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NOTES ON LINEAR THEORY OF COUPLED PARTICLE BEAMS WITH EQUAL EIGENEMITTANCES

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Abstract

We consider some aspects of the linear theory of coupled particle beams with equal eigenemittances and compare them with the one dimensional Courant-Snyder theory.

INTRODUCTION AND PRELIMINARIES

The property of the beam matrix to be proportional to the matrix which is simultaneously symmetric positive definite and symplectic is the characteristic property of the particle beams with equal eigenemittances, and the multidimensional theory of such particle beams can be developed in almost complete analogy with the famous one dimensional Courant-Snyder approach [1]. The purpose of this paper is to present some additional aspects of the linear theory of coupled particle beams with equal eigenemittances, which, due to space limitation, were not mentioned in [2].

Beam Matrix and Its Transport

Let us consider a collection of points in $2n$-dimensional phase space (a particle beam) and let, for each particle,

$$z \;=\; (q, p)^{\top} \;=\; (q_1, \ldots, q_n, p_1, \ldots, p_n)^{\top} \tag{1}$$

be a vector of canonical coordinates q and momenta p. Then, as usual, the beam (covariance) matrix is defined as

$$\Sigma \;=\; \Sigma^{\top} = \left\langle (z - \langle z \rangle) \cdot (z - \langle z \rangle)^{\top} \right\rangle \stackrel{\mathrm{def}}{=} \begin{pmatrix} \Sigma_{qq} & \Sigma_{qp} \\ \Sigma_{pq} & \Sigma_{pp} \end{pmatrix}, \tag{2}$$

where the brackets $\langle \cdot \rangle$ denote an average over a distribution of the particles in the beam. By definition, the matrix Σ is symmetric positive semidefinite and in the following we will restrict our considerations to the situation when this matrix is nondegenerated and therefore positive definite.

Let s be the independent variable (time or path length along the design orbit), and let us assume that the particle dynamics is governed by the linear system of Hamiltonian equations

$$dz \,/\, ds \;=\; JH(s)\, z, \tag{3}$$

where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad H = H^{\top} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}, \tag{4}$$

and I is the $n \times n$ identity matrix. Then the beam matrix Σ satisfies the linear differential equations

$$d\Sigma \,/\, ds \;=\; JH\Sigma \;-\; \Sigma HJ \tag{5}$$

and evolves according to the congruence

$$\Sigma(s) \;=\; A(s)\, \Sigma(0)\, A^{\top}(s), \tag{6}$$

where the symplectic matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad A(0) = I \tag{7}$$

is the fundamental matrix solution of the Eq. (3).

Projected Emittances and Eigenemittances

Projected emittances ε_m are the rms phase space areas covered by projections of the particle beam onto each coordinate plane (q_m, p_m). They can be calculated as follows

$$\varepsilon_m^2 \,=\, \Sigma_{qq}(m,m) \cdot \Sigma_{pp}(m,m) \,-\, \Sigma_{qp}^2(m,m) \tag{8}$$

and are used to characterize transverse and longitudinal beam dimensions in the laboratory coordinate system (i.e. in the variables z). Note that the projected emittances are invariants under linear uncoupled (with respect to the laboratory coordinate system) symplectic transport.

In order to define concept of eigenemittances, let us consider the matrix ΣJ. This matrix is nondegenerated and is similar to the skew symmetric matrix $\Sigma^{1/2} J \Sigma^{1/2}$, which means that its spectrum is of the form

$$\pm i\epsilon_1, \;\ldots,\; \pm i\epsilon_n, \tag{9}$$

where all $\epsilon_m > 0$ and i is the imaginary unit. The quantities ϵ_m are called eigenemittances and are generalizations of the projected emittances to the fully coupled case [3].

Eigenemittances are quantities which give beam dimensions in the coordinate frame in which the beam matrix is uncoupled between degrees of freedom and are invariants under arbitrary (possibly coupled) linear symplectic transformations.

If the beam matrix is uncoupled already in the laboratory frame, then the set of projected emittances coincides with the set of eigenemittances, and if the beam matrix has correlations between different degrees of freedom, then these two sets are different.

Note that the problem, which conditions two sets of positive real numbers must satisfy in order to be realizable as eigenemittances and projected emittances of a beam matrix, was solved for the most practically important two and three degrees of freedom cases in the paper [4].

