)}]\right)\ . \end{cases} \tag{3.123}$$

These terms cannot be subtracted by covariant counterterms, nor by counterterms of the form (3.110). This fact implies that the traceless part of $k_{(1)ij}$ needs to admit an expansion in α of the form:

$$k_{(1)ij} - \frac{1}{2}\, g_{(0)ij}\mathrm{Tr}[g_{(0)}^{-1}k_{(1)}] \quad = \quad \frac{1}{\alpha^2}\left(\kappa_{[0]ij} + \mathcal{O}(\alpha^{<0})\right)\ , \tag{3.124}$$

with $\kappa_{[0]ij}$ independent of α. In other words, in three boundary dimensions, only field theory states dual to bulk solutions that admit the behaviour (3.124) in α result in finite vacuum expectation values in the limit $\alpha \to \infty$. The expression for $\kappa_{[0]ij}$ is given by the vev of the QFT stress tensor in the zero Λ limit. As discussed at the end of section 3.2.2, in this limit the coefficient $k_{(1)ij}$ replaces the normalisable mode $g_{(3)ij}$ as the integration constant of the equations of motion for the case $d = 2$. Notice then that the coefficient $g_{(3)ij}$ drops out of equation (3.106) for the expectation value of the spatial stress s_{ij} in the limit $\alpha \to \infty$ and the latter is mapped to the Lie derivative of $\kappa_{[0]ij}$ along $n^a_{(0)}$ in this limit. In this way, $\kappa_{[0]ij}$ is part of the asymptotic bulk data that is mapped to boundary data in the zero Λ limit.

Finally, we will not compute here the diffeomorphism Ward identity for the general case in $d + 2 = 4$ dimensions because the constraint equations for the

[18]Note that: $\alpha^2 g_{(0)ij}\mathrm{Tr}\left[g_{(0)}^{-1}(\partial_u - \pounds_{\sigma_{(0)}})k_{(1)}\right] = \alpha^2\left(\partial_u - \pounds_{\sigma_{(0)}}\right)\left(g_{(0)ij}\mathrm{Tr}[g_{(0)}^{-1}k_{(1)}]\right) + \mathcal{O}(\alpha^0)$.

metric are very tedious to solve at second subleading order, but we will verify it explicitly for the Kerr solution discussed below. However, we would still like to emphasize that the terms in the holographic energy tensor that arise from the anomalous counterterm should not contribute to the spatial component of the Ward identity. Indeed, if we use the second identity in equation (3.89) and the expressions for the components of $\langle T_{ab}\rangle_{new}$ given in equations (3.112)–(3.114), we find:

$$ {}^{(0)}\mathcal{D}_a \left(\sqrt{q_{(0)}}\ \langle T^a_{\ i}\rangle_{new} \right) \ = \ {}^{(0)}\mathcal{D}_a \left(\sqrt{q_{(0)}}\ \langle T^a_{\ i}\rangle \right) \ . \tag{3.125} $$

This is the statement that the anomalous counterterm does not break invariance under boundary diffeomorphisms (3.46) that are foliation preserving.[19] On the other hand, if we compute the time component of the divergence of $\langle T_{ab}\rangle_{new}$ using the second identity in (3.90), we find in general that it is not equal to that of $\langle T_{ab}\rangle$. This must necessarily be the case because the anomalous counterterm is not invariant under those boundary diffeomorphisms in which the time coordinate transforms as $u \to \tilde{u}(u, x^i)$, and therefore break the spatial foliation of the boundary.

3.4.3. Kerr solution

As an application of the results of the previous section, we would like to compute the expectation value of the QFT energy tensor evaluated on those states dual to the asymptotically flat Schwarzschild and Kerr spacetimes. For the case of Schwarzschild-AdS$_4$, the metric in the coordinate system (4.41) reads:

$$ ds^2 \ = \ \frac{\ell_{\rm o}^2}{z^2}\left(-\left(\frac{1}{\alpha^2} + \frac{z^2}{\ell_{\rm o}^2} - \frac{2MG_0}{\ell_{\rm o}^4} z^3 \right) du^2 - 2du dz + \ell_{\rm o}^2\, d\Omega^2 \right) \ , \tag{3.126} $$

with $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ the metric on the S^2 and where the cosmological constant $\Lambda = -3/(\alpha^2\ell_{\rm o}^2)$. In the limit $\alpha \to \infty$ the metric tends to four dimensional Schwarzschild. The expectation values of the components of the holographic energy tensor in this case become:

$$ \sqrt{q_{(0)}}\ \langle s_{ij}\rangle \ = \ \sqrt{g_{(0)}} \left(\frac{M}{8\pi\ell_{\rm o}^2}\, g_{(0)ij} \right) \ , \tag{3.127} $$

$$ \sqrt{q_{(0)}}\ \langle \varepsilon \rangle \ = \ \sqrt{g_{(0)}} \left(\frac{M}{4\pi\ell_{\rm o}^2} \right) \ , \tag{3.128} $$

$$ \langle j_i \rangle \ = \ 0 \ , \tag{3.129} $$

[19] These are essentially spatial diffeomorphisms plus a possible redefinition of the time coordinate.

where the spatial metric $g_{(0)ij}dx^i dx^j = \ell_{\text{o}}^2\, d\Omega^2$. These expressions still hold in the zero Λ limit. The average energy $\langle E\rangle$ as defined in (3.96) is then equal to M.

In the case of Kerr spacetime, the metric in Gaussian null coordinates is very complicated,[20] but we can deduce its asymptotics up to the desired order from the following considerations. The Kerr metric follows from the zero Λ limit of Kerr-AdS$_4$ and the latter is asymptotically exactly AdS$_4$ – with the cross section of the asymptotic boundary with a spacelike hypersurface topologically an S^2. In our coordinate system, Kerr-AdS$_4$ must therefore be of the form:

$$ds^2 = \frac{\ell_{\text{o}}^2}{z^2}\left(-\left(\frac{1}{\alpha^2}+\mathcal{O}(z^2)\right)du^2 - 2du dz + \left(g_{(0)ij}+\mathcal{O}(z)\right)\left(dx^i+\mathcal{O}(z)du\right)\left(dx^j+\mathcal{O}(z)du\right)\right)\ , \tag{3.130}$$

where $g_{(0)ij}dx^i dx^j = \ell_{\text{o}}^2\, d\Omega^2$. Since the lapse $N_{(0)} = 1$, from equation (3.36) we have that $\sigma^i_{(1)} = 0$. Furthermore, since $\sigma^i_{(0)} = 0 = \partial_u g_{(0)ij}$, we find from equation (3.32) that $g_{(1)ij} = 0$. From equations (3.38) and (3.37) we then find that $g_{(2)ij} = 0 = \sigma^i_{(2)}$. Also, the spatial Ricci scalar $R[g_{(0)}] = 2/\ell_{\text{o}}^2$, so from (3.35) we have $\varphi_{(2)} = \ell_{\text{o}}^{-2}$. In this way, Kerr-AdS$_4$ must be asymptotically of the form:

$$\begin{aligned} ds^2 = \frac{\ell_{\text{o}}^2}{z^2}\Bigg[&-\left(\frac{1}{\alpha^2}+\frac{z^2}{\ell_{\text{o}}^2}+\varphi_{(3)}z^3+\mathcal{O}(z^{>3})\right)du^2 - 2du dz \\ &+\left(g_{(0)ij}+z^3 g_{(3)ij}+\mathcal{O}(z^{>3})\right)\left(dx^i+\left(z^3\sigma^i_{(3)}+\mathcal{O}(z^{>3})\right)du\right)\left(dx^j+\left(z^3\sigma^j_{(3)}+\mathcal{O}(z^{>3})\right)du\right)\Bigg] \end{aligned} \tag{3.131}$$

The coefficients $\varphi_{(3)}, g_{(3)ij}$ and $\sigma^i_{(3)}$ are the normalisable modes $G_{(d+1)\mu\nu}$ and from the constraint equations (3.253)–(3.255), supplemented by (3.256), it follows that they satisfy:

$$\mathrm{Tr}[g^{-1}_{(0)}g_{(3)}] = 0\ , \tag{3.132}$$

$$\frac{1}{\alpha^2}\,{}^{(0)}\nabla_j (g^{-1}_{(0)}g_{(3)})^j{}_i = \partial_u \sigma_{(3)i} + \frac{1}{3}\,\partial_i\varphi_{(3)}\ , \tag{3.133}$$

$$\frac{3}{2\alpha^2}\,{}^{(0)}\nabla_i \sigma^i_{(3)} = -\partial_u\varphi_{(3)}\ . \tag{3.134}$$

[20] See [182, 183] for specific examples. Note that Bondi-Sachs coordinates are related to the Gaussian null gauge by a simple redefinition of the radial coordinate.

The holographic energy tensor so far reads:

$$\sqrt{q_{(0)}}\ \langle s_{ij}\rangle = \frac{\ell_{\rm o}^2}{8\pi G_0}\sqrt{g_{(0)}}\left(\frac{3}{2\alpha^2}\, g_{(3)ij} - \frac{1}{2}\,\varphi_{(3)}\, g_{(0)ij}\right)\ , \tag{3.135}$$

$$\sqrt{q_{(0)}}\ \langle \varepsilon\rangle = \frac{\ell_{\rm o}^2}{8\pi G_0}\sqrt{g_{(0)}}\left(-\varphi_{(3)}\right)\ , \tag{3.136}$$

$$\langle j_i\rangle = \frac{\ell_{\rm o}^2}{8\pi G_0}\left(-\frac{3}{2}\,\sigma_{(3)i}\right)\ . \tag{3.137}$$

By using the second identity in equations (3.89) and (3.90) it then follows from the above constraints that the energy tensor is covariantly conserved:

$$^{(0)}\mathcal{D}_a\left(\sqrt{q_{(0)}}\,\langle T^a{}_i\rangle\right) = 0 = n^b_{(0)}\ {}^{(0)}\mathcal{D}_a\left(\sqrt{q_{(0)}}\,\langle T^a{}_b\rangle\right)\ . \tag{3.138}$$

Note that, apart from the constraints, the normalisable modes are so far arbitrary. We then require that the solution be stationary and axi-symmetric, which results in the constraints:

$$\mathrm{Tr}[g^{-1}_{(0)} g_{(3)}] = 0\ , \tag{3.139}$$

$$\frac{1}{\alpha^2}\ {}^{(0)}\nabla_j (g^{-1}_{(0)} g_{(3)})^j{}_i = \frac{1}{3}\,\partial_i\varphi_{(3)}\ , \tag{3.140}$$

$$\frac{1}{\alpha^2}\ {}^{(0)}\nabla_i \sigma^i_{(3)} = 0\ , \tag{3.141}$$

where the modes now depend only on the boundary coordinate θ. These are the necessary conditions for Kerr-AdS$_4$. In the zero Λ limit, however, there will be a further constraint. Recall that the equation for a given coefficient $g_{(n)ij}$ is of the form (3.45) and, therefore, that it becomes a differential equation for $g_{(n-1)ij}$ in the limit $\alpha\to\infty$. For the particular case of $n = 4$ in $d + 2 = 4$ bulk dimensions, the equation for $g_{(4)ij}$ turns into a differential equation for the normalisable mode $g_{(3)ij}$ in the zero Λ limit. Therefore, if we solve the dynamical equation (3.29), together with (3.256), at order z^2 we find in the limit $\alpha\to\infty$:

$$4\,k_{(3)ij} - g_{(0)ij}\,\mathrm{Tr}[g^{-1}_{(0)} k_{(3)}] + \varphi_{(4)}\, g_{(0)ij} + 3\ {}^{(0)}\nabla_{(i}\sigma^{(3)}_{j)} = 0\ , \tag{3.142}$$

where we have used the fact that $g_{(1)ij} = g_{(2)ij} = \sigma_{(1)i} = \sigma_{(2)i} = 0$ in our case. The equation for the coefficient $\varphi_{(4)}$ follows from the dynamical equation (3.251) and (3.256) for φ:

$$\varphi_{(4)} - 2\mathrm{Tr}[g^{-1}_{(0)} k_{(1)}] - \frac{3}{2}\ {}^{(0)}\nabla_i \sigma^i_{(3)} = 0\ . \tag{3.143}$$

Replacing in (3.142), we find:

$$4\,k_{(3)ij} + g_{(0)ij}\,\mathrm{Tr}[g^{-1}_{(0)} k_{(3)}] + 3\left({}^{(0)}\nabla_{(i}\sigma^{(3)}_{j)} + \frac{1}{2} g_{(0)ij}\ {}^{(0)}\nabla_i\sigma^i_{(3)}\right) = 0\ . \tag{3.144}$$

Now, in our case we have:

$$\begin{aligned} k_{(3)ij} &= \frac{1}{2N_{(0)}}\big((\partial_u - \pounds_{\sigma_{(0)}})g_{(3)ij} - \pounds_{\sigma_{(1)}}g_{(2)ij} - \pounds_{\sigma_{(2)}}g_{(1)ij} - \pounds_{\sigma_{(3)}}g_{(0)ij}\big) \\ &= -\,{}^{(0)}\nabla_{(i}\sigma^{(3)}_{j)} \ . \end{aligned} \tag{3.145}$$

Replacing in equation (3.144) results in the following constraint for $\sigma_{(3)i}$:

$$ {}^{(0)}\nabla_{(i}\sigma^{(3)}_{j)} - \frac{1}{2}\,g_{(0)ij}\,{}^{(0)}\nabla_k\sigma^k_{(3)} \;=\; 0 \ . \tag{3.146}$$

Now, the constraint (3.141) for $\sigma^i_{(3)}$ holds for all values of $\alpha \in \mathbb{R}$, so we extend this to the limit $\alpha \to \infty$ so that the metric is continuous in α. If this were not the case, then $\sigma^i_{(3)}$ would contain terms proportional to $\delta_{\Lambda,0}$ and therefore Kerr would not follow from the zero Λ limit of Kerr-AdS$_4$. The same argument applies to the ϕ-component of the constraint (3.140). The constraint equations for the normalisable modes in the limit $\alpha \to \infty$ therefore become:

$$\mathrm{Tr}[g^{-1}_{(0)}g_{(3)}] = 0 = {}^{(0)}\nabla_j(g^{-1}_{(0)}g_{(3)})^j{}_{i=\phi} \ , \tag{3.147}$$

$$\partial_i\varphi_{(3)} = 0 \ , \tag{3.148}$$

$${}^{(0)}\nabla_i\sigma_{(3)j} + {}^{(0)}\nabla_j\sigma_{(3)i} = 0 \ , \tag{3.149}$$

where the modes depend only on θ. The coefficient $\sigma^i_{(3)}$ is therefore a Killing vector of the spatial metric $g_{(0)ij}$ on the S^2 and hence we choose: $\sigma^i_{(3)}\partial_i := a/\ell_{\mathrm{o}}^4\,\partial_\phi$, with a some dimensionless constant. Furthermore, $\varphi_{(3)}$ is constant, so we define: $\varphi_{(3)} := -2MG_0/\ell_{\mathrm{o}}^4$. Note also that in the limit $a \to 0$ we must recover the Schwarzschild metric, so $g_{(3)ij}$ must be proportional to the parameter a. The average energy and angular momentum of those states dual to asymptotically flat Kerr are then given by:

$$\langle E\rangle \;=\; \frac{1}{2T}\int_{-T}^{T} du \int d^2x\sqrt{q_{(0)}}\,\langle\varepsilon\rangle \;=\; M \ , \tag{3.150}$$

$$\langle J^i\rangle\partial_i \;=\; \int d^2x\sqrt{g_{(0)}}\,\langle j^{\,i}\rangle\partial_i \;=\; -\frac{3}{4G_0}\,a\,\partial_\phi \ . \tag{3.151}$$

More generally, for an asymptotically Minkowski spacetime we have that $g_{(1)ij} = 0$, so the energy density will be of the form (3.136). The average energy will then be given by:

$$\langle E \rangle = -\frac{1}{2T}\frac{\ell_{\mathrm{o}}^2}{8\pi G_0}\int d^2x\sqrt{g_{(0)}}\int_{-T}^{T} du\,\varphi_{(3)}$$
$$= \frac{1}{2T}\int_{-T}^{T} du\, M(u)\ , \tag{3.152}$$

where $M(u)$ is the Bondi mass (see *e.g.* [168]).

3.5. Null boundaries and corner terms

So far we considered a single timelike boundary $\{z = \epsilon\}$ for the spacetime and neglected all possible corner integrals evaluated on the codimension two surfaces $\{z = \epsilon, u = \pm\infty\}$ that may arise in the gravitational action. If one also considers null boundaries $\{u = u_\pm\}$ in the spacetime, where these surfaces can be at infinity, the original bare action (3.4) is not the appropriate one in the sense that the variational problem is not well-defined, and a further surface term is needed. Furthermore, the renormalized gravity action in each dimension will require corner counterterms at $\{z = \epsilon, u = u_\pm\}$ that ensure that the action is finite once the regulator ϵ is removed. In order to derive the correct bare action in general, we start by performing an ADM decomposition of the spacetime metric with respect to timelike hypersurfaces of constant z as:

$$ds^2_{d+2} = G_{\mu\nu}dx^\mu dx^\nu$$
$$= \beta^2 dz^2 + q_{ab}\left(dx^a + \beta^a dz\right)\left(dx^b + \beta^b dz\right)\ . \tag{3.153}$$

The inverse and determinant of the metric are given by:

$$G^{\mu\nu} = \begin{pmatrix} \frac{1}{\beta^2} & -\frac{1}{\beta^2}\,\beta^a \\ -\frac{1}{\beta^2}\,\beta^a & q^{ab} + \frac{1}{\beta^2}\,\beta^a\beta^b \end{pmatrix}\ , \tag{3.154}$$

$$\sqrt{G} = \beta\sqrt{q}\ . \tag{3.155}$$

The unit normal m^μ to the surfaces of constant z is given by:

$$m_\mu = \beta\partial_\mu z\ , \tag{3.156}$$
$$m^\mu\partial_\mu = \frac{1}{\beta}\left(\partial_z - \beta^a\partial_a\right)\ , \tag{3.157}$$
$$m^\mu m^\nu G_{\mu\nu} = 1\ . \tag{3.158}$$

The metric q_{ab} represents the induced metric of the hypersurfaces of constant z and we can extend it to a tensor in the whole spacetime by defining: $q^{\mu\nu} := G^{\mu\nu} - m^\mu m^\nu$. Next we perform an ADM decomposition of q_{ab} with respect to surfaces of constant u. In each submanifold $\{z = constant\}$, we define these surfaces to be spacelike:

$$\begin{aligned} ds^2_{d+1} &= q_{ab} dx^a dx^b \\ &= -N^2 du^2 + \gamma_{ij}(dx^i + \sigma^i du)(dx^j + \sigma^j du) \ . \end{aligned} \tag{3.159}$$

The determinant of this metric is given by: $\sqrt{q} = N\sqrt{\gamma}$, so that: $\sqrt{G} = \beta N \sqrt{\gamma}$. In each submanifold $\{z = constant\}$, the future-directed unit normal n^a to the surfaces of constant u is given by:

$$n_a = -N\partial_a u \ , \tag{3.160}$$

$$n^a \partial_a = \frac{1}{N}\left(\partial_u - \sigma^i \partial_i\right) \ , \tag{3.161}$$

$$n^a n^b q_{ab} = -1 \ . \tag{3.162}$$

We can extend this unit normal to a vector in the whole spacetime by defining:

$$n^\mu := q^{\mu\nu}\left(-N\partial_\nu u\right) \ . \tag{3.163}$$

We then find: $n^\mu n^\nu G_{\mu\nu} = -1$ and: $m^\mu n^\nu G_{\mu\nu} = 0$. Finally, with the two unit normals m^μ and n^μ we construct two null vectors $n^\mu_\pm$ defined as:

$$n^\mu_\pm := n^\mu \pm m^\mu \ . \tag{3.164}$$

We find that: $n^\mu_\pm n^\nu_\pm G_{\mu\nu} = 0$ and: $n^\mu_\pm m^\nu G_{\mu\nu} = \pm 1$. Given this general construction, we will now show that, if the surfaces $\{u = u_\pm\}$ are null in the spacetime, the bare gravitational action for which the variational problem is well-posed is given by:

$$\begin{aligned} 16\pi G_0\, S &= \int dz du d^d x \sqrt{G}\left(\frac{d(d+1)}{\alpha^2 \ell_0^2} + R[G]\right) \\ &+ 2\int\limits_{z=\epsilon} du d^d x \sqrt{q}\, Q - 2\int\limits_{u=u_-}^{u=u_+} dz d^d x\, \beta\sqrt{\gamma}\ \nabla_\mu n^\mu_+ \ , \end{aligned} \tag{3.165}$$

with $\nabla_\mu G_{\nu\alpha} := 0$, and where Q is the extrinsic curvature of the hypersurfaces of constant z as before, such that: $Q = \nabla_\mu m^\mu$. Also, the last integral represents the difference: $\int\limits_{u=u_-}^{u=u_+} := \int\limits_{u=u_+} - \int\limits_{u=u_-}$. In order to show that the variational problem is well-defined, we perform a Gauss-Codazzi decomposition of the Ricci scalar $R[G]$:

$$R[G] = R[q] + Q^2 - Q\cdot Q - 2\nabla_\mu\left(m^\mu\, \nabla\cdot m - m\cdot\nabla m^\mu\right) \ . \tag{3.166}$$

Replacing in (3.165) and integrating the total derivatives results in the action:

$$\begin{aligned} 16\pi G_0\, S &= \int dz du d^d x\, \beta\sqrt{q}\left(\frac{d(d+1)}{\alpha^2 \ell_o^2} + R[q] + Q^2 - Q\cdot Q\right) \\ &- 2\int_{u=u_-}^{u=u_+} dz d^d x\, \beta\sqrt{\gamma}\left(K + \left(1 + N m^\mu \partial_\mu u\right)\nabla\cdot m\right) , \end{aligned} \quad (3.167)$$

where $K = q^{ab}D_a n_b = q^{\mu\nu}\nabla_\mu\left(q_\nu{}^\alpha n_\alpha\right) = q^{\mu\nu}\nabla_\mu n_\nu$ is the extrinsic curvature of the surfaces of constant u in each submanifold $\{z = \mathit{constant}\}$, with $D_a q_{bc} := 0$. Now, from the decomposition (3.154) we find in particular that:

$$\partial_\mu u \partial_\nu u\, G^{\mu\nu} = q^{uu} + (\beta^u/\beta)^2 = -N^{-2} + (\beta^u/\beta)^2 \ . \quad (3.168)$$

If the surfaces $u = u_\pm$ are null in the spacetime, the left-hand side vanishes at $u = u_\pm$ and we find up to a sign: $\beta^u = \beta/N$ at $u = u_\pm$. If we choose the opposite sign, then we should replace the null vector n_+ in (3.165) by its dual n_-. Replacing this condition for β^u in equation (3.157) results in:

$$1 + N\, m^\mu \partial_\mu u = 0 \qquad (u = u_\pm)\ . \quad (3.169)$$

Note that this holds everywhere if the surfaces of constant u are everywhere null, and in such case the null vector n_+ is given by: $n_{+\mu} = -N\partial_\mu u$. Finally, using equation (3.169) in the action (3.167) yields our final result:[21]

$$16\pi G_0\, S = \int dz \left(\int du d^d x\, \beta\sqrt{q}\left(\frac{d(d+1)}{\alpha^2\ell_o^2} + R[q] + Q^2 - Q\cdot Q\right) - 2\int_{u=u_-}^{u=u_+} d^d x\, \beta\sqrt{\gamma}\, K\right) \ . \quad (3.170)$$

This is the correct action for which the variational problem is well-posed [179]. Taking variations with respect to the lapse, shift, and induced metric β, β^a and q_{ab}, and requiring only that the boundary configurations of the fields are fixed, results in the ADM equations of motion.

If the spacetime contains null boundaries, the holographic renormalization of the gravitational action (3.165) will result in corner counterterms as emphasized above. We will exemplify this for the particular case of $d + 2 = 3$ dimensions and derive the anomalous counterterm (3.54) discussed in section 3.3.3. Returning to our gauge-fixed metric (4.41) for generic d, if we evaluate on-shell the last integral

[21] Note that the Gibbons-Hawking surface term takes a minus sign because we have defined the unit normal n^a to be future-directed.

in the action (3.165), we obtain:

$$-2\int\limits_{u=u_-}^{u=u_+} dzd^dx\,\beta\sqrt{\gamma}\;\nabla_\mu n_+^\mu = -2\int\limits_{\substack{u=u_-\\z=\epsilon}}^{u=u_+} d^dx\,\sqrt{g}\left(\frac{\ell_{\rm o}}{\epsilon}\right)^d$$
$$+\,2\ell_{\rm o}^d\int\limits_{u=u_-}^{u=u_+} dzd^dx\,\sqrt{g}\left(z^{-(d+1)}-\frac{1}{2}\,z^{-d}\partial_z\log\varphi\right)\;. \tag{3.171}$$

Using our asymptotic solutions (3.29)–(3.31) we find that, for $d=1$, the divergences of this term are given by:

$$-2\int\limits_{u=u_-}^{u=u_+} dzdx\,\beta\sqrt{\gamma}\;\nabla_\mu n_+^\mu = -4\int\limits_{\substack{u=u_-\\z=\epsilon}}^{u=u_+} dx\,\sqrt{g_{(0)}}\left(\frac{\ell_{\rm o}}{\epsilon}\right)+2\alpha^2\ell_{\rm o}\int\limits_{\substack{u=u_-\\z=\epsilon}}^{u=u_+} dx\,\sqrt{g_{(0)}}\,k_{(0)}\log\epsilon\;+\mathcal{O}(\epsilon^0)\;. \tag{3.172}$$

The counterterm that subtracts these divergences is given by:

$$4\int\limits_{\substack{u=u_-\\z=\epsilon}}^{u=u_+} dx\,\sqrt{\gamma}-2\,\alpha\ell_{\rm o}\int\limits_{\substack{u=u_-\\z=\epsilon}}^{u=u_+} dx\,\sqrt{\gamma}\,K\log\epsilon\;. \tag{3.173}$$

If we also take into account the surface term (3.50) that we discarded and use the result we found in (3.51), we find that the renormalized gravitational action in $d+2=3$ spacetime dimensions in the presence of null boundaries $u=u_\pm$ is given by:

$$16\pi G_0\,S_{ren} = \int dzdudx\,\sqrt{G}\left(\frac{d(d+1)}{\alpha^2\ell_{\rm o}^2}+R[G]\right)+2\int\limits_{z=\epsilon} dudx\sqrt{q}\,Q-2\int\limits_{u=u_-}^{u=u_+} dzdx\,\beta\sqrt{\gamma}\;\nabla_\mu n_+^\mu$$
$$+\,\frac{2}{\alpha\,\ell_{\rm o}}\int\limits_{z=\epsilon} dudx\,\sqrt{q}+6\int\limits_{\substack{u=u_-\\z=\epsilon}}^{u=u_+} dx\,\sqrt{\gamma}-2\,\alpha\ell_{\rm o}\int\limits_{\substack{u=u_-\\z=\epsilon}}^{u=u_+} dx\,\sqrt{\gamma}\,K\log\epsilon\;. \tag{3.174}$$

The last corner integral is exactly the anomalous counterterm that we found in (3.54).

3.6. Non-backreacting matter

In the remainder of this chapter we will be interested in computing the zero Λ limit of the vacuum expectation value and two-point correlator of a QFT operator dual to a non-backreacting massive scalar field in AdS_{d+2}. The background metric we are interested in is pure AdS with the cross section of the asymptotic boundary with a spacelike hypersurface topologically $\mathbb{R}^d$. In our coordinate system the metric reads:

$$\begin{aligned} ds^2 &= G_{\mu\nu} dx^\mu dx^\nu \\ &= \frac{\ell_\mathrm{o}^2}{z^2}\left(-\frac{1}{\alpha^2}\, du^2 - 2du dz + d\vec{x}_d^{\,2}\right) . \end{aligned} \tag{3.175}$$

In the limit $\alpha \to \infty$ the spacetime is a subset of Minkowski space, with $z = 0$ representing future null infinity. The bulk action for the scalar field ϕ in this background is given by:

$$S = \frac{1}{2}\int d^{d+2}x\sqrt{G}\left(G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \left(\frac{m}{\alpha}\right)^2\phi^2\right) . \tag{3.176}$$

The mass of the field is defined to be $M = m/\alpha$ so that it becomes massless in the zero Λ limit. This is a necessary condition associated with the fact that the asymptotic boundary is null in this limit and with the statement that only massless particles reach null infinity in Minkowski space.[22] For the moment we will keep the mass parameter m arbitrary, but we will see below that the conformal weight of the dual operator will be finite in the limit $\alpha \to \infty$ only if $m = \mathcal{O}(\alpha^0)$.

3.6.1. Solution and asymptotics

The equations of motion for the scalar in our background are given by:

$$\begin{aligned} \left(\frac{m}{\alpha}\right)^2\phi &= \Box_G\,\phi \\ &= \frac{z^{\Delta-k+2}}{\ell_\mathrm{o}^2}\left[\frac{1}{\alpha^2}\left(\varphi'' - \frac{k-1}{z}\varphi'\right) - 2\partial_u\varphi' + \frac{k-1}{z}\partial_u\varphi + \vec{\nabla}^2\varphi + \frac{\Delta(\Delta-(d+1))}{\alpha^2 z^2}\,\varphi\right] , \end{aligned} \tag{3.177}$$

where we defined $\varphi := z^{k-\Delta}\phi$ for some constant Δ, and $k := 2\Delta - (d+1)$. Also, $\varphi' := \partial_z\varphi$ and $\vec{\nabla}^2 = \delta^{ij}\partial_i\partial_j$. We will be interested in computing the correlation functions of the QFT operator in Euclidean signature, so we define the Euclidean

[22] Strictly speaking, massive scalar fields have an essential singularity at null infinity.

boundary time $\bar{u} := iu$. The dynamical equation then becomes:

$$\frac{1}{\alpha^2}\left(\varphi'' - \frac{k-1}{z}\varphi'\right) - 2i\dot{\varphi}' + i\frac{k-1}{z}\dot{\varphi} + \vec{\nabla}^2\varphi + \frac{\Delta(\Delta-(d+1)) - \ell_{\rm o}^2 m^2}{\alpha^2 z^2}\varphi = 0 \;, \tag{3.178}$$

where $\dot{\varphi} := \partial_{\bar{u}}\varphi$. We then define Δ as the highest root of the equation:

$$\Delta(\Delta-(d+1)) = \ell_{\rm o}^2 m^2 \;.$$

We also Fourier transform the dynamical equation in the coordinates $\bar{u}$ and x^i and obtain:

$$\frac{1}{\alpha^2}\left(\hat{\varphi}'' - \frac{k-1}{z}\hat{\varphi}'\right) + 2\omega\hat{\varphi}' - \omega\,\frac{k-1}{z}\hat{\varphi} - \vec{p}^{\,2}\hat{\varphi} \;\; = \;\; 0 \;, \tag{3.179}$$

where:

$$\hat{\varphi}(z,\omega,p^i) = \int d\bar{u}\, d^dx\, e^{-i\omega\bar{u}} e^{-i\vec{p}\cdot\vec{x}}\, \varphi(z,\bar{u},x^i) \;. \tag{3.180}$$

The solution for $\hat{\varphi}$ can be written in terms of Bessel functions as:

$$\hat{\varphi}(z,\omega,p) = \; e^{-\alpha^2\omega z}\, z^{k/2}\left[A(\omega,p)\, K_{k/2}(z\,\alpha\sqrt{\vec{p}^{\,2}+\alpha^2\omega^2}) + B(\omega,p)\, I_{k/2}(z\,\alpha\sqrt{\vec{p}^{\,2}+\alpha^2\omega^2})\right] \;, \tag{3.181}$$

where the coefficients $A(\omega,p)$ and $B(\omega,p)$ are arbitrary, and where $K_{k/2}(y)$ and $I_{k/2}(y)$ are the modified Bessel functions of the first and second kind. These admit the following asymptotics as $y \to 0$:

$$K_{k/2}(y) = 2^{k/2-1}\,\Gamma(k/2)\, y^{-k/2}\Bigg(1 + \frac{(iy)^2}{2(k-2)} + \frac{(iy)^4}{2(k-2)4(k-4)} + \ldots + a_k\, y^k + \tilde{a}_k\, y^k \log y^2 + \mathcal{O}(y^{>k})\Bigg) \;, \tag{3.182}$$

$$I_{k/2}(y) = \frac{2^{-k/2}}{\Gamma(k/2+1)}\, y^{-k/2}\left(y^k + \mathcal{O}(y^{>k})\right) \;, \tag{3.183}$$

with $\Gamma(a)$ the gamma function and a_k a k-dependent constant. The coefficient $\tilde{a}_k$ is non-vanishing only if $k/2$ is an integer and in such case is given by:

$$\tilde{a}_k \;\; = \;\; -\frac{(-1)^{k/2}\; 2^{-k}}{\Gamma(1+k/2)\Gamma(k/2)} \qquad : \quad k/2 \in \mathbb{N} \;. \tag{3.184}$$

The solution for $\hat{\varphi}$ therefore admits the expansion:

$$\begin{aligned}\hat{\varphi}(z,\omega,p) = e^{-\alpha^2\omega z}\Bigg[&\left(1-\frac{\alpha^2(\vec{p}^{\,2}+\alpha^2\omega^2)}{2(k-2)}\,z^2+\frac{\alpha^4(\vec{p}^{\,2}+\alpha^2\omega^2)^2}{8(k-2)(k-4)}\,z^4+\ldots\right)\hat{\varphi}_{(0)}(\omega,p)\\ &+\, b(\omega,p)\,z^k+\tilde{\hat{\varphi}}_{(k)}(\omega,p)\,z^k\log z+\mathcal{O}(z^{>k})\Bigg]\\ = \hat{\varphi}_{(0)}+z\,\hat{\varphi}_{(1)}&+z^2\,\hat{\varphi}_{(2)}+z^3\,\hat{\varphi}_{(3)}+\ldots+z^k\,\hat{\varphi}_{(k)}+z^k\log z\,\tilde{\hat{\varphi}}_{(k)}+\mathcal{O}(z^{>k})\ ,\end{aligned} \tag{3.185}$$

where we wrote the function $A(\omega,p)$ as:

$$A(\omega,p) = \frac{2^{1-k/2}}{\Gamma(k/2)}\left(\alpha\sqrt{\vec{p}^{\,2}+\alpha^2\omega^2}\right)^{k/2}\hat{\varphi}_{(0)}(\omega,p)\ . \tag{3.186}$$

The coefficients $\hat{\varphi}_{(0)}(\omega,p)$ and $\hat{\varphi}_{(k)}(\omega,p)$ are arbitrary functions in ω and $\vec{p}^{\,2}$ and the coefficients $\hat{\varphi}_{(n<k)}$ are given up to $n=3$ by:

$$\hat{\varphi}_{(1)} = -\alpha^2\omega\,\hat{\varphi}_{(0)}\ , \tag{3.187}$$

$$\hat{\varphi}_{(2)} = \left(\frac{1}{2}\,\alpha^4\omega^2-\frac{\alpha^2(\vec{p}^{\,2}+\alpha^2\omega^2)}{2(k-2)}\right)\hat{\varphi}_{(0)}\ , \tag{3.188}$$

$$\hat{\varphi}_{(3)} = \left(-\frac{1}{6}\,\alpha^6\omega^3+\frac{\alpha^4\omega(\vec{p}^{\,2}+\alpha^2\omega^2)}{2(k-2)}\right)\hat{\varphi}_{(0)}\ . \tag{3.189}$$

The coefficient $\tilde{\hat{\varphi}}_{(k)}$ of the inhomogeneous term is given by:

$$\tilde{\hat{\varphi}}_{(k)} = 2\,\tilde{a}_k\left(\alpha\sqrt{\vec{p}^{\,2}+\alpha^2\omega^2}\right)^{k}\hat{\varphi}_{(0)}\ . \tag{3.190}$$

The full solution $\phi(z,\bar{u},x^i)$ for the scalar field is then given by:

$$\begin{aligned}\phi(z,\bar{u},x) &= z^{\Delta-k}\int d\omega d^dp\,e^{i\omega\bar{u}}\,e^{i\vec{p}\cdot\vec{x}}\,\hat{\varphi}(z,\omega,p)\\ &= z^{\Delta-k}\Big(\varphi_{(0)}+z\,\varphi_{(1)}+z^2\,\varphi_{(2)}+z^3\,\varphi_{(3)}+\ldots+z^k\,\varphi_{(k)}+z^k\log(\mu z)\,\tilde{\varphi}_{(k)}+\mathcal{O}(z^{>k})\Big)\ ,\end{aligned} \tag{3.191}$$

where we introduced a scale μ of dimension L^{-1} so that the argument of the logarithm is dimensionless. The coefficients $\varphi_{(0)}=\varphi_{(0)}(\bar{u},x)$ and $\varphi_{(k)}=\varphi_{(k)}(\bar{u},x)$ are arbitrary functions and represent the standard non-normalisable and normalisable modes in the AdS/CFT correspondence. The boundary configuration $\varphi_{(0)}$ is the source for the scalar operator $\mathcal{O}$ in the dual QFT and $\varphi_{(k)}$ will be mapped to the vacuum expectation value of $\mathcal{O}$. The coefficients $\varphi_{(n<k)}$ together with the inhomogeneous term $\tilde{\varphi}_{(k)}$ are local functionals of the source for the case of α finite.

