PROCEEDINGS OF SCIENCE

Cargese lectures: a string phenomenology primer

Steven Abel*
Institute for Particle Physics Phenomenology, Durham
E-mail: s.a.abel@durham.ac.uk

A pedagogical introduction to string theory and string phenomenology is given. The classical and quantum strings are introduced using a pedagogical and pragmatic approach for a reader familiar with supersymmetric field theory. The emphasis is on deriving and understanding the phenomenological properties, chiefly the spectrum and interactions of the 5 perturbative supersymmetric theories. An overview of some recent phenomenological applications and developments is also given.

Cargèse Summer School: Cosmology and Particle Physics Beyond the Standard Models
June 30-August 11 2007
Cargese, France

*Speaker.

PoS(cargese)001

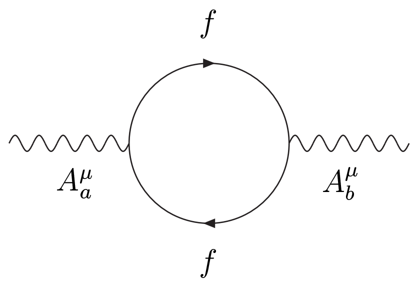


Figure 1: Vacuum polarization in field theory.

1. Introduction: what is string phenomenology anyway?

In this brief set of lectures I give an overview of String theory and its phenomenology. The lectures will be aimed at the newcomer familiar with field theory and my aim is to be pedagogical rather than comprehensive. My goal is to introduce the basic physical concepts, in enough depth that the interested reader will be able to progress to the many textbooks on the subject, but not in so much depth that I simply end up repeating these already excellent texts [1, 2, 3]. The concepts I develop will be sufficient to then go on to sketch the more recent developments in phenomenology.

To start, I would like to discuss what string *phenomenology* actually means, or at least what it means to me, because I believe the emphasis has changed in recent years. There is no doubt that in the early days of string theory (during the first string revolution if you will) the expectation was that string theory would yield a unique or at most small number of possible models one of which would closely resemble the Minimal Supersymmetric Standard Model (MSSM) which has been eloquently outlined to you by Hitoshi Murayama in these lectures. Indeed Heterotic $E_8 \times E_8$ (I will summarize the properties of the 5 supersymmetric theories shortly but for the moment you just need to know that this is one of them) seemed to give tantilising support to this idea, yielding as it does GUT models as well as MSSM-like models. In recent years however, and especially since the second string revolution, a large number of new model building techniques have evolved based on so called D-branes in type II models. These can also yield MSSM-like models and there is little indication in string theory as to which is the correct route. The search for yet more MSSM-like models then seems a less productive enterprise than it once did.

Because of this I think it important to lay out what I believe to be the two important ways that string phenomenology influences our thinking in less specific ways. The first important property of string theory is that it remains our only candidate for a theory of quantum gravity. Thus even though the final theory may or may not resemble string theory, there is no doubt that string theory has shown us how theories of quantum gravity deal with various problems that cannot be addressed in field theory. The most obvious example is of course the taming of Ultra-Violet (UV) divergences. Consider the vacuum polarization diagram and its string equivalent (in an open string theory) which includes the annulus, as shown in figures (1) and (2) respectively.

The UV divergence corresponds to the "loop becoming small". At energies much higher than the string scale the loop is much smaller than the typical string length and the diagram turns into a tiny cylinder. However, the conformal properties of string theory – which I'll get to later – ensure

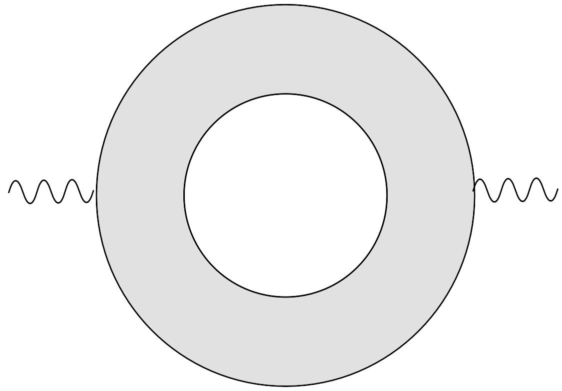


Figure 2: Vacuum polarization in an open string theory

that the overall size of the cylinder is meaningless: only the ratio of string-width to annulus-radius is physically meaningful. The diagram can then be seen as a tree level propagation of a closed string formed by the combination of two open ones. UV in the "open string channel" corresponds to long range Infra-Red (IR) propagation in the "closed string channel". IR divergences are much easier to understand (and cancel) since they depend on the global properties of the theory. The general lesson here is that in a theory of quantum gravity, one expects pathologies of the field theory, such as UV divergences, to be cured by new physical degrees of freedom (in this case closed strings) which pop-up in the theory at Planckian (or more precisely stringy) energies.

1.1 On large extra dimensions

The second important area where string theory has influenced our ideas is in its interplay with field theory. A case in point is again extra dimensional field theory. The proposal by Arkani-Hamed et al in ref.[4] that large extra dimensions can be an explanation for the apparently large Planck scale, was supported by the fact the such a construction, in which matter fields are confined to a subspace of a higher dimensional model, have a natural realization in string theory. Many ideas that are now common, such as large extra dimensions, have arisen from string theory or at least been inspired by it. Conversely ideas couched purely in terms of for example extra dimensional field theory have often guided subsequent string theory developments.

Consider how we used to estimate the fundamental scale of quantum-gravity. The familiar estimate is a dimensional one, based on measured constants of nature

$$\left.\begin{array}{l} G = 6.673\times 10^{-11} m^3\,\mathrm{kg}^{-1}\,s^{-2} \\ h = 1.055\times 10^{-34} Js \\ c = 2.997\times 10^{8} ms^{-1} \end{array}\right\} \rightarrow L_{\mathrm{Pl}} = \sqrt{Gh/c^3} = 1.61\times 10^{-33}\,\mathrm{cm}$$

The resulting Planck length ($\equiv M_{\mathrm{Pl}} = 1.22\times 10^{19}\,\mathrm{GeV}$) is the scale at which we used to think quantum-gravity effects would first make themselves felt.

What can go wrong with this estimate? The crucial point, emphasized in Ref.[4], is that the energy scale at which we measure G_N is vastly different from M_{Pl} itself. (This is possible because, alone among the forces, the effect on gravity of adding extra masses is always positive.) The implicit assumption is that in between the two scales there are no abnormally large parameters entering into the physics. In particular for this discussion, in theories which have extra dimensions that are much larger than the fundamental scale, the measured Newton's constant can be much weaker than expected because the gravitational force is diluted by the extra volume. Indeed the naive relation is

$$G_4 \sim V_{D-4}^{-1} G_D \tag{1.1}$$

where V_{D-4} is the volume of whatever extra dimensions our theory happens to have. (Note that we will for simplicity only consider flat extra dimensions.) If for example we have a fundamental scale of $M_s \sim 1\,\mathrm{TeV}$ then $V_{D-4} \sim 10^{32}$ (in fundamental lengths) gives the required enhancement factor of 10^{16} to the Planck mass. If $D = 10$ then we would require the extra dimensions to be of order few $\times 10^5\,\mathrm{TeV}^{-1}$.

On the other hand gauge forces cannot consistently be allowed to feel the same extra dimensional volumes. This is because gauge couplings are dimensionless so that the extra volume would just lead to either nonperturbatively large or immeasurably small couplings. (They could feel *some* large volumes however, in which case there is some rescaling required and the relationships become a little more complicated but similar.)

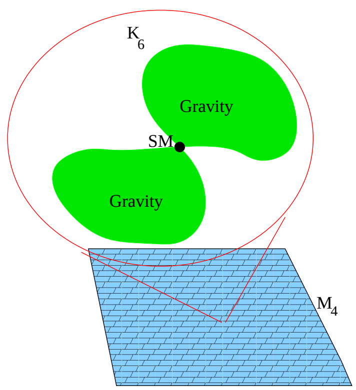


Figure 3: Brane world picture with 4 large flat dimensions represented as a plane and extra small dimensions determining the different scales of nature.

The generic picture for significantly changing the scale of quantum gravity is therefore as shown in Figure 3. The large flat 4 dimensional space in which we apparently live is shown as the flat plane. Blowing up any portion of it reveals an internal space that determines all of the physics (supersymmetry, particle content and so on). The fundamental scale can be much lower than the Planck scale if gravity feels a large internal volume (denoted by green blobs), with the Standard Model (SM) fields being confined to some restricted subvolume.

This type of set up is a natural possibility in string theory with its 6 extra dimensions, but *large* extra dimensions are a reasonable thing to consider only because of a feature of string theory that we used to regard as a problem, namely the *vacuum degeneracy problem*. To summarize, the problem is that string theory gives no hints as to the shape or size of the compactified vacua, or even the number of compactified dimensions. So for example we have no explanation as to why there are 4 large flat dimensions. More specifically this can be stated as follows. The size and shape of a particular compactification manifold can be specified by various parameters (for example the various radii), known collectively as moduli. Choosing a particular compactification radius corresponds to fixing these parameters. Since they determine the 4 dimensional physics they should of course be the same (i.e. Figure 3 should look the same) at every point in M_4. However these parameters correspond to the VEVs of fields in the spectrum that are left over from the higher dimensional metric. These fields turn out to be massless, and indeed their potential is completely flat to all orders in perturbation theory. (In terms of Figure 3, if for example we perturb the compactification manifold at a particular point in M_4 then all the neighbouring manifolds are perturbed and so on, and a signal radiates out at the speed of light in M_4; these are the massless particles.) In addition we are at liberty to set the compactification to be as large as we like, with the hope that our preferred choice will at some stage be explained by a non-perturbative contribution to the moduli potential. So when it comes to lowering the fundamental scale, the vacuum degeneracy problem is seen as a virtue.

2. On energy scales and model building

We now turn to how this idea has been realized in stringy set-ups. For this we first need a “road-map" of string theory in order to orient ourselves; we begin with the canonical layout of 10 dimensional string theory plus supergravity shown in Figure 4.

Five of the labelled points represent the various perturbative regimes (i.e. different kinds of string theory) that can be written down in 10 dimensions. These are Heterotic, and type IIA/B, all of which are theories of closed strings, and type I which is an $SO(32)$ theory of open strings. In addition the diagram includes a sixth point representing 11D supergravity. The triumph of the 2nd string revolution was to demonstrate that by applying successive duality transformations it is possible to get from any of the 6 perturbative points on this diagram to any other. The conjecture is therefore that the perturbative theories are simply limits of some nonperturbative underlying theory which encompasses the whole of this diagram, for which the search continues. In the meantime one can consider the phenomenological possibilities for the 6 theories where we can do perturbation theory.

Later in this review I will discuss how phenomenology has taken us to all the different corners of this road map. The itinerary is determined by the value of the string scale in the different models, starting with the most conservative case of a string scale of the order of the Planck mass in weakly coupled heterotic models down to GUT string scale (strongly coupled heterotic), so-called intermediate scale models (type I and II models) and finally discussing the radical idea of a TeV string scale (in non-supersymmetric models with D-branes intersecting at non-trivial angles). Before doing so however, I will review the construction of the 5 fundamental string theories. The properties of these models are summarized in the following table

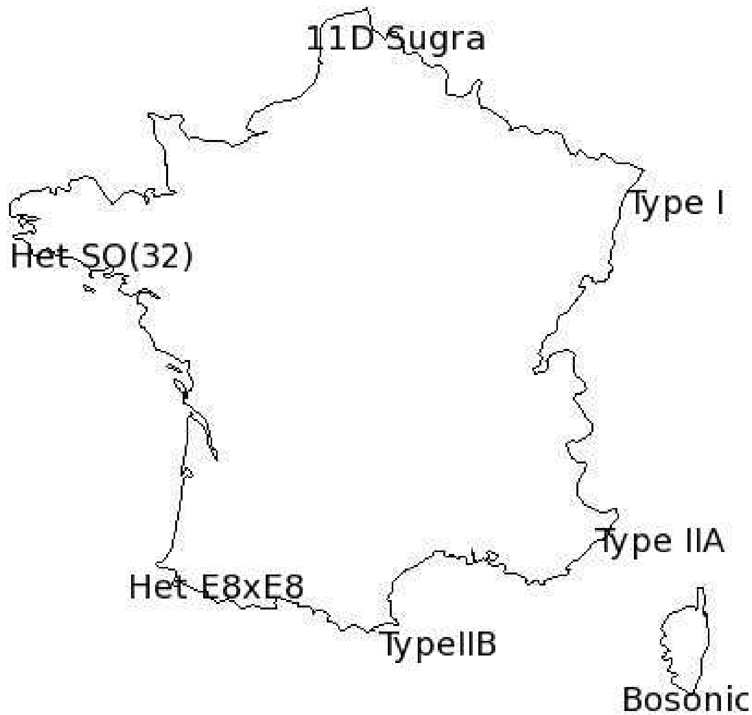


Figure 4: Le pays mystérieux. The 5 well-behaved incarnations of string theory, plus 11D supergravity are well connected by easy-to-follow routes. The central region is highly nonperturbative and mysterious (to outsiders). There is also a badly behaved (but very welcoming) 26 dimensional bosonic theory, whose connections with the rest of the theory are tenuous.

Type	Open/Closed	Dp-branes?
IIA	Closed	$p = 0,2,4,6,8$ allowed
IIB	Closed	$p = 1,3,5,7,9$ allowed
I	Open and Closed	$p = 1,5,9$ allowed
Heterotic $SO(32)$	Closed	
Heterotic $E_8 \times E_8$	Closed	

3. The classical point particle and geodesic motion

Many of the concepts that will be important can be understood intuitively at the classical level and indeed many of the most important model building issues are geometrical and classical. For example the number of generations is given by the number of fixed points in Heterotic closed string models, or the number of intersections in intersecting brane models, both classical properties. For this reason I shall spend longer than usual, in this and the following sections, on developing the classical behaviour of strings, and then move on to consider how one can derive the spectrum of the 5 classical string theories. I should state before beginning this exposition that my approach will be pragmatic and brief.

Let us begin by going back to the classical point particle in general relativity. By now I hope you all have a good idea of what the ideas are. We wish to describe a particle *falling* under the influence of a background gravitational metric $g_{\mu\nu}(X)$.

I will using the labels $\mu, \nu = 0 \dots D-1$ for the spacetime degrees of freedom. The metric itself is generally a function of the spacetime coordinates which I'll call X^μ. The particle follows *geodesic motion;* the explicit definitions can vary (you may have seen it defined in terms of parallel transport of the velocity 4-vector for example) but for us the convenient definition is that it is motion that minimizes the length of the world line.

We should sort out what this means in terms of invariant physical observables. All observers will agree for example that a clock falling from point A to point B will have showed the same time when it passed B. This is the *length of the world-line* given by

$$\int ds \tag{3.1}$$

where in special relativity

$$ds^2 = dX_0^2 - dX_1^2 - ...dX_{D-1}^2 \tag{3.2}$$

and in GR we have

$$\begin{aligned} g_{\mu\nu} &= (-+++...) \\ ds^2 &= -g_{\mu\nu} dX^\mu dX^\nu \end{aligned} \tag{3.3}$$

Geodesic motion is motion that minimizes the proper time,

$$\delta \int ds = 0, \tag{3.4}$$

and the *postulate* of GR is that particles follow this motion. Unfortunately s is a rather inconvenient parameter, and instead we can define a *world-line parameter* τ (a set of arbitrary "notches" on the world-line) and use

$$\delta \int \sqrt{-\dot{X}.\dot{X}}\, d\tau = 0 \tag{3.5}$$

instead, which is obviously equivalent by eq.3.3. In the above I am using the shorthand: $\dot{X}.\dot{X} = g_{\mu\nu} \frac{dX^\mu}{d\tau} \frac{dX^\nu}{d\tau}$.

3.1 The action for geodesic motion

What should we take for the action? Hamilton's principle suggests $-\int \sqrt{-\dot{X}.\dot{X}}\, d\tau$ with τ playing the role of time as the action, since it is already something that is minimized. Unfortunately this does not have the right units: in units where $c = h = 1$ then $[s] \equiv [X^0] \equiv [X^i]$[1] and we need to make the action dimensionless by multiplying with something that has dimensions of $[s^{-1}] \equiv [mass]$. The only other invariant parameter available to us is the rest mass of the object, m so our guess for the action takes the form

$$\begin{aligned} S &= -m \int ds \\ &= -m \int \sqrt{-\dot{X}.\dot{X}}\, d\tau \end{aligned} \tag{3.6}$$

[1]Note that greek indices $\mu = 0 \dots D-1$ refer to spacetime whereas latin ones $i = 1 \dots D-1$ refer to space only.

This is remarkably simple but not quite yet in the usual form since τ is just a parameter, whereas we are used to using the actual time, $X^0 \equiv t$, in whatever coordinate system I feel like using.

Reparameterization invariance

This is easily achieved because the Lagrangian is *manifestly reparameterization invariant* (as it should be since the physics should not depend on the parameterization). That is I can redefine a new parameterization of the world-line by

$$\tilde{\tau} = \tilde{\tau}(\tau) \tag{3.7}$$

and the action should look the same: by the chain rule

$$g_{\mu\nu}\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau} = g_{\mu\nu}\frac{dX^\mu}{d\tilde{\tau}}\frac{dX^\nu}{d\tilde{\tau}}(\frac{d\tilde{\tau}}{d\tau})^2 \tag{3.8}$$

and since $d\tilde{\tau} = (\frac{d\tilde{\tau}}{d\tau})d\tau$ we can indeed just replace $\tau \to \tilde{\tau}$ everywhere in the action. In the *gauge* where $\tau = t$, (which for some not very obvious reason is called the *physical gauge*) the action now takes the more comforting form

$$S = -\int_{t_A}^{t_B} m\sqrt{1 - v.v}dt. \tag{3.9}$$

Note that one of the $X's$ (i.e. X^0) has been removed; the Lagrangian is a function of the "fields" $X^{i=1..D-1}$, and t is the time parameter.

3.2 Equations of motion

It is worth deriving the equations of motion because it will be good practice for what we have to do later on with strings: the Euler-Lagrange equations are

$$\dot{P}_\mu - \frac{\partial \mathscr{L}}{\partial X^\mu} = 0 \tag{3.10}$$

where

$$P_\mu = \frac{\partial \mathscr{L}}{\partial \dot{X}^\mu} \tag{3.11}$$

and dots imply differentiation w.r.t. τ. Note that the X^μ look a bit like a set of D fields in a one dimensional (τ) field theory. This is more than a pedantic observation; when we come later to identify string theory as a CFT it will be "field" theory in this sense, and the X^μ will be the fields in question. I want you to keep this correspondence in mind (whenever I mention fields this is what I mean).

Consider special relativity: here we have

$$\mathscr{L} = -m\sqrt{(\dot{X}^0)^2 - \sum(\dot{X}^i)^2} \tag{3.12}$$

This gives us

$$\begin{aligned} P_\mu &= m\frac{1}{\sqrt{(\dot{X}^0)^2 - \sum(\dot{X}^i)^2}}(-\dot{X}^0, \dot{X}^i) \\ &= m\gamma(-1, v^i) \end{aligned} \tag{3.13}$$

where $v^i = \frac{\partial X^i}{\partial X^0}$ and $\gamma = 1/\sqrt{(1-v.v)}$ and we have used the chain rule. The EL equations give us

$$\dot{P}^\mu = 0. \tag{3.14}$$

Note that this is not quite the statement of constant momentum because we have differentiation w.r.t. τ not the physical time X^0; to get the latter we just have to multiply by $1/\dot{X}^0$. We can apply the above to get the geodesic equations of motion in a general setting, which should be familiar to you. Recall that they are

$$\ddot{x}^\mu + \Gamma^\mu_{\rho\sigma}\dot{x}^\rho \dot{x}^\sigma = 0 \tag{3.15}$$

where

$$\Gamma^\rho_{\nu\sigma} = \frac{g^{\rho\mu}}{2}(g_{\mu\nu,\sigma} + g_{\mu\sigma,\nu} - g_{\sigma\nu,\mu}) \tag{3.16}$$

is the Christoffel symbol. Getting this result is good practice for manipulating metric differentiation.

Exercises A:

1. ***Derive the general geodesic equation of motion***

3.3 Symmetries and conserved currents

τ *reparameterization invariance*

As an exercise now read appendices A and B in order to recap how conserved currents arise. Following the notes there, (dropping the x-coordinate, and replacing $u \to X^\mu$) we can work out the Hamiltonian associated with the reparameterization invariance;

$$\begin{aligned} \mathscr{H} &= P_i v^i - \mathscr{L} \\ &= m\gamma v.v + m/\gamma \\ &= m\gamma \end{aligned} \tag{3.17}$$

which is the usual expression for the relativistic energy.

Lorentz invariance

Consider a flat metric $\eta_{\mu\nu}$. This has a full Poincarśymmetry. The Lorentz symmetry is generated by rotations on just the X^μ coordinates of the form

$$X^\mu \to \Lambda^\mu_\nu X^\nu \tag{3.18}$$

that leaves $X.X$, $\dot{X}.\dot{X}$ invariant and hence the Lagrangian is invariant even if the metric remains the same (it's an *isometry*). Consider an infinitessimal transformation $\Lambda^\mu_\nu = \delta^\mu_\nu + \varepsilon^\mu_\nu$; then invariance of the dot product requires

$$\begin{aligned} \eta_{\mu\nu}\delta(X^\mu X^\nu) &= 2X^\mu X^\rho \eta_{\mu\nu}\varepsilon^\nu_\rho \\ &= 2X_\mu X_\rho \varepsilon^{\mu\rho} = 0 \end{aligned} \tag{3.19}$$

which is satisfied if $\varepsilon^{\mu\rho} + \varepsilon^{\rho\mu} = 0$; that is rotations generated by antisymmetric matrices generate the Lorentz symmetry group. Now consult the appendices; converting from the general formalism, the equivalence is $t \equiv \tau$, $u \equiv X^\mu$, and $\delta u \equiv \varepsilon^\mu_\nu X^\nu$. The boundary term that should vanish is

$$\begin{aligned} \int \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial u_t}\delta u\right) dt \equiv & \int \frac{d}{d\tau}\left(\frac{\partial \mathscr{L}}{\partial \dot{X}^\mu}\delta X^\mu\right) d\tau \\ = & \int \frac{d}{d\tau}\left(P_\mu X_\nu \varepsilon^{\mu\nu}\right) d\tau \\ = & \frac{\varepsilon^{\mu\nu}}{2}\left[P_\mu X_\nu - P_\nu X_\mu\right]_{t_A}^{t_B} \end{aligned} \tag{3.20}$$

so that

$$M_{\mu\nu} = P_\mu X_\nu - P_\nu X_\mu \tag{3.21}$$

are a set of conserved currents. (Strictly speaking only the angular momenta M_{ij} are conserved.)

Space-shift invariance

This is an additional symmetry of constant shifts of the X^μ coordinates, and together with the Lorentz rotations the whole forms the Poincaré group. The shift is

$$X^\mu \rightarrow X^\mu + \varepsilon^\mu \tag{3.22}$$

The Lagrangian is trivially invariant under this shift since ε^μ is constant. The action is invariant as well if the boundary term vanishes;

$$\int \frac{d}{d\tau}\left(\frac{\partial \mathscr{L}}{\partial \dot{X}^\mu}\delta X^\mu\right) d\tau = \varepsilon^\mu \left[P_\mu\right]_{t_A}^{t_B} \tag{3.23}$$

identifying the momentum P_μ as the conserved current associated with shift symmetry.

3.4 Including gauge fields

When we reach D-branes much later, it will also be useful to know how to include gauge fields in this formalism. In classical electrodynmics the rate of change of momentum of a charged particle in an em field is given by the Lorentz force law which in 4 dimensions can be written;

$$\frac{dP_i}{dt} = q(E_i + \varepsilon_{ijk} v^j B^k) \tag{3.24}$$

where $\varepsilon_{123} = 1$ and ε_{ijk} is antisymmetric under exchange of any two indices (the Levi-Cevita symbol), so that $\varepsilon_{ijk} v^j B^k \equiv \overline{v} \times \overline{B}$. We wish to write this in a covariant formalism, so we need

$$F^{\mu\nu} = \begin{pmatrix} 0 & E^i \\ -E^j & B^{ij} \end{pmatrix} \tag{3.25}$$

where $B^{ij} = \partial^i A^j - \partial^j A^i$ is the generalization of the 4D magnetic field which has $B_z \equiv B^{12}$, $B_y = -B^{13}$ and $B_x = B^{23}$ or more succinctly $B_i = \varepsilon_{ijk} B^{jk}$. The Lorentz force law is then

$$\frac{dP_i}{dt} = q(E_i + F_{ij} v^j) \tag{3.26}$$

Now take the physical gauge, where $\dot{X}^\mu = (1, v^i)$. The above is clearly the space parts of

$$\dot{P}_\mu = qF_{\mu\nu}\dot{X}^\nu \tag{3.27}$$

Since the derivatives are the same on both sides it is now trivial to get to any other gauge. It can be shown (see exercise 3 below) that this equation results from the action

$$\begin{aligned} S_{em} &= -q\int A_\mu \dot{X}^\mu d\tau \\ &= -q\int A.dl \end{aligned} \tag{3.28}$$

where dl indicates the line-integral along the particle's path. You may have seen this action previously in the Aharonov-Bohm effect.

3.5 The einbein formalism

String theory has an intimate connection with conformal field theory. In order to take advantage of this it is helpful to use the so-called einbein formalism. In this formalism we introduce a new non-propagating worldsheet field $e(\tau)$. The action becomes

$$S_P = \frac{1}{2}\int d\tau \frac{\dot{X}^2}{e} - em^2. \tag{3.29}$$

Applying the equations of motion for e,

$$\frac{\delta S_P}{\delta e} = \dot{X}^2 + e^2 m^2 = 0 \tag{3.30}$$

we recover the previous action. The (classical) physics is entirely equivalent if we use the X equations of motion with the einbein equation of motion imposed as a constraint. The two formalisms are useful in different situations. The advantage of the einbein formalism for the particle case is that we avoid rather tricky square roots, and that we can treat massless particles ($m = 0$). For strings, the equivalent formalism proves vital when it comes to quantization.

Exercises B:

1. ***Check that Maxwell's equations correspond to the equation*** $\partial_\nu F^{\mu\nu} = j^\mu$
2. ***What does the "zero" component of the force law equation correspond to?***
3. ***Check that the action in eq.3.28 yields the force law eq.3.27***

4. The classical string and Nambu-Goto action

We now want to adapt the previous discussion to the string. We are essentially going to follow the steps of the point particle generalizing to a one dimensional object falling under gravity. First I'll describe the generalization. We have seen that point particles follow geodesics that minimize the world-line. A string is an extended object that must also have an action that is invariant. The generalization of the world-line for the particle is the *world-sheet*, the two dimensional area that is swept out in space-time by the string as it moves along. The action is therefore postulated as

$$S = -T \int dA \tag{4.1}$$

where A is the proper world-sheet area (as determined by a collection of falling clocks and rods attached to the string) T has dimensions $mass^2$ to counter the dimensions of area. In order to render this action in a usable form, we need to parameterize the world-sheet as we did the world line. In order to do this we introduce another parameter as well as τ which I'll call $\sigma \in [0, 2\pi]$. The strings can either be *closed* (i.e. a closed loop with with no endpoints) or *open* (with two endpoints); the σ parameter, if you imagine a static string, takes us all the way from one end (at 0) to the other (at 2π) or if the string is a closed loop takes us all the way around it. I'll now make a few definitions:

- T: *string tension*. That this constant really plays the roll of the tension of the string will be shown explicitly later
- X^μ: The *target space* coordinate, the position in space time of the string
- $\sigma^{\alpha=0,1} = (\tau, \sigma)$: *world-sheet coordinates*

Next we need to find an expression for the *proper area.* First consider the world-sheet coordinates. τ is a time-like parameter whereas σ is something that measures the distance around the string so it is space-like. In order to get a better picture it helps to go to a Euclidean signature by defining

$$\begin{aligned} \chi^\alpha &= (i\tau, \sigma) \\ Y^\mu &= (iX^0, X^i) \end{aligned} \tag{4.2}$$

Now consider an area element $d\hat{A}$ that is mapped from a small rectangle of parameters with lengths $\delta\chi^0$ and $\delta\chi^1$. The actual element $d\hat{A}$ is a parallelogram with sides given by vectors

$$\begin{aligned} &\delta\chi^0 \partial_0 Y^\mu \\ &\delta\chi^1 \partial_1 Y^\mu \end{aligned} \tag{4.3}$$

The area of a parallelogram with two sides a b with angle θ between them is $ab \sin\theta = \sqrt{a^2 b^2 - (a.b)^2}$. We can insert the above into this expression

$$\begin{aligned} dA &= \sqrt{|\delta\chi^0 \partial_0 Y^\mu|^2 |\delta\chi^1 \partial_1 Y^\mu|^2 - (\delta\chi^0 \partial_0 Y^\mu \delta\chi^1 \partial_1 Y_\mu)^2} \\ &= \delta\chi^0 \delta\chi^1 \sqrt{(\partial_0 Y.\partial_0 Y)(\partial_1 Y.\partial_1 Y) - (\partial_0 Y.\partial_1 Y)^2} \end{aligned} \tag{4.4}$$

Where the inner product is the Euclidean one over D dimensions of Y^μ. Now we can easily return to the Minkowski variables;

$$dA = -id\hat{A} = \delta\tau\delta\sigma\sqrt{(\dot{X}.X')^2 - (\dot{X}.\dot{X})(X'.X')} \tag{4.5}$$

where I have had to absorb an overall factor of i in the area in going to the Minkowski signature and where for short hand I have defined

$$\dot{X}^\mu = \partial_\tau X^\mu;\ X'^\mu = \partial_\sigma X^\mu \tag{4.6}$$

The nett effect of having a Minkowski signature is just the minus sign inside the square-root. Finally the action is given by the integration over all the elements;

$$S_{NG} = -T\int\sqrt{(\dot{X}.X')^2 - (\dot{X}.\dot{X})(X'.X')}\,d\tau d\sigma \tag{4.7}$$

4.1 Symmetries and conserved currents

Reparameterization (Diffeomorphism or Diff) invariance

Before getting the equations of motion we should check that this action also has reparameterization invariance. In order to do this we first need to define the *induced metric.* Consider a line element *on the world sheet* of length ds given by

$$ds^2 = -g_{\mu\nu}dX^\mu dX^\nu. \tag{4.8}$$

We can rewrite this in world sheet elements as

$$ds^2 = -(\partial_a X.\partial_b X)\,d\sigma^a d\sigma^b \tag{4.9}$$

where $a,b \in \{0,1\}$ are indices for the world sheet. This defines what is called the induced metric

$$h_{ab} = \partial_a X.\partial_b X \tag{4.10}$$

which is the metric required for line elements that are constrained to lie in the world sheet. It is also known as the pullback of the space-time metric onto the world-sheet. Written out explicitly we have

$$h_{ab} = \begin{pmatrix} \dot{X}.\dot{X} & \dot{X}.X' \\ \dot{X}.X' & X'.X' \end{pmatrix} \tag{4.11}$$

and now we see that the action is actually remarkably simple;

$$S_{NG} = -T\int\sqrt{-h}\,d\tau d\sigma \tag{4.12}$$

where $h = \det h_{ab}$. This action is quite easily seen to be reparameterization invariant; indeed if I redefine the coordinates to $\tilde{\tau}(\tau,\sigma),\tilde{\sigma}(\tau,\sigma)$ then invariance of the line element means that

$$ds^2 = -h_{ab}\frac{\partial\sigma^a}{\partial\tilde{\sigma}^c}\frac{\partial\sigma^b}{\partial\tilde{\sigma}^d}\,d\tilde{\sigma}^c d\tilde{\sigma}^d \tag{4.13}$$

so we have

$$\begin{aligned}\tilde{h}_{cd} &= h_{ab}\frac{\partial\sigma^a}{\partial\tilde{\sigma}^c}\frac{\partial\sigma^b}{\partial\tilde{\sigma}^d}\\ \tilde{h} &= h\,\det\left(\frac{\partial\sigma^a}{\partial\tilde{\sigma}^c}\right)^2\end{aligned} \tag{4.14}$$

and the action can be written

$$\begin{aligned}S_{NG} &= -T\int\sqrt{-\tilde{h}}\,\det\left(\frac{\partial\tilde{\sigma}^a}{\partial\sigma^c}\right)d\tau d\sigma\\ &= -T\int\sqrt{-\tilde{h}}\,d\tilde{\tau}d\tilde{\sigma}\end{aligned} \tag{4.15}$$

where in the last line I used the Jacobian for the integration

$$d\tilde{\tau}d\tilde{\sigma} = \det\left(\frac{\partial\tilde{\sigma}^a}{\partial\sigma^c}\right)d\tau d\sigma \tag{4.16}$$

Poincaré invariance

In a Minkowski background the theory has Poincaré invariance just as for the classical point particle. Under infinitessimal transformations of the form

$$X^\mu \to \Lambda^\mu_\nu X^\nu + \varepsilon^\mu \tag{4.17}$$

one finds the following conserved currents;

$$\begin{aligned}p_\mu &= \int d\sigma P^\tau_\mu\\ M_{\mu\nu} &= \int d\sigma(X_\mu P^\tau_\nu - X_\mu P^\tau_\nu)\end{aligned} \tag{4.18}$$

where the canonical momentum is defined as

$$P^\alpha_\mu = \frac{\partial\mathscr{L}}{\partial(\partial_\alpha X^\mu)}\,. \tag{4.19}$$

Note that P^τ_μ plays the role of momentum density on the string. The conserved currents are the corresponding point particle quantities, integrated over the length of the string.

4.2 Equations of motion

We now impose the equations of motion and I will be a little more careful than in the point particle case, because of the boundary terms;

$$\delta S_{NG} = T\int d\sigma d\tau(\partial_a P^a_\mu)\delta X^\mu - T\int d\tau[P^\sigma_\mu\delta X^\mu]^{\sigma=\pi}_{\sigma=0}\,. \tag{4.20}$$

The Euler-Lagrange equations are given by the local contribution on the world-sheet

$$\partial_a P^a_\mu = 0\,. \tag{4.21}$$

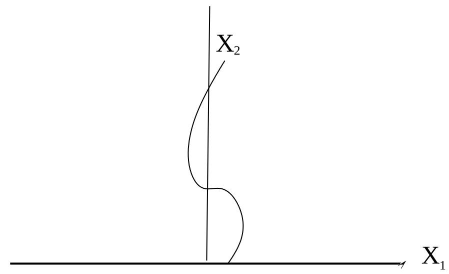


Figure 5: Open string. The lower end has ND boundary conditions, the upper end NN.

In addition we must impose a boundary condition at the enpoints of the string

$$[P^{\sigma}_{\mu}\delta X^{\mu}]^{\sigma=\pi}_{\sigma=0} = 0\,. \tag{4.22}$$

This can be satisfied by imposing either Neumman boundary condition, $P^{\sigma}_{\mu} = 0$, or Dirichlet ones, $X^{\mu} = const$ along each dimension. Note that there is of course nothing particular about the choice of coordinate system. The statement is simply that the momentum vector $\eta^{\mu\nu}P^{\sigma}_{\nu}$ is orthogonal to the "velocity" (i.e. $\partial_{\tau}X^{\mu}$) of the string at its endpoints.

4.3 Some examples of classical physics

Using the equations of motion and boundary conditions above, it is fun to think of some examples of classical physics to convince oneself that the classical string behaves pretty much like a piece of elastic (with zero mass when the string is stationary). One can pretty easily prove the following results:

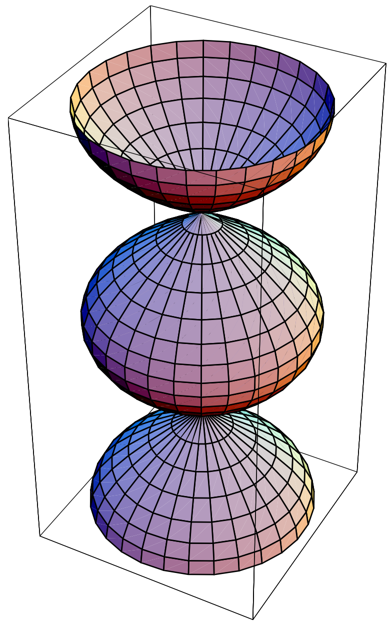
- Strings with only Neumann ends move at the speed of light.
- Spinning open strings have an angular momentum (i.e. M_{12} if the string is spinning in the 12 plane) that is proportional to the mass-squared (i.e. $(p^0)^2$). The constant of proportionality is known as the Regge-slope.
- A static open string stretched to length L has an energy $p_0 = LT$, so the T plays the role of tension.
- The following set-ups tell us a little more about the physics of strings and D-branes. Consider a circular open string of radius R_0 released from rest at time $t = 0$. The radius of the string has a time dependence given by

$$R(t) = R_0\cos(t/R_0)\,, \tag{4.23}$$

and the corresponding world sheet is shown below. It executes oscillations with period $2\pi R_0$. Note that this period is also the time taken for a light signal to go round the circumference of the string, so it is consistent with causality. The momentum is found to be

$$p_{\mu} = \frac{2\pi RT}{\sqrt{1-R_t^2}}\,(1,0,0,\ldots)\,, \tag{4.24}$$

where $R_t \equiv \frac{dR}{dt}$. Note that initially, the zeroth component is simply the stretching energy $2\pi R_0 T$, so that T is indeed playing the role of tension. As the string accelerates to the centre, $R_t \to 1$, and its energy diverges with a relativistic γ factor. The worldsheet is shown below:



- Consider now the situation shown in figure.(6). A circular arc of string in the 12 plane is again released from rest with radius R_0, but this time given Dirichlet boundary conditions on two branes intersecting at angle ϑ as shown. Using the formulae above it is not hard to show that the motion is as for the closed string, $R(t) = R_0 \cos(t/R_0)$. Now however the momentum is found to be
$$p_\mu = \frac{2\pi RT}{\sqrt{1-R_t^2}}\left(\frac{\vartheta}{2\pi}, \frac{R_t}{2\pi}\sin\vartheta\,, \frac{R_t}{2\pi}(1-\cos\vartheta), \ldots\right). \tag{4.25}$$
For $\vartheta = 2\pi$ we of course recover the previous closed loop result. For general angles, the momentum in the 12 plane is no longer zero and is time dependent. Momentum in these directions is now transferred to the D-branes as the string oscillates. Moreover the string is not free to leave the intersection; the string "lives at the intersection" as it is common to say. The classical stretching energy provides a potential keeping the string at the intersection, and when the strings are quantized we will find a sector of intersection states here as one would intuitively expect, in addition to whatever states exist in the "bulk".

4.4 The Polyakov action

The square root in the Nambu-Goto action makes it difficult to work with. An action which does not contain the square root may be obtained by analogy with the introduction of the einbein for the particle. For the string we introduce an independent world-sheet metric $\gamma_{ab}(\tau,\sigma)$ and writing
$$S_P[X,\gamma] = -\frac{T}{2}\int d\tau d\sigma\, \sqrt{-\gamma}\gamma^{ab}h_{ab}\,, \tag{4.26}$$

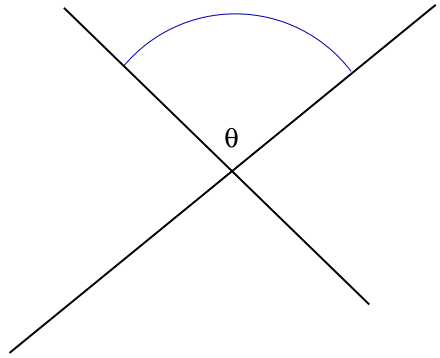


Figure 6: Arc of open string on D-branes. Momentum is not conserved in the 12 plane.

where $\gamma = \det \gamma_{ab}$. This is known as the Polyakov action. Using $\delta\gamma = -\gamma\gamma_{ab}\delta\gamma^{ab}$, the Euler-Lagrange equation for γ^{ab} reads

$$T_{ab} \equiv h_{ab} - \frac{1}{2}\gamma_{ab}\gamma^{cd}h_{cd} = 0 \tag{4.27}$$

which may be recast as

$$\frac{h_{ab}}{\sqrt{-h}} = \frac{\gamma_{ab}}{\sqrt{-\gamma}}\,, \tag{4.28}$$

allowing the Nambu-Goto action to be recovered from the Polyakov. Again the classical physics is identical if we use the Polyakov action and impose the γ equation of motion as a constraint.