BEAM MATRIX WITH EQUAL EIGENEMITTANCES

Let us assume that the matrix Σ has all eigenemittances equal to each other and equal to the value $\epsilon > 0$. Then the matrix

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$$W \stackrel{\text{def}}{=} \Sigma \,/\, \epsilon \tag{10}$$

is simultaneously symmetric positive definite and symplectic. We call it the Twiss matrix and parametrize it as follows

$$W = \begin{pmatrix} \beta & -\alpha \\ -\alpha^{\top} & \gamma \end{pmatrix}, \tag{11}$$

where the $n \times n$ submatrices $\beta = \beta^{\top}$, α and $\gamma = \gamma^{\top}$ are the natural matrix generalizations of the corresponding 1D scalar Twiss parameters [1,2]. Due to symplecticity of the matrix W the matrix Twiss parameters satisfy the relations

$$\beta\,\gamma = I + \alpha^2, \tag{12a}$$

$$\alpha\,\beta = \beta\,\alpha^{\top}, \tag{12b}$$

$$\gamma\,\alpha = \alpha^{\top}\,\gamma. \tag{12c}$$

NORMALIZED VARIABLES

As it was earlier shown in [2], there exists a lower block triangular symplectic matrix T satisfying the relation

$$W = T^{-1}\,T^{-\top}, \tag{13}$$

and any such matrix can be represented in the form

$$T = \begin{pmatrix} r^{\top}\beta^{-1/2} & 0 \\ r^{\top}\beta^{-1/2}\,\alpha & r^{\top}\beta^{1/2} \end{pmatrix}. \tag{14}$$

where $\beta^{1/2}$ denotes an unique positive definite symmetric square root of the matrix β, and r is an arbitrary $n \times n$ orthogonal matrix.

Substituting (13) into (6) one obtains

$$(T(s)A(s)T^{-1}(0)) \cdot (T(s)A(s)T^{-1}(0))^{\top} = I, \tag{15}$$

which means that the $2n \times 2n$ matrix

$$R(s) = T(s)\,A(s)\,T^{-1}(0) \tag{16}$$

is orthosymplectic (i.e. orthogonal and symplectic simultaneously) and therefore can be partitioned into the form

$$R = \begin{pmatrix} C & S \\ -S & C \end{pmatrix}, \quad CS^{\top} = SC^{\top}, \quad CC^{\top} + SS^{\top} = I. \tag{17}$$

The equality (16), when written in the form

$$A(s) = T^{-1}(s)\,R(s)\,T(0), \tag{18}$$

gives us a (familiar in 1D) parametrization of the beam transfer matrix $A(s)$, and if we will introduce normalized variables z_n by means of the equation

$$z(s) = T^{-1}(s)\,z_n(s), \tag{19}$$

then the dynamics in the variables z_n is simply a rotation

$$z_n(s) = R(s)\,z_n(0) \tag{20}$$

and is governed by the system

$$dz_n \,/\, ds = JH_n(s)\,z. \tag{21}$$

In this system

$$H_n = \begin{pmatrix} A_n & B_n \\ -B_n & A_n \end{pmatrix}, \quad A_n = A_n^{\top}, \quad B_n = -B_n^{\top}, \tag{22}$$

and the $n \times n$ matrices A_n and B_n can be calculated as follows

$$A_n = r^{\top}\beta^{-1/2}H_{22}\beta^{-1/2}r, \tag{23}$$

$$B_n = r^{\top}\cdot dr/ds + \tfrac{1}{2}r^{\top}\beta^{-1/2}[\beta H_{12} - H_{21}\beta + H_{22}\alpha^{\top}$$
$$-\alpha H_{22} + d\beta^{1/2}/ds\cdot\beta^{1/2} - \beta^{1/2}\cdot d\beta^{1/2}/ds]\beta^{-1/2}r. \tag{24}$$

One sees that by utilizing the arbitrariness of the matrix r it is possible to make the matrix B_n equal to zero, but it does not seems to be a good choice because it makes the decomposition (13) non-local and, in general, non-symmetric with respect to the spacial variables q. So, our preferred choice is to take $r = I$, unless other, more important reasons will appear.

BETATRON MISMATCH

Let us consider two Twiss matrices W_1 and W_2 propagating along the same beamline. If one expresses the matrix A in (18) using T_1 and R_1 associated with the matrix W_1 and substitute this representation into the transport equation for the matrix W_2, then one obtains

$$(T_1W_2T_1^{\top})(s) = R_1(s)\cdot(T_1W_2T_1^{\top})(0)\cdot R_1^{\top}(s). \tag{25}$$

Because $R_1^{\top} \equiv R_1^{-1}$, the eigenvalues of the matrix $T_1W_2T_1^{\top}$ are invariants. This matrix is symplectic and symmetric positive definite. Thus its eigenvalues (i.e. mismatch amplitudes) can be arranged in the sequence

$$\lambda_1, \ldots, \lambda_n, \lambda_1^{-1}, \ldots, \lambda_n^{-1}, \quad (\lambda_1 \geq \ldots \geq \lambda_n \geq 1). \tag{26}$$

Moreover, there exists orthosymplectic matrix Q_λ such that

$$(T_1W_2T_1^{\top})(0) = Q_\lambda\,\Delta_\lambda\,Q_\lambda^{\top}, \tag{27}$$

where

$$\Delta_\lambda = \mathrm{diag}(\Delta,\, \Delta^{-1}), \quad \Delta = \mathrm{diag}(\lambda_1, \ldots, \lambda_n). \tag{28}$$

From (25) and (27) it follows that

$$(T_1W_2T_1^{\top})(s) = (R_1(s)\,Q_\lambda)\cdot\Delta_\lambda\cdot(R_1(s)\,Q_\lambda)^{\top}, \tag{29}$$

which expresses the dynamics of one Twiss matrix through the known dynamics of the other Twiss matrix plus knowledge of the mismatch amplitudes and phases, Δ_λ and Q_λ.