Their expressions are given by:

$$\frac{1}{\alpha^2}\,\varphi_{(n)} = \frac{1}{n(k-n)}\Big(i(k+1-2n)\,\dot{\varphi}_{(n-1)} + \vec{\nabla}^2\varphi_{(n-2)}\Big) \quad : \ 0 < n < k \ , \tag{3.192}$$

$$\frac{1}{\alpha^2}\,\tilde{\varphi}_{(k)} = \begin{cases} \frac{1}{k}\Big(i(k-1)\dot{\varphi}_{(k-1)} - \vec{\nabla}^2\varphi_{(k-2)}\Big) & : \ k/2 \in \mathbb{N} \ , \\ 0 \quad \text{otherwise} \ , \end{cases} \tag{3.193}$$

where $\varphi_{(-1)} := 0$. The above is exactly the asymptotic solution one would obtain by solving the dynamical equation (3.178) in powers of z in a neighbourhood of $z = 0$. In the case $\alpha^{-1} = 0$, the coefficients are non-local functionals of the sources in the same fashion as the coefficients $g_{(n)ij}$ in the asymptotic expansion (3.29) of the metric that we found in section 3.2.2. For the case of α finite, the source and the mode $\varphi_{(0)}$ and $\varphi_{(k)}$ are arbitrary, so there will be solutions for the scalar field in AdS that diverge in the limit $\alpha \to \infty$. We are interested in those configurations for the field that result in well-defined solutions of the equations of motion in Minkowski space in this limit, so we henceforth restrict our space of solutions in AdS to the subspace of those that admit the limit. This discussion mimics that in section 3.2.2 for the spacetime metric. This is enforced by requiring that the coefficients in the asymptotics (3.191) be non-divergent as $\alpha \to \infty$. Since the modes $\varphi_{(n<k)}$ and $\tilde{\varphi}_{(k)}$ are functionals of $\varphi_{(0)}$, this requirement imposes constraints on the behaviour in α of the derivatives of the source. For k non-odd, these will be constraints on the time derivatives. As an example, from $n = 1, 2, 3$ it follows that:

$$\begin{aligned} \dot{\varphi}_{(0)} &= \mathcal{O}(\alpha^{-2}) \ , && (3.194) \\ \ddot{\varphi}_{(0)} &= \frac{1}{\alpha^2}\left(\frac{1}{k-3}\,\vec{\nabla}^2\varphi_{(0)}\right) + \mathcal{O}(\alpha^{-4}) \ , && (3.195) \\ \dddot{\varphi}_{(0)} &= \frac{1}{\alpha^4}\left(\frac{3}{k-5}\,\vec{\nabla}^2\big(\alpha^2\dot{\varphi}_{(0)}\big)\right) + \mathcal{O}(\alpha^{-6}) \ . && (3.196) \end{aligned}$$

On the other hand, for odd values of k there will be a further constraint, this time on the spatial derivatives: $\vec{\nabla}^{k-1}\varphi_{(0)} = \mathcal{O}(\alpha^{-(k-1)})$. As in section 3.2.2, we find again that the well-definedness of the bulk solutions in the zero Λ limit translates into a statement about the sources and states on the dual QFT and, in particular, that the existence of the limit is connected with the behaviour in α of the time and spatial derivatives of the source.

3.6.2. Renormalization and vacuum expectation values

In this section we will renormalize holographically the bulk action for the scalar field in the AdS background (3.175), analyse the limit $\alpha \to \infty$ and compute the vev of the dual operator. Under this limit the spacetime becomes Minkowski space and the solution in AdS is mapped to a solution of the scalar field equations in Minkowski. As in section 3.3.1, we proceed by replacing the asymptotic boundary of the spacetime by a regulating surface $z = \epsilon$ and evaluate (3.176) on-shell:

$$iS^{on-shell} = \frac{\ell_{\rm o}^d}{2\alpha^2} \int\limits_{z=\epsilon} d\bar{u} d^d x \, \epsilon^{-k} \Big((\Delta - k)\, \varphi^2 + \epsilon \, \varphi \, \varphi' \Big) - \frac{\ell_{\rm o}^d}{4} \int\limits_{z=\epsilon} d\bar{u} d^d x \, \epsilon^{-k+1} \partial_{\bar{u}} \varphi^2 . \tag{3.197}$$

The integrand in the last integral is a total derivative and therefore can be removed from the on-shell action in the absence of null boundaries $\{u = constant\}$ for the spacetime. We then use the asymptotic solution (3.191) to replace for φ and find those terms that diverge if we take the limit $\epsilon \to 0$. For finite α these will be local functionals of the source $\varphi_{(0)}$ and therefore, up to anomalies, can be rewritten covariantly as described in section 3.3.1. The resulting divergent terms can then be subtracted by a covariant counterterm action S_{ct} consisting of minus such terms. The renormalized action S_{ren} will then be given by $S_{ren} = S + S_{ct}$. The number of counterterms increases with k, so we will focus separately on the cases $k = 2$ and $k = 4$.

k=2

In this case the procedure described above results in the following renormalized action:

$$\begin{aligned} iS_{ren} &= \frac{1}{2} \int d^{d+2}x \sqrt{G} \left(G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \left(\frac{m}{\alpha} \right)^2 \phi^2 \right) \\ &+ \frac{1}{2} \int\limits_{z=\epsilon} d^{d+1}x \, \sqrt{q} \left(- \frac{\Delta - k}{\alpha \ell_{\rm o}} \, \phi^2 + (\alpha \ell_{\rm o}) \, \phi \Box_q \phi \, \log \epsilon \right) , \end{aligned} \tag{3.198}$$

where q_{ab} is the induced metric on the regulating surface:

$$\begin{aligned} q_{ab} dx^a dx^b &= \frac{\ell_{\rm o}^2}{\epsilon^2} \left(\frac{1}{\alpha^2} \, d\bar{u}^2 + d\vec{x}_d^2 \right) \\ &= \frac{\ell_{\rm o}^2}{\epsilon^2} \, q_{(0)ab} dx^a dx^b \, , \end{aligned} \tag{3.199}$$

and where $\Box_q$ is the Laplacian with respect to q_{ab} and $q_{(0)ab}$ is the QFT metric. The resulting couterterms are precisely the canonical ones from standard holographic renormalization in the AdS/CFT correspondence (see *e.g.* [78]). This is expected because the canonical counterterm action is covariant up to the anomaly in $\log \epsilon$. The latter breaks invariance of the action under specific bulk diffeomorphisms involving the radial coordinate z, but our background (3.175) is mapped to the Poincaré patch of AdS by the boundary diffeomorphism $u \to \alpha^2(u-z)$, $x^i \to \alpha\, x^i$. The surfaces of constant z are therefore preserved by the diffeomorphism and hence the canonical counterterm action is not affected by the transformation.

The next step is to determine whether the counterterms spoil the zero Λ limit of the renormalized on-shell action. For that purpose we evaluate S_{ren} on-shell, take the limit $\epsilon \to 0$ and look for those terms proportional to positive powers of α as described in section 3.3.1. In the simple case of $k=2$ no such terms survive once the regulator is removed and therefore the couterterm action does not spoil the zero Λ limit. As we increase the value of k we will see that further counterterms are needed apart from the canonical ones in order to restore the well-behaved-ness of the action in the limit $\alpha \to \infty$.

Vacuum expectation value

The variation of the renormalized on-shell action is given by:

$$i\,\delta S_{ren}^{on-shell} = \int\limits_{z=\epsilon} d^{d+1}x\,\sqrt{q}\left(\frac{\epsilon}{\alpha\ell_{\mathrm{o}}}\left(\phi' - i\alpha^2\dot{\phi}\right) - \frac{\Delta-k}{\alpha\ell_{\mathrm{o}}}\phi + (\alpha\ell_{\mathrm{o}})\,\Box_q\phi\,\log\epsilon\right)\delta\phi\ . \tag{3.200}$$

Using the AdS/CFT prescription, the one-point function of the dual operator $\mathcal{O}$ is then given by:[23]

$$\begin{aligned}\sqrt{q_{(0)}}\,\langle\mathcal{O}\rangle &= \frac{i\delta S_{ren}^{on-shell}}{\delta\varphi_{(0)}} = \lim_{\epsilon\to 0}\left(\epsilon^{\Delta-k}\,\frac{i\delta S_{ren}^{on-shell}}{\delta\phi}\right)\\ &= \frac{\ell_{\mathrm{o}}^d}{\alpha^2}\left(2\varphi_{(2)} - \tilde{\varphi}_{(2)}\right) - \ell_{\mathrm{o}}^d\,\vec{\nabla}^2\varphi_{(0)}\ .\end{aligned} \tag{3.201}$$

We therefore find that the vev is mapped to the normalisable mode $\varphi_{(2)}$ for finite α as expected. The term proportional to $\tilde{\varphi}_{(2)}$ is unphysical in the sense that it can

[23] Recall from section 3.3.4 that the well-defined observables are always the tensor densities, in this case $\sqrt{q_{(0)}}\,\langle\mathcal{O}\rangle$. By construction, the n-point functions themselves are divergent in the zero Λ limit because the boundary lapse vanishes in this limit. In particular for the 1-point function: $(1/\sqrt{q_{(0)}})\ i\delta S_{ren}^{on-shell}/\delta\varphi_{(0)} = \alpha\left(1/(N_{(0)}\sqrt{g_{(0)}})\ i\delta S_{ren}^{on-shell}/\delta\varphi_{(0)}\right)$, which diverges as $\alpha \to \infty$, where in this case $N_{(0)} = 1$ and $g_{(0)ij} = \delta_{ij}$.

be subtracted from the expectation value by adding to the renormalized action the finite covariant counterterm (finite both in ϵ and α):

$$-\frac{\alpha\ell_{\rm o}}{4}\int\limits_{z=\epsilon} d^{d+1}x\,\sqrt{q}\,\phi\Box_q\phi\ . \tag{3.202}$$

The variation of this term is then proportional to $\tilde{\varphi}_{(2)}$:

$$\begin{aligned}\frac{i\delta}{\delta\varphi_{(0)}}\left(-\frac{\alpha\ell_{\rm o}}{4}\int\limits_{z=\epsilon} d^{d+1}x\,\sqrt{q}\,\phi\Box_q\phi\right) &= \lim_{\epsilon\to 0}\left(-\frac{\alpha\ell_{\rm o}}{4}\,\epsilon^{\Delta-k}\frac{i\delta}{\delta\phi}\int\limits_{z=\epsilon} d^{d+1}x\,\sqrt{q}\,\phi\Box_q\phi\right)\\ &= \frac{\ell_{\rm o}^d}{\alpha^2}\,\tilde{\varphi}_{(2)}\ .\end{aligned} \tag{3.203}$$

The term proportional to the spatial Laplacian of the source cannot be subtracted without partially breaking diffeomorphism invariance of the bulk action. The finite counterterm that subtracts this term is given by:

$$-\frac{\alpha\ell_{\rm o}}{4}\int\limits_{z=\epsilon} d^{d+1}x\,\sqrt{q}\,\phi\vec{\nabla}^2_\gamma\phi\ , \tag{3.204}$$

where $\vec{\nabla}^2_\gamma$ is the Laplacian with respect to the spatial metric $\gamma_{ij}dx^i dx^j = \ell_{\rm o}^2/\epsilon^2\, d\vec{x}_d^2$ and, therefore, breaks invariance under diffeomorphisms that are not foliation preserving. This is the same type of anomalous counterterm that we found in section 3.4. However, there is no need for a counterterm of this type in the present case. It may seem that the spatial Laplacian of the source in the vev (3.201) will give rise to contact terms proportional to the spatial Laplacian of delta functions and, therefore, that partially break diffeomorphism invariance of the two-point correlator computed by taking the variation of the vev. However, this will not be the case because the variation of the normalisable mode $\varphi_{(2)}$ will provide a contribution that precisely cancels these so that the two-point function is completely covariant for finite α. We will see that this is indeed the case in section 3.6.3.

Finally, note that the vev admits a well-behaved zero Λ limit. If we switch off the source and take the limit $\alpha\to\infty$, the expectation value of the operator vanishes identically. In other words, any scalar operator of conformal dimension $\Delta = 1 + (d+1)/2$ evaluated on QFT states dual to gravity solutions with $\Lambda = 0$ necessarily has a vanishing expectation value in the absence of the source.

k=4

In this case the renormalized action is given by:

$$\begin{aligned} iS_{ren} = {} & \frac{1}{2}\int d^{d+2}x\sqrt{G}\left(G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \left(\frac{m}{\alpha}\right)^2\phi^2\right) \\ & + \frac{1}{2}\int\limits_{z=\epsilon} d^{d+1}x\,\sqrt{q}\left(-\frac{\Delta-k}{\alpha\ell_{\rm o}}\,\phi^2 - \frac{\alpha\ell_{\rm o}}{k-2}\,\phi\Box_q\phi + \frac{(\alpha\ell_{\rm o})^3}{4}\,\phi(\Box_q)^2\phi\,\log\epsilon\right)\,, \end{aligned} \tag{3.205}$$

where the counterterm action again coincides with the canonical one. Let us now verify whether the counterterms spoil the zero Λ limit of the action. If we evaluate S_{ren} on-shell, take the limit as the regulator $\epsilon \to 0$ and look for those terms proportional to positive powers of α, we find:

$$\lim_{\epsilon\to 0}\, iS^{on-shell}_{ren} = -\frac{\ell_{\rm o}^d}{4}\int\limits_{z=0} d\bar{u}\,d^dx\left(\alpha^2\varphi_{(0)}\,\ddot{\varphi}_{(2)} + \alpha^2\varphi_{(1)}\,\ddot{\varphi}_{(1)} + \alpha^2\varphi_{(2)}\,\ddot{\varphi}_{(0)}\right) + \mathcal{O}(\alpha^0)\,. \tag{3.206}$$

The second and third terms are of order $\mathcal{O}(\alpha^0)$. This is so because from equation (3.192) for $n=1,2$ we have:

$$\dot{\varphi}_{(0)} = \mathcal{O}(\alpha^{-2}) \;\Rightarrow\; \varphi_{(2)}\,\ddot{\varphi}_{(0)} = \mathcal{O}(\alpha^{-2})\,, \tag{3.207}$$

$$\dot{\varphi}_{(1)} = -i\left(\frac{4}{\alpha^2}\,\varphi_{(2)} - \vec{\nabla}^2\varphi_{(0)}\right) \Rightarrow \ddot{\varphi}_{(1)} = -i\left(\frac{4}{\alpha^2}\,\dot{\varphi}_{(2)} + \frac{i}{\alpha^2}\,\vec{\nabla}^2\varphi_{(1)}\right) \Rightarrow \varphi_{(1)}\,\ddot{\varphi}_{(1)} = \mathcal{O}(\alpha^{-2})\,. \tag{3.208}$$

On the other hand, the first term is of order α^2. If we use again equation (3.192) but for $n=3$, we find:

$$\begin{aligned} & \dot{\varphi}_{(2)} = i\left(\frac{3}{\alpha^2}\,\varphi_{(3)} - \vec{\nabla}^2\varphi_{(1)}\right) \;\Rightarrow\; \ddot{\varphi}_{(2)} = i\left(\frac{3}{\alpha^2}\,\dot{\varphi}_{(3)} + i\left(\frac{4}{\alpha^2}\,\vec{\nabla}^2\varphi_{(2)} - \vec{\nabla}^4\varphi_{(0)}\right)\right) \\ & \Rightarrow\; \varphi_{(0)}\,\ddot{\varphi}_{(2)} = \vec{\nabla}^4\varphi_{(0)} + \mathcal{O}(\alpha^{-2})\,. \end{aligned} \tag{3.209}$$

In this way we find that the zero Λ limit of the action is spoiled by the counterterm action:

$$\lim_{\epsilon\to 0}\, iS^{on-shell}_{ren} = -\alpha^2\,\frac{\ell_{\rm o}^d}{4}\int\limits_{z=0} d\bar{u}\,d^dx\,\varphi_{(0)}(\vec{\nabla}^2)^2\varphi_{(0)} + \mathcal{O}(\alpha^0)\,. \tag{3.210}$$

This divergence in α is subtracted by the finite counterterm (finite in ϵ):

$$\frac{(\alpha\ell_{\rm o})^3}{4}\int\limits_{z=\epsilon} d\bar{u}d^dx\,\sqrt{q}\,\phi\,(\vec{\nabla}^2_\gamma)^2\phi\,, \tag{3.211}$$

where $\vec{\nabla}^2$ is the Laplacian with respect to the spatial metric $\gamma_{ij}dx^i dx^j = \ell_{\rm o}^2/\epsilon^2 d\vec{x}_d^2$ as before. Unlike the case of $k=2$, this new counterterm is needed in order to restore the well-behaved-ness of the action in the zero Λ limit. This is done, however, at the expense of breaking invariance of the renormalized action under diffeomorphisms that are not foliation preserving. Since this counterterm is finite with respect to the regulator, it is associated to a choice of scheme on the QFT side. This means that a renormalization scheme that breaks invariance of the QFT under transformations that do not preserve the spacelike foliation of the boundary is a necessary requirement, so that the QFT states result in finite expectation values and correlators once the QFT limit associated to the zero Λ limit is taken. The final renormalized action is then given by:

$$
\begin{aligned}
iS_{ren} &= \frac{1}{2}\int d^{d+2}x\sqrt{G}\left(G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \left(\frac{m}{\alpha}\right)^2\phi^2\right) \\
&+ \frac{1}{2}\int\limits_{z=\epsilon} d^{d+1}x\,\sqrt{q}\left(-\frac{\Delta-k}{\alpha\ell_{\rm o}}\,\phi^2 - \frac{\alpha\ell_{\rm o}}{k-2}\,\phi\Box_q\phi + \frac{(\alpha\ell_{\rm o})^3}{4}\,\phi\,(\Box_q)^2\phi\,\log\epsilon\right) \\
&+ \frac{1}{2}\int\limits_{z=\epsilon} d^{d+1}x\,\sqrt{q}\,\left(\frac{(\alpha\ell_{\rm o})^3}{2}\,\phi\,(\vec{\nabla}_\gamma^2)^2\phi\right)\,. \qquad (3.212)
\end{aligned}
$$

Vacuum expectation value

The variation of the on-shell action is given by:

$$
\begin{aligned}
i\,\delta S_{ren}^{on-shell} = \int\limits_{z=\epsilon} d^{d+1}x\,\sqrt{q}\bigg(&\frac{\epsilon}{\alpha\ell_{\rm o}}\left(\phi' - i\alpha^2\dot{\phi}\right) - \frac{\Delta-k}{\alpha\ell_{\rm o}}\phi - \frac{\alpha\ell_{\rm o}}{k-2}\,\Box_q\phi \\
&+ \frac{(\alpha\ell_{\rm o})^3}{2}\,(\vec{\nabla}_\gamma^2)^2\phi + \frac{(\alpha\ell_{\rm o})^3}{4}\,(\Box_q)^2\phi\,\log\epsilon\bigg)\delta\phi\,. \qquad (3.213)
\end{aligned}
$$

The vacuum expectation value of the dual QFT operator is then given by:

$$
\begin{aligned}
\sqrt{q_{(0)}}\,\langle\mathcal{O}\rangle &= \frac{i\delta S_{ren}^{on-shell}}{\delta\varphi_{(0)}} = \lim_{\epsilon\to 0}\left(\epsilon^{\Delta-k}\,\frac{i\delta S_{ren}^{on-shell}}{\delta\phi}\right) \\
&= \frac{\ell_{\rm o}^d}{\alpha^2}\left(4\varphi_{(4)} - \frac{7}{3}\,\tilde{\varphi}_{(4)}\right) + \frac{2\ell_{\rm o}^d}{3}\,\vec{\nabla}^2\varphi_{(2)}\,. \qquad (3.214)
\end{aligned}
$$

For finite α, the vev is again mapped to the normalisable mode $\varphi_{(4)}$. The term proportional to $\tilde{\varphi}_{(4)}$ can be subtracted by adding the finite covariant counterterm to the action (finite both in ϵ and α):[24]

$$-\frac{7}{96}\,(\alpha\ell_{\rm o})^3 \int\limits_{z=\epsilon} d^{d+1}x\,\sqrt{q}\,\phi(\Box_q)^2\phi\ . \qquad (3.215)$$

The term proportional to the spatial Laplacian of $\varphi_{(2)}$, however, remains. Note then that the expectation value admits a well-behaved zero Λ limit. For finite α, the coefficient $\varphi_{(2)}$ is a functional of $\varphi_{(0)}$, so setting the source to zero and then taking the limit $\alpha \to \infty$ results in a vanishing vev for the operator. On the other hand, in the case $\alpha^{-1} = 0$ the coefficient $\varphi_{(2)}$ is a non-local functional of $\varphi_{(0)}$. From equation (3.192) for $n = 2, 3$ with $\alpha^{-1} = 0$ we find that $\varphi_{(2)}$ is defined by the differential equation: $\ddot{\varphi}_{(2)} = (\vec{\nabla}^2)^2\varphi_{(0)}$. In this way, setting first $\alpha^{-1} = 0$ in the vev and then switching off the source results in a non-trivial expectation value for the operator: $\sqrt{q_{(0)}}\,\langle\mathcal{O}\rangle \sim \vec{\nabla}^2\varphi_{(2)}$, where $\ddot{\varphi}_{(2)} = 0$. We expect this type of behaviour to be reproduced for generic values of $k \geq 4$.

3.6.3. Two-point correlator

In this last section we will compute the 2-point function for the scalar operator with $k = 2, 4$ and analyse its zero Λ limit. This is done by choosing a full solution of the equations of motion that is well-behaved in the bulk interior and then taking a first-order variation of the vevs (3.201) and (3.234) in the presence of the source. If we return to equation (3.181) for the Fourier transform $\hat{\varphi}$ of the scalar field and look at the behaviour of the Bessel functions as $z \to \infty$, we find that $\hat{\varphi}$ diverges as $z \to \infty$ unless we set the coefficient $B(\omega, p) = 0$. In this way, the solution that is well-behaved in the interior is given by:

$$\phi(z,\bar{u},\vec{x}) = \frac{2^{1-k/2}}{\Gamma(k/2)}\, z^{\Delta-k/2} \int d\omega d^dp\, e^{i\omega\bar{u}}\, e^{i\vec{p}\cdot\vec{x}}\, e^{-\alpha^2\omega z}\, \hat{\varphi}_{(0)}(\omega,\vec{p})\,(\alpha|p|)^{k/2}\, K_{k/2}(\alpha z|p|)\ , \qquad (3.216)$$

where we used the expression (3.186) for the coefficient $A(\omega, p)$, and where $|p| := \sqrt{\vec{p}^2 + \alpha^2\omega^2}$. The solution can be rewritten as an integration in position space by defining:

$$\varphi_{(0)}(\bar{v},\vec{y}) \;=\; \int d\omega d^dp\, e^{i\omega\bar{v}}\, e^{i\vec{p}\cdot\vec{y}}\, \hat{\varphi}_{(0)}(\omega,\vec{p})\ , \qquad (3.217)$$

[24] As a technical point, the fact that the integrand is finite in α follows from the discussion at the end of section 3.6.1. From equation (3.193) with $k = 4$ it follows that: $\Box^2_{q_{(0)}}\varphi_{(0)} = -(16/\alpha^4)\,\tilde{\varphi}_{(4)}$. The coefficient $\tilde{\varphi}_{(4)}$ is non-divergent in α by definition (recall that we resctricted the space of solutions in AdS to the subspace where the coefficients are well-behaved as $\alpha \to \infty$, *i.e.* we focus only on those solutions in AdS that result in solutions in Minkowski space in this limit). This implies that $\Box^2_{q_{(0)}}\varphi_{(0)} = \mathcal{O}(\alpha^{-4})$.

and using the identity (1.111):

$$\int d^{d+1}X \, \frac{e^{-ip\cdot X}}{(\epsilon^2+|X|^2)^{\Delta}} \;=\; a(k)\,\epsilon^{-k/2}\,|p|^{k/2}K_{k/2}(\epsilon|p|) \;, \tag{3.218}$$

where $k = 2\Delta - (d+1)$, $|X|^2 = X_0^2 + X^i X^i$, $|p| = \sqrt{\vec{p}^2+\alpha^2\omega^2}$, and $a(k)$ is a proportionality constant that depends only on k. The solution (3.216) can then be rewritten as:

$$\phi(z,\bar{u},\vec{x}) = \frac{2^{1-k/2}}{\Gamma(k/2)}\,\frac{\alpha^{k-\Delta-1}}{a(k)}\int d\bar{v}d^dy\,\varphi_{(0)}(\bar{v},\vec{y})\,\frac{(\alpha z)^{\Delta}}{\Big((\alpha z)^2+\big(\frac{\bar{u}-\bar{v}}{\alpha}+i\alpha z\big)^2+|\vec{x}-\vec{y}|^2\Big)^{\Delta}} \;. \tag{3.219}$$

This is precisely the expression one would obtain by solving the scalar field equation in Euclidean AdS$_{d+2}$ in Poincaré coordinates as we did in section 1.3.2, requiring that the solution be well-behaved in the bulk interior and finally transforming the scalar field to the coordinate system (3.175). From this representation we can immediately read the bulk-to-boundary propagator and obtain the expression for the unrenormalized two-point function. If we use the identity (1.101):

$$\lim_{z\to 0}\frac{(\alpha z)^{\Delta}}{\Big((\alpha z)^2+\big(\frac{\bar{u}-\bar{v}}{\alpha}+i\alpha z\big)^2+|\vec{x}-\vec{y}|^2\Big)^{\Delta}} \;\sim\; \alpha\, b(k)\,(\alpha z)^{\Delta-k}\delta(\bar{u}-\bar{v})\delta^d(\vec{x}-\vec{y}) \;, \tag{3.220}$$

with $b(k)$ a constant that depends only on k, then the on-shell bare action (3.176) is given by:

$$\begin{aligned} S^{on-shell} &= \frac{1}{2}\int\limits_{z=\epsilon} d^{d+1}x\,\sqrt{G}\,\phi\,G^{z\mu}\partial_\mu\phi \\ &= \frac{\alpha^{k-3}\ell_o^d}{\tilde{b}(k)}\int\limits_{z=\epsilon} d\bar{u}d^dx\int d\bar{v}d^dy\;\frac{\varphi_{(0)}(\bar{u},\vec{x})\,\varphi_{(0)}(\bar{v},\vec{y})}{\Big(\big(\frac{\bar{u}-\bar{v}}{\alpha}\big)^2+|\vec{x}-\vec{y}|^2\Big)^{\Delta}}\,(1+\mathcal{O}(z)) \;, \end{aligned} \tag{3.221}$$

with $\tilde{b}(k)$ a dimensionless constant. Taking the variations of the on-shell action with respect to the source and absorbing the overall proportionality constant in the normalisation of the operator results in the unrenormalized two-point correlator:

$$\begin{aligned} \sqrt{q_{(0)}}^{\,2}\langle\mathcal{O}(\bar{v},\vec{y})\mathcal{O}(\bar{u},\vec{x})\rangle &= \frac{i\delta^2 S^{on-shell}}{\delta\varphi_{(0)}(\bar{v},\vec{y})\delta\varphi_{(0)}(\bar{u},\vec{x})} \\ &= \frac{1}{\Big(\big(\frac{\bar{u}-\bar{v}}{\alpha}\big)^2+|\vec{x}-\vec{y}|^2\Big)^{\Delta}} \;. \end{aligned} \tag{3.222}$$

In the zero Λ limit and away from coincident points, this results in the correct expression for the two-point function of a scalar operator of weight Δ but in d dimensions.

In order to compute the renormalized correlator, we return to our original representation (3.216) for the physical solution and use the expansion (3.182) around $z = 0$ for the Bessel function with $k = 2, 4$ to find:

$$\phi(z, \bar{u}, \vec{x}) \quad = \quad z^{\Delta - k} \left(\varphi_{(0)} + ... + z^k \varphi_{(k)} + z^k \log(\mu z)\, \tilde{\varphi}_{(k)} + ...\right) \ , \quad (3.223)$$

where the normalisable mode $\varphi_{(k)}$ for $k = 2, 4$ is given in terms of the source by:

$$\varphi_{(k=2)} = \frac{\alpha^2}{4}\left(\vec{\nabla}^2 - \alpha^2 \partial_{\bar{u}}^2 - \left(2\gamma_E - 2\log 2 + \log\left(-\frac{\alpha^2}{\mu^2}\Box_{q_{(0)}}\right)\right)\Box_{q_{(0)}}\right)\varphi_{(0)} \ , \quad (3.224)$$

$$\varphi_{(k=4)} = \frac{\alpha^4}{24}\left(-3\alpha^2\partial_{\bar{u}}^2\vec{\nabla}^2 - 2\alpha^4\partial_{\bar{u}}^4 - \frac{3}{4}\left(2\gamma_E - \frac{3}{2} - 2\log 2 + \log\left(-\frac{\alpha^2}{\mu^2}\Box_{q_{(0)}}\right)\right)\Box_{q_{(0)}}^2\right)\varphi_{(0)} \ , \quad (3.225)$$

with γ_E the Euler constant and $\Box_{q_{(0)}} = \alpha^2 \partial_{\bar{u}} + \vec{\nabla}^2$ the Laplacian with respect to the QFT metric. At the end of section 3.6.1 we found that the requirement that the coefficients $\varphi_{(n<k)}$ and $\tilde{\varphi}_{(k)}$ in the asymptotics be well-defined in the limit $\alpha \to \infty$ results in constraints on the behaviour in α of the time derivatives of the source. Since the normalisable mode for each k is also well-defined in the limit $\alpha \to \infty$ by definition, and from the above expressions (3.224) for the physical solution we have that $\varphi_{(k)}$ is now a functional of the source, we find that the requirement that the solution be well-behaved in the interior results in a further constraint on the source for each value of k. The constraint will be on the behaviour in α of the spatial derivatives. From equations (3.193) and (3.192) for $k = 2, 4$ we have in particular that:

$$\Box_{q_{(0)}}^{k/2}\varphi_{(0)} \quad = \quad -\frac{k^{k/2}}{\alpha^k}\,\tilde{\varphi}_{(k)} \ = \ \mathcal{O}(\alpha^{-k}) \ . \quad (3.226)$$

It then follows from equation (3.224) that the non-normalisable mode of the physical solution for $k = 2, 4$ needs to satisfy:

$$\vec{\nabla}^k \varphi_{(0)} \quad = \quad \mathcal{O}(\alpha^{-2}) \ . \quad (3.227)$$

For $k = 2$ this implies that the vev (3.201) evaluated on such a solution is identically zero in the zero Λ limit. For $k = 4$ it implies that the bulk action (3.205) evaluated on such a solution is well-defined in the zero Λ limit, as well as the vev for the dual QFT operator, without the need for the anomalous counterterm. Nonetheless, the renormalization should hold for any solution of the bulk equations of motion, so in general the anomalous conterterm is needed to restore the well-behaved-ness of the zero Λ limit of the bulk action.

Case k=2

If we take the variation of the one-point function (3.201) (with $\tilde{\varphi}_{(2)}$ subtracted) with respect to the source $\varphi_{(0)}$ and use the expression (3.224) for the coefficient $\varphi_{(2)}$, we obtain:

$$\begin{aligned}\sqrt{q_{(0)}}^{\,2}\langle\mathcal{O}(\bar{v},\vec{y})\mathcal{O}(\bar{u},\vec{x})\rangle &= \frac{\delta}{\delta\varphi_{(0)}(\bar{v},\vec{y})}\left(\sqrt{q_{(0)}}\,\langle\mathcal{O}(\bar{u},\vec{x})\rangle\right)\\ &= -\frac{\ell_{\text{o}}^{d}}{2}\Big(1+2\gamma_E-2\log 2+2\log\alpha\Big)\Box_{q_{(0)}}\delta(\bar{u}-\bar{v})\delta^{d}(\vec{x}-\vec{y})\\ &\quad-\frac{\ell_{\text{o}}^{d}}{2}\log\left(-\mu^{-2}\Box_{q_{(0)}}\right)\Box_{q_{(0)}}\delta(\bar{u}-\bar{v})\delta^{d}(\vec{x}-\vec{y})\ .\end{aligned} \tag{3.228}$$

The first term proportional to the Laplacian on the delta functions is scheme dependent and it can be removed by adding a finite and *local* counterterm to the action proportional to (3.202). The scheme-independent piece is then:

$$\sqrt{q_{(0)}}^{\,2}\langle\mathcal{O}(\bar{v},\vec{y})\mathcal{O}(\bar{u},\vec{x})\rangle = -\frac{\ell_{\text{o}}^{d}}{2}\log\left(-\mu^{-2}\Box_{q_{(0)}}\right)\Box_{q_{(0)}}\delta(\bar{u}-\bar{v})\delta^{d}(\vec{x}-\vec{y})\ . \tag{3.229}$$

If we use the identity [77]:

$$\int d^{d+1}X\frac{e^{ip\cdot X}}{|X|^{d-1}}\log\left(\tilde{\mu}^2|X|^2\right) \;=\; -\frac{c}{|p|^2}\log\left(\mu^{-2}|p|^2\right)\ , \tag{3.230}$$

with $\tilde{\mu}=\gamma_E\,\mu/2$ and c a proportionality constant that depends only on d, and Fourier transform it, we find:

$$\Box^{n+1}\frac{\log\left(\tilde{\mu}^2|X|^2\right)}{|X|^{d-1}} \;=\; c\log\big(-\mu^{-2}\Box\big)\Box^{n}\,\delta^{d+1}(X)\ . \tag{3.231}$$

If we apply this identity to the right-hand side of (3.229) we obtain:

$$\begin{aligned}\sqrt{q_{(0)}}^{\,2}\langle\mathcal{O}(\bar{v},\vec{y})\mathcal{O}(\bar{u},\vec{x})\rangle &= -\frac{\ell_{\text{o}}^{d}}{2\alpha c}\Box^2_{q_{(0)}}\frac{\log\left(\tilde{\mu}^2\left[\left(\frac{\bar{u}-\bar{v}}{\alpha}\right)^2+|\vec{x}-\vec{y}|^2\right]\right)}{\left|\left(\frac{\bar{u}-\bar{v}}{\alpha}\right)^2+|\vec{x}-\vec{y}|^2\right|^{(d-1)/2}}\\ &= \tilde{c}\,\mathcal{R}\frac{1}{\left|\left(\frac{\bar{u}-\bar{v}}{\alpha}\right)^2+|\vec{x}-\vec{y}|^2\right|^{\Delta}}\ ,\end{aligned} \tag{3.232}$$

where $\Delta = 1 + (d+1)/2$. The proportionality constant $\tilde{c}$ can be absorbed in a normalisation of $\mathcal{O}$. The term $\mathcal{R}\left(1/|X|^{2\Delta}\right)$ on the right-hand side is the renormalized version of the correlator $1/|X|^{2\Delta}$ and it coincides with the latter away from coincident points [68]. In the zero Λ limit we find:

$$\lim_{\alpha\to\infty} \sqrt{q_{(0)}}^{\,2} \langle \mathcal{O}(\bar{v},\vec{y})\mathcal{O}(\bar{u},\vec{x})\rangle \quad = \quad \mathcal{R}\frac{1}{|\vec{x}-\vec{y}|^{2\Delta}} \ , \tag{3.233}$$

which is the renormalized version of the correlator that we found in (3.222) in the zero Λ limit.