The Polyakov formalism allows us to make an important link with conformal field theory as follows. Note that (4.28) is unchanged by a Weyl transformation,

$$\gamma_{ab}\left(\tau,\sigma\right) \to e^{2\omega(\tau,\sigma)}\gamma_{ab}\left(\tau,\sigma\right)\,, \tag{4.29}$$

and so Weyl-equivalent metrics correspond to the same embedding in spacetime. The diff invariance allows the three degrees of freedom in γ_{ab} to be replaced with just one;

$$\gamma_{ab} = e^{\phi(\tau,\sigma)}\eta_{ab}\,, \tag{4.30}$$

where $\eta_{ab} = \mathrm{diag}\left(-1,+1\right)$. The Weyl invariance allows this to be further reduced to

$$\gamma_{ab} = \eta_{ab}\,. \tag{4.31}$$

In this gauge, the action reads

$$S_{PC}[X] = -\frac{T}{2}\int d\sigma d\tau\; \eta^{ab}\partial_a X^\mu \partial_b X_\mu\,. \tag{4.32}$$

Varying the X^μ yields

$$\delta S_{PC}\left[X\right] = T\int d\sigma d\tau\;\left(\partial^a\partial_a X_\mu\right)\delta X^\mu - T\int d\tau\;\left(\partial_\sigma X_\mu\right)\delta X^\mu\Big|_{\sigma=0}^{\sigma=\pi} = 0\,. \tag{4.33}$$

The first term constrains the X fields to obey the one-dimensional wave equation,

$$\partial^a\partial_a X^\mu = 0\,, \tag{4.34}$$

whereas vanishing of the second term sets the boundary conditions on the string. We may choose

- Periodic boundary conditions, $X^{\mu}(\tau,0) = X^{\mu}(\tau,\pi)$.

  This choice leads to closed strings. Defining $\sigma^{\pm} = \tau \pm \sigma$, general solutions to the wave equation may be written as a superposition of right- and left-moving fields,

$$X^{\mu}(\tau,\sigma) = X^{\mu}_{-}\left(\sigma^{-}\right) + X^{\mu}_{+}\left(\sigma^{+}\right) . \tag{4.35}$$

  The solution may be written in terms of a Fourier series,

$$\begin{aligned} X^{\mu}_{-}\left(\sigma^{-}\right) &= \tfrac{1}{2}x^{\mu} + \alpha' p^{\mu}\sigma^{-} + i\sqrt{\tfrac{\alpha'}{2}}\textstyle\sum_{n\neq 0}\tfrac{1}{n}\alpha^{\mu}_{n}e^{-2in\sigma^{-}} \\ X^{\mu}_{+}\left(\sigma^{+}\right) &= \tfrac{1}{2}x^{\mu} + \alpha' p^{\mu}\sigma^{+} + i\sqrt{\tfrac{\alpha'}{2}}\textstyle\sum_{n\neq 0}\tfrac{1}{n}\tilde{\alpha}^{\mu}_{n}e^{-2in\sigma^{+}} \end{aligned} \tag{4.36}$$

  with x^{μ} and p^{μ} being the centre of mass and momentum of the string and α^{μ}_{n}, $\tilde{\alpha}^{\mu}_{n}$ being right- and left-moving Fourier coefficients, where we have defined

$$\alpha^{\mu}_{0} = \tilde{\alpha}^{\mu}_{0} = \sqrt{\frac{\alpha'}{2}}p^{\mu} . \tag{4.37}$$

  The reality condition $X^{\mu} = (X^{\mu})^{*}$ implies

$$(\alpha^{\mu}_{n})^{*} = \alpha^{\mu}_{-n} \qquad \text{and} \qquad (\tilde{\alpha}^{\mu}_{n})^{*} = \tilde{\alpha}^{\mu}_{-n} . \tag{4.38}$$

- Neumann boundary conditions, $\partial_{\sigma}X|_{\sigma=0} = \partial_{\sigma}X|_{\sigma=\pi} = 0$.

  This choice of boundary conditions describes open strings, where the left- and right-movers combine to give a standing wave:

$$X^{\mu}(\tau,\sigma) = x^{\mu} + 2\alpha' p^{\mu}\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha^{\mu}_{n}e^{-in\tau}\cos(n\sigma) . \tag{4.39}$$

  This time,

$$\alpha^{\mu}_{0} = \sqrt{2\alpha'}p^{\mu} . \tag{4.40}$$

  Again, $X^{\mu} = (X^{\mu})^{*}$ implies $\left(\alpha^{\mu}_{n}\right)^{*} = \alpha^{\mu}_{-n}$.

- Branes intersecting at an angle $\pi\theta$ [5].

  As discussed above (note that ϑ has been replaced by $\pi\theta$ for convenience). Consider branes intersecting in the 12 plane, which we can complexify;

$$Z = X^{1} + iX^{2} . \tag{4.41}$$

  In addition without loss of generality, lie the $\sigma = 0$ end of the open string along $X^{2} = 0$. The boundary conditions are

$$\begin{aligned} \sigma = 0 \ &; \ \partial_{\sigma}(Re(Z) = Im(Z) = 0 \\ \sigma = \pi \ &; \ \partial_{\sigma}(Re(e^{i\pi\theta}Z) = Im(e^{i\pi\theta}Z) = 0 . \end{aligned} \tag{4.42}$$

The mode expansion satisfying these boundary conditions is

$$Z(\tau,\sigma) = i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbf{Z}} \frac{\alpha_{n-\theta}}{n-\theta} e^{i(n-\theta)\sigma^+} + \frac{\alpha^\dagger_{n+\theta}}{n+\theta} e^{i(n+\theta)\sigma^-} \tag{4.43}$$

where α, $\alpha^\dagger$ are now a suitable independent combination of α^1 and α^2. Again, the string is localized at the intersection point, and there is no zero mode. The case often quoted in the pedagogical literature is open strings on parallel branes, corresponding to $\theta = 1$ when the exponentials combine into $e^{-in\tau}\sin(n\sigma)$.

4.5 The RNS superstring

We now consider classical supersymmetric string theory. There are two approaches available; the Ramond-Neveu-Schwarz formalism, which introduces supersymmetry on the worldsheet directly and then extends to spacetime, and the Green-Schwarz formalism which introduces spacetime supersymmetry explicitly. We work in the RNS formalism. Our starting point is the ungauged Polyakov action, which we supplement with D massless Majorana spinors on the worldsheet:

$$S = -\frac{T}{2}\int d\tau d\sigma\, \sqrt{-\gamma}\left(\gamma^{ab}\partial_a X^\mu \partial_b X_\mu + i\overline{\Psi}^\mu \rho^a \partial_a \Psi_\mu\right). \tag{4.44}$$

The ρ^a are two-dimensional gamma-matrices satisfying the usual Clifford algebra $\left\{\rho^a, \rho^b\right\} = 2\eta^{ab}$,

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{4.45}$$

and we have defined $\overline{\Psi} \equiv \Psi^T \rho^0$. This action is invariant under the global supersymmetry transformation

$$\delta X^\mu = \overline{\xi}\Psi^\mu, \qquad \delta\Psi^\mu = -i\rho^a \partial_a X^\mu \xi \tag{4.46}$$

for an arbitrary Majorana spinor ξ. Promoting $\xi \to \xi\,(\tau,\sigma)$ requires the addition of a gravitino χ_a. Then,

$$\begin{aligned} S_L = -\frac{T}{2}\int d\tau d\sigma\, \sqrt{-\gamma}\Big(&\gamma^{ab}\partial_a X^\mu \partial_b X_\mu + i\overline{\Psi}^\mu \rho^a \partial_a \Psi_\mu \\ &+2\overline{\chi}_a \rho^b \rho^a \Psi^\mu \partial_b X_\mu + \frac{1}{2}\overline{\Psi}_\mu \psi^\mu \overline{\chi}_a \rho^b \rho^a \chi_b\Big), \end{aligned} \tag{4.47}$$

which is invariant under the local supersymmetry transformation

$$\begin{aligned} \delta X^\mu &= \overline{\xi}\Psi^\mu \\ \delta\Psi^\mu &= -i\rho^a \xi\left(\partial_a X^\mu - \overline{\Psi}^\mu \chi_a\right) \\ \delta\chi_a &= \partial_a \xi \\ \delta e^a_b &= -2i\overline{\xi}\rho^a \chi_b\,, \end{aligned} \tag{4.48}$$

where e^a_b satisfies $\gamma_{ab} = e^c_a e^d_b \eta_{cd}$. Also present is a superconformal symmetry,

$$\delta\chi_a = i\rho_a \varepsilon \tag{4.49}$$

for an arbitrary Majorana spinor $\varepsilon(\tau,\sigma)$.

Just as for the bosonic string, (4.47) has a *diff*$\times$*Weyl* invariance which may be used to select $\gamma_{ab}=\eta_{ab}$. Furthermore, the local super- and superconformal symmetries are enough to select a gauge with $\chi_a=0$. Finding Euler-Lagrange equations for X^μ and Ψ^μ before selecting this *covariant gauge* recovers the one-dimensional wave equation (4.34), plus the Dirac equation

$$i\rho^a\partial_a\Psi^\mu=0\,. \tag{4.50}$$

The Euler-Lagrange equations for γ^{ab} and χ_a read, in covariant gauge,

$$T_{ab}\equiv\partial_aX^\mu\partial_bX_\mu+\frac{i}{4}\overline{\Psi}^\mu(\rho_a\partial_b+\rho_b\partial_a)\Psi_\mu-\frac{1}{2}\eta_{ab}\left(\partial^cX^\mu\partial_cX_\mu+\frac{i}{2}\overline{\Psi}^\mu\rho^c\partial_c\Psi_\mu\right)=0$$
$$J^a\equiv\frac{1}{2}\rho^b\rho^a\Psi^\mu\partial_bX_\mu=0\,. \tag{4.51}$$

These are known as super-Virasoro constraints. Note that $\partial_aJ^a=0$; the supercurrent J^a is the conserved quantity associated with the local symmetry (4.48).

As in the previous section, we have the proviso that the surface terms in the variation of S_L must vanish. For the X^μ, the requirements are identical to those of the previous section. To find boundary conditions on the fermionic fields, split Ψ^μ into right- and left-moving fields,

$$\Psi^\mu=\begin{pmatrix}\Psi^\mu_-\\ \Psi^\mu_+\end{pmatrix}\,. \tag{4.52}$$

With $\partial_\pm=\frac{1}{2}(\partial_\tau\pm\partial_\sigma)$, the Dirac equation for Ψ^μ reads

$$\partial_+\Psi^\mu_-=0 \qquad\text{and}\qquad \partial_-\Psi^\mu_+=0, \tag{4.53}$$

so that $\Psi^\mu_-$and Ψ^μ_+ describe right- and left-moving fermionic worldsheet fields respectively. The condition for surface terms to vanish is

$$[\Psi_-\cdot\delta\Psi_--\Psi_+\cdot\delta\Psi_+]^{\sigma=\pi}_{\sigma=0}=0\,. \tag{4.54}$$

Then,

- For closed strings, Periodic (Ramond, or just R) and anti-periodic (Neveu-Schwarz, or NS) boundary conditions may be chosen independently for right- and left-movers,

$$\Psi^\mu_-(\tau,\pi)=\pm\Psi^\mu_-(\tau,0)$$
$$\Psi^\mu_+(\tau,\pi)=\pm\Psi^\mu_+(\tau,0) \tag{4.55}$$

  giving four sectors in total. The mode expansions are

$$\Psi^\mu_-=\sum_r\psi^\mu_re^{-2ir\sigma^-} \qquad\qquad \Psi^\mu_+=\sum_r\tilde{\psi}^\mu_re^{-2ir\sigma^+} \tag{4.56}$$

  with r being integer moded in an R sector, and half-integer moded in an NS sector. The Majorana condition $\Psi^\mu_\pm=\left(\Psi^\mu_\pm\right)^*$ constrains the Fourier coefficients ψ^μ_r and $\tilde{\psi}^\mu_r$:

$$(\psi^\mu_r)^*=\psi^\mu_{-r} \qquad\text{and}\qquad (\tilde{\psi}^\mu_r)^*=\tilde{\psi}^\mu_{-r}\,. \tag{4.57}$$

- For open strings, left- and right-movers are related. Possible boundary conditions are

$$\begin{aligned}\Psi^{\mu}_{-}(\tau,\pi) &= \Psi^{\mu}_{+}(\tau,0)\\ \Psi^{\mu}_{-}(\tau,\pi) &= \pm\Psi^{\mu}_{+}(\tau,\sigma)\,.\end{aligned}\tag{4.58}$$

There are now only two sectors; Ramond and Neveu-Schwarz, with mode expansions

$$\Psi^{\mu}_{-} = \frac{1}{\sqrt{2}}\sum_r \psi^{\mu}_r e^{-ir\sigma^-} \qquad\qquad \Psi^{\mu}_{+} = \frac{1}{\sqrt{2}}\sum_r \psi^{\mu}_r e^{-ir\sigma^+}\tag{4.59}$$

Again, $r\in\mathbb{Z}$ in the R sector, and $r\in(\mathbb{Z}+\frac{1}{2})$ in the NS sector.

5. Canonical quantization

In the canonical quantization procedure, one imposes equal-time commutation relations on the X^{μ} and their canonical momenta $P^{\mu} = T\dot{X}^{\mu}$,

$$\left[P^{\mu}\left(\tau,\sigma\right), X^{\nu}\left(\tau,\sigma'\right)\right] = i\delta\left(\sigma-\sigma'\right)\eta^{\mu\nu}\,,\tag{5.1}$$

with other commutators zero. One also imposes equal-time anticommutation relations on $\Psi^{\mu}_{\pm}$ and their canonical momenta $\frac{1}{2}iT\Psi^{\mu}_{\pm}$,

$$\frac{1}{2}iT\left\{\Psi^{\mu}_{\pm}\left(\tau,\sigma\right), \Psi^{\mu}_{\pm}\left(\tau,\sigma'\right)\right\} = i\delta\left(\sigma-\sigma'\right)\eta^{\mu\nu}\tag{5.2}$$

with other anticommutators zero.

In all that follows, subscripts $\{n,m\}$ should be implicitly understood to be integer valued, whilst $\{r,s\}$ should be understood to take integer values in the R sector and half-integer values in the NS sector. Inserting the mode expansions (4.39) into our relations leads to

$$\begin{aligned}[x^{\mu},\, p^{\nu}] &= i\eta^{\mu\nu}\\ [\alpha_n,\, \alpha_m] = [\tilde{\alpha}_n,\, \tilde{\alpha}_m] &= m\delta_{m+n}\eta^{\mu\nu}\end{aligned}\tag{5.3}$$

with other commutators zero. Hence,

$$a^{\mu\dagger}_n = \frac{1}{\sqrt{n}}\alpha^{\mu}_{-n} \qquad a^{\mu}_n = \frac{1}{\sqrt{n}}\alpha^{\mu}_n \qquad (n>0)\tag{5.4}$$

are a set of D creation/annhiliation operators for right-moving modes. Similarly, inserting (4.59) gives

$$\{\psi^{\mu}_r,\, \psi^{\nu}_s\} = \{\tilde{\psi}^{\mu}_r,\, \tilde{\psi}^{\nu}_s\} = \delta_{r+s}\eta^{\mu\nu}\tag{5.5}$$

with other anticommutators zero.

5.1 The super-Virasoro algebra

Now, let us begin to examine the physical spectrum of the theory. One may write the classical constraint equations (4.51) as

$$T_{++} = T_{--} = J_+ = J_- = 0 \tag{5.6}$$

with

$$\begin{aligned} T_{\pm\pm} &\equiv \frac{1}{2}(T_{00} \pm T_{01}) = \partial_\pm X^\mu \partial_\pm X_\mu + \frac{i}{2}\psi^\mu_\pm \partial_\pm \psi_{\pm\mu} \\ J_\pm &\equiv \frac{1}{2}(J_0 \pm J_1) = \Psi^\mu_\pm \partial_\pm X_\mu \,. \end{aligned} \tag{5.7}$$

It is useful to define Fourier components of T_{--} and J_-:

$$\begin{aligned} L_0 &\equiv \frac{1}{4\pi\alpha'}\int_0^\pi d\sigma T_{--} + a \\ &= \frac{1}{2}\alpha_0^\mu \alpha_{0\mu} + \sum_{n>0} \alpha^\mu_{-n}\alpha_{\mu n} + \sum_{r>0} r\psi^\mu_{-r}\psi_{\mu r} + a \\ L_m &\equiv \frac{1}{4\pi\alpha'}\int_0^\pi d\sigma e^{2im\sigma^-} T_{--} \\ &= \frac{1}{2}\sum_n \alpha^\mu_{m-n}\alpha_{\mu n} + \frac{1}{2}\sum_r \left(\frac{1}{2}m - r\right)\psi^\mu_{m-r}\psi_{\mu r} \qquad (m \neq 0) \\ G_r &\equiv \frac{1}{4\pi\alpha'}\int_0^\pi d\sigma e^{2ir\sigma^-} J_- \\ &= \frac{1}{2}\sum_n \psi^\mu_{r-n}\alpha_{\mu n} \end{aligned} \tag{5.8}$$

Notice that we have treated L_0 separately, as we have a problem in this case; the raising and lowering operators do not commute, so in which order should we write them? The convention is that the lowering operators go to the right, and the (infinite) zero-point energy a is left to be dealt with later.

Our operators obey the super-Virasoro algebra,

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + A_m \delta_{m+n} \\ [L_m, G_r] &= \left(\frac{1}{2}m - r\right) G_{m+r} \\ \{G_r, G_s\} &= 2L_{r+s} + B_r \delta_{r+s} \end{aligned} \tag{5.9}$$

with sector-dependent anomaly terms

$$\begin{aligned} A_m &= \frac{1}{8}Dm^3 & B_r &= \frac{1}{8}Dr^2 & &\text{(R)} \\ A_m &= \frac{1}{8}Dm\left(m^2 - 1\right) & B_r &= \frac{1}{8}D\left(r^2 - \frac{1}{4}\right). & &\text{(NS)} \end{aligned} \tag{5.10}$$

In terms of these super-Virasoro operators, the Virasoro constraints (4.51) applied to physical states $|\varphi\rangle$ are

$$\begin{aligned} (L_0 - a)|\varphi\rangle &= 0 \\ L_m|\varphi\rangle &= 0 \qquad (m > 0) \\ G_r|\varphi\rangle &= 0 \qquad (r > 0)\,. \end{aligned} \tag{5.11}$$

Because of the anomaly terms (5.10), it is inconsistent to impose these conditions for both positive and negative m, r; in other words, it is not possible to implement the Virasoro contraints fully at the quantum level.

For open strings, only the above algebra is present; for closed strings, the Fourier components of T_{++} give operators $\tilde{L}_m$, $\tilde{G}_r$. These are exactly similar to equations (5.8), but written in terms of the left-moving operators $\tilde{\alpha}_n$ and $\tilde{\psi}_r$. Hence these operators obey a copy of the algebra (5.9).

5.2 The light-cone gauge

For convenience, in order to examine the physical degrees of freedom we can use the so-called light-cone gauge. This removes an infinite over counting (which manifests itself as unphysical negative norm states in the spectrum called ghosts). Consider first the purely bosonic string. First we define light cone coordinates

$$\begin{aligned} X^+ &= \frac{1}{\sqrt{2}}(X^0 + X^{D-1}) \\ X^- &= \frac{1}{\sqrt{2}}(X^0 - X^{D-1}). \end{aligned} \tag{5.12}$$

It is not hard to see that a pair of vectors are contracted as

$$V^\mu W_\mu = V^i W^i - V^+ W^- - V^- W^+ . \tag{5.13}$$

The X^+ coordinate corresponds to the time coordinate seen in a frame in which the string is moving with infinite momentum. The light cone gauge is usually expressed by saying

$$X^+(\sigma, \tau) = x^+ + 2\alpha' p^+ \tau. \tag{5.14}$$

There is no oscillator dependence, and it is the frame in which every point on the string is at the same value of "time" (X^+). It is now possible to use the string equation of motion to eliminate the X^- coordinate as well; i.e. we use the mode expansion

$$X_-^- = \frac{x^-}{2} + \alpha' p^- \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \alpha_n^- e^{-2in\sigma^-} \tag{5.15}$$

and its equivalent for X_+^-. Inserting these into the equations of motion we find a linear equation for each α_n^- and $\tilde{\alpha}_n^-$, which are completely determined.

Exercise: Use the string equations of motion to determine X^-

This removes all degrees of freedom that are "longitudinal" to the state and leaves only the $D-2$ oscillator modes $\alpha^{i=1..D-1}$. The nett effect when evaluating physical operators such as the Hamiltonian,is to leave only the contributions of the $D-2$ physical transverse degrees of freedom. And the spectrum is generated by these $D-2$ oscillators on the vacuum.

5.3 The open string spectrum

For open strings, after setting $L_0 = a$ in (5.8) and applying (4.40), the mass-shell condition reads

$$m^2 = -p^\mu p_\mu = \frac{1}{\alpha'}(N+a)\,, \tag{5.16}$$

where

$$N = \sum_{n>0} \alpha^\mu_{-n}\alpha_{\mu n} + \sum_{r>0} r\psi^\mu_{-r}\psi_{\mu r} \tag{5.17}$$

counts the number of states present at each level. We still have the issue of infinite zero-point energy a, which we must regularize. A convenient approach is so-called Riemann zeta regularization. Equivalently one can use the result

$$\begin{aligned}\sum_{n=1}^{\infty}(n-\theta) &= -\lim_{\varepsilon\to 0}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\sum_n e^{-\varepsilon(n-\theta)}\right] \\ &= \lim_{\varepsilon\to 0}\left[\frac{1}{\varepsilon^2}-\frac{1}{12}-\frac{1}{2}\theta(\theta-1)+\mathscr{O}(\varepsilon)\right] \\ &\equiv -\frac{1}{12}+\frac{1}{2}\theta(1-\theta)\,,\end{aligned} \tag{5.18}$$

where in the intervening steps I have summed the geometric series, expanded the result in ε and thrown away the leading infinity. The normal-ordering constants are then found by multiplying the usual contribution of $\frac{1}{2}$ for the harmonic oscillator by the number of states at each level and the number of contributing dimensions,

$$\begin{aligned}a_{\mathrm{X}} &= \frac{D-2}{2}\sum_{n=1}^{\infty} n = -\frac{D-2}{24} \\ a_\psi &= \begin{cases} -\frac{D-2}{2}\sum_{r=0}^{\infty} r = -\frac{D-2}{2}\sum_{r=1}^{\infty} r = \frac{D-2}{24} & \text{(R)} \\ -\frac{D-2}{2}\sum_{r=\frac{1}{2}}^{\infty} r = -\frac{D-2}{2}\sum_{r=1}^{\infty}(r-\frac{1}{2}) = -\frac{D-2}{48} & \text{(NS)}\end{cases}\end{aligned} \tag{5.19}$$

Therefore, we have

$$a = a_{\mathrm{X}} + a_\psi = \begin{cases} 0 & \text{(R)} \\ -\frac{D-2}{16} & \text{(NS)}\end{cases} \tag{5.20}$$

The factor of $D-2$ (rather than D) comes in because only transverse excitations of the string contribute – this may be seen explicitly by going to the light-cone gauge as described above, in which the *diff* $\times$ *Weyl* redundancy of the action is elimated. A suitable value for D may be found by examining the two sectors of the open-string spectrum. Of particular interest to a string phenomenologist are the states which are massless at the string level.

NS sector

Here, the ground state $|0;k\rangle$ has

$$\alpha' m^2 = -\frac{D-2}{16} \tag{5.21}$$

which is tachyonic for all $D > 2$ – we will deal with this problem shortly. The first excited state $\psi^\mu_{-\frac{1}{2}}|0;k\rangle$ transforms as a vector under the Lorentz group, and has $D-2$ transverse degrees of freedom. It is therefore a candidate for a spacetime boson A^{μ}, transforming under $SO(D-2)$. If this is so, then it ought to be a massless state. Since

$$\alpha' m^2 = \frac{1}{2} - \frac{D-2}{16}\,, \tag{5.22}$$

this constrains us to the value $D = 10$, in which $\psi^{\mu}_{-\frac{1}{2}}|0;k\rangle$ is an $\mathbf{8_v}$ of $SO(8)$. In fact, states in the NS sector are spacetime bosons at each mass level.

R sector

In the R sector, the ψ^{μ}_0 are massless. Furthermore, they obey a Clifford algebra

$$\left\{\sqrt{2}\psi^{\mu}_0,\sqrt{2}\psi^{\nu}_0\right\} = 2\eta^{\mu\nu}\,, \tag{5.23}$$

implying that $\Gamma^{\mu} = \sqrt{2}\psi^{\mu}_0$ are ten-dimensional gamma matrices. Let us define a set of raising and lowering operators by

$$\begin{aligned} b^{0\pm} &= \frac{1}{2}\left(\pm\Gamma^0 + \Gamma^1\right) \\ b^{a\pm} &= \frac{1}{2}\left(\Gamma^{2a} \pm i\Gamma^{2a+1}\right)\,, \qquad a = 1,\ldots,4 \end{aligned} \tag{5.24}$$

which obey

$$\left\{b^{a+},b^{b-}\right\} = \delta^{ab} \tag{5.25}$$

with other anticommutators zero. Beginning from a lowest weight state satisfying $b^{a-}|\zeta\rangle = 0$, a representation of dimension $2^5 = 32$ may be created by acting on $|\zeta\rangle$ in all possible ways with the b^{a+}. These 32 states may be denoted as

$$|\mathbf{s}\rangle = \left|\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2}\right\rangle \tag{5.26}$$

where $|\zeta\rangle$ is the state with $-\frac{1}{2}$ in each position.

The utility of this definition may be seen by noting that the generators of the $SO(9,1)$ Lorentz algebra are

$$M^{\mu\nu} = -\frac{i}{4}\left[\Gamma^{\mu},\Gamma^{\nu}\right] \tag{5.27}$$

which may be written in terms of the raising and lowering operators as

$$S_a \equiv i^{\delta_a} M^{2a,2a+1} = b^{a+}b^{a-} - \frac{1}{2} \tag{5.28}$$

where $|\mathbf{s}\rangle$ is an eigenvector of S_a with eigenvalues $s_a = \pm\frac{1}{2}$. Therefore, the spinors $|\mathbf{s}\rangle$ form the so-called Dirac representation of the Lorentz algebra, and the ground states ψ^{μ}_0 are seen to form a ten-dimensional spacetime fermion. Defining a ten-dimensional chirality operator,

$$\Gamma^{11} = \Gamma^0\Gamma^1\cdots\Gamma^9\,, \tag{5.29}$$

the Dirac representation may be reduced into two inequivalent Weyl representations of $SO(9,1)$ depending upon the value of $\Gamma^{11}|\mathbf{s}\rangle = \pm 1$;

$$\mathbf{32} = \mathbf{16} + \mathbf{16}' . \tag{5.30}$$

Not all possibilities for $|\mathbf{s}\rangle$ survive the physical state conditions (5.11). In particular,

$$G_0|\mathbf{s}\rangle = 0 \implies k \cdot \Gamma|\mathbf{s}\rangle = 0 \tag{5.31}$$

which is the Dirac equation. Choosing the (massless) frame $\mathbf{k} = (-k_1, k_1, 0, \ldots, 0)$, we see that

$$k \cdot \Gamma|\mathbf{s}\rangle = 2k_1\Gamma^0 \left(S_0 - \frac{1}{2}\right)|\mathbf{s}\rangle = 0 \tag{5.32}$$

so that only states with $s_0 = +\frac{1}{2}$ survive. Now, the two Weyl representations decompose under $SO(9,1) \longrightarrow SO(1,1) \times SO(8)$ as

$$\begin{aligned} \mathbf{16} &\longrightarrow \left(+\frac{1}{2}, \mathbf{8_S}\right) + \left(-\frac{1}{2}, \mathbf{8_C}\right) \\ \mathbf{16}' &\longrightarrow \left(+\frac{1}{2}, \mathbf{8_C}\right) + \left(-\frac{1}{2}, \mathbf{8_S}\right) . \end{aligned} \tag{5.33}$$

Therefore, surviving physical ground states in the R sector fall into either an $\mathbf{8_S}$ or $\mathbf{8_C}$ of $SO(8)$.

The GSO projection

As we saw, the lowest-lying state in the NS sector is tachyonic. A prescription which removes the tachyon is the Gliozzi-Scherk-Olive (GSO) projection, in which physical states $|\varphi\rangle$ have a projection operator applied,

$$|\varphi\rangle \longrightarrow P_{\mathrm{GSO}}|\varphi\rangle . \tag{5.34}$$

The prescription seems at first sight a little ad-hoc. However there are deep reasons why the projection is necessary. At the level of the spectrum, it can be seen to imbue our theory with space-time supersymmetry, which of course is not guaranteed by the presence of world-sheet supersymmetry. On a deeper level it can be seen to arise as a result of the requirement of so-called modular invariance. In the following section we shall see this in detail, but for the moment let us consider the prescription for applying the GSO projection.

In the NS sector the operator P_{GSO} is given by,

$$P_{\mathrm{GSO}} = \frac{1}{2}\left[1 - (-1)^{N_F}\right] \tag{5.35}$$

whilst the *fermion number operator* N_F is defined as

$$N_F = \sum_{r>0} \psi^{\mu}_{-r}\psi_{\mu r} . \tag{5.36}$$

The GSO projection in the NS sector then acts to remove states with an even number of ψ oscillator excitations, deleting the tachyon from the spectrum.

Sector	Type	$SO(8)$ rep.	Corresponding massless fields
NS-NS	bosonic	$\mathbf{8_V}\otimes\mathbf{8_V}=\mathbf{35}\oplus\mathbf{28}\oplus\mathbf{1}$	graviton $g_{\mu\nu}$, B-field $B_{\mu\nu}$, dilaton Φ
NS-R	fermionic	$\mathbf{8_V}\otimes\mathbf{8_S}=\mathbf{8_S}\oplus\mathbf{56_S}$	gravitino Ψ_μ, dilatino λ
R-NS	fermionic	$\mathbf{8_S}\otimes\mathbf{8_V}=\mathbf{8_S}\oplus\mathbf{56_S}$	gravitino Ψ'_μ, dilatino λ'
R-R	bosonic	$\mathbf{8_S}\otimes\mathbf{8_S}=p$-forms	Ramond-Ramond fields

Table 1: Massless states of the closed type IIB string.

In the R sector, the definition is modified to include the chirality operator Γ^{11}:

$$P^{\pm}_{\mathrm{GSO}}=\frac{1}{2}\left[1\mp\Gamma^{11}(-1)^{N_F}\right]. \tag{5.37}$$

The projection P^{+}_{GSO} now acts to delete states with an odd number of ψ oscillator excitations in the $\mathbf{8_S}$ of $SO(8)$, and an even number of ψ oscillator excitations in the $\mathbf{8_C}$. P^{-}_{GSO} acts in the opposite fashion, but as there is no absolute definition of chirality the choice of $P^{\pm}_{\mathrm{GSO}}$ is irrelevant for open strings.

The true importance of the GSO projection lies in its ability to create a string spectrum which has spacetime supersymmetry. After applying the projection, there are an equal number of degrees of freedom in both the NS and R sector ground states: these form a $\mathbf{8_V}\oplus\mathbf{8_S}$ vector supermultiplet of the $D=10$, $\mathcal{N}=1$ supersymmetry algebra. In fact, the GSO projection ensures spacetime supersymmetry between NS sector bosons and R sector fermions at each mass level.

5.4 The closed string spectrum of type II models

The closed-string spectrum is obtained by taking tensor products of left- and right-moving states, each of which is very similar in form to the states found in the previous section. The physical state conditions $(L_0-a)|\varphi\rangle=(\tilde{L}_0-a)|\varphi\rangle=0$ lead to the level-matching requirement that there be an equal number of excitations of left- and right-movers, so that we are constrained to glueing together only those states with the property

$$m_L^2=m_R^2. \tag{5.38}$$

At each mass level, there are four possible sectors, summarized in table 1.

When we perform the GSO projection, the relative choice of $P^{\pm}_{\mathrm{GSO}}$ for the left- and right-movers is now important.

- Taking the opposite projection on both sides leads to a spectrum

$$(\mathbf{8_V}\oplus\mathbf{8_S})\otimes(\mathbf{8_V}\oplus\mathbf{8_C}), \tag{5.39}$$

  in which the spinors have opposite chiralities on either side. This is known as a *type IIA* theory. The spectrum of states is the same as that of a *non-chiral* ten-dimensional $\mathcal{N}=2$ supergravity theory.

- Taking the same projection on both sides leads to a spectrum

$$(\mathbf{8_V} \oplus \mathbf{8_S}) \otimes (\mathbf{8_V} \oplus \mathbf{8_S}) \tag{5.40}$$

(or equivalently, $(\mathbf{8_V} \oplus \mathbf{8_C}) \otimes (\mathbf{8_V} \oplus \mathbf{8_C})$), in which the spinors have the same chirality on either side. This is a *type IIB* theory, and the resulting spectrum of states is that of a *chiral* ten-dimensional $\mathcal{N} = 2$ supergravity theory.

Exercises D:

1. ***A p-brane theory is based on p-dimensional fundamental objects. Derive the expression for the Nambu-Goto action using a "world-volume" parameterization*** $\tau, \sigma^{1..p}$**.**

6. The heterotic string and modular invariance

So far we have seen how the spectrum of the simplest supersymmetric models in 10 dimensions can be derived. However these models are unrealistic. Most obviously we would like to have 4-dimensions, and also chiral models, which requires $\mathcal{N} = 1$ supersymmetry. The first step in this direction was the construction of so-called heterotic models, which were the first models that were seriously considered for phenomenology [6]. They perturbatively include large GUT groups ($E_8, E_6, SO(32)$) and gravity. In this section we shall develop these models, in a way which allows us to return to the question of consistency of closed string models.

The heterotic string is a curious combination of supersymmetric left-movers and bosonic right-movers as follows:

Left Movers	Right Movers
$X_+^{\mu=0..9}(\sigma_+)$	$X_-^{\mu=0..9}(\sigma_-)$
$\Psi_+^{\mu=0..9}(\sigma_+)$	$X_-^{J=1..16}(\sigma_-)$

In 2 dimensions bosons and complex fermions can be inter-converted (called fermionization or bosonization – of which more in section 12). The relation is given by,

$$: e^{iX_R^J} := \Psi_-^{2J-1} + i\Psi_-^{2J} \tag{6.1}$$

It proves useful to combine the ψ's into complex fermions

$$\lambda_-^J = \frac{1}{\sqrt{2}}(\Psi_-^{2J-1} + i\Psi_-^{2J}) \,. \tag{6.2}$$

Using the appropriate operator product expansions, one can show that λ_+^J satisfies the correct commutation relations for fermions given above. The right-movers have sixteen real bosons, or equivalently sixteen complex fermions, that do not correspond to space time degrees of freedom. (For example, in the spectrum we will only find a metric $g^{\mu\nu}$ with indices for the first 0..9 indices. These extra bosons can be regarded as an extra contribution to the theory that makes it consistent, i.e. free of conformal anomalies. One example of many consistency checks is that without them there is no Lorentz invariance in space-time signalled by the lack of massless gravitons in the spectrum.)

The full action therefore combines the bosonic and supersymmetric actions. In the conformal and light-cone gauges

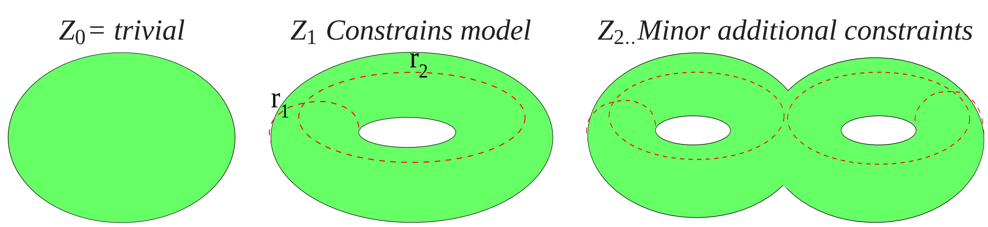
$$S_{LC} = -\frac{T}{2}\int d^2\sigma \left(\eta^{ab}\partial_a X^j \partial_b X^j + i\overline{\Psi}^j_+ \rho^a \partial_a \Psi^j_+ + i\overline{\lambda}^J_- \rho^a \partial_a \lambda^J_-\right) \tag{6.3}$$

where $J = 1\ldots16$ counts the complex right-moving fermions, and $j = 1\ldots8$ counts the left-moving transverse degrees of freedom. It is not hard to see that the appropriate constraint equations $T_{ab} = G_a = 0$ must be the sum of the bosonic contribution from the right movers and the supersymmetric contribution from the left movers.

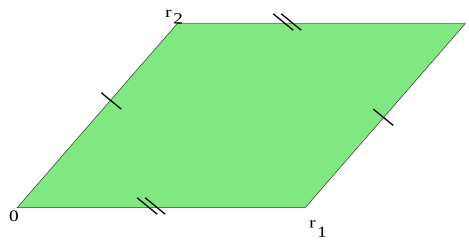
The technique of constructing the string models with all the additional degrees of freedom expressed as world-sheet fermions is known as the fermionic formulation. It was developed in refs.[7, 8, 9]. In this discussion I shall use the notation of ref.[8]. It is important to realize that the consistent models in 10-D are of course independent of the formalism (i.e. fermionic or bosonic) used to derive them. The fermionic formulation can also be used to develop 4-D models and this in fact was the point of the original papers. There it gives a slightly unusual viewpoint for model building; it disgards the geometrical interpretation of the 4-D models as compactified 10-D models, and regards the world-sheet fermions simply as extra degrees of freedom thrown in to cancel the conformal anomaly. Later I shall return to the 4-D models in this formalism, but for the moment let us concentrate on our task of finding the consistent models in 10 dimensions.

6.1 Modular Invariance - the tool to tell us which models are consistent

We now turn to the question that I alluded to at the end of the previous section, namely how to determine the consistent models. The trick is to start doing some perturbation theory. If we go to complicated enough diagrams, some putative model will give inconsistent answers (for example more than one answer for the same physical amplitude) whereupon it can be discarded. In fact we only need to go as far as vacuum→vacuum amplitudes (one loop partition functions) with no vertex operators to determine all the consistent 10 dimensional models. The relevant diagram are shown below.



The reason that the one loop diagram is so constraining is that it must be modular invariant. Consider the one loop diagram for a particular shape (i.e. given by the length of the two cycles) of torus. First recall that going to the conformal gauge ($\gamma^{ab} = e^{\phi}\eta^{ab}$) leaves a Weyl invariance in the metric (since there is no ϕ dependence). This allows one by a suitable rescaling to go to a flat metric. Now consider the integration region itself: this is now planar, so the world sheet integral is over the region shown in the diagram



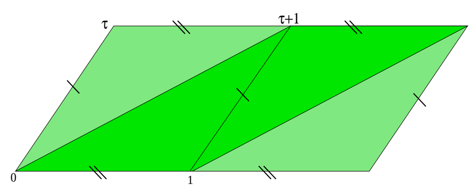
The torus is defined by two complex parameters

$$z = z + \tau_1 n + \tau_2 m \tag{6.4}$$

where n, m are integers. Lines with strokes are identified. But we can still use the Weyl invariance to get rid of one of the parameters. i.e. $z \to \lambda z$ is still a symmetry of the 2D theory and we can reduce it to

$$z = z + 2\pi n + 2\pi m \tau \tag{6.5}$$

so that any point is defined by the coordinates $\sigma_1, \sigma_2 \in (0, 2\pi]$ where $z = \sigma_1 + \tau \sigma_2$. The parameter τ defining the torus is called the Teichmüller parameter: it should not be confused with the world-sheet coordinate τ. There is an additional invariance under large reparameterizations. Any reparameterization that describes the same torus has to be moded out to avoid over-counting.