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PROPAGATION OF TWISS PARAMETER

It is clear that the matrix W satisfies the same differential equations as the matrix Σ, i.e. the equations

$$dW/ds \;=\; JHW - WHJ, \tag{30}$$

but now they has to be treated as differential equations with constraints or differential equations on the manifolds defined by the relations (12).

The purpose of this section is to show how one can obtain the differential equation involving only the matrix β. We will see that it is possible under assumption that the sub-matrix H_{22} in (4) is positive definite.

First, one can exclude easily from (30) equation for the matrix γ using (12a) and obtain

$$d\beta/ds = H_{21}\,\beta \,+\, \beta\,H_{12} \,-\, \alpha\,H_{22} \,-\, H_{22}\,\alpha^{\top}, \tag{31a}$$

$$d\alpha/ds = \beta H_{11} - \alpha H_{21} + H_{21}\alpha - H_{22}\beta^{-1}(I + \alpha^2). \tag{31b}$$

As the next step one needs to express the matrix α as a single-valued function of the matrix β and its derivative using Equations (31a) and (12b). Under assumption that $H_{22} > 0$ it becomes possible and the unique solution for the matrix α can be expressed as follows

$$\alpha \;=\; \tfrac{1}{2} C H_{22}^{-1}$$

$$+\tfrac{1}{2} H_{22} \int_0^{\infty} D(\xi)\,(\beta H_{22}^{-1} C - C H_{22}^{-1}\beta)\,D(\xi)\,d\xi, \tag{32}$$

where

$$C \;=\; H_{21}\,\beta \,+\, \beta\,H_{12} \,-\, d\beta/d\tau, \tag{33}$$

$$D(\xi) \;=\; H_{22}^{-1/2}\,\exp\left(-H_{22}^{-1/2}\beta H_{22}^{-1/2}\,\xi\right)\,H_{22}^{-1/2}. \tag{34}$$

Finally, the desired differential equation involving only the matrix β can be obtained by substituting (32) into the Eq. (31b).

The given expression (32) for the matrix α requires computations of the matrix exponential and evaluation of a matrix integral, which looks rather inconvenient. But for particular values of n one can do that and, for example, for $n = 2$ the result is as follows

$$\alpha \,=\, \tfrac{1}{2}\left[C + \left(\beta H_{22}^{-1} C - C H_{22}^{-1}\beta\right)/\mathrm{tr}\left(\beta H_{22}^{-1}\right)\right] H_{22}^{-1}. \tag{35}$$

SIEGEL UPPER HALF SPACE AND GENERALIZED MOBIUS MAP

In the paper [5] it was shown that with an appropriate parametrization the linear transport of the Twiss parameters in 1D can be viewed as a bilinear (or Mobius) map of the upper complex half-plane (which is the Poincare upper half plane) into itself. In this section we show that for an arbitrary n the linear dynamics of the Twiss parameters of the particle beams with equal eigenemittances can be represented as a generalized Mobius map acting on the so-called Siegel upper half space, which is the natural generalization of the Poincare upper half plane for the higher dimensions.

The Siegel upper half space $\mathbb{H}_n$ of order n is the set of all $n \times n$ complex symmetric matrices Z with positive definite imaginary part

$$\mathbb{H}_n \;=\; \left\{ Z = X + iY \in \mathbb{C}^{(n,n)} \,|\, Z^{\top} = Z,\; Y > 0 \right\}. \tag{36}$$

So, if one defines the one to one correspondence

$$X = \beta^{-1}\alpha, \quad Y = \beta^{-1} \tag{37}$$

between the matrix Twiss parameters β and α, and the elements of the Siegel upper half space Z, then the transport of the Twiss parameters in the variables Z takes on the form of a generalized Mobius transformation

$$Z(s) \;=\; -\left[A_{22}(s)Z(0) - A_{21}(s)\right]$$

$$\cdot\left[A_{12}(s)Z(0) - A_{11}(s)\right]^{-1}, \tag{38}$$

and, therefore, the dynamics of the Twiss parameters can be considered from the point of view of the symplectic geometry on $\mathbb{H}_n$ [6].

Note that the variables Z also satisfy the matrix differential Riccati equation

$$dZ/ds = H_{11} \,-\, H_{12}Z \,-\, ZH_{21} + ZH_{22}Z. \tag{39}$$

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