Case k=4

In this case the one-point function for the QFT operator receives a contribution from the anomalous counterterm (3.211). This term renders the vacuum expectation value finite in the zero Λ limit, but it introduces contact terms in the two point function. In order to verify this more explicitly, we isolate the contribution from this term in the vev:

$$\sqrt{q_{(0)}}\,\langle\mathcal{O}\rangle = \left(\frac{4\ell_{\rm o}^d}{\alpha^2}\,\varphi_{(4)} + \frac{2\ell_{\rm o}^d}{3}\,\vec{\nabla}^2\varphi_{(2)} - \frac{\alpha^2\ell_{\rm o}^d}{2}\,\vec{\nabla}^4\varphi_{(0)}\right) + \frac{\alpha^2\ell_{\rm o}^d}{2}\,\vec{\nabla}^4\varphi_{(0)} \ , \tag{3.234}$$

where the last term represents the contribution from the anomalous counterterm. We have also subtracted the term proportional to $\tilde{\varphi}_{(4)}$ which is scheme dependent. If we use the expression (3.225) for the normalisable mode $\varphi_{(4)}$ and take the variation of the one-point function with respect to the source, we obtain:

$$\begin{aligned}\sqrt{q_{(0)}}^{\,2}\langle\mathcal{O}(\bar{v},\vec{y})\mathcal{O}(\bar{u},\vec{x})\rangle &= \frac{\delta}{\delta\varphi_{(0)}(\bar{v},\vec{y})}\left(\sqrt{q_{(0)}}\,\langle\mathcal{O}(\bar{u},\vec{x})\rangle\right)\\ &= -\frac{\alpha^2\ell_{\rm o}^d}{6}\left(2+\frac{3}{4}\left(2\gamma_E - 2\log 2 - \frac{3}{2} + 2\log\alpha\right)\right)\Box^2_{q_{(0)}}\delta(\bar{u}-\bar{v})\delta^d(\vec{x}-\vec{y})\\ &\quad -\frac{\alpha^2\ell_{\rm o}^d}{8}\,\log\left(-\mu^{-2}\,\Box_{q_{(0)}}\right)\Box^2_{q_{(0)}}\,\delta(\bar{u}-\bar{v})\delta^d(\vec{x}-\vec{y})\\ &\quad +\frac{\alpha^2\ell_{\rm o}^d}{2}\,\delta(\bar{u}-\bar{v})\,\vec{\nabla}^4\delta^d(\vec{x}-\vec{y}) \ .\end{aligned} \tag{3.235}$$

The first term proportional to the square of the Laplacian can be removed by adding a finite and local counterterm to the action proportional to (3.215). The last term arising from the anomalous counterterm is a contact term that diverges when the operators are defined at equal time $\bar{u} = \bar{v}$. This piece cannot be removed from the correlator by a counterterm without spoiling the zero Λ limit of the bulk action. This type of contact terms spoils the behaviour of the correlator at coincident points in time and will always appear in the two-point functions for values of $k \geq 4$ if we simultaneously require that the bulk action be well-defined in the zero Λ limit. At non-coincident points, if we subtract the scheme-dependent term and use the identity (3.231), we find:

$$\begin{aligned} \sqrt{q_{(0)}}^{\,2} \langle \mathcal{O}(\bar{v},\vec{y}) \mathcal{O}(\bar{u},\vec{x}) \rangle &= -\frac{\alpha \ell_{\text{o}}^d}{8c} \Box^3_{q_{(0)}} \frac{\log\left(\tilde{\mu}^2 \left[\left(\frac{\bar{u}-\bar{v}}{\alpha}\right)^2 + |\vec{x}-\vec{y}|^2 \right]\right)}{\left| \left(\frac{\bar{u}-\bar{v}}{\alpha}\right)^2 + |\vec{x}-\vec{y}|^2 \right|^{(d-1)/2}} \\ &= \tilde{c}\, \mathcal{R} \frac{1}{\left| \left(\frac{\bar{u}-\bar{v}}{\alpha}\right)^2 + |\vec{x}-\vec{y}|^2 \right|^{\Delta}} \qquad (\bar{u} \neq \bar{v}) \ . \end{aligned} \tag{3.236}$$

where $\Delta = 2 + (d+1)/2$. If we absorb the constant $\tilde{c}$ in the normalisation of the operator and take the limit $\alpha \to \infty$, we again find the renormalized version of the correlator that we obtained in (3.222) in this limit.

3.7. Appendix: Gaussian null coordiantes

In this section we will derive our coordinate system by performing a brief ADM analysis of the spacetime metric $G_{\mu\nu}$. For a thorough treatment see the original works in [159, 160, 162]. We introduce coordinates $x^\mu = (u, x^A) = (u, r, x^i) = (r, x^a)$ and define the surfaces of constant u to be null. We then do an ADM decomposition of $G_{\mu\nu}$ with respect to these surfaces as:

$$ds^2 = -\alpha^2 du^2 + h_{AB} \left(dx^A + \alpha^A du\right) \left(dx^B + \alpha^B du\right) \ . \tag{3.237}$$

We also decompose the induced metric h_{AB} with respect to the surfaces of constant r as:

$$h_{AB} dx^A dx^B = \beta^2 dr^2 + \gamma_{ij} \left(dx^i + \beta^i dr\right) \left(dx^j + \beta^j dr\right) \ , \tag{3.238}$$

and define the spatial metric γ_{ij} to be positive-definite. Since the surfaces of constant u are null by definition, the induced metric h_{AB} must be degenerate. Since the determinant $\sqrt{h} = \beta\sqrt{\gamma}$ and $\gamma_{ij} > 0$, the degeneracy of h_{AB} implies that $\beta = 0$ everywhere. With this condition, we rewrite $G_{\mu\nu}$ without loss of generality as:

$$ds^2 = -\phi\, du^2 + 2M du dr + \gamma_{ij} \left(dx^i + \sigma^i du + \beta^i dr\right) \left(dx^j + \sigma^j du + \beta^j dr\right) \tag{3.239}$$

$$= N^2 dr^2 + q_{ab} \Big(dx^a + N^a dr\Big) \left(dx^b + N^b dr\right) \ , \tag{3.240}$$

where (N, N^a) are the lapse and shift of the radial foliation in r and where the induced metric q_{ab} is given by:

$$q_{ab}dx^a dx^b = -\phi\, du^2 + \gamma_{ij}\left(dx^i + \sigma^i du\right)\left(dx^j + \sigma^j du\right)\ . \tag{3.241}$$

Let us then perform an ADM decomposition of the Einstein-Hilbert Lagrangian with respect to the radial foliation (3.240):

$$\mathcal{L} = \sqrt{G}\,R[G] = N\sqrt{q}\left(R[q] + Q^2 - Q\cdot Q - 2\nabla_\mu v^\mu\right)\ , \tag{3.242}$$

where $Q_{ab} = 1/(2N)\,(\partial_r - \pounds_N)\,q_{ab}$ is the extrinsic curvature of the surfaces of constant r and: $v^\mu = Qn^\mu - a^\mu$, with n^μ and a^μ the unit normal and acceleration of these surfaces, respectively. The last term in the Lagrangian is a total derivative and thus will be discarded. The decomposed Lagrangian is now a functional of the lapse, shift and induced metric N, N^a and q_{ab}. A quick inspection of $\mathcal{L}$ then reveals that only q_{ab} contains radial derivatives and therefore the equations of motion for the metric will be second order differential equations in r for q_{ab} only. This indicates as usual that N and N^a do not represent true degrees of freedom and therefore can be gauge-fixed, *i.e.* can be brought to any configuration by diffeomorphisms near a surface of constant r. If we then return to (3.241) we find that q_{ab} depends only on ϕ, σ^i and γ_{ij}. This means that the Lagrangian does not contain radial derivatives of the functions M and β^i that appear in (3.239) and therefore these can be gauge-fixed by diffeomorphisms. The simplest gauge we can choose is the Gaussian gauge $\left(M = 1, \beta^i = 0\right)$ in which the spacetime metric assumes the final form:

$$ds^2 = -\phi\, du^2 + 2dudr + \gamma_{ij}\left(dx^i + \sigma^i du\right)\left(dx^j + \sigma^j du\right)\ , \tag{3.243}$$

with determinant $\sqrt{G} = \sqrt{\gamma}$. In the particular case of black hole spacetimes in gaussian null coordinates, the horizon is defined to consist of the surface $r = 0$. Then note that it is still possible to use a further diffeomorphism of the form $x^i \to x^i + f^i(x, u)$ in (3.243) and choose the set of functions f^i such that:

$$\sigma^i \quad\to\quad r^\alpha\tilde{\sigma}^i(r, u, x) \quad : \quad \alpha > 0\ ,\ \tilde{\sigma}^i = \mathcal{O}(r^{\geq 0})\ . \tag{3.244}$$

Also, since the horizon is a null surface, we find that the function ϕ must behave near $r = 0$ at least as:

$$\phi \quad = \quad r^\beta\varphi(r, u, x) \quad : \quad \beta > 0\ ,\ \varphi = \mathcal{O}(r^{\geq 0})\ . \tag{3.245}$$

In most cases the equations of motion near the horizon then fix the exponents $\alpha, \beta = 1$ for a non-degenerate horizon, and $\alpha = 1, \beta = 2$ for a degenerate one.

3.8. Appendix: Ricci tensor

In this section we provide the decomposition of the Ricci tensor of our gauge-fixed metric:

$$\begin{aligned} ds^2_{d+2} &= G_{\mu\nu}dx^\mu dx^\nu \\ &= \frac{\ell_{\rm o}^2}{z^2}\Big(-\varphi N_{(0)}du^2 - 2N_{(0)}dudz + g_{ij}\left(dx^i+\sigma^i du\right)\left(dx^j+\sigma^j du\right)\Big)\ , \end{aligned} \tag{3.246}$$

where $N_{(0)} = N_{(0)}(u,x^i)$ and the remaining components of the metric depend on all coordinates. The inverse and determinant of the metric are given by:

$$G^{\mu\nu} = \left(\frac{z}{\ell_{\rm o}}\right)^2 N_{(0)}^{-1}\begin{pmatrix} 0 & -1 & 0 \\ -1 & \varphi & \sigma^i \\ 0 & \sigma^i & N_{(0)}g^{ij}\end{pmatrix}\ , \tag{3.247}$$

$$\sqrt{G} = (\ell_{\rm o}/z)^{d+2}N_{(0)}\sqrt{g}\ . \tag{3.248}$$

Define:

$$k_{ij} := \frac{1}{2N_{(0)}}\left(\partial_u - \pounds_\sigma\right)g_{ij}\ . \tag{3.249}$$

The decomposition of the Ricci tensor $R_{\mu\nu}[G]$ is then given by [159, 168]:

$$\begin{aligned} 2R_{zi}[G] &= \frac{1}{N_{(0)}}\left(-(g\cdot\sigma')'_i + \frac{d}{z}\Big((g\cdot\sigma')_i - \partial_i N_{(0)}\Big) - \frac{1}{2}\,\mathrm{Tr}[g^{-1}g']\Big((g\cdot\sigma')_i - \partial_i N_{(0)}\Big)\right) \\ &\quad + \nabla_j (g^{-1}g')^j{}_i - \partial_i \mathrm{Tr}[g^{-1}g']\ , \end{aligned} \tag{3.250}$$

$$\begin{aligned} 2\Big(R_{zu}[G] - \sigma^i R_{zi}[G]\Big) &= \varphi'' - \frac{d+2}{z}\,\varphi' + \frac{2(d+1)}{z^2}\,\varphi + \mathrm{Tr}[g^{-1}g']\left(\frac{1}{2}\,\varphi' - \frac{1}{z}\,\varphi\right) \\ &\quad - \nabla_i\Big(\sigma'^i - g^{ij}\partial_j N_{(0)}\Big) - \frac{1}{N_{(0)}}\,\sigma'^i\Big((g\cdot\sigma')_i - \partial_i N_{(0)}\Big) \\ &\quad - N_{(0)}\left(2\,\mathrm{Tr}[g^{-1}k]' - \frac{2}{z}\,\mathrm{Tr}[g^{-1}k] + (k\cdot g')\right)\ , \end{aligned} \tag{3.251}$$

$$
\begin{aligned}
2R_{ij}[G] \;=\; & 2R_{ij}[g] + \frac{1}{N_{(0)}}\Bigg[-(\varphi\, g'_{ij})' + \frac{d}{z}\,\varphi\, g'_{ij} + \frac{2}{z}\,\varphi' g_{ij} - \frac{2(d+1)}{z^2}\,\varphi\, g_{ij} \\
& + \varphi\left(\frac{1}{z}\, g_{ij} - \frac{1}{2}\, g'_{ij}\right)\mathrm{Tr}[g^{-1}g'] + \varphi\,(g'\cdot g')_{ij} \\
& + 2\nabla_{(i}\Big(\left((g\cdot\sigma')_{j)} - \partial_{j)}N_{(0)}\right) - N_{(0)}^{-1}(g\cdot\sigma')_i\,(g\cdot\sigma')_j\Bigg] + \partial_i \log N_{(0)}\partial_j \log N_{(0)} \\
& + 4\,k'_{ij} - \frac{2d}{z}\,k_{ij} + \mathrm{Tr}[g^{-1}g']k_{ij} + \left(g'_{ij} - \frac{2}{z}\,g_{ij}\right)\mathrm{Tr}[g^{-1}k] - 4(k\cdot g')_{(ij)}\ ,
\end{aligned}
\tag{3.252}
$$

$$
2R_{zz}[G] \;=\; -\mathrm{Tr}[g^{-1}g''] + \frac{1}{2}\,(g'\cdot g')\ , \tag{3.253}
$$

$$
\begin{aligned}
& 2\Big(R_{ui}[G] - \sigma^j R_{ij}[G] - \varphi\, R_{zi}[G]\Big) = \\
& \qquad (\partial_u - \pounds_\sigma)\,\left[\frac{1}{N_{(0)}}\left((g\cdot\sigma')_i - \partial_i N_{(0)}\right)\right] + \mathrm{Tr}[g^{-1}k]\,\left((g\cdot\sigma')_i - \partial_i N_{(0)}\right) \\
& \qquad + 2\,(g^{-1}k)^j{}_i\,\partial_j N_{(0)} - (g^{-1}g')^j{}_i\,\partial_j\varphi + \partial_i\varphi' + N_{(0)}\left(-\frac{d}{z} + \frac{1}{2}\,\mathrm{Tr}[g^{-1}g']\right)\partial_i\left(\varphi/N_{(0)}\right) \\
& \qquad + 2N_{(0)}\left(\nabla_j(g^{-1}k)^j{}_i - \partial_i \mathrm{Tr}[g^{-1}k]\right) - \varphi\left(\nabla_j(g^{-1}g')^j{}_i - \partial_i \mathrm{Tr}[g^{-1}g']\right)\ ,
\end{aligned}
\tag{3.254}
$$

$$
\begin{aligned}
& \frac{2}{N_{(0)}}\Big[R_{uu}[G] - 2\,\sigma^i R_{ui}[G] + \sigma^i\sigma^j R_{ij}[G] - \varphi\left(R_{zu}[G] - \sigma^i R_{zi}[G]\right)\Big] = \\
& \qquad \left(-\frac{d}{z} + \frac{1}{2}\,\mathrm{Tr}[g^{-1}g']\right)(\partial_u - \pounds_\sigma)\,(\varphi/N_{(0)}) - 2\,(\partial_u - \pounds_\sigma)\,\mathrm{Tr}[g^{-1}k] \\
& \qquad + \varphi\left(2\,\mathrm{Tr}[g^{-1}k]' + (k\cdot g')\right) - \varphi'\,\mathrm{Tr}[g^{-1}k] - 2N_{(0)}\big(k\cdot k\big) + \nabla^i\nabla_i\varphi \\
& \qquad + g^{ij}\,\partial_i\varphi\,\partial_j \log N_{(0)} + \frac{1}{N_{(0)}}\left(\varphi\nabla_i\sigma'^{\,i} - \sigma'^{\,i}\partial_i\varphi\right)\ ,
\end{aligned}
\tag{3.255}
$$

where the prime denotes differentiation with respect to z, the trace and inner product are taken with respect to g_{ij}, and where $\nabla_i g_{jk} := 0$. When replaced by the Einstein equations:

$$
R_{\mu\nu}[G] \;=\; -\frac{d+1}{\alpha^2\ell_{\mathrm{o}}^2}\,G_{\mu\nu}\ , \tag{3.256}
$$

we find that equations (3.250)–(3.252) represent the dynamical equations for the metric components σ^i, φ and g_{ij}, respectively, whereas equations (3.254) and (3.255) are constraint equations since they do not contain second order derivatives in z. After (3.252) is solved, equation (3.253) can also be seen as a constraint equation because it can be replaced by an equation without second order derivatives in z if we use the trace of (3.252).

3.9. Appendix: Terms X_{ij} and X_i

The algebraic expressions for the terms X_{ij} and X_i that appear in equations (3.106) and (3.108) depend on the coefficient $g_{(1)ij}$ and vanish if the boundary metric is static. In general, the expressions are given by:

$$\begin{aligned} X_{ij} = &\frac{1}{4\alpha^2}\, g_{(1)ij} \left(\mathrm{Tr}^2[g_{(0)}^{-1} g_{(1)}] + \left(g_{(1)} \cdot g_{(1)}\right)\right) - \frac{3}{4}\,\mathrm{Tr}[g_{(0)}^{-1} g_{(1)}]\, k_{(1)ij} - \frac{5}{4}\,\mathrm{Tr}[g_{(0)}^{-1} k_{(1)}]\, g_{(1)ij} \\ &+ \frac{1}{2}\, R_{(0)}\, g_{(1)ij} - \frac{3}{2}\, g_{(1)(i}{}^{k}\, {}^{(0)}\nabla_{j)} \partial_k \log N_{(0)} + \frac{1}{4}\left({}^{(0)}\nabla_i \partial_j \mathrm{Tr}[g_{(0)}^{-1} g_{(1)}] - {}^{(0)}\Box g_{(1)ij}\right) \\ &+ \frac{1}{4N_{(0)}}\left(\mathrm{Tr}[g_{(0)}^{-1} g_{(1)}]\, {}^{(0)}\nabla_i \partial_j N_{(0)} + g_{(1)ij}\, {}^{(0)}\Box N_{(0)} - g_{(0)ij}\,\mathrm{Tr}[g_{(0)}^{-1} g_{(1)}]\, {}^{(0)}\Box N_{(0)}\right) \ , \end{aligned} \tag{3.257}$$

$$\begin{aligned} X_i = &\frac{3}{8}\left(g_{(1)} \cdot g_{(1)}\right)_i{}^j \partial_j \log N_{(0)} + \frac{1}{2}\, g_{(1)ij}\, {}^{(0)}\nabla_k g_{(1)}^{kj} + \frac{1}{2}\, {}^{(0)}\nabla_k \left(g_{(1)} \cdot g_{(1)}\right)^k{}_i \\ &- \frac{3}{4}\, g_{(1)i}^{\ j} \partial_j \mathrm{Tr}[g_{(0)}^{-1} g_{(1)}] + \frac{1}{16}\,\partial_i \mathrm{Tr}^2[g_{(0)}^{-1} g_{(1)}] - \frac{5}{16}\,\partial_i \left(g_{(1)} \cdot g_{(1)}\right) \ . \end{aligned} \tag{3.258}$$

To obtain these expressions we made use of the matrix identity:

$$(AB^{-1}A)_{ij} - \frac{1}{2} B_{ij} \mathrm{Tr}[B^{-1}AB^{-1}A] = \quad \mathrm{Tr}[B^{-1}A]\left(A_{ij} - \frac{1}{2} B_{ij} \mathrm{Tr}[B^{-1}A]\right) \ , \tag{3.259}$$

for any 2x2 matrices A and B such that $\det B \neq 0$.

Chapter 4

Aspects of Ricci-flat Holography - II

4.1. Introduction

In the previous chapter we discussed the zero Λ limit of vacuum expectation values and correlation functions in AdS/CFT at a formal level, the associated issues and attempted to address them. We found that the analysis requires a suitable foliation of the spacetime and we derived the mapping between bulk and boundary data in the associated coordinate system. We focused specifically on the case of the bulk spacetime metric and a non-backreacting scalar field, determined their unique asymptotics, computed correlators of the dual operators and discussed the necessary conditions for the correspondence between the near-boundary asymptotics and the vevs to admit a well-behaved zero Λ limit. We found that the existence of the limit essentially translates into a statement about the sources and states of the boundary theory. The most important open problem, however, is to understand more precisely the nature of the zero Λ limit from the point of view of string theory, and also holographically from the dual field theory perspective. In this final chapter we will follow a different viewpoint and attempt to gain some insight on the nature of flat space holography by formulating the problem, not as a limit of AdS/CFT, but as an extension of the AdS/CFT ingredients to gravitational theories with exactly vanishing cosmological constants. More specifically, we will construct a correspondence between certain Ricci-flat geometries that naturally generalize Minkowski spacetime and families of conformal field theories at the null boundary of the manifolds.

From the results of the previous chapter we have learned that the zero Λ limit of AdS/CFT implies in particular that putative field theories dual to Ricci-flat spacetimes should be defined at the conformal boundary of the spaces. In the case of a vanishing cosmological constant, this boundary is a null manifold and a direct application of the AdS/CFT recipe therefore requires a dual quantum field theory on a degenerate manifold.[1] We found in section 3.6.3 that the holographic 2-point function in the zero Λ limit was consistent with this fact: the form of the correlator was consistent with that of an operator defined in two dimensions less. The main problem in establishing a duality between a gravitational theory and a quantum field theory on a codimension two submanifold lies in the reconstruction of two bulk dimensions from quantum field theory data, which in addition cannot be both simultaneously spacelike. In AdS/CFT, the conformal boundary of the Einstein spaces is timelike and any quantum field theory at the boundary need only to contain enough information to allow for the reconstruction of one spatial (radial) direction. Such information is indeed captured in the dynamics and kinematic constraints of the boundary theory from which the bulk fields can be reconstructed. In our case, on the other hand, the null nature of the conformal boundary requires the behaviour of bulk fields along the timelike direction to be captured as well by the dual field theory. Time evolution in the bulk must therefore be of central importance in such a holographic description of Ricci-flat spacetimes.[2]

In the work initially developed by de Boer and Solodukhin [188, 189] it was proposed that fields on *(d+2)*–dimensional Minkowski space could be reconstructed from conformal field theory data on a *d*–dimensional conformal manifold representing the boundary of the Minkowski lightcone. The key observations are a) that the interior of the lightcone is naturally foliated by conformally compact, hyperbolic (or Euclidean AdS) hypersurfaces whose boundaries all degenerate to the boundary of the lightcone and b) that the isometry group of (an asymptotically) Minkowski spacetime contains a subgroup that acts on such boundary as the conformal group. Since each leaf of the foliation admits a holographic description in terms of a conventional Euclidean *d*–dimensional CFT on its conformal boundary, the authors asked whether the interior of the lightcone could be described holographically in terms of a family of CFTs on this codimension two submanifold

[1] See also the discussion in [184] and the preliminary works [185, 186, 187] on quantum field theories defined on null manifolds.

[2] It should be stressed that Ricci-flat spaces with Euclidean signature cannot have a conformal boundary. It is simple to show that the Ricci scalar of conformally compact Riemannian manifolds cannot vanish asymptotically, hence conformal compactness necessarily requires the Ricci-flat spaces to be Lorentzian, which also follows from the fact that the conformal boundary must be null. This implies in particular that one cannot have simultaneously a static conformal embedding and a time-independent defining function, otherwise a simple Wick rotation would violate the above statement. See appendix B.1 for further details.

to which all boundaries converge. The problem raised by this approach regards to the reconstruction of the bulk timelike direction from field theory data. The AdS/CFT dictionary allows one to reconstruct the radial direction on each slice through the dynamics of the respective dual quantum field theory. The authors then left open the non-trivial possibility that evolution along the extra timelike direction defined by, and orthogonal to, the foliation could also be reconstructed from the infinite set of CFTs that reside on the boundary of the lightcone. Although the AdS/CFT duality guarantees that each CFT encodes radial evolution along each slice, it does not necessarily imply the family of CFTs should encode time evolution orthogonal to the slices.

The purpose of this chapter is to elaborate on such proposal and provide evidence supporting the conjecture that bulk fields on a specific class of Ricci-flat spacetimes can be reconstructed out of conformal field theory data on a codimension two conformal manifold representing the boundary of a null surface in the bulk that extends to null infinity. We will do so in the case of pure gravity[3] by showing that the bulk spacetime metric can be reconstructed from the conformal structure of null infinity and from the expectation values of a family of conformal field theory stress tensors. Our results will be consistent with a dual description of the time evolution of the Ricci-flat metric by a family of conformal field theories. The procedure will follow the standard AdS/CFT programme at the full non-linear level for our class of spacetimes by finding the most general spacetime asymptotics towards the conformal boundary, holographically renormalizing the gravitational action, computing the expectation values and Ward identities of the stress tensors of the field theories and mapping these to the data necessary to the reconstruction of the bulk metric. The results obtained for the holographic Weyl anomalies in even dimensions then imply that, for each CFT, the bulk timelike coordinate plays the role of the CFT central charge(s) in a gauge invariant way, with the bulk Planck length as the characteristic length. We elaborate more on this aspect in section 4.2.2 and then mainly in 4.4.3.

In the next section we review the foliation of Minkowski space that motivates our framework and describe the generalisation to a specific class of asymptotically Ricci-flat manifolds. We then outline the approach taken to deducing the most general asymptotics of such spacetimes and which is based on the initial value for-

[3]Recall that in AdS/CFT, pure bulk gravity is describing a conformal field theory with vanishing vacuum expectation values and correlators of every gauge-invariant operator with the exception of the CFT stress-energy tensor. This is indeed the picture that arises by working with the full supergravity action, performing the holographic computations and in the end setting the bulk matter configurations to zero. The field theory is therefore in a state in which no operator has dynamics but the stress-energy tensor.

mulation of general relativity. In section 4.3 we briefly review the latter formalism and apply it to our class of manifolds. Solving the equations of motion within such framework will allow us to obtain in a unique way the asymptotic behaviour of the metric for such spacetimes and to find its relation to the ambient metric of Fefferman and Graham [190]. In section 4.4 we renormalize holographically the gravitational action and compute the vacuum expectation values and Ward identities of the family of dual field theories. The last section represents a generalisation of the previous formalism. Our class of asymptotically Ricci-flat manifolds will be generalised further by including subleading corrections to the spacetime asymptotics. This will allow us to obtain different expectation values for different field theories in this family. In this chapter's appendix we provide a few definitions that are necessary to our formalism together with several technical results.

4.2. Preliminaries

4.2.1. Foliation of Minkowski space

Let $(\mathcal{M}, G_{\mu\nu})$ be *(d+2)*–dimensional Minkowski space. In spherical coordinates:

$$ds^2_{d+2} = -dT^2 + dr^2 + r^2 d\Omega_d^2 \,. \tag{4.1}$$

Let one introduce null coordinates ($v := T + r\,,\, u := T - r$):

$$ds^2_{d+2} = -dvdu + v^2 \left(\frac{1-u/v}{2}\right)^2 d\Omega_d^2 \,, \tag{4.2}$$

such that infinity is represented by the union of the regions: $\Im^+ = \{v = +\infty\,, |u| < \infty\}$, $\Im^- = \{u = -\infty\,, |v| < \infty\}$, $i_\pm = \{v = u = \pm\infty : |v-u| < \infty\}$ and: $i_0 = \{v = -u = +\infty : |v{+}u| < \infty\}$. To bring these regions to finite values of the coordinates, one introduces Penrose-type null coordinates ($v' := \arctan v\,,\, u' := \arctan u$):

$$ds^2_{d+2} = \frac{1}{\cos^2 v' \, \cos^2 u'} \left(-dv'\,du' + \frac{1}{4}\sin^2(v'-u')\,d\Omega_d^2\right) := \rho^{-2}(x)\,d\tilde{s}^2_{d+2} \,, \tag{4.3}$$

where infinity is represented by the region where the defining function $\rho(x) := \cos v' \, \cos u'$ vanishes. Since $r > 0$ in (4.1), then $v' > u'$ and the flattened Penrose diagram for the conformal embedding $(\tilde{\mathcal{M}}, \tilde{G}_{\mu\nu} = \rho^2 G_{\mu\nu})$ is given in figure 4.1, where each point represents a S^d (with the exception of the corners $i_{\pm,0}$).[4]

[4]More precisely, the conformal embedding is obtained from $\tilde{\mathcal{M}}$ by deleting the corners $i_{\pm,0}$. Notice that $d\rho = 0$ and $\tilde{G}$ is degenerate in those regions, hence the triple $(\tilde{\mathcal{M}}, \tilde{G}, \rho)$ does not represent an asymptote unless the corners are removed from the conformal embedding. See appendix B.1 for further details.

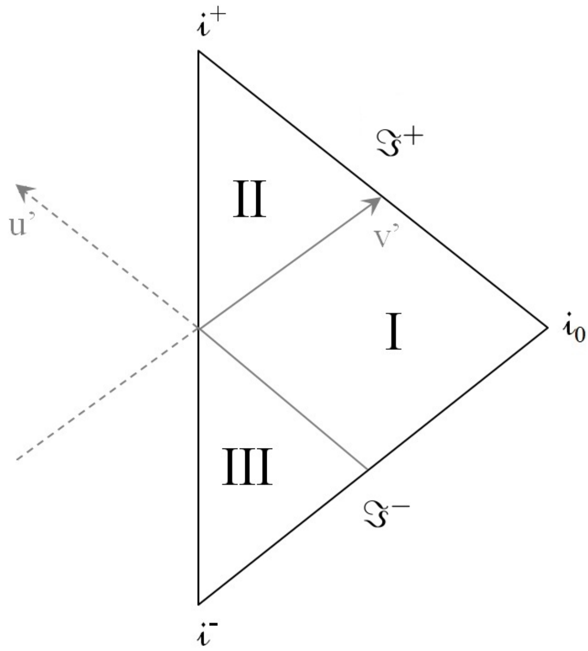


Figure 4.1: Penrose diagram for Minkowski space

Let one now return to the non-compact coordinate system (4.2). We are interested in looking at the region near $\Im^+$ in two particular charts. We introduce Rindler-type coordinates (z,t) defined as: $\big(z\,e^t := u\ ,\ z^{-1}e^t := v\big)$ such that:

$$ds^2_{\mathrm{II}} = \frac{e^{2t}}{z^2}\left(dz^2 - z^2dt^2 + \left(\frac{1-z^2}{2}\right)^2 d\Omega_d^2\right)\,. \tag{4.4}$$

This coordinate system covers only region II of Minkowski space, where $v,u>0$. Notice that, due to the coordinate singularities at $z=\pm1,0$, these coordinates are only defined in the interval: $z\in]0,1[$. In order to cover region I (only) we analytically continue t and z to complex values: $(z\to iz\,,t\to t-i\pi/2)$, which is equivalent to defining: $\big(z\,e^t := u\ ,\ z^{-1}e^t := -v\big)$ in the original coordinate system (4.2):

$$ds^2_{\mathrm{I}} = \frac{e^{2t}}{z^2}\left(-\,dz^2 + z^2dt^2 + \left(\frac{1+z^2}{2}\right)^2 d\Omega_d^2\right)\,. \tag{4.5}$$

In this case, the coordinate z is defined in the interval $z\in]-\infty,0[$. The Penrose diagram in each region in the new coordinates is given in figure 4.2, with z_0 and z_1 a positive and a negative constant, respectively. It is relevant to notice that worldlines in region II orthogonal to the surfaces of constant t, *i.e.* with tangent

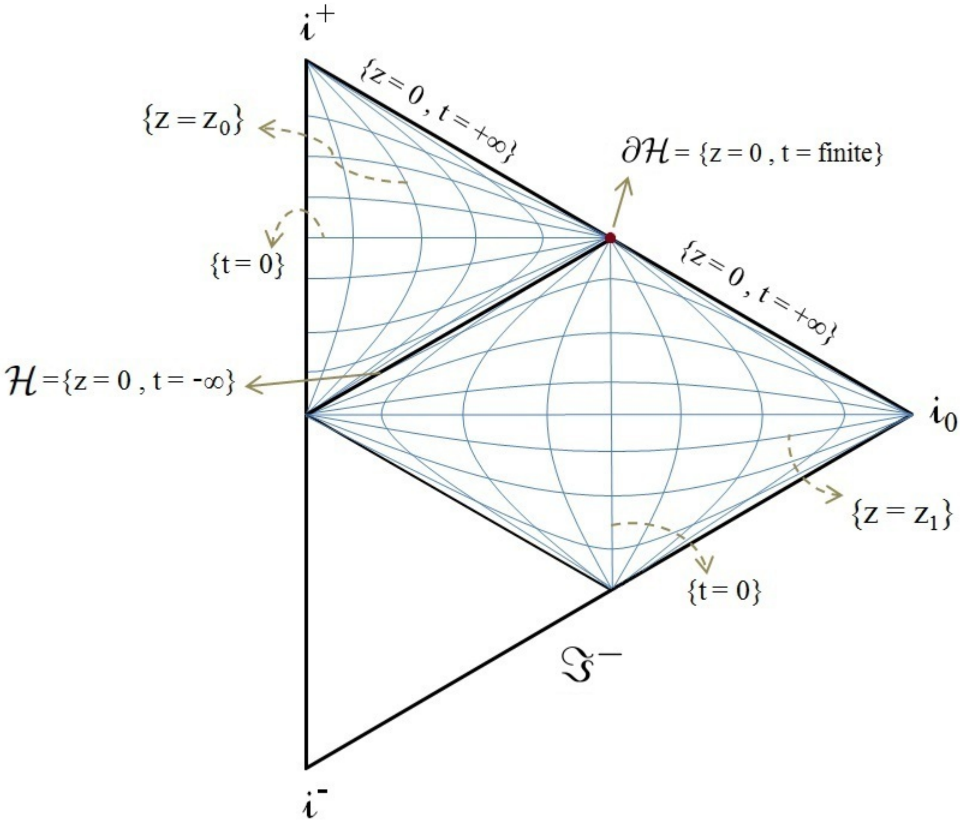


Figure 4.2: Foliation of Minkowski space

vector the unit normal $n = e^{-t}\partial_t$ to such surfaces (also called Eulerian worldlines), define geodesic observers. In Minkowski coordinates (4.1):

$$n \;=\; \frac{1}{\sqrt{1-V^2}}\,\Big(\partial_T + V\,\partial_r\Big) \;:\; V = dr/dT \;, \tag{4.6}$$

which is the standard velocity of inertial particles in radial motion, where the constant relative velocity V is related to the (constant) coordinate z as: $V = \frac{1-z^2}{1+z^2}$. This feature will be revisited later in section 4.4.

Future null infinity in each region is given by: $\Im^+ = \{z = 0 : t \neq -\infty\}$ and the defining function in the new coordinates becomes:

$$\rho(x)^2 = \frac{z^2}{e^{2t}}\,\left(1 + 2\,z^2\cosh(2t) + z^4\right) \;. \tag{4.7}$$

This implies in particular that the conformal embedding for region II approaches Rindler space times S^d as $z \to 0$:

$$d\tilde{s}^2_{\mathrm{II}} \sim dz^2 - z^2 dt^2 + \frac{1}{4} d\Omega^2_d \;, \tag{4.8}$$

with Rindler horizon $\{z = 0\}$ represented by the union $\Im^+ \cup \mathcal{H}$ in region II with bifurcation point $\partial\mathcal{H} \subset \Im^+$. The null surface $\mathcal{H} = \{u = 0\,, 0 < v < \infty\}$ $= \{z = 0\,, t = -\infty : 0 < ze^{-t} < \infty\}$ represents the boundary of the past domain of dependence of any partial Cauchy surface in region II and is therefore the past Cauchy horizon of this region. This horizon defines the future lightcone of Minkowski space with respect to an inertial observer at the origin $\{r = 0, T = 0\}$.

4.2.2. Holographic foliation and generalisation

As we move into region II from region I, the timelike surfaces of constant t asymptote to $\mathcal{H}$ and become spacelike as we cross it. These surfaces are hyperbolic manifolds ($\mathbb{H}_{d+1}$) in region II and de Sitter in region I, as well as conformally compact. The (future) conformal boundary of each such surface converges to a common region $\partial\mathcal{H} = \{u = 0\,, v = +\infty\} = \{z = 0\, : z\,e^t = 0 = z\,e^{-t}\}$ representing the boundary of $\mathcal{H}$. Notice that the coordinate t degenerates on $\partial\mathcal{H}$ where it can assume any value. According to the AdS/CFT correspondence, each $\mathbb{H}_{d+1}$ surface admits a dual description in terms of a d–dimensional Euclidean conformal field theory on its conformal boundary at $\partial\mathcal{H}$ and in particular, fields on each surface (including the induced metric) can be reconstructed out of CFT data at $\partial\mathcal{H}$. Given a family of CFTs at this boundary, one is therefore able to reconstruct a collection of hyperbolic hypersurfaces and their fields, but not necessarily able to reconstruct fields in the flat spacetime foliated by such surfaces. The AdS/CFT dictionary allows one to reconstruct from CFT data the evolution along the z-direction of the pullback of bulk fields to each slice, but it does not determine the evolution of bulk fields along the time direction orthogonal to the slices. In order to obtain a dual description of such behaviour, one needs to find asymptotically (near $\partial\mathcal{H}$) the most general time evolution of a given class of bulk fields and then to determine how the bulk data necessary to the reconstruction of such evolution is mapped to data in the family of CFTs.