$\tau \to \tau + 1$ redefines torus :
$\tau \to -1/\tau$ swops σ_1 and σ_2 and just reorients torus

These two transformation generate the modular group, PSL(2,Z)

$$\tau \to \frac{a\tau + b}{c\tau + d} \quad a, b, c, d \in Z \ ; \ ad - bc = 1 \tag{6.6}$$

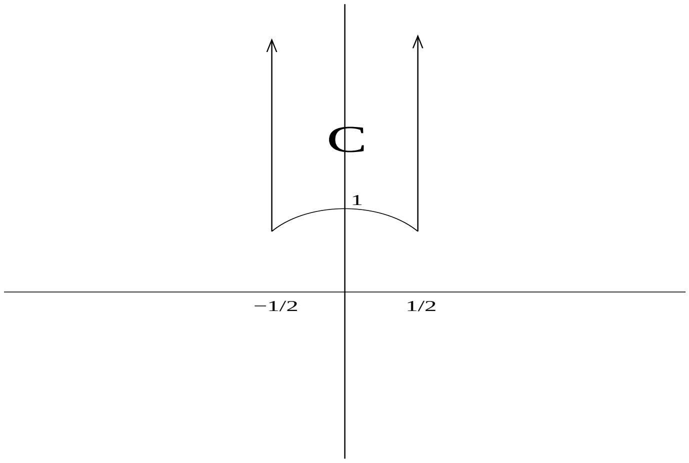
For a particular value of τ we get a corresponding $Z_1(\tau)$. The total one loop partition function then requires us to integrate over all independent values of this parameter

$$Z_1 = \int_{\mathscr{C}} \frac{d^2\tau}{Im(\tau)^2} Z_1(\tau) \tag{6.7}$$

where $\mathscr{C}$ is the fundamental region (i.e. the region of τ left after moding out the modular transformations). The measure of the integration renders the integration modular invariant, and so in order to make sense our integrand should itself be modular invariant.

Exercise: using the transformations above show that $d\tau d\bar{\tau}/Im(\tau)^2$ is modular invariant.

The fundamental region is shown below. The Dehn twist maps all values of τ into the region $-1/2 \leq Re(\tau) \leq 1/2$. The second transformation maps $|\tau| < 1$ values into the $|\tau| > 1$ region. Any point outside the fundamental region shown below can therefore be mapped into it. All independent tori are therefore included by integrating over this region.



Modular invariance is then the condition that this region is equivalent to any of the other regions we could have chosen;

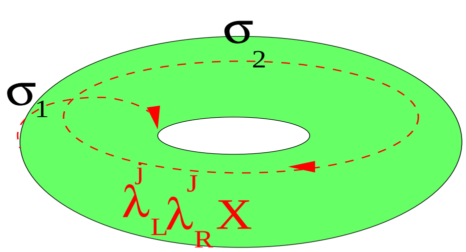
$$Z_1(\tau) = Z_1(-1/\tau) = Z_1(\tau+1) \tag{6.8}$$

It constrains the possible 10-D theories - (e.g. gauge groups) as we will now show. The calculation is rather intricate, so at the end of 1.3.1 (where we talk about how we define different models) we summarize how this happens and then give a set of rules for model building, so that eventually one can short-circuit most of the interim calculation which is included in Appendix C.

6.2 World sheet boundary conditions, and hamiltonians

6.2.1 What do we mean by 'different models'?

The world-sheet fermions do not need to be single valued for consistency. All that we require is that the action S_{LC} be single valued and this leaves some freedom in the phases that the fields can acquire as they are transported around the world-sheet. Any particular model is defined by the boundary conditions (phases) of the world sheet bosons and fermions as they are transported around the world sheet.



For the world-sheet to be embedded sensibly in target space-time we must have a single-valued $X^j_\pm$. We can define the other boundary conditions for the fermions with vectors v_j u_j v_J u_J ($j=1..8$, $J=1..16$) and for the moment these define arbitrary phases. Defining the torus by coordinates $\sigma_{12}\in[0,2\pi]$ then the boundary conditions can be written

$$\begin{aligned} X^j(\sigma_1+2\pi,\sigma_2) &= X^j(\sigma_1,\sigma_2) \\ X^j(\sigma_1,\sigma_2+2\pi) &= X^j(\sigma_1,\sigma_2) \\ \Psi^j_+(\sigma_1+2\pi,\sigma_2) &= e^{-2\pi i v_j}\Psi^j_+(\sigma_1,\sigma_2) \\ \Psi^j_+(\sigma_1,\sigma_2+2\pi) &= e^{-2\pi i u_j}\Psi^j_+(\sigma_1,\sigma_2) \\ \lambda^J_-(\sigma_1+2\pi,\sigma_2) &= e^{-2\pi i v_J}\lambda^J_-(\sigma_1,\sigma_2) \\ \lambda^J_-(\sigma_1,\sigma_2+2\pi) &= e^{-2\pi i u_J}\lambda^J_-(\sigma_1,\sigma_2) \end{aligned} \tag{6.9}$$

It is trivial to see that the action is invariant for arbitrary v_J,u_J. The left-handed world-sheet fermions which have space-time indices have to have the same phase as the gravitino which can be ±1 or $v_j,u_j=0$ or $\frac{1}{2}$. We can pair the left-moving Majorana fermions into complex fermions, which I'll denote λ_+, and we can then club the left and right-mover phases into vectors with

$$\begin{aligned} V &= [v_1,v_2,v_3,v_4\;\; v_5,..v_{20}] \\ U &= [u_1,u_2,u_3,u_4\;\; u_5,..u_{20}], \end{aligned} \tag{6.10}$$

where $v_1=v_2=v_3=v_4=0,\pi$ and similar for u_j. For later use I'll define the inner product as

$$V.U=\sum_{J=1}^{16}v_Ju_J-\sum_{j=1}^{4}v_ju_j\,. \tag{6.11}$$

So for example we might have

$$V=[(\frac{1}{2})^4\;\;(0)^8(\frac{1}{2})^8]. \tag{6.12}$$

In general the phases on the complex fermions can be arbitrary, however it turns out that all cases for the 10 dimensional models are equivalent to taking entries of $0,\frac{1}{2}$ only (i.e. phases of $0,\pi$). When the boundary conditions are $0,\frac{1}{2}$ it sometimes proves convenient to treat a complex fermion as two real ones in which case we just double the entries (and multiply the real elements in the dot product by $\frac{1}{2}$). Hopefully it should be clear when this is the case.

6.2.2 Mode expansions for arbitrary boundary condition phases

The partition function is an amplitude for propagation through time τ, and for that we need the hamiltonian for the world sheet fields. The bosonic contribution is simply a model independent product of modular (Dedekind-η) functions that is included in Appendix C. For the complex fermions which are of interest here, we have the normal mode expansion for a single left-moving particle on the world sheet labelled canonically by $\sigma\,\tau$ (we will for convenience drop the j index for the moment)

$$\lambda_+(\sigma,t)=\sum_{n=1}^{\infty}b_{n+v-1}e^{-2i(n+v-1)\sigma^+}+d^\dagger_{n-v}e^{2i(n-v)\sigma^+} \tag{6.13}$$

I will henceforth (for the rest of the heterotic discussion) drop the tilde on the left-moving (σ^+) excitations as it should be obvious from the context and the index (i.e. i or I) whether the mode is left or right moving. We will take $\sigma \in [0,\pi]$ (in contrast with ref.[7]). As long as the phases $v_i \in \{0, \frac{1}{2}\}$ we may always translate the notation to real fermions. To be explicit, it is not hard to see that

$$\begin{aligned}\Psi^{2j-1}_{r>0} &= \frac{1}{\sqrt{2}}(b_r + d_r) \\ \Psi^{2j}_{n>0} &= \frac{1}{\sqrt{2}i}(b_r - d_r)\,. \end{aligned} \tag{6.14}$$

Also note for later use that there is no d_0, so in the Ramond sector b_0 is precisely the raising operator for space-time fermions defined in eq.(5.25).

The quantization condition for the 2D fermions is given by

$$\{b^\dagger_a, b_b\} = \{d^\dagger_a, d_b\} = \delta_{ab}. \tag{6.15}$$

The hamiltonian at some time t is needed for the partition function (see Appendix C). It is given by integrating the world sheet hamiltonian over the σ at time t,

$$\begin{aligned} H_v(t) &= \int d\sigma \frac{i}{2}\overline{\lambda}\partial_+\lambda + h.c. \\ &= \int d\sigma \sum_{n,m}\left(b^\dagger_{n+v-1} b_{m+v-1}(n+v-1)e^{i(m-n)\sigma_+} - d_{n-v}d^\dagger_{m-v}(n-v)e^{i(m-n)\sigma_+}\right) \end{aligned} \tag{6.16}$$

Doing the integration, rearranging the d's for normal ordering, and regularizing the infinite contribution gives

$$H_v(t) = \sum_n \left((n+v-1)b^\dagger_{n+v-1}b_{n+v-1} + (n-v)d^\dagger_{n-v}d_{n-v}\right) + a_v \tag{6.17}$$

where the properly regularized vacuum energy is given by[2]

$$a_v = \sum_{n=1}(n-v) \equiv \frac{1}{2}(v^2 - v + \frac{1}{6}). \tag{6.18}$$

Exercise: Prove this expression using the procedure described in the footnote.

The particle spectrum is given by exciting the v vacuum (with energy a_v) with one or none fermion operators of each kind (i.e. index and excitation number),

$$|\phi\rangle = d^{\dagger\, n_1}_{1-v} d^{\dagger\, n_2}_{2-v} ... b^{\dagger\, m_1}_{v} ... |0\rangle \tag{6.19}$$

where $m_i, n_i = 0, 1$ are the individual excitation numbers. The fermion number operator which counts the total nett number of excitations is then (by definition and the quantization rules) given by

$$N_v(t) = \sum_{n=1}\left(b^\dagger_{n+v-1}b_{n+v-1} - d^\dagger_{n-v}d_{n-v}\right). \tag{6.20}$$

[2]Again, as a short cut to this result, one can express the sum as $\mathrm{Lim}_{\varepsilon\to 0}\sum(n-v)e^{-\varepsilon(n-v)}$, and perform the sum. Throwing away the leading $1/\varepsilon^2$ divergence (corresponding to the infinite contribution from all the anti-particles) leaves the v dependent a_v as the leading constant term.

Finally we can find the charge which is the fermion density integrated over σ;

$$\begin{aligned} Q = & \int d\sigma \overline{\lambda}\lambda \\ = & \, N_v + \sum_n (n-v)^0 \\ = & \, N_v + v - \tfrac{1}{2}. \end{aligned} \tag{6.21}$$

The extra piece is known as the vacuum charge. The charges here will correspond to the actual charges of the space time gauge groups.

We can already see the types of massless states that will occur. The NS-NS sector is when the boundary conditions are all $v_j = v_J = \frac{1}{2}$. Each complex fermion above contributes $-1/24$ to the vacuum energy. The 8 transverse real bosons each contribute $-1/24$, so that in total we have

$$a_V = [-\frac{1}{2}, -1] \tag{6.22}$$

where the two entries are left and right energies respectively. Massless states in this sector can be built from excitations such as

$$\begin{aligned} &(b^{i\dagger}_{\frac{1}{2}} \oplus d^{i\dagger}_{\frac{1}{2}})|0\rangle_L \times (b^{I\dagger}_{\frac{1}{2}} \oplus d^{I\dagger}_{\frac{1}{2}})(b^{J\dagger}_{\frac{1}{2}} \oplus d^{J\dagger}_{\frac{1}{2}})|0\rangle_R \\ &(b^{i\dagger}_{\frac{1}{2}} \oplus d^{i\dagger}_{\frac{1}{2}})|0\rangle_L \times \alpha^{j\dagger}_1 |0\rangle_R \,. \end{aligned} \tag{6.23}$$

The former is a gauge boson the latter is a transverse field that includes the 10-D graviton. Note that we could equally have used the real fermion notation. For example gauge bosons would look like

$$\psi^{i'\dagger}_{\frac{1}{2}}|0\rangle_L \times \psi^{I'\dagger}_{\frac{1}{2}}\psi^{J'\dagger}_{\frac{1}{2}}|0\rangle_R \,, \tag{6.24}$$

where here $I',J' = 1\ldots 32$. By the commutation relation of the real fermions, these are seen to form the ($32.31/2 = \mathbf{496}$) adjoint of $SO(32)$ (with the left-moving factor providing the transverse Lorentz index for the gauge boson). The complex notation gives its decomposition under the $SU(16)$ subgroup; i.e. $\psi^{I'\dagger}_{\frac{1}{2}}\psi^{J'\dagger}_{\frac{1}{2}} \equiv (b^{I\dagger}_{\frac{1}{2}} b^{J\dagger}_{\frac{1}{2}} + d^{I\dagger}_{\frac{1}{2}} d^{J\dagger}_{\frac{1}{2}} + b^{I\dagger}_{\frac{1}{2}} d^{J\dagger}_{\frac{1}{2}})$ corresponds to

$$\mathbf{496} \equiv \mathbf{16}.\mathbf{15}/2 + \mathbf{16}.\mathbf{15}/2 + \mathbf{16}.\mathbf{16}\,. \tag{6.25}$$

At this point we can see that getting massless gravitons and gauge bosons, and hence preserved Lorentz invariance, is a good check of D=10 or 26 dimensions. In D dimensions, the vacuum energy is

$$a_V = [-\frac{3(D-2)}{48}, -\frac{D-2}{24}] \tag{6.26}$$

so the graviton mass is

$$m_{grav} = [\frac{1}{2}, 1] + a_V = \left[\frac{10-D}{16}, \frac{26-D}{24}\right]. \tag{6.27}$$

Of course here 16 of the bosonic dimensions are regarded as 'internal' as they have no left moving counterpart.

Exercise: Using the expression for a_v verify this result - i.e. that the graviton is massless in D=10 with 16 internal boson.

The partition function is given by analogy with $\langle e^{iHt}\rangle$. For a single fermion with boundary condition u,v the fermion acquires the phase factor $e^{2\pi iv}$ when propagated through the complex time $2\pi\tau \equiv t$. Propagation in the σ direction gives the phase factor $e^{2\pi iu}$. The contribution to the partition function is then

$$Z_u^v(\tau) = Tr\left(q^{H_v} e^{2\pi i(\frac{1}{2}-u)N_v}\right) \tag{6.28}$$

where $q = e^{2\pi i\tau}$. We leave the detailed discussion of the partition function and modular invariance calculation to Appendix C. Here I'll just describe how it works to determine the allowed 10D models. A particular set of boundary conditions is called a sector. A model can have different sectors and in each sector we find a particular partition function Z_U^V. We must have a sum over all possible sectors in the partition function (by definition, since the PF sums over all possibilities), so that it should look like

$$Z_1(\tau) = bosonic\, contribution \times \sum_{\{all\, U,V\}} C_U^V Z_U^V(\tau) \tag{6.29}$$

The C_U^V are coefficients that have to be chosen to give the desired modular invariance. Any model is defined by the complete set of possible boundary conditions, i.e. the set of allowed $\{U,V\}$.

One finds (in Appendix C) that modular invariance constrains the possible phases and hence the allowed models. The end result is a set of rules for model building. In brief one first chooses a set of basis vectors to generate all the allowed sectors. The modular invariance constraint then projects out states so that not all the states that we can write down as above survive in the end. For example it will project out tachyons and result in a (10D space time) supersymmetric model. This is precisely the GSO projection which we earlier put in by hand! (Again, note that world-sheet supersymmetry does not automatically mean a space time supersymmetric theory). Let's now go straight to the rules and then do some example models.

Exercise: Read Appendix C, and verify the following rules for model building

6.3 Rules for model building

Defining the model: Choose a set of basis vectors of W_i to generate the different sectors (the V boundary conditions). The sectors are given by linear combinations

$$V = \alpha_a W_a \tag{6.30}$$

where these phases are mod(1) and we sum over a. We must consider each sector in turn given by integers $\alpha_a = 1..m_a$ where $m_a W_a = \underline{0}$ mod(1) . For example the two basis vectors

$$\begin{aligned} W_0 &= [(\frac{1}{2})^4\ (\frac{1}{2})^8(\frac{1}{2})^8], \\ W_1 &= [(0)^4\ (\frac{1}{2})^8(\frac{1}{2})^8]. \end{aligned} \tag{6.31}$$

define a model with $m_0 = m_1 = 2$. We then have to consider the sectors $\underline{0}, W_0, W_1, W_0+W_1$ where

$$W_0 + W_1 = [(\frac{1}{2})^4\ (0)^8(0)^8]. \tag{6.32}$$

Structure constants: Choose a set of 'structure' constants k_{ab} that obey the following conditions *mod(1)* ;

$$\begin{aligned} m_b k_{ab} &= 0 \\ k_{ab} + k_{ba} &= W_a.W_b \\ k_{aa} + k_{a0} + w_a^1 - \frac{1}{2} W_a.W_a &= 0 \end{aligned} \tag{6.33}$$

Spectrum: In every sector $V = \alpha_a W_a$ find the spectrum of states at a given level. The states must satisfy the level matching condition, that the left- and right-moving Hamiltonians match, $H_V = L_0 - a_V^- = \tilde{H}_V = \tilde{L}_0 - a_V^+$.

Projections: In every sector $V = \alpha_a W_a$ apply the modular invariance projection on the states

$$W_a.N_V = k_{ab}\alpha_b + w_a^1 + k_{0a} - W_a.V \tag{6.34}$$

where we sum over b. (It is the sum over the $U = \beta_a W_a$ boundary condition that has resulted in this projection as in Appendix C.)

6.4 Examples

All models can be constructed by applying the rules above in the order given:

1) Define the model (i.e. choose the set of basis vectors)

2) Choose the structure constants

3) In each sector find the vacuum energy[3]

4) Then find the massless states that satisfy the modular invariance projection

The end result is a massless spectrum which can be examined for supersymmetry, gauge group, particle content and so on.

6.4.1 A tachyonic SO(32) model with W_0

Lets use these rules to look at some possible models. All models need the NS sector given by $V = W_0$ so the simplest case is to choose a model with just this vector in the basis.

Structure constants: Have $m_0 = 2$ and $W_0.W_0 = 3$, so that $k_{00} = 0$ *or* $\frac{1}{2}$.

Vacuum energies:

[3]Using our previous expression for a_v, the total vacuum energy including the bosonic contribution is given by

$$a_V = \left[-\frac{1}{2} + \frac{1}{2}\sum_j^4 \left(v^j - \frac{1}{2} \right)^2, -1 + \frac{1}{2}\sum_J^{16} \left(v^J - \frac{1}{2} \right)^2 \right] \tag{6.35}$$

sector $V =$	a_V
$\underline{0} = [(0)^4, (0)^8(0)^8]$	[0,1]
$W_0 = [(\frac{1}{2})^4, (\frac{1}{2})^8(\frac{1}{2})^8]$	$[-\frac{1}{2}, -1]$

The $\underline{0}$ sector gives only massless states and will not be considered further.

States allowed by projections in the W_0 sector: We have already seen that that massless states are of the form

$$\psi^i_{-\frac{1}{2}}|0\rangle_L \times \psi^I_{-\frac{1}{2}}\psi^J_{-\frac{1}{2}}|0\rangle_R \;\;,\;\; \psi^i_{-\frac{1}{2}}|0\rangle_L \times \alpha^j_{-1}|0\rangle_R \tag{6.36}$$

where again since we only have phases 0 or π, I have written each complex fermion as two real ones: in the above we therefore have the transverse space time indices $i, j = 1..8$ and $I, J = 1..32$. The projection is given by

$$W_0.N = w_0^1 = \frac{1}{2} \; mod(1) \tag{6.37}$$

But

$$W_0.N = \frac{1}{2}\left(\sum_{J=1}^{32} N^J - \sum_{j=1}^{8} N^j\right) \tag{6.38}$$

which just means that the difference in oscillator numbers between left and right movers is odd. All the states above satisfy this so none are projected out. This completes the massless spectrum. However note that there is also a tachyon state with negative mass squared

$$|0\rangle_L \times \psi^J_{-\frac{1}{2}}|0\rangle_R \,. \tag{6.39}$$

The first massless states above are gauge bosons. Massless physical states appear in representations of SO(D-2) as required. Restoring the longitudinal degrees of freedom they would be written as A^μ_{IJ} in an effective action. The fermionic excitations ψ^I anticommute, so that $I \neq J$. There are therefore $31 \times 32/2 = 496$ antisymmetric bosons as above. The indices act on fundamentals of 32's so that this is the 496 of SO(32). The remaining massless states are gravitational. They can be written ϕ^{ij}. This can be decomposed into irreducible representations (antisymmetric, traceless symmetric and trace) of the transverse rotation group SO(8) as follows;

$$\begin{aligned}\phi^{ij} \;&=\; \phi^{[i,j]} + \left(\phi^{\{i,j\}} - \frac{1}{D-2}\delta^{ij}\phi^{kk}\right) + \frac{1}{D-2}\delta^{ij}\phi^{kk} \\ &\equiv B^{ij} + G^{ij} + \Phi.\end{aligned} \tag{6.40}$$

Hence we find spin-2 particle $G^{\mu\nu}$ (graviton) an antisymmetric tensor $B^{\mu\nu}$ and a scalar which is the dilaton.

At the first excited level we find massive excitations such as

$$\begin{aligned}&\psi^i_{-\frac{1}{2}}\psi^j_{-\frac{1}{2}}|0\rangle_L \times \psi^I_{-\frac{1}{2}}\psi^J_{-\frac{1}{2}}\psi^K_{-\frac{1}{2}}|0\rangle_R \\ &\qquad \alpha^i_{-1}|0\rangle_L \times \psi^I_{-\frac{1}{2}}\psi^J_{-\frac{1}{2}}\psi^K_{-\frac{1}{2}}|0\rangle_R\end{aligned} \tag{6.41}$$

Together these form a physical state $A^{\mu\nu}_{IJK}$. Counting the space time excitations above we find 28+8=36, so the physical state is the antisymmetric representation of SO(9) not SO(8), as is appropriate for a massive state. The world-sheet bosonic excitation provides the required extra longitudonal degrees of freedom.

6.4.2 A supersymmetric SO(32) model with W_0,W_1

Consider adding an additional basis vector

$$W_1 = [(0)^4, (\frac{1}{2})^8(\frac{1}{2})^8] \tag{6.42}$$

Structure constants: Have $m_0 = m_1 = 2$ and $W_a.W_b = \begin{pmatrix} 3 & 4 \\ 4 & 4 \end{pmatrix}$, so that we have four possibilities; $k_{11} = k_{10} = k_{01} = 0, \frac{1}{2}$ and $k_{00} = 0, \frac{1}{2}$

Vacuum energies:

sector $V =$	a_V
$W_0 = [(\frac{1}{2})^4, (\frac{1}{2})^8(\frac{1}{2})^8]$	$[-\frac{1}{2}, -1]$
$W_1 = [(0)^4, (\frac{1}{2})^8(\frac{1}{2})^8]$	$[0, -1]$

The $\underline{0}$ and $W_0 + W_1$ sectors are massive.

States allowed by projections in $V = W_0$ sector: Initially the spectrum is as before. However now we have two projections

$$\begin{aligned} W_0.N_{W_0} &= w_0^1 = \frac{1}{2} \qquad\qquad mod(1) \\ W_1.N_{W_0} &= k_{10} + w_1^1 + k_{01} - W_1.W_0 = 0 \;\; mod(1) \end{aligned} \tag{6.43}$$

But

$$\begin{aligned} W_0.N &= \frac{1}{2}\left(\sum_{J=1}^{32} N^J - \sum_{j=1}^{8} N^j\right) \\ W_1.N &= \frac{1}{2}\sum_{J=1}^{32} N^J. \end{aligned} \tag{6.44}$$

The tachyon state does not satisfy the second condition since it has an odd number of right moving excitations so is projected out. Note that this is the equivalent of the GSO projection, but we have derived it from the requirement of modular invariance!

Exercise: Using the projection rules verify that the tachyon is projected out.

States allowed by projections in $V = W_1$ sector: These will turn out to be the fermionic superpartners of the states in the W_0 sector. The right moving excitation can be as before, i.e. space time, or antisymmetric internal. However now the left moving side can have any number of b_0^i excitations and still be massless. Recall that in the complex fermion notation, there are no zero modes for the d's, so we have to be careful here to use the indices of complex fermions running from $i = 1..4$. That is the massless excitations are

$$\begin{aligned} &(1 + b_0^i + b_0^i b_0^{j\neq i} + b_0^i b_0^{j\neq i} b_0^{k\neq i,j} + \ldots b_0^1 b_0^2 b_0^3 b_0^4)|0\rangle_L \\ &\qquad \times \left(\psi^I_{-\frac{1}{2}}\psi^J_{-\frac{1}{2}}|0\rangle_R + \alpha^j_{-1}|0\rangle_R\right) \end{aligned} \tag{6.45}$$

In this sector there is no possibility of tachyons. The projections turn out to be as before

$$
\begin{aligned}
W_0.N_{W_1} &= k_{01} + w_0^1 + k_{11} - W_0.W_0 = \frac{1}{2} \qquad mod(1) \\
W_1.N_{W_1} &= k_{11} + w_1^1 + k_{01} - W_1.W_0 = 0 \qquad mod(1)
\end{aligned} \tag{6.46}
$$

We see that the W_1 constraint acts only on the right-movers so projects out exactly the states with odd numbers of right-moving excitations (there are no massless ones above of course). Again the W_0 constraint requires only odd numbers of left movers so we are left with the states

$$
(b_0^i + b_0^i b_0^{j\neq i} b_0^{k\neq i,j})|0\rangle_L \times \left(\psi^I_{-\frac{1}{2}} \psi^J_{-\frac{1}{2}} |0\rangle_R + \alpha^j_{-1}|0\rangle_R\right) \tag{6.47}
$$

Counting the space time excitations, we initially had $2^4 = 16$ (i.e. there can only be one or no b_0 oscillator for every index because they anticommute) states, which are reduced to 8 by this projection. Adding back the longitudinal degrees of freedom (b_0^0 excitations) gives 16 degrees of freedom in each state. These are chiral 10 dimensional fermions which we denote $|a\rangle$. (c.f. 4 dimensional fermions which would have just $i = 0,1$ (complex) excitations and hence 4 degrees of freedom – each chirality then has two elements.)

So, the state

$$
\left(|a\rangle_L \times \alpha^j_{-1}|0\rangle_R\right) \tag{6.48}
$$

is a single 10D gravitino. We can count the number of gravitinos to see how much supersymmetry we have. Here the single gravitino implies that we have just one supersymmetry, $\mathcal{N} = 1$.

6.4.3 Digression on fermions and chirality

As we have seen, space-time fermions appear in Ramond sectors (whenever there are zeroes in the boundary condition), and these are of the form

$$
\begin{aligned}
&(1 + b_0^i + b_0^i b_0^{j\neq i} + b_0^i b_0^{j\neq i} b_0^{k\neq i,j} + \dots b_0^1 b_0^2 b_0^3 b_0^4)|0\rangle_L \\
&\qquad\qquad \times right - moving\ stuff
\end{aligned} \tag{6.49}
$$

The b_0 excitations can be written to make the fermionic properties clearer in terms of Γ's as described in section 5.3. Now consider the modular invariance projections above. It required

$$
\sum_{j=1}^{4} N^j = \sum_{j=1}^{4} b_0^{j\dagger} b_0^j + irrelevant\,stuff = odd \tag{6.50}
$$

But we can show

$$
\sum_{j=1}^{4} b_0^{j\dagger} b_0^j = 2 + \sum_j s_j \tag{6.51}
$$

where

$$
\sum_j s_j = -\frac{i}{4}\sum_j [\Gamma^{2j}, \Gamma^{2j+1}] = \sum_j M^{2j\,(2j+1)} \tag{6.52}
$$

is the spin-matrix of the state, so the modular invariance condition is actually a condition on the spin of the transverse modes, and hence a chirality projection. To make that explicit, write the condition on the space-time fermions (which here I'll denote by the generic symbol χ) as follows;

$$
(-1)^{\sum_{i=1}^4 b_0^{i\dagger} b_0^i} \chi = -\chi \tag{6.53}
$$

Using the above relation and the Clifford algebra this becomes

$$exp[\frac{i\pi}{2}\sum_j \Gamma^{2j}\Gamma^{2j+1}]\chi = -\chi\,. \tag{6.54}$$

Taylor expanding the exponential, and using the Clifford algebra term by term gives

$$\prod_j(\cos\frac{\pi}{2} - \Gamma^{2j}\Gamma^{2j+1}\sin\frac{\pi}{2})\chi = \prod_j \Gamma^{2j}\Gamma^{2j+1}\chi = -\chi \tag{6.55}$$

or

$$(1+\Gamma^{11})\chi = 0 \tag{6.56}$$

giving the 10 dimensional chirality projection. As above, we usually denote the fermionic representation as

$$(b_0^i + b_0^i b_0^{j\neq i} b_0^{k\neq i,j})|0\rangle_L = |s_0,s_1,s_2,s_3,s_4\rangle_L = |a\rangle_L \tag{6.57}$$

where the spins s_i in the representation are $\pm\frac{1}{2}$ and their sum satisfies the chirality projection above. It is not hard to see that the projection at general levels corresponds precisely to $P_{GSO} = (1+\Gamma_{11}(-1)^{N_F})$.

Exercise: Using the Clifford algebra of the gamma matrices, verify the above expansion of $exp[\frac{i\pi}{2}\sum_j \Gamma^{2j}\Gamma^{2j+1}]$

6.4.4 A supersymmetric $E_8 \times E_8$ model with W_0,W_1,W_2

Consider adding an additional basis vector to the previous model

$$W_2 = [(0)^4,\, (\frac{1}{2})^8(0)^8] \tag{6.58}$$

Structure constants: We have $m_0 = m_1 = m_2 = 2$ and $W_a.W_b = \begin{pmatrix} 3 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 2 \end{pmatrix}$, so that there are sixteen possibilities; $k_{12} = k_{21} = 0, \frac{1}{2}$ and $k_{22} = k_{20} = k_{02} = 0, \frac{1}{2}$ and $k_{11} = k_{10} = k_{01} = 0, \frac{1}{2}$ and $k_{00} = 0, \frac{1}{2}$

Vacuum energies:

sector $V =$	a_V
$W_0 = [(\frac{1}{2})^4,\, (\frac{1}{2})^8(\frac{1}{2})^8]$	$[-\frac{1}{2}, -1]$
$W_0 + W_2 = [(\frac{1}{2})^4,\, (0)^8(\frac{1}{2})^8]$	$[-\frac{1}{2}, 0]$
$W_0 + W_1 + W_2 = [(\frac{1}{2})^4,\, (\frac{1}{2})^8(0)^8]$	$[-\frac{1}{2}, 0]$
$W_1 = [(0)^4,\, (\frac{1}{2})^8(\frac{1}{2})^8]$	$[0, -1]$
$W_1 + W_2 = [(0)^4,\, (0)^8(\frac{1}{2})^8]$	$[0, 0]$
$W_2 = [(0)^4,\, (\frac{1}{2})^8(0)^8]$	$[0, 0]$

The $\underline{0}$ and $W_0 + W_1$ sectors are massive.

States allowed by projections in $V = W_0$ sector: Initially the spectrum is as before. However now we have three projections

$$\begin{aligned} W_0.N_{W_0} &= w_0^1 = \frac{1}{2} \qquad\qquad\qquad mod(1) \\ W_1.N_{W_0} &= k_{10} + w_1^1 + k_{01} - W_1.W_0 = 0 \;\; mod(1) \\ W_2.N_{W_0} &= k_{20} + w_2^1 + k_{02} - W_2.W_0 = 0 \;\; mod(1) \end{aligned} \tag{6.59}$$

But

$$\begin{aligned} W_0.N &= \frac{1}{2}\left(\sum_{J=1}^{32} N^J - \sum_{j=1}^{8} N^j\right) \\ W_1.N &= \frac{1}{2}\sum_{J=1}^{32} N^J \\ W_1.N &= \frac{1}{2}\sum_{J=1}^{8} N^J \end{aligned} \tag{6.60}$$

Again the tachyon is projected out. However now we also require an even number of excitations in the 1st or 2nd 8. (Again I am discussing real world sheet fermions and simply doubling the indices.) We therefore have gauge boson states

$$\psi^i_{-\frac{1}{2}}|0\rangle_L \times \psi^I_{-\frac{1}{2}}\psi^J_{-\frac{1}{2}}|0\rangle_R \tag{6.61}$$

where $I \neq J$ come from both from the first 16 real fermions or both from the second 16. That gives us a $16.15/2 = 120$ possibilities in each half giving a **120** gauge bosons of $SO(16)$ and a **120'** of a second $SO(16')$. (See Appendix E.)

States allowed by projections in $V = W_0 + W_2$ and $V = W_0 + W_1 + W_2$ sectors: There are two other sectors that give gauge bosons $W_0 + W_2$ and $W_0 + W_1 + W_2$ and from our discussion above we see that these will be in the fermionic representations of $SO(16)$ and $SO(16')$ respectively. Consider for example the $W_0 + W_2$ sector. The would-be gauge boson states are

$$\psi^i_{-\frac{1}{2}}|0\rangle_L \times |a\rangle_R \tag{6.62}$$

It is worth looking at the projections briefly to make an observation about the chiralities. The projections are

$$\begin{aligned} W_0.N_{W_0+W_2} &= k_{00} + k_{02} + w_0^1 + k_{00} - W_0.W_0 - W_0.W_2 = \frac{1}{2} \;\; mod(1) \\ W_1.N_{W_0+W_2} &= k_{10} + k_{12} + w_1^1 + k_{01} - W_1.W_0 - W_1.W_2 = 0 \;\; mod(1) \\ W_2.N_{W_0+W_2} &= k_{20} + k_{22} + w_2^1 + k_{02} - W_2.W_0 - W_2.W_2 = 0 \;\; mod(1) \end{aligned} \tag{6.63}$$

The chirality projection should be the same for these states to survive the projection. Since the LHS of the first projection has an extra $1/2$ contribution from the right mover $\psi_{-1/2}$ excitation, this requires that $k_{02} = k_{12} = k_{22}$. One of these conditions is already given by the preliminary modular invariance constraints on the k_{ab} above so that the new condition for these excitations to survive is

$$k_{02} = k_{12} \tag{6.64}$$

For the equivalent states in the $V = W_0 + W_1 + W_2$ sector to survive the projection, we have the same condition (from looking at just the W_2 projection). If this is not satisfied then we have just the $SO(16) \times SO(16')$ model. If it is, then we have additional $2^8/2$ =128 bosons in the fermion representations of $SO(16)$ and $SO(16')$. This gives an $E_8 \times E_8'$ model (the gauge bosons of E_8 can be decomposed as $\mathbf{120} + \mathbf{128}$ under $SO(16)$).

States allowed by projections in $V = W_1 + \ldots$ sectors: Again these are the fermionic superpartners of the states in all the sectors above with a single chirality projection, since the right moving excitation can be as before, i.e. space-time, or antisymmetric internal. Consider the W_1 sector which again gives rise to the 10D gravitino and gauginos. The projections are now

$$\begin{aligned} W_0.N_{W_1} &= k_{01} + w_0^1 + k_{00} - W_0.W_1 = \frac{1}{2} \;\; mod(1) \\ W_1.N_{W_1} &= k_{11} + w_1^1 + k_{01} - W_1.W_1 = 0 \;\; mod(1) \\ W_2.N_{W_1} &= k_{21} + w_2^1 + k_{02} - W_2.W_1 = 0 \;\; mod(1) \end{aligned} \tag{6.65}$$

The would-be gravitino state is as before

$$|a\rangle_L \times \alpha^j_{-1}|0\rangle_R \, . \tag{6.66}$$

Now we require the same chirality projection on the left movers for this state to exist, so we must have

$$k_{01} + k_{00} = k_{11} + k_{01} = k_{21} + k_{02} \tag{6.67}$$

From our preliminary constraints on k_{ab} above, we have that the middle expression is equal to zero, and hence the new conditions are $k_{12} = k_{02}$ and $k_{01} = k_{00}$. The first of these is the same as the above condition to have E_8 gauge groups rather than $SO(16)$.

6.4.5 Summary of heterotic models

gauge group	SUSY
$SO(32)$	No, tachyonic
$SO(32)$	$\mathcal{N} = 1$
$SO(16) \times SO(16')$	No, no tachyons
$E_8 \times E_8'$	$\mathcal{N} = 1$

7. Type II and 0 models, and the link with D-branes

Having developed the formalism for writing down general modular invariant models, I will briefly now revisit the type II models to show firstly how the GSO projections emerge from modular invariance and also the existence of another class of related nonsupersymmetric models known as type 0 models.

7.1 Reinstating supersymmetry on both sides

It is straightforward to adapt the fermionic formalism to incorporate supersymmetry in both the left- and right-movers on the world-sheet, indeed the construction of type II models is much easier than the heterotic case. The action in the light cone gauge is

$$S_{LC} = -\frac{T}{2}\int d^2\sigma \left((\partial_a X^j)^2 + i\Psi^j_+\rho^+\partial_+\Psi^j_+ + i\Psi^j_-\rho^-\partial_-\Psi^j_-\right) \tag{7.1}$$

where $j = 1..8$ labels the transverse degrees of freedom on each side. Again, we can if we wish complexify the real (Majorana) fermions into $\lambda_\pm$. Now our entire discussion of modular invariance goes through unaltered, apart from the fact that we now have sectors defined by boundary condition phases for just 4 transverse complex fermions on each side;

$$\begin{aligned} V &= [v_1, v_2, v_3, v_4 \;\; v_5, ..v_8] \\ U &= [u_1, u_2, u_3, u_4 \;\; u_5, ..u_8], \end{aligned} \tag{7.2}$$

In addition, as for the left movers in the heterotic case, the phases of the fermions must be degenerate on each side since (as before) they should have the same boundary condition as the 2D world sheet gravitino. This leaves only 4 possible sectors

$$\begin{aligned} V &= [(\tfrac{1}{2})^4, (\tfrac{1}{2})^4] \\ V &= [(0)^4, (\tfrac{1}{2})^4] \\ V &= [(\tfrac{1}{2})^4, (0)^4] \\ V &= [(0)^4, (\tfrac{1}{2})^4] \end{aligned} \tag{7.3}$$

which correspond to the labels NS-NS, R-NS, NS-R and R-R respectively. Thus the only two basis vectors required are

$$\begin{aligned} W_0 &= [(\tfrac{1}{2})^4, (\tfrac{1}{2})^4] \\ W_1 &= [(0)^4, (\tfrac{1}{2})^4] \end{aligned} \tag{7.4}$$

and there are only two independent models, one given by W_0 only, and one with both W_0 and W_1.

The modular invariance constraints work as before, but with the added feature that both left- and right-movers determine the space-time statistics (since they both carry Lorentz indices), so that now the projection becomes

$$W_a.N_V = k_{ab}\alpha_b + (w_a^1 - w_a^5) + k_{0a} - W_a.V \tag{7.5}$$

with an additional w_a^5 dependence.

7.2 The type 0A and 0B models with W_0

In the former model, there are only two sectors, and both can give massless states

Structure constants: We have $m_0 = 2$ and $W_0.W_0 = 0$, so that $k_{00} = 0$ *or* $\frac{1}{2}$.