Our main goal will be to obtain such asymptotics for the spacetime metric and to show that it is possible to reconstruct its evolution near $\partial\mathcal{H}$ from data belonging to a family of conformal field theories at this boundary, in particular from the conformal structure at $\partial\mathcal{H}$ and from the expectation values of the stress tensors of each field theory. We will also find in the case of even dimensions that the bulk time coordinate t defining the leaves of the foliation essentially plays the role of the central charges of the CFTs. This feature is not at all surprising: each leaf of the foliation dually described by a unique CFT is uniquely defined by a hypersurface condition $\{t = constant\}$ and the time dependence of the metric (in our gauge (4.4) the factor e^t) represents, on each slice, the AdS radius ℓ of the hyperbolic hypersurface, which from AdS/CFT is mapped to the central charges of the respective dual field theory [14, 124, 175].

In order to reconstruct holographically the spacetime metric, we need to generalise the procedure developed in the previous section to a larger class of spacetimes. We therefore generalise Minkowski space by any conformally compact, asymptotically Ricci-flat manifold that admits an asymptotically hyperbolic hypersurface of constant mean curvature. In section 4.5 we will analyse how the mean curvature condition may be relaxed. Since the conformal boundary of such manifolds is necessarily null, the hypersurface must extend to null infinity (as opposed to a spatial infinity). Such hypersurfaces are called hyperboloidal (see *e.g.* [191, 192, 193, 194]) and represent the natural generalisation of the hyperbolic leaves in the previous foliation of Minkowski space. Our starting point will be such initial hypersurface and we will then generate the Ricci-flat embedding near the hypersurface by time evolving it as follows.

If a Ricci-flat space admits a spacelike hypersurface, then the Ricci-flat neighbourhood of the surface can be identified with its "time evolution" in the ADM sense [195], *i.e.* the vacuum Einstein equations in this region are completely equivalent to the (gauge-fixed) Gauss-Codazzi equations for the surface, also called ADM, or initial value equations. The solution to the latter equations represents the time evolution of the induced metric of the hypersurface. Given such solution, together with the lapse function and the shift vector, one can then construct the most general metric for the Ricci-flat embedding near the surface as explained in the next section. The embedding will then be foliated by the different instances of the time-evolved hypersurface.

If the initial surface Σ is in particular asymptotically hyperbolic of constant mean curvature, then it is possible to solve asymptotically the Gauss-Codazzi equations and therefore to obtain the most general asymptotics of such Ricci-flat embeddings. Furthermore, since Σ extends to null infinity $\Im^+$ by definition, the region $\partial\mathcal{H}$ in our previous case of Minkowski space will now represent that where Σ intersects $\Im^+$ and we will then verify that the boundary of each time slice converges to $\partial\mathcal{H}$. Moreover, $\partial\mathcal{H}$ also represents the intersection of some null surface with null infinity. Such null surface will then be the past Cauchy horizon of the (generated) embedding and represents the generalisation of the null horizon $\mathcal{H}$ that we found in the case of Minkowski space (see figure 4.3. Notice that the region to the right of $\mathcal{H}$ contains future-inextendible causal curves that do not intersect Σ).

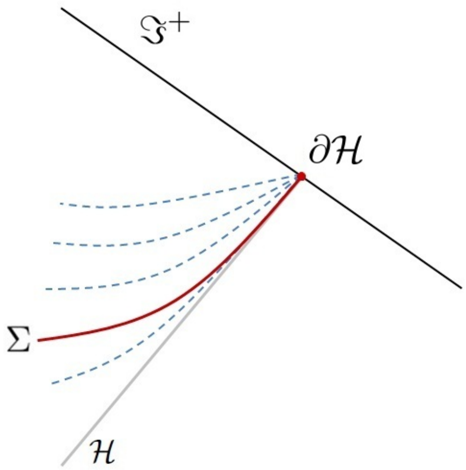


Figure 4.3: Ricci-flat embedding near the initial hypersurface Σ. The dashed lines represent the different time slices obtained by time evolving Σ and the generated embedding represents the region between some initial and final slices (which can be taken to be infinitesimally close to $\mathcal{H}$ and to $\Im^+$, respectively).

4.2.3. Exterior of the future lightcone

So far we have restricted our attention to a generalisation of region II of Minkowski space, the future domain of dependence of $\mathcal{H}$. Region I represents the exterior of the lightcone in which the leaves of the foliation are de Sitter hypersurfaces with future conformal boundary converging to $\partial\mathcal{H}$, whereas region III is the interior of the past lightcone with a hyperbolic foliation as in region II. In the particular case of Minkowski space, all the results obtained for region II may be extended to region I or III by analytic continuation: $(z \to iz\,, t \to t - i\pi/2)$, or radial and time reversal: $(z \to -z\,, t \to -t)$, respectively. In this chapter we will mainly restrict our attention to a single Ricci-flat region generated by time evolving an initial Cauchy surface as explained above and which will be realised in the next section. The generalisation of regions I and III can then be obtained from such solution by the above continuations.[5]

In the next section we briefly review the initial value formulation of relativity which will allow us to generate the Ricci-flat embeddings of initial data hypersurfaces. We will then use the existence and uniqueness properties of solutions to the Cauchy problem in order to obtain the spacetime asymptotics.

4.3. Spacetime asymptotics

4.3.1. Initial value formulation

Since we are only interested in (asymptotically) Ricci-flat embeddings, for the sake of simplicity we will focus on the vacuum version of the initial value formulation. A recent review of the subject can be found in [196].

An initial data set is a triple $\left(\Sigma, h_{[0]}, K_{[0]}\right)$ consisting of a Riemannian manifold Σ with a positive-definite metric $h_{[0]ab}$ and a symmetric tensor field $K_{[0]ab}$ satisfying the *constraint equations*:

$$R[h_{[0]}] + (K^a_{[0]a})^2 - K_{[0]ab}K^{ab}_{[0]} \quad = \quad 0 \quad , \tag{4.9}$$

$$^{[0]}D_b K^b_{[0]a} - {}^{[0]}D_a K^b_{[0]b} \quad = \quad 0 \quad , \tag{4.10}$$

where $R_{ab}[h_{[0]}]$ and $^{[0]}D_a$ are the Ricci tensor and the covariant derivative associated to $h_{[0]}$. Given an initial data set, one introduces a flow parameter t called the development time and evolves the initial data in time by specifying a scalar

[5] It should be emphasized, however, that the generalisation of region I would involve generating a Ricci-flat region by evolving a timelike surface along a spatial direction. While the above analytic continuation should result in the appropriate generalisation of region I, it must be emphasized that the initial data surface is not Cauchy in such case. This implies that the uniqueness property of solutions to the initial value problem in the case of Cauchy initial data surfaces proved by Choquet-Bruhat and reviewed in the next section do not necessarily carry over to this case, hence the unique spacetime asymptotics that we will later obtain may no longer be unique when analytically continued. See also appendix A of [78] on the analytic continuation of solutions in the context of AdS to dS/CFT.

and vector fields N and A^a and solving the first order differential equations:

$$\partial_t h_{ab} = \pounds_A h_{ab} + 2NK_{ab} \quad , \tag{4.11}$$

$$\partial_t K_{ab} = \pounds_A K_{ab} + D_a D_b N - N\Big(R_{ab}[h] - 2K^c_{\ a} K_{cb} + KK_{ab}\Big) \ , \tag{4.12}$$

subject to the initial value conditions:

$$h_{ab}\Big|_{t=0} = h_{[0]ab} \quad , \quad K_{ab}\Big|_{t=0} = K_{[0]ab} \ , \tag{4.13}$$

with D_a the covariant derivative with respect to h_{ab}. After a solution is found, one constructs the metric tensor $G_{\mu\nu}$ of a Lorentzian manifold $\mathcal{M} = I \times \Sigma$, where $I \in \mathbb{R}$ is the interval over which the time evolution is carried, according to the formula:

$$\begin{aligned} ds^2 &= G_{\mu\nu} dx^\mu dx^\nu \\ &= -N^2 dt^2 + h_{ab}\left(dx^a + A^a dt\right)\left(dx^b + A^b dt\right) \ . \end{aligned} \tag{4.14}$$

The manifold $(\mathcal{M}, G)$ is called the development of the Cauchy surface Σ and will satisfy the vacuum Einstein equations. Notice that every Ricci-flat space can be generated in this way because the equations (4.9)–(4.13) simply represent the Gauss-Codazzi identities[6] for a Ricci-flat space cast as a Cauchy problem. The development represents an embedding of the initial Cauchy surface $\Sigma = \{t = 0\}$ and is foliated by a one-parameter family of spacelike surfaces Σ_t representing the time evolution of Σ, each defined by the condition $t = \textit{constant}$ with future-directed unit normal $n = -Ndt$. The induced metric and extrinsic curvature of such surfaces are given by:

$$h_{\mu\nu} = G_{\mu a} G_{\nu b} h^{ab} = G_{\mu\nu} + n_\mu n_\nu \quad , \quad K_{\mu\nu} = G_{\mu a} G_{\nu b} K^{ab} \ , \tag{4.15}$$

where $K^{ab} = (h^{-1} K h^{-1})^{ab}$. Finally, a choice of *lapse function* N and *shift vector* A^a as above is called a gauge choice and it represents how one chooses to foliate $\mathcal{M}$ and how to propagate the coordinate system of Σ in $\mathcal{M}$. In other words, it represents a choice of coordinates for $\mathcal{M}$ near Σ and one is free to choose a lapse and a shift without changing the physical spacetime in such neighbourhood.

Choquet-Bruhat showed [197] that there always exists a solution to the initial value problem for smooth initial data and that such solution is unique in a neighbourhood of the initial Cauchy surface. Choquet-Bruhat and Geroch [198] also

[6] By virtue of the Bianchi identities, the constraint equations have the fundamental property that they hold for all t if they hold at a given t. In the conventional approach to relativity, equation (4.11) represents the definition of extrinsic curvature $K_{\mu\nu} = 1/2\pounds_n h_{\mu\nu}$ projected onto the surfaces of constant t.

showed global existence and uniqueness of a maximal development, the element in the set of solutions into which every other solution can be isometrically mapped. See the review in [195]. These properties are the main reason behind our choice of approach to finding the spacetime asymptotics. We will make use of the existence and uniqueness of a solution to the Cauchy problem in order to deduce the most general asymptotics of the developments of initial data sets that asymptote to hyperboloidal sets. The latter are defined as follows.

An asymptotically hyperboloidal initial data set [191, 199, 200, 201] (see also [202, 203]) is defined as any initial data set $(\Sigma, h_{[0]}, K_{[0]})$ such that $(\Sigma, h_{[0]})$ is a conformally compact, asymptotically hyperbolic manifold[7] and $K_{[0]}$ asymptotically covariantly conserved.

This last condition is equivalent to $K_{[0]}$ being asymptotically equal to $h_{[0]}$ up to a proportionality constant. This can be proved as follows (we work in a sufficiently small neighbourhood of the conformal boundary). If $K_{[0]}$ is equal to $h_{[0]}$, then it is covariantly conserved. Reciprocally, if $K_{[0]}$ is covariantly conserved, then it follows from the diffeomorphism constraint equation (4.10) that the trace of $K_{[0]}$ is constant. Now, since Σ is Einstein with negative scalar curvature, its Ricci tensor is proportional to the metric:

$$R_{ab}[h_{[0]}] = -\frac{d}{\ell^2}\, h_{[0]ab} \ , \tag{4.16}$$

with $d+1$ the dimension of Σ and ℓ a real constant. If the trace of $K_{[0]}$ vanishes, then the Hamiltonian constraint equation (4.9) cannot be satisfied because $K_{[0]ab}K_{[0]}^{ab} \geq 0$ ($h_{[0]}$ is positive definite and $K_{[0]}$ is symmetric and real, see below). Hence, the trace of $K_{[0]}$ must be a non-zero constant and it can then always be normalised (*e.g.* by rescaling the metric) such that:

$$\mathrm{Tr}[h_{[0]}^{-1}K_{[0]}] = \pm\frac{d+1}{\ell} \ . \tag{4.17}$$

If we then decompose $K_{[0]}$ in terms of the shear tensor σ_{ab} and the mean curvature:

$$K_{[0]ab} = \sigma_{ab} + \frac{1}{d+1} h_{[0]ab}\mathrm{Tr}[h_{[0]}^{-1}K_{[0]}] \ , \tag{4.18}$$

it then follows from the Hamiltonian constraint equation (4.9) that: $\sigma_{ab}\sigma^{ab} = 0$. Now, since $h_{[0]ab}$ is positive definite and σ_{ab} Hermitian, then there exists an invertible matrix Q such that (see *e.g.* [204]):

$$Q^\dagger h_{[0]} Q = \mathbb{1} \quad , \quad Q^\dagger \sigma Q = D \ , \tag{4.19}$$

[7] See appendix B.1 about the equivalence between such manifolds and conformally compact, asymptotically Einstein Riemannian manifolds of negative scalar curvature.

with $D = \text{diag}(\lambda_1, .., \lambda_{d+1}) : \lambda_i \in \mathbb{R}$ and where $\dagger$ denotes the Hermitian conjugate. In this way:

$$0 = \text{Tr}[h_{[0]}^{-1}\sigma h_{[0]}^{-1}\sigma] = \text{Tr}[QQ^\dagger\,(Q^\dagger)^{-1}DQ^{-1}\,QQ^\dagger\,(Q^\dagger)^{-1}DQ^{-1}] = \text{Tr}[DD] = \sum \lambda_i^2\ . \tag{4.20}$$

Hence: $D = 0$, which implies that σ_{ab} vanishes and therefore, by (4.18), that $K_{[0]}$ is equal to $h_{[0]}$ up to a proportionality constant.[8]

Since $K_{[0]}$ represents the extrinsic curvature of the initial data surface $(\Sigma, h_{[0]})$ in the embedding, and as remarked above the diffeomorphism constraint implies that Σ is of constant mean curvature, then the developments of asymptotically hyperboloidal sets are the asymptotically Ricci-flat spacetimes that we have introduced in the previous section.[9] In this way, by finding the unique solution to the initial value problem with such initial data sets, we are able to find the asymptotics of such embeddings. We will begin by developing exact hyperboloidal sets and in section 4.5 allow deviations of $K_{[0]}$ away from its asymptotic value by considering arbitrary subleading contributions.

4.3.2. Ricci-flat asymptotics

Let $(\Sigma, h_{[0]}, K_{[0]})$ be an asymptotically hyperboloidal initial data set as introduced in the previous section, with $(\Sigma, h_{[0]})$ of dimension $d+1$ and normalised such that: $R_{ab}[h_{[0]}] = -d\,h_{[0]ab}$ and: $K_{[0]} = h_{[0]}$ in a sufficiently small neighbourhood of the conformal boundary of Σ. It was found in [205, 190], see also [123], that coordinates can be found in which the most general asymptotics of this initial Cauchy surface takes the form:

$$ds^2_{d+1} \;=\; h_{[0]ab}dx^a dx^b \;\sim\; \frac{1}{z^2}\left(dz^2 + g_{ij}dx^i dx^j\right)\ , \tag{4.21}$$

[8]The vanishing of the shear can also be seen by finding coordinates at a given point such that $h_{[0]}$ is locally the Euclidean metric. Then, since σ_{ab} is real and symmetric, it can be diagonalised at each point by an orthogonal matrix. Replacing both conditions in the trace equation $\sigma_{ab}\sigma^{ab} = 0$ implies $\sigma_{ab} = 0$ locally and hence everywhere since the equation is tensorial.

[9]There is a technical point regarding conformal compactness. If the embedding of some Cauchy surface is conformally compact, then it is possible to show that the surface is also conformally compact (see *e.g.* section 2 of [194]). The reciprocal, however, is not necessarily true. In the above, we have only demanded that the initial data surface be conformally compact, whereas in the previous section we required conformal compactness of the Ricci-flat embedding. This means that, after finding the solution to our initial value problem, we will have to verify that the embedding is indeed conformally compact.

with the conformal boundary $\partial\Sigma = \{z = 0\}$ and where $g_{ij}(z, x)$ is given asymptotically by the power series:

$$g_{ij}(z,x) \;=\; g_{(0)ij}(x) + z^2 g_{(2)ij}(x) + ... + z^d g_{(d)ij}(x) + z^d \log z \, \tilde{g}_{(d)ij}(x) + \mathcal{O}(z^{>d}) \;, \tag{4.22}$$

where only even powers of z arise below the order z^d. The *non-normalisable mode* $g_{(0)}$ is an arbitrary field and each coefficient $g_{(n<d)}$, as well as $\tilde{g}_{(d)}$, is a local functional of $g_{(0)}$. For odd values of d, or for $d = 2$, the logarithmic term vanishes and for even values it is traceless and divergenceless (with respect to $g_{(0)}$). The *normalisable mode* $g_{(d)}$ is undetermined up to its trace and divergence, which vanish for odd values of d and are local functionals of $g_{(0)}$ for even values. See [123] for the explicit expressions of these functionals.

In order to evolve this initial data in time and generate a Ricci-flat development we need to prescribe a lapse function and a shift vector and we do so by choosing the geodesic normal gauge: $(N = 1, A^a = 0)$, also called synchronous gauge, or Gaussian normal coordinates.[10] In this gauge and sufficiently close to the Cauchy surface $\Sigma = \{\hat{t} = 0\}$, the metric tensor (4.14) of the development $\mathcal{M} = I \times \Sigma$ takes the form:

$$ds^2_{d+2} = -d\hat{t}^{\,2} + h_{ab} dx^a dx^b \;. \tag{4.23}$$

Since the constraint equations are trivially satisfied by the initial data, the only equation left to solve is the dynamical equation obtained by replacing equation (4.11) in (4.12):

$$2R_{ab}[h] + \ddot{h}_{ab} + \frac{1}{2}\dot{h}_{ab}\mathrm{Tr}[h^{-1}\dot{h}] - \left(\dot{h}h^{-1}\dot{h}\right)_{ab} = 0 \;, \tag{4.24}$$

subject to the initial value conditions (4.13) and where $\dot{h} := \partial_{\hat{t}}\, h$. The unique solution to this initial value problem is now very simple to find. One can easily begin by verifying that the ansatz:

$$ds^2_{d+2} = -d\hat{t}^{\,2} + h_{ab}(\hat{t}, x) dx^a dx^b = -d\hat{t}^{\,2} + \left(1 + \hat{t}\,\right)^2 h_{[0]ab}(x) dx^a dx^b \;, \tag{4.25}$$

is a solution to the dynamical equation, which is simply the well-known result that the Lorentzian cone of an Einstein space of negative curvature is Ricci-flat. Furthermore, since the extrinsic curvature on the surfaces of constant $\hat{t}$ is given by: $K_{ab} = \frac{1}{2}\partial_{\hat{t}}\, h_{ab}$, one finds that:

$$h_{ab}\big|_{\hat{t}=0} = h_{[0]ab} \quad , \quad K_{ab}\big|_{\hat{t}=0} = h_{[0]ab} \;. \tag{4.26}$$

[10] Recall that any metric can be written in this gauge in a sufficiently small neighbourhood of a (non-null) hypersurface. Regarding the notation from the previous section, we will relabel our development time $t \to \hat{t}$.

The metric (4.25) is therefore a solution to this initial value problem and hence the unique solution. By performing the transformation of coordinates: $e^t := 1+\hat{t}$, our solution becomes:

$$ds^2_{d+2} = e^{2t}\left(-dt^2 + h_{[0]ab}dx^a dx^b\right) \ , \tag{4.27}$$

where the Cauchy surface $\Sigma = \{t = 0\}$. Let us now denote by $\partial\mathcal{H}$ the region in the development[11] described by the conformal boundary $\partial\Sigma$ under time evolution over the interval $I \in \mathbb{R}$ (*i.e.* the portion of null infinity foliated by the leaves $t = constant$). Above we have found that, near $\partial\Sigma = \{z = 0\}$, $h_{[0]}$ takes the form (4.21). In this way, the development (4.27) in a neighbourhood of $\partial\mathcal{H}$ takes the asymptotic form:[12]

$$ds^2_{d+2} = \frac{e^{2t}}{z^2}\left(dz^2 - z^2 dt^2 + g_{ij}dx^i dx^j\right) \ , \tag{4.28}$$

with $g_{ij}(z,x)$ given asymptotically by the expansion (4.22). In order to verify that the slices of constant time converge to $\partial\Sigma$, and therefore that $\partial\mathcal{H}$ coincides with $\partial\Sigma$, we bring in future null infinity to finite affine parameter distances by a suitable conformal compactification. We define coordinates ($u := z\,e^t$, $\rho := z\,e^{-t}$) such that:

$$ds^2_{d+2} \ = \ \frac{1}{\rho^2}\left(d\rho du + g_{ij}dx^i dx^j\right) \ , \tag{4.29}$$

where:[13] $g_{ij} = g_{(0)ij} + \rho\,u\,g_{(2)ij} + \mathcal{O}(\rho^2)$. Since $\tilde{G}_{\mu\nu} := \rho^2 G_{\mu\nu}$ is at least C^2 (or C^1 for $d = 3$) and non-degenerate, then $(\tilde{\mathcal{M}}, \tilde{G})$ defines a conformal compactification with conformal boundary $\{\rho = 0\}$. The Penrose diagram near the boundary with the spatial coordinates x^i suppressed (note that $g_{(0)} = g_{(0)}(x^i)$) is then given by figure 4.4.

Returning to our asymptotic solution (4.28), if the Ricci-flat development is in particular Minkowski space as in (4.4), we have the expansion:

$$g_{ij} = g_{(0)ij} + z^2 g_{(2)ij} + z^4 g_{(4)ij} \ : \ \begin{cases} g_{(0)ij}dx^i dx^j = \frac{1}{4}d\Omega^2_d \ , \\ g_{(2)ij} = -2g_{(0)ij} \ , \\ g_{(4)ij} = g_{(0)ij} \ . \end{cases} \tag{4.30}$$

[11] Not strictly *in* the development, but in some appropriate conformal embedding.

[12] In other words, this is the solution we would have found had we time evolved directly the asymptotic metric (4.21).

[13] For $d = 3$, we have instead $\mathcal{O}(\rho^{3/2})$. Recall that the logarithmic term in (4.22) vanishes for $d = 2, 3$.

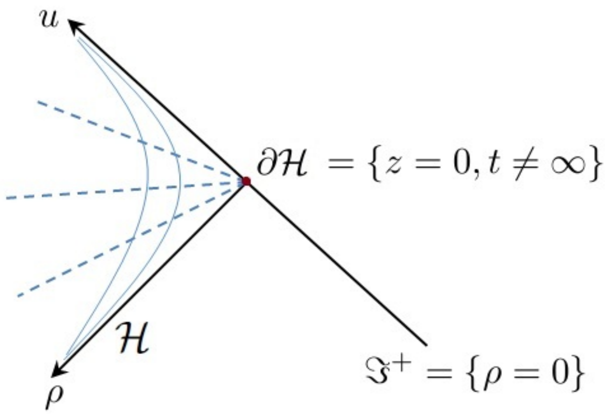


Figure 4.4: Conformal embedding near future null infinity. The dashed lines represent different surfaces of constant t, while the solid lines are surfaces of constant z. The null surface $\mathcal{H} = \{z = 0, t = -\infty\}$ is the past Cauchy horizon of the development.

More generally, in Minkowski space g_{ij} can be expanded as:[14]

$$g_{ij}(z,x) = g_{(0)ij} + z^2 g_{(2)ij} + z^4 g_{(4)ij} \quad : \quad \begin{cases} g_{(2)ij} = -\frac{1}{d-2}\left(R_{ij}[g_{(0)}] - \frac{1}{2(d-1)} g_{(0)ij} R[g_{(0)}]\right) \ , \\ g_{(4)ij} = \frac{1}{4}\left(g_{(2)} g_{(0)}^{-1} g_{(2)}\right)_{ij} \ , \end{cases} \tag{4.31}$$

where $g_{(0)}$ is any conformally flat metric. This can be easily seen by recalling that Minkowski space $\mathbb{R}^{1,d+1}$ is the Lorentzian cone of the hyperbolic space $\mathbb{H}_{d+1}$. It was found in [206] that $\mathbb{H}_{d+1}$ in Poincaré coordinates is given by:

$$ds^2_{d+1} = \frac{1}{z^2}\left(dz^2 + g_{ij} dx^i dx^j\right) \ , \tag{4.32}$$

where g_{ij} is given by the expansion (4.31) with $g_{(0)}$ any conformally flat metric. Hence, the respective cone is Minkowski. This implies in particular that the solution $g_{ij} = \delta_{ij}$ in (4.28) is also Minkowski and the direct transformation of coordinates is given by:

$$X_0 := \frac{e^t}{2}\left(z + \frac{1+\vec{x}^2}{z}\right) \ , \ Z := \frac{e^t}{2}\left(z + \frac{-1+\vec{x}^2}{z}\right) \ , \ X^i := \frac{x^i}{z} e^t \ , \tag{4.33}$$

[14]For the special case $d = 2$ we have that $g_{(2)} - g_{(0)}\mathrm{Tr}[g_{(0)}^{-1} g_{(2)}]$ is the stress tensor of the Liouville field. See [206] for further details.

such that:

$$
\begin{aligned}
ds^2_{d+2} &= \frac{e^{2t}}{z^2}\left(dz^2 - z^2dt^2 + dx^i dx^i\right) \\
&= -dX_0^2 + dZ^2 + dX^i dX^i \ .
\end{aligned}
\tag{4.34}
$$

It is not difficult to show that the Penrose diagram for (4.34) corresponds to that of region II in figure 4.2 with the surfaces $t = constant$ and $z = constant$ still as in this diagram. Although the spacetimes (4.4) and (4.34) are both Minkowski and therefore diffeomorphic, we will find in the next section that they do not yield the same expectation values for the holographic stress tensors. These are computed from the (renormalized) gravitational action and vanish for the latter solution with future null infinity $\mathbb{R}\times\mathbb{R}^d$, whereas are non-vanishing for the former with null infinity $\mathbb{R}\times S^d$ in the same fashion as in AdS holography [123]. This is associated to the fact that the holographic renormalization scheme that we will employ later breaks invariance of the gravitational action with respect to bulk diffeomorphisms[15] that result in a conformal transformation at $\partial\mathcal{H}$ and therefore spacetimes related by such transformations, such as (4.4) and (4.34), will not necessarily result in the same renormalized expectation values as pointed out in [123].

The asymptotics (4.28) for our class of Ricci-flat spacetimes is the Lorentzian cone of $(\Sigma, h_{[0]})$ near $\partial\Sigma$ and represents the desired generalisation of region II of Minkowski space discussed in section 4.2.2 near null infinity. In section 4.2.3 we have emphasized that region I of Minkowski can be obtained from region II by the analytic continuation: $(z\to iz\,,\, t\to t-i\pi/2)$. By applying the same continuation to our solution (4.28), we obtain the Riemannian cone of an Einstein space of positive curvature and which represents our generalisation of region I near null infinity:

$$
ds^2_{d+2} = \frac{e^{2t}}{z^2}\left(-dz^2 + z^2dt^2 + g_{ij}dx^i dx^j\right) , \tag{4.35}
$$

with $g_{ij}(z,x)$ given asymptotically by the expansion (4.22) under the substitutions:[16]

$$
g_{(4n+2)ij} \to -g_{(4n+2)ij} \ : \ n\in\mathbb{N}_0 \ . \tag{4.36}
$$

[15] The bulk diffeomorphisms that preserve the asymptotic functional form (4.28) of the metric contain a subgroup that generates conformal transformations at $\partial\mathcal{H}$. The proof of this fact is sketched in section 6 of [188], where our solution represents a particular case of the asymptotic metric analysed in this reference. A conformal transformation at $\partial\mathcal{H}$ is therefore realised in the bulk as an asymptotic "isometry". See [174] about the relation between bulk diffeomorphisms and conformal transformations at the boundary in the context of AdS/CFT.

[16] Recall that the normalisable mode $g_{(d)}$ is undetermined up to its trace and divergence and these vanish for odd values of d. In this way, $g_{(d)}$ does not suffer any transformation for d odd.

An interesting example of such Ricci-flat spacetimes is the 5-dimensional Riemannian cone of de Sitter-Schwarzschild:

$$ds^2 = d\hat{t}^2 + \left(\hat{t}/\ell\right)^2 \left(-\left(1 - \frac{r^2}{\ell^2} - \frac{2M}{r}\right) d\tau^2 + \frac{dr^2}{1 - \frac{r^2}{\ell^2} - \frac{2M}{r}} + r^2 d\Omega^2\right) \ . \quad (4.37)$$

This solution contains two curvature singularities: a spacelike singularity at $r = 0$ hidden behind a Killing horizon (for $M \leq \ell/3\sqrt{3}$) of topology $\mathbb{R}^3$, and a null and therefore trivially naked singularity at $\hat{t} = 0$. In the coordinate system (4.35), this solution reads as:[17]

$$ds^2 = \frac{e^{2t}}{z^2}\left(-dz^2 + z^2 dt^2 + \left(1 - \frac{1}{2}\frac{M}{\ell}\left(\frac{z}{\rho}\right)^3\right)^{4/3}\left(\left(\frac{1 + \frac{1}{2}\frac{M}{\ell}\left(\frac{z}{\rho}\right)^3}{1 - \frac{1}{2}\frac{M}{\ell}\left(\frac{z}{\rho}\right)^3}\right)^2 d\rho^2 + \rho^2 d\Omega^2\right)\right) \ . \quad (4.38)$$

In this case, g_{ij} admits the expansion:

$$g_{ij} = g_{(0)ij} - z^2 g_{(2)ij} + z^3 g_{(3)ij} + \mathcal{O}(z^6) \ : \ \begin{cases} g_{(0)ij}dx^i dx^j = d\rho^2 + \rho^2 d\Omega_d^2 \ , \\ g_{(2)ij} = 0 \ , \\ g_{(3)ij}dx^i dx^j = \frac{4}{3}\frac{M}{\ell}\rho^{-3}\left(d\rho^2 - \frac{1}{2}\rho^2 d\Omega^2\right) \ . \end{cases} \quad (4.39)$$

In section 4.4.3 we will use the above expansions of g_{ij} for Minkowski and the cone of dS_4–Schwarzschild as an exercise to compute the holographic stress tensors of the respective dual theories.

So far we have found a coordinate system in which the most general asymptotics of our class of asymptotically Ricci-flat spacetimes assumes the form (4.28). It is now simple to see that our solution is diffeomorphic to the ambient metric of Fefferman and Graham [190] by recalling that the ambient construction represents the Lorentzian cone of an Einstein Riemannian manifold in coordinates adapted to the study of the past Cauchy horizon [203]. Indeed, by introducing coordinates $\left(r := z^2, v := z^{-1}e^t\right)$, our solution assumes the form:

$$ds^2_{d+2} = -r dv^2 - v dv dr + v^2 g_{ij} dx^i dx^j \ , \quad (4.40)$$

which represents the ambient metric with $g_{ij}(r,x)$ expanded as in (4.22) with $z = \sqrt{r}$. For v finite, this represents an expansion away from the Cauchy horizon

[17] The transformation of coordinates is given by: $e^t = \hat{t}$, $-\ell\log(z/\ell) = \tau - \int dr \frac{A(r)}{A(r)^2-1}$, $\ell\log(\rho/\ell) = -t - \frac{2}{3}\ell\log 2 + \int dr \frac{A(r)^3}{A(r)^2-1}$, where: $A(r)^{-2} = r^2/\ell^2 + 2M/r$. Notice in particular that: $r/\ell = B_-^{2/3}\rho/z$ and also that: $r^2/\ell^2 + 2M/r = B_-^{4/3}(B_+/B_-)^2\rho^2/z^2$, where: $B_\pm = 1 \pm \frac{1}{2}(M/\ell)\, z^3/\rho^3$. The curvature singularities are now given by: $e^t B_- = 0$.

$\mathcal{H} = \{z = 0, t = -\infty\} = \{r = 0, v \neq \infty\}$, whereas for $v = \infty$ it is an expansion away from $\partial\mathcal{H} = \{z = 0, t \neq \infty\} = \{r = 0, v = +\infty\}$. The coordinate system (4.40) is therefore well-adapted to the study of the former region, but unsuited to the study of the latter and the other way around with respect to (4.28). Since we are rather interested in the spacetime asymptotics, a correct choice of coordinates in our case is given by our original solution (4.28).[18] As a matter of fact, rather than relying on a particular time coordinate, any coordinate system of the form:

$$ds^2_{d+2} = -N(t)^2 dt^2 + \frac{\beta(t)^2}{z^2}\left(dz^2 + g_{ij}dx^i dx^j\right) \quad : \quad \beta(t) := \int dt N(t) \ , \ \forall N(t) \neq 0 \ , \tag{4.41}$$

with $N(t)$ smooth, is suited to the study of the spacetime asymptotics and is related to our previous coordinates by: $e^t \to \int dt N(t)$. We will find more useful to keep in this way the lapse function $N(t)$ arbitrary. In particular, we can now notice that the scalar field $\beta(t)$, which will play a central role in the remainder of this chapter, is a gauge invariant quantity, the coordinate invariant part of the lapse, and measures propertime distances along the so-called Eulerian worldlines, the timelike curves with tangent vector the future-directed unit normal $n = -Ndt = N^{-1}\partial_t$ to the constant time slices.[19] Indeed, the line element for such curves reduces to:

$$ds^2_{d+2}\Big|_{z,x^i=const.} = -d\beta(t)^2 + \frac{\beta(t)^2}{z^2}\left(dz^2 + g_{ij}dx^i dx^j\right)\Big|_{z,x^i=const.} = -d\beta(t)^2 \ . \tag{4.42}$$

For later use in the holographic renormalization of the action, it is also useful to rewrite such relation in the form:

$$1 = \dot{\beta}/N = n^\mu \partial_\mu \beta = -\partial^\mu \beta \, \partial_\mu \beta \ . \tag{4.43}$$

These Eulerian observers can in turn be defined as those for whom our constant time hypersurfaces represent locally the set of events that are simultaneous.[20] In our case, such Eulerian worldlines are in fact geodesics with affine parameter β. Indeed, the acceleration of such worldlines is given by [196]:

$$a^\mu = n \cdot \nabla n^\mu = h^{\mu\nu}\partial_\nu \log N(t) = 0 \ , \tag{4.44}$$

[18] Furthermore, notice that the metric (4.28), or (4.40), represents the most general spacetime asymptotics near the boundary $\partial\mathcal{H}$ of $\mathcal{H}$, but not necessarily the most general Ricci-flat metric near the entire $\mathcal{H}$ unless we restrict further our class of spacetimes to those in which the metric in a neighbourhood of $\mathcal{H}$ is identically equal to its asymptotic form. Such spacetimes are defined by the ambient metric defining conditions 1)–3) in section 2 of [190] (see also the conditions a)–d) in Problem 5.1 of this reference).

[19] For the interpretation of β as a thermodynamic variable, see [207, 208] and references therein.

[20] See also section 3.3 of [196] for further details.

where $h^{\mu\nu} = G^{\mu\nu} + n^\mu n^\nu$ is the induced metric of the surfaces of constant t and hence its contraction with the gradient of the lapse vanishes since the latter is a pure function of the time coordinate. In the case of Minkowski space, such geodesics coincide with those found in (4.6) defining inertial particles in flat space. The scalar $\beta(t)$ therefore measures the invariant distance between points on different time slices connected by geodesics orthogonal to the slices.