Vacuum energies:

sector $V =$	a_V
$\underline{0} = [(0)^4, (0)^4]$	$[0,0]$
$W_0 = [(\frac{1}{2})^4, (\frac{1}{2})^4]$	$[-\frac{1}{2}, -\frac{1}{2}]$

States allowed by projections in the W_0 (NS-NS) sector: This sector allows a tachyon state;

$$|0\rangle_L \times |0\rangle_R. \tag{7.6}$$

There are also massless states of the form

$$\tilde{\psi}^i_{-\frac{1}{2}}|0\rangle_L \times \psi^j_{-\frac{1}{2}}|0\rangle_R \tag{7.7}$$

where again writing the complex fermion as real ones we have $i, j = 1..8$. These contribute the same gravitational states to the spectrum as the heterotic string case, including the graviton. The projection is given by

$$W_0.N = w_0^1 - w_0^5 = 0 \ mod(1) \tag{7.8}$$

which is trivially satisfied in both cases since there are the same numbers of excitations from the left and right movers;

$$W_0.N_{W_0} = \frac{1}{2}\left(\sum_{j=1}^{8} \tilde{N}^j - \sum_{j=1}^{8} N^j\right) = 0. \tag{7.9}$$

States allowed by projections in the $\underline{0}$ (R-R) sector: The massless states are fermionic on both sides, of the form

$$|a\rangle_L \times |\tilde{a}\rangle_R. \tag{7.10}$$

Note that, these states are space-time *bosons* (they are like a mesonic bound-state of two fermions and have bosonic statistics). The rules are then as before, with w_a^1 replaced by $w_a^1 - w_a^5$ everywhere. For example projection is given by

$$W_0.N_0 = w_0^1 - w_0^5 + k_{00} = 0, \frac{1}{2} \ mod(1) \tag{7.11}$$

which produces a chiral projection on each side, which is the same or opposite for $k_{00} = 0, \frac{1}{2}$ respectively. The model where the left-mover and right-mover chiralities are the same is customarily called type 0B, and when they are both different it is a type 0A model (analogously to type IIB and IIA). Both models have a tachyon and no space-time fermions.

7.2.1 The type IIA/B models with W_0 and W_1

The type II models have four possible sectors, all of which can give massless states. The modular invariance can be verified relatively easily.

Exercise: Work through Appendix D which specializes the modular invariance to the simple type II case. It will help you understand the more general rules.

Structure constants: We have $m_0 = m_1 = 2$ and $W_a.W_b = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. The structure constant constraints leave us with two possibilities, $k_{01} = k_{10} = k_{11} = 0, \frac{1}{2}$ and $k_{00} = 0, \frac{1}{2}$.

Vacuum energies:

sector $V =$	a_V
$\underline{0} = [(0)^4, (0)^4]$	$[0,0]$
$W_0 = [(\frac{1}{2})^4, (\frac{1}{2})^4]$	$[-\frac{1}{2}, -\frac{1}{2}]$
$W_0 + W_1 = [(\frac{1}{2})^4, (0)^4]$	$[-\frac{1}{2}, 0]$
$W_1 = [(0)^4, (\frac{1}{2})^4]$	$[0, -\frac{1}{2}]$

States allowed by projections in the W_0 (NS-NS) sector: Again this sector has a would-be tachyon state;

$$|0\rangle_L \times |0\rangle_R. \tag{7.12}$$

and massless states of the form

$$\tilde{\psi}^i_{-\frac{1}{2}}|0\rangle_L \times \psi^j_{-\frac{1}{2}}|0\rangle_R \tag{7.13}$$

however now the projections are

$$\begin{aligned} W_0.N_{W_0} &= w_0^1 - w_0^5 = 0 \ \ mod(1) \\ W_1.N_{W_0} &= w_1^1 - w_1^5 = \frac{1}{2} \ mod(1) \end{aligned} \tag{7.14}$$

That is only the states with an odd number of excitations on each side survive the projection, and the tachyon is projected out.

States allowed by projections in the $\underline{0}$ (R-R) sector: Again the massless states are fermionic on both sides, of the form

$$|a\rangle_L \times |\tilde{a}\rangle_R. \tag{7.15}$$

Note that, these states are space-time *bosons* (they are like a bound states of two fermions so have bosonic statistics). The projection is given by

$$\begin{aligned} W_0.N_0 &= k_{00} + w_0^1 - w_0^5 = k_{00} = 0, \frac{1}{2} \ \ mod(1) \\ W_1.N_0 &= k_{01} + w_1^1 - w_1^5 = k_{01} + \frac{1}{2} = 0, \frac{1}{2} \ mod(1) \end{aligned} \tag{7.16}$$

which produces a chiral projection on each side, which is the same or opposite for $k_{00} = 0, \frac{1}{2}$ respectively. (The value of k_{01} then selects the chirality on each side.) The model where the left-mover and right-mover chiralities are the same is our type IIB model, and when they are both different it is the type IIA model. As for the heterotic string, it is simple to show that these projections correspond to $P^{\pm}_{GSO}$ depending on the choice of structure constants.

States allowed by projections in the W_1 and $W_0 + W_1$ (R-NS) and (NS-R) sectors: As both sectors are similar consider the former. The lowest lying states are massless space time fermions with a lorentz index (i.e. gravitinos);

$$|a\rangle_L \times \psi^j_{-\frac{1}{2}}|0\rangle_R. \tag{7.17}$$

The projection is given by

$$\begin{aligned} W_0.N_{W_1} &= k_{01} + w_0^1 - w_0^5 + k_{00} = k_{01} + k_{00} = 0, \frac{1}{2} \;\; mod(1) \\ W_1.N_{W_1} &= k_{11} + w_1^1 - w_1^5 + k_{01} = k_{11} + k_{01} + \frac{1}{2} = \frac{1}{2} \; mod(1) \end{aligned} \tag{7.18}$$

The W_1 projection is automatically satisfied for this state since there is one right-moving excitation. The chirality projection of $|a\rangle_L$for this state is given by $k_{01} + k_{00} + \frac{1}{2}$, and subtracting the W_0 and W_1 projections on the R-R states shows that the chirality projection on the left-moving part is also given by $k_{00} + k_{01} + \frac{1}{2}$. i.e. the gravitinos have the same chirality as the left-moving half of the R-R states.

A second gravitino

$$\tilde{\psi}^j_{-\frac{1}{2}}|0\rangle_L \times |\tilde{a}\rangle_R \tag{7.19}$$

arises from the $W_0 + W_1$ sector. Here the RHS of the W_0, W_1 projections are $k_{00} + k_{01} + k_{00} = k_{01}$ and $k_{10} + k_{11} + k_{01} + \frac{1}{2} = k_{11} + \frac{1}{2} mod(1)$ respectively. The latter can be compared with the chirality projection on the right-moving half of the R-R states which is given by $k_{01} + \frac{1}{2}$, again giving the same chirality.

7.2.2 Summary of type 0 and II models

model	SUSY
0A (same chirality)	No, tachyonic
IIA (same chirality)	$\mathcal{N} = 2$
0B (different chirality)	No, tachyonic
IIB (different chirality)	$\mathcal{N} = 2$

7.3 Inferring the existence of D-branes from R-R fields

There is an important connection between the fields in the R-R sector and the existence of D-branes (for a review see for example [10]). A Dp-brane has a natural coupling to $p+1$ forms (i.e. tensor fields with $p+1$ lorentz indices, $C_{\mu_1..\mu_{p+1}}$). The coupling in question is

$$I_{WZ} = \rho_p \int d^{p+1}\sigma \hat{C}^{p+1} \tag{7.20}$$

where

$$\hat{C}^{p+1} = C_{\mu_1..\mu_{p+1}} \partial X^{\mu_1} ... \partial X^{\mu_{p+1}}. \tag{7.21}$$

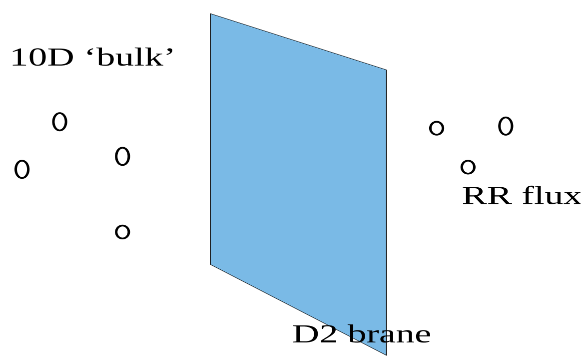
$C_{\mu_1..\mu_{p+1}}$is the pullback of $\hat{C}^{p+1}$ onto the world volume.

In this equation, the integral is over the p space coordinates of the brane plus time, hence the integral over the world-volume is a $p+1$ dimensional integral. The world-volume field $X^\mu(\sigma)$ defines the position of the brane volume in space-time exactly as it did for the strings. Now if in the volume there exists some $p+1$ form field such as a field strength, it can couple to the world volume in an invariant way as above, with the constant ρ_p being a 'charge density'.

The simplest example is the 0 brane (particle). It can couple to the one form gauge field A_μ as in eq.(3.28). This is a well known type of coupling that leads to the phenomena of Wilson lines in QCD or the Aharonov-Bohm phases in QED. A one dimensional example comes from fundamental strings. Here the coupling is of the form

$$\int d\sigma d\tau B_{\mu\nu}\partial_\sigma X^\mu \partial_\tau X^\nu, \tag{7.22}$$

where $B_{\mu\nu}$ is the same antisymmetric tensor that we derived in the gravitational spectra of the 10D string models. The point is now that, if we find a new two form field $C_{\mu\nu}$ in the R-R sector, then we can expect it to couple to a 1 dimensional object, and for R-R fields this object turns out to be D1 brane (a so-called D-string as opposed to fundamental string). The picture is as below, with Dp-branes existing in a 10D bulk, having an R-R charge and producing an R-R flux.



In the case at hand the new bulk R-R fields arose in the R-R sector and can be written as tensors as follows. The fields we found were bispinors so we need to do a little work before we can see the connection directly. First write the bispinors directly as

$$H_{\alpha\beta} \tag{7.23}$$

where α,β are spinor indices going from 1..32 so we initially have 2^{10} degrees of freedom. As we saw there were two projections on chirality (from the W_0 and W_1 conditions) so we ended up with $2^4\times 2^4 = 256$ states.

Now recast these states as tensors keeping track of the numbers of degrees of freedom. The relative chirality projection removed half of the degrees of freedom, and was that for type IIB strings the left and right moving chiralities is the same and for type IIA it is opposite. For type IIB for example we therefore have

$$(\Gamma_{11})_{\gamma\alpha}H_{\gamma\beta} = H_{\alpha\gamma}(\Gamma_{11})_{\beta\Gamma} \tag{7.24}$$

or using $\Gamma_{11}^T = -\Gamma_{11}$

$$\Gamma_{11}H = -H\Gamma_{11}. \tag{7.25}$$

Likewise for type IIA we have

$$\Gamma_{11}H = H\Gamma_{11}. \tag{7.26}$$

In order to extract the equivalent tensors we can first decompose the bispinors as follows

$$H_{\alpha\beta} = \sum_{n=0}^{10} \frac{i^n}{n!} H_{\mu_1...\mu_n}\Gamma^0\Gamma^{[\mu_1}..\Gamma^{\mu_n]} \tag{7.27}$$

where the $[\mu_1..\mu_2]$ means antisymmetrized in all the indices[4]. This has the full 2^{10} degrees of freedom (i.e. ${}_0C_{10} + {}_2C_{10} + ... + {}_{10}C_{10}$), but the chirality condition above will remove half of these. An additional duality constraint we have is that $H_{\mu_1..\mu_n} = \frac{1}{(10-n)!}\varepsilon^{\mu_1..\mu_{10}}H_{\mu_{n+1}..\mu_{10}}$. We verify this by inserting it into the sum above and using the 10D identities for $\varepsilon^{\mu_1..\mu_D}$ in terms of antisymmetric gamma products, yielding

$$H_{\alpha\beta} = \sum_{n=0}^{10} \frac{i^n}{(10-n)!} H_{\mu_{n+1}...\mu_{10}}\Gamma^0\Gamma^{[\mu_{n+1}}..\Gamma^{\mu_{10}]}. \tag{7.28}$$

This is equivalent to the W_0 projection. So $H_{\alpha\beta}$ defined this way has only 512 degrees of freedom (i.e. ${}_0C_{10} + {}_1C_{10} + {}_2C_{10} + {}_3C_{10} + {}_4C_{10} + \frac{1}{2}{}_5C_{10}$) which are reduced to 256 by the chirality projection. The tensors that survive this projection in the type IIA and type IIB cases are different. We can work them out using the usual anticommutation rule of Γ_{11} (c.f. Γ_5 for 4D Dirac fermions) which is $\{\Gamma_{11}, \Gamma_\mu\} = 0$. Applying $\Gamma_{11}H = -H\Gamma_{11}$ to the above decomposition, we find that only the odd n terms survive the projection in type IIB and applying $\Gamma_{11}H = H\Gamma_{11}$ leaves only the even n terms in type IIA. So our 256 fields are

type IIB	$H_\mu, H_{\mu_1\mu_2\mu_3}, H_{\mu_1..\mu_5}$
type IIA	$H, H_{\mu_1\mu_2}, H_{\mu_1..\mu_4}$

Exercise: Derive the rule $\{\Gamma_{11}, \Gamma^\mu\} = 0$ *using the Clifford algebra* ($\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$). *Use it to show that the chirality projection leaves only odd tensors in type IIB and even in type IIA.*

Bearing in mind that the indices are antisymmetric, the counting of states for type IIB is ${}_1C_{10} + {}_3C_{10} + \frac{1}{2}{}_5C_{10} = 256$ where the factor $\frac{1}{2}$ is because the duality above relates the components of $H_{\mu_1..\mu_5}$ to themselves. For type IIB we have ${}_0C_{10} + {}_2C_{10} + {}_4C_{10} = 256$.

So we have identified the tensorial equivalent of the bispinors appearing in the R-R sector. However these are not yet the physical propagating fields. This is because the massless bispinors satisfy the Dirac equation (in both the left and right moving sectors);

$$\Gamma.pH = H\Gamma.p = 0 \tag{7.29}$$

[4] So that for example $\Gamma^{[\mu_1}..\Gamma^{\mu_3]} = \frac{1}{6}(\Gamma^{\mu_1}\Gamma^{\mu_2}\Gamma^{\mu_3} + \Gamma^{\mu_2}\Gamma^{\mu_3}\Gamma^{\mu_1} + \Gamma^{\mu_3}\Gamma^{\mu_1}\Gamma^{\mu_2} - \Gamma^{\mu_2}\Gamma^{\mu_1}\Gamma^{\mu_3} - \Gamma^{\mu_1}\Gamma^{\mu_3}\Gamma^{\mu_2} - \Gamma^{\mu_3}\Gamma^{\mu_2}\Gamma^{\mu_1})$

or equivalently in terms of the tensor fields

$$p^{[\mu}H^{\nu_1..\nu_n]} = p_\mu H^{\mu\nu_2..\nu_n} = 0 \tag{7.30}$$

Exercise: Show that these two equations are equivalent

The first of these relations implies that the H field can be written locally as a derivative - i.e. like the field strength in QED it is the covariant derivative of a potential A_μ. In this case we can write

$$H_{\mu_1..\mu_n} = \frac{1}{n-1}\partial_{[\mu_1}C_{\mu_2..\mu_n]} \tag{7.31}$$

So a rank $p+2$ tensor gives rise to a set of $p+1$ form electric potentials which naturally couples to a p-brane;

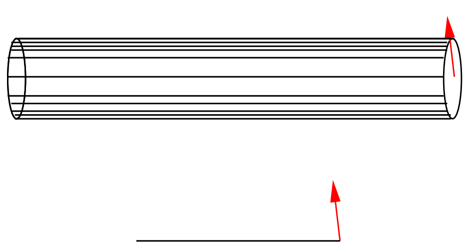
type IIB	C, $C_{\mu_1\mu_2}$,$C_{\mu_1..\mu_4}$, $C_{\mu_1..\mu_6}$, $C_{\mu_1..\mu_8}$	D-branes; $p = -1, 1, 3, 5, 7, (9)$
type IIA	$H^{(0)}$, C_{-1}, C_{μ_1},$C_{\mu_1..\mu_3}$, $C_{\mu_1..\mu_5}$, $C_{\mu_1\mu_7}$,$C_{\mu_1..\mu_9}$	Dbranes; $p = 0, 2, 4, 6, 8$

The additional potentials correspond to magnetic potentials which have been obtained from the Hodge dual of $H^{(n)}$ (i.e. $H_{\mu_{n+1}..\mu_D}$) . It turns out that (almost) all of these branes can be built as 'lumps of field', monopole-like solutions of the field equations. These solutions were Note that in addition to the above, as well as the fundamental string which couples to the $B_{\mu\nu}$ field, we should find an object that couples to its dual $B_{\mu_1..\mu_6}$. This is a 5 brane soliton of the NS-NS sector. For a review of the constructions of these classical solutions the reader is referred to ref.[11].

8. Compactification: obtaining D=4

8.1 Background: Kaluza-Klein models

The 10D models are not much use for describing the everyday world, so I now turn to how to get 4 dimensions and $\mathcal{N} = 1$ supersymmetry. The most geometric formulation is compactification. It relies on the fact that a higher dimensional space can be rolled up. At large distances or, equivalently, as we reduce the compactification radius, the object appears to have fewer dimensions as in the figure.



The idea of extra dimensions was first suggested by Kaluza and Klein in the 1920's as a way of unifying gravity and electromagnetism. The idea works as follows. Consider a 5 dimensional space time, with metric g_{MN} with $M,N = 0..4$. If we compactify on a "small" circle of radius R then the massless states decompose as $g_{MN} = g_{\mu\nu} + A_{\mu 5} + \phi$ so that the off-diagonal element plays the role of the electromagnetic field, and indeed couples to the charged fields in the right way. This is shown in the figure as a $U(1)$ rotation symmetry (arrow) of the compactified space that remains in the massless theory even when we take the limit of small R. The total space is then a product of an internal space which I'll call K and the usual 4 dimensional non-compact space. The direct way to tell if physics is like this is to probe down to scales R (by for example using colliders that can access equivalent energies $1/R$.)

At these scales one would see a typical spectrum of "Kaluza-Klein" states. These are the residue of the continuous 5D spectrum of momenta. Before compactification, the momentum in the 5th dimension is continuous. After compactification, the momentum values are constrained because the fields must be single valued. So for example a generic field which can be written as $\phi(x^\mu, x^5)$ must obey

$$\phi(x^\mu, x^5 + 2\pi R) = \phi(x^\mu, x^5). \tag{8.1}$$

We can expand this field in modes in the 5th dimension

$$\phi(x^\mu, x^5) = \sum_{n=-\infty}^{\infty} \phi_n(x^\mu)\, e^{i\frac{n}{R}y} \tag{8.2}$$

The continuous 5D spectrum becomes an infinite but discrete tower of 4D states ϕ_n which only becomes continuous in the $R \to \infty$ limit. Now consider the Klein-Gordon equation for the 5D state;

$$\begin{aligned}(\partial_\mu \partial^\mu + \partial_5 \partial^5)\phi(x^\mu, x^5) &= 0 \\ &= \sum (\partial_\mu \partial^\mu - \frac{n^2}{R^2})\phi_n\, e^{i\frac{n}{R}y}\end{aligned} \tag{8.3}$$

The n-modes have an effective mass n/R and from the 4D point of view we find a Kaluza-Klein spectrum, an infinite tower of states whose quantum numbers are the same, but with equally spaced masses.

Broadly speaking the same situation obtains in string theory. We begin with a 10D theory but reduce it to 4D by compactifying on an internal small 6 dimensional manifold, which I called K_6 in the Introduction. The situation is as in figure.3 with the 10D space being decomposed as

$$M_{10} = M_4 \times K_6. \tag{8.4}$$

Every point in the 4D space has its own internal 6D space that whose properties give rise to the observed properties of the low energy theory, such as remaining supersymmetry, gauge groups etc. These have to be derived from the 10D string theory which is assumed to exist before the compactification.

So, the properties of the compactified space determine the low energy physics that we see. In addition one expects a Kaluza-Klein-ish spectrum to appear at the compactification scale, with an infinite spectrum of particles appearing above it as we "open" the higher dimensional space. In a sense we have already seen this in the spectrum derived in the heterotic string with its 16 internal right-moving bosonic degrees of freedom. For example the massless graviton is accompanied by an infinite tower of states such as

$$\tilde{\psi}^i_{-\frac{1}{2}-n}|0\rangle_L \times \alpha^J_{-n}\alpha^j_{-1}|0\rangle_R \;\; n = 1,2.. \tag{8.5}$$

The extra massive states have masses of order the Planck scale $\approx 10^{19} GeV$, so are not of relevance to phenomenology, so for the most part we will be concerned only with the massless spectrum.

8.2 Toroidal heterotic compactifications: $\mathcal{N} = 4$ SUSY

Compactification tends to leave internal gauged symmetries in the effective subspace (indeed this was the original point of Kaluza and Klein). The simplest compactifications we can imagine, a torus, is a flat compactified K_6 with coordinated identified under translations. For convenience let's collect the compact coordinates together into 3 complex ones which I'll call

$$Z^j = X^{2j} + iX^{2j+1} \quad j = 2..4 \tag{8.6}$$

so that j is the same index we were using before to label the complex space-time coordinates. $j = 1$ labels the two transverse components of the non-compact space in the light cone gauge.

Now the 6 dimensional torus, T_6, is identified by

$$Z^j = Z^j + a^j \tag{8.7}$$

where a_j is a complex constant. The metric of the compact space is Euclidean everywhere, so we don't expect to lose any states in this compactification. This is also reasonable since the main constraints on the model come from the one-loop partition function which is itself a torus. The diagram has a one-to-one mapping to the compact space so the projections on states will remain the same.

The low energy theory relevant for phenomenology (i.e. the spectrum of massless states in the string spectrum), are unaffected, and it is easy to see that this leads to $\mathcal{N} = 4$ supersymmetry simply by counting the number of 4D gravitinos that are contained in the 10D gravitino we derived earlier. Recall that the single chiral 10D state was of the form

$$(b^i_0 + b^i_0 b^{j\neq i}_0 b^{k\neq i,j}_0)|0\rangle_L \times \alpha^j_{-1}|0\rangle_R \tag{8.8}$$

giving ${}_1C_4 + {}_3C_4$ = 8 transverse degrees of freedom. However in 4D only the first excitation, b^1_0 is related to the 4D space-time fermion, and the rest are just internal excitations. To see this decompose the left-moving part into pieces that have a b^1_0 excitation and pieces that do not;

$$\begin{aligned} 1 \times (b^{i\neq 1}_0 + b^{i\neq 1}_0 b^{j\neq 1,i}_0 b^{k\neq 1,i,j}_0)|0\rangle_L \\ + b^1_0 \times (1 + b^{i\neq 1}_0 b^{j\neq 1,i}_0)|0\rangle_L \end{aligned} \tag{8.9}$$

From the 4D point of view we have the two transverse fermionic degrees of freedom (i.e. the 1 or the b^1_0) each with 4 internal degrees of freedom in the spinor representation of $SO(6)$.

Exercise: count the internal excitations above and confirm that there are 4 of each chirality. $SO(6)$ is isomorphic to $SU(4)$. Show that the fermionic representations above are a fundamental and anti-fundamental ($4+\bar{4}$) of $SU(4)$. Hint; use the same identification as earlier, i.e. $b^j_0 = \frac{1}{2}\left(\Gamma^{2j} + i\Gamma^{2j+1}\right)$, The indices of $\Gamma_{\alpha\beta}$ then label the elements of $SU(4)$.

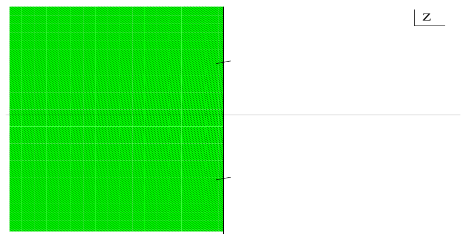
8.3 Getting $\mathscr{N}=1$ SUSY in $D=4$: Orbifolds

$\mathscr{N}=4$ models are not good candidates for low energy physics as they do not have a chiral spectrum (for one thing): the 4D gravitinos above had both chiralities (i.e. both even and odd numbers of b_0^1 excitations), and this will be true for all 4D fermions that come out of this theory. But the Standard Model is chiral, the left-handed particles coupling to $SU(2)_L$ interact differently from right-handed ones: we need a theory that is chiral.

8.3.1 Origami with 2 dimensional orbifolds - the cone

One way to do this is to have K_6 that is an "orbifold" of T^6. An orbifold is the quotient of a manifold by a subgroup G of its isometries[5] In the case of T^6 the group G consists of point groups (rotations in 6 dimensions) and space groups (shifts) that leave the torus invariant. In the phenomenological context they were first developed in refs.[12, 13]. For a review see ref.[14].

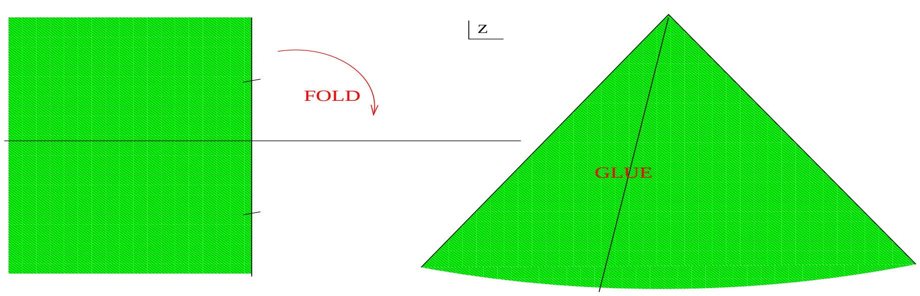
To begin with a more simple example consider a cone which is an orbifold of R^2. Consider for example R^2/Z_2 shown in the figure



We describe R^2 by with the single complex coordinate Z. The orbifold is defined by the Z_2 equivalence relation

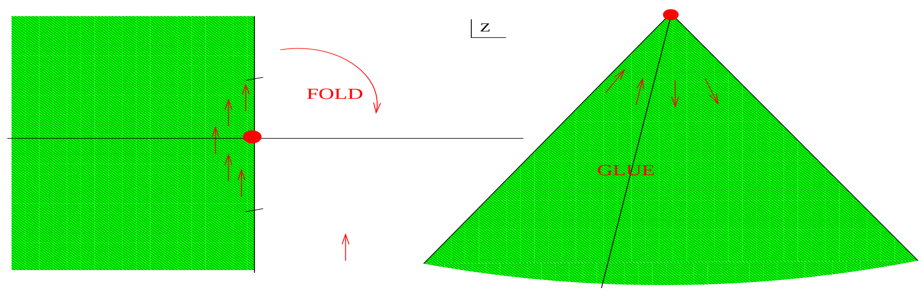
$$Z=-Z\,, \tag{8.10}$$

leavingthe fundamental domain on the diagram shown in green. The lines with strokes are identified since they are mapped into each other under $Z\to -Z$ and also the origin is a fixed point since it is mapped into itself - marked with a red circle below. We can fold the sheet and glue the axis to itself to form a cone. The curvature is everywhere zero except at the origin which forms a point (where the curvature is ill-defined).



[5]This is the "physicist's definition". More correctly an orbifold generalizes the notion of manifold, to allow for the presence of points whose neighbourhood is diffeomoerphic to the quotient of $\mathbf{R}^n$ by a finite group.

The effect of the "infinite curvature" at the origin is to cause a rotation in vectors that are parallel transported around it. We show this in the next figure. The red vector is transported around the origin. When we fold the sheet up, it reverses direction at the join.



We can make different cones by instead identifying $Z = e^{2\pi i/n}Z$ where n is an integer. This divides the complex plane into n identical segments. A single segment has the edges identified as above, except the deficit angle is now not π but $2\pi - \pi/n$. (The deficit angle is the angle of the wedge that is *cut out* of the plane.) The parallel transported vector is now rotated by the deficit angle.

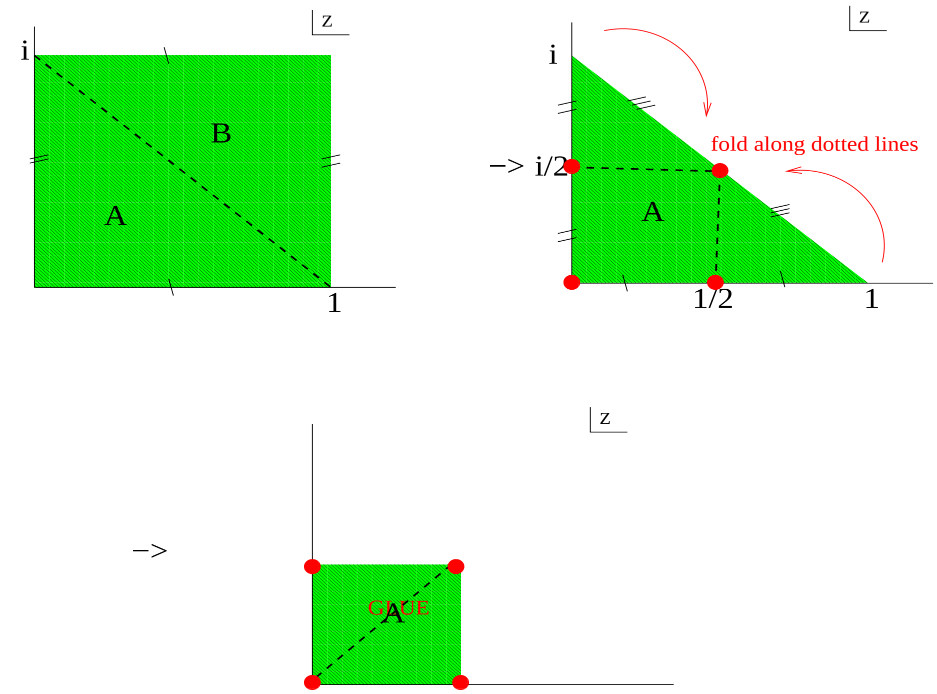
Exercise: convince yourself that parallel transported vectors are rotated by the deficit angle.

8.3.2 T^2/Z_2 : the pillowcase

A slightly less trivial example is provided by T^2/Z_2. We can define a simple 2D torus by identifying

$$\begin{aligned} Z &= Z + i \\ Z &= Z + 1, \end{aligned} \tag{8.11}$$

and the Z_2 projection is again the identification $Z = -Z$. The torus is shown in the first figure below where lines with equal numbers of strokes are identified. The Z_2 equivalence maps half the torus (the B region) into the A region, so the area of the fundamental region is reduced by half (this is generally true, i.e. a Z_n moding reduces the fundemantal region by a factor of n).



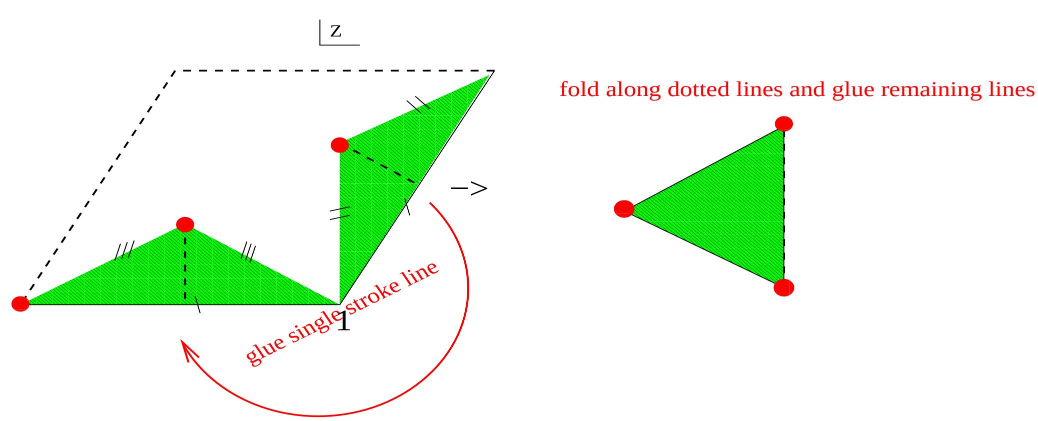
Next we can show that a combination of Z_2 operations and translations in the torus (i.e. $Z \to Z+n+im$, where n,m are integers) identifies the lines as shown, leaving 4 fixed points shown as a red dot. Joining the identified lines leaves a 'pillowcase' with 4 corners.

8.3.3 T^2/Z_3: the three point cushion

For the final example in 2D we show the torus defined by

$$\begin{aligned} Z &= Z + e^{\pi i/3} \\ Z &= Z + 1, \end{aligned} \tag{8.12}$$

divided by the Z_3 projection $Z = e^{2\pi i/3}Z$. We show this in the figure below. The torus is a parallelogram with angles of $\pi/3$ and $2\pi/3$. As we have seen, the Z_3 moding reduces the fundamental region by a factor of three. The remaining region is shown in the next figure. Folding as shown leaves a three pointed cushion.



8.4 T^6/Z_3: a "realistic" $\mathcal{N}=1$ heterotic model

The Z_3 case is interesting for phenomenology because one can directly construct an $\mathcal{N}=1$ model as we will shortly see. First complexify the six internal coordinates to three complex Z^i defining T^6 as described earlier. The simplest example is to keep the complex (a_i) directions determining the torus orthogonal so that we just repeat the 2D example 3 times;

$$\begin{aligned} Z^i &= Z^i + e^{\pi i/3} \\ Z^i &= Z^i + 1. \end{aligned} \tag{8.13}$$

Then project it out with the single Z_3 identification

$$\begin{aligned} Z^1 &\to e^{2\pi i/3} Z^1 \\ Z^2 &\to e^{2\pi i/3} Z^2 \\ Z^3 &\to e^{-4\pi i/3} Z^3 \end{aligned} \tag{8.14}$$

(all at the same time) which is usually written more succinctly as

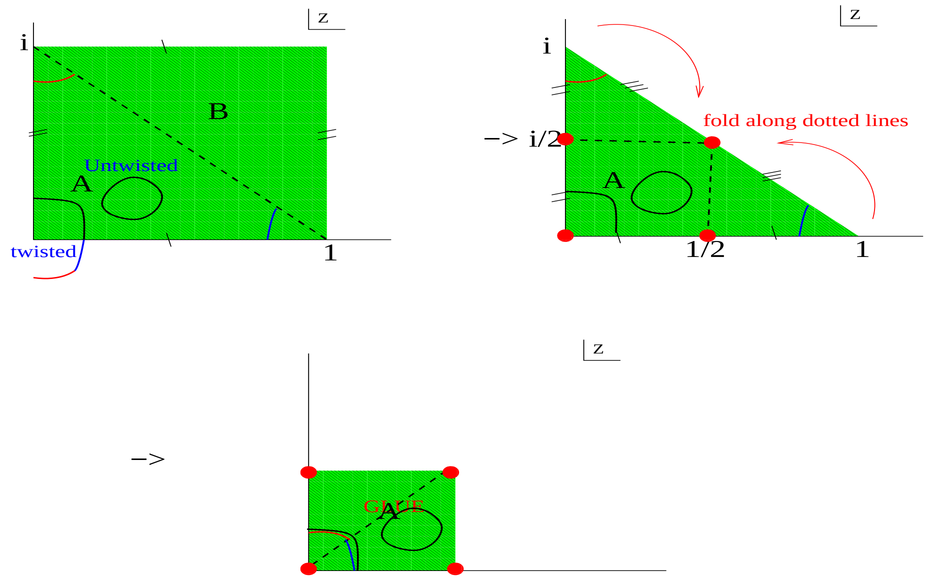
$$Z^i = e^{2\pi i v_i} Z^i \; ; \; v = (\frac{1}{3}, \frac{1}{3}, \frac{-2}{3}). \tag{8.15}$$

The Z_3 acts slightly differently on one of the internal degrees freedom in order to satisfy modular invariance constraints (i.e. for consistency of one-loop amplitudes again) as we shall see shortly.

Now let's impose this compact structure on 6 space dimensions of the $E_8 \times E_8$ heterotic model of section 6.4.4. Note that (from the invariance of the supercurrent term in the action) on the supersymmetric side the fermions have to transform in the same way as the bosons,

$$\lambda_+^i = e^{2\pi i v_i} \lambda_+^i \; ; \; v = (\frac{1}{3}, \frac{1}{3}, \frac{-2}{3}). \tag{8.16}$$

In this model, for each of the sectors labelled NS or R generated by $W_{0,1,2}$, we can add additional sectors with twists of $e^{2\pi i v_i}$, $e^{4\pi i v_i}$ (which adds additional phases to the boundary conditions of the 3 complex internal space-time fermions). These sectors give "twisted" states whose endpoints are related by a Z_3 transformation. The states in the spectrum can then be divided into "untwisted" or "twisted" as in the next figure for the Z_2 orbifold (which is simpler to draw).



The orbifold in the figure has Z_2 twists for some of the compact dimensions, so we show it exactly like the 2D example. Twisted states will have boundary condition $Z(\sigma+\pi) = -Z(\sigma)$ so the ends do not meet before the orbifolding, and these states do not exist on the torus. After the orbifolding, however, we cut away half the torus and fold it up. The figure shows a twisted state at the origin (which is a fixed point of the Z_2), with the string endpoints separated by a Z_2 transformation. The black, red and blue regions show portions of the string that are mapped into different bits of the fundamental region upon orbifolding (i.e. use a combination of Z_2 and translations to map the red and blue parts into the region remaining in the second figure). When the fundamental region is folded up, the all string endpoints rejoin forming a closed string around the fixed point. This is shown in the third figure, where the black region is on the bottom surface and the red and blue regions are on the top. Also shown is an untwisted state which is unaffected by all our folding and gluing. Thus the twisted states live at fixed points, whereas untwisted ones can move throughout the compactified space.

As we have seen untwisted states are the original states before the orbifolding. In addition to the introduction of new twisted sectors, the effect of orbifolding is to project some of these states out. As we shall now see, this can break the supersymmetry to $\mathcal{N}=1$ as desired. We return to the Z_3 model and consider the gravitinos. Begin with the $\mathcal{N}=4$ gravitino multiplet we found on the torus coming from the states in the R-NS sector. To recap and simplify a little they can be written

$$(b_0^0)^{N_1}(b_0^1)^{N_2}(b_0^2)^{N_3}(b_0^3)^{N_4}\,|0\rangle_R \times \alpha^{\hat{\mu}}_{-1}|0\rangle_L = |a\rangle_R \times \alpha^{\hat{\mu}}_{-1}|0\rangle, \qquad (8.17)$$

where the excitation numbers can be $N_i = 0,1$ and b_0^0 is the transverse zeroth mode of the 4-D space time degrees of freedom, $b_0^{1,2,3}$ correspond to the 3 complex internal dimensions (6-real), and $\hat{\mu}$ here is the 4D space-time index since we are interested in the number of 4D gravitinos (it corresponds to just the $\mu = 0,1,2,3$ indices of the 10D gravitino). (Note for this discussion, the labelling on the b_0^i's has changed from $i = 1..4$ to $i = 0..3$) Initially we had $2^4 = 16$ gravitino degrees of freedom which gave us 4 gravitinos in 4 dimensions, but after the GSO (W_0) projection we had to impose a chirality projection,

$$N_1 + N_2 + N_3 + N_4 = odd \tag{8.18}$$

leaving only an odd number of $b's$ so this became 8. We then interpreted this as four 4-D gravitinos in the spinor representation of $SO(6)_{K_6}$. However let's now apply the condition that the states are also invariant under the Z_3 orbifold action. The $\alpha^{\hat{\mu}}_{-1}$ does not transform under the Z_3, but the b^i_0 are all rotated by the phase factor $e^{2\pi i v_i}$, so that the only Z_3 invariant states that remain that in addition satisfy the GSO projection have

$$\begin{aligned} N_2 = N_3 = N_4 &= 0,1 \\ N_0 &= 1,0 \end{aligned} \tag{8.19}$$

This leaves only the 2 degrees of freedom

$$(b^0_0 + b^1_0 b^2_0 b^3_0)|0\rangle_L \times \alpha^{\hat{\mu}}_{-1}|0\rangle_R \tag{8.20}$$

of $\mathcal{N} = 1$ supersymmetry.

When we add an orbifolding, modular invariance provides an additional constraint on how the right-handed (gauge side) behaves (to ensure the torus diagram is invariant under the orbifold action). In fact the constraints can be satisfied by again embedding the Z_3 projection in the gauge side (there are other possibilities). That is we separate out 3 bosonic coordinates from the gauge side and fermionize them into complex fermions with, $\lambda_-^{i=1..3} = e^{iX_i}$. We then apply the Z_3 projection simultaneously on both sides. That is the total action of the Z_3 is

$$(Z, \lambda_-, \lambda_+) \to e^{2\pi i(v,v,\tilde{v})}(Z, \lambda_-, \lambda_+). \tag{8.21}$$

This projects out some of the gauge bosons and results in the breaking $E_8 \times E_8' \to SU(3) \times E_6 \times E_8'$. Indeed this is evident from the fact that the Z_3 rotation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{8.22}$$

is one of the Cartan generators of $SU(3)$ and hence commutes with it. To see it explicitly, consider the gauge boson degrees of freedom of the E_8, which came from two sectors W_0 and $W_0 + W_2$ (the former gave adjoints of $SO(16)$, the latter a spinor **128** of $SO(16)$ altogether giving **248** of E_8). In the NS-NS sector, the states are of the form

$$\psi^i_{-\frac{1}{2}}|0\rangle_L \times \psi^I_{-\frac{1}{2}}\psi^{J\neq I}_{-\frac{1}{2}}|0\rangle_R \tag{8.23}$$

where $I = 1..16$. Now the Z_3 operation means that we must leave the first 6 rightmoving fermions as three complex ones, so only the ten remaining ones can be written this way, giving the $10.9/2 = \mathbf{45}$ adjoint of $SO(10)$. The 3 complex right-moving fermions we have singled out must be written as complex fermions. They can appear in excitations as

$$\psi^i_{-\frac{1}{2}}|0\rangle_L \times (\tilde{b}^{I\dagger}_{\frac{1}{2}}\tilde{b}^{J\neq I\dagger}_{\frac{1}{2}} + \tilde{d}^{I\dagger}_{\frac{1}{2}}\tilde{d}^{J\neq I\dagger}_{\frac{1}{2}} + \tilde{b}^{I\dagger}_{\frac{1}{2}}\tilde{d}^{J\dagger}_{\frac{1}{2}}|0\rangle_R \tag{8.24}$$

where $I,J = 1..3$. Note that without the orbifolding this gives 3+3+9=15 states which is just the adjoint of $SO(6)$ and so far we haven't done anything except separate out this factor. Now we must apply the invariance under Z_3 however. The fermions all have a phase factor $e^{2\pi i/3}$ (since the phases on the physical states $\in [0,2\pi]$ so that $-4\pi i/3 \equiv 2\pi i/3$ when constructing the states). Noting that (by the definition of the mode expansion of the $\lambda_\pm$) the d's transform with the opposite phases to the b's under the orbifold action, we find that all $\tilde{b}\tilde{b}$ and $\tilde{d}\tilde{d}$ states are projected out, but for $\tilde{b}\tilde{d}$ any I,J gives a Z_3 invariant state, and we have 9 states in all. 8 of these form the adjoint of $SU(3)$. The trace combination (i.e. the sum of the $U(1)$'s) must be orthogonal to $SU(3)$ since $SU(3)$ generators are traceless, hence it is an extra $U(1)$, often called $U(1)_X$. From the W_0+W_2 sector Z_3 invariance leaves only $(1+\tilde{b}_0^1\tilde{b}_0^2\tilde{b}_0^3)(\tilde{b}^{4N_4}...\tilde{b}^{8N_8})|0\rangle_R$ on the gauge side where again $N_4,..N_8 = 0,1$ corresponding to a b_0 excitation or not. There are then 2×2^5 possibilities, but the GSO (W_0) projection removes half of these, leaving two chiralities (i.e. a $\mathbf{16}+\overline{\mathbf{16}}$) of spinor representations of $SO(10)$. The 1+$\mathbf{16}+\overline{\mathbf{16}}+\mathbf{45}$ gauge bosons of $U(1)_X\times SO(10)$ together form the **78** gauge bosons of E_6. The charges of the states can be calculated using the expression $Q^j = N^j + v^j - \frac{1}{2}$ derived earlier, where $j = 1,2,3$ labels the three complex world sheet fermions. The states have the correct charges.

Exercise: calculate the charges $Q^j = N^j + v^j - \frac{1}{2}$ under the three $U(1)$'s corresponding to the first three complex fermions. Define $Q_X = \frac{1}{3}(Q^1+Q^2+Q^3)$ to be the charges under $U(1)_X$. Verify that they are $0,1,-\frac{1}{2}$ for the gauge bosons of E_6.

8.4.1 Further gauge breaking Wilson lines

Once we have achieved an $\mathcal{N}=1$ theory with chiral fermions, the next task is to break the gauge group down to the Standard Model one. A particularly useful trick that one can use on non-simply connected manifolds is Wilson line breaking. Since it will crop up from time to time I will give a brief outline.

Consider the gauge group E_6. In general, when we construct the path-ordered product of gauge elements around a loop, C, we have a gauge rotation

$$U \sim Pe^{\int_C A_\mu dx^\mu} \tag{8.25}$$

where U is matrix valued. This quantity is gauge invariant and consquently should be independent of the path C. However, on a small path the Wilson loop becomes (by Stokes' law)

$$U \to e^{F_{\mu\nu}\Delta^{\mu\nu}} \tag{8.26}$$

where $\Delta^{\mu\nu}$ is the area tensor of the closed path. Thus, on simply connected manifolds, when the curvature vanishes, we expect Wilson lines to be unity since we can always contract the path to a point. This is not the case on non-simply connected manifolds, where we can have vanishing curvature tensors, but Wilson loops that are not equal to unity for paths that are homotopically equivalent to non-trivial cycles. The remaining gauge group is the subgroup that commutes with U.

The situation is precisely the same as when a cosmic string is formed. Far away from the string the gauge field strength vanishes. However the gauge fields themselves go though a gauge rotation if we take a test particle on a path around the cosmic string. The vacuum manifold is consequently not simply connected, the obstruction being the cosmic string itself (i.e. we cannot continuously contract the path to zero through the cosmic string).

Since the Wilson lines are in the internal (K_6) space, we can identify the higgs fields that are responsible for the breaking from the 4-D point of view. Schematically they arise as follows. The Yang-Mills terms are functions of the covariant derivatives

$$(\delta_b^c \partial^\mu - f_{ab}^c A^{a\mu}) A_\nu^b, \tag{8.27}$$

where the indices run over all ten space time indices. Let there be a Wilson line along internal direction y such that $A_\mu^a(y) = e^{i\lambda^b f_{bc}^a y} A_\mu^c(0)$ (i.e. $A_y^b = i\lambda^b$). Thus $\partial^y A_\nu^a \propto i\lambda^b f_{bc}^a A_\nu^c \neq 0$ for the 6-internal dimensions. The 4-D Lagrangian now gets a mass-squared term

$$\left(i\lambda^b f_{bc}^a A_\nu^c\right)^2 \tag{8.28}$$

which is non-zero for the gauge fields that do not commute with U. The masses of the broken gauge bosons are proportional to the VEV of the component of the gauge field along the Wilson line (in this case A_y^b) which is playing the role of the higgs field.

As an example, let us see how Wilson lines can break the E_6 gauge group down to $SU(3)_c \times SU(3)_L \times SU(3)_R$ and its subgroups. We can specify the Wilson line element in terms of the $SU(3)^3$ subgroup;

$$U_0 = (\alpha) \times \begin{pmatrix} \beta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta^{-2} \end{pmatrix} \times \begin{pmatrix} \gamma & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \gamma^{-1}\delta^{-1} \end{pmatrix}. \tag{8.29}$$

We choose them to be of this form because at the very least we need to keep the Standard Model gauge group, $SU(3)_c \times SU(2)_L \times U(1)_Y$. When $\alpha, \beta, \gamma, \delta$ are all cube roots of unity, the three factors clearly all commute with the three $SU(3)$ subgroups and we have $SU(3)_c \times SU(3)_L \times SU(3)_R = E_6/Z_3$. If we instead choose only α to be a cube root of unity (which is an element of $SU(3)$) but $\gamma, \beta\ \delta$ to be n'th roots of unity we find the gauge group $SU(3)_c \times SU(2)_L \times U(1) \times U(1) \times U(1)$. Note that the Wilson lines have not reduced the *rank* of the group which contains superfluous $U(1)$ factors. This is a common feature of heterotic theories. They must be eliminated with some other mechanism. If for example they are anomalous then they get Stückelberg masses from the Green-Schwarz anomaly cancellation mechanism. Of course one might also try to use them to one's advantage by for example trying to implement some kind of Froggatt-Nielsen mechanism for the Yukawa couplings.

9. Phenomenology of heterotic models

9.1 Weakly coupled models

At first sight (i.e. perturbatively) only the Heterotic models seemed to be of much use for model building so let us first discuss those models from a more phenomenological point of view. These models seem to be singled out for phenomenology because (before the possibility of using D-branes was appreciated), they alone seemed to contain both quantum gravity and gauge fields.

Gravity, being a spin 2 field, requires closed strings which rules out type I. However the type II models are also ruled out because they *only* contain gravity multiplets and no gauge fields. Heterotic theories are also closed strings, but as they are a curious combination of supersymmetric and bosonic string theories they contain fermions and both gauge and gravity degrees of freedom.

Let us summarize and generalize the phenomenological properties that we have deduced for our quasi-realistic Z_3 example. As we have seen the 16 additional internal degrees of freedom in the bosonic half become gauge degrees of freedom in the effective theory; hence the gauge groups in 10 dimensions end up being rank 16. (Indeed anomaly cancellation alone is enough to restrict them still further to be either $E_8 \times E_8'$ or $\mathrm{SO}(32)$ as derived above – the latter turns out to be dual to the SO(32) of the type I models).

Model building in heterotic strings concentrated on the $E_8 \times E_8'$ gauge group. In order to get $\mathcal{N} = 1$ supersymmetry in 4 dimensions, the the most general requirement on the compactification manifold K_6 is that it has to be of a certain type (namely Calabi-Yau) [15]. The orbifold compactifications of the previous section are singular limits of Calabi-Yau manifolds. This identification is useful because it means that many properties of the effective theory (the nett number of generations for example) can be derived from the topological properties of the Calabi-Yau. In addition the great advantage of orbifold models is that because they are essentially flat spaces with some singularities, one is still able to use conformal field theory techniques to calculate scattering amplitudes. As well as the Z_3 orbifold of the previous section, the possible orbifoldings are Z_4, Z_6, Z_6', Z_7, Z_8, Z_8', Z_{12}, Z_{12}'. The prime indicates the same point group but a different compactification lattice [16]. In addition product orbifold groups $Z_N \times Z_M$ are possible.

The modular invariance conditions then require a breaking of the gauge group by the compactification. One attractive route of gauge breaking is to adopt the approach used for the Z_3 orbifold example discussed above, namely to embed the geometrical orbifold action on the space-time into the gauge degrees of freedom. This leads to a gauge breaking such as

$$E_8 \times E_8' \longrightarrow G \times E_6 \times E_8' \longrightarrow \mathrm{MSSM} \times \mathrm{hidden}$$

The precise gauge symmetry breaking pattern (i.e. the subgroup G) depends on the orbifold in question. This route became known as the “standard embedding” and the possibilities are relatively restricted. Standard embedding generates the so-called (2,2) models (where the numbers indicate the supersymmetry of the world-sheet CFT on the left and right sides respectively). In addition far less restricted asymmetric embeddings which still have $\mathcal{N} = 1$ space-time supersymmetry ((2,0) models) can be constructed. The further symmetry breaking by Wilson lines down to something resembling the Standard Model is extremely unconstrained. Phenomenologically, in the simpler embeddings the first E_6 factor is already a potential Grand Unified group whereas the second E_8' factor forms a hidden sector group. The latter is a potential source of supersymmetry breaking by for example the condensing of the gaugino of some hidden sector group at a high mass scale (much like the condensation that takes place in QCD leading to a Λ_{QCD} breaking) [17]. Early applications to string motivated scenarios were discussed in ref.[18]. The remarkable thing about this part of the story is that the effect of gaugino condensation is a non-perturbatively induced contribution to the superpotential that can be determined to all orders (thanks to holomorphy) in the effective firld theory [19]. (For a review of supersymmetry breaking see ref.[20].)

Let us now turn to the question of the fundamental scale. In heterotic models, all degrees of freedom in the perturbative model are the result of excitations of closed strings. All closed strings can travel everywhere in the compact space and so both gauge and gravity degrees of freedom necessarily feel the same compact volume, V_6 say. The Planck scale and the gauge couplings can then be simply computed from the dimensional reduction of the 10-dimensional theory . In terms of the string scale, M_s, and the heterotic string coupling, λ_H, they read

$$M_{\rm Pl}^2 \sim \frac{V_6}{\lambda_H^2} M_s^8, \tag{9.1}$$

and

$$\alpha_{YM} \sim \frac{\lambda_H^2}{V_6 M_s^6}. \tag{9.2}$$

These expressions, together with the experimental fact that $\alpha_{YM} \lesssim 1$, imply, in the case that the heterotic string remains weakly coupled (*i.e.* $\lambda_H \lesssim 1$), the following relations between the compactification, string and Planck scales [21];

$$M_s \sim M_{Pl} \sim V_6^{1/6}. \tag{9.3}$$

The models that arise from the weakly coupled heterotic string with simple embeddings therefore suffer from the problem that the natural unification scale for them is the Planck scale, but that the unification scale as derived from the RG running of the gauge couplings is of order (assuming the MSSM with a desert between the weak and GUT scales) $M_{GUT} \sim 3 \times 10^{16}$GeV).

9.2 Strongly coupled models

One way to address this problem is to go to the strongly coupled limit of heterotic string theory. The strongly coupled $E_8 \times E_8'$ heterotic theory is only tractable thanks to the fact that, as Horava and Witten showed [22], it is described by 11-dimensional supergravity compactified on an S^1/Z_2 orbifold. Based on anomaly cancellation arguments they argued that an E_8 gauge group lives on each of the two 10-dimensional orbifold fixed planes whereas gravity lives in the 11-dimensional bulk as sketched in Fig. 7. In the case of strong coupling, the radius of the orbifold R_{11} is larger than the compactification scale of the 6 extra dimensions. It is therefore possible to consider the compactification of this theory down to 4 dimensions in two steps, with an intermediate 5-dimensional model compactified on an orbifold.

The 11-dimensional action takes the form

$$S = \frac{1}{2\kappa_{11}^2} \int \mathrm{d}^{11}x \sqrt{g} R - \sum_i \frac{3^{1/3}}{4\pi(2\pi\kappa_{11}^2)^{2/3}} \int \mathrm{d}^{10}x \sqrt{g} \mathrm{Tr} F_i^2 + \dots, \tag{9.4}$$

where κ_{11} is the 11-dimensional gravitational constant and i runs over the two 10-dimensional fixed planes where the two E_8 groups live. Compactifying down to five dimensions (with a compact volume V_6) and then to four dimensions we can write the fundamental 11-dimensional constant, $M_{11} = 2\pi(4\pi\kappa_{11}^2)^{-1/9}$ and the radius of the 11-th dimension, R_{11}, in terms of 4-dimensional quantities,

$$M_{11} = (2\alpha_{GUT} V_6)^{-1/6}, \quad R_{11}^2 = \left(\frac{\alpha_{GUT}}{2}\right)^3 V_6 M_{\rm Pl}^4. \tag{9.5}$$

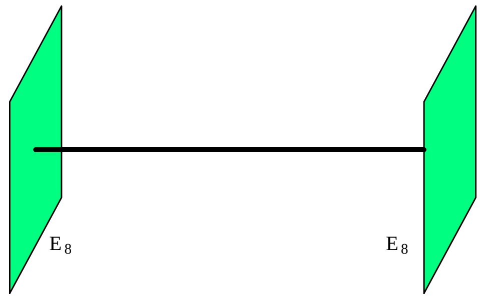


Figure 7: Horava-Witten construction for the strong coupling limit of the $E_8 \times E_8$ heterotic string. The green planes represent the 10-dimensional boundaries of the orbifold S^1/Z_2 where each E_8 factor lives, 11-dimensional supergravity propagates in the bulk.

It is now possible to have

$$M_{11} \sim V_6^{-1/6} \sim M_{\rm GUT} \sim 10^{16} \text{ GeV}, \tag{9.6}$$

and therefore $R_{11}^{-1} \sim 10^{13} - 10^{15}$ GeV.

Thus the heterotic string can accommodate both a fundamental scale of the order of the Planck mass in the weak coupling limit, and of the GUT scale in the strong coupling limit.

At first sight this seems to offer an easy way of uniting the string scale (i.e. the fundamental scale of gravity) with the apparently successful unification prediction. However such a low unification scale appears to be too low to be consistent with proton decay limits at least for the rather minimal models that would be consistent with the HW set-up (see ref.[23]).

9.3 New orbifold GUT models

There has in the past couple of years been renewed interest in the possibilities for orbifold model building, mainly focussing on attempts to solve the problems presented by proton decay (or the lack thereof) by using non-standard embeddings. Let us summarize the problems we have encountered with the simplest attempts to incorporate GUT model building into the heterotic string.

First I should re-emphasize the apparent success of the minimal MSSM, namely the apparent unification of the string, weak and hypercharge forces at about $M_{GUT} \approx 3 \times 10^{16}$ GeV. This accurate unification looks too precise to be just be coincidence. In addition the multiplets of the Standard Model fall promisingly into **16**'s of $SO(10)$ which also looks too good to be a coincidence. The identification of the SM particles is

$$\begin{aligned} \mathbf{16}: &\ Q, U^c, D^c, L, E^c, N^c \\ \mathbf{10}: &\ H_U, H_D, T, T^c \end{aligned} \tag{9.7}$$

where T, T^c are undesirable Higgs triplet superfields. On the other hand any kind of meaningful complete and reasonably minimal unification at that scale results in too rapid proton decay. There is an additional serious problem for unification, the doublet-triplet mass splitting problem; namely how to drive the mass scale of the triplet higgs fields to be of order the GUT scale while keeping the doublet components light, another form of hierarchy problem. Solving this entails extremely complicated higgs sectors.

These apparent contradictions have been addressed in a variety of new constructions based on orbifold models [24, 25, 26]. The central observation is that in orbifold models, the spectrum retains a "memory" of the underlying $E_8 \times E_8'$ structure after orbifolding. Thus one naturally finds a theory that contains "bulk" sectors with larger symmetry together with various "GUT" sectors located at the orbifold fixed points which fall into representations of the larger (unified groups), but which are missing troublesome states such as higgs triplets. This structure can explain why an apparently unified theory may lack the GUT mass states that mediate proton decay, and is directly analogous to (and was inspired by) the extra-dimensional orbifold constructions in field theory.

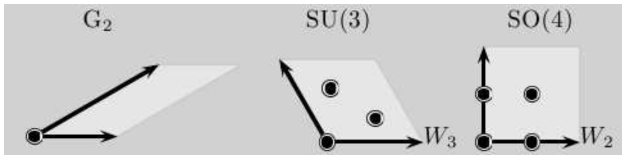


Figure 8: Compactification lattice for the model of ref.[26]. The first two torus factors are compactified at the string scale. The third torus is compactified at order the GUT scale.

As an example consider the model of ref.[26]. The compactification lattice for this model is the T^6/Z_{6-II} orbifold of ref.[24]. The T^6 compactification lattice is the product of G_2, $SU(3)$ and $SO(4)$ root lattices as shown in figure (8) [6]. Specifically, the orbifold action is given by the vector

$$v = \left(-\frac{1}{6}, -\frac{1}{3}, \frac{1}{2}\right) , \tag{9.8}$$

which generates both Z_3 and Z_2 twists;

$$v_3 = 2v \, , \; v_2 = 3v \, . \tag{9.9}$$

The Z_3 twist leaves the $SO(4)$ plane invariant and the Z_2 twist leaves the $SU(3)$ plane invariant. The orbifolding is embedded into the gauge degrees of freedom with a simultaneous phase shift (using the fermionic formulation for the internal gauge degrees of freedom). In order to do this we bosonize the 16 internal right-moving degrees of freedom. The orbifold twist is embedded into the gauge side by the vector

$$V = \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{3}\right)\left(\frac{17}{6}, \left(-\frac{5}{2}\right)^6, \frac{5}{2}\right) . \tag{9.10}$$

The Wilson lines are given by a set of 16-vectors W_a such that $U_a = \text{diag}\,\{exp(2\pi i W_a)\}$. In the model of ref.[26] there are two Wilson linaes W_2 and W_3 (the suffix indicating the order of the Wilson line) in the $SO(4)$ and $SU(3)$ planes respectively, given by

$$\begin{aligned} W_2 &= \left(-\frac{1}{2}, 0, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0^3\right)\left(\frac{23}{4}, -\frac{25}{4}, -\frac{21}{4}, -\frac{19}{4}, -\frac{25}{4}, -\frac{21}{4}, -\frac{17}{4} \frac{17}{4}\right) \\ W_3 &= \left(-\frac{1}{6}, \frac{1}{2}, \left(-\frac{1}{6}\right)^5\right)\left(0, -\frac{2}{3}, \frac{1}{3}, \frac{4}{3}, -1, 0^3\right) . \end{aligned} \tag{9.11}$$

[6] taken from ref.[26] with kind permission of the authors

Note that these models are non-standard embeddings in the sense that the twists and shifts act on both E_8 factors. In order to accommodate the successful prediction of gauge unification, the third torus is compactified at a scale M_{GUT}^{-1}. The first two tori are compactified at the fundamental scale. Thus above the GUT scale and below the string scale the theory can be described with an effective 6-dimensional T_2/Z_2 theory. This effective orbifold GUT theory is shown in figure 9.

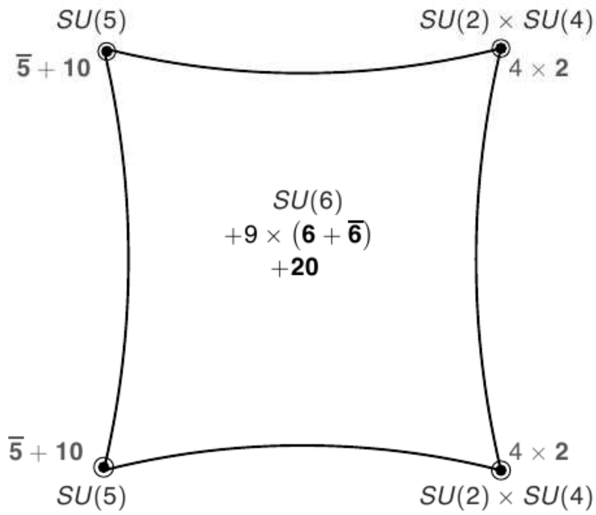


Figure 9: The effective 6-dimensional field theory approximation to the model of ref.[26] above the GUT scale.

The bulk contains an enhanced $SU(6)$ symmetry. The fixed points have the symmetry further projected by the twisting of the third torus. The intersection of all the different preserved symmetries is the SM gauge group with some extra $U(1)$ factors that are anomalous and hence heavy by the Green-Schwarz mechanism. Some additional SM gauge singlets are present and get VEVs to generate the required Yukawa couplings of the SM via the Froggat-Nielsen mechanism.

9.4 The fermionic construction in 4D

Before closing this introduction to closed string phenomenology, I would like to give an honorable mention to the so-called fermionic formulation of the heterotic string. In this one extends the fermionic formalism which I used to derive the 5 perturbative models in 10 dimensions. Compactification to 4 dimensions leaves 6 superfluous bosonic degrees of freedom on both left- and right-moving sides. These can be fermionized into 12 real (or 6 complex) fermions. Including the degrees of freedom that were already there in 10 dimensions, the different models are defined by boundary vectors (for complex fermions) of the form

$$\begin{aligned} V &= [v_1, ..v_{10}\,;\; v_{11}, ..v_{32}] \\ U &= [u_1, ..u_{10}\,;\; u_{11}, ..u_{32}]\,. \end{aligned} \tag{9.12}$$

Again there is a Lorentzian convention for dot-products. Apart from the change in numbers, the rules for model building are essentially unchanged from the 10D ones (indeed since those rules were derived from the requirement of modular invariance of products of conformal field theories, they could hardly be anything else). There is one additional requirement: the models should correspond to compactified 10D supersymmetric models. In order to ensure this it is enough to specify that the choice of boundary condition leaves the supercurrent invariant. For this it is more convenient to express the theory in terms of real fermions. The boundary condition vectors are then expressed rather laboriously as

$$\begin{aligned} V &= [(v_1,v_2),(v_3,v_4,v_5),(v_6,v_7,v_8),(v_9,v_{10},v_{11}),(v_{12},v_{13},v_{14}),(v_{15},v_{16},v_{17}),(v_{18},v_{19},v_{20});\\ &\qquad v_{21},..v_{64}] \\ U &= [(u_1,u_2),(u_3,u_4,u_5),(u_6,u_7,u_8),(u_9,u_{10},u_{11}),(u_{12},u_{13},u_{14}),(u_{15},u_{16},u_{17}),(u_{18},u_{19},u_{20});\\ &\qquad u_{21},..u_{64}]\,. \end{aligned} \tag{9.13}$$

The two first entries on the left-moving side are the two transverse degrees of freedom of spacetime. The remaining fermions on the left-moving side are grouped into threes. The 10D world-sheet supercurrent is $J_+ = \psi_i \partial X^i$ and transforms into plus or minus itself on parallel transport around the world-sheet in the R or NS sectors respectively. When a single boson X is fermionized into two real fermions ψ_1 and ψ_2 say, we have an SCFT identification $i\psi_1\psi_2 \sim \partial X$, so that the supercurrent is

$$J_+ = \sum_{i=1,2} \psi_i \partial X^i + i\sum_{i=1}^{6} \psi_{3i}\psi_{3i+1}\psi_{3i+2}\,.$$

The model building rules are therefore augmented with the “triplet constraint” that

$$v_1 = v_2 = v_{3i} + v_{3i+1} + v_{3i+2} \mod(1) \qquad \forall i = 1..6\,. \tag{9.14}$$

A particularly fruitful choice of boundary vectors is the set of ref.[27], which consists of five vectors usually denoted $\{\mathbf{1}, S, b_1, b_2, b_3\}$ in the language of ref.[9]; in order to be consistent with our earlier introduction, I will continue with the notation of ref.[7];

$$
\begin{aligned}
W_0 &= \Big[\frac{1}{2}^2, (\frac{1}{2},\frac{1}{2},\frac{1}{2})(\frac{1}{2},\frac{1}{2},\frac{1}{2})(\frac{1}{2},\frac{1}{2},\frac{1}{2})(\frac{1}{2},\frac{1}{2},\frac{1}{2})(\frac{1}{2},\frac{1}{2},\frac{1}{2})(\frac{1}{2},\frac{1}{2},\frac{1}{2});\\
&\qquad (\frac{1}{2},\frac{1}{2},\frac{1}{2})(\frac{1}{2},\frac{1}{2},\frac{1}{2})(\frac{1}{2},\frac{1}{2},\frac{1}{2})(\frac{1}{2},\frac{1}{2},\frac{1}{2})(\frac{1}{2},\frac{1}{2},\frac{1}{2})(\frac{1}{2},\frac{1}{2},\frac{1}{2}), (\frac{1}{2})^{10}(\frac{1}{2})^{16}\Big]\\
W_1 &= \big[0^2, (0,0,0)(0,0,0)(0,0,0)(0,0,0)(0,0,0)(0,0,0);\\
&\qquad (\frac{1}{2},\frac{1}{2},\frac{1}{2})(\frac{1}{2},\frac{1}{2},\frac{1}{2})(\frac{1}{2},\frac{1}{2},\frac{1}{2})(\frac{1}{2},\frac{1}{2},\frac{1}{2})(\frac{1}{2},\frac{1}{2},\frac{1}{2})(\frac{1}{2},\frac{1}{2},\frac{1}{2}), (\frac{1}{2})^{10}(\frac{1}{2})^{16}\Big]\\
W_2 &= \Big[0^2, (0,\frac{1}{2},\frac{1}{2})(0,\frac{1}{2},\frac{1}{2})(\frac{1}{2},0,\frac{1}{2})(\frac{1}{2},0,\frac{1}{2})(\frac{1}{2},0,\frac{1}{2})(\frac{1}{2},0,\frac{1}{2});\\
&\qquad (0,\frac{1}{2},\frac{1}{2})(0,\frac{1}{2},\frac{1}{2})(\frac{1}{2},0,\frac{1}{2})(\frac{1}{2},0,\frac{1}{2})(\frac{1}{2},0,\frac{1}{2})(\frac{1}{2},0,\frac{1}{2}),\ (0)^{10}(\frac{1}{2})^{16}\Big]\\
W_3 &= \Big[0^2, (\frac{1}{2},0,\frac{1}{2})(\frac{1}{2},0,\frac{1}{2})(0,\frac{1}{2},\frac{1}{2})(0,\frac{1}{2},\frac{1}{2})(\frac{1}{2},\frac{1}{2},0)(\frac{1}{2},\frac{1}{2},0);\\
&\qquad (\frac{1}{2},0,\frac{1}{2})(\frac{1}{2},0,\frac{1}{2})(0,\frac{1}{2},\frac{1}{2})(0,\frac{1}{2},\frac{1}{2})(\frac{1}{2},\frac{1}{2},0)(\frac{1}{2},\frac{1}{2},0),\ (0)^{10}(\frac{1}{2})^{16}\Big]\\
W_4 &= \Big[0^2, (\frac{1}{2},\frac{1}{2},0)(\frac{1}{2},\frac{1}{2},0)(\frac{1}{2},\frac{1}{2},0)(\frac{1}{2},\frac{1}{2},0)(0,\frac{1}{2},\frac{1}{2})(0,\frac{1}{2},\frac{1}{2});\\
&\qquad (\frac{1}{2},\frac{1}{2},0)(\frac{1}{2},\frac{1}{2},0)(\frac{1}{2},\frac{1}{2},0)(\frac{1}{2},\frac{1}{2},0)(0,\frac{1}{2},\frac{1}{2})(0,\frac{1}{2},\frac{1}{2}),\ (0)^{10}(\frac{1}{2})^{16}\Big]
\end{aligned}
\tag{9.15}
$$

The theory with just $W_{1,2}$ is a supersymmetric $SO(44)$ model. The three additional projection vectors cut down the gauge group (without cutting the rank) into the the obvious subgroups. Note that the space-time side is embedded in the gauge side in the sense that the 18 “compactified” degrees of freedom are simply copied over to the right-movers. This is analogous to the standard embedding of the orbifold models. In addition 10 of the world-sheet fermions on the gauge side play the same role as the transverse fermions on the left-movers; the equivalent of $\mathcal{N}=1$ spacetime supersymmetry on the gauge side is an E_6 gauge group factor. This is broken by the projections to $SO(10)$. Finally the last 16 fermions generate an E_8b group factor. The gauge group is then $SO(10)\times SO(6)^3\times E_8$. The untwisted sector gives rise to vector-like pairs of **10**s of $SO(10)$ which can play the role of Higgses in the Standard Model. The 3 sets of twisted sectors lead to 48 multiplets in **16**s of $SO(10)$ (16 from each sector) – these will eventually lead to matter.

The second stage of the model adds at least 3 more vectors to project the $SO(10)$ down to one of its subgroups, a particularly interesting possibility being then $SU(3)\times SU(2)\times U(1)^2$ [28]. I will not show the vectors explicitly, but suffice to say that the 10 fermionic degrees of freedom are complexified (into 5 complex fermions) and given boundary conditions of e.g. $(\frac{1}{4})^5$ in the final boundary vector. The final models can be very close to the Standard Model with 3 matter generations remaining from the original 48, naturally heavy (i.e. string scale) Higgs triplets and suitable Yukawa couplings. For more details and a complete discussion of the phenomenology see ref.[29].

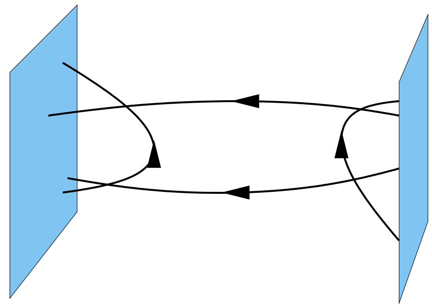


Figure 10: D-brane realization of a U(2) gauge group.

10. Open string models: string at singularities

The arrival of the large extra dimension idea stimulated interest in the other variants of string theory as model building tools. In particular attention turned to the type I and type II theories which have in their nonperturbative spectrum objects known as Dirichlet branes [30, 31]. These can be built like monopoles from the effective field theory, and are membrane-like and fully dynamical, with a typical surface tension and a width of order the fundamental scale (divided by the string coupling). As we argued earlier based on the types of R-R fields in the particular models, they have p dimensions on their world volume where $p = 1,3,5,7,9$ for type IIB, $1,5,9$ for type I and $0,2,4,6,8$ for type IIA. The interesting feature of D-branes from a model builder's point of view is that open strings can end on them and this can generate gauge groups in the following way. Associated with an open string end point is an index, the Chan-Paton index. If there are a few branes together, the index simply labels the branes to which the open string is attached. If we consider two branes for example, the endpoints can be attached in one of 4 ways as in Figure 10.

What do we see when we observe this from 4 dimensions? Remember that from the 4 dimensional point of view we need to arrange things such that the compactified space is the same everywhere. In particular the brane must be lying in the large M_4 space that we observe in order for the open string to be able to travel along it (otherwise it would be stuck at a single point in M_4. So the branes must have $p \geq 3$. (If $p = 3$ the branes appear as points in the compactified space.) Given this, the open strings may freely propagate in M_4 but have 4 internal degrees of freedom corresponding to the adjoint of U(2). As we shall see later, these degrees of freedom are included in the perturbation theory by adding "Chan-Paton indices" on the vertex operator which represents the emission of such a state.

It also turns out that the strings have to have an excitation from the brane volume giving them a Lorentz (gauge boson) or internal (matter field) index. Finally a remarkable feature of D-branes is that they break only half the supersymmetry (i.e. they are BPS). The original theory which as we saw has $N = 8$ supersymmetry in 4 dimensions (if the compactified space is toroidal) ends up being $N = 4$. We thus end up with an $N = 4$ theory with $U(2)$ gauge group. In order to reach a more phenomenogically interesting $N = 1$ configuration, the compactified space K_6 can be chosen in such a way that the supersymmetry is already partially broken before the D-branes are added. The simplest (i.e. most calculable) way to achieve this is to use orbifolds as the background.

Before we start throwing branes together at random, we need to take care of some consistency conditions. The most important of these for D-branes are the famous Ramond-Ramond tadpole conditions. As we saw every D-brane has a “Ramond-Ramond” (RR) charge, and couples to the Ramond-Ramond fields of the closed string spectrum. Since these are closed string states they do not care about the presence or otherwise of the D-branes. In a toroidal compactification they propagate throughout the entire compactified volume. Curvature singularities, for example when the compactified space is an orbifold, introduce a second type of “twisted” RR fields that are confined to the fixed point. The RR fields behave like gravitons and dilatons and form part of the gravitational spectrum. However they differ in the respect that flux lines of Ramond-Ramond fields must be absorbed in a compact space otherwise the theory is inconsistent. One has to be careful therefore to choose the arrangements of D-branes such that the flux lines are all absorbed. Once this requirement is satisfied, other requirements such as anomaly cancellation are usually satisfied as well.

These requirements led to an approach to model building which became known as “bottom-up” [32]. Consider what are the important features of any model from the point of view of phenomenology. The leading factors are those things that have to do with the gauge groups, particle content, number of generations and so on. Secondary factors are things that have to do with supersymmetry breaking, the cosmological constant etc. The latter are things whose eventual properties are intertwined with gravity. As such their influence on phenomenology is less important. In a large extra dimension set-up, the correspondence with the configuration in the compactified space is rather direct. The primary factors have to do with the local arrangements of D-branes around, for example, some orbifold fixed point, whereas the secondary factors are all associated with objects far away in the bulk of the compactified space. For example a “hidden” sector can be included consisting of a collection of branes at some *other* fixed point far away in the compactified space. The communication to the visible sector then has to be through the bulk, and will get the same volume suppression as that felt by gravity. This is shown schematically in Figure 11. The points represent for example D3 branes localized at some point in the compactified space with twisted RR flux cancelled locally. These are chosen in such a way that the visible sector is the MSSM. Gravity and the untwisted RR fields live in the bulk of the compactified space. These details and in particular the details of untwisted RR flux cancellation are less well determined.

The bottom-up approach begins therefore by focussing on the local MSSM configuration. We assume an intermediate fundamental scale of

$$M_I \sim \sqrt{M_W M_{Pl}} \sim 10^{11}\, GeV. \tag{10.1}$$

This scale is familiar from the hidden sector supersymmetry breaking communicated by gravity and had been suggested earlier on more general grounds to do with supersymmetry breaking and mediation by gravity [33]. First a set of D-branes is included at some fixed point of K_6 with all the necessary elements to make up the standard model gauge group and leave $N = 1$ supersymmetry in the visible sector. This can for example be a set of D3-branes lying on top of each other at a single point in K_6, but with their world volumes filling the whole of M_4 (as of course we require if the open strings on their world volumes are able to travel anywhere in M_4). We then need to satisfy the requirements of local RR-tadpole cancellation. That is we need to add in additional branes (D7 branes for example) such that the “twisted” RR-tadpoles cancel but locally supersymmetry is preserved. This puts a constraint on the angles at which the branes can interesect (for example

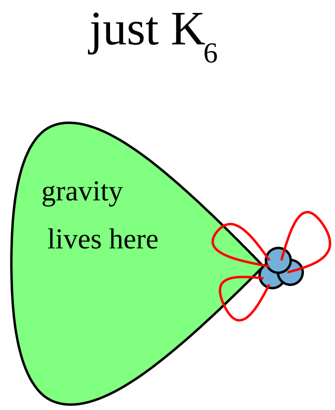


Figure 11: Schematic picture of the bottom-up approach. The small blue points represent the local configuration of D-branes leading to the MSSM whereas the large green blob represent the global structure, less important from a phenomenological viewpoint.

that the D7 branes intersect at right angles). This arrangement takes care of the local consistency conditions, however one should also take care of the global RR-tadpoles and make sure those fluxes cancel as well. This can be done by adding other D-branes and anti-D branes elsewhere in the bulk or may be done in some other way. From the point of view of 4D phenomenology therefore, the particular way in which the global tadpoles are cancelled affects only the hidden sector, and consequently the soft supersymmetry breaking and cosmological constant. A consistent set-up is shown schematically in Figure 12. This figure shows the global RR flux being absorbed by anti-branes, but the set-up can be entirely different away from the visible sector without affecting the MSSM set-up directly.

The reason for the particular choice of the intermediate scale can now be made clear. The additional ingredients required to ensure global tadpole cancellation generally break supersymmetry. Since it is only the global configuration that breaks supersymmetry, the net effect is the same as hidden sector supersymmetry breaking communicated by gravity and we must choose the fundamental scale accordingly. In other words, the volume of the bulk can be responsible for the large Planck scale and the dilution of supersymmetry breaking effects only if $M_s \sim M_I$. The precise dependences on volumes can be derived from the reduction of the effective 10 dimensional type I action to 4 dimensions [34]. We begin with the Planck mass relation to the total compact volume

$$V_{K_6} = \lambda_I^2 \frac{M_P^2}{M_s^2}, \tag{10.2}$$

where λ_I is the string coupling. To get an idea of what this has to be, we can look at the effective gauge coupling α_p on a p-dimensional brane. The gauge interactions are proportional to the string coupling but are diluted by the volume of the branes in the compactified space, V_{p-3}, since the gauge bosons are free to roam anywhere in this volume. Hence

$$\alpha_p \sim \frac{\lambda_I}{V_{p-3}}. \tag{10.3}$$

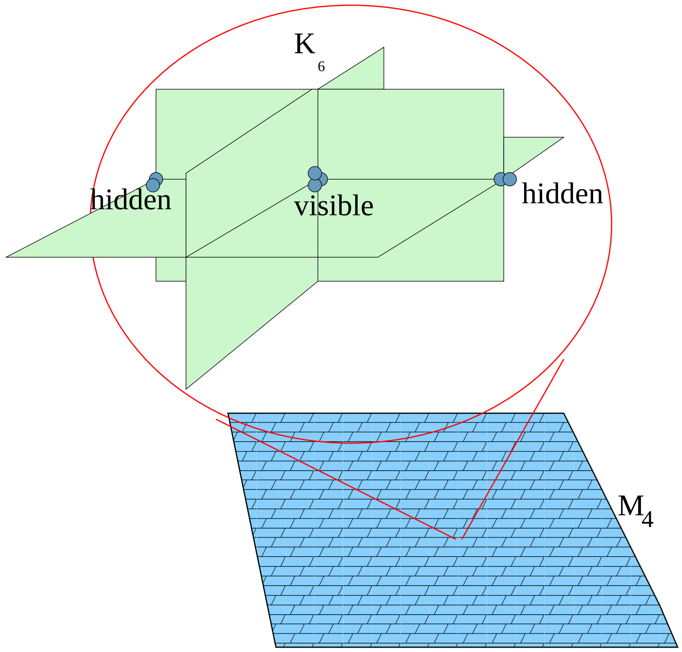


Figure 12: Set-up for the bottom up approach. The visible sector consists of 3-branes at a fixed point in K_6. D7 branes have to be included passing through this fixed point to cancel local RR-tadpoles. Global absence of tadpoles requires additional branes and/or anti-branes in the bulk, or possibly something else entirely.