4.4. Holographic reconstruction of spacetime

4.4.1. Outline

In the previous section we obtained our spacetime asymptotics in the form (4.41) with the expansion (4.22). In other words, after fixing the gauge freedom associated to a choice of coordinates, we found asymptotically the time evolution of our class of metrics. By construction, our bulk solution is foliated by conformally compact, asymptotically Einstein hypersurfaces of negative scalar curvature with a conformal boundary at $\partial\mathcal{H}$. Each such surface admits a dual description in terms of a d–dimensional Euclidean conformal field theory at $\partial\mathcal{H}$ and we would like to identify in this family of field theories the data necessary to the reconstruction of the bulk metric (4.41). In the previous section we found that, after choosing our time coordinate represented by a choice of $N(t)$, our spacetime asymptotics in the neighbourhood of the initial Cauchy surface is determined by the conformal structure of $\partial\mathcal{H}$ (the conformal class $[g_{(0)}]$) up to order z^d, excluding. Moreover, we found that it is possible to move past such order and reconstruct the bulk metric near the surface up to very high order[21] from the knowledge of the normalizable mode $g_{(d)}$. On each time slice, this mode is associated to the holographic energy tensor of the respective dual field theory on the conformal boundary [123]. As we now briefly review for convenience, this result follows from the standard AdS/CFT prescription (1.106), which identifies the supergravity partition function in asymptotically locally (Euclidean) AdS spaces with the generating functional of QFT correlation functions. The former is a functional of the boundary configurations $\phi_{(0)}$ of bulk fields and these are identified as sources for gauge-invariant operators $\mathcal{O}$ in the dual field theory. For a weakly coupled gravitational theory, one can work in a saddle-point approximation and take the gravitational on-shell action as

[21] The obstacle to the reconstruction up to all orders is associated to the Fefferman-Graham coordinates (4.21) on the surface. Once we have gauge-fixed our coordinate system on the surface, the gauge-fixing condition is valid only in a thickening near the boundary of the surface. This issue is analogous to the Gribov ambiguity in gauge theories and the absence of global gauge conditions, where gauge choices only hold in a neighbourhood of a gauge orbit.

the generating functional W of connected QFT correlation functions:

$$W[\phi_{(0)}] \;=\; \log \langle \exp\left(-\int_{\partial\mathcal{H}} \phi_{(0)}\mathcal{O}\right)\rangle_{QFT} \;=\; \log Z_{\mathrm{SUGRA}}[\phi_{(0)}] \;\sim\; -S^{\mathrm{onshell}}[\phi_{(0)}]\,, \tag{4.45}$$

where S is the gravitational action in an asymptotically (E)AdS space and $\partial\mathcal{H}$ its conformal boundary. One can then obtain in particular the expectation value and correlation functions of the QFT energy tensor by functionally differentiating S^{onshell} with respect to the induced metric at $\partial\mathcal{H}$ representing the boundary configuration of the asymptotically AdS metric tensor. From the Hamilton-Jacobi theory, it follows that the on-shell action is a Hamilton principal functional and therefore that its first variation with respect to the induced metric results in the canonical momentum of the boundary, also known as the Brown-York quasi-local energy tensor [179]. This tensor, however, in its bare form is ill-defined since both sides of (4.45) suffer from divergences. One then proceeds by introducing a regulating boundary in the asymptotically AdS space and renormalizing the gravitational action with a set of covariant counterterms [123]. Finally, since the conformal boundary is a region of the conformal embedding, one computes the Brown-York tensor of the regulating surface in the embedding from the renormalized action and in the end removes the regulator by taking the limit as the surface tends to the boundary. The renormalized Brown-York tensor obtained in this way corresponds by construction to the vacuum expectation value of the dual field theory energy tensor as computed at strong coupling from the renormalized generating functional W^{ren} of the QFT. An explicit computation then shows that such one-point function indeed coincides exactly with the normalizable mode $g_{(d)}$ for odd d and is equal to $g_{(d)}$ plus local functionals of the source $g_{(0)}$ for even values of d. See [123] for further details.

The data necessary to the reconstruction of the bulk metric therefore consists of the modes $g_{(0)}$ (or any representative of the conformal structure $[g_{(0)}]$ at $\partial\mathcal{H}$) and $g_{(d)}$. While the former should be identified as the source for each conformal field theory stress tensor, the latter should be mapped to the expectation values of the stress tensors and in this section we will consider replacing the action on the right hand side of (4.45) by the gravitational action for our class of Ricci-flat spacetimes in an attempt to confirm these entries in the holographic dictionary. By following the steps just outlined above, we will show that it is possible to reproduce the expectation values and Ward identities of the stress tensors of the field theories that reside at $\partial\mathcal{H}$ and to identify in this family the data necessary to the reconstruction of our spacetime asymptotics. Such results seem to support a non-trivial extension of the prescription (4.45) to Ricci-flat embeddings of asymptotically hyperbolic manifolds. We will begin with the holographic renormalization of the gravitational on-shell action and then deduce the renormalized Brown-York tensor at $\partial\mathcal{H}$ which should correspond, for each fixed value of t, to the expectation value of the stress tensor of each field theory.

4.4.2. Renormalization

In order to renormalize the gravitational action, we begin by considering our spacetime as the region bounded by a (regulating) timelike hypersurface $\{z = \epsilon\}$. Such regulating boundary corresponds to one of the surfaces of constant z in figure 4.4 and in the end we will take the limit $\epsilon \to 0$. In our coordinate system (4.41), we approach $\partial\mathcal{H}$ under such limit for finite, non-zero values of $\beta(t)$ (recall that $(\Im^+, \mathcal{H}) = \{z = (0,0)\,, \beta(t) = (+\infty, 0)\}$ and $\beta(t) \in]0, +\infty[$). The gravitational action is then given by:

$$2\kappa_{d+2}^2\, S \;=\; \int_{\mathcal{M}} d^{d+2}x\,\sqrt{G}\,R[G] + 2\int_{z=\epsilon} d^{d+1}x\sqrt{q}\,Q^A_A\;, \tag{4.46}$$

where: $2\kappa_{d+2}^2 = 16\pi G_N$ and where q_{AB} and Q_{AB} are the induced metric and the extrinsic curvature on the regulating boundary with coordinates $x^A = (t, x^i)$. Given a spacetime with a timelike boundary $\{z = \epsilon\}$, the Brown-York tensor, or the canonical momentum of the boundary, is obtained from the action as follows. One begins with a canonical decomposition of the spacetime metric $G_{\mu\nu}$:

$$ds_{d+2}^2 \;=\; G_{\mu\nu}dx^\mu dx^\nu \;=\; M^2 dz^2 + q_{AB}\left(dx^A + U^A dz\right)\left(dx^B + U^B dz\right)\;. \tag{4.47}$$

In our particular case we would have: $M = \beta(t)/z$ and $U^A = 0$. The extrinsic curvature $Q_{AB} = (2M)^{-1}\left(\partial_z - \pounds_U\right) q_{AB}$ and its extension to the spacetime is given in the same fashion as in (4.15). One then proceeds by rewriting the action in the canonical form using the Gauss-Codazzi identities (see *e.g.* [195]):

$$\begin{aligned}2\kappa_{d+2}^2\, S &= \int_{\mathcal{M}} d^{d+2}x\, M\sqrt{q}\Big(R[q] + Q^2 - Q\cdot Q - 2\nabla_\mu\left(m^\mu Q - a^\mu\right)\Big) + 2\int\limits_{z=\epsilon} d^{d+1}x\sqrt{q}\,Q \\ &= \int_{\mathcal{M}} d^{d+2}x\, M\sqrt{q}\Big(R[q] + Q^2 - Q\cdot Q\Big) \;=\; 2\kappa_{d+2}^2\int dz\, L[q, q', M, U]\;,\end{aligned} \tag{4.48}$$

where the acceleration vector $a^\mu = m\cdot\nabla m^\mu$ and where $m = M dz$ is the unit normal to the surfaces of constant z. Note that $a^\mu m_\mu = 0$. Also, $q' := \partial_z q$ and L is the canonical Lagrangian. We emphasize that it is the on-shell action in the canonical form that is a Hamilton principal functional and therefore a functional of the boundary configuration of the spacetime metric. Furthermore:

$$\delta G_{\mu\nu} \;=\; \Big(2\, m_\mu m_\nu/M\Big)\delta M + \Big(2\, m_{(\mu}q_{\nu)A}/M\Big)\delta U^A + \Big(q^A_\mu q^B_\nu\Big)\,\delta q_{AB}\;, \tag{4.49}$$

and hence the action $S = S[G]$ is a functional of the canonical fields M, U^A and q_{AB}. The variation of S is then given by:

$$\begin{aligned}\delta S &= \int dz \int d^{d+1}x \left(\frac{\delta L}{\delta q_{AB}}\, \delta q_{AB} + \frac{\delta L}{\delta q'_{AB}}\, \delta q'_{AB} + ... \right) \\ &= \int dz \int d^{d+1}x \left(\frac{\delta L}{\delta q_{AB}} - \frac{d}{dz} \frac{\delta L}{\delta q'_{AB}} \right) \delta q_{AB} + ... + \int\limits_{z=\epsilon} d^{d+1}x \frac{\delta L}{\delta q'_{AB}} \delta q_{AB} \ ,\end{aligned} \tag{4.50}$$

where the ellipses denote the variation of the Lagrangian with respect to the lapse and shift M and U^A which are Lagrange multipliers in the canonical formalism. Since $\delta L/\delta q'_{AB}(z=\epsilon)$ is by definition the canonical momentum π^{AB} of the regulating boundary that is conjugate to q_{AB}, and since the remaining terms in (4.50) represent the equations of motion and the Hamiltonian and diffeomorphism constraints, we find:

$$\delta S\Big|_{\mathrm{on-shell}} = \int_{z=\epsilon} d^{d+1}x\, \pi^{AB} \delta q_{AB} \ . \tag{4.51}$$

A quick computation using (4.48) shows that $\pi^{AB} = -\sqrt{q}\left(Q^{AB} - q^{AB}Q\right)/2\kappa^2_{d+2}$ and hence:

$$\frac{2\kappa^2_{d+2}}{\sqrt{q}} \frac{\delta S^{\mathrm{onshell}}}{\delta q^{AB}(z=\epsilon)} = Q_{AB} - q_{AB}Q \ , \tag{4.52}$$

which represents the Brown-York tensor of the regulating boundary. An explicit computation of this tensor using our solution (4.41) reveals that it diverges under the limit $\epsilon \to 0$ and therefore we need to renormalize the action (4.48).[22] In order to do so, we use our solution (4.41) and the asymptotic series (4.22) and begin by evaluating the integral on-shell[23] and reading the terms that diverge as $\epsilon \to 0$. We find that the divergent terms are of the form:

$$\int_{z=\epsilon} d^{d+1}x \left(\sum_{n>0}^{d} A_n\, \epsilon^{-n} + \mathcal{A} \log \epsilon \right) \ , \tag{4.53}$$

[22]It should be emphasized that it is the canonical action (4.48), as opposed to (4.46), that is renormalized, since it is the former that is the functional of the boundary configuration of the metric. If the lapse $M = M(z)$, then the acceleration a^μ is identically zero and therefore both forms of the action exhibit the same divergences as evaluated by an examination of the asymptotic form of the integrands. In such case, it is irrelevant which of the two actions is renormalized. Indeed, the gauge choice $M = M(z)$ is the standard gauge in conventional AdS/CFT and hence one can renormalize directly the gravitational action without any canonical decomposition. For a different gauge choice, a canonical decomposition of the action is required (or then, the addition of a total derivative $-2\nabla_\mu a^\mu$ to the undecomposed action). In the language of AdS/CFT, the UV divergences of the QFT partition function are mapped to the IR divergences of the on-shell gravitational action in the canonical form. In our case, from the spacetime asymptotics (4.41) we find that $M = M(z,t)$ and hence $a^\mu = -q^{\mu\nu}\partial_\nu \log M = n^\mu/\beta \neq 0$. See also [121, 139].

[23]The integration limits in the z-integral are as follows: $\int_{z_0}^{\epsilon} dz$, for some constant $z_0 > \epsilon$.

where the coefficients A_n and $\mathcal{A}$ are local functionals of $g_{(0)ij}$ and β. We then invert the expansion (4.22) in order to express $g_{(0)ij}$ order by order in ϵ in terms of g_{ij} and therefore covariantly in terms of $\gamma_{ij} := \beta(t)^2/\epsilon^2 g_{ij}$, where γ is the induced metric on the surfaces $\{z = \epsilon\,,\, t = constant\}$, and replace the inverted expansion $g_{(0)} = g_{(0)}[\gamma, \epsilon, \beta]$ in the functionals A_n and $\mathcal{A}$. The next step is to introduce the projector $\gamma_{AB} := q_{AB} + n_A n_B$ onto the surfaces of constant time with unit normal n_A and use the standard identities from the theory of embedded hypersurfaces to extend the scalar functionals of γ_{ij} to scalar functionals of γ_{AB} on the submanifold $\{z = \epsilon\}$. The divergent terms written in this way can then be minimally subtracted from the action by introducing a preliminary counterterm integral at $z = \epsilon$ consisting of minus such divergent terms. These counterterms are given by:

$$
\begin{aligned}
2\kappa^2_{d+2}\, S^{pre-CT} &= \int_{z=\epsilon} d^{d+1}x\sqrt{q}\,\Bigg(2(d-1)\beta^{-1} + \frac{\beta}{d-2}\, R^A_A[\gamma] \\
&\quad + \frac{\beta^3}{(d-4)(d-2)^2}\left(R_{AB}[\gamma]R^{AB}[\gamma] - \frac{d}{4(d-1)}\left(R^A_A[\gamma]\right)^2\right) + \ldots + \beta^{d-1}\mathcal{A}_{(d)}\log\epsilon\Bigg) \\
&= \int_{z=\epsilon} d^{d+1}x\sqrt{q}\left(\sum_{n=0}^{[d/2-1]} C_n\,\beta^{2n-1} + \beta^{d-1}\mathcal{A}_{(d)}\log\epsilon\right)\,, \qquad (4.54)
\end{aligned}
$$

where we have written explicitly the counterterms C_n up to $d = 6$. The notation $[d/2-1]$ represents the integer value of $d/2-1$ rounded up. The coefficient $\mathcal{A}_{(d)}$ will be proportional to the conformal anomalies of the field theories, it vanishes for odd d and for even d up to $d = 4$ is given by:

$$
\mathcal{A}_{(2)} = -R^A_A[\gamma] \quad , \quad \mathcal{A}_{(4)} = -\frac{1}{8}\left(R_{AB}[\gamma]R^{AB}[\gamma] - \frac{1}{3}\left(R^A_A[\gamma]\right)^2\right)\,. \qquad (4.55)
$$

Notice that, on the surfaces of constant t, these preliminary counterterms agree with those found in [123] in the context of AdS/CFT. Furthermore, and as in AdS holographic renormalization, these counterterms break invariance of the action with respect to diffeomorphisms involving the radial coordinate z due to the explicit dependence on the regulator ϵ and this will be associated as usual to the conformal anomalies of the field theories.

Due to the dependence of S^{pre-CT} on $\beta(t)$, this counterterm action is not yet fully covariant. We can then covariantise it by introducing a boundary Lagrange multiplier λ for β. The simplest choice is to use the identity (4.43) and add to (4.54) the Lagrange multiplier:[24]

$$
2\kappa^2_{d+2}\, S^{\lambda} = \int_{z=\epsilon} d^{d+1}x\sqrt{q}\,\lambda\left(1 + q^{AB}\partial_A\beta\partial_B\beta\right)\,. \qquad (4.56)
$$

[24]The action (4.56) for β is identical to the action for a pressureless perfect fluid, or dust, at the boundary, where λ is proportional to the fluid's rest mass density and β plays the role of

We then treat β as a dynamical field at the boundary with respect to which the variation of the renormalized action $S^{ren} := S + S^{pre-CT} + S^{\lambda}$ should vanish. Such variation then results in the configuration for λ:

$$\frac{\delta S^{ren}}{\delta\lambda} = 0 \quad \Leftrightarrow \quad q^{AB}\partial_A\beta\partial_B\beta \;=\; -1 \;, \tag{4.57}$$

$$\begin{aligned}\frac{\delta S^{ren}}{\delta\beta} = 0 \quad \Leftrightarrow \quad \partial_A\left(\sqrt{q}\,\lambda\,\partial^A\beta\right) \;=&\; \frac{1}{2}\sqrt{q}\left(\sum_{n=0}^{[d/2-1]}(2n-1)\,C_n\,\beta^{2(n-1)}\right.\\ &\left. +\,(d-1)\beta^{d-2}\mathcal{A}_{(d)}\log\epsilon\right) \;.\end{aligned} \tag{4.58}$$

By using our solution (4.41), now written as:

$$ds^2_{d+2} = -N(t)^2dt^2 + \frac{\bar{\beta}(t)^2}{z^2}\left(dz^2 + g_{ij}dx^idx^j\right) \quad : \quad \bar{\beta}(t) := \int dtN(t) \;, \tag{4.59}$$

and looking for solutions $\beta = \beta(t)$, we obtain:

$$\beta \;=\; \bar{\beta}(t) \;, \tag{4.60}$$

$$2\,\lambda \;=\; -\sum_{n=0}^{[d/2-1]}\frac{2n-1}{d-1}\,C_n\,\bar{\beta}(t)^{2n-1} - \bar{\beta}(t)^{d-1}\mathcal{A}_{(d)}\log\epsilon \;, \tag{4.61}$$

where we have set the integration constants to zero. The final renormalized action is therefore given by:

$$\begin{aligned}2\kappa^2_{d+2}\,S^{ren} \;=&\; \int_{\mathcal{M}} d^{d+2}x\,M\sqrt{q}\left(R[q] + Q^2 - Q\cdot Q\right)\\ &+\int_{z=\epsilon} d^{d+1}x\sqrt{q}\left(\sum_{n=0}^{[d/2-1]}C_n\,\beta^{2n-1} + \beta^{d-1}\mathcal{A}_{(d)}\log\epsilon\right)\\ &+\int_{z=\epsilon} d^{d+1}x\sqrt{q}\,\lambda\left(1+|\partial\beta|^2\right) \;.\end{aligned} \tag{4.62}$$

the fluid's propertime [125]. This feature simply follows from the fact that the set of Eulerian worldlines described at the end of section 4.3.2 for which β is the propertime behaves as a congruence of dust particles. Equation (4.58) is a continuity equation for the rest mass current and, in the absence of the counterterm action (4.54), it expresses the conservation of the total mass.

The on-shell value of δS^{ren} is then given by:

$$
\begin{aligned}
2\kappa_{d+2}^2\, \delta S^{ren}\Big|_{\text{on-shell}} = & \int_{z=\epsilon} d^{d+1}x\, \sqrt{q}\, \Big(Q_{AB} - q_{AB}Q\Big)\delta q^{AB} \\
& - \frac{1}{2} \int_{z=\epsilon} d^{d+1}x \sqrt{q}\, \Bigg(\sum_{n=0}^{[d/2-1]} C_n\, \bar{\beta}^{2n-1} + \bar{\beta}^{d-1} \mathcal{A}_{(d)} \log\epsilon \Bigg) q_{AB}\, \delta q^{AB} \\
& + \int_{z=\epsilon} d^{d+1}x \sqrt{q}\, \Bigg(\sum_{n=0}^{[d/2-1]} \bar{\beta}^{2n-1} \frac{\partial C_n}{\partial q^{AB}} + \bar{\beta}^{d-1}\, \log\epsilon\, \frac{\partial \mathcal{A}_{(d)}}{\partial q^{AB}} \Bigg) \delta q^{AB} \\
& + \int_{z=\epsilon} d^{d+1}x \sqrt{q}\, \lambda \left(\partial_A \bar{\beta} \partial_B \bar{\beta}\right) \delta q^{AB} \ . \qquad (4.63)
\end{aligned}
$$

Using equation (4.61) for λ and the identity: $\partial_A \bar{\beta} = -n_A$, the last integral in (4.63) is given by:

$$
\begin{aligned}
\int_{z=\epsilon} d^{d+1}x \sqrt{q}\, \lambda \left(\partial_A \bar{\beta} \partial_B \bar{\beta}\right) \delta q^{AB} \ = \ -\frac{1}{2} \int_{z=\epsilon} d^{d+1}x \sqrt{q}\, \Bigg(& \sum_{n=0}^{[d/2-1]} \frac{2n-1}{d-1}\, C_n\, \bar{\beta}^{2n-1} \\
& + \bar{\beta}^{d-1} \mathcal{A}_{(d)} \log\epsilon \Bigg) n_A n_B\, \delta q^{AB} \ . \qquad (4.64)
\end{aligned}
$$

In this way, the renormalized Brown-York tensor is given by (we drop the bar notation over β from now on):

$$
\begin{aligned}
\frac{2\kappa_{d+2}^2}{\sqrt{q}} \frac{\delta S^{ren}_{onshell}}{\delta q^{AB}(z=\epsilon)} = & \ Q_{AB} - q_{AB}Q + \sum_{n=0}^{[d/2-1]} \beta^{2n-1} \left(\frac{\partial C_n}{\partial q^{AB}} - \frac{1}{2}\, C_n \left(q_{AB} + \frac{2n-1}{d-1}\, n_A n_B \right)\right) \\
& + \beta^{d-1} \left(\frac{\partial \mathcal{A}_{(d)}}{\partial q^{AB}} - \frac{1}{2}\, \mathcal{A}_{(d)} \gamma_{AB} \right) \log\epsilon \ . \qquad (4.65)
\end{aligned}
$$

As an exercise, for $d+2=4$ we obtain:

$$
\frac{2\kappa_4^2}{\sqrt{q}} \frac{\delta S^{ren}_{onshell}}{\delta q^{AB}(z=\epsilon)} \ = \ Q_{AB} - q_{AB}Q - 2\beta^{-1} q_{AB} + \beta^{-1}\gamma_{AB} \ , \qquad (4.66)
$$

where the term $(R_{AB}[\gamma] - \frac{1}{2}\gamma_{AB} R[\gamma]) \log\epsilon$ vanishes identically for $d=2$.

4.4.3. Holographic stress tensors

Following [179], we now decompose the Brown-York tensor into the spatial stress tensor s_{ij} and the momentum and energy densities j_i and ε:

$$s_{ij} := \frac{2}{N\sqrt{\gamma}}\frac{\delta S^{ren}_{onshell}}{\delta\gamma^{ij}} = \gamma_i^A\gamma_j^B\left(\frac{2}{\sqrt{q}}\frac{\delta S^{ren}_{onshell}}{\delta q^{AB}}\right) , \tag{4.67}$$

$$j_i := \frac{1}{\sqrt{\gamma}}\frac{\delta S^{ren}_{onshell}}{\delta V^i} = \gamma_i^A n^B\left(\frac{2}{\sqrt{q}}\frac{\delta S^{ren}_{onshell}}{\delta q^{AB}}\right) , \tag{4.68}$$

$$\varepsilon := -\frac{1}{\sqrt{\gamma}}\frac{\delta S^{ren}_{onshell}}{\delta N} = -n^A n^B\left(\frac{2}{\sqrt{q}}\frac{\delta S^{ren}_{onshell}}{\delta q^{AB}}\right) , \tag{4.69}$$

where: $q_{AB}dx^A dx^B = -N^2dt^2 + \gamma_{ij}\left(dx^i + V^i dt\right)\left(dx^j + V^j dt\right)$ and: $\sqrt{q} = N\sqrt{\gamma}$. For our spacetime solution, the non-trivial components are the spatial stress and energy density; the contraction of the Brown-York tensor (4.65) with $\gamma_i^A n^B$ vanishes identically, resulting in a vanishing momentum j_i.

The expectation value of the stress tensor of each field theory is now obtained by computing the spatial components s_{ij} of the renormalized Brown-York tensor in the conformal embedding and taking the limit as the regulating surface $\{z = \epsilon\}$ tends to the boundary $\{z = 0\}$. Recall from (4.29) that our defining function $\rho = z/\beta$. By factorising the latter, our spacetime solution reads as:

$$ds^2_{d+2} = \frac{\beta(t)^2}{z^2}\left(dz^2 - z^2N^2_{(0)}dt^2 + (g_{(0)ij} + \mathcal{O}(z^2)\,)\,dx^i dx^j\right) , \tag{4.70}$$

with $N_{(0)} := N(t)/\beta(t)$. The metric $\tilde{G}$ of the conformal embedding is given by $\tilde{G}_{\mu\nu} = (z/\beta)^2 G_{\mu\nu}$ and the expectation value of the operator dual to $g_{(0)}$ is therefore obtained as:[25]

$$\langle T_{ij}\rangle = \frac{2}{N_{(0)}\sqrt{g_{(0)}}}\frac{\delta S^{ren}_{onshell}}{\delta g^{ij}_{(0)}} = \lim_{\epsilon\to 0}\left(\beta\,(\beta/\epsilon)^{d-2}\,s_{ij}\right) . \tag{4.71}$$

From the perspective of the standard AdS/CFT dictionary, we cannot interpret $N_{(0)}$ in the usual sense as a source term for some dual operator in each field theory (note that $N_{(0)}$ cannot be switched off). We cannot interpret it as a source for the time-component of the energy tensor at $z = 0$ either, due to the different asymptotic behaviours of the spatial and time components of the metric. Nevertheless, we can still define the renormalized quantity:

$$\mathcal{E} := -\frac{1}{\sqrt{g_{(0)}}}\frac{\delta S^{ren}_{onshell}}{\delta N_{(0)}} = \lim_{\epsilon\to 0}\left(\beta\,(\beta/\epsilon)^{d}\,\varepsilon\right) . \tag{4.72}$$

[25] Here and in equation (4.72) we have performed the intermedium step: $\delta S/\delta\tilde{q}^{AB} = (z/\beta)^2\delta S/\delta q^{AB}$, where: $\tilde{q}_{AB} := (z/\beta)^2 q_{AB}$, and performed the decomposition of $\delta/\delta\tilde{q}^{AB}$ in the same way as in (4.67)–(4.69).

An explicit computation of $\langle T_{ij}\rangle$ and $\mathcal{E}$ using our spacetime solution results in the following expectation values:[26]

$$\langle T_{ij}\rangle \;=\; \frac{d\,\beta(t)^d}{2\kappa^2_{d+2}}\left(g_{(d)ij}+X_{ij}[g_{(0)}]\right)\ , \tag{4.73}$$

$$\mathcal{E} \;=\; \frac{1}{d-1}\,\langle T^i_i\rangle\ , \tag{4.74}$$

where X_{ij} is a functional of $g_{(0)}$, it vanishes for odd values of d and its explicit expression for even values depends on d. Also, the trace of $\langle T_{ij}\rangle$ is taken with respect to $g_{(0)}$. Equation (4.73) represents the expectation value of the spatial stress tensor of each field theory described holographically by a hypersurface of constant t and it coincides with the holographic stress tensor found in [123] in the context of AdS/CFT. Equation (4.74) identifies $\mathcal{E}$ with the Weyl anomaly of each field theory.[27] The holographic Ward identities for the field theories are given by:

$${}^{(0)}\nabla_j\,\langle T^j{}_i\rangle \;=\; 0 \quad \forall d, \tag{4.76}$$

$$\langle T^i_i\rangle \;=\; 0 \quad : \; d=2n+1\ ,\ n\in\mathbb{N}\ , \tag{4.77}$$

where ${}^{(0)}\nabla_j$ is the covariant derivative associated to $g_{(0)}$. For even values of d, the trace of the stress tensor depends on d. For $d=2$, we find:[28]

$$\langle T_{ij}\rangle \;=\; \frac{\beta(t)^2}{\kappa_4^2}\left(g_{(2)ij}-g_{(0)ij}\,\mathrm{Tr}[g^{-1}_{(0)}g_{(2)}]\right)\ . \tag{4.78}$$

The holographic Weyl anomaly in this case is given by:

$$\langle T^i_i\rangle \;=\; \frac{\beta(t)^2}{2\kappa_4^2}\,R[g_{(0)}] \;=\; \frac{c}{24\pi}\,R[g_{(0)}]\ , \tag{4.79}$$

[26] Notice that the term Q_{AB} as well as: $\partial C_n/\partial q^{AB}$ and: $(\partial\mathcal{A}_{(d)}/\partial q^{AB}-\frac{1}{2}\mathcal{A}_{(d)}\gamma_{AB})\log\epsilon$ in (4.65) are purely spatial and therefore vanish when contracted with n^A. The expression for $\mathcal{E}$ follows from an explicit computation of (4.72) for each value of d and has been verified up to $d=6$.

[27] As an observation, by using the definitions (4.71) and (4.72), equation (4.74) can be rewritten as:

$$\left(-2\,g^{ij}_{(0)}\,\frac{\delta}{\delta g^{ij}_{(0)}}+(1-d)\,N_{(0)}\,\frac{\delta}{\delta N_{(0)}}\right)S^{ren}_{onshell} \;=\; 0, \tag{4.75}$$

and which represents the holographic Callan-Symanzik equation for a anomaly-free CFT deformed by a source $N_{(0)}$ for a relevant scalar operator of dimension one. The Weyl anomaly of each field theory can therefore also be seen as that created by a non-vanishing vacuum expectation value of such an operator. Notice that, in such case and by using (4.76), the diffeomorphism Ward identity would also be satisfied since $\partial_i N_{(0)}=0$ when on-shell.

[28] If $d+2=4$, then our (asymptotic) solution (4.41) represents the Lorentzian cone of a three-dimensional Einstein Riemannian manifold of negative scalar curvature. The latter is therefore the hyperbolic 3-space, up to possible global identifications, and hence (4.41) is diffeomorphic to Minkowski spacetime.

where the central charge of each 2-dimensional CFT is related to $\beta(t)$ as:

$$\beta(t) \;=\; \ell'_P\sqrt{\frac{c}{6}} \qquad : \quad \ell'_P \;:=\; 2\sqrt{\frac{\hbar G_N}{c_0^3}} \;, \tag{4.80}$$

where we have reinserted[29] the factors of $\hbar$ and c_0 and defined ℓ'_P as twice the usual convention for the Planck length in four bulk dimensions. As an exercise, for four dimensional Minkowski space (4.4) in the cone coordinate (also known as Milne's space):

$$ds^2 = -dt^2 + t^2\left(\frac{dz^2}{z^2} + \frac{1}{z^2}\left(\frac{1-z^2}{2}\right)^2 d\Omega^2\right) \;, \tag{4.81}$$

we obtain:

$$\langle T_{ij}\rangle \;=\; \frac{t^2}{2\kappa_4^2}\left(4g_{(0)ij}\right) \;, \tag{4.82}$$

$$c/6 \;=\; \left(t/\ell'_P\right)^2 \;, \tag{4.83}$$

with $4g_{(0)ij}dx^i dx^j = d\Omega^2$ the metric on the S^2.

Equation (4.80) associates the spectrum of central charges to the bulk time coordinate essentially in a gauge-invariant way, *i.e.* it does not depend on a particular choice of time coordinate due to the invariant meaning of $\beta(t)$, and it requires, either a notion of locality in the spectrum, or a discretisation of time distances in units of the Planck length. More generally, in even dimensions, the time coordinate is mapped to the coefficients, or central charges, of the Euler density and Weyl invariant in the conformal anomalies, also known as type A and B anomalies [209].[30] In AdS/CFT, the renormalization group equations are local in the energy scale and this property is consistent with the notion of (coarse) locality in the bulk radial direction, hence we obtain a precise matching on both sides of the duality. In our case, on the other hand, the interrelationship between the different field theories in the family, and in particular between their central charges, is not clear. Consistency of our framework therefore requires a notion of locality in the spectrum so that locality in the time direction is recovered. In order to understand more comprehensively the correlation between the field theories required by the duality, as well as the role of the time coordinate in the family, it seems necessary to study the full group of diffeomorphisms in the bulk that preserves the form of the metric (4.41). Such diffeomorphisms contain a subgroup involving the time

[29] Recall that $\hbar$ arises from the partition function (4.45).

[30] Recall that, holographically and in the absence of higher curvature corrections, these coefficients are related, see *e.g.* [124, 175]

coordinate [188] which should act at $\partial\mathcal{H}$ as a particular type of transformation between the different field theories. By construction, this family should be invariant under such transformations and an identification of such symmetry group should allow a better understanding of the way the field theories are connected.

In section 4.3.2 we performed the analytic continuation of our spacetime asymptotics to the Riemannian cone (4.35) representing (asymptotically) our generalisation of region I of Minkowski space. The regulating boundary $\{z = \epsilon\}$ for these spacetimes is now spacelike and it can be verified that, under such continuation, the renormalized action is given by (4.62) under the substitution $Q \to iQ$ and with the counterterm action multiplied by -1.[31] In this case, the renormalized Brown-York tensor is then given by the right-hand side of (4.65) multiplied by -1. This implies that the expectation value of the spatial stress tensors is still given by formula (4.73) for $d = 4n + 2\ :\ n \in \mathbb{N}_0$, and given by (4.73) multiplied by minus one otherwise. As an exercise, for $d = 3$, we have:

$$\langle T_{ij}\rangle = -\frac{3\beta(t)^3}{2\kappa_5^2}\, g_{(3)ij} \quad : \quad \langle T^i_i\rangle = 0 = {}^{(0)}\nabla_j \,\langle T^j_{\ i}\rangle \ . \tag{4.84}$$

For the cone (4.38) of dS_4–Schwarzschild, we find:

$$\langle T_{ij}\rangle dx^i dx^j = \frac{M}{\ell}\,\frac{\left(e^t/\rho\right)^3}{\kappa_5^2}\left(-2d\rho^2 + \rho^2 d\Omega^2\right) \ . \tag{4.85}$$

Before ending this section and as an exercise, we can consider a simple example of a family of two-dimensional CFTs with a continuous parametrisation of the central charge and which we will take to be a family of Liouville field theories. Classical Liouville theory is the theory of two-dimensional conformal manifolds, or Riemann surfaces. Suppose we have a two-dimensional Riemannian manifold with metric:

$$ds^2 \ = \ \tilde{g}_{ij} dx^i dx^j \ := \ e^{2b\phi} g_{(0)ij} dx^i dx^j \ , \tag{4.86}$$

where $\tilde{g}$ is a representative of the conformal structure $[\tilde{g}]$ of some Riemann surface, b is a dimensionless constant and $g_{(0)}$ is an arbitrary "background" metric. The field $\phi(x)$ is known as the Liouville field. The scalar curvature of $\tilde{g}$ and $g_{(0)}$ are then related as: $R[\tilde{g}] = e^{-2b\phi}\left(R[g_{(0)}] - 2b\Box\phi\right)$, where $\Box$ is the Laplacian with respect to $g_{(0)}$. Since the Riemann surface is endowed with a conformal structure, we can take the representative $\tilde{g}$ to be a metric of constant curvature and write: $R[\tilde{g}] = -8\pi\mu b^2\ :\ \mu > 0$. In this way, we find: $\Box\phi = 4\pi\mu b e^{2b\phi} + (2b)^{-1}R[g_{(0)}]$.

[31] Recall also the transformation (4.36) for the coefficients in the expansion of g_{ij}.

The action for this theory is the classical Liouville action [210, 211]:

$$S \;=\; \frac{1}{4\pi}\int d^2x\sqrt{g_{(0)}}\left(g_{(0)}^{ij}\partial_i\phi\partial_j\phi + 4\pi\mu e^{2b\phi} + QR[g_{(0)}]\phi\right)\;, \tag{4.87}$$

where the so-called background charge $Q = b^{-1}$. The continuous parameter b is either real of purely imaginary, corresponding respectively to a metric $\tilde{g}$ of negative or positive curvature.[32] Since we are only interested in the conformal structure $[\tilde{g}]$, we require that ϕ transforms as: $\phi \to \phi - (Q/2)\log\Omega$ under the conformal transformation: $g_{(0)} \to \Omega\, g_{(0)}$, such that $\tilde{g} \to \tilde{g}$. The equation of motion for ϕ is therefore invariant under the combined transformations, which implies that the action is also invariant up to boundary terms, as long as $Q = b^{-1}$. Since every two-dimensional metric is conformally flat, we can now set $g_{(0)} = \mathbb{1}$ (which just corresponds to a redefinition of ϕ in (4.86)) and then change to complex coordinates: $\left(z = x^1 + ix^2\,,\, \bar{z} = x^1 - ix^2\right)$ such that:

$$ds^2 \;=\; e^{2b\phi}dzd\bar{z}\;. \tag{4.88}$$

In the gauge (4.88), the non-vanishing components of the holomorphic Liouville energy tensor (obtained from $\delta S/\delta g_{(0)}$) are given by:

$$T(z) \;=\; -\left(\partial\phi\right)^2 + Q\partial^2\phi\;, \tag{4.89}$$

$$\bar{T}(\bar{z}) \;=\; -\left(\bar{\partial}\phi\right)^2 + Q\bar{\partial}^2\phi\;, \tag{4.90}$$

where we have used the equation of motion for ϕ. Under the holomorphic transformation: $z \to \omega(z)$, $\phi \to \phi - (Q/2)\log|\partial\omega|^2$, the energy tensor transforms as:

$$T(z) \to (\partial\omega)^{-2}\left(T(z) - \frac{Q^2}{2}\left(\frac{\partial^3\omega}{\partial\omega} - \frac{3}{2}\left(\frac{\partial^2\omega}{\partial\omega}\right)^2\right)\right)\;. \tag{4.91}$$

A comparison with the standard transformation law for $T(z)$ under a holomorphic conformal transformation yields the "classical" central charge:

$$c \;=\; 6\,Q^2\;. \tag{4.92}$$

This result can also be obtained by considering Liouville's theory on a cylinder and verifying that the Fourier components of the energy tensor satisfy the Virasoro Poisson bracket algebra with the above central charge (see *e.g.* [212] with a slightly different notation). From the transformation law for the Liouville field, it follows that the fields $e^{2\alpha\phi}$ are primary with "classical" dimension $\Delta = \alpha Q$, for some constant α. When the Liouville theory is quantised, the background charge Q receives corrections, as well as the central charge and the dimensions Δ. After

[32] In the latter case, the Liouville field is redefined as $\phi \to -i\phi$, resulting in a negative sign in the kinetic term in the action.

normal ordering the (off-shell) energy tensor, its Fourier components only satisfy the Virasoro commutator algebra (with the canonical commutation relations imposed on the Liouville field and momentum, see [213]) if $Q = b^{-1} + b$. The Virasoro algebra then yields the central charge:

$$c \;=\; 6\,Q^2 + 1\;. \tag{4.93}$$

The classical limit corresponds to $b \to 0$. Finally, by considering the OPE of the operators $e^{2\alpha\phi}$ with the energy tensor, it follows that these are primary, now with dimension:

$$\Delta = \alpha Q - \alpha^2\;. \tag{4.94}$$

In conventional AdS/CFT, a Liouville theory at the conformal boundary can be realised holographically by an asymptotically hyperbolic 3-space with a scalar field of mass $m^2 = \alpha(Q-\alpha)\,(\alpha(Q-\alpha)-2)$ for each dynamical operator $e^{2\alpha\phi}$. Motivated by our results, we also expect that, by switching on some scalar field in our spacetime in four dimensions, we are able to capture holographically the dynamics of scalar operators in a family of two-dimensional CFTs at $\partial\mathcal{H}$. In [188] it was brought to the attention that, in general, each (non-backreacting) scalar field in the bulk decomposes into an infinite set of massive scalars on each surface of constant time. This feature poses several problems to the holographic computation of correlators of scalar operators, in particular because we know that each massive scalar on a constant time slice is dual to a single scalar operator of definite scaling dimension in the dual field theory. Since we do not have a definite mass (or a unique scalar field) on a slice of constant time associated to a scalar field on the spacetime, we lose the correspondence between a bulk field and a single operator per field theory just as we had between the spacetime metric and the stress tensors. It seems possible, however, to bypass this issue and reproduce the correlators of one operator per field theory by setting up an initial value problem for the bulk scalar in a manner similar to the approach that we followed in the case of the spacetime metric. In this way, we restrict the solutions of the bulk wave equation to a subclass that suffices to capture the dynamics of the operators in the field theories.[33] Such particular solutions reduce on each slice to a single massive scalar and in this case, the mass of the field on a slice, and therefore the dimension of the dual operator, will be associated to the time-dependence of the bulk spacetime

[33] Of course that, in this way, we are not able to reconstruct holographically any solution of the bulk wave equation, but rather a subclass of such solutions. In the same way, we are not able to reconstruct holographically any asymptotically Ricci-flat metric, but rather the subclass of such metrics that admit an asymptotically hyperbolic hypersurface of constant mean curvature. It seems that such general classes of solutions contain more information than that that just a family of field theories in two dimensions less can provide. On the other hand, if we are only interested in reproducing the correlators of the operators of each field theory in the family, then the above approach should be sufficient.

field, *i.e.* to its derivatives with respect to $\beta(t)$. With a correspondence between a bulk solution and a single operator per field theory, we should be able to reproduce holographically the correlators of operators in the family of field theories.

Returning to the case at hand, and in particular for a family of Liouville field theories at $\partial\mathcal{H}$, we have deduced that pure gravity in the bulk captures the dynamics of the stress tensors of each theory. By using equations (4.80) and (4.93), it then follows that $\beta(t)$ is associated to the continuous background charge of the family as:

$$\beta(t) \;=\; \ell'_P\sqrt{Q^2+1/6} \;\sim\; \ell'_P\, Q \qquad (b\to 0)\ . \tag{4.95}$$

Equation (4.94) then implies that the scaling dimensions of the primary operators of each field theory are also expressed in terms of $\beta(t)$. If a scalar field in the bulk is dual to a family of primary operators (each belonging to a different field theory) of the same scaling dimension Δ, then different operators in this family must have different α's. From equation (4.94) we find that: $\alpha(t) = Q/2 \pm \sqrt{(Q/2)^2-\Delta}$, where the time dependence of α is obtained from the relation (4.95). The bulk scalar would therefore be dual to the family operator: $exp\Big[\Big(Q(t) \pm \sqrt{Q(t)^2-4\Delta}\Big)\,\phi_t\Big]$, where ϕ_t is the operator of the field theory with central charge $c(t)$. In the limit as $b\to 0$ (or at late times $\beta(t)\gg 1$), the family operator asymptotes to $exp\,(2\beta\phi/\ell'_P)$ or to the identity operator.

In the next section we will generalise our class of asymptotically Ricci-flat spacetimes, deduce their asymptotics and compute the modifications to the expectation values of the stress tensors. This is done for the following reason. From equation (4.73) it follows that, up to a constant factor, different CFTs have the same expectation values of the stress tensors.[34] This implies that the reconstruction of the spacetime asymptotics only requires the knowledge of the holographic stress tensor of a single CFT, besides the conformal structure $[g_{(0)}]$. In order to obtain different vevs for different field theories, a relative time-dependence is needed in the expressions (4.73) for the expectation values which would discriminate between different CFTs. This could be achieved by requiring for example that $\partial_t g_{ij} \neq 0$ in the spacetime metric (4.41). Indeed, we can obtain different expectation values of the stress tensors for different field theories by considering subleading contributions to the asymptotic value of the initial data $K_{[0]}$. The solution to such Cauchy problem will be similar to our solution (4.41), but with a particular time-dependent g_{ij}. The reconstruction of the spacetime asymptotics in

[34]The n-point correlators should also be the same. These are obtained by functionally differentiating $\langle T_{ij}\rangle$ with respect to $g_{(0)}$ for an exact bulk solution and, since both the source and the normalisable mode $g_{(d)}$ are t-independent, such differentiation should not introduce any additional time dependence which would discriminate between different CFTs.

such case will then require the knowledge of a family of different holographic stress tensors. In the next section we will therefore allow an arbitrary initial extrinsic curvature $K_{[0]}$ away from the conformal boundary of the initial data surface such that it approaches $h_{[0]}$ only asymptotically.

4.5. Corrections to the holographic stress tensors

4.5.1. Generalisation of the initial data

As in section 4.3.2, we start from an asymptotically hyperboloidal initial data set $(\Sigma, h_{[0]}, K_{[0]})$, with $(\Sigma, h_{[0]})$ of dimension $d+1$, and consider the first subleading orders of $K_{[0]}$ as we move away from the conformal boundary $\{z=0\}$ of the initial Cauchy surface. The asymptotic form of our initial data is therefore:

$$R_{ab}[h_{[0]}] = -\frac{d}{\ell^2}\, h_{[0]ab} \quad , \tag{4.96}$$

$$K_{[0]ab} = \frac{1}{\ell}\, h_{[0]ab} + z^{-2+\alpha}\left(\frac{d}{2}\,\ell\,\mathcal{T}_{ab}\right) \;:\; \mathcal{T}_{ab}(z,x) = \mathcal{T}_{(0)ab}(x) + \mathcal{O}(z^{>0}) \quad , \tag{4.97}$$

where we have now kept for convenience the constant curvature radius ℓ of Σ arbitrary as in equations (4.16) and (4.17). Also, $\alpha \in \mathbb{R}$ and $\mathcal{T}_{ab}$ is so far an arbitrary symmetric tensor on Σ with the above asymptotic expansion representing the first subleading corrections to the asymptotic value of $K_{[0]}$. As before, we will take the first condition (4.96) to be asymptotically equivalent to the solution (4.21):

$$h_{[0]ab}dx^a dx^b \sim \frac{\ell^2}{z^2}\left(dz^2 + g_{ij}dx^i dx^j\right) \ , \tag{4.98}$$

with the Fefferman-Graham asymptotic expansion (4.22). Furthermore, since $h_{[0]} = \mathcal{O}(z^{-2})$ and we require the initial data set to be asymptotically hyperboloidal, we find that $\alpha > 0$.

The next step is to analyse the constraints on $\mathcal{T}_{ab}$ imposed asymptotically by the initial data constraint equations (4.9) and (4.10). For our purposes, we will only need the constraints imposed on the leading order $\mathcal{T}_{(0)}$. By using the asymptotic form (4.98) of $h_{[0]}$ we find that the constraint equations in a neighbourhood of the conformal boundary take the form:

$$\mathcal{T}_{zz} + \mathrm{Tr}[g^{-1}\mathcal{T}] + \frac{1}{4}\, z^{\alpha} \left(2\, \mathcal{T}_{zz}\, \mathrm{Tr}[g^{-1}\mathcal{T}] + \mathrm{Tr}^2[g^{-1}\mathcal{T}] - 2\, \mathcal{T}_{zi}\, g^{ij} \mathcal{T}_{jz} - \mathrm{Tr}[g^{-1}\mathcal{T} g^{-1}\mathcal{T}]\right) = 0 \ , \tag{4.99}$$

$$\nabla_i \left(g^{ij}\mathcal{T}_{jz}\right) \;=\; \frac{d}{z}\left(\mathcal{T}_{zz} + \frac{\alpha-1}{d}\, \mathrm{Tr}[g^{-1}\mathcal{T}]\right) + \mathrm{Tr}[g^{-1}\mathcal{T}'] - \frac{1}{2}\, \mathcal{T}_{zz} \mathrm{Tr}[g^{-1}g'] - \frac{1}{2}\, \mathrm{Tr}[g^{-1}g' g^{-1}\mathcal{T}] \ , \tag{4.100}$$

$$\nabla_j \left(g^{-1}\mathcal{T}\right)^{j}_{\;i} - \partial_i\, \mathrm{Tr}[g^{-1}\mathcal{T}] \;=\; \frac{d+1-\alpha}{z}\, \mathcal{T}_{zi} - \mathcal{T}'_{zi} - \frac{1}{2}\, \mathcal{T}_{zi}\, \mathrm{Tr}[g^{-1}g'] + \partial_i \mathcal{T}_{zz} \ , \tag{4.101}$$

where ∇_i is the covariant derivative associated to g_{ij} and $\mathcal{T}' := \partial_z \mathcal{T}$. From the leading order of the three constraint equations above we find the following three conditions for $\alpha \neq d+1$:

$$\mathcal{T}_{(0)zz} = 0 \quad , \quad \mathrm{Tr}[g_{(0)}^{-1}\mathcal{T}_{(0)}] = 0 \quad , \quad \mathcal{T}_{(0)zi} = 0 \ . \tag{4.102}$$

We also find the following additional condition for $\alpha = d$ from the first subleading order of the third constraint equation above:

$$^{(0)}\nabla_j \mathcal{T}^{j}_{(0)i} \;=\; 0 \ , \tag{4.103}$$

where $^{(0)}\nabla_i$ is the covariant derivative associated to $g_{(0)}$ and where the indices are raised with $g_{(0)}$. These four particular conditions will play an important role in the analysis that follows next. With these constraints identified, the approach we now take towards the initial value problem is to make a choice of lapse function and shift vector and solve asymptotically the evolution equations (4.11) and (4.12) in powers of z up to some desired order. If we find a solution to this problem up to some order in z, then such solution is the unique solution up to that order. We will analyse the three possible situations: when the power α is greater than, equal to or less than d and we will find that the relevant case is when $\alpha = d$.

As in the previous sections, we begin by choosing to evolve the initial data in time in the gauge $(N = 1, A^a = 0)$ in which the metric tensor (4.14) of the Ricci-flat development in a neighbourhood of $\Sigma = \{\hat{t} = 0\}$ assumes the form (4.23):

$$ds^2_{d+2} = -d\hat{t}^{\,2} + h_{ab} dx^a dx^b \ . \tag{4.104}$$

The extrinsic curvature (4.11) on the surfaces of constant $\hat{t}$ in this gauge is given by $K_{ab} = \frac{1}{2}\partial_{\hat{t}}\, h_{ab}$. We then introduce a new coordinate t as: $e^t := 1+\hat{t}/\ell$, in which

$\Sigma = \{t = 0\}$, and define $\tilde{h}_{ab} := \ell^{-2} e^{-2t} h_{ab}$. The development's metric therefore becomes:

$$ds^2_{d+2} = \ell^2 e^{2t} \left(-dt^{\,2} + \tilde{h}_{ab} dx^a dx^b \right) \, . \tag{4.105}$$

The extrinsic curvature on Σ becomes:

$$K_{[0]ab} \;=\; K_{ab}(t=0) \;=\; \left[\ell\, e^t \left(\tilde{h}_{ab} + \frac{1}{2}\, \partial_t \tilde{h}_{ab} \right) \right]_{t=0} \;=\; \ell\, \tilde{h}_{ab}\Big|_{t=0} + \frac{\ell}{2}\, \partial_t \tilde{h}_{ab}\Big|_{t=0} \, . \tag{4.106}$$

In this way, the asymptotic initial conditions (4.96) and (4.97) in this coordinate system read respectively as:

$$\tilde{h}_{ab} dx^a dx^b\Big|_{t=0} \;=\; \frac{1}{z^2} \left(dz^2 + g_{ij} dx^i dx^j \right) \, , \tag{4.107}$$

$$\partial_t \tilde{h}_{ab}\Big|_{t=0} \;=\; d\, z^{-2+\alpha}\, \mathcal{T}_{ab} \, , \tag{4.108}$$

with g_{ij} given by equation (4.22). We then solve asymptotically the dynamical equation obtained by replacing equation (4.11) in (4.12) subject to the above initial value conditions. This equation reads as:

$$2 \left(R_{ab}[\tilde{h}] + d\, \tilde{h}_{ab} \right) + \ddot{\tilde{h}}_{ab} + d\, \dot{\tilde{h}}_{ab} + \tilde{h}_{ab}\, \mathrm{Tr}[\tilde{h}^{-1} \dot{\tilde{h}}] + \frac{1}{2}\, \dot{\tilde{h}}_{ab}\, \mathrm{Tr}[\tilde{h}^{-1} \dot{\tilde{h}}] - \left(\dot{\tilde{h}} \tilde{h}^{-1} \dot{\tilde{h}} \right)_{ab} \;=\; 0 \, , \tag{4.109}$$

where $\dot{\tilde{h}} := \partial_t \tilde{h}$.

4.5.2. Asymptotics and expectation values

We begin by analysing the simplest case $\alpha > d$. Let one start with the following ansatz:

$$\tilde{h}_{ab} dx^a dx^b \;=\; \frac{1}{z^2} \left(dz^2 + \tilde{g}_{ij} dx^i dx^j \right) + \mathcal{O}(z^{>-2+d}) \, , \tag{4.110}$$

$$\tilde{h}_{zi} \;=\; (1 - e^{-dt}) A_{[d]i}(z,x) + \mathcal{O}(z^{>-1+d}) \;:\; A_{[d]i} = \mathcal{O}(z^{>-2+d}) \, , \tag{4.111}$$

$$\tilde{g}_{ij}(z,x) \;:=\; g_{(0)ij}(x) + z^2 g_{(2)ij}(x) + \ldots + z^d g_{(d)ij}(x) + z^d \log z\, \tilde{g}_{(d)ij}(x) \, , \tag{4.112}$$

where only even powers in z arise below the order z^d in the above finite series[35] and where each coefficient $g_{(n)}$ is defined to be equal to the coefficient $g_{(n)}$ in the Fefferman-Graham expansion (4.22). This ansatz satisfies the dynamical equation (4.109) up to order z^{-2+d} (see appendix 4.6.1 for the complete treatment), and

[35] It should be emphasized that, unlike the expansion (4.22), the above expansion (4.112) is defined to be finite and to terminate at order z^d.

since $\alpha > d$, it also satisfies the initial conditions (4.107) and (4.108) up to order z^{-2+d}. In this way, the unique solution to this initial value problem must coincide with our ansatz up to order z^{-2+d}. Such solution receives corrections in all components at higher orders, but we will find that we do not need the asymptotic form of $\tilde{h}_{ab}$ beyond the order z^{-2+d}. Finally, we reinstate the lapse function $N(t)$ by redefining our time coordinate $t \to \log \int dt\, N(t)/\ell$ such that the development's metric becomes:

$$ds^2_{d+2} \;=\; -N(t)^2 dt^2 + \beta(t)^2 \tilde{h}_{ab} dx^a dx^b \;, \tag{4.113}$$

with $\dot{\beta} := N(t)$ as before.

Given the above asymptotic solution for the Ricci-flat development (equation (4.113) with (4.110)) we then proceed as previously by renormalizing the gravitational action. If we regularize and evaluate the action (4.48) on-shell, we find that the divergences arise only down to order ε^{-d} as in (4.53) and that these only involve the asymptotic form of $\tilde{h}_{ab}$ up to order z^{-2+d}. Hence, the counterterm action will be exactly the same as in (4.62) because our solution is the same as in (4.41) up to that order. We also find that the expectation values of the holographic stress tensors are equal to those found in (4.73) because only the terms up to z^{-2+d} in the asymptotic expansion of $\tilde{h}_{ab}$ survive under the limit $\epsilon \to 0$ taken in (4.71). In this way, the case $\alpha > d$ does not yield new results.

The next step is to analyse the power $\alpha = d$. In this case, the four conditions (4.102)–(4.103) on the leading term $\mathcal{T}_{(0)ab}$ hold. Let one start with the following ansatz:

$$\tilde{h}_{ab} dx^a dx^b = \frac{1}{z^2}\left[dz^2 + \left(\tilde{g}_{ij} + z^d\left(\Delta_{ij} - \Delta_{[0]ij}\right)\right)dx^i dx^j\right] + \mathcal{O}(z^{>-2+d}) \;, \tag{4.114}$$

with $\tilde{g}$ and $\tilde{h}_{zi}$ as defined in (4.111)–(4.112) and where:

$$\Delta_{ij} := \Delta_{ij}(t,x) \quad , \quad \Delta_{[0]ij} := \Delta_{ij}(t=0,x) \;. \tag{4.115}$$

If we replace this ansatz in the dynamical equation (4.109), we find that this equation is solved up to order z^{-2+d} if Δ_{ij} satisfies the second order differential equation in t:

$$\ddot{\Delta}_{ij} + d\,\dot{\Delta}_{ij} = 0 \quad \Rightarrow \quad \Delta_{ij}(t,x) = \Delta_{[0]ij}(x) + \left(1 - e^{-dt}\right)\Delta_{[d]ij}(x) \;, \tag{4.116}$$

subject to the conditions that the integration constant $\Delta_{[d]ij}$ be traceless and covariantly conserved with respect to $g_{(0)}$ (see appendix 4.6.2). In this way, the ansatz:

$$\tilde{h}_{ab}dx^a dx^b = \frac{1}{z^2}\left[dz^2 + \left(\tilde{g}_{ij} + z^d\left(1-e^{-dt}\right)\Delta_{[d]ij}(x)\right)dx^i dx^j\right] + \mathcal{O}(z^{>-2+d}) \ , \tag{4.117}$$

$$\mathrm{Tr}[g_{(0)}^{-1}\Delta_{[d]}] = 0 \quad , \quad {}^{(0)}\nabla_j \Delta^j_{[d]i} = 0 \ , \tag{4.118}$$

solves the dynamical equation up to order z^{-2+d}. If we then compute the initial values $\tilde{h}_{ab}(t=0)$ and $\partial_t \tilde{h}_{ab}(t=0)$, we find that these coincide with the initial conditions (4.107) and (4.108) up to order z^{-2+d} (recall that $\mathcal{T}_{(0)za} = 0$) if we identify $\Delta_{[d]ij}$ with $\mathcal{T}_{(0)ij}$:

$$\Delta_{[d]ij} \ = \ \mathcal{T}_{(0)ij} \ . \tag{4.119}$$

Recall that the leading order $\mathcal{T}_{(0)ij}$ is also traceless and covariantly conserved. In this way, with the above identification, we find that our ansatz (4.117) solves the initial value problem up to order z^{-2+d} and therefore the unique solution to this problem must coincide with our ansatz up to this order. Finally, we reinstate the lapse function $N(t)$ as in (4.113) to obtain:

$$ds^2_{d+2} = -N(t)^2dt^2 + \beta(t)^2\left(\frac{dz^2}{z^2} + \frac{1}{z^2}\left[\tilde{g}_{ij} + z^d\left(1-\left(\beta(t)/\ell\right)^{-d}\right)\mathcal{T}_{(0)ij}\right]dx^i dx^j + \mathcal{O}(z^{>-2+d})\right) , \tag{4.120}$$

with $\dot{\beta} = N(t)$. Given the above asymptotic solution we then proceed to compute the holographic stress tensors. The counterterm action will again be the same as in (4.62) because our asymptotic solution for the Ricci-flat development only differs from the previous solution (4.41) at order z^{-2+d}. The divergences of the on-shell action involve the asymptotic form of the Ricci-flat metric below this order and the trace of $\tilde{h}_{ij}$ with respect to $g_{(0)}$ at order z^{-2+d}. However, since the trace of $\mathcal{T}_{(0)ij}$ vanishes, the trace of $\tilde{h}_{ij}$ is the same as that computed from (4.41) up to order z^{-2+d} and hence we find the same divergent terms as previously, as well as the same counterterms. If we then compute the holographic stress tensors according to formula (4.71), we find them to be of the same form as in (4.73) plus an additional contribution:

$$\langle T_{ij}\rangle \ = \ \frac{d\,\beta(t)^d}{2\kappa^2_{d+2}}\left(g_{(d)ij} + \mathcal{T}_{(0)ij} + X_{ij}[g_{(0)}]\right) - \frac{d\,\ell^d}{2\kappa^2_{d+2}}\mathcal{T}_{(0)ij} \ . \tag{4.121}$$

Notice that we recover the result (4.73) for the initial hypersurface $\Sigma = \{\beta(t) = \ell\}$. The diffeomorphism Ward identities are still as in equation (4.76) and the trace Ward identities are also as in (4.77) for odd values of d. For even values the trace depends on d. For $d+2=4$, we have:

$$\begin{aligned} \langle T_{ij} \rangle &= \frac{\beta(t)^2}{\kappa_4^2}\left(g_{(2)ij} + \mathcal{T}_{(0)ij} - g_{(0)ij}\mathrm{Tr}[g_{(0)}^{-1}g_{(2)}]\right) - \frac{\ell^2}{\kappa_4^2}\mathcal{T}_{(0)ij} \ , & (4.122) \\ \langle T^i_i \rangle &= \frac{\beta(t)^2}{2\kappa_4^2} R[g_{(0)}] \ = \ \frac{c}{24\pi} R[g_{(0)}] \ . & (4.123) \end{aligned}$$

For small and large values of the central charge, we find the asymptotic behaviours:

$$\langle T_{ij} \rangle \ = \ \begin{cases} -\frac{\ell^2}{\kappa_4^2}\mathcal{T}_{(0)ij} + \mathcal{O}(\beta^2) & : \ \beta \ll 1 \ , \\ \frac{\beta^2}{\kappa_4^2}\left(g_{(2)ij} + \mathcal{T}_{(0)ij} - g_{(0)ij}\mathrm{Tr}[g_{(0)}^{-1}g_{(2)}]\right) + \mathcal{O}(\beta^0) & : \ \beta \gg 1 \ . \end{cases} \qquad (4.124)$$

Different field theories in the family, identified by their central charges, have therefore different expectation values of the stress tensors and, in order to reconstruct the bulk spacetime metric from CFT data in this case, we need such expectation values in the regimes of large and small central charges.

The last case to be analysed is for $\alpha < d$. In this case, only the three conditions (4.102) on the leading term $\mathcal{T}_{(0)ab}$ hold. The strategy is again to start from an ansatz as in (4.114), where the power z^d is now replaced by z^α, and find the differential equation and constraints that Δ_{ij} needs to satisfy in order for the dynamical equation to be solved up to order $z^{-2+\alpha}$. We then compute the initial values $\tilde{h}(t=0)$ and $\partial_t \tilde{h}(t=0)$ and find the relations between the integration constants in the solution for Δ_{ij} and the leading order $\mathcal{T}_{(0)ij}$ in order for the initial conditions to be satisfied up to order $z^{-2+\alpha}$. The unique solution to this initial value problem must then coincide with our refined ansatz up to order $z^{-2+\alpha}$. The next step is to attempt to renormalize the gravitational action (4.48). If we regularize it as previously and evaluate it on-shell, we find that the divergences involve the asymptotic form of the Ricci-flat metric up to order z^{-2+d} as before. However, we have just deduced that this asymptotic form involves terms already at order $z^{-2+\alpha}$ (and beyond) that are not pure functionals of $g_{(0)ij}$. Such terms are rather functionals of the unspecified leading and subleading orders $\mathcal{T}_{(n)ab}$ in the asymptotic expansion of $\mathcal{T}_{ab}$. In this way, the divergences will be functionals of the undetermined terms $g_{(0)}$ and $\mathcal{T}_{(n)}$. This implies that we cannot rewrite the divergences covariantly in terms of the induced metric $\gamma_{ij} = h_{ij}$ (or γ_{AB}) as in equation (4.54). We can invert the asymptotic series for $\tilde{h}_{ij}$ as before in order to rewrite $g_{(0)ij}$ covariantly in terms of γ_{ij}, but now only up to order $z^{-2+\alpha}$, which is below z^{-2+d}. Furthermore, the divergences involving the terms $\mathcal{T}_{(n)}$ cannot be rewritten in a covariant fashion because the asymptotic expansion for $\mathcal{T}_{ab}$ is undetermined and unrelated to γ_{ij} and hence cannot be inverted in order to express the terms $\mathcal{T}_{(n)}$ as functionals of γ_{ij}. In the language of holographic renormalization, this type of divergent terms are said to be non-local in the sources. We therefore find that if the exponent $\alpha < d$, the Ricci-flat development is non-renormalizable holographically in the sense that the gravitational action for the development cannot be renormalized with local covariant counterterms and the only perturbations of the initial data $K_{[0]}$ away from the hyperboloidal one that result in renormalizable developments must occur at order equal to or greater than z^{-2+d}.

4.6. Appendix: Asymptotic solutions of the dynamical equation

4.6.1. Solution for $\alpha > d$

In this section we show that the ansatz (4.110)–(4.112) solves the dynamical equation (4.109) up to order z^{-2+d}. Let us begin by performing an ADM decomposition of $\tilde{h}_{ab}$ with respect to surfaces of constant z:

$$\tilde{h}_{ab}dx^a dx^b = M^2 dz^2 + \gamma_{ij}(dx^i + A^i dz)(dx^j + A^j dz) \; , \tag{4.125}$$

with unit normal $n_a = M\partial_a z$. The inverse of $\tilde{h}_{ab}$ is given by:

$$\tilde{h}^{ab} = \begin{pmatrix} M^{-2} & -M^{-2}A^i \\ -M^{-2}A^j & \gamma^{ij} + M^{-2}A^i A^j \end{pmatrix} \; . \tag{4.126}$$

We then start from the following ansatz:

$$\begin{cases} M \;=\; z^{-1} + \mathcal{O}(z^{>-1+d}) \quad , \\ A_i \;=\; (1-e^{-dt})A_{[d]i}(z,x) + \mathcal{O}(z^{>-1+d}) \quad : \quad A_{[d]i} = \mathcal{O}(z^{>-2+d}) \quad , \\ \gamma_{ij} \;=\; z^{-2}g_{ij} \quad : \quad g_{ij}(t,z,x) = \tilde{g}_{ij}(z,x) + \mathcal{O}(z^{>d}) \; , \; \tilde{g}_{ij} = \mathcal{O}(z^0) \quad , \end{cases} \tag{4.127}$$

such that:

$$\tilde{h}_{ab}dx^a dx^b = \frac{1}{z^2}\left(dz^2 + \tilde{g}_{ij}dx^i dx^j\right) + \mathcal{O}(z^{>-2+d}) \; . \tag{4.128}$$

The reason for the above dependence of the shift $A_i = \gamma_{ij}A^j$ on t at orders between $\mathcal{O}(z^{>-2+d})$ and $\mathcal{O}(z^{-1+d})$ will become clear later. If we replace this ansatz in the dynamical equation:

$$2\left(R_{ab}[\tilde{h}] + d\,\tilde{h}_{ab}\right) + \ddot{\tilde{h}}_{ab} + d\,\dot{\tilde{h}}_{ab} + \tilde{h}_{ab}\,\mathrm{Tr}[\tilde{h}^{-1}\dot{\tilde{h}}] + \frac{1}{2}\,\dot{\tilde{h}}_{ab}\,\mathrm{Tr}[\tilde{h}^{-1}\dot{\tilde{h}}] - \left(\dot{\tilde{h}}\tilde{h}^{-1}\dot{\tilde{h}}\right)_{ab} = 0 \; , \tag{4.129}$$

we find:

$$R_{ij}[\tilde{h}] + d\,\tilde{h}_{ij} + \mathcal{O}(z^{>-2+d}) \;=\; 0 \quad , \tag{4.130}$$

$$n^a n^b R_{ab}[\tilde{h}] + d + \mathcal{O}(z^{>d}) \;=\; 0 \quad , \tag{4.131}$$

$$n^a R_{ai}[\tilde{h}] + \frac{1}{2}\,n^a\left(\ddot{\tilde{h}}_{ai} + d\,\dot{\tilde{h}}_{ai}\right) + \mathcal{O}(z^{>-1+2d}) \;=\; 0 \quad . \tag{4.132}$$

We begin by analysing the spatial components (i,j). In order to do so, we need the Gauss-Codazzi identity:

$$K'_{ij} = \pounds_A K_{ij} - D_i D_j M + M\left(R_{ij}[\gamma] + 2(K\gamma^{-1}K)_{ij} - K_{ij}\mathrm{Tr}[\gamma^{-1}K] - R_{ij}[\tilde{h}]\right)\,, \tag{4.133}$$

$$K_{ij} = \frac{1}{2M}\left(\gamma'_{ij} - \pounds_A\gamma_{ij}\right)\ , \tag{4.134}$$

where prime denotes differentiation with respect to z and $D_i\gamma_{jk} := 0$. If we use our ansatz in (4.134), we find:

$$K_{ij} = -z^{-2}g_{ij} + \frac{1}{2}z^{-1}g'_{ij} + \mathcal{O}(z^{>-2+d})\ . \tag{4.135}$$

With this result, the Gauss-Codazzi identity becomes:

$$\begin{aligned} 2\left(R_{ij}[\tilde{h}] + \frac{d}{z^2}\,g_{ij}\right) \;&=\; 2R_{ij}[g] - g''_{ij} + \frac{d-1}{z}\,g'_{ij} + \frac{1}{z}\,g_{ij}\mathrm{Tr}[g^{-1}g'] - \frac{1}{2}\,g'_{ij}\mathrm{Tr}[g^{-1}g'] \\ &\quad + \left(g'g^{-1}g'\right)_{ij} + \mathcal{O}(z^{>-2+d})\ . \end{aligned} \tag{4.136}$$

If we then replace this identity in (4.130), we obtain:

$$2R_{ij}[g] - g''_{ij} + \frac{d-1}{z}\,g'_{ij} + \frac{1}{z}\,g_{ij}\mathrm{Tr}[g^{-1}g'] - \frac{1}{2}\,g'_{ij}\mathrm{Tr}[g^{-1}g'] + \left(g'g^{-1}g'\right)_{ij} + \mathcal{O}(z^{>-2+d}) = 0\,. \tag{4.137}$$

The above is the Fefferman-Graham (FG) equation plus additional contributions at order $z^{>-2+d}$ and therefore is solved by the FG solution up to order z^d in g_{ij}:

$$g_{ij} = \underbrace{g_{(0)ij} + z^2 g_{(2)ij} + \ldots + z^d \log z\,\tilde{g}_{(d)ij} + z^d g_{(d)ij}}_{\tilde{g}} + \mathcal{O}(z^{>d})\ , \tag{4.138}$$

with $g_{(n)}$ the FG coefficients. In this way, our ansatz (4.110)–(4.112) solves the spatial components of the dynamical equation up to order z^{-2+d}. The same procedure can be applied to the remaining components (4.131) and (4.132). For the former, we need the Gauss-Codazzi identity:

$$-R[\gamma] + \mathrm{Tr}^2[\gamma^{-1}K] - \mathrm{Tr}[\gamma^{-1}K\gamma^{-1}K] \;=\; n^a n^b R_{ab}[\tilde{h}] - \gamma^{ij}R_{ij}[\tilde{h}]\ , \tag{4.139}$$

where $R[\gamma]$ is the Ricci-scalar of γ_{ij} and $n^a = \tilde{h}^{ab} n_b \; : \; n_b = M \partial_b z$. If we use our ansatz (4.127) in this identity, together with the expression (4.135) for K_{ij} and the previous Gauss-Codazzi identity (4.136), we find:

$$n^a n^b R_{ab}[\tilde{h}] = -d - \frac{z^2}{2} \left(\mathrm{Tr}[g^{-1} g''] - \frac{1}{z} \mathrm{Tr}[g^{-1} g'] - \frac{1}{2} \mathrm{Tr}[g^{-1} g' g^{-1} g'] \right) + \mathcal{O}(z^{>d}) \; . \tag{4.140}$$

If we then replace this identity in the equation (4.131), we obtain:

$$\mathrm{Tr}[g^{-1} g''] - \frac{1}{z} \mathrm{Tr}[g^{-1} g'] - \frac{1}{2} \mathrm{Tr}[g^{-1} g' g^{-1} g'] + \mathcal{O}(z^{>-2+d}) = 0 \; . \tag{4.141}$$

The above is the FG trace equation plus additional contributions at order $z^{>-2+d}$ and is solved by the FG solution up to order z^d in g_{ij}. Hence, the components (4.131) of the dynamical equation are also solved by our ansatz (4.110)–(4.112) up to order z^{-2+d} in $\tilde{h}_{ab}$. Finally, for the remaining components (4.132) we need the last Gauss-Codazzi identity:

$$D_j \left(\gamma^{-1} K \right)^j_{\ i} - D_i \mathrm{Tr}[\gamma^{-1} K] = n^a R_{ai}[\tilde{h}] \; . \tag{4.142}$$

If we use again the ansatz (4.127) in this identity, we find:

$$n^a R_{ai}[\tilde{h}] = \frac{z}{2} \left(\nabla_j \left(g^{-1} g' \right)^j_{\ i} - \nabla_i \mathrm{Tr}[g^{-1} g'] \right) + \mathcal{O}(z^{>d}) \; , \tag{4.143}$$

where $\nabla_i g_{jk} := 0$. By replacing this identity in (4.132), we obtain:

$$\nabla_j \left(g^{-1} g' \right)^j_{\ i} - \nabla_i \mathrm{Tr}[g^{-1} g'] + z^{-1} n^a \left(\ddot{\tilde{h}}_{ai} + d \, \dot{\tilde{h}}_{ai} \right) + \mathcal{O}(z^{>-1+d}) = 0 \; . \tag{4.144}$$

Now notice that $n = z \partial_z + \mathcal{O}(z^{>1+d})$ and hence that:

$$z^{-1} n^a \left(\ddot{\tilde{h}}_{ai} + d \, \dot{\tilde{h}}_{ai} \right) = \ddot{\tilde{h}}_{zi} + d \, \dot{\tilde{h}}_{zi} + \mathcal{O}(z^{>-2+2d}) \; . \tag{4.145}$$

Since $\tilde{h}_{zi} = A_i$, we obtain:[36]

$$z^{-1} n^a \left(\ddot{\tilde{h}}_{ai} + d \, \dot{\tilde{h}}_{ai} \right) = \mathcal{O}(z^{>-1+d}) \; , \tag{4.146}$$

and hence equation (4.144) becomes:

$$\nabla_j \left(g^{-1} g' \right)^j_{\ i} - \nabla_i \mathrm{Tr}[g^{-1} g'] + \mathcal{O}(z^{>-1+d}) = 0 \; . \tag{4.147}$$

This last equation is the FG divergence equation plus additional contributions at order $z^{>-1+d}$ and is also solved by the FG solution up to order z^d in g_{ij}. Hence, we conclude that our ansatz (4.110)–(4.112) solves the dynamical equation (4.109) up to order z^{-2+d} in $\tilde{h}_{ab}$.

[36]In general we can have $A_i = A_{[0]i}(z,x) - e^{-dt} A_{[d]i}(z,x) + \mathcal{O}(z^{>-1+d}) \; : \; A_{[0,d]i} = \mathcal{O}(z^{>-2+d})$, but we set $A_{[0]} = A_{[d]}$ so that our ansatz satisfies the initial condition (4.107).