Substituting Eq. (10.3) into Eq. (10.2) gives us

$$\alpha_p M_P^2 = M_s^2 \frac{V_{9-p}}{V_{p-3}}, \tag{10.4}$$

where V_{9-p} is the co-volume (i.e. the volume orthogonal to the p brane). Any process we care to calculate that breaks supersymmetry, such as a contribution to the scalar mass-squareds communicated via closed string modes from an anti-brane, feels the same volume dependence

$$m_{SUSY}^2 \sim M_s^2 \frac{V_{p-3}}{V_{9-p}}. \tag{10.5}$$

The dilution due to the co-volume V_{9-p} is obvious. The V_{p-3} enhancement factor arises from the sum over Kaluza-Klein (momentum) modes in the brane volume and is essentially the same factor as that arising in $1/\alpha_p$. Essentially this is like a phase space factor. (As a rule-of-thumb, one can use the fact that if we invert a radius, $R_i \to 1/R_i$, we also turn that dimension from a brane dimension into a dimension orthogonal to the brane or vice-versa, and also change the dimensionality of the brane, $p \to p \pm 1$. Hence the volumes must appear as the ratio of brane volume to co-volume, V_{p-3}/V_{9-p}.) There is no $1/\lambda_I$ contribution as there is in the tree level Yang-Mills terms (hence the equation for α_p) because the diagrams that contribute to M_{SUSY} are one-loop and λ_I acts like a loop expansion parameter.

Now, for reasonable phenomenology we would like $M_{SUSY} \sim M_W$ so that from the above, and assuming that we have $\alpha_p \sim 1$ we need

$$M_s^2 \sim M_W M_P$$

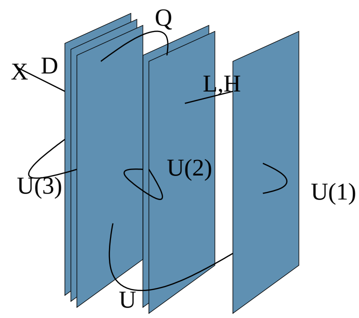


Figure 13: Local arrangement of states on D3-branes leading to the MSSM.

as expected, and consequently a volume ratio

$$\frac{V_{p-3}}{V_{9-p}} \sim \frac{M_W}{M_P} \tag{10.6}$$

The beauty of the bottom-up approach is that is allows us to disregard those parts of the construction that are not vital to phenomenology. For example there is a question of global validity of these models due to the fact that there are uncancelled tadpoles of another kind, namely NS-NS tadpoles. These however can be absorbed dynamically by adjusting the background (i.e. K_6) and their presence does not automatically render the theory inconsistent [35]. Although this effect may make the theory intractible on a global scale, it may still be a reasonable approximation to assume a nice (tractable) flat or orbifold background near the visible sector branes, where we can still calculate, for example, interactions.

Let us turn briefly to the local arrangement of branes that yields the visible sector particle content and gauge group. This is often represented as in Figure 13. The Figure shows the arrangement of $D3$ branes at a particular fixed point in K_6. The branes are extended in M_4 and fixed in K_6 so that two of the dimensions shown are in M_4 and the dimension orthogonal to the branes should be in K_6. In addition the branes are on top of each other. (Any separation of branes translates into a mass for the relevant states due to the stretching energy.) There are three stacks of branes corresponding to a gauge group $U(3) \times U(2) \times U(1)$. The gauge states are those strings with ends attached on a single stack of branes. The matter states correspond to strings stretched between different stacks of branes and consequently appear (in this simple example) in the bifundamental. Thus we can identify strings stretched between the $U(3)$ and $U(2)$ stacks with left handed quarks, Q_L, between the $U(2)$ and $U(1)$ branes with left handed leptons and higgses, and between the $U(3)$ and $U(1)$ branes with right handed quarks. The gauge groups contain too many $U(1)$ factors, and the final reduction down to a single $U(1)_Y$ of hypercharge comes about because there is only one linear combination of $U(1)'s$ that is anomaly free. Of course string theory is a consistent theory, and there should be no anomalies at all. But the way in which string theory cancels the anomalies makes the *naively* anomalous $U(1)'s$ massive, and one expects that the anomalous combinations will be broken. Remarkably the states turn out to have the hypercharge assignments of the SM. This crucial stringy anomaly cancellation (the Green-Schwarz mechanism) is represented schematically in figure

The bottom up approach has a number of advantages, many of which were outlined in Refs.[32, 36]. For example the prediction of an intermediate fundamental scale is interesting for a number of

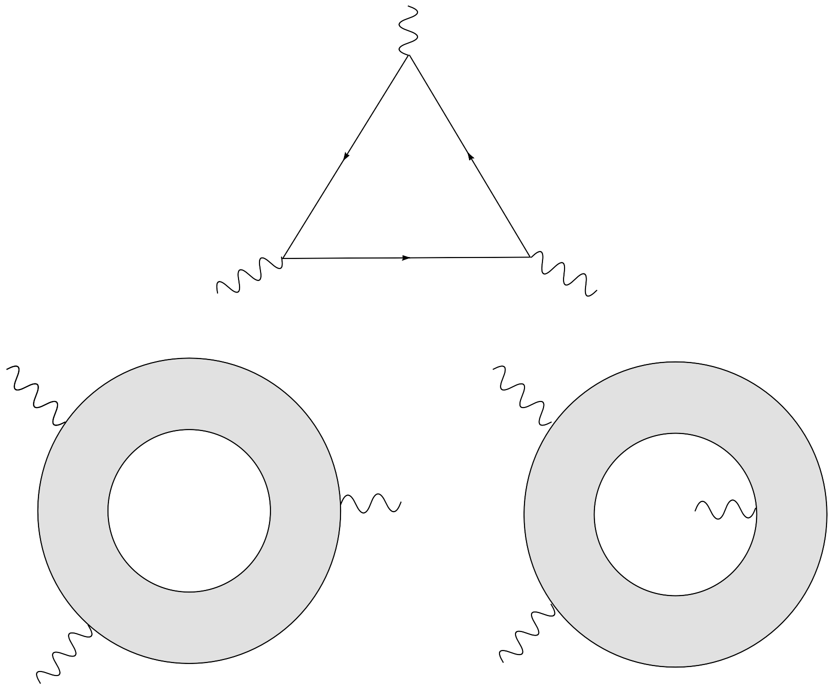


Figure 14: Schematic representation of the Green-Schwarz anomaly cancellation mechanism which is required to understand the cancellation of $U(1)$ anomalies. The upper diagram is the usual field theoretic diagram. In the string theory the anomaly contributions are built out of the left lower diagram (which has a field theory limit equivalent to the upper diagram), and the right lower diagram which is "stringy". The latter corresponds to the coupling of an open string $U(1)$ photon, to closed string modes which then emit two open string gauge bosons. Note that this process is a *tree-level* propagation of a closed string.

reasons. It is a natural realization of hidden sector supersymmetry breaking communicated by gravity. The model provides axions with just the right Peccei-Quinn scale to allow an axion solution to the string CP problem. In addition the see-saw mechanism for neutrino masses is consistent with a fundamental intermediate scale, and so on. One of the disadvantages of the bottom-up approach is that, by its very nature it is difficult to make concrete predictions of phenomenological implications. This is because the approach begins with a visible sector that resembles the MSSM and, by construction, aspects such as supersymmetry breaking have to do with the global configuration over which we assume very little control.

11. Intersecting branes

As we saw earlier, classical strings can be trapped at the intersection of two branes. If one imagines D-branes of some dimensonality wrapping a compact space, there is therefore the possibility that upon quantization these intersection states could lead to interesting low energy spectra. In particular, D-branes are (BPS) and so preserve half the supersymmetries, however D-branes intersecting at different angles will preserve *different* supersymmetries, and so one may hope to get chiral $\mathcal{N}=1$ spectra, and even break all the supersymmetry this way.

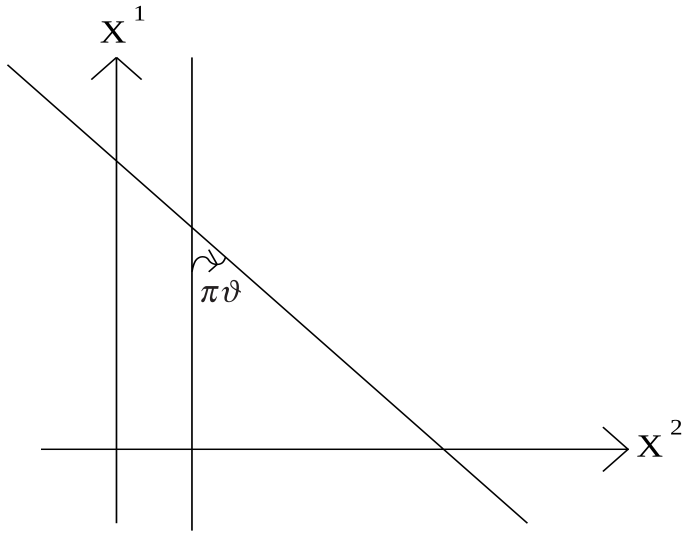


Figure 15: A 'twisted' open string state - the angle is $\pi\vartheta$.

11.1 Quantizing the intersection

In order to carry out this program it is first necessary to understand what happens at the intersection of two branes a little better. It turns out that the states here are very similar to twisted states on orbifolds.

Let us recap and extend what we saw in section 4.4. This will give me a chance to introduce a more convenient complex worldsheet coordinate to replace σ and τ. An open string stretched between two D-branes intersecting at an angle $\pi\vartheta$, as depicted in figure 15, has the boundary conditions,

$$\begin{aligned} &\partial_\tau X^2(0) = \partial_\sigma X^1(0) = 0, \\ &\partial_\tau X^1(\pi) + \partial_\tau X^2(\pi)\cot(\pi\vartheta) = 0, \\ &\partial_\sigma X^2(\pi) - \partial_\sigma X^1(\pi)\cot(\pi\vartheta) = 0. \end{aligned} \tag{11.1}$$

Thus the correct holomorphic solutions to the string equation of motion are,

$$\begin{aligned} \partial X(z) &= \textstyle\sum_n \alpha_{k-\vartheta} z^{-n+\vartheta-1}, \\ \partial \overline{X}(z) &= \textstyle\sum_n \overline{\alpha}_{n+\vartheta} z^{-n-\vartheta-1}, \end{aligned} \tag{11.2}$$

where I have intriduced $z = -e^{\tau - i\sigma}$ as the worldsheet coordinate with domain the upper-half complex plane. (Note I am using X to stand for the complex coordinate $X_1 + iX_2$ rather than the Z of section 4.4, to avoid confusion.)

This domain is often extended to the entire complex plane using the 'doubling trick', i.e. we define,

$$\partial X(z) = \begin{cases} \partial X(z) & \mathrm{Im}(z) \geq 0 \\ \overline{\partial}\overline{X}(\overline{z}) & \mathrm{Im}(z) < 0 \end{cases}, \tag{11.3}$$

and similarly for $\partial\overline{X}(z)$.

Now the mode expansion of a closed string state in the presence of a $\mathbb{Z}_N$ orbifold twist field, is identical to (11.2) with the replacement $\vartheta = \frac{1}{N}$. Hence, we see that there is a natural correspondence

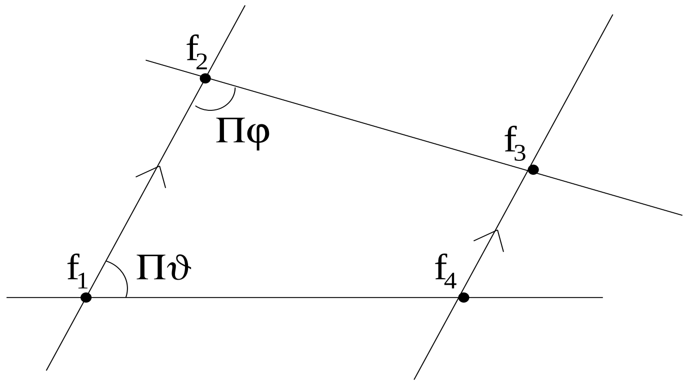


Figure 16: A set of D-branes which can lead to a nonperturbative 4 point interaction. This configuration has two independent angles (i.e. two parallel branes). In principle there could be three independent angles.

between open strings stretched between intersecting branes and a twisted closed string state on an orbifold. (To take account of this correspondence, we must introduce what's called a twist field $\sigma_\vartheta(w,\overline{w})$ into the vertex operator which represents the emission of the open string (see later). This field's job is to change the boundary conditions of X to be those of eq.(11.1), where the intersection point of the two D-branes is at $X(w,\overline{w})$.) The mode expansion for X in these coordinates is then,

$$X(z,\overline{z}) = \sqrt{\frac{\alpha'}{2}}\sum_n \left(\frac{\alpha_{n-\vartheta}}{n-\vartheta}z^{-n+\vartheta} + \frac{\tilde{\alpha}_{n+\vartheta}}{n+\vartheta}\overline{z}^{-n-\vartheta}\right), \tag{11.4}$$

with the right and left moving modes being mapped into upper and lower half planes. A similar mode expansion is obtained for the fermions with the obvious addition of $\frac{1}{2}$ to the boundary conditions for NS sectors.

Quantization then proceeds in the usual manner. In particular the spectrum can be written as follows. Introduce a twist vector ϑ_i with 4 entries representing 4 complex coordinates (three internal and one transverse space time). Introduce a lattice of excitations $r_i \in \mathbf{Z}, \mathbf{Z}+\frac{1}{2}$ for **NS** or **R** sectors respectively. Then the GSO projected spectrum of open string stretched between two branes is given by

$$\alpha' M^2 = \frac{L^2}{4\pi^2\alpha'} + N_{bosonic} + \frac{(r+\vartheta)^2}{2} - \frac{1}{2} + a_\vartheta \tag{11.5}$$

where

$$a_\vartheta = \sum_i \frac{1}{2}|\vartheta_i|(1-\vartheta_i), \tag{11.6}$$

and where $N_{bosonic}$ represents the obvious contribution from bosonic oscillators. Here L is the displacements between the branes (of course in more than two dimensions, they can be separated but at angles). This is the classical stretching energy; i.e. the groundstate is the stretched string which already has classical energy $LT = L/2\pi\alpha'$.

The correspondence with the spectrum of twisted states on orbifolds can be understood geometrically as in figure 17. This figure shows two identical three point diagrams which are sewn together at their edges. An open string living at the intersection is doubled up to form a twisted closed string. As a result we expect to find a open string $\sim \sqrt{\text{closed string}}$ relation. However, we

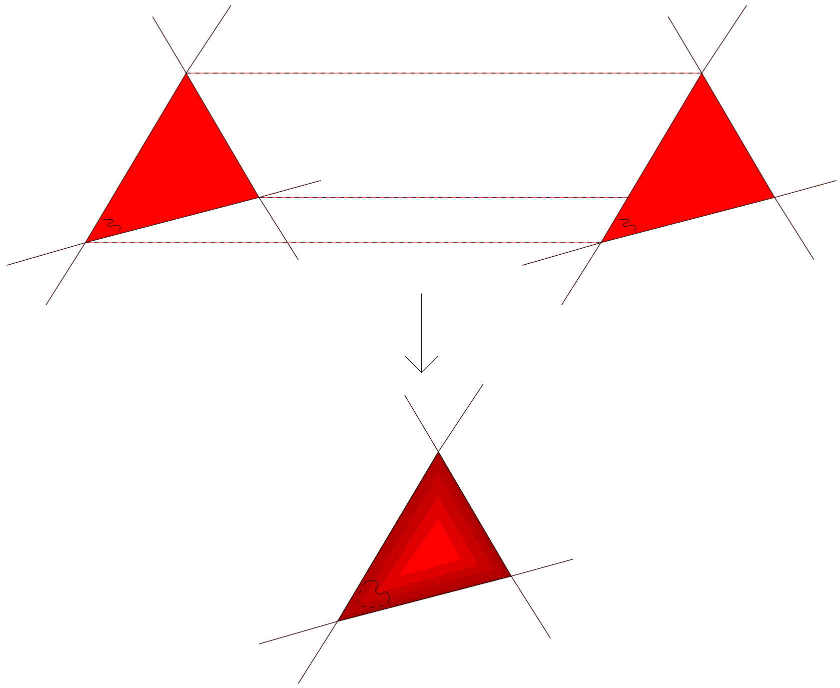


Figure 17: Identifying open strings to form closed strings

also note that the intersection angles in this case are more general than the rather restrictive ones found in supersymmetric orbifolds of closed strings.

11.2 $M_s \sim$TeV: Branes at angles

Having understood the open string spectrum, at least a little, we will discuss now a class of models that represent, within a bottom-up approach, realistic string models with many of the features of the SM, allowing in principle for a very low string scale. Our main aim in this review is to account for their phenomenological features, their realistic structure and, especially, their flavour structure, which, as it turns out, provides the deepest probe of this kind of models and the most stringent constraints on the string scale as well.

Models with D-branes intersecting at non-trivial angles [5] (see [37] for an earlier application of the same idea, in the dual version of branes with fluxes, to supersymmetry breaking), have a number of very appealing phenomenological features such as for instance four-dimensional chirality or a reduced amount of symmetries (both gauge and supersymmetries) among many others. One particularly important feature that these models have is an attractive explanation for family replication. Specifically the matter fields correspond to the string states at the intersections that are stretched between two branes. There are then three generations simply because the branes are wrapped so that each type of intersection appears three times, with a repeated set of multiplets stretched between the branes at the intersections.

In particular, configurations with branes at angles typically break all the supersymmetries (supersymmetric configurations have been constructed [38] but they are very constrained and minimal models are very difficult to obtain) and therefore a very low string scale $\sim$ TeV is required. The first semi-realistic models were constructed in [39] and soon after in [40] and [41] (see [42] for some related technical developments). These initial models presented additional gauge symmetries or matter content beyond the ones in the SM. The first models containing just the SM were presented in [43]. Since then, a great deal of effort has gone into into the study of the consistency and stability [44] and phenomenological implications of intersecting brane models, from the construction of supersymmetric models [38], gauge symmetry breaking [45], GUT or realistic SM constructions [46] to cosmological implications [47]. In the following we will review some of these developments paying particular attention to their flavour structure [48, 49, 50] and its profound experimental implications.

For the sake of clarity we will concentrate here on one very particular model [48] that exemplifies most of the interesting properties as well as some of the possible problems of models with branes intersecting at angles. It is an orientifold compactification of type II theory with four stacks of D6-branes wrapping factorizable 3-cycles on the compact dimensions. This mouthfull is displayed in Fig. 18 which shows just the compactified space, K_6. The compactified space is a compact factorizable 6-Torus

$$T^2 \times T^2 \times T^2,$$

and the orientifold projection is given by $\Omega\mathscr{R}$ where Ω is the world-sheet parity and $\mathscr{R}$ is a reflection about the horizontal axis of each of the three 2-tori,

$$\mathscr{R} Z_I = \bar{Z}_I.$$

We have denoted the coordinates of the tori by complex coordinates $Z_I = X_{2I+2} + \mathrm{i} X_{2I+3}$, $I = 1,2,3$, so the three boxes in the figure represent each 2 torus, with the edges being identified. Recall that the 6 branes must lie in M_4 so that there are only three dimensions of each $D6$-brane that will appear in K_6. The branes therefore appear as just lines in each T_2. The nett effect of the orientifold projection is to introduce mirror images of the branes in each T_2 (in the plane running horizontally). The images do not add any new states so we have not included them in the diagram.

This particular model contains at low energies just the particle content and symmetries of the MSSM. In order to get that we include four stacks of D6-branes, called *baryonic* (a), *left* (b), *right* (c), and *leptonic* (d). Three of the dimensions of each D6-brane wrap a 1-cycle on each of the three 2-tori, with wrapping numbers denoted by (n_k^I, m_k^I), *i.e.* the stack k wraps n_k^I times the horizontal dimension of the $I-$th torus and m_k^I times the vertical direction. We have to include for consistency their orientifold images with $(n_k^I, -m_K^I)$ wrapping numbers. The number of branes in each stack, their wrapping numbers and the gauge groups they give rise to are shown in Table 3 and a subset of them, together with some of the relevant moduli, are displayed in Fig. 18.

The open string light spectrum in these models consists of the following fields:

- $(p+1)$-dimensional gauge bosons (for the case of a stack of N Dp-branes) corresponding in general to the group $\mathrm{U}(N) \sim \mathrm{SU}(N) \times \mathrm{U}(1)$ live in the world volume of the corresponding branes. In our particular configuration, we have seven-dimensional gauge bosons corresponding to the gauge group $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)_a \times \mathrm{U}(1)_c \times \mathrm{U}(1)_d$ (see Table 3) [7]. Of

[7] Note that the left stack of branes consists of just one brane that gives rise directly to a $\mathrm{USp}(2) \sim \mathrm{SU}(2)$ gauge group instead of the usual $\mathrm{U}(1)$ due to the orientifold projection [31].

Stack	N_k	Gauge group	wrapping numbers
a	3	$SU(3) \times U(1)_a$	(1,0);(1,3);(1,-3)
b	1	$SU(2)$	(0,1);(1,0);(0,-1)
c	1	$U(1)_c$	(0,1);(0,-1);(1,0)
d	1	$U(1)_d$	(1,0);(1,3);(1,-3)

Table 3: Number of branes, gauge groups and wrapping numbers for the different stacks in the models discussed in the text.

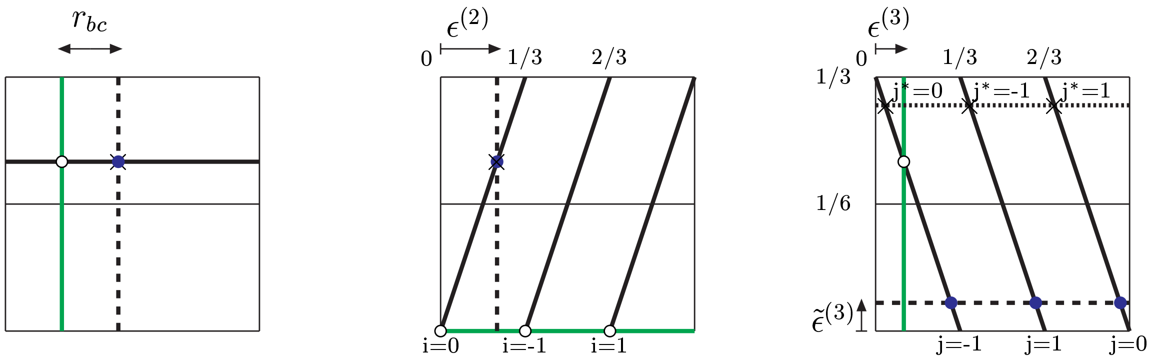


Figure 18: Brane configuration in the model discussed in the text. The leptonic sector is not represented while the baryonic, left, right and orientifold image of the right are respectively the dark solid, faint solid, dashed and dotted. The intersections corresponding to the quark doublets ($i = -1, 0, 1$), up type singlets ($j = -1, 0, 1$) and down type singlets ($j^* = -1, 0, 1$) are denoted by an empty circle, full circle and a cross, respectively. All distance parameters are measured in units of $2\pi R$ with R the corresponding radius (except $\tilde{\varepsilon}^{(3)}$ which is measured in units of $6\pi R$).

the several abelian groups, every anomalous linear combination receives a mass through the Green-Schwartz mechanism, whereas anomaly-free combinations can remain massless or not, depending on the particular brane configuration. This is indeed a salient feature of this class of models that allow non-anomalous gauge bosons to couple to the RR two-form fields acquiring a mass of the order of the string scale in this form [43]. The phenomenology of these extra massive $U(1)'$s has been studied in [51] finding a bound on the string scale $M_s \gtrsim 1$ TeV. Interestingly enough, these gauge symmetries remain at the perturbative level as unbroken global symmetries [43]. Quite generally these new global symmetries correspond to baryon, lepton, or Peccei-Quinn like symmetries, preventing proton decay even in low scale models. In our particular example, the anomaly free massless combination corresponding to the hypercharge is

$$Q_Y = \frac{1}{6} Q_a - \frac{1}{2}(Q_c + Q_d).$$

- Four-dimensional chiral massless fermions living on the intersections of two branes and transforming as bi-fundamentals of the corresponding gauge groups. Their number depend on a topological invariant, the intersection number, which in the case of factorizable cycles on a factorizable torus is simply

$$\mathscr{I}_{ab} = \prod_{I=1}^{3} (n_a^I m_b^I - m_a^I n_b^I),$$

with different signs corresponding to different chiralities. The fact that these branes wrap compact dimensions naturally provide intersection numbers greater than one and therefore replication of fermions with the same quantum numbers. It should be mentioned here that in the case of lower-dimensional branes, like D5 or D4-branes, chirality is not automatic and locating the whole configuration at orbifold singularities is required in order to get it [41].

- Four-dimensional scalars, also localized at the intersections, with masses that depend on the particular configurations of the branes. They can be seen as the (generally massive when SUSY is broken by the intersection) superpartners of the fermions at the intersections. In realistic models, scalars with the quantum numbers of the (MS)SM Higgs boson also exist. In the example we are considering the configuration is such that the same supersymmetry is preserved at each of the intersections and massless scalars, superpartners of the corresponding fermions completing the matter spectrum of the MSSM live at the intersections.

The massive spectrum comprises, apart from the usual winding and KK modes and string excitations not related to the intersections normally present in string models, a set of massive vector-like fermions, the so-called *gonions* [41], localized near the intersections and with angle-dependent masses. Although a purely effective field theory study shows that relatively light vector-like fermions, especially when they mix with the top quark, are the most likely source of modifications of trilinear couplings [52], the presence of Flavour Changing Neutral Currents in these models overcomes in general any phenomenological relevance of these states.

We have therefore seen that at the level of the light spectrum, models with intersecting branes have a number of nice features, namely four-dimensional chiral fermions, natural family replication and local and global symmetries and matter content of the SM (or simple extensions thereof). As we have seen, the closed string sector, which lives in the full ten-dimensional target space, contains among other fields the graviton. These models thus have a natural hierarchy of dimensionalities, with gravity propagating in ten dimensions, gauge interactions in seven and matter in four. As we sketched in the introduction, this will allow us to reduce the string scale down to observable levels.

In our particular example, as can be seen in Fig. 18, there are no dimensions transverse to *all* the branes and therefore no transverse volume can be made large enough to account for the large effective four-dimensional Planck mass with a small string scale. The thing that is stopping us are of course the gauge couplings which would receive the same volume suppression seen in Eq.10.3 and become extremely small. This problem can be circumvented in several ways, the simplest one is to connect our small torus to a large volume manifold without affecting the brane structure [53], for instance, cutting a hole and sewing and large volume manifold in a region away from the branes [8]. This approach is in spirit quite similar to the bottom-up approach. A second possibility

[8]There is a conceptual difficulty in this construction that can be phrased as why in such a large volume manifold, the relevant physics occurs in such a tiny region. This difficulty is in one way or another always present in the large extra dimensions approach to the hierarchy problem but, as we have emphasized, the vacuum degeneracy problem makes this possibility at least conceivable in a stringy set-up.

is to consider lower-dimensional branes, for which transverse dimensions to all branes do exist. Realistic examples with D5-branes and a string scale as low as *few* TeV have been constructed in [54]. (See also [55] for other examples with extra vector-like fermions.) In these models the effective four-dimensional Planck mass reads

$$M_P = \frac{2}{\lambda_{II}} M_s^4 \sqrt{V_4 V_2}, \tag{11.7}$$

where $V_{4,2}$ stand for the volume of the four-dimensional manifold where the branes wrap and the volume of the two-dimensional one transverse to all the branes and λ_{II} is the string coupling and M_s is the string scale. In this situation it is possible to have all scales of order TeV but the transverse dimensions then have to be $\sim mm$ [4].

Gauge couplings can be simply computed from a dimensional reduction of the Yang-Mills theory living on the world-volume of the stack of branes. As expected, it is suppressed by the volume of the compact dimensions of the brane,

$$\frac{1}{g_a^2} = \frac{M_s^3}{16\pi^4 \lambda_{II}} V_a, \tag{11.8}$$

where we have considered the case at hand, *i.e.* D6-branes wrapping 3-cycles on the compact space and considered the gauge coupling of an $\mathrm{SU}(N_a)$ group. Reasonable values for the couplings are obtained if the relevant volume for the brane is $V_a \sim M_P^3 \sim \mathrm{TeV}^3$. Contrary to the original expectation, under certain mild assumptions, gauge coupling unification can be obtained [56] (see also [57] for a study of gauge threshold corrections in intersecting brane models).

Models with intersecting branes therefore allow in principle for a very low string scale, $M_s \sim 1$ TeV, while keeping the Planck mass (11.7) and the gauge couplings (11.8) at the observed values. Notice as well that in the case of non-supersymmetric models, a low string scale is preferred to avoid large corrections to the Higgs vev.

11.3 Globally consistent models

We have not yet elaborated on the details of the construction and their consistency conditions such as the absence of Ramond-Ramond tadpoles or the presence of unbroken supersymmetries. These conditions greatly restrict the number of possibilities, usually requiring the presence of more complicated spaces by further orbifolding and orientifolding the toroidal structure we have discussed. Nice reviews are given in refs. [58, 59].

12. Interactions, esp. Yukawa couplings

In order to do much meaningful phenomenology, one needs information about the superpotential. In string models in flat backgrounds it is rather satisfying that the interactions in the superpotential can be computed with standard CFT techniques. Often these can yield finite results for amplitudes that in the equivalent extra dimensional field theory would be badly behaved. The techniques are common to both closed and open strings and I will concentrate on the latter.

12.1 Perturbation theory

In order to discuss the calculation of couplings, let us briefly return to the basics, and develop the formalism for doing perturbation theory in flat backgrounds. In particular we need to be able to describe the emission and absorption of physical states. Just as in field theory, we may define a perturbation series expansion for string scattering amplitudes. Perturbation theory with strings is potentially superior to perturbation theory with particles for two reasons. Firstly, since the different elements of a string worldsheet contain a multiple of string states simultaneously, a single string diagram contains many Feynman diagrams: string theory is potentially much more efficient than field theory. Second, there is no unique point in spacetime at which all observers will agree that a string interaction takes place. In field theory, propagators coming together at a well-defined interaction point lead to ultraviolet divergences, but since the interaction point in string diagrams is in this sense 'smeared out' over spacetime, UV divergences are avoided.

In field theory, terms in the perturbation expansion of a scattering amplitude are ordered topologically according to the number of loops in the Feynman diagram, and the expansion parameter is taken to be the coupling strength of the field. In string theory, terms are also ordered topologically: we add a term $\lambda\chi$ to the action (4.44), where χ is the *Euler number*,

$$\chi = \frac{1}{4\pi}\int_M d\sigma d\tau\sqrt{-\gamma}R + \frac{1}{2\pi}\int_{\partial M} dsk\,. \tag{12.1}$$

Here, R is the Ricci scalar for a given worldsheet M with boundary ∂M, and k is the extrinsic curvature of the worldsheet. This term is not dynamical, and does not affect the spectrum found above: instead, its effect is to weight the action by a factor which depends only on the topology of the worldsheet. The perturbation expansion parameters are taken to coupling strength of open and closed strings, g_O and g_C respectively.

The Euler number may also be expressed as

$$\chi = 2 - 2h - b - c\,, \tag{12.2}$$

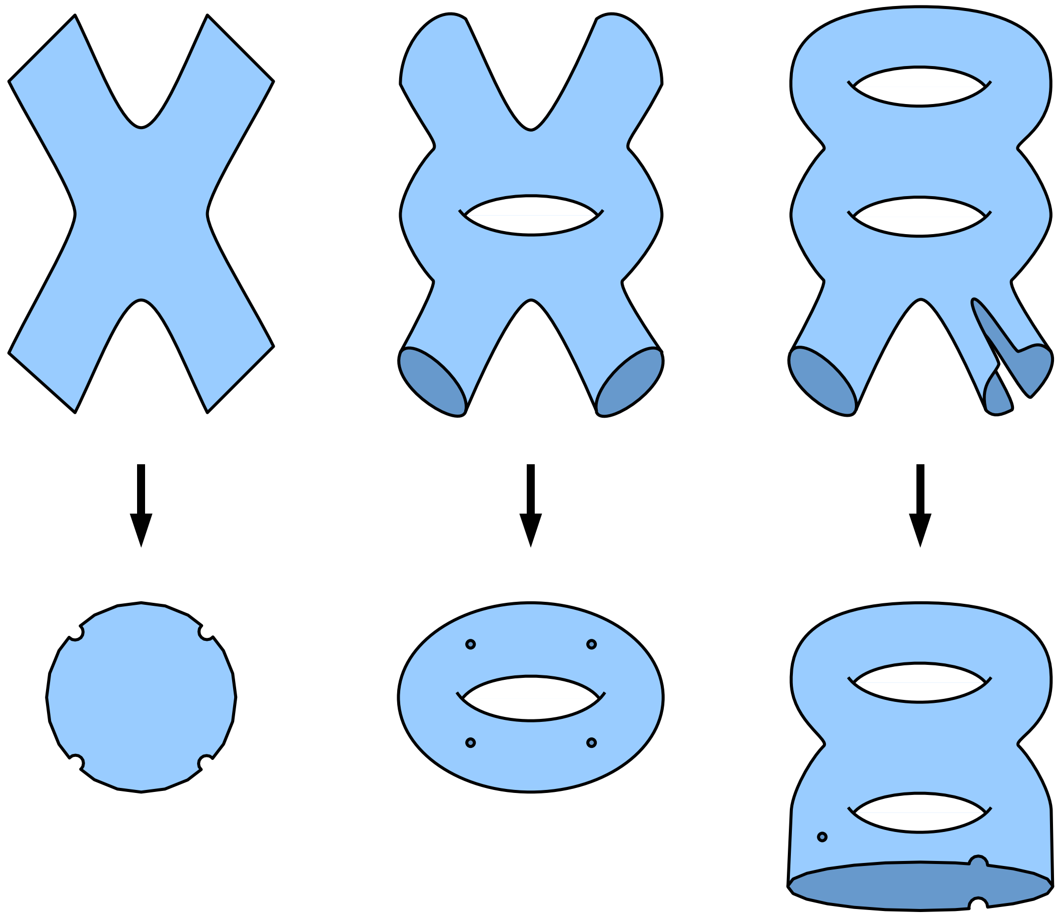
where h is the number of handles a given worldsheet has, b is the number of boundaries it has and c is the number of crosscaps present. Some example diagrams showing worldsheets with boundaries and handles are presented in figure 19. In the third diagram, we see that one closed string can be replaced by two open strings. Therefore, making a closed string 'costs' the same as making two open strings, so that the couplings have the relation

$$g_O^2 \sim g_C\,. \tag{12.3}$$

Cross-caps occur only in unoriented theories, in which only those states described in sec. 5.3 which are preserved under the worldsheet parity operation

$$\Omega: \quad \sigma \longrightarrow \pi - \sigma \tag{12.4}$$

are retained, as described earlier. To make a cross-cap, we cut a small hole in the worldsheet, and then glue together all diametrically opposed edges.



Open strings, tree-level $\chi = 1$ | Closed strings, one-loop $\chi = 0$ | Mixed, two-loop $\chi = -3$

Figure 19: Example diagrams in string perturbation theory. In the second row, the external states have been conformally mapped to points, to be replaced by vertex operators.

The Polyakov path-integral

To obtain the $\mathscr{S}$-matrix for string theory, one should imagine that the incoming and outgoing (asymptotic) states are taken off to infinity, just as one does in field theory. The Weyl invariance (4.29) may then be used to map the external states to local disturbances on the worldsheet, as shown in the figure, which are then replaced by local *vertex operators* $\mathscr{V}(k,\tau,\sigma)$. The general procedure for calculating scattering amplitudes is then to consider a particular topology, insert vertex operators onto it, calculate the probability of the diagram spontaneously occuring, and sum over all physically distinct cases.

There are three complications associated with this procedure. First, to avoid overcounting, we must account for the diff×Weyl gauge invariance of the action. Second, some of the topologies have *moduli* associated with them, describing different embeddings of the worldsheet into spacetime. For the torus, for instance, one may imagine tori of different 'fatness' and 'oval-ness' – and indeed, as we saw in section 6, the torus is defined by a complex modulus τ, the Teichmüller parameter. All values of the moduli associated with a particular topology must be taken into account, so whatever the diagram, one would always expect to have to integrate over the fundamental region of some moduli space, as we did for the one-loop partition function. Third, topologies may have *Conformal Killing Vectors* (CKVs) associated with them. These isometries lead to worldsheets which are mathematically distinct but have the same physical embedding in spacetime, and as such we should divide out by them. Taking the torus as an example again, the CKVs can be thought of as the two ways in which a (regular) torus may be rotated whilst leaving it physically unchanged. For a given topology, the number of moduli μ and CKVs κ are related to the Euler number by the Riemann-Roch theorem,

$$\mu - \kappa = -3\chi \,. \tag{12.5}$$

There are two general approaches to calculating amplitudes in a manner consistent with the above: the operator approach, as typified by [1], and the (Polyakov) path-integral approach, as typified by [2]. The operator approach is not without its merits, but the algebra involved is tiresome. Therefore, we generally make use of the path-integral formalism in this thesis. Here, one first Euclideanizes the worldsheet,

$$(\tau,\sigma) \longrightarrow (-iy,x) \tag{12.6}$$

after which one may write down a well-defined path integral,

$$\mathscr{S} = \sum_{\chi,\alpha\beta} \int \frac{\mathscr{D}X\,\mathscr{D}\psi\,\mathscr{D}g}{V_{\text{diff}\times\text{Weyl}}} e^{-S_E - \lambda\chi} \prod_{i=1}^{n} \int d^2 z\,\sqrt{g}\mathscr{V}_i(k_i,z_i)\,, \tag{12.7}$$

where S_E is the Euclideanized version of the action (4.44) and the $z_i = x_i + iy_i$ are points on the Euclideanized worldsheet. The sum is over topologies χ and also *spin-structures* $\alpha\beta$, which are all possible ways in which we may choose periodic and anti-periodic boundary conditions for the fermions ψ on a particular topology.

A gauge in the diff×Weyl space is then fixed by a Faddeev-Popov procedure [2], in which one fixes the coordinates of κ of the vertex operators, and integrates over the positions of those that remain. The moduli and CKVs are accounted for by introducing anticommuting *ghost* fields b, c and on the worldsheet: one b ghost is introduced for each modulus, and one c ghost for each CKV. In practice, the contributions of the ghosts can be simply determined by operator methods.

We now discuss some of the specific topologies which play a role in perturbation theory, beginning with tree level where $\chi > 0$. There are three possible topologies to consider, none of which have any moduli associated with them:

- The sphere S_2, with $\chi = 2$. The Riemann-Roch result (12.5) tells us that $\kappa = 6$ CKVs are present. We may use these to completely fix the positions of three vertex operators on the worldsheet.

- The disk D_2, which has one boundary. Here, $\chi = 1$ and so $\kappa = 3$ by eq. (12.5). As the vertex operators must be on the worldsheet boundary, this is again enough to fix the positions of three vertex operators.