4.6.2. Solution for $\alpha = d$

We now show that the ansatz (4.117)–(4.118) solves the dynamical equation (4.109) up to order z^{-2+d}. We perform again the ADM decomposition (4.125) of $\tilde{h}_{ab}$ and start from the ansatz (4.127), but now do not impose the dependence of g_{ij} on t to arise at order $z^{>d}$:

$$\begin{cases} M \;=\; z^{-1} + \mathcal{O}(z^{>-1+d}) \quad , \\ A_i \;=\; (1-e^{-dt})A_{[d]i}(z,x) + \mathcal{O}(z^{>-1+d}) \quad : \quad A_{[d]i} = \mathcal{O}(z^{>-2+d}) \quad , \\ \gamma_{ij} = z^{-2} g_{ij}(t,z,x) \;:\; g_{ij} = \mathcal{O}(z^0) \;. \end{cases} \tag{4.148}$$

If we replace such ansatz in the dynamical equation (4.129) and use again the Gauss-Codazzi identities as in the previous section, we obtain:

$$\begin{aligned} &-\ddot{g}_{ij} - d\,\dot{g}_{ij} - g_{ij}\mathrm{Tr}[g^{-1}\ddot{g}] - \frac{1}{2}\,\dot{g}_{ij}\mathrm{Tr}[g^{-1}\dot{g}] + \left(\dot{g}g^{-1}\dot{g}\right)_{ij} = \\ &z^2\left(2R_{ij}[g] - g''_{ij} + \frac{d-1}{z}\,g'_{ij} + \frac{1}{z}\,g_{ij}\,\mathrm{Tr}[g^{-1}g'] - \frac{1}{2}\,g'_{ij}\mathrm{Tr}[g^{-1}g'] + \left(g'g^{-1}g'\right)_{ij}\right) + \mathcal{O}(z^{>d}) \;, \end{aligned} \tag{4.149}$$

$$\mathrm{Tr}[g^{-1}\ddot{g}] \;=\; z^2\left(\mathrm{Tr}[g^{-1}g''] - \frac{1}{z}\,\mathrm{Tr}[g^{-1}g'] - \frac{1}{2}\,\mathrm{Tr}[g^{-1}g'g^{-1}g']\right) + \mathcal{O}(z^{>d}) \;, \tag{4.150}$$

$$\nabla_j\left(g^{-1}g'\right)_i^{\;j} - \nabla_i\mathrm{Tr}[g^{-1}g'] + +\mathcal{O}(z^{>-1+d}) \;=\; 0 \;. \tag{4.151}$$

These are the analogues of equations (4.137), (4.141) and (4.147). We then seek for a solution of the form:

$$\begin{aligned} g_{ij}(t,z,x) \;=\;& g_{(0)ij}(x) + z^2 g_{(2)ij}(x) + ... + z^d\left(g_{(d)ij}(x) + \Delta_{ij}(t,x) - \Delta_{[0]ij}(x)\right) \\ &+ z^d\log z\,\tilde{g}_{(d)ij}(x) + \mathcal{O}(z^{>d}) \;, \end{aligned} \tag{4.152}$$

where only even powers in z arise below the order z^d and where each coefficient $g_{(n)}$ is defined to be the FG coefficient in the solution (4.138). Also, $\Delta_{[0]} := \Delta(t=0)$. If we replace for g_{ij} in (4.149)–(4.151), we find that the equations are satisfied up to order z^d in g_{ij} if Δ_{ij} obeys the equations:

$$\ddot{\Delta}_{ij} + d\dot{\Delta}_{ij} + g_{(0)ij}\mathrm{Tr}[g_{(0)}^{-1}\ddot{\Delta}] + d\,g_{(0)ij}\mathrm{Tr}[g_{(0)}^{-1}\left(\Delta - \Delta_{[0]}\right)] = 0 \;, \tag{4.153}$$

$$\mathrm{Tr}[g_{(0)}^{-1}\ddot{\Delta}] - d(d-2)\mathrm{Tr}[g_{(0)}^{-1}\left(\Delta - \Delta_{[0]}\right)] = 0 \;, \tag{4.154}$$

$${}^{(0)}\nabla_j\left(g_{(0)}^{-1}\left(\Delta - \Delta_{[0]}\right)\right)_i^{\;j} - \partial_i\mathrm{Tr}[g_{(0)}^{-1}\left(\Delta - \Delta_{[0]}\right)] = 0 \;, \tag{4.155}$$

where ${}^{(0)}\nabla_j g_{(0)ik} := 0$. Suppose that $\mathrm{Tr}[g^{-1}_{(0)}\dot{\Delta}] = 0$. Then, equation (4.154) is solved (recall that $\dot{g}_{(0)} = 0$) and equation (4.153) becomes:

$$\ddot{\Delta}_{ij} + d\,\dot{\Delta}_{ij} = 0 \quad \Rightarrow \quad \Delta_{ij}(t,x) = \Delta_{[0]ij}(x) + \left(1 - e^{-dt}\right)\Delta_{[d]ij}(x)\ . \tag{4.156}$$

If we insert this solution in the remaining equation (4.155), we obtain:

$${}^{(0)}\nabla_j \left(g^{-1}_{(0)}\Delta_{[d]}\right)^j_{\ i} = 0\ . \tag{4.157}$$

In this way, the ansatz:

$$\begin{aligned} g_{ij}(t,z,x) \;&=\; g_{(0)ij}(x) + z^2 g_{(2)ij}(x) + ... + z^d \left(g_{(d)ij}(x) + \left(1 - e^{-dt}\right)\Delta_{[d]ij}(x)\right) \\ &+\; z^d \log z\, \tilde{g}_{(d)ij}(x) + \mathcal{O}(z^{>d})\ : \\ \mathrm{Tr}[g^{-1}_{(0)}\Delta_{[d]}] = 0\ ,\ \ {}^{(0)}\nabla_j \Delta^j_{[d]i} = 0\ , \end{aligned} \tag{4.158}$$

is a solution to the equations (4.149)–(4.151) up to order z^d in g_{ij} and therefore our ansatz (4.117)–(4.118), or (4.148) with g_{ij} as above, solves the dynamical equation (4.109) up to order z^{-2+d} in $\tilde{h}_{ab}$.

Appendices

Appendix A

A.1. Diffeomorphisms

A homeomorphism is a map f between two topological spaces:

$$f : M \to N \ , \tag{A.1}$$

such that f and f^{-1} are continuous.

If such map exists, then M and N necessarily have the same dimension and are said to be homeomorphic to each other. Homeomorphisms provide an equivalence relation between topological spaces and therefore M and N are said to have the same topology if they are homeomorphic. Heuristically, M and N are "equivalent" as topological spaces, *i.e.* homeomorphic, if we can deform one into the other continuously without tearing them.

A diffeomorphism is a map φ between two manifolds:

$$\begin{aligned} \varphi : M \quad &\to \quad N \\ p \quad &\to \quad q = \varphi(p) \ , \end{aligned} \tag{A.2}$$

such that φ and φ^{-1} are smooth.

If such map exists, then M and N are said to be diffeomorphic to each other. Clearly, diffeomorphisms are homeomorphisms and provide a more stringent equivalence relation between manifolds. Two manifolds are said to have the same manifold structure if they are diffeomorphic to each other. Heuristically, M and N are "equivalent" as manifolds, *i.e.* diffeomorphic, if we can deform one into the other smoothly.[1]

[1]Caveat: In General Relativity, two manifolds diffeomorphic to each other are physically

An automorphism in the category of differentiable manifolds is a diffeomorphism $\varphi : M \to M$ from a manifold M to itself. The set of automorphisms on M is a group denoted as Diff(M) and called the diffeomorphism group of M.

If we are given a coordinate system x^μ and $\bar{x}^\mu$ on (some open subset of) M and N, respectively, we may express the diffeomorphism (A.2) as:

$$\begin{aligned} \varphi : M &\to N \\ x^\mu &\to \bar{x}^\mu = \varphi^\mu(x) \ . \end{aligned} \tag{A.3}$$

Diffeomorphisms are *active* transformations and map points between manifolds, or move the points of a manifold in which case $N = M$. If φ is an automorphism, however, one may adopt a *passive* viewpoint and rather see the action of φ as resulting in an ordinary coordinate reparametrization on M from some coordinate system x^μ to a new coordinate system $\bar{x}^\mu := \varphi^\mu(x)$. This is so because, under an automorphism φ, a tensor T at x is mapped to the pushed-forward tensor $\varphi_\star T$ at $\bar{x} = \varphi(x)$ and the components of $\varphi_\star T$ are equal to the components of T in the new coordinate system $\bar{x}^\mu$.[2] A simple example is that of T a vector field $V(x) \in T_xM$. Under the action of the automorphism it transforms as:

$$V^\mu(x) \quad \xrightarrow{\varphi} \quad \bar{V}^\mu(\bar{x}) = (\varphi_\star V)^\mu(\bar{x}) = \frac{d\varphi^\mu}{dx^\alpha}\, V^\alpha(\varphi^{-1}(\bar{x})) = \frac{d\varphi^\mu}{dx^\alpha}\, V^\alpha(x) \ . \tag{A.4}$$

Automorphisms can therefore be seen as reparametrizations and, for this reason, the diffeomorphism group Diff(M) is also called the group of reparametrizations

equivalent only if they are also (locally) isometric – *i.e.* if there exists a diffeomorphism between the two that preserves distances, such that the pushed-forward metric of M is equal to the metric of N *c.f.* section A.2 – . A simple example of two diffeomorphic manifolds that are not physically equivalent is that of the Clifford torus (with a flat metric induced from the embedding in $\mathbb{R}^4$) and the ring torus (with a curved metric induced from the embedding in $\mathbb{R}^3$). They are the "same" manifold, *i.e.* diffeomorphic, but with two physically inequivalent metrics.

[2] The same discussion applies to the case of T the metric tensor, but extra care needs to be taken in this case. If we are given a metric $g_{\mu\nu}dx^\mu dx^\nu$ on M and perform an ordinary transformation of coordinates by introducing $\bar{x}^\mu := \varphi^\mu(x)$, we measure distances with the metric components in the new coordinate system: $\bar{g}_{\mu\nu}d\bar{x}^\mu d\bar{x}^\nu$, and the distance between two points p and q remains the same. On the other hand, if we perform the diffeomorphism (A.3), the metric $g_{\mu\nu}$ is mapped to a bilinear form with components $(\varphi_\star g)_{\mu\nu} = \bar{g}_{\mu\nu}$ in the target manifold. Unless φ is an isometry, however, this bilinear form is different than the metric of the target manifold with which we measure distances (and which is part of the initial data). The distance between two points $P = \varphi(p)$ and $Q = \varphi(q)$ therefore will not be the same as the distance between p and q unless φ is an isometry. So, even though $g_{\mu\nu}$ is indeed mapped to the pushed-forward form $\varphi_\star g$ with components equal to $\bar{g}_{\mu\nu}$, the metric used to measure distances in each scenario is different. To avoid possible ambiguities, it is common practice in the high-energy literature to implicitly define the metric of the target manifold to be equal to the pushed-forward metric, so that distances after a transformation of coordinates or after a corresponding diffeomorphism are measured by the same metrics. In conclusion, diffeomorphisms can be truly seen as reparametrizations only if they are also isometries. See also section A.2.

of M.

Finally, we define compact and closed manifolds as follows. A compact manifold, in the sense of limit point compactness, is any manifold in which every sequence of points in the manifold has a limit point in the manifold. Examples are the sphere and the disk and counterexamples are the open disk and the plane. A closed manifold is any compact manifold without boundary. An example is the sphere and a counterexample is the disk.

A.2. Conformal and Weyl Transformations

Two metrics $g_{\mu\nu}$ and $\widetilde{g}_{\mu\nu}$ on some manifold M are said to be conformally equivalent, or to belong to the same conformal class, if they are equal up to a positive local factor:

$$\widetilde{g}_{\mu\nu}(x) = e^{2\phi(x)} g_{\mu\nu}(x) \ . \tag{A.5}$$

Let (M, g) and (N, h) be two manifolds diffeomorphic to each other. A conformal transformation (or conformal isometry) is a diffeomorphism $\varphi : M \to N$ such that the pulled-back metric $(\varphi^\star h)(x)$ is conformally equivalent to $g(x)$:

$$\varphi^\star\big(h(\varphi(x))\big) = e^{2\phi(x)} g(x) \ . \tag{A.6}$$

If the conformal factor $e^{2\phi(x)} = 1$ the conformal transformation is called in particular an isometry and if $\phi(x)$ is constant it is called a scale transformation. Isometries are therefore diffeomorphisms that preserve distances, whereas conformal transformations in general only preserve angles. It should be emphasized that it is common practice to restrict conformal transformations to be automorphisms such that $(N, h) = (M, g)$. One also defines the conformal group (resp. isometry group) of (M, g) as the set of all automorphisms $\varphi : M \to M$ that are also conformal transformations such that $\varphi^\star g \propto g$ (resp. $\varphi^\star g = g$).

The simplest example of a conformal transformation is the map between the cylinder $M = \mathbb{R} \times S^1$ and the plane $N = \mathbb{R}^2$. If we parametrise (M, g) and (N, h) as:

$$g_{\mu\nu} dx^\mu dx^\nu = d\tau^2 + d\sigma^2 \ , \tag{A.7}$$

$$h_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu = dr^2 + r^2 d\theta^2 \ , \tag{A.8}$$

with $\sigma, \theta \in [0, 2\pi[$, and define the diffeomorphism:

$$\begin{aligned} \varphi : \quad & M \to N \\ & (\tau, \sigma) \to (r, \theta) = (e^\tau, \sigma) \ , \end{aligned} \tag{A.9}$$

we find that: $(\varphi^\star h)_{\mu\nu} = e^{2\tau} g_{\mu\nu}$, and therefore φ is a conformal transformation. Since the metric $h_{\mu\nu}$ of the target manifold is not equal to the pushed-forward metric $(\varphi_\star g)_{\mu\nu}$, the distance between two points $P = \varphi(p)$ and $Q = \varphi(q)$ on N is not the same as the distance between p and q, but the angles between vectors are preserved. In contrast, if $M = \mathbb{R}^2$ parametrised as above with $\tau, \sigma \in \mathbb{R}$, an isometry would be the diffeomorphism $\varphi : (\tau, \sigma) \to (r, \theta) = (\sqrt{\tau^2 + \sigma^2}\,, \arctan(\tau/\sigma))$, such that: $(\varphi^\star h)_{\mu\nu} = g_{\mu\nu}$. In this case, distances between points are preserved by the diffeomorphism.

A Weyl transformation is closely related to a conformal transformation: given a manifold (M, g), it is a local rescaling of the metric of the form: $g_{\mu\nu}(x) \to \Omega^2(x) g_{\mu\nu}(x)$. Incidentally, this is called a conformal transformation in some of the literature, but we will not use this terminology. Note that the conformal factor $\Omega^2(x)$ in a Weyl rescaling can be any positive analytic function, whereas the set of conformal factors obtained by considering all conformal transformations $\varphi : M \to M$ is much smaller than the set of all positive analytic functions.

Finally, an expression (*e.g.* an action) is said to be conformally invariant if it is invariant under a two-step transformation:[3]

i) A conformal transformation: $$x^\mu \to \bar{x}^\mu = \varphi^\mu(x) \;:\; (\varphi^\star g)_{\mu\nu} = \Omega^2(x) g_{\mu\nu} \,, \tag{A.10}$$

ii) A Weyl transformation: $$g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu} \,. \tag{A.11}$$

We will call the combined transformations a conformal-Weyl transformation. Under such a transformation the metric components transform as:

$$g_{\mu\nu}(x) \stackrel{\text{conf.}}{\longrightarrow} \bar{g}_{\mu\nu}(\bar{x}) = (\varphi_\star g)_{\mu\nu}(\bar{x}) = \Omega^{-2}(\varphi^{-1}(\bar{x}))\, g_{\mu\nu}(\varphi^{-1}(\bar{x})) = \Omega^{-2}(x)\, g_{\mu\nu}(x) \stackrel{\text{Weyl}}{\longrightarrow} g_{\mu\nu}(x) \,,$$

and therefore the metric components remain invariant. On the other hand, the line element ds^2 is a scalar and therefore transforms as:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \stackrel{\text{conf.}}{\longrightarrow} \bar{g}_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu = g_{\mu\nu} dx^\mu dx^\nu \stackrel{\text{Weyl}}{\longrightarrow} \Omega^2 g_{\mu\nu} dx^\mu dx^\nu = \Omega^2 ds^2 \,. \tag{A.12}$$

Since conformal transformations are a subset of coordinate reparametrizations, every expression invariant under general coordinate transformations is invariant under conformal transformations and therefore will be conformally invariant iff it is Weyl invariant. It can be shown that Weyl invariance of a theory and the tracelessness of its stress-energy tensor are equivalent properties. See also section A.4 on the relation between conformal invariance and conformal dimensions.

[3]Note the misnomer: invariance under conformal transformations alone (or Weyl transformations alone) does not necessarily imply conformal invariance.

A.3. Conformal and Superconformal Group

Let us then study the set of conformal transformations in flat spacetime, *i.e.* the conformal group $\mathrm{Conf}(\mathbb{R}^{1,d-1})$. Suppose we perform an infinitesimal automorphism on a manifold (M, g) generated by a vector field $\xi(x)$:

$$x^\mu \to \bar{x}^\mu = \varphi^\mu(x) = x^\mu + \varepsilon \xi^\mu(x) + \mathcal{O}(\varepsilon^2) \ . \tag{A.13}$$

The pulled-back metric reads:

$$\begin{aligned}
(\varphi^\star g)_{\mu\nu}(x) &= \frac{d\varphi^\alpha}{dx^\mu}\frac{d\varphi^\beta}{dx^\nu}\, g_{\alpha\beta}(\varphi(x)) \\
&= g_{\mu\nu}(\varphi(x)) + \varepsilon\left(g_{\mu\beta}(\varphi(x))\partial_\nu\xi^\beta + g_{\alpha\nu}(\varphi(x))\partial_\mu\xi^\alpha\right) + \mathcal{O}(\varepsilon^2) \\
&= g_{\mu\nu}(x) + \varepsilon\left(\xi^\alpha\partial_\alpha g_{\mu\nu}(x) + g_{\mu\beta}(x)\partial_\nu\xi^\beta + g_{\alpha\nu}(x)\partial_\mu\xi^\alpha\right) + \mathcal{O}(\varepsilon^2) \\
&= g_{\mu\nu}(x) + \varepsilon \pounds_\xi g_{\mu\nu}(x) + \mathcal{O}(\varepsilon^2) \ .
\end{aligned} \tag{A.14}$$

If this is a conformal transformation we find that the generator ξ satisfies:

$$\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu \ = \ \pounds_\xi g_{\mu\nu} \ = \ 2\phi(x) g_{\mu\nu} \ , \tag{A.15}$$

where: $\phi(x) = d^{-1}\nabla_\mu\xi^\mu$. The generator ξ is called a Killing (resp. conformal Killing) vector field of $g_{\mu\nu}$ if $\phi = 0$ (resp. $\phi \neq 0$). If the metric $g_{\mu\nu}$ is the Minkowski metric $\eta_{\mu\nu}$, we can solve this equation explicitly for ξ and obtain:

$$i\xi^\mu\partial_\mu = a^\mu P_\mu + \omega^{\mu\nu}L_{\mu\nu} + \lambda D + b^\mu K_\mu \ , \tag{A.16}$$

where a^μ, λ and b^μ are constants, ω is an antisymmetric constant matrix, and the operators P, L, D and K are defined as:

$$\begin{aligned}
P_\mu = i\partial_\mu \quad &, \qquad L_{\mu\nu} = -i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right) \ , \\
D = ix^\mu\partial_\mu \quad &, \qquad K_\mu = i\big(2x_\mu x^\nu - (x\cdot x)\delta^\nu_\mu\big)\partial_\nu \ .
\end{aligned} \tag{A.17}$$

The conformal Killing ξ corresponds to a linear combination of two Killing vector fields and two conformal Killings, respectively, that generate the infinitesimal transformations:

$$\begin{aligned}
&x^\mu \to (1 - i\varepsilon\, a\cdot P)\, x^\mu = x^\mu + \varepsilon\, a^\mu \ , \\
&x^\mu \to (1 - i\varepsilon\, \omega\cdot L)\, x^\mu = x^\mu + \varepsilon\, \omega^\mu{}_\nu x^\nu \ , \\
&x^\mu \to (1 - i\varepsilon\, \lambda D)\, x^\mu = x^\mu + \varepsilon\, \lambda\, x^\mu \ , \\
&x^\mu \to (1 - i\varepsilon\, b\cdot K)\, x^\mu = x^\mu + \varepsilon\left(2(b\cdot x)x^\mu - (x\cdot x)b^\mu\right) \ .
\end{aligned}$$

Each transformation can be exponentiated (*i.e.* consider an infinite sequence of infinitesimal transformations) and results in the finite conformal transformations:

Diffeomorphisms		Generator	Conf. Factor
translations	$x^\mu \to x^\mu + a^\mu$	P_μ	1
rotations and boosts	$x^\mu \to \Lambda^\mu{}_\nu x^\nu$	$L_{\mu\nu}$	1
dilatations	$x^\mu \to e^\lambda\, x^\mu$	D	$e^{2\lambda}$
special conformal	$x^\mu \to \dfrac{x^\mu - b^\mu(x\cdot x)}{1-2(x\cdot b)+(b\cdot b)(x\cdot x)}$	K_μ	$\dfrac{(x-bx^2)^4}{x^4}$

The first two transformations correspond to isometries of the Minkowski metric, with $SO(1,d-1) \ni \Lambda = \exp(-i\omega\cdot L)$ in the fundamental representation, and form the Poincaré isometry group, while the third corresponds to a scale transformation. The special conformal transformation corresponds to a sequence of diffeomorphisms given by an inversion $x^\mu \to x^\mu/(x\cdot x)$, a translation $x^\mu \to x^\mu + b^\mu$ and again another inversion and results in the pulled-back metric $(\varphi^\star\eta)_{\mu\nu} = \big(1-2(x\cdot b)+(b\cdot b)(x\cdot x)\big)^2\,\eta_{\mu\nu}$.

The differential operators (A.17) form a representation of the generators of the conformal algebra in the space of functions, an infinite-dimensional representation space, and satisfy the Lie algebra:

$$
\begin{aligned}
& i[L_{\mu\nu},L_{\alpha\beta}] = \eta_{\nu\alpha}L_{\mu\beta}+\eta_{\mu\beta}L_{\nu\alpha}-\eta_{\mu\alpha}L_{\nu\beta}-\eta_{\nu\beta}L_{\mu\alpha}\ , && i[D,P_\mu] = P_\mu\ ,\\
& i[L_{\mu\nu},P_\alpha] = \eta_{\nu\alpha}P_\mu-\eta_{\mu\alpha}P_\nu\ , && i[D,K_\mu] = -K_\mu\ ,\\
& i[L_{\mu\nu},K_\alpha] = \eta_{\nu\alpha}K_\mu-\eta_{\mu\alpha}K_\nu\ , && i[P_\mu,K_\alpha] = -2L_{\mu\alpha}-2\eta_{\mu\alpha}D\ ,
\end{aligned}
$$

with all other commutators vanishing. This algebra is isomorphic to $\mathfrak{so}(2,d)$. This can be seen by introducing $J_{MN} : M,N = -1,0,1,...,d$ defined as (recall that $\mu,\nu = 0,..,d-1$):

$$J_{\mu\nu} := L_{\mu\nu}\ , \qquad J_{-1\mu} := \frac{1}{2}\left(P_\mu - K_\mu\right)\ , \qquad J_{d\mu} := \frac{1}{2}\left(P_\mu + K_\mu\right)\ , \qquad J_{-1d} := D\ .$$

The above commutation relations can then be rewritten as the algebra of $SO(2,d)$:

$$i[J_{MN},J_{AB}] = \eta_{NA}J_{MB}+\eta_{MB}J_{NA}-\eta_{MA}J_{NB}-\eta_{NB}J_{MA}\ . \tag{A.18}$$

The conformal algebra can be extended to the superconformal case by adding the corresponding fermionic generators. The above bosonic algebra is a subalgebra of the latter and we have in addition the commutators and anticommutators with the Poincaré supercharges Q^i_α and conformal supercharges S^j_β. These supercharges generate translations and conformal transformations in superspace. The indices

$i, j = 1, ..., \mathcal{N}$ transform in the fundamental of the $SU(\mathcal{N})_R$ R-symmetry group, while α, β are the spinor indices. The exact form of the superconformal algebra depends on the spacetime dimension and R-symmetry group and for this reason we will give the commutation relations with the supercharges only schematically. Together with the above bosonic subalgebra, the relevant part of the superconformal algebra is given by :

$$\begin{aligned} i[D,Q] &= \tfrac{1}{2}Q \ , & \{Q,Q\} &\simeq P \ , \\ i[D,S] &= -\tfrac{1}{2}S \ , & \{S,S\} &\simeq K \ , \\ i[K,Q] &\simeq S \ , & \{Q,S\} &\simeq L + D + R \ , \\ i[P,Q] &\simeq Q \ , \end{aligned}$$

where R^i_j are the generators of the R-symmetry group and we are omitting the commutators involving these generators.

A.4. Deformations of CFTs

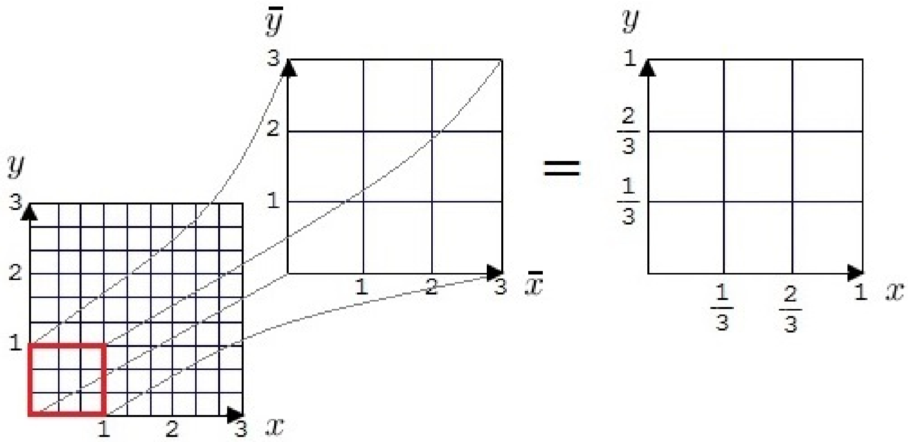
A way to discuss deformations of conformal field theories without going too much through the systematics of the renormalization group is to work with the notions of physical dimension and conformal dimension and to use the equivalence between UV regulators Λ and lattice spacings $a \sim 1/\Lambda$ in quantum field theory. The three principles we will need in this case are a) the action of a given theory is the phase of the exponential in the theory's generating functional, b) the action is always physically dimensionless (in units $\hbar = 1$), c) the coupling constants are always conformally dimensionless (they are conformal tensors that transform as in equation (1.72) with $\Delta = 0$ and $n = 0$).

Suppose we have a UV-finite conformal field theory in d dimensions (such as pure $\mathcal{N} = 4, d = 4$ SYM) with a conformally invariant action S_0. Suppose we deform the theory with some operator insertion and UV-regulate it by considering the theory on a lattice such that the total action is given by:

$$S[\varphi_0] \quad = \quad S_0 + \int_a d^d x \sqrt{g}\, \varphi_0 \mathcal{O}_\Delta(x) \ . \tag{A.19}$$

The field $\mathcal{O}_\Delta$ is a conformal scalar of conformal dimension Δ that transforms as in equation (1.72) under a conformal-Weyl transformation, whereas φ_0 is a coupling constant. The paramenter a is the lattice spacing such that the generating functional of the theory is a sum over field configurations with momenta $|k| < \Lambda = 1/(2a)$:

$$\mathcal{Z} = \int [d\phi]_\Lambda \, e^{-S[\varphi_0]} \quad , \quad [d\phi]_\Lambda = \prod_{|k|<\Lambda} d\phi(k) \ . \tag{A.20}$$



$$ds^2 = dx^2 + dy^2\ ; \qquad ds^2 = d\bar{x}^2 + d\bar{y}^2 = 9\left(dx^2 + dy^2\right)\ .$$

Figure A.1: Scale transformation on a lattice: $(x, y) \to (\bar{x}, \bar{y}) = \lambda(x, y)$ with $g_{\mu\nu}$ fixed ($\lambda = 3$). The lattice spacing is increased from $a = 1/3$ to 1. Since wavelengths shorter than twice the lattice spacing are suppressed, after the scale transformation we suppress all wavelengths shorter than 2 (and therefore we lose resolution). The scale factor 9 in the final line element is then transferred to the coupling constant so that the final theory is on the original lattice, just with a different coupling.

By definition, the physical mass dimensions are given by:

$$[x] = M^{-1}\ , \quad [\mathcal{O}_\Delta] = M^{\Delta}\ , \quad [\varphi_0] = M^{\varepsilon}\ , \tag{A.21}$$

where ε is some constant. Since $[S] = M^0$, we find that: $\Delta = d - \varepsilon$.

Next, let us perform a rescaling of the coordinates: $x^\mu \to \lambda x^\mu$. Note that this rescaling is a scale-Weyl transformation as discussed at the end of section A.2: a scale transformation $x^\mu \to \bar{x}^\mu = \lambda x^\mu$ followed by a constant Weyl transformation $g_{\mu\nu} \to \lambda^2 g_{\mu\nu}$. Since $g_{\mu\nu} \to g_{\mu\nu}$ under the combined transformations, it is common practise to express the transformation simply as $x^\mu \to \lambda x^\mu$ with the metric components $g_{\mu\nu}$ kept fixed. The regime $\lambda >> 1$ corresponds to the infrared, or low-energy regime: the lattice spacing increases to λa and we integrate out all wavelengths shorter than $2\lambda a$, see Figure A.1. On the other hand, the limit $\lambda \to 0$ corresponds to the ultra-violet, or high-energy limit where the lattice spacing shrinks to zero size and therefore we have to consider arbitrarily high momenta on the lattice. Since $\mathcal{O}_\Delta$ has conformal dimension Δ and the coupling φ_0

is conformally dimensionless, the action transforms as:

$$\begin{aligned} S[\varphi_0] &\to S_0^{\text{eff}} + \int_{\lambda a} d^d\bar{x}\sqrt{g}\,(\varphi_0 + \delta\varphi_0)\,\bar{\mathcal{O}}_\Delta(\bar{x}) \\ &= S_0 + \lambda^{d-\Delta}\int_a d^dx\sqrt{g}\,(\varphi_0 + \delta\varphi_0)\,\mathcal{O}_\Delta(x) \\ &= S[\bar{\varphi}_0]\ , \end{aligned} \tag{A.22}$$

where $\bar{\varphi}_0 = \lambda^{d-\Delta}(\varphi_0 + \delta\varphi_0) = \lambda^{\varepsilon}(\varphi_0 + \delta\varphi_0)$. The possible effective (or Wilsonian) correction $\delta\varphi_0$ to the coupling arises in the case $\lambda > 1$ after the rescaling because we are simultaneously integrating out the degrees of freedom with momenta $|k|$ in the interval $\Lambda/\lambda < |k| < \Lambda$ so that we don't double count momenta in the generating functional (wavelengths shorter than $2\lambda a$ are equivalent to wavelengths longer than $2\lambda a$ because of the periodicity of the new lattice):[4]

$$\mathcal{Z} \quad\to\quad \int [d\phi]_{\Lambda/\lambda}\,\exp\left(-S_0^{\text{eff}} - \int_{\lambda a} d^d\bar{x}\sqrt{g}\,(\varphi_0 + \delta\varphi_0)\,\bar{\mathcal{O}}_\Delta(\bar{x})\right)\ . \tag{A.23}$$

On the other hand, the effective action $S_0^{\text{eff}} = S_0$ because the undeformed theory is conformally invariant and free of divergences, so it does not require renormalization and therefore does not receive corrections. The final action is scale invariant iff $\varepsilon = 0$ and therefore we find that scale invariant theories cannot have physically dimensionful coupling constants. More importantly, we conclude that the effect of the lattice rescaling has been transferred into the coupling constant.[5] In this way, there are three possible cases to consider according to the dimension Δ:

i) $\Delta < d$: $\bar{\varphi}_0 \to 0$ in the UV, $\bar{\varphi}_0 \to \infty$ in the IR $\Rightarrow$ IR-relevant ,

ii) $\Delta > d$: $\bar{\varphi}_0 \to \infty$ in the UV, $\bar{\varphi}_0 \to 0$ in the IR $\Rightarrow$ IR-irrelevant ,

iii) $\Delta = d$: $\bar{\varphi}_0 = \varphi_0 + O(\delta\varphi_0)$ $\Rightarrow$ Marginal .

Marginal deformations result in another conformal field theory since the action remains conformally invariant (*c.f.* section A.2), whereas relevant deformations of a CFT result in a field theory which is conformally invariant only in the UV (the deformation vanishes in the high-energy limit but not elsewhere). Irrelevant deformations become dominant in the UV and therefore spoil conformal invariance

[4]Note that $\delta\varphi_0$ typically depends on λ because the degrees of freedom over which we integrate in the generating functional to obtain the effective theory are defined in the interval $\Lambda/\lambda < |k| < \Lambda$, so the integration intervals will depend on λ. For this reason, the marginal case $\Delta = d$ discussed next can still result in a flow of the coupling with λ. However, in the vicinity of the fixed point S_0 of the transformation such that φ_0 is infinitesimal we have that $\delta\varphi_0$ is second (or higher) order in φ_0 and therefore can be ignored, so that $\bar{\varphi}_0 = \lambda^{d-\Delta}\varphi_0$.

[5]In terms of renormalization flows, in which case $\lambda > 1$, we find that the theory described by (A.19) when the cut-off is Λ will effectively look like (A.22) at energy scales below Λ/λ.

at high-energies, but lead to the original conformal field theory in the IR. In terms of renormalization theory, while conformal field theories that remain conformal at the quantum level (*i.e.* the beta-functions vanish) do not require renormalization, it can be shown using a counting of the superficial degree of divergence that irrelevant deformations of such CFTs result in UV divergences in the Feynman diagrams that cannot be renormalized (require infinitely many counterterms), whereas relevant or marginal deformations in general introduce UV divergences but which are renormalizable by a finite number of counterterms [214] .

Appendix B

B.1. Conformal compactness and AdS asymptotia

In this appendix we deduce a few implications of conformal compactness that are relevant to our work. We begin with the standard definition.

A manifold $(\mathcal{M}, G)$ is defined to be $C^{n\geq 0}$ conformally compact if there exists an *asymptote* $(\tilde{\mathcal{M}}, \tilde{G}, \rho)$ consisting of a defining function $\rho(x) \in [0, +\infty[$ and a manifold-with-boundary $(\tilde{\mathcal{M}}, \tilde{G})$ with boundary $\partial\tilde{\mathcal{M}}$ satisfying the following properties [215, 216, 217]:

1) $\mathcal{M} = \text{int}\, \tilde{\mathcal{M}} = \{p \in \tilde{\mathcal{M}} : \exists \text{ open set } p \ni U \subset \tilde{\mathcal{M}}\}$,

2) $\tilde{G}_{\mu\nu} = \rho^2(x)\, G_{\mu\nu} \;\; : \;\; \tilde{\mathcal{M}} = \{\rho \geq 0\}$, $\partial\tilde{\mathcal{M}} = \{\rho = 0\}$,

3) $d\rho \neq 0$ on $\partial\tilde{\mathcal{M}}$,

with $\rho(x)$ of class C^∞ and $\tilde{G}$ non-degenerate and of class $C^{n\geq 0}$ in $\tilde{\mathcal{M}}$. The region $\{\rho = 0\}$ of $\tilde{\mathcal{M}}$ is referred to as the conformal boundary of $\mathcal{M}$ and $\tilde{\mathcal{M}}$ as the conformal embedding.

We are now interested in showing that a conformally compact, asymptotically Einstein manifold of negative scalar curvature is asymptotically locally AdS. In order to do so, we need the following result.

Proposition Let $(\mathcal{M}, G)$ be a conformally compact manifold with an asymptote $(\tilde{\mathcal{M}}, \tilde{G}, \rho)$. Then in the limit $\rho \to 0$, the Riemann tensor behaves asymptotically as:

$$R_{abcd} = -|\tilde{\nabla}\rho|^2 \left(G_{ac}G_{bd} - G_{ad}G_{bc}\right) + \mathcal{O}(\rho^{>-4}) \quad , \tag{B.1}$$

where: $|\tilde{\nabla}\rho|^2 := \tilde{G}^{ab}\partial_a\rho\partial_b\rho$ and where $\mathcal{O}(\rho^{>-4})$ denotes terms that diverge slower than ρ^{-4}.

Proof By using the transformation law of the Riemann tensor under a Weyl transformation, the Riemann of G and that of $\tilde{G}$ are related as:

$$R_{abcd} = \rho^{-2}\tilde{R}_{abcd} + \left(\rho^{-3}\tilde{G}\circ\tilde{\nabla}\tilde{\nabla}\rho - \frac{1}{2}\rho^{-4}|\tilde{\nabla}\rho|^2\tilde{G}\circ\tilde{G}\right)_{abcd} \quad , \tag{B.2}$$

where: $(A\circ B)_{abcd} := A_{ac}B_{bd} - A_{ad}B_{bc} + B_{ac}A_{bd} - B_{ad}A_{bc}$, and where $\tilde{\nabla}_a$ is the covariant derivative with respect to $\tilde{G}$. From the third condition in the definition of conformal compactness, we can introduce the defining function ρ as a coordinate in the neighbourhood of $\rho = 0$. In this way, since $\tilde{G}$ is at least C^0 in $\tilde{\mathcal{M}}$ and the Riemann tensor of $\tilde{G}$ contains at most second derivatives of $\tilde{G}$ with respect to ρ, then if $\tilde{R}_{abcd}$ diverges as $\rho \to 0$, it must do so slower than ρ^{-2}. Also, the term $\tilde{\nabla}_a\tilde{\nabla}_b\rho$ contains at most first derivatives of $\tilde{G}$ with respect to ρ and hence it must diverge slower than ρ^{-1}. Hence, the first two terms in (B.2) diverge slower than ρ^{-4} and thus we find:

$$\begin{aligned} R_{abcd} &= -\rho^{-4}|\tilde{\nabla}\rho|^2\left(\tilde{G}_{ac}\tilde{G}_{bd} - \tilde{G}_{ad}\tilde{G}_{bc}\right) + \mathcal{O}(\rho^{>-4}) \\ &= -|\tilde{\nabla}\rho|^2\left(G_{ac}G_{bd} - G_{ad}G_{bc}\right) + \mathcal{O}(\rho^{>-4}) \quad . \end{aligned} \tag{B.3}$$

q.e.d.

Now, from equation (B.1), the Ricci tensor of G behaves asymptotically as:

$$R_{ab} \;=\; -d\,\rho^{-2}|\tilde{\nabla}\rho|^2\tilde{G}_{ab} + \mathcal{O}(\rho^{>-2}) \;\sim -d\,|\tilde{\nabla}\rho|^2 G_{ab} \;, \tag{B.4}$$

where $d+1$ is the dimension of $\mathcal{M}$. If $(\mathcal{M}, G)$ is in particular asymptotically Einstein of negative scalar curvature, then from the above we find that $|\tilde{\nabla}\rho|^2$ must be a positive constant by definition. From equation (B.1) it then follows that the Riemann tensor is asymptotically equal to that of *AdS* and therefore $(\mathcal{M}, G)$ is asymptotically isometric to *AdS* space up to global properties such as the topology of a neighbourhood of the boundary. On the other hand, if $(\mathcal{M}, G)$ is asymptotically Ricci-flat, then from (B.4) we find that $|\tilde{\nabla}\rho|^2$ must vanish asymptotically as $\rho \to 0$. Since $d\rho \neq 0$ by definition, this implies that the conformal boundary $\{\rho = 0\}$ of $\mathcal{M}$ is null. We therefore define an asymptotically locally flat space as any conformally compact, asymptotically Ricci-flat manifold. This definition essentially coincides with that of asymptotic flatness at null infinity [195] if we relax any conditions on the topology of the conformal boundary.

We also show that the Ricci scalar of a conformally compact *Riemannian* manifold $(\mathcal{M}, G)$ of dimension $d+1 > 1$ cannot vanish asymptotically. From equation

(B.1), the Ricci scalar of G behaves asymptotically as:

$$R = -d(d+1)|\tilde{\nabla}\rho|^2 + \mathcal{O}(\rho^{>0}) \quad . \tag{B.5}$$

If $(\mathcal{M}, G)$ is Riemannian, one can write without loss of generality the positive-definite metric tensor G_{ab} near $\rho = 0$ as:

$$\begin{aligned} ds^2 &= G_{ab}dx^a dx^b \\ &= M^2 d\rho^2 + \gamma_{ij}\left(dx^i + B^i d\rho\right)\left(dx^j + B^j d\rho\right) \\ &= \rho^{-2}\left(\tilde{M}^2 d\rho^2 + \tilde{\gamma}_{ij}\left(dx^i + B^i d\rho\right)\left(dx^j + B^j d\rho\right)\right) \\ &= \rho^{-2}\tilde{G}_{ab}dx^a dx^b \quad . \end{aligned} \tag{B.6}$$

From this decomposition, we find:

$$|\tilde{\nabla}\rho|^2 = \tilde{G}^{\rho\rho} = \tilde{M}^{-2} \quad . \tag{B.7}$$

Since $\tilde{G}_{ab}$ is at least of class C^0 and $\tilde{G}_{\rho\rho} = \tilde{M}^2 + \tilde{\gamma}_{ij}B^iB^j$, then $\tilde{M}^{-2}$ must be supported in $\tilde{\mathcal{M}}$, otherwise $\tilde{G}_{\rho\rho}$ would not be C^0. Notice that $\tilde{\gamma}_{ij}$, being positive-definite, prevents the term $\tilde{\gamma}_{ij}B^iB^j$ from cancelling $\tilde{M}^2$ in $\tilde{G}_{\rho\rho}$. In this way, $|\tilde{\nabla}\rho|^2$ is non-vanishing and thus the Ricci scalar cannot vanish asymptotically.

In this section we have shown in particular that conformal compactness together with Einstein's equations with a negative cosmological constant implies that a manifold is asymptotically locally AdS. Fefferman and Graham [190] proved in particular a reciprocal statement: that Einstein's equations with a negative cosmological constant imply that the manifold is conformally compact. We will do so in section B.4.

B.2. Gauss-Codazzi decomposition

The Gauss-Codazzi relations are identities between geometric invariants of hypersurfaces and the respective embeddings and show that the Riemannian geometry of a manifold is completely encoded in the intrinsic and extrinsic geometry of embedded hypersurfaces. An introductory review of the subject in the context of the $3+1$ formalism of general relativity can be found in the notes [196].

Let $(\mathcal{M}, G_{\mu\nu})$ be a manifold of dimensional $d+1$ such that $\mathcal{M} = I \times \Sigma : I \in \mathbb{R}$ and let $(\Sigma_z) : z \in \mathbb{R}$ be a foliation of $\mathcal{M}$ by a family of hypersurfaces $(\Sigma_z, \gamma_{ab}(z,x))$.

The Gauss, Codazzi and Ricci identities are respectively:[1]

$$\begin{aligned} R_{abcd}[G] &= R_{abcd}[\gamma] - \varepsilon\left(K_{ac}K_{bd} - K_{ad}K_{bc}\right) \ , && \text{(B.8)}\\ R_{abc\mu}[G]n^\mu &= D_a K_{bc} - D_b K_{ac} \ , && \text{(B.9)}\\ R_{a\mu b\nu}[G]n^\mu n^\nu &= -\pounds_n K_{ab} + (K\cdot K)_{ab} + D_{(a}a_{b)} - \varepsilon\, a_a a_b \ , && \text{(B.10)} \end{aligned}$$

where n is the (future-directed) unit normal to the leaves of the foliation such that: $n^2 = \varepsilon = \pm 1$, the acceleration $a^\mu = n\cdot\nabla n^\mu$ and the extrinsic curvature $K_{\mu\nu} = \frac{1}{2}\pounds_n \gamma_{\mu\nu} = \gamma_\mu{}^\sigma \nabla_\sigma n_\nu$, where the induced metric $\gamma_{\mu\nu} = G_{\mu\nu} - \varepsilon\, n_\mu n_\nu$. Notice that: $\gamma^{\mu\nu} = \gamma^{ab}\partial_a x^\mu \partial_b x^\nu$, which follows from: $\gamma^{\mu\nu}n_\nu = 0$. Also: $D_a \gamma_{bc} := 0$. In terms of an ADM decomposition of the metric:

$$ds^2 = G_{\mu\nu}dx^\mu dx^\nu = \varepsilon N^2 dz^2 + \gamma_{ab}(dx^a + N^a dz)(dx^b + N^b dz) \ , \quad \text{(B.11)}$$

$$G^{\mu\nu} = \begin{pmatrix} \varepsilon N^{-2} & -\varepsilon N^{-2}N^a \\ -\varepsilon N^{-2}N^a & \gamma^{ab} + \varepsilon N^{-2}N^a N^b \end{pmatrix} \ , \quad \text{(B.12)}$$

we have: $n_\mu = \varepsilon N \partial_\mu z$, $n^\mu\partial_\mu = N^{-1}\left(\partial_z - N^a\partial_a\right)$ and: $a^\mu = -\varepsilon\,\gamma^{\mu\nu}\partial_\nu \log N$. We also have: $K_{ab} = \frac{1}{2N}\left(\partial_z - \pounds_N\right)\gamma_{ab}$.
From the above identities, we then find:[2]

$$\begin{aligned} n^\mu n^\nu R_{\mu\nu}[G] &= \gamma^{ab}\left(R_{a\mu b\nu}[G]n^\mu n^\nu\right) \\ &= -n^\mu\partial_\mu K - K\cdot K + \nabla_\mu a^\mu \\ &= K^2 - K\cdot K - \nabla_\mu(n^\mu K - a^\mu) \ , && \text{(B.13)}\\ n^\mu R_{\mu b}[G] &= \gamma^{ac}\left(R_{abc\mu}[G]n^\mu\right) \\ &= D_c K^c{}_b - \partial_b K \ , && \text{(B.14)}\\ R_{bd}[G] &= \gamma^{ac}R_{abcd}[G] + \varepsilon\left(R_{c\mu d\nu}[G]n^\mu n^\nu\right) \\ &= R_{bd}[\gamma] + \varepsilon\left(-\pounds_n K_{bd} - KK_{bd} + 2(K\cdot K)_{bd} + D_{(b}a_{d)}\right) - a_b a_d \ . \\ & && \text{(B.15)} \end{aligned}$$

From the trace of (B.15):

$$\gamma^{ab}R_{ab}[G] = R[\gamma] - \varepsilon\nabla_\mu\left(n^\mu K - a^\mu\right) \ . \quad \text{(B.16)}$$

[1] Note that: $\pounds_n K_{ab} = n^\mu\partial_\mu K_{ab} + 2K_{\mu(a}\partial_{b)}n^\mu = n^\mu\nabla_\mu K_{ab} + 2(K\cdot K)_{ab}$.
[2] Notice that: $D_b a_d = \gamma^\mu_b\gamma^\nu_d\nabla_\mu a_\nu$ and therefore: $D_c\, a^c - \varepsilon\, a_c\, a^c = \nabla_\mu a^\mu$.

In this way, we find:

$$R[G] = (\gamma^{\mu\nu} + \varepsilon\, n^\mu n^\nu) R_{\mu\nu}[G] = R[\gamma] + \varepsilon \left(K^2 - K \cdot K \right) - 2\varepsilon\, \nabla_\mu \left(n^\mu K - a^\mu \right) \ , \tag{B.17}$$

$$2\, n^\mu n^\nu \Big(R_{\mu\nu}[G] - \frac{1}{2} G_{\mu\nu} R[G] \Big) = -\varepsilon R[\gamma] + K^2 - K \cdot K \ . \tag{B.18}$$

When replaced by the Einstein's equations: $R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R + \Lambda G_{\mu\nu} = T_{\mu\nu}$, the identities (B.14), (B.15) and (B.18) become the ADM equations that follow from a Hamiltonian formulation of General Relativity [195]. The identities (B.18) and (B.14) become respectively the Hamiltonian and Diffeomorphism constraints and the identity (B.15) becomes the dynamical equation:

$$-\, \varepsilon R[\gamma] + K^2 - K \cdot K \ = 2\, \varepsilon \Lambda + 2\, T_{\mu\nu} n^\mu n^\nu \ , \tag{B.19}$$

$$D_c K^c{}_a - \partial_a K = T_{a\mu} n^\mu \ , \tag{B.20}$$

$$\varepsilon \left(-\pounds_n K_{bd} - K K_{bd} + 2 (K \cdot K)_{bd} + D_{(b} a_{d)} \right) - a_b a_d + R_{bd}[\gamma] = \frac{1}{d-1} \Big(2\Lambda \gamma_{bd} + T_{bd} - \gamma_{bd} T \Big) \ . \tag{B.21}$$

B.3. On ADM and the choice of Lapse and Shift

Let $(\mathcal{M}, G_{\mu\nu})$ be a generic spacetime with coordinates (z, x^a) and a metric decomposition of the form (B.11). For simplicity take ∂_z to be spacelike. Let $\Sigma = \{\rho(z, x) = 0\}$ be any timelike hypersurface in $\mathcal{M}$. Let $\alpha(z, x)$ be any smooth function supported in $\mathcal{M}$ and $\beta^a(z, x)$ any smooth set of functions. Then, in a sufficiently small neighbourhood of Σ it is always possible to introduce coordinates $r = r(z, x^b), y^a = y^a(z, x^b)$ in order to bring the metric (B.11) to the form:[3]

$$ds^2 = \alpha^2 dr^2 + h_{ab} (dy^a + \beta^a dr)(dy^b + \beta^b dr) \ , \tag{B.22}$$

with $\Sigma = \{r = 0\}$. Note that different choices of (α, β^b) require different transformation of coordinates and hence will result in different coordinates (r, y^a) (they will be different functions of z and x^a). Therefore, for each transformation of coordinates there exists a (α, β^a) and for each (α, β^a) there exists a transformation of coordinates such that the correspondence is surjective.

Since we can always bring (B.11) to the form (B.22) near Σ for any such α and β^a that we wish, we say that we can choose directly $N = \alpha$ and $N^a = \beta^a$ in (B.11). This is the same as to relabel $r \to z$, $y^a \to x^a$ and $h_{ab} \to \gamma_{ab}$ in

[3]Note that we have $d+1$ functions $\{z(r, y^b), x^a(r, y^b)\}$ to eliminate $d+1$ variables $\{N, N^a\}$, with $d+1$ the spacetime dimension.

(B.22). Note, however, that the new z, x^a and γ_{ab} in general are not the old ones and therefore different choices of (N, N^a) in (B.11) are associated with different coordinates z, x^a and induced metric γ_{ab}, although this is usually left implicit. Furthermore, by fixing (N, N^a) in this way we are foliating the spacetime near Σ by surfaces of constant r, *i.e.* by surfaces of constant z^{new} (but not! by surfaces of constant z^{old}) and therefore different choices of (N, N^a) correspond to different foliations since z^{new} varies with the choice of (N, N^a).

Suppose now that we start from a generic metric (B.11), choose some α and β^a and bring the metric to the form (B.22) by a transformation of coordinates. Then, we solve Einstein's equations for h_{ab} subject to some initial conditions prescribed on Σ and obtain a solution $G_{\mu\nu}$. Next, return to the generic metric (B.11), choose another α and β^a, say $\bar{\alpha}$ and $\bar{\beta}^a$, bring it to the form (B.22) by another transformation of coordinates and solve Einstein's equations for $\bar{h}_{ab}$ subject to the same initial conditions prescribed on Σ to obtain a solution $\bar{G}_{\mu\nu}$. Then $G_{\mu\nu}$ and $\bar{G}_{\mu\nu}$ will be locally isometric (*i.e.* related by a coordinate reparametrization). Spacetimes that satisfy the same Einstein's equations subject to the same initial conditions but have different lapses N and shifts N^a are related by a transformation of coordinates near Σ.[4] This is the statement that general relativity has a well-posed initial value formulation such that a solution to an initial-value problem is unique up to symmetries of the theory, which in this case are coordinate reparametrizations. For this reason, and since the diffeomorphism group is the gauge group of general relativity, we call a choice of lapse and shift a choice of gauge. The synchronous gauge, or Gaussian normal coordinates, corresponds to the choice $(\alpha = 1, \beta^a = 0)$:

$$ds^2 = dr^2 + h_{ab} dx^a dx^b \ . \tag{B.23}$$

In general, we cannot bring the metric of a spacetime to the form (B.22) everywhere in a single coordinate system. The fact that this procedure in general is only valid near a given hypersurface Σ is associated with the tendency of geodesics to cross and end in singularities. This issue is analogous to the Gribov problem in non-abelian gauge theories and the absence of global gauge conditions, where a choice of gauge is only valid near a gauge orbit. See *e.g.* [218] for an example of how to construct normal coordinates and why such construction may eventually break down sufficiently far away from Σ.

The fact that a choice of lapse and shift represents a choice of gauge can be seen from a similar perspective – and more closely related to the Hamiltonian

[4]It should be clarified that, since h_{ab} and $\bar{h}_{ab}$ are a priori arbitrary, then $G_{\mu\nu}$ and $\bar{G}_{\mu\nu}$ are not necessarily locally isometric before they are required to satisfy Einstein's equations subject to the *same* initial conditions. Without the latter requirement, $G_{\mu\nu}$ and $\bar{G}_{\mu\nu}$ in general will not be isometric and an example of that is Minkowski space and the Schwarzschild solution, both in Gaussian normal coordinates.

treatment of general relativity – by performing an ADM decomposition of the Einstein-Hilbert Lagrangian:

$$\mathcal{L} = \sqrt{G}\, R[G] = N\sqrt{\gamma}\left(R[\gamma] + K^2 - K\cdot K - 2\nabla_\mu\left(Kn^\mu - a^\mu\right)\right) , \qquad \text{(B.24)}$$

where we use the notation of the previous section. An inspection of the Lagrangian shows that only the induced metric γ_{ab} has derivatives in z and therefore the equations of motion will be second order differential equations in z for γ_{ab} only. The lapse and shift do not have any z-derivatives in $\mathcal{L}$ and therefore their conjugate momenta vanishes. This means that they are not dynamical variables and that are part of the initial data in an initial value problem.

B.4. Einstein metrics and conformal compactness

Given the results of the previous sections it is now a simple matter to show that any solution of Einstein's equations with a negative cosmological constant is conformally compact with a timelike conformal boundary. We will work with Gaussian coordinates (B.23) in $d+1$ dimensions:

$$ds^2_{d+1} = G_{\mu\nu}dx^\mu dx^\nu = dr^2 + \gamma_{ab}dx^a dx^b \ . \qquad \text{(B.25)}$$

The Einstein equations (B.19)–(B.21) in this coordinate system become:

$$R[\gamma] - \frac{1}{4}(\gamma^{-1}\gamma')^2 + \frac{1}{4}(\gamma^{-1}\gamma'\gamma^{-1}\gamma') = \frac{d(d-1)}{\ell^2} , \qquad \text{(B.26)}$$

$$D_c(\gamma^{-1}\gamma')^c{}_a - \partial_a(\gamma^{-1}\gamma') = 0 , \qquad \text{(B.27)}$$

$$R_{ab}[\gamma] - \frac{1}{2}\gamma''_{ab} - \frac{1}{4}(\gamma^{-1}\gamma')\gamma'_{ab} + \frac{1}{2}(\gamma'\gamma^{-1}\gamma')_{ab} = -\frac{d}{\ell^2}\,\gamma_{ab} , \qquad \text{(B.28)}$$

where the cosmological constant $\Lambda = -d(d-1)/(2\ell^2)$ and $\gamma' = \partial_r\gamma$. If we solve the equations asymptotically as $r \to \infty$ we obtain:

$$ds^2_{d+1} = dr^2 + e^{2r/\ell}\left(g_{(0)ab}(x) + \mathcal{O}(e^{-2r/\ell})\right)dx^a dx^b , \qquad \text{(B.29)}$$

where $g_{(0)ab}$ is a non-degenerate but otherwise arbitrary function of the coordinates x^a.[5] We can now define: $z := \ell\, e^{-r/\ell}$ to obtain the metric in Poincaré coordinates:

$$ds^2_{d+1} = \frac{\ell^2}{z^2}\left(dz^2 + \left(g_{(0)ab} + \mathcal{O}(z^2)\right)dx^a dx^b\right) . \qquad \text{(B.30)}$$

The function $\rho(x) = z/\ell$ and the metric $\tilde{G}_{\mu\nu} = (z/\ell)^2 G_{\mu\nu}$ satisfy the properties of section B.1 and therefore the set $(\tilde{\mathcal{M}}, \tilde{G}, \rho)$ forms an asymptote. In this way, the

[5] Non-degeneracy follows from the fact that: $\sqrt{G} \sim e^{dr/\ell}\sqrt{g_{(0)}}$ as $r \to \infty$ and the metric $G_{\mu\nu}$ is by definition non-degenerate.

manifold $(\mathcal{M}, G)$ is conformally compact with defining function z/ℓ and conformal boundary $\{z = 0\}$.

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Samenvatting

In de laatste vijftien jaren zijn er grote vooruitgangen geboekt in ons begrip van gravitationele fysica. Voorbije studies van zwarte gaten als kwantumsystemen suggereerden dat zwaartekracht fundamenteel holografisch is. Een semi-klassieke analyse van zwarte gaten door Hawking onthulde dat deze oplossingen een intrinsieke notie van entropie met zich mee dragen en dat deze entropie proportioneel is aan de oppervlakte van hun horizon. Samen met de tweede wet van de thermodynamica impliceren deze resultaten de Bekenstein grens $S_{\text{max}} = A/4G_0$ die de maximale entropie in een gebied van een ruimte die zwaartekracht bevat relateert tot de oppervlakte van het gebied. Deze eigenschap is in scherp contrast met de entropiegrenzen van locale kwantumveldentheorieën, waar het aantal vrijheidsgraden in een gebied typisch schaalt met de volume van het gebied, en dit toont aan dat een kwantumtheorie van zwaartekracht geen gewone veldentheorie van een massaloos spin-2 deeltje kan zijn. Een interpretatie van deze grens heeft op natuurlijke wijze geleid tot het holografisch principe van 't Hooft en Susskind [5, 6, 7], die stelt dat de toestanden van eender welke kwantumgravitatie theorie in feite bevat zijn in een theorie zonder zwaartekracht die op de rand van de ruimte gedefinieerd is. Het is de moeite om op dit punt een schijnbare incompatibiliteit te vermelden van dit principe met het Weinberg-Witten theorema [8] in kwantumveldentheorie (KVT). Een paar subtiliteiten negerend, stelt deze dat een KVT met een behouden energie-momentum tensor geen toestanden kan bevatten voor massaloze interagerende deeltjes van spin $j > 1$. Dit impliceert in het bijzonder dat zulke KVTs geen graviton toestanden kunnen bevatten. Het holografisch principe omzeilt de axioma's van dit theorema door het graviton in een ruimte te plaatsen die anders is dan die van de theorie op de rand, nl. in een ruimte met extra dimensies.

Dit principe wordt ons gepresenteerd als een fundamentele eigenschap van kwantumgravitatie, maar voor vele jaren is het vooral conceptueel gebleven, vooral door gebrek aan een precies kader om het idee computationeel te implementeren. De meest veelbelovende kandidaat leek snaartheorie, waar ruimtegebieden die zwaartekracht bevatten, worden beschreven als ensembles van kwantumtoestanden.

Bovendien konden Strominger en Vafa in de late jaren '90 de Bekenstein-Hawking entropie van extremale zwarte gaten reproduceren door bepaalde supersymmetrische oplossingen van snaartheorie te beschouwen en een stochastische telling te doen van hun microtoestanden [9]. Deze procedure was enkel mogelijk na de ontdeking van D-branen en hun belang in snaartheorie door Polchinski [10] en medewerkers [11, 12]. Deze microscopische telling is een berekening in de lager dimensionale wereldvolume-theorie van de zwarte gaten (of D-branen) en toonde aan dat snaartheorie in staat is om zwaartekracht als een holografische theorie voor te stellen. Vroegere ideeën over mogelijke lager-dimensionale beschrijvingen van zwaartekracht in snaartheorie waren besproken door Thorn en medewerkers [13].

Een gerelateerd maar onafhankelijk resultaat van Brown en Henneaux [14] in de late jaren '80 suggereerde dat bepaalde theorieën van zwaartekracht, met name die met Anti-de Sitter (AdS) asymptotica, intiem verbonden zijn met lager dimensionale conforme velden theorieën op een holografische manier. Hun analyse van drie-dimensionale Einstein zwaartekracht met AdS randvoorwaarden toonde aan dat de asymptotische symmetriegroep van AdS_3 op de rand van de ruimte werkt als de twee-dimensionale conforme groep. De algebra van de overeenkomstige behouden ladingen is een centraal-uitgebreide Virasoro algebra, die voor het eerst voorkwam in de context van snaartheorie. Dit werk erkende voor het eerst het belang van asymptotische randvoorwaarden van AdS ruimtes in een mogelijke implementatie van het holografische principe in het geval van AdS zwaartekracht. In het bijzonder impliceerde het dat elke veldentheorie gedefinieerd op de rand van Anti-de Sitter een conform invariante KVT zou zijn, maar het wees geen concreet voorstel aan die de randtheorie tot de gravitationele fysica vanbinnen zou relateren en voor vele jaren bleef het slechts een interessante nieuwsgierigheid. Desalniettemin heeft Strominger, gebaseerd op deze resultaten, aangetoond dat de microtoestanden van zwarte gaten met een AdS_3 nabij-horizon geometrie bevat zijn in een conforme veldentheorie gedefinieerd op de rand van deze geometrie [15]. Dit is aangetoond door gebruik van de formule van Cardy [16] die de groei geeft van toestanden in twee-dimensionale conforme veldentheorieën, die de entropie van deze zwarte gaten reproduceert.

De analyse van Brown en Henneaux en ook van Strominger is uitgevoerd zonder direct contact met snaartheorie en is daarom ook geldig voor elke theorie die bij lage energieën reduceert tot Einstein zwaartekracht. De ontdekking van D-branen als oplossingen van supergravitatie suggereerde echter dat snaartheorie eindelijk de holografische principes van deze en gerelateerde werken zou kunnen realiseren. Supergravitatie is de lage-energie limiet van snaartheorie en oplossingen daarvan beschrijven de dynamica van massaloze gesloten snaren bij lage energieën. Daarentegen zijn D-branen oppervlakten waar open snaren op eindigen en hun dynamica is beschreven door wereldvolume theorieën van het Born-Infeld type. Gesloten snaartheorieën zijn theorieën die zwaartekracht bevatten, terwijl open snaartheorieën essentieel ijktheorieën zijn die de dynamica van de D-branen beschrijven. De ontdekking [10] dat D-branen de bronnen van elektrische en magnetische (Ramond-Ramond) flux zijn in supergravitatie – m.a.w. dat ze bronnen zijn van gesloten snaren – leidde tot hun identificatie met supersymmetrische oplossingen van supergravitatie die gekend waren als extremale zwarte branen. Deze klassieke oplossingen beschrijven daarom de terugkoppeling van D-branen in de inbedding geometrie in de benadering van lage energieën en zijn zwart omdat ze waarnemingshorizons bevatten. De nabij-horizon geometrie van deze zwarte D-branen is in vele gevallen een product van een Anti-de Sitter ruimte en een compacte ruimte. Tegelijkertijd zijn de lage energie wereldvolume theorieën die leven op de branen ijktheorieën (die kwantumveldentheorieën zijn zonder zwaartekracht) met conforme symmetrie. Dit brengt de mogelijkheid naar boven dat de lager-dimensionale conforme veldentheorieën op de D-branen het holografisch beeld kunnen zijn van de gravitationele theorieën die leven in de overeenkomstige nabij-horizon geometrieën. Significant bewijs dat dit het geval is volgde uit berekeningen van D-braan verstrooiingsamplitudes [17, 18, 19], die toonden dat de mate van absorptie door D-branen van gesloten snaren even goed berekend kon worden door supergravitatie als door de wereldvolume theorieën.

De verzameling van deze resultaten wees naar het feit dat de holografische aspecten van zwaartekracht waarschijnlijk in snaartheorie zou kunnen gerealiseerd worden als een type van dualiteit tussen open snaren (of D-branen) en gesloten snaren, en culmineerde in de late jaren ’90 met het voorstel van Maldacena [20] van een concrete gelijkwaardigheid tussen bepaalde theorieën van gesloten snaren in AdS ruimte en conform invariante ijktheorieën in mindere dimensies. In daaropvolgende werken hebben Witten en medewerkers [21, 22] beargumenteerd dat deze ijktheorieën (de fundamentele theorieën, zonder Wilsoniaanse vrijheidsgraden uit te integreren) op de rand van AdS ruimten leven en hebben verder ook aangetoond dat observabelen in snaartheorie berekend kunnen worden vanuit de randtheorie. Om deze redenen is het voorstel van Maldacena, ook bekend als de AdS/CFT correspondentie, een exacte realisatie van het holografische principe in snaartheorie, waarbij alle gravitationele fysica vermoedelijk gecodeerd is op de rand van de ruimte. Witten en Susskind [23] beargumenteerden zelfs dat de AdS/CFT correspondentie de Bekenstein grens satureert die karakteristiek is van holografische theorieën, door aan te tonen dat de gravitationele theorie (die equivalent is aan de grenstheorie door de AdS/CFT dualiteit) precies één vrijheidsgraad per Planck grensoppervlakte heeft.

De ijktheorieën die voorkomen in AdS/CFT zijn Yang-Mills of niet-Abelse kwantumveldentheorieën. Dit zijn de type theorieën die de interactie van elementaire deeltjes in het standaardmodel van deeltjesfysica beschrijven (ook al zijn de veldentheorieën die voorkomen in AdS/CFT een idealisering van deze theorieën).

De electrozwakke theorie die zwakke kerninteracties en quantum electrodynamica beschrijft is een Yang-Mills theorie gebaseerd op de ijkgroep $SU(2) \times U(1)$, terwijl quantum chromodynamica (QCD) de sterke kernkracht beschrijft en gebaseerd is op $SU(3)$. QCD is een speciaal type ijktheorie. Terwijl bij de elektrozwakke theorie de interactiesterkte verzwakt bij lagere energieën, neemt bij QCD de interactiesterkte toe bij lage energieën, waardoor de theorie sterk gekoppeld wordt. Omdat de meeste berekeningen in kwantumveldentheorie gebaseerd zijn op perturbatietheorie, belet deze eigenschap ons om het lage-energie regime van QCD bijhorende karakteristieke fenomenen zoals kleuropsluiting te bestuderen met standaardmethoden. Het was de poging om de sterk gekoppelde fysica van QCD te begrijpen die voor de eerste keer leidde tot het idee dat snaartheorieën eigenlijk ijktheorieën in vermomming zouden kunnen zijn. In de jaren '70 suggereerde 't Hooft [24] dat QCD benaderd kan worden door een ijktheorie met ijkgroep $SU(N) : N \gg 1$. In deze grote N idealisering versimpelt de theorie aanzienlijk en is het vatbaar voor perturbatietheorie in $1/N$. Het is toen dat er gerealiseerd werd dat de perturbatieve expansie van de ijktheorie in Feynman diagrammen eigenlijk een expansie is van topologieën van snaartheorie wereldvellen en dat daardoor deze expansie een definitie van snaartheorie zou kunnen geven. Wij weten nu dat deze verrassende relatie tussen grote N Yang-Mills theorieën en snaartheorieën een speciaal geval is van de AdS/CFT correspondentie, waar de $1/N$ expansie van de ijktheorie overeenkomt met de snaartheorie perturbatietheorie in de wereldvel koppelingsconstante g_s.

Een verdere eigenschap van de correspondentie tussen ijk- en snaartheorieën als gegeven door de AdS/CFT dualiteit betreft de relatie tussen de ('t Hooft) koppelingsconstante van Yang-Mills theorie – die de sterkte van de ijktheorie interacties determineert – en de lengteschaal van de snaar, oftewel de inverse snaarspanning, die in het bijzonder de sterkte van het gravitationeel veld bepaalt in de duale snaartheorie. Het blijkt dat deze relatie een sterk/zwakke dualiteit is. Dit impliceert dat wanneer de ijktheorie in zijn sterk gekoppelde regime ligt, de snaartheorie goed benaderd kan worden door klassieke zwaartekracht. Zoals hierboven in het geval van QCD vermeld, zijn sterk gekoppelde veldentheorieën heel moeilijk om te bestuderen en daarom is de AdS/CFT correspondentie een uiterst nuttig werktuig om kwantumveldentheorieën bij sterke koppeling te begrijpen omdat het moeilijke problemen in de veldentheorieën vertaalt in gemakkelijkere problemen in klassieke zwaartekracht.

Door deze sterk/zwakke eigenschap van de dualiteit, hebben vele auteurs snel na de ontdekking van AdS/CFT toepassingen van de correspondentie voorgesteld in de theorie van gecondenseerde materie. Veel systemen in gecondenseerde materie fysica zijn moeilijk om te bestuderen met enkel veldentheoretische methoden. In het bijzonder zijn deze systemen typisch sterk gekoppeld, conform (of schaal) invariant, en sterk gecorreleerd nabij kwantumkritische punten, waar een transitie tussen verschillende kwantumtoestanden van materie plaatsvindt. Via de AdS/CFT dualiteit hebben verschillende sectoren van snaartheorie het potentieel om te dienen als holografische modellen van deze sterk gekoppelde systemen en kunnen in het bijzonder gebruikt worden om inzicht te verkrijgen in bepaalde kritische fenomenen in de theorie van gecondenseerde materie.

Dit laatste aspect van de AdS/CFT correspondentie is een van de voornaamste onderwerpen van deze thesis en in hoofdstuk 2 zullen we toepassingen van de dualiteit onderzoeken in de holografische beschrijvingen van bepaalde kwantumveldentheorieën. Deze laatsten hebben de specifieke eigenschap dat ze niet-relativistisch zijn en gebruikt kunnen worden om in de theorie van gecondenseerde materie systemen te beschrijven met een bepaalde type van anisotropie en schaalinvariantie. We zullen begrijpen hoe de symmetrieën van zulke systemen gerealiseerd worden in de veldentheorie modellen en hoe we de gravitationele dualen kunnen construeren. We zullen de vorm bespreken van de correlatiefuncties van deze veldentheorieën en ze holografisch berekenen gebruik makend van specifieke gravitationele modellen.

In hoofdstukken 3 en 4 zullen we ons richten op een ander aspect van de AdS/CFT correspondentie. Momenteel is het een centraal probleem in holografie om te begrijpen hoe snaartheorie in ruimtetijden met asymptotica die niet AdS zijn, geformuleerd kan worden in termen van veldentheorieën op mindere dimensies. Deze onderzoeksrichting heeft wat succes gekend voor het geval van niet-conforme braan-achtergronden, maar minder voor ruimtetijden met de Sitter randvoorwaarden. Asymptotisch vlakke ruimten, daarentegen, blijft de meest belangrijke klasse van gravitationele achtergronden waarin snaartheorie nog een holografische formulering mist. De laatste twee hoofdstukken richten zich op het inwinnen van enig inzicht in dit probleem en bevatten twee benaderingen tot vlakke ruimte holografie die verschillende perspectieven volgen, één gebaseerd op de vlakke ruimte limiet van AdS/CFT en een andere op het concept van holografische bedekkingen. De eerste benadering formuleert het probleem als een limiet van AdS/CFT waar de AdS kromming Λ verdwijnt. We zullen de limiet van Λ gaande naar nul bestuderen van vacuümsverwachtingswaarden en correlatiefuncties in AdS/CFT en verschillende noodzakelijke voorwaarden bespreken die nodig zijn zodat de correspondentie tussen de bulk en de randfysica een limiet heeft die goed gedefinieerd is. We zullen bewijs vinden dat vermeende veldentheorieën duaal aan snaartheorieën in AdS in de limiet van Λ gaande naar nul in essentie gedefinieerd zijn in twee dimensies minder, een eigenschap consistent met het feit dat de asymptotische rand van de AdS ruimten null worden in deze limiet.

De tweede benadering is gebaseerd op de observatie dat asymptotische Minkowski ruimten altijd kunnen worden bedekt met Euclidsche *AdS* (of hyperbolische) hypervlakken nabij de null-oneindigheid. Deze bedekking convergeert op natuurlijke wijze asymptotisch naar een codimensie twee oppervlakte op de rand van de ruimte. Omdat elk lid van de bedekking een AdS ruimte is, zullen we dit kenmerk onderzoeken en postuleren dat een asymptotisch Minkowski ruimte een holografische beschrijving toelaat in termen van een oneindige familie van (conform invariante) veldentheorieën die leven op de gedegenereerde rand van de bedekking. We zullen vinden dat het inderdaad mogelijk is om de asymptotica van zulke ruimtetijden te reconstrueren met observabelen die behoren tot een één-parameter familie van conforme veldentheorieën in twee dimensies minder. In het geval van tweedimensionale veldentheorieën is deze parameter de centrale lading van deze theorie en meet het in de zwaartekrachttheorie de ijkinvariante afstand tussen de verschillende AdS oppervlakten.

In het volgende hoofdstuk beginnen we met de AdS/CFT correspondentie te bespreken vanaf de eerste beginselen. We zullen beginnen met de grote N limiet van Yang-Mills theorieën te bespreken en hun relatie met snaartheorieën zoals hierboven vermeld. We zullen dan een beknopt overzicht geven van de aspecten van snaartheorie die relevant zijn voor ons werk en in detail de originele afleiding bespreken van de correspondentie van D-braan fysica en supergravitatie. We zullen dan een significant deel van dit eerste hoofdstuk wijden aan de correspondentie tussen toestanden en operatoren aan beide kanten van de dualiteit. Tenslotte zullen we de berekening van correlatiefuncties in kwantumveldentheorieën in snaartheorie bespreken.

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