- The projective plane RP_2, which has one cross-cap and hence also $\chi = 1$, $\kappa = 3$.

At one-loop level, $\chi = 0$. There are four possible topologies,

- The torus T_2, with $\mu = \kappa = 2$.
- The cylinder/annulus C_2, with $\mu = \kappa = 1$.
- The Klein bottle K_2, with $\mu = \kappa = 2$.
- The Möbius strip M_2, with $\mu = \kappa = 1$.

The simplest way to construct each of these is to identify various regions of the complex plane and integrate over just the fundamental domain in the amplitude. The classic example is of course the fundamental region of the one-loop partition function which we already met in section 6.1, where the moduli are represented by the complex Teichmüller parameter τ.

Vertex operators

Mathematically, the state-operator correspondence may be described using the tools of conformal field theory. After the Euclideanization eq.12.6, the closed-string mode expansions (4.36) may be written (defining $z = e^{2i\sigma_-}$ and hence, on the Euclidianized worldsheet, $\overline{z} = e^{2i\sigma_+}$) as

$$\partial X_-^{\mu}(z) = -i\sqrt{\frac{\alpha'}{2}}\sum_n \alpha_n^{\mu} z^{-n-1} \qquad\qquad \overline{\partial} X_+^{\mu}(\overline{z}) = -i\sqrt{\frac{\alpha'}{2}}\sum_n \tilde{\alpha}_n^{\mu}\overline{z}^{-n-1}\,, \tag{12.8}$$

where $\partial \equiv \partial_z$, $\overline{\partial} \equiv \partial_{\overline{z}}$. Notice that the left-moving (*holomorphic*) fields are written in terms of z, whilst the right-moving (*antiholomorphic*) fields are in terms of $\overline{z}$. These expressions invert to

$$\alpha_n^{\mu} = \sqrt{\frac{2}{\alpha'}}\oint_C \frac{dz}{2\pi} z^n \partial X_-^{\mu}(z) \qquad\qquad \tilde{\alpha}_n^{\mu} = -\sqrt{\frac{2}{\alpha'}}\oint_C \frac{d\overline{z}}{2\pi}\overline{z}^n \overline{\partial} X_+^{\mu}(\overline{z})\,, \tag{12.9}$$

with the contour C taken to enclose the origin of the complex plane anti-clockwise. Applying the residue theorem gives the state-operator correspondence for the X fields,

$$\alpha_{-n}^{\mu} \longrightarrow i\sqrt{\frac{2}{\alpha'}}\frac{1}{(n-1)!}\partial^n X^{\mu}(0) \qquad\qquad \tilde{\alpha}_{-n}^{\mu} \longrightarrow i\sqrt{\frac{2}{\alpha'}}\frac{1}{(n-1)!}\overline{\partial}^n X^{\mu}(0)\,. \tag{12.10}$$

This result is valid for operators inserted at the origin; for operators at arbitrary points z, the fields are simply translated. Now, an operator which localizes the string to a particular point X in spacetime is

$$\int d^2z\, \delta^{10}\left(X - X(z,\overline{z})\right)\,, \tag{12.11}$$

and the (tachyonic) ground state is the spacetime Fourier transform of this operator:

$$|0;k\rangle \longrightarrow \int d^2z\, e^{ik\cdot X}(z)\,. \tag{12.12}$$

Excited states are then constructed using eq.12.10; for instance, the first excited state of the closed string (the graviton, $g^{\mu\nu}$) has the vertex operator

$$\alpha_{-1}^{\mu}\tilde{\alpha}_{-1}^{\nu}|0;k\rangle \longrightarrow \int d^2z\, \partial X^{\mu}\overline{\partial}\,\overline{X}^{\nu} e^{ik\cdot X}(z)\,. \tag{12.13}$$

For the open string, the procedure is analagous, except that we only have one set of operators α_n^μ.

The fermionic oscillators ψ_r^μ may be treated in a similar fashion to the α_n^μ. Here, the mode expansions (4.56) become

$$\Psi^\mu(z) = \sum_r \psi_r^\mu z^{-r-\frac{1}{2}} \qquad \tilde{\Psi}^\mu(\bar{z}) = \sum_r \tilde{\psi}_r^\mu \bar{z}^{-r-\frac{1}{2}}. \tag{12.14}$$

And the state-operator correspondence is

$$\psi_r^\mu \longrightarrow \frac{1}{\left(r-\frac{1}{2}\right)!}\partial^{r-\frac{1}{2}}\Psi^\mu(0) \qquad \tilde{\psi}_r^\mu \longrightarrow \frac{1}{\left(r-\frac{1}{2}\right)!}\bar{\partial}^{r-\frac{1}{2}}\tilde{\Psi}^\mu(0). \tag{12.15}$$

This is all the information required to construct vertex operators of NS-sector states. R-sector states, built up from $|\mathbf{s}\rangle$, are potentially more complicated since the expansion in eq.12.14 has a branch-cut, and the state-operator correspondence is not simple. The solution lies in bosonization: first group the fields Ψ^μ into complex pairs as

$$\Psi = \frac{1}{\sqrt{2}}\left(\Psi^1 + i\Psi^2\right) \qquad \overline{\Psi} = \frac{1}{\sqrt{2}}\left(\Psi^1 - i\Psi^2\right). \tag{12.16}$$

The behaviour of these fields as they come together at a point on the worldsheet is determined by their *operator product expansion* (OPE):

$$\Psi(w)\overline{\Psi}(z) \sim \frac{1}{w-z}. \tag{12.17}$$

If we introduce a complex bosonic field H obeying

$$H(w)H(z) \sim -\log(w-z) \tag{12.18}$$

then the identification

$$\Psi(z) = e^{iH(z)} \qquad \overline{\Psi}(z) = e^{-iH(z)} \tag{12.19}$$

is consistent with the OPE (12.17); as such, all physics is unchanged by the identification. The antiholomorphic fields $\tilde{\Psi}(\bar{z})$ may be bosonized in an analogous manner. Bosonising the ten dimensions of the string into five complex pairs of the form (12.16) and introducing a set of five bosonic fields $\mathbf{H}$, the open string R-sector ground state is then identified as

$$|\mathbf{s}\rangle \longrightarrow \int dz e^{i\mathbf{s}\cdot\mathbf{H}} \tag{12.20}$$

where $\mathbf{s}$ is the vector (5.26) and the integration is over the worldsheet boundary. For the closed string, an symmetric operator in $\tilde{\mathbf{H}}$ is added, and the integration is taken over d^2z.

As an example, the vertex operator for the open string photon, $\psi^\mu_{-\frac{1}{2}}|0;k\rangle$, is

$$\mathcal{V}^\mu_{-1}(k,z) = g_O\lambda^a e^{-\phi}\Psi^\mu e^{ik\cdot X}(z). \tag{12.21}$$

This operator should be understood to be integrated over the worldsheet boundary, and a factor of the open string coupling g_O has been explicitly inserted. Two other points about this expression deserve further comment.

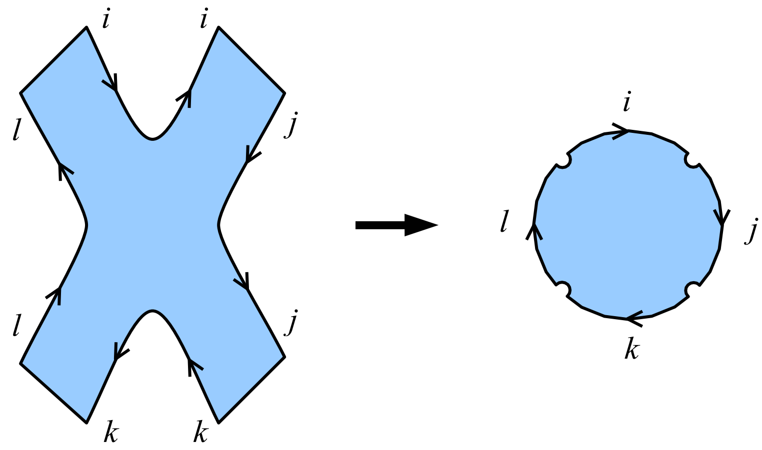


Figure 20: Scattering amplitudes with orientated worldsheets contain traces of Chan-Paton factors.

Picture-changing

First, notice that the subscript -1 has been attached to eq.12.21, and an operator $e^{-\phi}$ included. The argument for its presence goes as follows; firstly, we use the Virasoro constraints (4.51) to define a stress-energy tensors for X and Ψ fields on the worldsheet. In general, the OPE of this tensor with a vertex operator $\mathscr{V}$ takes the form

$$T(w)\mathscr{V}(z) = \frac{h}{(w-z)^2}\mathscr{V}(z) + \ldots \tag{12.22}$$

where h is the *conformal weight* of $\mathscr{V}$. To offset the factor of dz which appears together with eq.12.21, it turns out that $\mathscr{V}$ must have a total conformal weight of one. The conformal weights of Ψ and $e^{ik\cdot X}$ are $-\frac{1}{2}$ and $\frac{\alpha'}{4}k^2$ respectively and $k^2 = 0$, so we have a problem. The solution is to add commuting *superconformal ghost* fields β, γ onto the worldsheet, which may be bosonized in terms of the field ϕ. The operator $e^{a\phi}$ then has weight $-\frac{1}{2}a(a+2)$ [3], so that the composite operator (12.21) correctly has unit weight.

To avoid an anomaly in the $\beta\gamma$ CFT, it is necessary that the total superghost charge in a particular amplitude sums to the Euler number χ of a particular topology. In general then, we will need some prescription for changing the ϕ-charge, or *picture*, of our vertex operators. Such a prescription is the picture-changing operation,

$$\mathscr{V}_{i+1}(k,z) = \lim_{w\to z} e^{\phi}\partial X^{\mu}\psi_{\mu}(w)\mathscr{V}_i(z)\,. \tag{12.23}$$

Chan-Paton factors

Second, a *Chan-Paton* factor λ^a has been introduced into (12.21). This is a non-dynamical quantity which may be associated with the endpoints of strings. The idea is to write the general open string state $|a;k\rangle$ in the basis

$$|a;k\rangle = \sum_{i,j} |ij;k\rangle\, \lambda^a_{ij}\,. \tag{12.24}$$

Then, as figure 20 demonstrates, open-string scattering amplitudes must contain a factor

$$\sum_{i,j,k,l} \lambda^1_{ij}\lambda^2_{jk}\lambda^3_{kl}\lambda^4_{li} = \mathrm{tr}\left(\lambda^1\lambda^2\lambda^3\lambda^4\right)\,. \tag{12.25}$$

Since the trace is cyclic, scattering amplitudes are invariant under the gauge symmetry

$$\lambda^a \longrightarrow U\lambda^a U^\dagger \tag{12.26}$$

when $U \in U(N)$. Under this symmetry, one end i of the string transforms in the $\mathbf{N}$ of $U(N)$, whilst the other end (due to the relative orientation reversal) transforms in the $\overline{\mathbf{N}}$. Therefore, the open string vertex operator $\mathscr{V}^\mu_{-1}$ transforms in the adjoint $\mathbf{N}\otimes\overline{\mathbf{N}}$ representation, which supports our identification of it as a gauge boson.

12.2 Yukawas and flavour in open strings

Among the many phenomenological implications of low scale models, flavour physics is one of the most pressing, so it is to flavour that we now turn. Flavour experiments are typically able to probe mass scales much higher than the energy of current experiments and as we will see shortly this is particularly true in the case of intersecting brane models. The flavour structure of these models is not restricted to Yukawa couplings but flavour violating four-fermion contact interactions are also present at the classical level, giving them a uniquely rich structure. Nonetheless, since both sources of flavour violation are intimately related we shall start with the description of Yukawa couplings.

The leading contribution to Yukawa couplings between two fermions and a scalar, each living at a different intersection, is due to world-sheet instantons [41]. One can think of this as the classical action for a stretched string leaving an intersection

(with one end on each brane) and travelling to the opposite corners of the Yukawa triangle. The action for a string is the worldsheet area, and therefore the amplitude should depend on the area the string sweeps out;

$$Y_{ijk} \sim \mathrm{e}^{-A_{ijk}/\alpha'}, \tag{12.27}$$

where A_{ijk} is the area of the minimal area worldsheet with vertices at the three intersections, bounded by the corresponding branes. (See Fig. 21.) A more detailed study of Yukawa couplings, using calibrated geometry [48], and confirmed later by conformal field theory techniques [60], showed that when the compact space is a factorizable torus and the branes wrap factorizable cycles, the relevant area is the sum of the projected areas of the triangle over each sub-torus. The final result, including the quantum part reads

$$Y = \sqrt{2}\lambda_{II} 2\pi \sum_{I=1}^{3} \sqrt{\frac{4\pi B(\nu_I, 1-\nu_I)}{B(\nu_I,\theta_I)F(\nu_I, 1-\nu_I-\theta_I)}} \sum_m \mathrm{e}^{-\frac{A_I(m)}{2\pi\alpha'}}, \tag{12.28}$$

where we have neglected the presence of non-zero B field and Wilson lines and B is the Euler Beta function, I runs over the three tori, ν_I and θ_I are the angles at the fermionic intersections, m runs

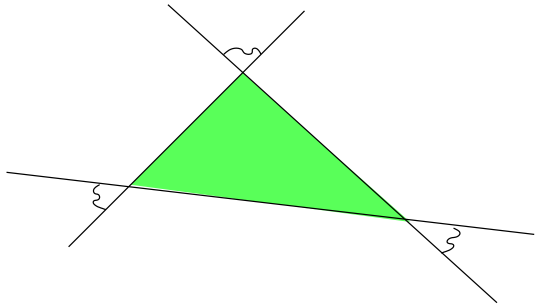


Figure 21: World-sheet instanton contribution to the Yukawa couplings. At each intersection a fermion or a scalar is localized.

over all possible triangles connecting the three vertices on each of the three tori (there is an infinite number of them due to the toroidal periodicity) and $A_I(m)$ is the projected area of the $m-$th triangle on the $I-$th torus.

This exponential dependence has been claimed as a nice feature of these models since it is expected to naturally give a hierarchical pattern of fermion masses. As we shall see, in practice this does not hold, at least in the simplest models. The reason is that in many cases, the dynamics of left-handed and right-handed fermions turns out to occur in different tori and the property that only the projected triangles are relevant translates into a factorization of the Yukawa couplings. An example is the very model we have been discussing in this section and displayed in detail in Fig. 18. Left-handed quarks live at different points only in the second torus while they live at the same unique intersection in the third one. The opposite happens for right-handed quarks. This results in the following factorizable form of the Yukawa couplings

$$Y^u_{ij} = a_i b^u_j, \quad Y^d_{ij} = a_i b^d_j, \tag{12.29}$$

where we have only explicitly written the classical part, including this time the presence of non-zero $B-$field and Wilson lines. The coefficients are

$$a_i \equiv \vartheta \begin{bmatrix} \frac{i}{3} + \varepsilon^{(2)} \\ \theta^{(2)} \end{bmatrix} \left(\frac{3J^{(2)}}{\alpha'}\right), \tag{12.30}$$

$$b^u_j \equiv \vartheta \begin{bmatrix} \frac{j}{3} + \varepsilon^{(3)} + \tilde{\varepsilon}^{(3)} \\ \theta^{(3)} + \tilde{\theta}^{(3)} \end{bmatrix} \left(\frac{3J^{(3)}}{\alpha'}\right), \tag{12.31}$$

$$b^d_j \equiv \vartheta \begin{bmatrix} \frac{j^*}{3} + \varepsilon^{(3)} - \tilde{\varepsilon}^{(3)} \\ \theta^{(3)} - \tilde{\theta}^{(3)} \end{bmatrix} \left(\frac{3J^{(3)}}{\alpha'}\right), \tag{12.32}$$

where $i, j, j^* = -1, 0, 1$, $J^{(k)}$ denotes the complex Kähler structure of the $k-$th torus, $\theta^{(2)}, \theta^{(3)}, \tilde{\theta}^{(3)}$ parameterize the Wilson lines and ϑ is the complex theta function with characteristics, defined as

$$\vartheta \begin{bmatrix} \delta \\ \phi \end{bmatrix} (\kappa) = \sum_{l \varepsilon Z} \exp[\pi \mathrm{i} (\delta + l)^2 \kappa + 2\pi \mathrm{i} (\delta + l) \phi]. \tag{12.33}$$

This factorizable form of the Yukawa couplings, Eq. (12.29), is too simple to lead to a realistic fermion spectrum. It is a rank one matrix with one massive and two massless eigenvalues. There are of course different ways out of this, either by using a more complicated (non-factorizable) compact manifold or by looking for configurations of branes in which the left and right dynamics occur at the same torus. An example of the latter has been provided recently in [50], where a three Higgs model with democratic rather than hierarchical Yukawas is studied. There is however another feature of these very simple models that makes the naive assertion above invalid when quantum corrections are taken into account. This new feature is the presence of flavour changing neutral couplings that propagate through quantum loops to the otherwise trivial structure of Yukawa couplings, providing them with enough complexity to give rise to a realistic set of fermion masses and mixing angles [9].

12.3 Flavour Changing Neutral Currents

We have emphasized in this review that, after the second string revolution, string theory greatly influenced (and in turn received some degree of inspiration from) field theory investigations, particularly in the area of models with extra dimensions. We shall see a salient example of the complementarity between string and field theory in extra dimensions in this section. Models with intersecting D-branes are a stringy realization of the brane world idea, in which four-dimensional fermions live in the boundaries of extra dimensions where gauge bosons are allowed to propagate, these latter dimensions being a further restriction to a submanifold of the full space-time where gravity lives [4, 61]. One well known property of brane worlds in which the different fermions live in separate points of the extra dimensions, the split fermion scenario [62], is the appearance of flavour changing neutral currents that tightly constraint the compactification scale $M_C \gtrsim 10^{2-3}$ TeV in the case of flat extra dimensions [63] [10]. (See also [65] for a model with light vector-like fermions, relevant for phenomenology despite this very large compactification scale.) The origin of these FCNC can be simply traced to the fact that Kaluza-Klein modes of the multi-dimensional gauge bosons, having a non-trivial profile in the extra dimensions, couple in a different way to the fermions localized at the different positions. Family non-universal gauge bosons then induce FCNC in the fermion mass eigenstate basis [66]. Gauge boson KK generated FCNC are therefore expected from a purely field theory viewpoint in models with intersecting D-branes. A string calculation of the tree level four-fermion amplitude, which can be performed [60] using an extension of the conformal field theory techniques developed for the heterotic orbifolds [16], indeed reproduces the field theory expectation. In addition, though, it reveals a new purely stringy source of flavour violation in these models mediated by string instantons [49]. These are simply worldsheets that directly connect four fermions of different generations living at different intersections in the same way that the Yukawas connected the higgs to two fermions. Again the suppression goes roughly as the area, so that one would expect the FCNC effect from this source to increase as the compactification length and hence worldsheet area decrease.

The full amplitude is a bit of a beast to work out but for completeness I will present it:

$$
\begin{aligned}
A(1,2,3,4) \;=\; & -g_s\alpha'(\lambda^1\lambda^2\lambda^3\lambda^4+\lambda^4\lambda^3\lambda^2\lambda^1)\int_0^1 dx\, x^{-1-\alpha' s}(1-x)^{-1-\alpha' t}\,\frac{1}{\prod_m^3 |J^m|^{1/2}} \\
& \times\left[\bar{u}^{(2)}\gamma_\mu u^{(1)}\bar{u}^{(4)}\gamma^\mu u^{(3)}\right]\sum e^{-S_{cl}(x)}\,.
\end{aligned}
\tag{12.34}
$$

[9] Although not necessary for the generation of fermion masses, these FCNC also affect the model in [50] as well, and therefore similar bounds on the string scale apply.

[10] The particular localization properties of KK modes in warped scenarios make the bounds in that case milder [64]).

The functions $|J|$ are;

$$|J| = \left(\frac{x^{1-\vartheta_1-\vartheta_2}}{(1-x)^{1-\vartheta_2-\vartheta_3}} \frac{\Gamma(1-\vartheta_1)\Gamma(\vartheta_3)}{\Gamma(\vartheta_3+\vartheta_4)\Gamma(\vartheta_2+\vartheta_3)} \,{}_2F_1[1-\vartheta_1, \vartheta_3, \vartheta_2+\vartheta_3; 1-x]\,{}_2F_1[1-\vartheta_1, \vartheta_3, \vartheta_3+\vartheta_4; x] + \frac{(1-x)^{1-\vartheta_2-\vartheta_3}}{x^{1-\vartheta_1-\vartheta_2}} \frac{\Gamma(\vartheta_1)\Gamma(1-\vartheta_3)}{\Gamma(\vartheta_1+\vartheta_2)\Gamma(\vartheta_1+\vartheta_4)} \,{}_2F_1[\vartheta_1, 1-\vartheta_3, \vartheta_1+\vartheta_4; 1-x]\,{}_2F_1[\vartheta_1, 1-\vartheta_3, \vartheta_1+\vartheta_2; x] \right) \tag{12.35}$$

where ${}_2F_1$ are the standard hypergeometric functions. Each $|J^m|$ is the contribution from the m'th internal complex dimension and one must use the relevant angles for that T_2 sub-torus. The λ's are the famous Chan-Paton factors, and $s = -(k_1+k_2)^2$, $t = -(k_2+k_3)^2$, $u = -(k_1+k_3)^2$ are the usual Mandlestam variables. The classical action S_{cl} (which is of course the world-sheet area in the full 6D internal space) turns out to be the sum of the projected world-sheet areas in the three T_2 tori when, as in this case, the compactification manifold is factorizable.

The point of displaying this lengthy expression is that it allows me to demonstrate one of the beauties of string theory: grotesque as it may be, this expression contains all of the necessary pole structure to generate the correct field theory behaviour. For example Higgs exchange (which I shall discuss presently) can be extracted from the situation shown in fig.22: The world-sheet areas are

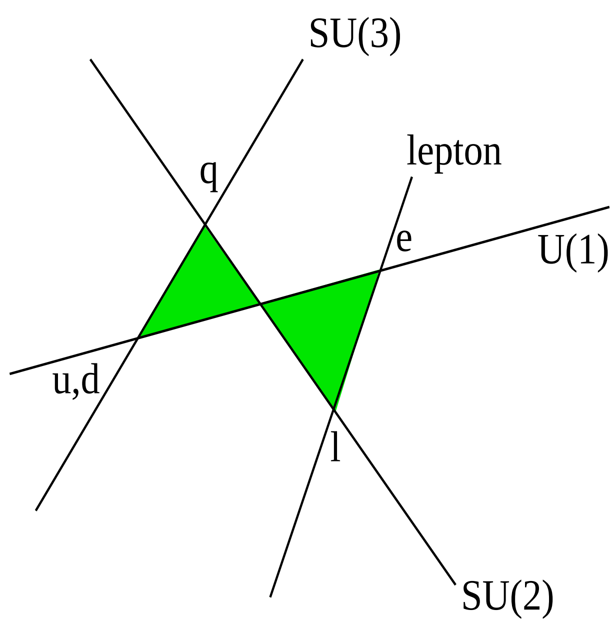


Figure 22: Higgs exchange as a “double instanton”

two Yukawa couplings and the Higgs field is the intersection state in middle.

The KK mediated flavour violating four fermion interactions come from diagrams with $S_{cl} = 0$ – i.e. two fermions annihilate, produce an open string KK mode with both ends on one brane, which propagates to a different intersection where it produces two new open string states. These contributions are of the form,

$$O_{LL}^{(\vec{n})} = \frac{(c_{LL}^{(\vec{n})})_{abcd}}{M_n^2} (\overline{\psi}_{aL}\gamma^\mu \psi_{bL})(\overline{\psi}_{cL}\gamma^\mu \psi_{dL}), \tag{12.36}$$

with the following dependence of the coefficient

$$(c^{(\vec{n})}_{LL})_{abcd} \sim \delta^{-M^2_{\vec{n}}/M^2_s} \sum_{ij} (U^\dagger_L)_{ai}(U_L)_{ib}(U^\dagger_L)_{cj}(U_L)_{jd} \cos\left[\vec{M}_{\vec{n}} \cdot (\vec{y}^L_i - \vec{y}^L_j)\right]. \tag{12.37}$$

U_L are the corresponding unitary matrices rotating current eigenstates into mass eigenstates and δ is an order one (but always larger) number that depends on the specific brane configurations and represents the string smoothing of the KK contribution at high energies which is generally divergent in the field theory calculation of the same effect. (Essentially, the string smoothing arises because the branes have a finite width of order the string length, and are therefore unable to excite modes of a shorter wavelength than this.) We have only written the Left-Left contribution, the case of Right-Right and Left-Right contributions is a straight-forward generalization of this. Note that in order to have FCNC it is essential that current and mass eigenstates are not aligned (so that the rotation matrices are non-trivial) and the different generations are localized at separate points of the extra dimension ($y_i - y_j \neq 0$). The exponential smoothing provided by the string dynamics, which is crucial in the case of more than one extra dimensions where the sums over KK modes typically diverge, has to be introduced by hand in a field-theory approach. String theory automatically cuts-off the contribution of KK modes heavier than the string scale. Therefore the larger the ratio R_c/L_s, the bigger the number of KK modes that contribute and the larger the effect is.

On the other hand, string instanton flavour changing neutral couplings depend very much on the chiralities of the external fermions (through the difference in the number of independent angles). Four-fermion interactions with all fermions of the same chirality (either all LH or all RH) correspond to a parallelogram with only one independent angle. Given the factorization property of the model we are discussing, the only non-vanishing world-sheet area occurs in one torus and the result is of the form

$$O^{str}_{LL} = \frac{(c^{(\vec{n})}_{LL})_{abcd}}{M^2_s} (\overline{\psi}_{aL}\gamma^\mu \psi_{bL})(\overline{\psi}_{cL}\gamma^\mu \psi_{dL}), \tag{12.38}$$

with the following dependence of the coefficient

$$(c^{str}_{LL})_{abcd} \sim \mathrm{e}^{-\frac{A}{2\pi L^2_s}} \sum_i (U^\dagger_L)_{ai}(U_L)_{(i+1)b}(U^\dagger_L)_{c(i+1)}(U_L)_{(i+2)d}, \tag{12.39}$$

where A is the area of the corresponding parallelogram (which is $\sim (4\pi^2 R^2_c)/3$) and $L_s = 1/M_s$ is the string scale. Already in this chirality preserving interaction we observe several differences with respect to the field theory case. The first one is that there are FCNC even in the case of Yukawa couplings aligned with gauge couplings (*i.e.* $U = 1$). Secondly, the exponential dependence on the ratio of string and compactification scales is opposite to that coming from the KK modes, the larger the ratio R_c/L_S, (*i.e.* the larger the area in string units) the stronger the suppression. Notice however that it is still necessary to have different generations living at separate points in order to have FCNC. The opposite dependence of the KK and string instanton contributions on the ratio of compactification and string scales allows us to put a lower bound on the string scale, independently of this ratio. An estimation of this bound [49], using the KK contribution to $|\varepsilon_K|$ and the string instanton contribution to $\tau \to ee\mu$ and relatively small mixing angles, leads to the bound $M_s \gtrsim 100$ TeV as shown in Fig. 23.

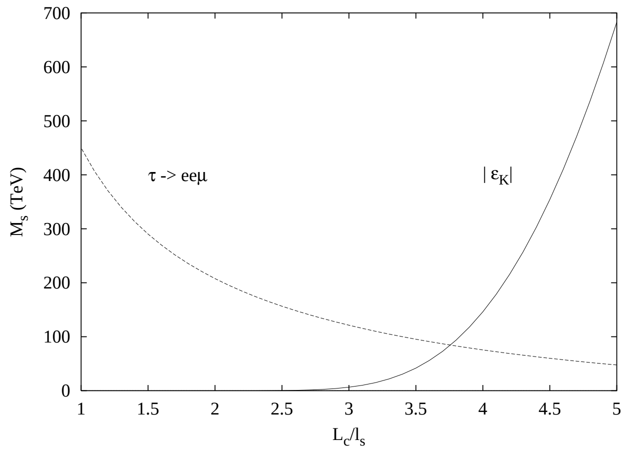


Figure 23: Bound on the string scale as a function of the ratio L_c/L_s from the KK contribution to $|\varepsilon_K|$ and the string instanton contribution to $\tau \to ee\mu$. A global bound $M_s \gtrsim 100$ TeV is found.

The chirality changing four-fermion interactions, connecting two left-handed and two right-handed fermions, is a bit more involved but far more interesting. We will give the final expressions here and outline the reasons for the new features without entering into the intricacies of the calculation. The main new feature is the absence of L-R factorization in the amplitude (except in some limiting cases). The reason is that now in general there are non-zero contributions in more than one 2-torus and the classical action is no longer the sum of the areas of each of the quadrangles (incidentally, this does not happen for the Yukawa couplings because in the three-point amplitude we can fix all three vertices using $SL(2,R)$ invariance whereas in the four-point one we have to integrate over the position of the fourth vertex, see.) As we shall see soon, this introduces enough flavour violation to generate, through loop corrections, a semi-realistic pattern of fermion masses and mixing angles.

Another nice feature with possible important phenomenological implications is related to Higgs-mediated like processes. Let us consider the situation displayed in Fig. 24. The Higgs mediated process can be obtained as the field theory limit of a string propagating from the vertices 2 and 3 down to the Higgs vertex and then back to the vertices 1 and 4. This contribution goes, in the t channel, like

$$\frac{\mathrm{e}^{-A_{23H}/2\pi L_s^2}\mathrm{e}^{-A_{14H}/2\pi L_s^2}}{t-M_H^2} \sim \frac{Y_{23}Y_{14}}{t-M_H^2}, \tag{12.40}$$

where M_H is the Higgs mass. On the other hand there is another, purely stringy contribution (not expected on field theory grounds) that can be very much enhanced for a low string scale and corresponds to a string sweeping out the area of the quadrangle between the four vertices 1,2,3,4 without going through the Higgs vertex (shaded area in the Figure). In this case if all the flavour dynamics happens on a single torus the amplitude goes as

$$\frac{\mathrm{e}^{-A_{1234}/2\pi L_s^2}}{M_s^2} \sim \frac{Y_{23}/Y_{14}}{M_s^2}. \tag{12.41}$$

If the flavour dynamics happens in more than one torus the detailed result depends on the particular configuration due to the non-factorization property of this four point amplitude alluded to above, but is roughly the same. A more detailed study is necessary before making any statement about the phenomenological implications of this property but it seems that a general feature of models with intersecting branes is the presence of Higgs-like processes enhanced (as opposite to the usual expected suppression) by light Yukawas.

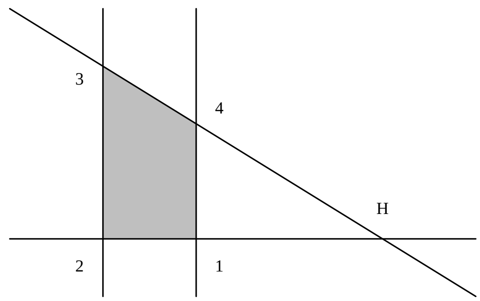


Figure 24: Higgs vs string instanton mediation of the process $(\bar{q}_{aL}q_{bR})(\bar{q}_{cR}q_{dL})$

Let us now concentrate on the relevant amplitude for the generation of fermion masses and mixing angles. In particular we will consider the quark sector and are interested on the $(\bar{q}_{aL}q_{bR})(\bar{q}_{cR}q_{dL})$ amplitude. The full expressions are intricate and do not admit a simple analytical form. In order to give some feeling of what happens we will consider a simplified case in which the relevant angles are the same on each sub-torus. In this case the classical action turns out to be [60]

$$S_{cl} = \frac{1}{4\pi\alpha'}\frac{\sin\pi\vartheta_2\sin\pi\vartheta_3}{\sin(\pi\vartheta_2+\pi\vartheta_3)}\sqrt{\sum_m (v_{23}^m - v_{14}^m)^2 \sum_n (v_{23}^n - v_{14}^n)^2}, \tag{12.42}$$

where $\theta_{2,3}$ are the (independent) angles at the corresponding intersections and v_{23}, v_{14} are the distances between the relevant intersections. From this expression it is clear that only in the trivial case (when $a = d$ or $b = c$) or in the degenerate case (when distances in all sub-tori are equal) the amplitude $\sim \exp(-S_{cl})$ factorizes.

Acknowledgements

It is a pleasure to thank Oleg Lebedev, Manel Masip, Anthony Owen and Ben Schofield for their comments, discussion and collaboration. Special thanks to Ben Schofield who kindly allowed the incorporation of some of his notes.

Appendix A: Conserved currents in 1+1D

Conserved currents are an important concept so it is worth recapping them here. I'll use a two dimensional (1 noncompact space x plus time, with a generic field u) field theory to do this. (Note that in string theory the obvious analogy is $t \equiv \tau$, $x \equiv \sigma$, $u \equiv X^{\mu}$ although there σ is always compact). We will use the lagrangian formalism;

$$S[u] = \int_{t_1}^{t_2} \int_{\infty}^{\infty} \mathscr{L}(u, u_t, u_x, u_{xx}, u_{xxx}, \ldots)\, dxdt \tag{12.43}$$

For example the Sine-Gordon equation has

$$\mathscr{L} = \frac{u_t^2}{2} - \frac{u_x^2}{2} - (1 - \cos u) \tag{12.44}$$

with the Euler-Lagrange equation giving

$$u_{tt} - u_{xx} + \sin u = 0. \tag{12.45}$$

Here for later reference note that $\mathscr{P} = 1 - \cos u$ is playing the role of the potential. Assume that two functions X and T can be assembled from the u u_x such that

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0 \tag{12.46}$$

Assume further that X remains constant at $x \rightarrow \pm\infty$

$$X \rightarrow c \tag{12.47}$$

Then eq.12.46 means that

$$\begin{aligned} \frac{d}{dt} \int_{-\infty}^{\infty} T\, dx &= \int_{-\infty}^{\infty} \frac{\partial T}{\partial t}\, dx \\ &= -\int_{-\infty}^{\infty} \frac{\partial X}{\partial x}\, dx \\ &= -[X]_{-\infty}^{\infty} \\ &= 0 \end{aligned} \tag{12.48}$$

so

$$Q = \int_{-\infty}^{\infty} T\, dx \tag{12.49}$$

is a conserved quantity. Note that the RHS of eq.(12.48) is the net inflow of charge at the boundaries.

Appendix B: Symmetries and conservation laws in 1+1D

Emmy Noether was the first to elucidate the deep connection between *symmetries* and conservation laws. The Euler-Lagrange equations derive from Hamilton's principle: that if we let $u \to u + \delta u$ then $\delta S = 0$ implies a set of locally obeyed equations of motion. One thing which is important though are the boundary terms. Assume that the Lagrangian depends only on u, u_t, u_x. (This will suffice for perturbative string theory, although there are many famous examples that depend on the higher derivatives.) Making the variation we actually get

$$\begin{aligned}\delta S = 0 &= \int\int \mathscr{L}(u+\delta u, u_t+\delta u_t, u_x+\delta u_x) - \mathscr{L}(u,u_t,u_x)\, dtdx \\ &= \int\int \frac{\partial \mathscr{L}}{\partial u}\delta u + \frac{\partial \mathscr{L}}{\partial u_t}\delta u_t + \frac{\partial \mathscr{L}}{\partial u_x}\delta u_x\, dtdx \\ &= \int\int \left(\frac{\partial \mathscr{L}}{\partial u} - \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial u_t} - \frac{d}{dx}\frac{\partial \mathscr{L}}{\partial u_x}\right)\delta u\, dtdx \\ &\quad + boundary\, terms \end{aligned} \tag{12.50}$$

(Here for example d/dt means use the chain rule with $u(x,t)$ and $u_x(x,t)$ but do not differentiate with respect to x.) Setting everything in brackets to zero gives the E-L equations

$$\frac{\partial \mathscr{L}}{\partial u} - \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial u_t} - \frac{d}{dx}\frac{\partial \mathscr{L}}{\partial u_x} = 0 \tag{12.51}$$

However to get the last two terms I integrated by parts once and generated some "boundary terms" (i.e. complete derivatives); in fact I used

$$\begin{aligned}\frac{d}{dx}\left(\frac{\partial \mathscr{L}}{\partial u_x}\delta u\right) &= \delta u \frac{d}{dx}\left(\frac{\partial \mathscr{L}}{\partial u_x}\right) + \delta u_x \left(\frac{\partial \mathscr{L}}{\partial u_x}\right) \\ \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial u_t}\delta u\right) &= \delta u \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial u_t}\right) + \delta u_t \left(\frac{\partial \mathscr{L}}{\partial u_t}\right)\end{aligned} \tag{12.52}$$

and the additional boundary terms are

$$\int\int \frac{d}{dx}\left(\frac{\partial \mathscr{L}}{\partial u_x}\delta u\right) + \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial u_t}\delta u\right)\, dxdt; \tag{12.53}$$

since they are total derivatives they make no difference to the equations of motion which are obeyed locally.

If the theory has only time, t, then things get trivial since we drop the x coordinate and insisting that $\delta S = 0$ gives

$$\delta S = \int \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial u_t}\delta u\right)\, dt = \tag{12.54}$$

$$\left[\frac{\partial \mathscr{L}}{\partial u_t}\delta u\right]_{t_A}^{t_B} = 0 \tag{12.55}$$

and if there is an invariance under $u \to \delta u$ we immediately find a conserved current

$$Q = \frac{\partial \mathscr{L}}{\partial u_t}\delta u\,. \tag{12.56}$$

The Hamiltonian density: time translation symmetry

Now let's find the current conserved under time translation; recall we seek $X$and T such that

$$\begin{aligned} X_x + T_t &= 0 \\ X|_{x=\pm\infty} &= 0 \end{aligned} \tag{12.57}$$

so that $d_t \int T dx = \int T_t dx = -\int X_x dx = 0$. Consider the infinitessimally small constant shift $t \to t+\varepsilon$. If S is invariant under this shift it is called a *time-translation symmetry* of the action, and there is a conserved current associated with that given by the boundary terms. We see it as follows: under the shift we have by Taylor expanding that

$$\begin{aligned} t &\to t+\varepsilon \\ u(x,t) &\to u(x,t+\varepsilon) = u(x,t) + \varepsilon u_t \\ u_t(x,t) &\to u_t(x,t+\varepsilon) = u_t(x,t) + \varepsilon u_{tt} \\ u_x(x,t) &\to u_x(x,t+\varepsilon) = u_x(x,t) + \varepsilon u_{xt} \end{aligned}$$

The extra bits on the RHS I will call δu, δu_t, δu_x, δu_{xx} *etc.* Now look at the shift in the action, or rather everything inside the integral. Since the E-L equations are satisfied locally, all we have left are the boundary terms. These are (assuming for the moment that $\mathscr{L} = \mathscr{L}(u, u_t, u_x)$ only and does not depend on u_{xx} or very importantly explicitly on x or t)

$$\begin{aligned} \delta\mathscr{L} &= \frac{d}{dx}\left(\frac{\partial \mathscr{L}}{\partial u_x}\delta u\right) + \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial u_t}\delta u\right) \\ &= \varepsilon\frac{d}{dx}\left(\frac{\partial \mathscr{L}}{\partial u_x}u_t\right) + \varepsilon\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial u_t}u_t\right) \end{aligned} \tag{12.58}$$

dividing by ε and taking the $\varepsilon \to 0$ limit to get $\frac{\delta\mathscr{L}}{\varepsilon} \to \frac{d\mathscr{L}}{dt}$ we have the relation

$$\frac{d}{dx}\left(\frac{\partial \mathscr{L}}{\partial u_x}u_t\right) + \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial u_t}u_t - \mathscr{L}\right) = 0 \tag{12.59}$$

This relation is in precisely the form $X_x + T_t = 0$ with

$$T \equiv \mathscr{H} = \frac{\partial \mathscr{L}}{\partial u_t}u_t - \mathscr{L} \tag{12.60}$$

so $\int \mathscr{H} dx$ is a conserved current; you may recognize this as the Hamiltonian (with $p \equiv \partial\mathscr{L}/\partial u_t$) hence the name $\mathscr{H}$. *So the hamiltonian is an expression of time-translation invariance. Conversely an explicit time dependence in the Lagrangian would break time translation invariance, and the current would no longer be conserved.* For the SG equation

$$\begin{aligned} \mathscr{H} = \frac{\partial \mathscr{L}}{\partial u_t}u_t - \mathscr{L} &= u_t^2 - \frac{u_t^2}{2} + \frac{u_x^2}{2} + (1-\cos u) \\ &= \frac{u_t^2}{2} + \frac{u_x^2}{2} + (1-\cos u) \end{aligned} \tag{12.61}$$

which is clearly the kinetic plus potential energy (densities) of the system.

Space translation symmetry

Now consider the infinitessimally small constant shift $x \longrightarrow x+\varepsilon$. If S is invariant under this shift it is called a *space-translation symmetry* of the action, and again there is a conserved current associated with that given by the boundary terms. Under the shift we have

$$\begin{aligned}
x &\longrightarrow x+\varepsilon \\
u(x,t) &\longrightarrow u(x,t+\varepsilon) = u(x,t)+\varepsilon u_x \\
u_t(x,t) &\longrightarrow u_t(x,t+\varepsilon) = u_t(x,t)+\varepsilon u_{tx} \\
u_x(x,t) &\longrightarrow u_x(x,t+\varepsilon) = u_x(x,t)+\varepsilon u_{xx}
\end{aligned}$$

To get the conserved current we can read of from the boundary terms that

$$\frac{d}{dx}\left(\frac{\partial \mathscr{L}}{\partial u_x}u_x - \mathscr{L}\right) + \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial u_t}u_x\right) = 0 \tag{12.62}$$

The only difference from the time-translation case is that the $-\mathscr{L}$ went in the d/dx term. In this case the conserved current is related to the momentum density

$$P = \frac{\partial \mathscr{L}}{\partial u_t}u_x\,. \tag{12.63}$$

Appendix C: Modular invariance in detail

C.1 The partition function for the complex fermions

First consider a single complex world sheet fermion with boundary conditions v, u. This means that the fermion acquires a phase factor $e^{2\pi iv}$ when propagated through the complex time $2\pi\tau \equiv t$. Propagation in the other direction on the torus must give the phase factor $e^{2\pi iu}$ for each fermion. The partition function (c.f. $\langle e^{iHt}\rangle$) is then

$$Z_u^v(\tau) = Tr\left(q^{H_v} e^{2\pi i(\frac{1}{2}-u)N_v}\right) \tag{12.64}$$

where $q = e^{2\pi i\tau}$ and where the final factor includes a phase from every world sheet fermion excitation in a particular physical state. To take the trace we sum over all states; that is we sum over all possible excitations (i.e. all possibilities one or zero numbers of each fermionic excitation)

$$\begin{aligned} Tr(\hat{O}) \;=\; & \langle 0|\hat{O}|0\rangle + \langle 0|b_v\hat{O}b_v^\dagger|0\rangle + \langle 0|b_{1+v}\hat{O}b_{1+v}^\dagger|0\rangle + \ldots\langle 0|b_v b_{1+v}\hat{O}b_{1+v}^\dagger b_v^\dagger|0\rangle + \ldots \\ & + \langle 0|d_{1-v}\hat{O}d_{1-v}^\dagger|0\rangle + \langle 0|d_{2-v}\hat{O}d_{2-v}^\dagger|0\rangle + \ldots \end{aligned}$$

The $b_{n+v-1}^\dagger$ and $d_{n-v}^\dagger$ can be commuted left through the H_v and N_v operators which then annihilate on the vacuum. The end result is

$$Z_u^v = q^{a_v}\prod_{n=1}^{\infty}(1+q^{n+v-1}e^{2\pi i(\frac{1}{2}-u)})(1+q^{n-v}e^{-2\pi i(\frac{1}{2}-u)}). \tag{12.65}$$

Conventionally this is expressed in terms of Jacobi theta functions

$$Z_u^v = e^{2\pi i(v-\frac{1}{2})(u-\frac{1}{2})}\frac{\theta\begin{bmatrix}\frac{1}{2}-v\\ u-\frac{1}{2}\end{bmatrix}}{\eta(\tau)} \tag{12.66}$$

where the Dedekind eta function is

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n). \tag{12.67}$$

The total contribution is then trivially given by the product of the individual contributions since they commute;

$$Z_U^V(\tau) = \prod_{j=1}^{4}\overline{Z}_{u_j}^{v_j}(\tau)\prod_{J=1}^{16}Z_{u_j}^{v_j}(\tau) \tag{12.68}$$

For the complete one loop partition function we must include bosons and in this case there are 8 real bosons for the left and right movers. In addition we sum over all possible sets of boundary conditions, and include a (very important) overall phase factor C_U^V (always allowed) when we add the contribution from different sectors

$$Z_1(\tau) = \sum_{\{all\,U,V\}}|\eta(\tau)|^{-16}Im(\tau)^{-4}(-1)^{f_\alpha}C_U^V Z_U^V(\tau). \tag{12.69}$$

where the $(-1)^{f_\alpha}$ is a factor of -1 for states that are space-time fermions. As we will now see, the phase factors C_U^V can be chosen to give modular invariance.

C.2 Modular properties of the partition function

Remember that the point was to now apply the modular invariance condition.

$$Z_1(\tau) = Z_1(-1/\tau) = Z_1(\tau+1) \tag{12.70}$$

The individual transformation properties of the eta and theta functions are well known

$$\begin{aligned}
\eta(\tau+1) &= e^{\pi i/12}\eta(\tau) \\
\eta(-1/\tau) &= \sqrt{-i\tau}\,\eta(\tau) \\
\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau+1) &= e^{-i\pi\alpha(\alpha-1)}\theta\begin{bmatrix}\alpha\\ \alpha+\beta-\frac{1}{2}\end{bmatrix}(\tau) \\
\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(-1/\tau) &= \sqrt{-i\tau}\,e^{2\pi i\alpha\beta}\theta\begin{bmatrix}\beta\\ -\alpha\end{bmatrix}(\tau)
\end{aligned}$$

Using these it is straightforward to show that

$$\begin{aligned}
Z_u^v(\tau+1) &= e^{\pi i(v^2-v+\frac{1}{6})}Z_{u-v}^u(\tau) \\
Z_u^v(-1/\tau) &= e^{2\pi i(u-\frac{1}{2})(v-\frac{1}{2})}Z_{-v}^u(\tau)
\end{aligned}$$

or collecting all the contributions together

$$\begin{aligned}
Z_U^V(\tau+1) &= e^{\pi i(V.V-W_0.V)}\,Z_{U-V}^U(\tau) \\
Z_U^V(-1/\tau) &= e^{2\pi i(V.U-W_0.(U+V))}\,Z_{-V}^U(\tau)
\end{aligned}$$

where

$$W_0 = [(\frac{1}{2})^4\ (\frac{1}{2})^8(\frac{1}{2})^8], \tag{12.71}$$

defines the NS sector which as we have seen gives us the graviton. (Note that all phases appearing in Z are to be taken mod(1) - e.g. $-1/3 \equiv 2/3$.)

Now consider the effect of $\tau \to -1/\tau$. This sends the fermionic part of the p.f. to

$$\sum_{\{all\,U,V\}} C_U^V Z_U^V(-1/\tau) = \sum_{\{all\,U,V\}} e^{2\pi i(V.U-W_0.(U+V))}C_U^V Z_{-V}^U(\tau). \tag{12.72}$$

But then since we sum over all boundary conditions, we can trivially write

$$\sum_{\{all\,U,V\}} C_U^V Z_U^V(\tau) = \sum_{\{all\,U,V\}} C_{-V}^U Z_{-V}^U(\tau). \tag{12.73}$$

Comparing the two expressions, the partition function is invariant if we choose C_V^U such that

$$C_{-V}^U = e^{2\pi i(V.U-W_0.(U+V))}C_U^V \tag{12.74}$$

for all sectors in the model. Likewise invariance under $\tau \longrightarrow \tau+1$ requires that

$$C_{U-V}^V = e^{2\pi i(\frac{1}{2}V.V-W_0.V)}C_U^V \tag{12.75}$$

for all sectors.

C.3 Proof of modular invariance

To show modular invariance for the models outlined in the text, we now adopt those rules. That is we define the sum over sectors by using a basis of vectors W_a and writing

$$V = \alpha_a W_a \ ; \ U = \beta_a W_a \tag{12.76}$$

where a is summed over. The fermionic part of the PF can be written

$$Z_{fermion} = \sum_{\{\alpha_a,\beta_a\}}^{m_a,m_a} (-1)^{f_\alpha} C^\alpha_\beta Z^V_U(\tau). \tag{12.77}$$

Consider where this expression came from (i.e. write it before evaluating the trace)

$$Z_{fermion} = \sum_{\{\alpha_a,\beta_a\}}^{m_a,m_a} (-1)^{f_\alpha}\, Tr\left(C^\alpha_\beta q^{H_V} e^{2\pi i(\delta_{c0}-\beta_c)W_c.N_V}\right), \tag{12.78}$$

where we sum over c. It is the sum over β_c that is giving us the projection in the text simply because we are summing over all β_c's and $\sum_{\beta_c=1}^{m_c} e^{2\pi i \beta_c \frac{l}{m_c}} = \delta_{l0}$ for integer l. The modular invariance projection in the rules means that we must have

$$C^\alpha_\beta = e^{2\pi i(\delta_{c0}-\beta_c)(k_{cb}\alpha_b + w^1_c + k_{0c} - W_c.V)} \tag{12.79}$$

To repeat the argument, the contribution to the partition function vanishes unless the term in the exponent summed over β's is zero, thereby enforcing the projection on states that we have presented in the text. It is a straighforward (but tedious) exercise to show, using the model building rules in the main text, that this expression for C^V_U satisfies the conditions for modular invariance derived above. The more masochistic reader may like to do this.

Appendix D: Simple modular invariance for type II

The type II theories allow a particularly simple proof of modular invariance, as there are only 4 sectors. Here we will verify that the rules in the text give modular invariance. First let us infer from the projection rules what the form of the partition function is. As in the text we define the sum over sectors by using a basis of vectors W_a and writing

$$V = \alpha_a W_a \; ; \; U = \beta_a W_a \tag{12.80}$$

where $a = 0,1$ is summed over, and $\alpha_a, \beta_a = 0,1$. Assume that the fermionic part of the PF can be written

$$Z_1(\tau) = \frac{1}{4} \sum_{\{\alpha_a,\beta_a\}}^{1,1} |\eta(\tau)|^{-16} Im(\tau)^{-4} (-1)^{f_\alpha} C^\alpha_\beta Z^V_U(\tau). \tag{12.81}$$

where the first piece is the bosonic contribution and $(-1)^{f_\alpha}$ is ± 1 for states that are space-time bosons/fermions. Consider the fermionic contribution written out before evaluating the traces;

$$Z_{fermion}(\tau) = \frac{1}{4} \sum_{\{\alpha_a,\beta_a\}}^{1,1} Tr\left((-1)^{f_\alpha} C^\alpha_\beta q^{H_V} e^{2\pi i(\delta_{c0}-\beta_c) W_c . N_V} \right), \tag{12.82}$$

where we sum over c. It is the sum over β_c that is giving us the projection in the text for the following reason. If we write $C^\alpha_\beta = e^{2\pi i(\delta_{c0}-\beta_c)\phi_c}$ and we have $W_c.N_V + \phi_c = odd/2$ for any c, then the sum $\beta_c = 0,1$ gives a factor $1-1=0$, and that particular state cannot contribute to the partition function and isn't in the spectrum. On the other hand states that satisfy $W_c.N_V + \phi_c = integer$ for all c contribute 1 to the partition function. The partition function is then doing its job of counting physical states, and we can interpret the rules in the text as implying

$$C^\alpha_\beta = e^{2\pi i(\delta_{c0}-\beta_c)(k_{cb}\alpha_b + w^1_c + k_{0c} - W_c.V)} \tag{12.83}$$

since this reproduces the right projection.

Now we just need to evaluate the partition function with this C^α_β and show it is modular invariant. For the type II models C^α_β simplifies. Indeed we can substitiute in the right hand side of the projections we worked out in the text, to find

$$\begin{aligned}
C^{0,0}_{1,0} &= 1 \\
C^{0,0}_{0,1} &= -(-1)^{2(k_{00}+k_{01})} \\
C^{1,0}_{1,0} &= 1 \\
C^{1,0}_{0,1} &= -1 \\
C^{0,1}_{1,0} &= 1 \\
C^{0,1}_{0,1} &= -(-1)^{2(k_{00}+k_{01})} \\
C^{1,1}_{1,0} &= 1 \\
C^{1,1}_{0,1} &= -(-1)^{2(k_{11}+k_{01})}
\end{aligned}$$

$$\begin{aligned}
C^{0,0}_{0,0} &= (-1)^{2k_{00}}\\
C^{0,0}_{1,1} &= -(-1)^{2k_{01}}\\
C^{1,0}_{0,0} &= 1\\
C^{1,0}_{1,1} &= -1\\
C^{0,1}_{0,0} &= (-1)^{2(k_{01}+k_{00})}\\
C^{0,1}_{1,1} &= -(-1)^{2(k_{11}+k_{01})}\\
C^{1,1}_{0,0} &= (-1)^{2k_{01}}\\
C^{1,1}_{1,1} &= -(-1)^{2k_{11}}
\end{aligned}$$

The different choices of k_{01} just give an overall definition of chirality and without loss of generality we can take $k_{01} = k_{11} = 0$.

We now need to evaluate the partition function. By performing the trace as described at the beginning of Appendix C, we express it in terms of Jacobi theta functions

$$Z^{v}_{u} = e^{2\pi i(v-\frac{1}{2})(u-\frac{1}{2})} \frac{\theta\begin{bmatrix}\frac{1}{2}-v\\ u-\frac{1}{2}\end{bmatrix}}{\eta(\tau)} \tag{12.84}$$

where the Dedekind eta function is

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n). \tag{12.85}$$

The total contribution is then given by the product of the individual contributions since they commute;

$$Z^{V}_{U}(\tau) = \prod_{j=1}^{4}\overline{Z}^{v_j}_{u_j}(\tau)\prod_{J=1}^{4}Z^{v_j}_{u_j}(\tau) \tag{12.86}$$

Performing the sums over α,s and β's and including the $(-1)^{f_\alpha}C^{\alpha}_{\beta}$ prefactors, gives an expression that can be factorized into left moving and right moving parts;

$$Z_1(\tau) = \frac{|\eta(\tau)|^{-16}Im(\tau)^{-4}}{4}\left[(-1)^{2k_{00}}(Z^0_0)^4 - (Z^{\frac{1}{2}}_0)^4 - (Z^0_{\frac{1}{2}})^4 + (Z^{\frac{1}{2}}_{\frac{1}{2}})^4\right]\times\left[(\overline{Z}^0_0)^4 - (\overline{Z}^{\frac{1}{2}}_0)^4 - (\overline{Z}^0_{\frac{1}{2}})^4 + (\overline{Z}^{\frac{1}{2}}_{\frac{1}{2}})^4\right], \tag{12.87}$$

where as we saw in the text, $k_{00} = 0, \frac{1}{2}$ for type IIB,A respectively. This expression should be modular invariant

$$Z_1(\tau) = Z_1(-1/\tau) = Z_1(\tau+1) \tag{12.88}$$

The modular properties of jacobi theta and dedekind eta functions can be looked up. They are written in Appendix C.2 and substituting them into the above expressions does indeed give invariance.

Exercise: substitute the modular transformations in Appendix C.2 into the above to prove modular invariance.

Appendix E: Some lie algebra definitions

The Lie algebra determines the local structure of the gauge group in our theories, and hence the number and structure of generators and so forth. The groups referred to in the text are ...

- $SO(n)$: the group of real orthogonal $n \times n$ matrices of determinant 1. The generators are antisymmetric Hermitian matrices. There are $n(n-1)/2$ independent entries in such a matrix, and hence the same number of generators

- $SU(n)$: the group of unitary matrices with determinant one. The generators are traceless hermitian $n \times n$ matrices, which have $n^2 - 1$ independent elements, and hence there are the same number of generators.

- $Sp(k)$: The symplectic groups are the groups consisting of unitary matrices, U, that satisfy $MUM^{-1} = (U^T)^{-1}$ where $M = i \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$ where I_k is the $k \times k$ identity matrix. They are generated by $2k \times 2k$ matrices, T, that satisfy $MTM^{-1} = T^T$.

- The generators of the exceptional groups E_6, E_7, E_8 can be decomposed into representations of the corresponding maximal subgroups. For example generators of E_8 consist of the adjoint $16(16-1)/2$ of $SO(16)$ plus a single chirality of the fermionic representation 2^7 giving 248 generators for E_8. For E_7 we can decompose it into representation of $U(1) \times SO(12)$; the single boson of $U(1) = 1$, both chiralities of $SO(12)$ fermions $= 2^6 = 64$ and the adjoint of $SO(12) = 12.11/2 = 66$, giving 133 generators in total. Finally we can decompose the generators of E_6 into representations of $U(1) \times SO(10)$ in exactly the same way $\rightarrow 1 + 45 + 32 = 78$ generators. The decomposition of E_8 was made evident in the text, so if nothing else you can use the heterotic string to help you decompose large groups!

References

[1] M. B. Green, J. H. Schwarz, and E. Witten, "Superstring Theory. vols. I and II", CUP, (1987)

[2] J. Polchinski, "String Theory", CUP (1998); M. Headrick, "Solutions to selected problems from Polchinski's String Theory", `http://www2.lns.mit.edu/ headrick/physics/polchinski/chapter7.ps`

[3] M. E. Peskin, "INTRODUCTION TO STRING AND SUPERSTRING THEORY", Lectures presented at the 1986 Theoretical Advanced Study Institute in Particle Physics, Santa Cruz, Calif., Jun 23 - Jul 19, (1986), SLAC-PUB-4251

[4] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, "The hierarchy problem and new dimensions at a millimeter," Phys. Lett. B **429** (1998) 263 [arXiv:hep-ph/9803315]; I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, "New dimensions at a millimeter to a Fermi and superstrings at a TeV," Phys. Lett. B **436** (1998) 257 [arXiv:hep-ph/9804398]; N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, "Phenomenology, astrophysics and cosmology of theories with sub-millimeter dimensions and TeV scale quantum gravity," Phys. Rev. D **59** (1999) 086004 [arXiv:hep-ph/9807344].

[5] M. Berkooz, M. R. Douglas and R. G. Leigh, "Branes intersecting at angles," Nucl. Phys. B **480** (1996) 265 [arXiv:hep-th/9606139].

[6] L. E. Ibáñez, H. P. Nilles and F. Quevedo, Phys. Lett. **B187** (1987), 25; L. E. Ibáñez, J. E. Kim, H. P. Nilles and F. Quevedo, Phys. Lett. **B191** (1987), 282; For reviews and references, see D. Bailin and A. Love, "Orbifold compactifications of string theory," Phys. Rept. **315** (1999) 285;

[7] H. Kawai, D. C. Lewellen and S. H. H. Tye, "Construction of Four-Dimensional Fermionic String Models," Phys. Rev. Lett. **57**, 1832 (1986) [Erratum-ibid. **58**, 429 (1987)].

[8] H. Kawai, D. C. Lewellen and S. H. H. Tye, "Construction of Fermionic String Models in Four-Dimensions," Nucl. Phys. B **288**, 1 (1987).

[9] I. Antoniadis, C. P. Bachas and C. Kounnas, "Four-Dimensional Superstrings," Nucl. Phys. B **289**, 87 (1987).

[10] C. P. Bachas, "Lectures on D-branes," arXiv:hep-th/9806199.

[11] D. Youm, "Black holes and solitons in string theory," Phys. Rept. **316**, 1 (1999) [arXiv:hep-th/9710046].

[12] L. J. Dixon, J. A. Harvey, C. Vafa and E. Witten, "Strings On Orbifolds," Nucl. Phys. B **261**, 678 (1985).

[13] L. J. Dixon, J. A. Harvey, C. Vafa and E. Witten, "Strings On Orbifolds. 2," Nucl. Phys. B **274**, 285 (1986).

[14] D. Bailin, "Introduction to string theory," Surveys High Energ. Phys. **11**, 97 (1997).

[15] P. Candelas, G. T. Horowitz, A. Strominger and E. Witten, "Vacuum Configurations For Superstrings," Nucl. Phys. B **258**, 46 (1985).

[16] L. J. Dixon, D. Friedan, E. J. Martinec and S. H. Shenker, "The Conformal Field Theory Of Orbifolds," Nucl. Phys. B **282** (1987) 13; S. Hamidi and C. Vafa, Nucl. Phys. B **279** (1987) 465.

[17] H. P. Nilles, "Dynamically Broken Supergravity And The Hierarchy Problem," Phys. Lett. B **115**, 193 (1982); S. Ferrara, L. Girardello and H. P. Nilles, "Breakdown Of Local Supersymmetry Through Gauge Fermion Condensates," Phys. Lett. B **125**, 457 (1983); J. P. Derendinger, L. E. Ibanez and

H. P. Nilles, "On The Low-Energy D = 4, N=1 Supergravity Theory Extracted From The D = 10, N=1 Superstring," Phys. Lett. B **155**, 65 (1985); M. Dine, R. Rohm, N. Seiberg and E. Witten, "Gluino Condensation In Superstring Models," Phys. Lett. B **156**, 55 (1985); C. Kounnas and M. Porrati, "Duality And Gaugino Condensation In Superstring Models," Phys. Lett. B **191**, 91 (1987).

[18] J.A. Casas, Z. Lalak, C. Muñoz and G.G. Ross, Nucl. Phys. B347 (1990) 243; A. Font, L. Ibáñez, D. Lüst and F. Quevedo, Phys. Lett. B245 (1990) 401; M. Cvetič, A. Font, L. Ibáñez, D. Lüst and F. Quevedo, Nucl. Phys. B361 (1991) 194; S. Ferrara, N. Magnoli, T.R. Taylor and G. Veneziano, Phys. Lett. B245 (1990) 409; H.P. Nilles and M. Olechowsky, Phys. Lett. B248 (1990) 268; P. Binétruy and M.K. Gaillard, Phys. Lett. B253 (1991) 119; D. Lüst and T.R. Taylor, Phys. Lett. B253 (1991) 335; B. de Carlos, J.A. Casas and C. Muñoz, Phys. Lett. B263 (1991) 248; B. de Carlos, J. A. Casas and C. Munoz, "Soft Susy Breaking Terms In Stringy Scenarios: Computation And Phenomenological Viability," Phys. Lett. B **299**, 234 (1993) [arXiv:hep-ph/9211266]; B. de Carlos, J. A. Casas and C. Munoz, "Supersymmetry breaking and determination of the unification gauge coupling constant in string theories," Nucl. Phys. B **399**, 623 (1993) [arXiv:hep-th/9204012];

[19] I. Affleck, M. Dine and N. Seiberg, "Dynamical Supersymmetry Breaking In Four-Dimensions And Its Phenomenological Implications," Nucl. Phys. B **256**, 557 (1985); I. Affleck, M. Dine and N. Seiberg, "Calculable Nonperturbative Supersymmetry Breaking," Phys. Rev. Lett. **52**, 1677 (1984); I. Affleck, M. Dine and N. Seiberg, "Dynamical Supersymmetry Breaking In Chiral Theories," Phys. Lett. B **137**, 187 (1984); I. Affleck, M. Dine and N. Seiberg, "Dynamical Supersymmetry Breaking In Supersymmetric QCD," Nucl. Phys. B **241**, 493 (1984).

[20] K. A. Intriligator and N. Seiberg, "Lectures on supersymmetric gauge theories and electric-magnetic duality," Nucl. Phys. Proc. Suppl. **45BC** (1996) 1 [arXiv:hep-th/9509066].

[21] P. H. Ginsparg, "Gauge And Gravitational Couplings In Four-Dimensional String Theories," Phys. Lett. B **197**, 139 (1987).

[22] P. Horava and E. Witten, Nucl. Phys. **B460** (1996) 506; Nucl. Phys. **B475** (1996) 94.

[23] See review by S. Raby in W. M. Yao *et al.* [Particle Data Group], "Review of particle physics," J. Phys. G **33** (2006) 1.

[24] T. Kobayashi, S. Raby and R.-J. Zhang, Phys. Lett. **B593** (2004), 262 [hep-ph/0403065]

[25] S. Förste, H. P. Nilles, P. K. S. Vaudrevange and A. Wingerter, Phys. Rev. **D70** (2004), 106008 [hep-th/0406208]; T. Kobayashi, S. Raby and R.-J. Zhang, Nucl. Phys. **B704** (2005), 3 [hep-ph/0409098]; A. Hebecker and M. Trapletti, "Gauge unification in highly anisotropic string compactifications," Nucl. Phys. B **713** (2005) 173 [hep-th/0411131]; W. Buchmüller, K. Hamaguchi, O. Lebedev and M. Ratz, Nucl. Phys. **B712** (2005), 139 [hep-ph/0412318]; W. Buchmüller, K. Hamaguchi, O. Lebedev and M. Ratz, [hep-ph/0512326]; W. Buchmüller, K. Hamaguchi, O. Lebedev and M. Ratz, Phys. Rev. Lett. **96** (2006), 121602 [hep-ph/0511035]; W. Buchmüller, K. Hamaguchi, O. Lebedev and M. Ratz, Nucl. Phys. B **785**, 149 (2007) [hep-th/0606187]; O. Lebedev, H. P. Nilles, S. Raby, S. Ramos-Sanchez, M. Ratz, P. K. S. Vaudrevange and A. Wingerter, "A mini-landscape of exact MSSM spectra in heterotic orbifolds," Phys. Lett. B **645** (2007) 88 [hep-th/0611095];

[26] W. Buchmuller, C. Ludeling and J. Schmidt, "Local SU(5) Unification from the Heterotic String," JHEP **0709**, 113 (2007) [arXiv:0707.1651 [hep-ph]].

[27] A. Faraggi, Nucl. Phys. B**387** (1992) 239; A. Faraggi and D.V. Nanopoulos, Phys. Rev.**D48** (1993) 3288.

[28] A. E. Faraggi, D. V. Nanopoulos and K. j. Yuan, “A Standard Like Model in the 4D Free Fermionic String Formulation,” Nucl. Phys. B **335** (1990) 347.

[29] A. E. Faraggi, E. Manno and C. Timirgaziu, “Minimal standard heterotic string models,” Eur. Phys. J. C **50** (2007) 701 [arXiv:hep-th/0610118].

[30] J. Polchinski, “Lectures on D-branes,” arXiv:hep-th/9611050; C.V.Johnson, D-branes, CUP (2003)

[31] E. G. Gimon and J. Polchinski, “Consistency Conditions for Orientifolds and D-Manifolds,” Phys. Rev. D **54** (1996) 1667 [arXiv:hep-th/9601038].

[32] G. Aldazabal, L. E. Ibanez, F. Quevedo and A. M. Uranga, “D-branes at singularities: A bottom-up approach to the string embedding of the standard model,” JHEP **0008**, 002 (2000) [arXiv:hep-th/0005067].

[33] K. Benakli, “Phenomenology of low quantum gravity scale models,” Phys. Rev. D **60**, 104002 (1999) [arXiv:hep-ph/9809582]; C. P. Burgess, L. E. Ibanez and F. Quevedo, “Strings at the intermediate scale or is the Fermi scale dual to the Planck scale?,” Phys. Lett. B **447**, 257 (1999) [arXiv:hep-ph/9810535].

[34] L. E. Ibanez, C. Munoz and S. Rigolin, “Aspects of type I string phenomenology,” Nucl. Phys. B **553** (1999) 43 [arXiv:hep-ph/9812397].

[35] W. Fischler and L. Susskind, “Dilaton Tadpoles, String Condensates And Scale Invariance,” Phys. Lett. B **171**, 383 (1986); Phys. Lett. B **173**, 262 (1986)

[36] S.A.Abel, B.C.Allanach, F.Quevedo, L.Ibanez, M.Klein, JHEP 0012:026,2000

[37] C. Bachas, “A Way to break supersymmetry,” arXiv:hep-th/9503030.

[38] M. Cvetic, G. Shiu and A. M. Uranga, “Three-family supersymmetric standard like models from intersecting brane worlds,” Phys. Rev. Lett. **87**, 201801 (2001) [arXiv:hep-th/0107143]; M. Cvetic, G. Shiu and A. M. Uranga, “Chiral four-dimensional N = 1 supersymmetric type IIA orientifolds from intersecting D6-branes,” Nucl. Phys. B **615**, 3 (2001) [arXiv:hep-th/0107166]; M. Cvetic, G. Shiu and A. M. Uranga, “Chiral type II orientifold constructions as M theory on G(2) holonomy spaces,” arXiv:hep-th/0111179; D. Cremades, L. E. Ibanez and F. Marchesano, “SUSY quivers, intersecting branes and the modest hierarchy problem,” JHEP **0207**, 009 (2002) [arXiv:hep-th/0201205]; M. Cvetic, P. Langacker and G. Shiu, “Phenomenology of a three-family standard-like string model,” Phys. Rev. D **66**, 066004 (2002) [arXiv:hep-ph/0205252]; R. Blumenhagen, L. Gorlich and T. Ott, “Supersymmetric intersecting branes on the type IIA T**6/Z(4) orientifold,” JHEP **0301** (2003) 021 [arXiv:hep-th/0211059]; M. Cvetic, I. Papadimitriou and G. Shiu, “Supersymmetric three family SU(5) grand unified models from type IIA orientifolds with intersecting D6-branes,” arXiv:hep-th/0212177; G. Honecker, “Chiral supersymmetric models on an orientifold of Z(4) x Z(2) with intersecting D6-branes,” Nucl. Phys. B **666**, 175 (2003) [arXiv:hep-th/0303015]; M. Cvetic and I. Papadimitriou, “More supersymmetric standard-like models from intersecting D6-branes on type IIA orientifolds,” Phys. Rev. D **67**, 126006 (2003) [arXiv:hep-th/0303197]; M. Cvetic, P. Langacker and J. Wang, “Dynamical supersymmetry breaking in standard-like models with intersecting D6-branes,” Phys. Rev. D **68**, 046002 (2003) [arXiv:hep-th/0303208]; K. Behrndt and M. Cvetic, “Supersymmetric intersecting D6-branes and fluxes in massive type IIA string theory,” arXiv:hep-th/0308045; C. Kokorelis, “N = 1 locally supersymmetric standard models from intersecting branes,” arXiv:hep-th/0309070.

[39] R. Blumenhagen, L. Goerlich, B. Kors and D. Lust, “Noncommutative compactifications of type I strings on tori with magnetic background flux,” JHEP **0010** (2000) 006 [arXiv:hep-th/0007024].

[40] S. Forste, G. Honecker and R. Schreyer, "Orientifolds with branes at angles," JHEP **0106**, 004 (2001) [arXiv:hep-th/0105208].

[41] G. Aldazabal, S. Franco, L. E. Ibanez, R. Rabadan and A. M. Uranga, "D = 4 chiral string compactifications from intersecting branes," J. Math. Phys. **42** (2001) 3103 [arXiv:hep-th/0011073]; G. Aldazabal, S. Franco, L. E. Ibanez, R. Rabadan and A. M. Uranga, "Intersecting brane worlds," JHEP **0102** (2001) 047 [arXiv:hep-ph/0011132].

[42] R. Blumenhagen, L. Gorlich, B. Kors and D. Lust, "Asymmetric orbifolds, noncommutative geometry and type I string vacua," Nucl. Phys. B **582** (2000) 44 [arXiv:hep-th/0003024]; S. Forste, G. Honecker and R. Schreyer, "Supersymmetric Z(N) x Z(M) orientifolds in 4D with D-branes at angles," Nucl. Phys. B **593** (2001) 127 [arXiv:hep-th/0008250].

[43] L. E. Ibanez, F. Marchesano and R. Rabadan, "Getting just the standard model at intersecting branes," JHEP **0111** (2001) 002 [arXiv:hep-th/0105155].

[44] R. Rabadan, "Branes at angles, torons, stability and supersymmetry," Nucl. Phys. B **620** (2002) 152 [arXiv:hep-th/0107036]; R. Blumenhagen, B. Kors, D. Lust and T. Ott, "The standard model from stable intersecting brane world orbifolds," Nucl. Phys. B **616** (2001) 3 [arXiv:hep-th/0107138]; G. Honecker, "Intersecting brane world models from D8-branes on (T(2) x T(4)/Z(3))/Omega R(1) type IIA orientifolds," JHEP **0201**, 025 (2002) [arXiv:hep-th/0201037]; R. Blumenhagen, D. Lust and T. R. Taylor, "Moduli stabilization in chiral type IIB orientifold models with fluxes," Nucl. Phys. B **663** (2003) 319 [arXiv:hep-th/0303016].

[45] D. Cremades, L. E. Ibanez and F. Marchesano, "Intersecting brane models of particle physics and the Higgs mechanism," JHEP **0207**, 022 (2002) [arXiv:hep-th/0203160].

[46] C. Kokorelis, "GUT model hierarchies from intersecting branes," JHEP **0208** (2002) 018 [arXiv:hep-th/0203187]; "New standard model vacua from intersecting branes," JHEP **0209** (2002) 029 [arXiv:hep-th/0205147]; "Exact standard model compactifications from intersecting branes," JHEP **0208** (2002) 036 [arXiv:hep-th/0206108]; "Exact standard model structures from intersecting D5-branes," arXiv:hep-th/0207234; "Deformed intersecting D6-brane GUTs. I," JHEP **0211** (2002) 027 [arXiv:hep-th/0209202]; "Deformed intersecting D6-brane GUTs. II," arXiv:hep-th/0210200; "Deformed intersecting D6-brane GUTs and N = 1 SUSY," arXiv:hep-th/0212281; M. Axenides, E. Floratos and C. Kokorelis, "SU(5) unified theories from intersecting branes," arXiv:hep-th/0307255;

[47] J. Garcia-Bellido, R. Rabadan and F. Zamora, "Inflationary scenarios from branes at angles," JHEP **0201** (2002) 036 [arXiv:hep-th/0112147]; R. Blumenhagen, B. Kors, D. Lust and T. Ott, "Hybrid inflation in intersecting brane worlds," Nucl. Phys. B **641** (2002) 235 [arXiv:hep-th/0202124]; M. Gomez-Reino and I. Zavala, "Recombination of intersecting D-branes and cosmological inflation," JHEP **0209** (2002) 020 [arXiv:hep-th/0207278].

[48] D. Cremades, L. E. Ibanez and F. Marchesano, "Yukawa couplings in intersecting D-brane models," JHEP **0307** (2003) 038 [arXiv:hep-th/0302105].

[49] S. A. Abel, M. Masip and J. Santiago, "Flavour changing neutral currents in intersecting brane models," JHEP **0304** (2003) 057 [arXiv:hep-ph/0303087].

[50] N. Chamoun, S. Khalil and E. Lashin, "Fermion masses and mixing in intersecting branes scenarios," arXiv:hep-ph/0309169.

[51] D. M. Ghilencea, L. E. Ibanez, N. Irges and F. Quevedo, "TeV-scale Z' bosons from D-branes," JHEP **0208** (2002) 016 [arXiv:hep-ph/0205083]; D. M. Ghilencea, "U(1) masses in intersecting D-brane SM-like models," Nucl. Phys. B **648** (2003) 215 [arXiv:hep-ph/0208205].

[52] F. del Aguila, M. Perez-Victoria and J. Santiago, "Effective description of quark mixing," Phys. Lett. B **492** (2000) 98 [arXiv:hep-ph/0007160]; F. del Aguila, M. Perez-Victoria and J. Santiago, "Observable contributions of new exotic quarks to quark mixing," JHEP **0009** (2000) 011 [arXiv:hep-ph/0007316].

[53] R. Blumenhagen, V. Braun, B. Kors and D. Lust, "Orientifolds of K3 and Calabi-Yau manifolds with intersecting D-branes," JHEP **0207** (2002) 026 [arXiv:hep-th/0206038]; A. M. Uranga, "Local models for intersecting brane worlds," JHEP **0212** (2002) 058 [arXiv:hep-th/0208014].

[54] D. Cremades, L. E. Ibanez and F. Marchesano, "Standard model at intersecting D5-branes: Lowering the string scale," Nucl. Phys. B **643** (2002) 93 [arXiv:hep-th/0205074].

[55] D. Bailin, G. V. Kraniotis and A. Love, "Standard-like models from intersecting D4-branes," Phys. Lett. B **530** (2002) 202 [arXiv:hep-th/0108131]; D. Bailin, G. V. Kraniotis and A. Love, "New standard-like models from intersecting D4-branes," Phys. Lett. B **547** (2002) 43 [arXiv:hep-th/0208103]; D. Bailin, G. V. Kraniotis and A. Love, "Standard-like models from intersecting D5-branes," Phys. Lett. B **553** (2003) 79 [arXiv:hep-th/0210219]. D. Bailin, G. V. Kraniotis and A. Love, "Intersecting D5-brane models with massive vector-like leptons," JHEP **0302** (2003) 052 [arXiv:hep-th/0212112].

[56] R. Blumenhagen, D. Lust and S. Stieberger, "Gauge unification in supersymmetric intersecting brane worlds," JHEP **0307** (2003) 036 [arXiv:hep-th/0305146].

[57] D. Lust and S. Stieberger, "Gauge threshold corrections in intersecting brane world models," arXiv:hep-th/0302221.

[58] A. M. Uranga, "Chiral four-dimensional string compactifications with intersecting D-branes," Class. Quant. Grav. **20**, S373 (2003) [arXiv:hep-th/0301032].

[59] R. Blumenhagen, M. Cvetic, P. Langacker and G. Shiu, "Toward realistic intersecting D-brane models," Ann. Rev. Nucl. Part. Sci. **55**, 71 (2005) [arXiv:hep-th/0502005].

[60] M. Cvetic and I. Papadimitriou, "Conformal field theory couplings for intersecting D-branes on orientifolds," Phys. Rev. D **68** (2003) 046001 [arXiv:hep-th/0303083]; S. A. Abel and A. W. Owen, "Interactions in intersecting brane models," Nucl. Phys. B **663** (2003) 197 [arXiv:hep-th/0303124]; S. A. Abel and A. W. Owen, "N-point amplitudes in intersecting brane models," Nucl. Phys. B **682**, 183 (2004) [arXiv:hep-th/0310257].

[61] V. A. Rubakov and M. E. Shaposhnikov, "Do We Live Inside A Domain Wall?," Phys. Lett. B **125** (1983) 136; I. Antoniadis, "A Possible New Dimension At A Few Tev," Phys. Lett. B **246** (1990) 377; I. Antoniadis and K. Benakli, "Limits on extra dimensions in orbifold compactifications of superstrings," Phys. Lett. B **326** (1994) 69 [arXiv:hep-th/9310151]; A. Pomarol and M. Quiros, "The standard model from extra dimensions," Phys. Lett. B **438** (1998) 255 [arXiv:hep-ph/9806263].

[62] N. Arkani-Hamed and M. Schmaltz, "Hierarchies without symmetries from extra dimensions," Phys. Rev. D **61** (2000) 033005 [arXiv:hep-ph/9903417].

[63] A. Delgado, A. Pomarol and M. Quiros, "Electroweak and flavor physics in extensions of the standard model with large extra dimensions," JHEP **0001** (2000) 030 [arXiv:hep-ph/9911252]; C. D. Carone, "Electroweak constraints on extended models with extra dimensions," Phys. Rev. D **61** (2000) 015008 [arXiv:hep-ph/9907362].

[64] S. J. Huber, "Flavor violation and warped geometry," Nucl. Phys. B **666** (2003) 269 [arXiv:hep-ph/0303183].

[65] F. Del Aguila and J. Santiago, "Signals from extra dimensions decoupled from the compactification scale," JHEP **0203** (2002) 010 [arXiv:hep-ph/0111047].

[66] P. Langacker and M. Plumacher, "Flavor changing effects in theories with a heavy Z' boson with family non-universal couplings," Phys. Rev. D **62** (2000) 013006 [arXiv:hep-ph/0001204].